# BOUNDARY STABILIZATION AND DISTURBANCE REJECTION FOR AN UNSTABLE TIME FRACTIONAL DIFFUSION-WAVE EQUATION

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**Abstract.** In this paper, we study boundary stabilization and disturbance rejection problem for an unstable time fractional diffusion-wave equation with Caputo time fractional derivative. For the case of no boundary external disturbance, both state feedback control and output feedback control *via* Neumann boundary actuation are proposed by the classical backstepping method. It is proved that the state feedback makes the closed-loop system Mittag-Leffler stable and the output feedback makes the closed-loop system asymptotically stable. When there is boundary external disturbance, we propose a disturbance estimator constructed by two infinite dimensional auxiliary systems to recover the external disturbance. A novel control law is then designed to compensate for the external disturbance in real time, and rigorous mathematical proofs are presented to show that the resulting closed-loop system is Mittag-Leffler stable and the states of all subsystems involved are uniformly bounded. As a result, we completely resolve, from a theoretical perspective, two long-standing unsolved mathematical control problems raised in Liang [*Nonlinear Dyn.* **38** (2004) 339–354] where all results were verified by simulations only.

## 1. INTRODUCTION

In this paper, we consider boundary stabilization and disturbance rejection problem for an unstable time fractional diffusion-wave equation with Neumann boundary control governed by

$$\begin{cases} {}^{C}_{0}D^{\alpha}_{t}u(x,t) = u_{xx}(x,t) + \lambda(x)u(x,t), \ x \in (0,1), t \geq 0, \\ u(0,t) = 0, \ \ u_x(1,t) = U(t), \ t \geq 0, \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ 0 \leq x \leq 1, \end{cases} \tag{1.1}$$

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where $\alpha \in (1, 2)$ is the order of the fractional derivative, $u(x, t)$ is the displacement of wave propagation, $\lambda(x)$ is a continuous function on $[0, 1]$, $U(t)$ is the control input, and ${}_0^C D_t^\alpha u(x, t)$ stands for the Caputo derivative with respect to time variable $t$. We consider system (1.1) in the state space $L^2(0, 1)$. The investigation on boundary stabilization problem of this class of fractional partial differential equations (PDEs) can be traced back to the 2004 paper [21], where a special case of system (1.1) with $\lambda(x) \equiv 0$ was studied. System (1.1) can be considered as a "special" cable, possibly made by some special smart materials, fixed at one end, and stabilized by a boundary controller at the other end. In [21], a boundary controller for system (1.1) with $\lambda(x) \equiv 0$ was designed as $U(t) = -ku_t(1, t)$ with $k > 0$, which is a classical negative velocity feedback control law based on the passive principle for the wave equation. In [22], the performance and properties of a fractional order boundary controller given by $U(t) = -k {}_0^C D_t^\mu u(1, t)$ with $k > 0$ for system (1.1) with $\lambda(x) \equiv 0$ were verified. However, we notice that the results in both [21] and [22] were only illustrated by numerical simulations without any theoretical proof. For system (1.1) with $\lambda(x) \equiv 0$, $\alpha = 2$ and $U(t) = -k {}_0^C D_t^\mu u(1, t), k > 0$ in [30], the asymptotic stability was proved by LaSalle's invariance principle and the fact that the input-output relationship of a class of generalized diffusion equation can be shown by fractional integrals. In the last a few years, stabilization for fractional time derivative PDEs has been attracted much attention. Recently, both the state feedback and output feedback for unstable time fractional reaction diffusion equations have been developed in [41] by the Riesz basis and backstepping methods. The output stabilization for a fractional reaction diffusion equation subject to spatially-varying diffusion coefficient has been developed in [4]. The Mittag-Leffler convergent observer based feedback control of a coupled semilinear subdiffusion system has been explored in [9]. For controllability aspect of fractional PDEs, an interesting result that fractional in time systems fail to be null controllable due to the long-tail effect of fractional derivative has been presented in [26]. However, the key techniques obtaining the stability in [4, 9, 41] are based on the fractional Lyapunov method established initially in [20] where the concept of Mittag-Leffler stability was introduced. A useful fractional derivative inequality: ${}_0^C D_t^\alpha x^2(t) \leq 2x(t) {}_0^C D_t^\alpha x(t)$ for $\alpha \in (0, 1)$ which is not known until 2014 in [1] is another powerful tool in establishing the stability. For the fractional Halanay inequality with time-varying delay, we refer to [18]. However, when $\alpha \in (1, 2)$, the inequality ${}_0^C D_t^\alpha x^2(t) \leq 2x(t) {}_0^C D_t^\alpha x(t)$ fails, which can be seen from a simple counter-example that when $\alpha = 1.5$ and $x(t) = t$, ${}_0^C D_t^\alpha x^2(t) = \frac{2}{\Gamma(1.5)} t^{1.5}$ and $x(t) {}_0^C D_t^\alpha x(t) \equiv 0$, which implies that ${}_0^C D_t^\alpha x^2(t) > 2x(t) {}_0^C D_t^\alpha x(t) = 0$ for $t > 0$. Therefore, the method used in [4, 10, 41] is not applicable for the fractional wave equation.

In most practical situations, systems operate in the environment suffering from different kinds of external disturbances or uncertainties. To maintain the performance of systems, the designed control law must be required to be robust against the disturbance or uncertainty in some extent. There have been many effective approaches to the control of systems with external disturbance or uncertainty, such as the adaptive control strategy for systems with unknown constants [6] or unknown amplitudes of harmonic disturbance [16], the sliding model control for one dimensional heat equation [37], wave equation with nonlinear boundary [25] and fractional order systems [2] that are all subject to boundary control matched disturbance, and the uncertainty and disturbance estimator (UDE)-based control for filtering the system input and state information [5]. Another remarkable powerful approach applied widely in dealing with external disturbance and uncertainty is the so-called active disturbance rejection control (ADRC) initially proposed by Han in the later 1980s [17]. Analogous to the adaptive control where the estimation/cancellation strategy is adopted, the ADRC can handle much more general disturbance that is estimated in real time by an extended state observer (ESO) and is compensated in the closed-loop by an ESO-based feedback control, which makes the control energy significantly reduced in practice [38]. Very recently, this emerging control technology has been shown to be highly efficient in stabilizing the one dimensional partial differential equations (PDEs) [7, 12], multi-dimensional PDEs [13, 39] and stochastic differential equations [14, 15], which are all subject to external disturbance, among many others. The investigation on ADRC for nonlinear fractional-order systems and fractional PDEs with fractional derivative order $\alpha \in (0, 1)$ also has been available in [8] and in our recent work [40], but ADRC for the case of $\alpha \in (1, 2)$ has not been yet studied. Disturbance rejection for fractional PDEs with $\alpha \in (1, 2)$ has been verified in [21] by numerical simulations without mathematical proof, which motivates this paper to establish the mathematical basis for this case.

We proceed as follows. The problem formulation and some preliminaries are presented in Section 2. The boundary state feedback control and boundary output feedback control for system (1.1) without disturbance are included in Sections 3 and 4 respectively. In Section 5, the boundary controller for system (1.1) subject to external disturbance is explored. A numerical simulation is presented in Section 6 followed up concluding remarks in Section 7.

## 2. Problem formation and preliminaries

In view of state of the arts of the research aforementioned in last section, the following two interesting and nontrivial problems arise naturally:

**Problem I:** Can we design an appropriate control law $U(t)$ to stabilize system (1.1) so that the closed-loop system is asymptotically stable?

When the boundary control $u_x(1, t) = U(t)$ of system (1.1) is replaced by $u_x(1, t) = U(t) + d(t)$ with $d(t)$ being the disturbance, we can then ask the second question:

**Problem II:** Can we propose an output feedback control law to system (1.1) by rejecting the disturbance $d(t)$ and maintaining the stability of the closed-loop system simultaneously?

Significantly, two long-standing unsolved problems left in the paper [21] are just the special cases of Problems I and II, where the plant is the system (1.1) with $\lambda(x) \equiv 0$ and the stability results have only been checked by numerical simulations without theoretical proofs.

Next, we present some preliminaries. The Caputo derivative of $w(x, t)$ with respect to time variable $t$ is defined by

$$
{}_0^C D_t^\alpha w(x, t) = I^{2-\alpha} \left[ \frac{\partial^2 w(x, t)}{\partial t^2} \right],
$$

where $I^{2-\alpha}$ is the Riemann-Liouville fractional integral operator given by

$$
I^{2-\alpha} w(x, t) = \frac{1}{\Gamma(2 - \alpha)} \int_0^t (t - s)^{1-\alpha} w(x, s) \mathrm{d}s.
$$

The time fractional diffusion-wave equation is perhaps one of the most important fractional order linear partial differential equations, which is expected to describe generalized processes which interpolate or extrapolate the classical phenomenon of diffusion [33]. In a physical model presented in [28], the fractional wave equation describes the propagation of mechanical diffusive waves in viscoelastic media. It is well known from the definition of the Caputo derivative that

$$
\lim_{\alpha \to 2^-} {}_0^C D_t^\alpha w(x, t) = \frac{\partial^2 w(x, t)}{\partial t^2}.
$$

This means that when $\alpha = 2$, the system (1.1) is reduced to the classic wave equations. To analyze the stability, we recall the two-parameter Mittag-Leffler function $E_{\delta, \gamma}(z)$ defined as

$$
E_{\delta, \gamma}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\delta k + \gamma)},
$$

where $\delta, \gamma > 0$ and $\Gamma(\cdot)$ is the Gamma function. Taking $\gamma = 1$, we define one-parameter Mittag-Leffler function $E_\delta(z) := E_{\delta, 1}(z)$. It is easy to see that $E_1(x) = e^x$, $E_2(-x^2) = \cos(x)$, and $E_{2,2}(-x^2) = \frac{\sin(x)}{x}$ for all $x \in \mathbb{R}$.

System (1.1) without control input is unstable when $\lambda(x)$ is sufficiently large, which can be seen from the following simple Example 2.1.

**Example 2.1.** Let $\lambda(x) = \frac{\pi^2}{2}$ and let the initial value be $u_0(x) = \sin(\frac{\pi}{2}x)$, $u_1(x) = \sin(\frac{\pi}{2}x)$. Then, system (1.1) without control admits a solution given by

$$u(x,t) = E_\alpha\left(\frac{\pi^2}{4}t^\alpha\right)\sin\left(\frac{\pi}{2}x\right) + tE_{\alpha,2}\left(\frac{\pi^2}{4}t^\alpha\right)\sin\left(\frac{\pi}{2}x\right),$$

which indicates that

$$\|u(\cdot,t)\|_{L^2(0,1)} = \frac{1}{2}\left(E_\alpha\left(\frac{\pi^2}{4}t^\alpha\right) + tE_{\alpha,2}\left(\frac{\pi^2}{4}t^\alpha\right)\right)^2$$

and

$$\|u_t(\cdot,t)\|_{L^2(0,1)} = \frac{1}{2}\left(t^{-1}E_{\alpha,0}\left(\frac{\pi^2}{4}t^\alpha\right) + E_\alpha\left(\frac{\pi^2}{4}t^\alpha\right)\right)^2,$$

where $t^{-1}E_{\alpha,0}(t^\alpha) \triangleq \sum_{k=1}^\infty \frac{t^{k\alpha-1}}{\Gamma(\alpha k)}, t \geq 0$. Obviously, from properties of the Mittag-Leffler function, it is easy to conclude that both $\|u(\cdot,t)\|_{L^2(0,1)} \to +\infty$ and $\|u_t(\cdot,t)\|_{L^2(0,1)} \to +\infty$ as $t \to \infty$.

It is seen from Example 2.1 that an appropriate control must be imposed at the control end to guarantee the asymptotic stability of the closed-loop of system (1.1). Generally speaking, by [29], the eigenvalues of the operator $\frac{\partial^2}{\partial x^2} + \lambda(x)$ with the certain domain lying outside the closed angular sector $|\arg(\frac{\partial^2}{\partial x^2} + \lambda(x))| > \frac{\pi}{2}\alpha$ will leads to instability.

The first aim of this paper is to design a control law $U(t)$ to stabilize (1.1) and to present a mathematical proof. Obviously, once this is achieved, by taking $\lambda(x) = 0$, we give a complete solution for problem raised in [21].

Significantly, two long-standing unsolved problems left in the paper [21] are just the special cases of Problems I and II, where the plant is the system (1.1) with $\lambda(x) \equiv 0$ and the stability results have only been checked by numerical simulations without theoretical proofs.

The following lemmas will be used in the proof of the stability of the closed-loop system.

**Lemma 2.2.** *([32], Thm. 1.6) Let $\delta \in (0,2)$ and $\gamma \in \mathbb{R}$. For $\frac{\pi}{2}\delta < \mu < \min\{\pi, \pi\delta\}$, there exists a constant $M = M(\delta, \gamma, \mu)$ such that*

$$|E_{\delta,\gamma}(z)| \leq \frac{M}{1+|z|} \text{ for all } z \in \mathbb{C} \text{ with } \mu \leq |\arg(z)| \leq \pi,$$

*where $E_{\delta,\gamma}(z)$ is a Mittag-Leffler function with double parameters.*

**Lemma 2.3.** *Let $\delta \in (0,2)$ and $\gamma \in \mathbb{R}$. For $\frac{\pi}{2}\delta < \mu < \min\{\pi, \pi\delta\}$, there exists a constant $M = M(\delta, \mu)$ such that for all $z \in \mathbb{C}$ with $\mu \leq |\arg(z)| \leq \pi$, there holds*

$$|E_{\delta,\delta-1}(z)| \leq \frac{M}{1+|z|^2}, \quad |E_{\delta,\delta}(z)| \leq \frac{M}{1+|z|^2}, \tag{2.1}$$

*where $E_{\delta,\gamma}(z)$ is a Mittag-Leffler function with double parameters.*

*Proof.* By Theorem 1.4 of [32] or Chapter 1, p. 43, 1.8.28 of [19], we have the following asymptotic behavior of $E_{\delta,\gamma}(z)$:

$$E_{\delta,\gamma}(z) = -\sum_{k=1}^p \frac{z^{-k}}{\Gamma(\gamma - k\delta)} + \mathcal{O}(|z|^{-p-1}) \text{ as } |z| \to +\infty, \text{ with } \mu \leq \arg(z) \leq \pi. \tag{2.2}$$

Noticing the fact $\frac{z^{-1}}{\Gamma(\gamma-\delta)} = 0$ for $\gamma = \delta - 1$ or $\gamma = \delta$, and taking $p = 2$ in (2.2), we obtain $E_{\delta,\delta-1}(z) = -\frac{1}{\Gamma(-\delta-1)z^2} + \mathcal{O}(z^{-3})$, $E_{\delta,\delta}(z) = -\frac{1}{\Gamma(-\delta)z^2} + \mathcal{O}(z^{-3})$, from which, it is seen that

$$\lim_{z\to\infty,\mu\leq\arg(z)\leq\pi} \frac{E_{\delta,\delta-1}(z)}{(1+|z|^2)^{-1}} = -\frac{1}{\Gamma(-\delta-1)},$$
$$\lim_{z\to\infty,\mu\leq\arg(z)\leq\pi} \frac{E_{\delta,\delta}(z)}{(1+|z|^2)^{-1}} = -\frac{1}{\Gamma(-\delta)}.$$

Hence, for any fixed $M_1 > 0$ satisfying $M_1 > \max\{|\frac{1}{\Gamma(-\delta-1)}|, |\frac{1}{\Gamma(-\delta)}|\}$, there exists a constant $T > 0$, such that for all $|z| \geq T$ with $\mu \leq \arg(z) \leq \pi$,

$$|E_{\delta,\delta-1}(z)| \leq \frac{M_1}{1+|z|^2}, \quad |E_{\delta,\delta}(z)| \leq \frac{M_1}{1+|z|^2}. \tag{2.3}$$

Since both $E_{\delta,\delta-1}(z)$ and $E_{\delta,\delta}(z)$ are continuous function with respect to $z$ on $\mathbb{C}$, we know $M_2 = \max\{(1+|z|^2)|E_{\delta,\delta-1}(z)| + (1+|z|^2)|E_{\delta,\delta}(z)| : |z| \leq T, \ \mu \leq |\arg(z)| \leq \pi\} < +\infty$, which, together with (2.3), implies that (2.1) with $M = \max\{M_1, M_2\}$. $\qquad\square$

**Remark 2.4.** The importance of Lemma 2.3 lies in that it gives a more precise asymptotic behavior estimation than Lemma 2.2 as $z \to \infty$. This precise estimation will be used in the proof of Theorem 4.2. Moreover, from the proof of Lemma 2.3, the constant $M$ depends on $\delta$, i.e., $M = M(\delta)$. It is worth stressing that $M(\delta) \to +\infty$ as $\delta \to 2^-$. Otherwise, $M(2) < +\infty$, since $E_{2,1}(-x) = \cos\sqrt{x}$ for all $x \geq 0$, we get

$$|\cos\sqrt{x}| = |E_{2,1}(-x)| \leq \frac{M(2)}{1+|x|^2}. \tag{2.4}$$

Taking $x = (2n\pi)^2$ in (2.4) and letting $n \to +\infty$, we obtain $1 \leq 0$, which is a contradiction. Also, from $E_{2,2}(-x) = \frac{\sin\sqrt{x}}{\sqrt{x}}$, we can see that $\frac{1+x^2}{\sqrt{x}}\sin\sqrt{x} \leq M(2)$. Taking $x = (2n\pi + \pi/2)^2$ and letting $n \to +\infty$ result in $+\infty < M(2) < +\infty$. This leads to a contradiction again.

## 3. Boundary stabilization *via* state feedback

In this section, a backstepping approach is adopted to construct a state feedback stabilizing control for system (1.1). Motivated by [35, 36, 41], we introduce an invertible transformation $u \to w$:

$$w(x,t) = (1 + \mathbb{P}_1)u(x,t) = u(x,t) - \int_0^x k(x,y)u(y,t)\mathrm{d}y, \tag{3.1}$$

to transform system (1.1) into the following equivalent system:

$$\begin{cases} {}_0^C D_t^\alpha w(x,t) = w_{xx}(x,t), & x \in (0,1), \ t \geq 0, \\ w(0,t) = 0, \quad w_x(1,t) = 0, \ t \geq 0, \\ w(x,0) = w_0(x), \ w_t(x,0) = w_1(x), \ 0 \leq x \leq 1, \end{cases} \tag{3.2}$$

which is asymptotically stable. Now we determine the kernel function $k(x,y)$ of (3.1). For this purpose, finding Caputo's fractional derivative for (3.1) and utilizing the first equation of (1.1), *via* performing the integration

by parts, we obtain

$$
\begin{aligned}
{}_0^C D_t^\alpha w(x,t) &= {}_0^C D_t^\alpha u(x,t) - \int_0^x k(x,y) {}_0^C D_t^\alpha u(y,t) \mathrm{d}y \\
&= {}_0^C D_t^\alpha u(x,t) - \int_0^x k(x,y) \big( u_{yy}(y,t) + \lambda(y) u(y,t) \big) \mathrm{d}y \\
&= {}_0^C D_t^\alpha u(x,t) - \big( k(x,x) u_x(x,t) - k(x,0) u_x(0,t) \\
&\quad - [k_y(x,x) u(x,t) - k_y(x,0) u(0,t)] \big) \\
&\quad - \int_0^x (k_{yy}(x,y) + \lambda(y) k(x,y)) u(y,t) \mathrm{d}y
\end{aligned}
\tag{3.3}
$$

and

$$
\begin{aligned}
w_{xx}(x,t) = u_{xx}(x,t) &- \frac{d}{dx}(k(x,x)) u(x,t) - k(x,x) \\
&\times u_x(x,t) - k_x(x,x) u(x,t) - \int_0^x k_{xx}(x,y) u(y,t) \mathrm{d}y.
\end{aligned}
\tag{3.4}
$$

Substituting (3.3) and (3.4) into (3.2), it is verified that the kernel function $k(x,y)$ should satisfy the following partial differential equation:

$$
\begin{cases}
k_{xx}(x,y) - k_{yy}(x,y) = \lambda(y) k(x,y), \\
k(x,0) = 0, \quad k(x,x) = -\dfrac{1}{2} \displaystyle\int_0^x \lambda(y) \mathrm{d}y.
\end{cases}
\tag{3.5}
$$

By Theorem 2.1 of [35], there exist a unique solution $k \in C^2(\bar{\mathcal{F}})$ for the PDE (3.5) where $\mathcal{F} := \{(x,y) \in \mathbb{R}^2 : 0 \le y \le x \le 1\}$. In particular, if $\lambda(x) = \lambda$ is a constant, then its unique explicit solution is given by $k(x,y) = -\lambda y \frac{I_1(\sqrt{\lambda(x^2-y^2)})}{\sqrt{\lambda(x^2-y^2)}}$ where $I_1(x)$ is a first-order modified Bessel function of the first kind given by $I_1(x) = \sum_{n=0}^\infty \frac{x^{2n+1}}{2^{2n+1} n!(n+1)!}$. In order to find the inverse transformation of (3.1), let

$$
u(x,t) = (1 + \mathbb{P}_1)^{-1} w(x,t) = w(x,t) + \int_0^x l(x,y) w(y,t) \mathrm{d}y.
\tag{3.6}
$$

Similarly, finding Caputo's fractional derivative for (3.6) and utilizing the first equation of (3.2) by performing the integration by parts, we obtain

$$
\begin{aligned}
{}_0^C D_t^\alpha u(x,t) &= {}_0^C D_t^\alpha w(x,t) + \int_0^x l(x,y) {}_0^C D_t^\alpha w(y,t) \mathrm{d}y \\
&= {}_0^C D_t^\alpha w(x,t) + \int_0^x l(x,y) w_{xx}(y,t) \mathrm{d}y \\
&= \big( l(x,x) w_x(x,t) - l(x,0) w_x(0,t) \\
&\quad - [l_y(x,x) w(x,t) - l_y(x,0) w(0,t)] \big) \\
&\quad + \int_0^x l_{yy}(x,y) w(y,t) \mathrm{d}y + w_{xx}(x,t),
\end{aligned}
\tag{3.7}
$$

and

$$
\begin{aligned}
u_{xx}(x,t) = w_{xx}(x,t) &+ \frac{d}{dx}(l(x,x)) w(x,t) + l(x,x) \\
&\times w_x(x,t) + l_x(x,x) w(x,t) + \int_0^x l_{xx}(x,y) w(y,t) \mathrm{d}y.
\end{aligned}
\tag{3.8}
$$

Substituting (3.6), (3.7) and (3.8) into (1.1), we see that the kernel function $l(x,y)$ satisfies the following partial differential equation:

$$\begin{cases} l_{xx}(x,y) - l_{yy}(x,y) = -\lambda(x)l(x,y), \\ l(x,0) = 0, \quad l(x,x) = -\frac{1}{2}\int_0^x \lambda(y)\mathrm{d}y. \end{cases} \tag{3.9}$$

Once again, by Theorem 2.2 of [35], the PDE (3.9) admits a unique solution $l \in C^2(\bar{\mathcal{F}})$. In particular, when $\lambda(x) = \lambda$ is a constant function, the unique solution is given explicitly by $l(x,y) = -\lambda y \frac{J_1(\sqrt{\lambda(x^2-y^2)})}{\sqrt{\lambda(x^2-y^2)}}$ where $J_1(x)$ is a first-order Bessel function that $J_1(x) = \sum_{n=0}^\infty (-1)^n \frac{x^{2n+1}}{2^{2n+1}n!(n+1)!}$.

Next, we show the asymptotic stability of the system (3.2).

**Lemma 3.1.** *Let* $\alpha \in (1,2)$. *For any initial value* $(w_0, w_1) \in [L^2(0,1)]^2$, *system* (3.2) *admits a unique solution* $w \in C(0,\infty; L^2(0,1))$, *and there exists a constant* $M > 0$ *such that*

$$\|w(\cdot,t)\|^2_{L^2(0,1)} \leq \frac{M}{1+t^{2\alpha-2}}\|(w_0,w_1)\|^2_{[L^2(0,1)]^2}. \tag{3.10}$$

*Moreover, when* $(w_0,w_1) \in [H^2(0,1)]^2$, *there exists a constant* $M' > 0$ *such that*

$$\|(w(\cdot,t), w_t(\cdot,t))\|^2_{[L^2(0,1)]^2} \leq \frac{M'}{1+t^{2\alpha-2}}\|(w_0,w_1)\|^2_{[H^2(0,1)]^2}. \tag{3.11}$$

*Proof.* Define the operator $A : D(A)(\subset L^2(0,1)) \to L^2(0,1)$ as follows:

$$\begin{cases} [Af](x) = f''(x), \\ D(A) = \{f \in H^2(0,1)| \ f(0) = 0, f'(1) = 0\}. \end{cases} \tag{3.12}$$

A direct computation shows that $A$ is self-adjoint in $L^2(0,1)$ with the eigen-pairs $\{\mu_j, e_j(x)\}$ given by

$$\mu_j = -\left(j + \frac{1}{2}\right)^2 \pi^2, \ e_j(x) = \sqrt{2}\sin\left(j + \frac{1}{2}\right)\pi x, j = 0, 1, 2, \ldots. \tag{3.13}$$

Since $\{e_j(x)\}$ forms an orthnormal basis for $L^2(0,1)$, we can express the solution of (3.2) as

$$w(x,t) = \sum_{j=0}^\infty \varphi_j(t)e_j(x) \tag{3.14}$$

with the initial state $w_0(x) = \sum_{j=0}^\infty a_j e_j(x), w_1(x) = \sum_{j=0}^\infty b_j e_j(x)$, where the coefficients $a_j, b_j$ are $a_j = \int_0^1 e_j(x)w_0(x)\mathrm{d}x, b_j = \int_0^1 e_j(x)w_1(x)\mathrm{d}x$. Since $w_0, w_1 \in L^2(0,1)$, one has $\{a_j\}, \{b_j\} \in l^2$. Moreover, $\|\{a_j\}\|_{l^2} = \|w_0\|_{L^2(0,1)}$ and $\|\{b_j\}\|_{l^2} = \|w_1\|_{L^2(0,1)}$. For each $j = 0, 1, 2, \cdots$, it is seen that $\varphi_j \in C(0,\infty; \mathbb{R})$ in (3.14) should satisfy the following linear fractional differential equation:

$${}^C_0 D^\alpha_t \varphi_j(t) = \mu_j \varphi_j(t), \ \varphi_j(0) = a_j, \ \varphi'_j(0) = b_j. \tag{3.15}$$

In view of Theorem 4.3 in [19], the solution of (3.15) is found to be

$$\varphi_j(t) = a_j E_\alpha(\mu_j t^\alpha) + b_j t E_{\alpha,2}(\mu_j t^\alpha).$$

Hence, the solution of (3.2) is finally derived by

$$w(x,t) = \sum_{j=0}^{\infty} \left[ a_j E_\alpha(\mu_j t^\alpha) + b_j t E_{\alpha,2}(\mu_j t^\alpha) \right] e_j(x). \tag{3.16}$$

From Lemma 2.2, there exists a consant $C_1 > 0$ such that for all $t \geq 0$,

$$E_\alpha(\mu_j t^\alpha) \leq \frac{C_1}{1 - \mu_j t^\alpha}, \quad t E_{\alpha,2}(\mu_j t^\alpha) \leq \frac{C_1 t}{1 - \mu_j t^\alpha}. \tag{3.17}$$

Since $\{e_j(x)\}$ forms an orthnormal basis for $L^2(0,1)$, it follows from (3.16) and (3.17) that

$$
\begin{aligned}
\|w(\cdot,t)\|_{L^2(0,1)}^2 &= \sum_{j=0}^{\infty} \left[ a_j E_\alpha(\mu_j t^\alpha) + b_j t E_{\alpha,2}(\mu_j t^\alpha) \right]^2 \\
&\leq 2 \sum_{j=0}^{\infty} \left[ a_j^2 \frac{C_1^2}{(1 - \mu_j t^\alpha)^2} + b_j^2 \frac{C_1^2 t^2}{(1 - \mu_j t^\alpha)^2} \right] \\
&\leq 2 \frac{C_1^2(1 + t^2)}{(1 - \mu_0 t^\alpha)^2} \sum_{j=0}^{\infty} [a_j^2 + b_j^2] = 2 \frac{C_1^2(1 + t^2)}{(1 - \mu_0 t^\alpha)^2} \\
&\times \|(w_0, w_1)\|_{[L^2(0,1)]^2}^2 \leq \frac{M}{1 + t^{2\alpha-2}} \|(w_0, w_1)\|_{[L^2(0,1)]^2}^2.
\end{aligned} \tag{3.18}
$$

In the last step of (3.18), we used the fact that $\frac{C_1^2(1+t^2)}{(1-\mu_0 t^\alpha)^2} \leq \frac{M}{1+t^{2\alpha-2}}$ for some $M > 0$ due to $\lim_{t\to\infty} 2\frac{C_1^2(1+t^2)}{(1-\mu_0 t^\alpha)^2}(1 + t^{2\alpha-2}) = 2C_1^2/\mu_0^2$.

Next, when $(w_0, w_1) \in [H^2(0,1)]^2$, then, $\{j^2 a_j\}, \{j^2 b_j\} \in l^2$. Moreover, from (3.13), there exists a constant $C_2 > 0$ such that

$$\sum_{j=0}^{\infty} \mu_j^2 a_j^2 \leq C_2 \|w_0\|_{H^2(0,1)}^2. \tag{3.19}$$

By Chapter 1, p. 49, 1.10.1, p. 50, 1.10.7 in [19], $\frac{\mathrm{d}}{\mathrm{d}t} E_\alpha(\mu_j t^\alpha) = t^{-1} E_{\alpha,0}(\mu_j t^\alpha)$, $\frac{\mathrm{d}}{\mathrm{d}t}(t E_{\alpha,2}(\mu_j t^\alpha)) = E_\alpha(\mu_j t^\alpha)$, where $t^{-1} E_{\alpha,0}(\mu_j t^\alpha) := \sum_{k=1}^{\infty} (\mu_j^k t^{\alpha k-1})/\Gamma(\alpha k)$ is a function defined on $[0, +\infty)$. This, together with (3.16), gives

$$w_t(x,t) = \sum_{j=0}^{\infty} \left[ a_j t^{-1} E_{\alpha,0}(\mu_j t^\alpha) + b_j E_\alpha(\mu_j t^\alpha) \right] e_j(x). \tag{3.20}$$

Since $t^{-1} E_{\alpha,0}(t^\alpha) \triangleq \sum_{k=1}^{\infty} \frac{t^{k\alpha-1}}{\Gamma(\alpha k)}$, which is continuous over $t \in [0, \infty)$, by Lemma 2.2, there exists a constant $C_3 > 0$ such that $|t^{-1} E_{\alpha,0}(-t^\alpha)| \leq \frac{C_3}{1+t^\alpha}$. Replacing $t$ by $(-\mu_j)^{1/\alpha} t$ gives

$$|t^{-1} E_{\alpha,0}(\mu_j t^\alpha)| \leq \frac{C_3(-\mu_j)^{1/\alpha}}{1 - \mu_j t^\alpha}. \tag{3.21}$$

By Lemma 2.2, there exists a constant $C_4 > 0$ such that

$$E_\alpha(\mu_j t^\alpha) \leq \frac{C_4}{1 - \mu_j t^\alpha}. \tag{3.22}$$

Since $(-\mu_j)^{2/\alpha}a_j^2 \leq (-\mu_j)^2 a_j^2$ owing to $\alpha \in (1,2)$ and $-\mu_j > 1$, it follows from (3.19), (3.20), (3.21) and (3.22) that

$$
\begin{aligned}
\|w_t(\cdot,t)\|_{L^2(0,1)}^2 &= \sum_{j=0}^{\infty} \left[ a_j t^{-1} E_{\alpha,0}(\mu_j t^\alpha) + b_j E_\alpha(\mu_j t^\alpha) \right]^2 \\
&\leq 2\sum_{j=0}^{\infty} \left[ a_j^2 \frac{C_3^2(-\mu_j)^{2/\alpha}}{(1-\mu_j t^\alpha)^2} + b_j^2 \frac{C_4^2 t^2}{(1-\mu_j t^\alpha)^2} \right] \\
&\leq 2\frac{C_3^2 + C_4^2 t^2}{(1-\mu_0 t^\alpha)^2} \sum_{j=0}^{\infty} [(-\mu_j)^{2/\alpha} a_j^2 + b_j^2] \\
&\leq 2\frac{C_3^2 + C_4^2 t^2}{(1-\mu_0 t^\alpha)^2} \|(w_0, w_1)\|_{[H^2(0,1)]^2}^2.
\end{aligned}
\tag{3.23}
$$

Since $\lim_{t\to\infty} 2\frac{C_3^2 + C_4^2 t^2}{(1-\mu_0 t^\alpha)^2}(1 + t^{2\alpha-2}) = 2C_4^2/\mu_0^2$, there exists a constant $M_1 > 0$ such that $2\frac{C_3^2 + C_4^2 t^2}{(1-\mu_0 t^\alpha)^2} \leq \frac{M_1}{1 + t^{2\alpha-2}}$. In view of (3.23),

$$
\|w_t(\cdot,t)\|_{L^2(0,1)}^2 \leq \frac{M_1}{1 + t^{2\alpha-2}} \|(w_0, w_1)\|_{[H^2(0,1)]^2}^2,
$$

which, together with (3.18), implies that (3.11) holds with $M' = M + M_1$. $\qquad\square$

**Remark 3.2.** From Lemma 3.1, for the stabilization of system (1.1) with $\lambda(x) \equiv 0$ considered in [21, 22], the controller can be taken as $U(t) \equiv 0$ if there is no disturbance. In other words, system (1.1) with $\lambda(x) \equiv 0$ is automatically asymptotically stable without control. In [23] where the controller for system (1.1) with $\lambda(x) \equiv 0$ was designed based on the Smith predictor by using delayed boundary measurement. By Lemma 3.1 again, there is no need to measure the boundary because system (1.1) with $\lambda(x) \equiv 0$ without disturbance is asymptotically stable. We can therefore say from Lemma 3.1 that the stability of system (1.1) with $\lambda(x) \equiv 0$ raised in [21] has been rigorously proved.

**Remark 3.3.** It is worth emphasizing that the asymptotic stability of (3.2) does not hold when the fractional order takes $\alpha = 2$. Actually, when $\alpha = 2$, system (3.2) becomes

$$
\begin{cases}
w_{tt}(x,t) = w_{xx}(x,t), & x \in (0,1),\ t \geq 0, \\
w(0,t) = 0, \quad w_x(1,t) = 0,\ t \geq 0, \\
w(x,0) = w_0(x),\ w_t(x,0) = w_1(x),\ 0 \leq x \leq 1,
\end{cases}
\tag{3.24}
$$

which is a classical wave equation. Since Mittag-Leffler function has the properties:

$$
E_2(-x^2) = \cos(x), \quad E_{2,2}(-x^2) = \frac{\sin(x)}{x}, \quad \forall x \geq 0,
$$

from (3.16), the solution of (3.24) is explicitly solved by

$$
w(x,t) = \sum_{j=0}^{\infty} \left[ a_j E_2(\mu_j t^2) + b_j t E_{2,2}(\mu_j t^2) \right] e_j(x) = \sum_{j=0}^{\infty} \left[ a_j \cos\left(j + \frac{1}{2}\right)\pi t + b_j \frac{\sin\left(j + \frac{1}{2}\right)\pi t}{(j + \frac{1}{2})\pi} \right] e_j(x).
$$

This implies the fact that in fractional case the influence of the natural frequency of the system vanishes over time, which reveals a damping characteristic and differs from the integer-order case $\alpha = 2$ in (3.24).

We propose the following state feedback control law:

$$U(t) = k(1,1)u(1,t) + \int_0^1 k_x(1,y)u(y,t)\mathrm{d}y, \tag{3.25}$$

under which, the closed-loop system of (1.1) becomes

$$\begin{cases} {}^C_0 D^\alpha_t u(x,t) = u_{xx}(x,t) + \lambda(x)u(x,t), x \in (0,1), t \geq 0, \\ u(0,t) = 0, \ t \geq 0, \\ u_x(1,t) = k(1,1)u(1,t) + \int_0^1 k_x(1,y)u(y,t)\mathrm{d}y, \ t \geq 0, \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ 0 \leq x \leq 1. \end{cases} \tag{3.26}$$

**Theorem 3.4.** *For any initial value* $(u_0, u_1) \in [H^2(0,1)]^2$, *the closed-loop system* (3.26) *admits a unique solution* $(u, u_t) \in C(0, \infty; [L^2(0,1)]^2)$. *Moreover, there exist two positive constants* $M, \mu > 0$ *such that*

$$\|(u(\cdot,t), u_t(\cdot,t))\|^2_{[L^2(0,1)]^2} \leq \frac{M}{1 + t^{2\alpha-2}}\|(u_0, u_1)\|^2_{[H^2(0,1)]^2}. \tag{3.27}$$

*Proof.* By the invertible transformation (3.1), the closed-loop system (3.26) is equivalent to system (3.2). From Lemma 3.1, there exists a constant $M_1 > 0$ such that

$$\|(w(\cdot,t), w_t(\cdot,t))\|^2_{[L^2(0,1)]^2} \leq \frac{M_1}{1 + t^{2\alpha-2}}\|(w_0, w_1)\|^2_{[H^2(0,1)]^2}. \tag{3.28}$$

On the other hand, by (3.1) and (3.6), there exist two constants $M_2, M_3 > 0$ such that

$$\begin{cases} \|(u(\cdot,t), u_t(\cdot,t))\|^2_{[L^2(0,1)]^2} \leq M_2 \|(w(\cdot,t), w_t(\cdot,t))\|^2_{[L^2(0,1)]^2}, \\ \|(w_0, w_1)\|^2_{[H^2(0,1)]^2} \leq M_3 \|(u_0, u_1)\|^2_{[H^2(0,1)]^2}. \end{cases} \tag{3.29}$$

The (3.27) then follows from (3.28) and (3.29) with $M = M_1 M_2 M_3 > 0$. □

**Remark 3.5.** Theorem 3.4 implies that (3.26) is Mittag-Leffler stable. Indeed, for any $\beta \in (0,1)$, $\Gamma(1-\beta) \geq \Gamma(1) = 1$ and $\frac{\Gamma(1-\beta)M}{\Gamma(1-\beta) + \Gamma(1-\beta)t^{2\alpha-2}} \leq \frac{\Gamma(1-\beta)M}{1 + \Gamma(1-\beta)t^{2\alpha-2}}$. By the well known inequality of Mittag-Leffler function ([34], Thm. 4) below

$$\frac{1}{1 + \Gamma(1-\alpha)x} \leq E_\alpha(-x) \leq \frac{1}{1 + \Gamma(1+\alpha)^{-1}x}, \quad \forall x \geq 0,$$

it follows from (3.27) that $\|(u(\cdot,t), u_t(\cdot,t))\|^2_{[L^2(0,1)]^2} \leq \Gamma(1-\beta)M E_\beta(-t^{2\alpha-2})\|(u_0, u_1)\|^2_{[H^2(0,1)]^2}$, which, together with the definition of Mittag-Leffler stability [20], leads to Mittag-Leffler stability of (3.26). It is worth noting that Mittag-Leffler stability implies the asymptotical stability and not conversely.

## 4. Observer-based output feedback for system (1.1)

In this section, assume the output measurement of (1.1) without disturbance is $u(1,t)$, we are devoted to design an output feedback control for system (1.1) and do stability analysis. Note that the output feedback stabilization for the system without disturbance is also interesting and nontrivial. To this purpose, an observer

for system (1.1) is firstly proposed as follows:

$$\begin{cases} {}_0^C D_t^\alpha \widehat{u}(x,t) = \widehat{u}_{xx}(x,t) + \lambda(x)\widehat{u}(x,t) + p_1(x)[\widehat{u}(1,t) - u(1,t)], \ \widehat{u}(0,t) = 0, \ t \geq 0, \\ \widehat{u}_x(1,t) = p_0[\widehat{u}(1,t) - u(1,t)] + U(t), \ t \geq 0, \\ \widehat{u}(x,0) = \widehat{u}_0(x), \ \widehat{u}_t(x,0) = \widehat{u}_1(x), \ 0 \leq x \leq 1, \end{cases} \tag{4.1}$$

where $x \in [0,1]$ and $\widehat{u}_0(\cdot), \widehat{u}_1(\cdot) \in H^2(0,1)$ are the initial states; $p_1(x)$ and $p_0$ are the observer gains to be chosen to guarantee that the observer error $\widehat{u}(x,t) - u(x,t)$ approaches zero as $t$ goes to infinity.

Let $\widetilde{u}(x,t) = \widehat{u}(x,t) - u(x,t)$. It is straightforward to verify that $\widetilde{u}(x,t)$ is governed by

$$\begin{cases} {}_0^C D_t^\alpha \widetilde{u}(x,t) = \widetilde{u}_{xx}(x,t) + \lambda(x)\widetilde{u}(x,t) + p_1(x)\widetilde{u}(1,t), \\ \widetilde{u}(0,t) = 0, \ \ \widetilde{u}_x(1,t) = p_0\widetilde{u}(1,t), \\ \widetilde{u}(x,0) = \widehat{u}_0(x) - u_0(x), \ \widetilde{u}_t(x,0) = \widehat{u}_1(x) - u_1(x). \end{cases} \tag{4.2}$$

To find the observer gains $p_1(x)$ and $p_0$, we look for the transformation:

$$\widetilde{u}(x,t) := (1 + \mathbb{P}_2)z(x,t) = z(x,t) - \int_x^1 p(x,y)z(y,t)\mathrm{d}y \tag{4.3}$$

that transforms (4.2) into the following target system:

$$\begin{cases} {}_0^C D_t^\alpha z(x,t) = z_{xx}(x,t), \\ z(0,t) = z_x(1,t) = 0, \\ z(x,0) = z_0(x), \ \ z_t(x,0) = z_1(x), \end{cases} \tag{4.4}$$

which, by (3.10) and Remark 3.5, is Mittag-Leffler stable. To find the differential equation satisfied by $p(x,y)$, we substitute (4.3) into (4.2) to give the equation satisfied by $z(x,t)$:

$$\begin{aligned} {}_0^C D_t^\alpha z(x,t) &= z_{xx}(x,t) + \int_x^1 p(x,y)z_{yy}(y,t)\mathrm{d}y \\ &- \left( \int_x^1 p(x,y)z(y,t)\mathrm{d}y \right)_{xx} - \int_x^1 \lambda(x)p(x,y)z(y,t)\mathrm{d}y + \lambda(x)z(x,t) + p_1(x)z(1,t) \\ &= z_{xx}(x,t) + \int_x^1 [p_{yy}(x,y) - p_{xx}(x,y) - \lambda(x)p(x,y)]z(y,t)\mathrm{d}y + \left(\lambda(x) + 2\frac{d}{dx}p(x,x)\right)z(x,t) \\ &+ z_x(1,t)p(x,1) + (p_1(x) - p_y(x,1))z(1,t), \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} z(0,t) &= \widetilde{u}(0,t) + \int_0^1 p(0,y)z(y,t)\mathrm{d}y = \int_0^1 p(0,y)z(y,t)\mathrm{d}y, \\ z_x(1,t) &= \widetilde{u}_x(1,t) - p(1,1)z(1,t) = \widetilde{u}_x(1,t) - p(1,1)\widetilde{u}(1,t) = (p_0 - p(1,1))\widetilde{u}(1,t). \end{aligned} \tag{4.6}$$

Comparing (4.5) and boundary conditions (4.6) with (4.4), we see that $p(x,y)$ satisfies the following partial differential equation:

$$\begin{cases} p_{yy}(x,y) - p_{xx}(x,y) = \lambda(x)p(x,y), \\ p(x,x) = -\dfrac{1}{2}\displaystyle\int_0^x \lambda(\xi)\mathrm{d}\xi, \ \ p(0,y) = 0, \end{cases} \tag{4.7}$$

and the observer gains should be selected as

$$p_1(x) = p_y(x, 1), \quad p_0 = p(1, 1). \tag{4.8}$$

In order to show the existence of the solution of (4.7) and the invertibility of (4.3), new variables are introduced as follows: $\bar{x} := y$, $\bar{y} := x$, $\bar{p}(\bar{x}, \bar{y}) := p(x, y)$. Then, (4.7) becomes

$$\begin{cases} \bar{p}_{\bar{x}\bar{x}}(\bar{x}, \bar{y}) - \bar{p}_{\bar{y}\bar{y}}(\bar{x}, \bar{y}) = \lambda(\bar{y})\bar{p}(\bar{x}, \bar{y}), \\ \bar{p}(\bar{x}, \bar{x}) = -\dfrac{1}{2} \displaystyle\int_0^{\bar{x}} \lambda(\xi) \mathrm{d}\xi, \quad \bar{p}(\bar{x}, 0) = 0. \end{cases} \tag{4.9}$$

It is seen that (4.9) is exactly the same as (3.5) for $k(x, y)$. Thus, (4.9) admits a unique solution $\bar{p} \in C^2(\bar{\mathcal{F}})$ and so does for (4.7), and the transformation (4.3) is invertible. Note that the invertibility of transformation (4.3) can be also obtained by Theorem 3.1 and Theorem 3.3 of [24], and the inverse transformation can be expressed by

$$z(x, t) = (1 + \mathbb{P}_2)^{-1}\widetilde{u}(x, t) = \widetilde{u}(x, t) + \int_x^1 q(x, y)\widetilde{u}(y, t)\mathrm{d}y, \tag{4.10}$$

where $q \in C^2(\bar{\mathcal{F}})$ is uniquely determined by (4.3). Moreover, similar to (3.6)–(3.9), using (4.2) and substituting (4.10) into (4.4), it is found that $q(x, y)$ satisfies

$$\begin{cases} q_{xx}(x, y) - q_{yy}(x, y) = \lambda(y)p(x, y), \\ q(x, x) = -\dfrac{1}{2} \displaystyle\int_0^x \lambda(\xi) \mathrm{d}\xi, \quad q(0, y) = 0, \end{cases} \tag{4.11}$$

and $q(1, 1) = p_0 = p(1, 1)$, $\int_x^1 q(x, y)p_y(y, 1)\mathrm{d}y + p(1, 1)q_y(x, 1) + p_y(x, 1) = q_y(x, 1)$.

With the transformation (4.3) and (4.10), from Lemma 3.1, the following convergence result for observer (4.1) can be obtained immediately.

**Lemma 4.1.** *For any control input $U \in L^2_{loc}(0, \infty)$ and initial state $(u_0, u_1, \widehat{u}_0, \widehat{u}_1) \in [L^2(0, 1)]^4$, the closed-loop system (4.2) admits a unique solution $\widetilde{u} \in C(0, \infty; L^2(0, 1))$ and there exists a constant $M$ such that*

$$\|\widetilde{u}(\cdot, t)\|^2_{L^2(0,1)} \leq \frac{M}{1 + t^{2\alpha - 2}}\|(\widehat{u}_0 - u_0, \widehat{u}_1 - u_1)\|^2_{[L^2(0,1)]^2}. \tag{4.12}$$

*Moreover, when $(u_0, u_1, \widehat{u}_0, \widehat{u}_1) \in [H^2(0, 1)]^4$, there exists a constant $M' > 0$ such that*

$$\|(\widetilde{u}(\cdot, t), \widetilde{u}_t(\cdot, t))\|^2_{[L^2(0,1)]^2} \leq \frac{M'}{1 + t^{2\alpha - 2}}\|(\widehat{u}_0 - u_0, \widehat{u}_1 - u_1)\|^2_{[H^2(0,1)]^2}. \tag{4.13}$$

Because an estimate $\widehat{u}(x, t)$ of the state $u(x, t)$ is obtained by observer (4.1), a natural output feedback control, in terms of the state feedback control (3.25), is designed naturally as

$$U(t) = k(1, 1)\widehat{u}(1, t) + \int_0^1 k_x(1, y)\widehat{u}(y, t)\mathrm{d}y. \tag{4.14}$$

This is an observer-based control law with $u(x,t)$ being replaced by $\widehat{u}(x,t)$. Under the output feedback (4.14), the resulting closed-loop system of (1.1) is

$$
\begin{cases}
{}^C_0 D^\alpha_t u(x,t) = u_{xx}(x,t) + \lambda(x)u(x,t), \\
u(0,t) = 0, \ t \geq 0, \\
u_x(1,t) = k(1,1)\widehat{u}(1,t) + \displaystyle\int_0^1 k_x(1,y)\widehat{u}(y,t)\mathrm{d}y, \\
{}^C_0 D^\alpha_t \widehat{u}(x,t) = \widehat{u}_{xx}(x,t) + \lambda(x)\widehat{u}(x,t) \\
\quad + p_1(x)[\widehat{u}(1,t) - u(1,t)], \ \widehat{u}(0,t) = 0, \\
\widehat{u}_x(1,t) = p_0[\widehat{u}(1,t) - u(1,t)] \\
\quad + k(1,1)\widehat{u}(1,t) + \displaystyle\int_0^1 k_x(1,y)\widehat{u}(y,t)\mathrm{d}y, \\
\widehat{u}(x,0) = \widehat{u}_0(x), \ \widehat{u}_t(x,0) = \widehat{u}_1(x), \\
u(x,0) = w_0(x), \ u_t(x,0) = u_1(x), \ 0 \leq x \leq 1.
\end{cases}
\tag{4.15}
$$

**Theorem 4.2.** *For any initial value $Z_0 := (u_0, u_1, \widehat{u}_0, \widehat{u}_1) \in [H^2(0,1)]^4$, the closed-loop system (4.15) admits a unique solution $(u, u_t, \widehat{u}, \widehat{u}_t) \in C(0, \infty; [L^2(0,1)]^4)$. Moreover, for some $M > 0$ the solution of (4.15) satisfies*

$$
\begin{aligned}
&\|(u(\cdot,t), \widehat{u}(\cdot,t))\|^2_{[L^2(0,1)]^2} \\
&\leq \left[ \frac{1}{t^{2\alpha-2}+1} + \left( \int_0^t \frac{(t-s)^{\alpha-1}}{1+\mu_0^2(t-s)^{2\alpha}}(1+s) \left( \sum_{k=0}^\infty \frac{1}{(1-\mu_k s^\alpha)^2} \right)^{1/2} \mathrm{d}s \right)^2 \right] M\|Z_0\|^2_{[H^2(0,1)]^4}
\end{aligned}
\tag{4.16}
$$

*and*

$$
\begin{aligned}
&\|(u_t(\cdot,t), \widehat{u}_t(\cdot,t))\|^2_{[L^2(0,1)]^2} \\
&\leq \left[ \frac{1}{t^{2\alpha}+1} + \left( \int_0^t \frac{(t-s)^{\alpha-2}}{1+\mu_0^2(t-s)^{2\alpha}}(1+s) \left( \sum_{k=0}^\infty \frac{1}{(1-\mu_k s^\alpha)^2} \right)^{1/2} \mathrm{d}s \right)^2 \right] M\|Z_0\|^2_{[H^2(0,1)]^4},
\end{aligned}
\tag{4.17}
$$

*where $\mu_k = -(k+\frac{1}{2})^2\pi^2$. Furthermore, $\lim_{t\to\infty} \|(u(\cdot,t), u_t(\cdot,t), \widehat{u}(\cdot,t), \widehat{u}_t(\cdot,t))\|_{[L^2(0,1)]^4} = 0$.*

*Proof.* Since $\widetilde{u}(x,t) = \widehat{u}(x,t) - u(x,t)$ is an observer error, by the invertible transformation (4.3), system (4.15) can be rewritten as an equivalent system as follows:

$$
\begin{cases}
{}^C_0 D^\alpha_t \widehat{u}(x,t) = \widehat{u}_{xx}(x,t) + \lambda(x)\widehat{u}(x,t) + p_1(x)\widetilde{u}(1,t), \\
\widehat{u}_x(1,t) = p_0\widetilde{u}(1,t) + k(1,1)\widehat{u}(1,t) + \displaystyle\int_0^1 k_x(1,y)\widehat{u}(y,t)\mathrm{d}y, \ \widehat{u}(0,t) = 0, \\
{}^C_0 D^\alpha_t z(x,t) = z_{xx}(x,t), \\
z(0,t) = z_x(1,t) = 0.
\end{cases}
\tag{4.18}
$$

By (4.3), $\widetilde{u}(1,t) = z(1,t)$. Under the invertible transformation

$$
\widehat{w}(x,t) = (1+\mathbb{P}_1)\widehat{u}(x,t) = \widehat{u}(x,t) - \int_0^x k(x,y)\widehat{u}(y,t)\mathrm{d}y,
\tag{4.19}
$$

where $k(x, y)$ is defined by (3.5), we further transform system (4.18) into

$$
\begin{cases}
{}_0^C D_t^\alpha \widehat{w}(x, t) = \widehat{w}_{xx}(x, t) + p_1(x) z(1, t), \\
\widehat{w}(0, t) = 0, \quad \widehat{w}_x(1, t) = p_0 z(1, t), \\
{}_0^C D_t^\alpha z(x, t) = z_{xx}(x, t), \\
z(0, t) = z_x(1, t) = 0.
\end{cases}
\tag{4.20}
$$

From the proof of Lemma 3.1, the solution of "$z$-part" of (4.20) is explicitly found to be

$$
z(x, t) = \sum_{j=0}^{\infty} \left[ a_{1j} E_\alpha(\mu_j t^\alpha) + a_{2j} t E_{\alpha,2}(\mu_j t^\alpha) \right] e_j(x),
\tag{4.21}
$$

with the initial state $z_0(x) = \sum_{j=0}^{\infty} a_{1j} e_j(x), z_1(x) = \sum_{j=0}^{\infty} a_{2j} e_j(x)$, where the coefficients are given by $a_{1j} = \int_0^1 e_j(x) z_0(x) \mathrm{d}x, a_{2j} = \int_0^1 e_j(x) z_1(x) \mathrm{d}x$. Since $z_0, z_1 \in H^2(0, 1)$ and $\{e_j(x)\}$ is an orthonormal basis for $L^2(0, 1)$, we have $\{j^2 a_{1j}\} \in l^2$, $\{j^2 a_{2j}\} \in l^2$, and for some $M_1 > 0$,

$$
\|\{\mu_j a_{1j}\}\|_{l^2} \le M_1 \|z_0\|_{H^2(0,1)}, \quad \|\{\mu_j a_{2j}\}\|_{l^2} \le M_1 \|z_1\|_{H^2(0,1)}
\tag{4.22}
$$

In order to show the well-posedness and Mittag-Leffler stability for system (4.20), a new variable is introduced as follows: $\widehat{z}(x, t) = \widehat{w}(x, t) - x p_0 z(1, t)$. This variable substitution is to make boundary conditions homogeneous. We can verify that $\widehat{z}(x, t)$ satisfies

$$
\begin{cases}
{}_0^C D_t^\alpha \widehat{z}(x, t) = \widehat{z}_{xx}(x, t) + p_1(x) \sum_{j=0}^{\infty} \left[ a_{1j} E_\alpha(\mu_j t^\alpha) + a_{2j} t E_{\alpha,2}(\mu_j t^\alpha) \right] e_j(1) \\
\qquad\qquad\qquad - x p_0 \sum_{j=0}^{\infty} \mu_j \left[ a_{1j} E_\alpha(\mu_j t^\alpha) + a_{2j} t E_{\alpha,2}(\mu_j t^\alpha) \right] e_j(1), \\
\widehat{z}(0, t) = \widehat{z}_x(1, t) = 0.
\end{cases}
\tag{4.23}
$$

Since $p_1(x), -x p_0 \in L^2(0, 1)$, we write $p_1(x), -x p_0 \in L^2(0, 1)$ as $p_1(x) = \sum_{j=0}^{\infty} b_{1j} e_j(x), -x p_0 = \sum_{j=0}^{\infty} b_{2j} e_j(x)$, where $b_{1j} = \int_0^1 e_j(x) p_1(x) \mathrm{d}x, b_{2j} = -\int_0^1 e_j(x) x p_0 \mathrm{d}x$. Since $\{e_j(x)\}$ forms an orthnormal basis for $L^2(0, 1)$, we can write the solution of (4.23) as $\widehat{z}(x, t) = \sum_{j=0}^{\infty} \phi_j(t) e_j(x)$ with the initial state $\widehat{z}_0(x) = \sum_{j=0}^{\infty} c_{1j} e_j(x), \widehat{z}_1(x) = \sum_{j=0}^{\infty} c_{2j} e_j(x)$, where the coefficients are given by $c_{1j} = \int_0^1 e_j(x) \widehat{z}_0(x) \mathrm{d}x$ and $c_{2j} = \int_0^1 e_j(x) \widehat{z}_1(x) \mathrm{d}x$. Since $\widehat{z}_0, \widehat{z}_1 \in H^2(0, 1)$, one has $\{j^2 c_{1j}\} \in l^2, \{j^2 c_{2j}\} \in l^2$ and $\|\{\mu_j c_{1j}\}\|_{l^2} \le M_2 \|\widehat{z}_0\|_{H^2(0,1)}, \|\{\mu_j c_{2j}\}\|_{l^2} \le M_2 \|\widehat{z}_1\|_{H^2(0,1)}$. Moreover, let $\psi_k(t) = a_{1k} E_\alpha(\mu_k t^\alpha) + a_{2k} t E_{\alpha,2}(\mu_k t^\alpha)$. Then, $\varphi_j \in C(0, \infty; \mathbb{R})$ satisfies the following linear fractional differential equation:

$$
{}_0^C D_t^\alpha \phi_j(t) = \mu_j \phi_j(t) + b_{1j} \sum_{k=0}^{\infty} \psi_k(t) e_k(1) + b_{2j} \sum_{k=0}^{\infty} \mu_k \psi_k(t) e_k(1)
\tag{4.24}
$$

with the initial value $\phi_j(0) = c_{1j}, \phi_j'(0) = c_{2j}$. From Theorem 4.3 of [19], we derive the solution of (4.24) as follows

$$
\begin{aligned}
\phi_j(t) &= c_{1j}E_\alpha(\mu_j t^\alpha) + c_{2j}tE_{\alpha,2}(\mu_j t^\alpha) \\
&+ b_{1j}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\mu_j(t-s)^\alpha)\left[\sum_{k=0}^\infty \psi_k(s)e_k(1)\right]ds \\
&+ b_{2j}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\mu_j(t-s)^\alpha)\left[\sum_{k=0}^\infty \mu_k\psi_k(s)e_k(1)\right]ds.
\end{aligned}
\tag{4.25}
$$

From Lemma 2.3, there exists $C_1 > 0$ such that for all $t \geq 0$, $|E_{\alpha,\alpha}(\mu_j t^\alpha)| \leq \frac{C_1}{1+\mu_j^2 t^{2\alpha}}$. By Lemma 2.2, there exist two constants $C_2, C_3 > 0$ such that for all $t \geq 0$,

$$
E_\alpha(\mu_k t^\alpha) \leq \frac{C_2}{1-\mu_k t^\alpha}, \quad tE_{\alpha,2}(\mu_k t^\alpha) \leq \frac{C_3 t}{1-\mu_k t^\alpha}.
\tag{4.26}
$$

Since $|e_k(1)| = \sqrt{2} < 2$, the third term of (4.25) is estimated as

$$
\begin{aligned}
&\left|\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\mu_j(t-s)^\alpha)\left[\sum_{k=0}^\infty \psi_k(s)e_k(1)\right]ds\right| \\
&\leq 2C_1\int_0^t \frac{(t-s)^{\alpha-1}}{1+\mu_j^2(t-s)^{2\alpha}}\sum_{k=0}^\infty \left[\frac{C_2|a_{1k}|}{1-\mu_k s^\alpha} + \frac{C_3|a_{2k}|s}{1-\mu_k s^\alpha}\right]ds.
\end{aligned}
\tag{4.27}
$$

The fourth term of (4.25) is estimated as

$$
\begin{aligned}
&\left|\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\mu_j(t-s)^\alpha)\left[\sum_{k=0}^\infty \mu_k\psi_k(s)e_k(1)\right]ds\right| \\
&\leq 2C_1\int_0^t \frac{(t-s)^{\alpha-1}}{1+\mu_j^2(t-s)^{2\alpha}}\sum_{k=0}^\infty \left[\frac{C_2|\mu_k a_{1k}|}{1-\mu_k s^\alpha} + \frac{C_3|\mu_k a_{2k}|s}{1-\mu_k s^\alpha}\right]ds.
\end{aligned}
\tag{4.28}
$$

Noticing that $|\mu_k| \geq |\mu_0| = \pi^2/4$ for all $k \in \mathbb{N}$, it follows from (4.25), (4.27) and (4.28) that

$$
\begin{aligned}
\|\widehat{z}(\cdot,t)\|_{L^2(0,1)}^2 &= \sum_{j=0}^\infty \phi_j^2(t) \leq 4\sum_{j=0}^\infty \left[c_{1j}^2\frac{C_2^2}{(1-\mu_j t^\alpha)^2} + c_{2j}^2\frac{C_3^2 t^2}{(1-\mu_k t^\alpha)^2}\right] \\
&+ 16\sum_{j=0}^\infty b_{1j}^2\left(C_1\int_0^t \frac{(t-s)^{\alpha-1}}{1+\mu_0^2(t-s)^{2\alpha}}\sum_{k=0}^\infty \left[\frac{C_2|a_{1k}|}{1-\mu_k s^\alpha} + \frac{C_3|a_{2k}|s}{1-\mu_k s^\alpha}\right]ds\right)^2 \\
&+ 16\sum_{j=0}^\infty b_{2j}^2\left(C_1\int_0^t \frac{(t-s)^{\alpha-1}}{1+\mu_0^2(t-s)^{2\alpha}}\sum_{k=0}^\infty \left[\frac{C_2|\mu_k a_{1k}|}{1-\mu_k s^\alpha} + \frac{C_3|\mu_k a_{2k}|s}{1-\mu_k s^\alpha}\right]ds\right)^2 \\
&\leq 8\frac{C_2^2+C_3^2 t^2}{(1-\mu_0 t^\alpha)^2}\sum_{j=0}^\infty c_j^2 + 32\left(C_1\int_0^t \frac{(t-s)^{\alpha-1}}{1+\mu_0^2(t-s)^{2\alpha}}\right. \\
&\times \left.\sum_{k=0}^\infty \frac{|\mu_k a_k|(1+s)}{1-\mu_k s^\alpha}ds\right)^2 \sum_{j=0}^\infty b_j^2,
\end{aligned}
\tag{4.29}
$$

where $a_j = \max\{C_2|a_{1j}|, C_3|a_{2j}|\}$, $b_j = \max\{|b_{1j}|, |b_{2j}|\}$, $c_j = \max\{|c_{1j}|, |c_{2j}|\}$. We claim that

$$\|\widehat{z}(\cdot, t)\|_{L^2(0,1)}^2 \leq \frac{M_3}{1 + t^{2\alpha-2}}\|(\widehat{z}_0, \widehat{z}_1)\|_{[H^2(0,1)]^2}^2 + \left(\int_0^t \frac{(t-s)^{\alpha-1}}{1 + \mu_0^2(t-s)^{2\alpha}}(1+s)\left(\sum_{k=0}^\infty \frac{1}{(1 - \mu_k s^\alpha)^2}\right)^{1/2} ds\right)^2 M_4\|(z_0, z_1)\|_{[H^2(0,1)]^2}^2 \tag{4.30}$$

for some $M_3, M_4 > 0$, and

$$\lim_{t\to\infty} \int_0^t \frac{(t-s)^{\alpha-1}(1+s)}{1 + \mu_0^2(t-s)^{2\alpha}}\left(\sum_{k=0}^\infty \frac{1}{(1 - \mu_k s^\alpha)^2}\right)^{1/2} ds = 0. \tag{4.31}$$

Indeed, by (4.22), it has $\sum_{k=0}^\infty (\mu_k a_k)^2 \leq M_1^2(C_2 + C_3)^2\|(z_0, z_1)\|_{[H^2(0,1)]^2}^2$. By Cauchy's inequality,

$$\begin{aligned}
&\int_0^t \frac{(t-s)^{\alpha-1}}{1 + \mu_0^2(t-s)^{2\alpha}}\sum_{k=0}^\infty \frac{|\mu_k a_k|(1+s)}{1 - \mu_k s^\alpha} ds \\
&\leq \int_0^t \frac{(t-s)^{\alpha-1}}{1 + \mu_0^2(t-s)^{2\alpha}}(1+s)\left(\sum_{k=0}^\infty \frac{1}{(1 - \mu_k s^\alpha)^2}\right)^{1/2} ds\left(\sum_{k=0}^\infty (\mu_k a_k)^2\right)^{1/2} \\
&\leq \int_0^t \frac{(t-s)^{\alpha-1}}{1 + \mu_0^2(t-s)^{2\alpha}}(1+s)\left(\sum_{k=0}^\infty \frac{1}{(1 - \mu_k s^\alpha)^2}\right)^{1/2} ds M_1^2(C_2 + C_3)^2\|(z_0, z_1)\|_{[H^2(0,1)]^2}^2.
\end{aligned} \tag{4.32}$$

By the fact $\lim_{t\to\infty}(1 + t^{2\alpha-2})\frac{C_2^2 + C_3^2 t^2}{(1 - \mu_0 t^\alpha)^2} = C_3^2$ for $\alpha > 1$, and $\sum_{j=0}^\infty c_j^2 \leq \|(\widehat{z}_0, \widehat{z}_1)\|_{[L^2(0,1)]^2}^2 \leq \|(\widehat{z}_0, \widehat{z}_1)\|_{[H^2(0,1)]^2}^2$, there exists $M_3 > 0$ such that

$$8\frac{C_2^2 + C_3^2 t^2}{(1 - \mu_0 t^\alpha)^2}\sum_{j=0}^\infty c_j^2 \leq \frac{M_3\|(\widehat{z}_0, \widehat{z}_1)\|_{[H^2(0,1)]^2}^2}{1 + t^{2\alpha-2}}. $$

This, together with (4.29) and (4.32), implies that (4.30) holds with $M_4 = 32M_1^2(C_2 + C_3)^2 \sum_{j=0}^\infty b_j^2$. Since $\alpha \in (1, 2)$ and $\mu_k = -(k + \frac{1}{2})^2\pi^2$, it is easy to verify that

$$C_4 := \int_0^\infty \frac{1}{(1 + s^\alpha)^2} ds, \quad C_5 := \int_0^\infty \frac{s^{\alpha-1}}{1 + \mu_0^2 s^{2\alpha}} ds < +\infty \tag{4.33}$$

and

$$C_6 := \sum_{k=0}^\infty \frac{1}{(-\mu_k)^{1/\alpha}}, \quad C_7 := \sum_{k=0}^\infty \frac{1}{(\mu_k)^2} < +\infty. \tag{4.34}$$

For any given $\varepsilon > 0$, take $T > 1$ so that $\frac{2}{T^{\alpha-1}} \leq \frac{\varepsilon}{C_5 C_7^{1/2}}$. Since $\lim_{t\to\infty}\max_{s\in[0,T]} \frac{2T^{1.5}(t-s)^{\alpha-1}}{1 + \mu_0^2(t-s)^{2\alpha}} = 0$, we can choose $T' > T$ such that for all $t > T'$,

$$\max_{s\in[0,T]} \frac{2T^{1.5}(t-s)^{\alpha-1}}{1 + \mu_0^2(t-s)^{2\alpha}} < \frac{\varepsilon}{C_4^{1/2}C_6^{1/2}}. \tag{4.35}$$

Denoting $h(s) = \sum_{k=0}^{\infty} \frac{1}{(1-\mu_k s^\alpha)^2}$ and noting that for all $s > 1$,

$$(s+1)(h(s))^{1/2} \le \frac{2}{s^{\alpha-1}} \left( \sum_{k=0}^{\infty} \frac{1}{(\mu_k)^2} \right)^{1/2}, \tag{4.36}$$

it follows from (4.33), (4.34) and (4.35) that for all $t > T'$,

$$\int_0^t \frac{(t-s)^{\alpha-1}}{1+\mu_0^2(t-s)^{2\alpha}} (1+s)(h(s))^{1/2} ds$$

$$\le \int_0^T \frac{(t-s)^{\alpha-1}}{1+\mu_0^2(t-s)^{2\alpha}} (1+s)(h(s))^{1/2} ds + \int_T^t \frac{(t-s)^{\alpha-1}}{1+\mu_0^2(t-s)^{2\alpha}} (1+s)(h(s))^{1/2} ds$$

$$\le \max_{s \in [0,T]} \frac{2T(t-s)^{\alpha-1}}{1+\mu_0^2(t-s)^{2\alpha}} \int_0^T (h(s))^{1/2} ds + \frac{2}{T^{\alpha-1}} \int_T^t \frac{(t-s)^{\alpha-1}}{1+\mu_0^2(t-s)^{2\alpha}} ds C_7^{1/2}$$

$$\le \max_{s \in [0,T]} \frac{2T^{1.5}(t-s)^{\alpha-1}}{1+\mu_0^2(t-s)^{2\alpha}} \left( \int_0^T h(s) ds \right)^{1/2} + \frac{2}{T^{\alpha-1}} \int_0^\infty \frac{s^{\alpha-1}}{1+\mu_0^2 s^{2\alpha}} ds C_7^{1/2}$$

$$\le \frac{\varepsilon}{C_4^{1/2} C_6^{1/2}} \left( \sum_{k=0}^{\infty} \frac{1}{(-\mu_k)^{1/\alpha}} \int_0^{(-\mu_k)^{1/\alpha} T} \frac{1}{(1+s^\alpha)^2} ds \right)^{1/2}$$

$$+ \frac{\varepsilon}{C_5} \int_0^\infty \frac{s^{\alpha-1}}{1+\mu_0^2 s^{2\alpha}} ds \le 2\varepsilon.$$

Hence, by the arbitrariness of $\varepsilon$, we obtain that (4.31) holds. Since $\widehat{z}(x,t) = \widehat{w}(x,t) - xp_0 z(1,t)$, it follows from (3.13), (4.26) and (4.21) that

$$\|\widehat{w}(\cdot,t)\|_{L^2(0,1)}^2 \le 2\|\widehat{z}(\cdot,t)\|_{L^2(0,1)}^2 + 4p_0^2 \left( \sum_{j=0}^{\infty} \psi_j(t) \right)^2$$

$$\le 2\|\widehat{z}(\cdot,t)\|_{L^2(0,1)}^2 + 8p_0^2 \left( \sum_{j=0}^{\infty} \frac{C_2^2}{(1-\mu_j t^\alpha)^2} \sum_{j=0}^{\infty} a_{1j}^2 + \sum_{j=0}^{\infty} \frac{C_3^2 t^2}{(1-\mu_j t^\alpha)^2} \sum_{j=0}^{\infty} a_{2j}^2 \right)$$

which, together with $\sum_{j=0}^{\infty} a_{1j}^2 + \sum_{j=0}^{\infty} a_{2j}^2 \le \|(z_0, z_1)\|_{[L^2(0,1)]^2}^2 \le \|(z_0, z_1)\|_{[H^2(0,1)]^2}^2$ and the fact that

$$\sum_{j=0}^{\infty} \frac{C_2^2}{(1-\mu_j t^\alpha)^2} \le \frac{1}{t^{2\alpha}+1} \sum_{j=0}^{\infty} \frac{C_2^2}{\mu_j^2} \le \frac{C_2^2 C_7}{t^{2\alpha-2}+1},$$
$$\sum_{j=0}^{\infty} \frac{C_3^2 t^2}{(1-\mu_j t^\alpha)^2} \le \frac{1}{t^{2\alpha-2}+1} \sum_{j=0}^{\infty} \frac{C_3^2}{\mu_j^2} \le \frac{C_3^2 C_7}{t^{2\alpha-2}+1}, \tag{4.37}$$

yields

$$\|\widehat{w}(\cdot,t)\|_{L^2(0,1)}^2 \le \frac{2M_3 + 8p_0^2(C_2^2 C_7 + C_3^2 C_7)}{1+t^{2\alpha-2}} \|(z_0, z_1, \widehat{z}_0, \widehat{z}_1)\|_{[H^2(0,1)]^4}^2$$

$$+ 2 \left( \int_0^t \frac{(t-s)^{\alpha-1}}{1+\mu_0^2(t-s)^{2\alpha}} (1+s) \left( \sum_{k=0}^{\infty} \frac{1}{(1-\mu_k s^\alpha)^2} \right)^{1/2} ds \right)^2 M_4 \|(z_0, z_1)\|_{[H^2(0,1)]^2}^2. \tag{4.38}$$

This, together with (4.31), gives $\lim_{t\to\infty} \|\widehat{w}(\cdot,t)\|^2_{L^2(0,1)} = 0$. From the bounded transformation

$$\begin{pmatrix} u(x,t) \\ \widehat{u}(x,t) \end{pmatrix} = \begin{pmatrix} (I+\mathbb{P}_1)^{-1} & -(I+\mathbb{P}_2) \\ (I+\mathbb{P}_1)^{-1} & 0 \end{pmatrix} \begin{pmatrix} \widehat{w}(x,t) \\ z(x,t) \end{pmatrix}, \tag{4.39}$$

and the simple fact that $\widehat{u}_0(x) = (I+\mathbb{P}_1)^{-1}\widehat{w}_0(x) = (I+\mathbb{P}_1)^{-1}(\widehat{z}_0(x) + xp_0z_0(1))$, $\widehat{u}_1(x) = (I+\mathbb{P}_1)^{-1}\widehat{w}_1(x) = (I+\mathbb{P}_1)^{-1}(\widehat{z}_1(x) + xp_0z_1(1))$, $u_0(x) = (I+\mathbb{P}_1)^{-1}\widehat{w}_0(x) - (I+\mathbb{P}_2)z_0(x)$, it follows from (4.38) and Lemma 3.1 that (4.16) holds with some $M > 0$ and $\lim_{t\to\infty} \|(u(\cdot,t), \widehat{u}(\cdot,t))\|_{[L^2(0,1)]^2} = 0$.

Next, we show that

$$\begin{aligned} \|\widehat{z}_t(\cdot,t)\|^2_{L^2(0,1)} &\leq \frac{M_5}{1+t^{2\alpha}} \|(\widehat{z}_0, \widehat{z}_1)\|^2_{[H^2(0,1)]^2} \\ &+ \left( \int_0^t \frac{(t-s)^{\alpha-2}}{1+\mu_0^2(t-s)^{2\alpha}}(1+s) \left( \sum_{k=0}^\infty \frac{1}{(1-\mu_k s^\alpha)^2} \right)^{1/2} \mathrm{d}s \right)^2 M_6 \|(z_0, z_1)\|^2_{[H^2(0,1)]^2}, \end{aligned} \tag{4.40}$$

for some $M_5, M_6 > 0$. Actually, by Chapter 1, p. 49, 1.10.1 of [19] and Chapter 1, p. 50, 1.10.7 of [19], we have $\frac{\mathrm{d}}{\mathrm{d}t}E_\alpha(\mu_j t^\alpha) = t^{-1}E_{\alpha,0}(\mu_j t^\alpha)$, $\frac{\mathrm{d}}{\mathrm{d}t}(tE_{\alpha,2}(\mu_j t^\alpha)) = E_{\alpha,1}(\mu_j t^\alpha)$ and $\frac{\mathrm{d}}{\mathrm{d}t}(t^{\alpha-1}E_{\alpha,\alpha}(\mu_j t^\alpha))) = t^{\alpha-2}E_{\alpha,\alpha-1}(\mu_j t^\alpha)$. This, combined with (4.25), gives

$$\begin{aligned} \dot{\phi}_j(t) &= c_{1j}t^{-1}E_{\alpha,0}(\mu_j t^\alpha) + c_{2j}E_{\alpha,1}(\mu_j t^\alpha) \\ &+ b_{1j}\int_0^t (t-s)^{\alpha-2}E_{\alpha,\alpha-1}(\mu_j(t-s)^\alpha) \left[ \sum_{k=0}^\infty \psi_k(s)e_k(1) \right] \mathrm{d}s \\ &+ b_{2j}\int_0^t (t-s)^{\alpha-2}E_{\alpha,\alpha-1}(\mu_j(t-s)^\alpha) \left[ \sum_{k=0}^\infty \mu_k\psi_k(s)e_k(1) \right] \mathrm{d}s. \end{aligned} \tag{4.41}$$

Noting that $t^{-1}E_{\alpha,0}(t^\alpha) = \sum_{k=1}^\infty \frac{t^{k\alpha-1}}{\Gamma(\alpha k)}$, which is continuous on $t \in [0,\infty)$, by Lemma 2.2 and $\arg(-t^\alpha) = \pi$, there exists a constant $C_8 > 0$ such that $|t^{-1}E_{\alpha,0}(-t^\alpha)| \leq \frac{C_8}{1+t^\alpha}$. This yields

$$|t^{-1}E_{\alpha,0}(\mu_j t^\alpha)| \leq \frac{C_8(-\mu_j)^{1/\alpha}}{1-\mu_j t^\alpha}. \tag{4.42}$$

It follows from Lemma 2.3 that there exists a constant $C_9 > 0$ such that

$$|E_{\alpha,\alpha-1}(\mu_j t^\alpha)| \leq \frac{C_9}{1+\mu_j^2 t^{2\alpha}}. \tag{4.43}$$

Similar to the estimation (4.29) for $\widehat{z}(x,t)$, by (4.41), (4.42) and (4.43), we get

$$\begin{aligned} \|\widehat{z}_t(\cdot,t)\|^2_{L^2(0,1)} &= \sum_{j=0}^\infty \dot{\phi}_j^2(t) \leq \frac{8}{(1-\mu_0 t^\alpha)^2} \sum_{j=0}^\infty \widehat{c}_j^2 \\ &+ 32 \left( C_9 \int_0^t \frac{(t-s)^{\alpha-2}}{1+\mu_0^2(t-s)^{2\alpha}} \sum_{k=0}^\infty \frac{|\mu_k a_k|(1+s)}{1-\mu_k s^\alpha} \mathrm{d}s \right)^2 \sum_{j=0}^\infty b_j^2 \end{aligned} \tag{4.44}$$

where $\widehat{c}_j = \max\{C_8(-\mu_j)^{1/\alpha}|c_{1j}|, C_2|c_{2j}|\}$. Since $\|\{\mu_j c_{1j}\}\|_{l^2} \le M_2\|\widehat{z}_0\|_{H^2(0,1)}, \|\{\mu_j c_{2j}\}\|_{l^2} \le M_2\|\widehat{z}_1\|_{H^2(0,1)}$, one has $\sum_{j=0}^{\infty} \widehat{c}_j^2 \le C_{10}\|(\widehat{z}_0,\widehat{z}_1)\|_{[H^2(0,1)]^2}^2$ for some $C_{10} > 0$. By the Cauchy inequality,

$$
\begin{aligned}
&\int_0^t \frac{(t-s)^{\alpha-2}}{1+\mu_0^2(t-s)^{2\alpha}} \sum_{k=0}^{\infty} \frac{|\mu_k a_k|(1+s)}{1-\mu_k s^{\alpha}} \mathrm{d}s \\
&\le \int_0^t \frac{(t-s)^{\alpha-2}}{1+\mu_0^2(t-s)^{2\alpha}}(1+s)\bigg(\sum_{k=0}^{\infty} \frac{1}{(1-\mu_k s^{\alpha})^2}\bigg)^{1/2} \mathrm{d}s \bigg(\sum_{k=0}^{\infty}(\mu_k a_k)^2\bigg)^{1/2} \\
&\le \int_0^t \frac{(t-s)^{\alpha-2}}{1+\mu_0^2(t-s)^{2\alpha}}(1+s)\bigg(\sum_{k=0}^{\infty} \frac{1}{(1-\mu_k s^{\alpha})^2}\bigg)^{1/2} \mathrm{d}s\, M_1^2(C_2+C_3)^2\|(z_0,z_1)\|_{[H^2(0,1)]^2}^2.
\end{aligned}
\tag{4.45}
$$

Now, we claim that

$$
\lim_{t\to\infty}\int_0^t \frac{(t-s)^{\alpha-2}(1+s)}{1+\mu_0^2(t-s)^{2\alpha}}\bigg(\sum_{k=0}^{\infty}\frac{1}{(1-\mu_k s^{\alpha})^2}\bigg)^{1/2}\mathrm{d}s = 0.
\tag{4.46}
$$

Indeed, from the proof of (4.31), and the fact

$$
\int_0^{\infty} \frac{s^{\alpha-2}}{1+\mu_0^2 s^{2\alpha}}\mathrm{d}s < +\infty, \alpha \in (1,2),
$$

we know that (4.46) holds. From $\frac{8}{(1-\mu_0 t^{\alpha})^2} \le \frac{32/\pi^2}{1+t^{2\alpha}}$, (4.45), (4.44) and (4.46), we get (4.40) with $M_5 = C_{10}32/\pi^2$, $M_6 = 32C_9 M_1^2(C_2+C_3)^2 \sum_{j=0}^{\infty} b_j^2$ and $\lim_{t\to\infty}\|\widehat{z}_t(\cdot,t)\|_{L^2(0,1)}^2 = 0$. Moreover, by $\widehat{z}_t(x,t) = \widehat{w}_t(x,t) - xp_0 z_t(1,t)$, (3.13), (4.26) and (4.21), denoting $\Psi_j(t) = a_{1j}t^{-1}E_{\alpha,0}(\mu_j t^{\alpha}) + a_{2j}E_{\alpha,1}(\mu_j t^{\alpha})$, we obtain

$$
\begin{aligned}
\|\widehat{w}_t(\cdot,t)\|_{L^2(0,1)}^2 &\le 2\|\widehat{z}_t(\cdot,t)\|_{L^2(0,1)}^2 + 4p_0^2\bigg(\sum_{j=0}^{\infty}\Psi_j(t)\bigg)^2 \\
&\le 2\|\widehat{z}_t(\cdot,t)\|_{L^2(0,1)}^2 + 8p_0^2\bigg(\sum_{j=0}^{\infty}\frac{C_2^2}{(1-\mu_j t^{\alpha})^2}\sum_{j=0}^{\infty}a_{2j}^2 + \sum_{j=0}^{\infty}\frac{C_8^2}{(1-\mu_j t^{\alpha})^2}\sum_{j=0}^{\infty}(-\mu_j)^{2/\alpha}a_{1j}^2\bigg).
\end{aligned}
$$

Similar to the inequality (4.38), by $\sum_{j=0}^{\infty}a_{1j}^2 + \sum_{j=0}^{\infty}(-\mu_j)^{2/\alpha}a_{2j}^2 \le \|(z_0,z_1)\|_{[H^2(0,1)]^2}^2$, we derive

$$
\begin{aligned}
\|\widehat{w}_t(\cdot,t)\|_{L^2(0,1)}^2 \le{}& \frac{2M_5 + 8p_0^2(C_2^2 C_7 + C_8^2 C_7)}{1+t^{2\alpha}}\|(z_0,z_1,\widehat{z}_0,\widehat{z}_1)\|_{[H^2(0,1)]^4}^2 \\
&+2\bigg(\int_0^t \frac{(t-s)^{\alpha-2}}{1+\mu_0^2(t-s)^{2\alpha}}(1+s)\bigg(\sum_{k=0}^{\infty}\frac{1}{(1-\mu_k s^{\alpha})^2}\bigg)^{1/2}\mathrm{d}s\bigg)^2 M_6\|(z_0,z_1)\|_{[H^2(0,1)]^2}^2.
\end{aligned}
\tag{4.47}
$$

In view of the bounded transformation

$$
\begin{pmatrix} u_t(x,t) \\ \widehat{u}_t(x,t) \end{pmatrix} = \begin{pmatrix} (I+\mathbb{P}_1)^{-1} & -(I+\mathbb{P}_2) \\ (I+\mathbb{P}_1)^{-1} & 0 \end{pmatrix}\begin{pmatrix} \widehat{w}_t(x,t) \\ z_t(x,t) \end{pmatrix},
\tag{4.48}
$$

and the simple fact that $\widehat{u}_0(x) = (I+\mathbb{P}_1)^{-1}\widehat{w}_0(x) = (I+\mathbb{P}_1)^{-1}(\widehat{z}_0(x) + xp_0 z_0(1))$, $\widehat{u}_1(x) = (I+\mathbb{P}_1)^{-1}\widehat{w}_1(x) = (I+\mathbb{P}_1)^{-1}(\widehat{z}_1(x) + xp_0 z_1(1))$, $u_0(x) = (I+\mathbb{P}_1)^{-1}\widehat{w}_0(x) - (I+\mathbb{P}_2)z_0(x)$, it follows from (4.38) and Lemma 3.1 that (4.17) holds with some $M > 0$ and $\lim_{t\to\infty}\|(u_t(\cdot,t),\widehat{u}_t(\cdot,t))\|_{[L^2(0,1)]^2} = 0$. This completes the proof of the theorem. $\square$

**Remark 4.3.** Different from the stability proof for fractional diffusion equation in [41], which is based on the fractional Lyapunov method and Riesz basis method, here it is difficult to use fractional Lyapunov method for the fractional order $\alpha \in (1,2)$ and the inequality $_{0}^{C}D_{t}^{\alpha}z^2(t) \leq 2z(t)_{0}^{C}D_{t}^{\alpha}z(t)$ with $\alpha \in (1,2)$, instead, the Riesz basis approach is adopted to prove Theorem 4.2 through estimating the explicit expression of solution.

**Remark 4.4.** Similar to Remark 3.3, we emphasize that Theorem 4.2 is true only for $\alpha \in (1,2)$ and the fractional order $\alpha$ in (4.15) cannot be $\alpha = 2$. This is because the proof of Theorem 4.2 is based on Lemma 2.1 and requires the condition on the boundedness of $M(\alpha)$ (given in Remark 2.4) which can be seen from (4.26) and (4.43). However, when $\alpha = 2$, $M(2) = +\infty$.

## 5. Disturbance rejection *via* state feedback

In this section, as addressed in [21], it is assumed that a disturbance force $d(t)$ comes from the same end where the boundary control input is imposed, that is, the stabilization and disturbance rejection is considered for a system described by

$$\begin{cases} {}_{0}^{C}D_{t}^{\alpha}u(x,t) = u_{xx}(x,t) + \lambda(x)u(x,t), \ x \in (0,1), t \geq 0, \\ u(0,t) = 0, \\ u_x(1,t) = U(t) + d(t), \ t \geq 0, \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ 0 \leq x \leq 1, \end{cases} \tag{5.1}$$

where $\alpha \in (1,2)$ is the order of the fractional derivative, $u(x,t)$ is the displacement of wave propagation, $\lambda(x) \in C[0,1]$, $U(t)$ is the control input, and $d(t)$ is the boundary disturbance.

Now, we come to the second objective of the paper to seek a control law $U(t)$ to stabilize (5.1) and reject the disturbance $d(t)$. Obviously, once this is done, by taking $\lambda(x) = 0$, we give a complete solution to problem raised in [21].

To achieve Mittag-Leffler stability for system (5.1) and reject the disturbance $d(t)$, the following assumption about $d(t)$ is assumed.

**Assumption 5.1.** The disturbance $d(t)$ satisfies $d \in L^{\infty}(0,\infty)$ and $_{0}^{C}D_{t}^{\alpha}d(\cdot) \in L^{\infty}(0,\infty)$.

The examples of such kinds of disturbances satisfying Assumption 5.1 include all finite sum of harmonic disturbances like $d(t) = a_j \sin(\omega_j t)$ with unknown amplitude $a_j$ and unknown frequency $\omega_j$. To see this, for any given frequency $\omega$, we prove that $_{0}^{C}D_{t}^{\alpha}\sin(\omega t), _{0}^{C}D_{t}^{\alpha}\cos(\omega t)$ are uniformly bounded for all $t \geq 0$, *i.e.*,

$$\sup_{t \geq 0}|_{0}^{C}D_{t}^{\alpha}\sin(\omega t)| < +\infty \quad \text{and} \quad \sup_{t \geq 0}|_{0}^{C}D_{t}^{\alpha}\cos(\omega t)| < +\infty.$$

Indeed, by Caputo's derivative, we have

$$|_{0}^{C}D_{t}^{\alpha}\sin(\omega t)| = \frac{\omega^2}{\Gamma(2-\alpha)}\left|\int_0^t (t-s)^{1-\alpha}\sin(\omega s)\mathrm{d}s\right|$$
$$= \frac{\omega^2}{\Gamma(2-\alpha)}\left|\int_0^t s^{1-\alpha}\sin(\omega(t-s))\mathrm{d}s\right| = \frac{\omega^2}{\Gamma(2-\alpha)}$$
$$\times \left|\sin(\omega t)\int_0^t s^{1-\alpha}\cos(\omega s)\mathrm{d}s - \cos(\omega t)\int_0^t s^{1-\alpha}\sin(\omega s)\mathrm{d}s\right|.$$

It follows from page 284, A.4.11 of [11] that

$$\int_0^\infty s^{1-\alpha}\cos(\omega s)\mathrm{d}s = \frac{\Gamma(2-\alpha)}{\omega^\alpha}\sin\left(\frac{\alpha\pi}{2}\right) \ \text{ and } \ \int_0^\infty s^{1-\alpha}\sin(\omega s)\mathrm{d}s = -\frac{\Gamma(2-\alpha)}{\omega^\alpha}\cos\left(\frac{\alpha\pi}{2}\right)$$

hold for $\alpha \in (1,2)$, which infers that both $\int_0^t s^{1-\alpha} \cos(\omega s) \mathrm{d}s$ and $\int_0^t s^{1-\alpha} \sin(\omega s) \mathrm{d}s$ are bounded over $[0, \infty)$. According to the boundedness of $\sin(\omega t)$ and $\cos(\omega t)$, we know that $\sup_{t \geq 0} |{}_0^C D_t^\alpha \sin(\omega t)| < +\infty$ and thus ${}_0^C D_t^\alpha \sin(\omega t) \in L^\infty(0, \infty)$. Similarly, one can show that ${}_0^C D_t^\alpha \cos(\omega \cdot) \in L^\infty(0, \infty)$.

Since any periodic signal can be expanded through Fourier decomposition and can be approximated by the finite sum of periodic harmonic signal [3], the disturbance signal satisfying Assumption 5.1 is much general, and Assumption 5.1 seems to be reasonable. It is worth pointing out that the noise $d(t)$ in [21] is assumed to be a sinusoidal disturbance signal with an unknown amplitude and phase but with a known frequency $\omega$. Clearly, Assumption 5.1 includes these sinusoidal signals and is much more general than the one of [21].

It should be noted that for the case of no disturbance, the collocated feedback control (3.25) will stabilize Mittag-Leffler the system (5.1). Nevertheless, this designed stabilizer (3.25) is not robust to the external disturbance. For instance, when $d(t) \equiv C \neq 0$ is a constant, from the transformation (3.1) and (3.6), it is easy to verify that the system (5.1) under the control (3.25) has a solution:

$$u(x,t) = Cx + C \int_0^x l(x,\xi)\xi \mathrm{d}\xi \in L^2(0,1) \tag{5.2}$$

with $l(x,y)$ being the solution of (3.9). So, when there is the external disturbance, the control must be re-designed.

Since a stabilizing control law for system without disturbance force $d(t)$ has been designed in Sections 3 and 4, it is natural to find a disturbance estimator to estimate the disturbance force $d(t)$ and compensate for the disturbance in the closed-loop.

To estimate the disturbance, following [7], we propose a disturbance estimator as follows:

$$\begin{cases} {}_0^C D_t^\alpha v(x,t) = v_{xx}(x,t) + \lambda(x)u(x,t), x \in (0,1), t \geq 0, \\ v(0,t) = 0, \quad v_x(1,t) = U(t), \ t \geq 0, \\ {}_0^C D_t^\alpha z(x,t) = z_{xx}(x,t), \quad x \in (0,1), \ t \geq 0, \\ z(0,t) = 0, \quad z(1,t) = v(1,t) - u(1,t), \ t \geq 0, \\ v(x,0) = v_0(x), \ v_t(x,0) = v_1(x), \ 0 \leq x \leq 1, \\ z(x,0) = z_0(x), \ z_t(x,0) = z_1(x), \ 0 \leq x \leq 1, \end{cases} \tag{5.3}$$

which is an infinite-dimensional system with the state consisting of the functions $v$, $z$, defined on $(0,1)$. It is seen that system (5.3) is completely determined by the displacement $u(x,t)$ and input $U(t)$. In other words, system (5.3) is a completely known system. Since the disturbance estimator (5.3) proposed here looks complicated, now let us explain why we make such a construction.

Firstly, a "$v$"-part of system (5.3) is to bring the disturbance from original system to a Mittag-Leffler stable system. Indeed, let $\widehat{v}(x,t) = v(x,t) - u(x,t)$, it is easy to verify that $\widehat{v}(x,t)$ satisfies the following time fractional wave equation:

$$\begin{cases} {}_0^C D_t^\alpha \widehat{v}(x,t) = \widehat{v}_{xx}(x,t), \quad x \in (0,1), \ t \geq 0, \\ \widehat{v}(0,t) = 0, \quad \widehat{v}_x(1,t) = -d(t), \ t \geq 0, \\ \widehat{v}(x,0) = \widehat{v}_0(x), \ \widehat{v}_t(x,0) = \widehat{v}_1(x), \ 0 \leq x \leq 1. \end{cases} \tag{5.4}$$

**Lemma 5.2.** *Let Assumption 5.1 hold and $\alpha \in (1,2)$. For any initial value $(\widehat{v}_0, \widehat{v}_1) \in [L^2(0,1)]^2$, there exists a unique solution to (5.4) such that $\widehat{v} \in C(0, \infty; L^2(0,1))$ satisfying $\sup_{t \geq 0} \|\widehat{v}(\cdot, t)\|_{L^2(0,1)} < +\infty$. Moreover, when $\lim_{t \to \infty} d(t) = \lim_{t \to \infty} {}_0^C D_t^\alpha d(t) = 0$, it has $\lim_{t \to \infty} \|\widehat{v}(\cdot, t)\|_{L^2(0,1)} = 0$. Furthermore, when $(\widehat{v}_0(\cdot), \widehat{v}_1(\cdot)) \in [H^2(0,1)]^2$, there holds $\sup_{t \geq 0} \|(\widehat{v}(\cdot, t), \widehat{v}_t(\cdot, t))\|_{L^2(0,1)} < +\infty$.*

*Proof.* Since the disturbance $d(t)$ in (5.4) is at the boundary, we define a function $g(x) = x$, $x \in [0,1]$ and introduce the variable $\widetilde{v}(x,t) = \widehat{v}(x,t) + g(x)d(t)$ to shift $d(t)$ from the boundary into the in-domain. Indeed,

we can verify that $\widetilde{v}(x,t)$ is governed by

$$\begin{cases} {}_0^C D_t^\alpha \widetilde{v}(x,t) = \widetilde{v}_{xx}(x,t) + g(x){}_0^C D_t^\alpha d(t), x \in (0,1), t \geq 0, \\ \widetilde{v}(0,t) = 0, \quad \widetilde{v}_x(1,t) = 0, \ t \geq 0, \\ \widetilde{v}(x,0) = \widetilde{v}_0(x), \ \widetilde{v}_t(x,0) = \widetilde{v}_1(x), \ 0 \leq x \leq 1, \end{cases} \tag{5.5}$$

where $\widetilde{v}_0(x) = \widehat{v}_0(x) + g(x)d(0)$ and $\widetilde{v}_1(x) = \widehat{v}_1(x) + g(x)\dot{d}(0)$. Since the function $g$ is in $L^2(0,1)$, we write $g(x)$ as $g(x) = \sum_{j=0}^\infty a_j e_j(x)$ where the Fourier's coefficient $a_j$ is given by $a_j = \int_0^1 e_j(x)x\mathrm{d}x$. Since $\{e_j(x)\}$ forms an orthnormal basis for $L^2(0,1)$, we write the solution of (5.5) as

$$\widetilde{v}(x,t) = \sum_{j=0}^\infty \phi_j(t)e_j(x). \tag{5.6}$$

The initial states are $\widetilde{v}_0(x) = \sum_{j=0}^\infty b_j e_j(x)$, $\widetilde{v}_1(x) = \sum_{j=0}^\infty c_j e_j(x)$, where $b_j = \int_0^1 e_j(x)\widetilde{v}_0(x)\mathrm{d}x$, $c_j = \int_0^1 e_j(x)\widetilde{v}_1(x)\mathrm{d}x$, with $\{b_j\}, \{c_j\} \in l^2$ and $\phi_j \in C(0,\infty;\mathbb{R})$ satisfying

$$ {}_0^C D_t^\alpha \phi_j(t) = \mu_j \phi_j(t) + a_j {}_0^C D_t^\alpha d(t), \tag{5.7}$$

and $\phi_j(0) = b_j$, $\phi_j'(0) = c_j$. In view of Theorem 4.3 in [19], the solution of (5.7) is calculated to be

$$\begin{aligned} \phi_j(t) &= b_j E_\alpha(\mu_j t^\alpha) + c_j t E_{\alpha,2}(\mu_j t^\alpha) \\ &\quad + a_j \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\mu_j(t-s)^\alpha){}_0^C D_t^\alpha d(s)\mathrm{d}s. \end{aligned} \tag{5.8}$$

Since $\{e_j(x)\}$ is an orthonormal basis for $L^2(0,1)$, by the inequality $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$ for any $a,b,c \in \mathbb{R}$, it follows from (5.6), (5.8) and (4.26) that

$$\begin{aligned} \|\widetilde{v}(\cdot,t)\|_{L^2(0,1)}^2 &= \sum_{j=0}^\infty \phi_j^2(t) \leq 3\sum_{j=0}^\infty \left[ b_j^2(E_\alpha(\mu_j t^\alpha))^2 + c_j^2(tE_{\alpha,2}(\mu_j t^\alpha))^2 \right. \\ &\quad \left. + a_j^2\left(\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\mu_j(t-s)^\alpha){}_0^C D_t^\alpha d(s)\mathrm{d}s\right)^2 \right]. \end{aligned} \tag{5.9}$$

It follows from Lemmas 2.2 and 2.3 that there exists a constant $M_1 > 0$ such that $E_\alpha(\mu_j t^\alpha) \leq \frac{M_1}{1-\mu_j t^\alpha}$, $tE_{\alpha,2}(\mu_j t^\alpha) \leq \frac{M_1 t}{1-\mu_j t^\alpha}$, $|E_{\alpha,\alpha}(\mu_j t^\alpha)| \leq \frac{M_1}{1+\mu_j^2 t^{2\alpha}}$. Since $\lim_{t\to\infty} \frac{M_1^2(1+t^2)}{(1-\mu_0 t^\alpha)^2} = 0$, it follows from (4.33) and (5.9) that

$$\begin{aligned} \|\widetilde{v}(\cdot,t)\|_{L^2(0,1)}^2 &\leq 3\sum_{j=0}^\infty \left[ b_j^2 \frac{M_1^2}{(1-\mu_j t^\alpha)^2} + c_j^2 \frac{M_1^2 t^2}{(1-\mu_j t^\alpha)^2} \right. \\ &\quad \left. + a_j^2 \|D_t^\alpha d(\cdot)\|_{L^\infty(0,\infty)}^2 \left(\int_0^t \frac{M_1^2(t-s)^{\alpha-1}}{1+\mu_j^2(t-s)^{2\alpha}}\mathrm{d}s\right)^2 \right] \\ &\leq 3\frac{M_1^2(1+t^2)}{(1-\mu_0 t^\alpha)^2} \sum_{j=0}^\infty [b_j^2 + c_j^2] \\ &\quad + \|D_t^\alpha d(\cdot)\|_{L^\infty(0,\infty)}^2 \left(\int_0^t \frac{M_1^2 s^{\alpha-1}}{1+\mu_0^2 s^{2\alpha}}\mathrm{d}s\right)^2 \sum_{j=0}^\infty a_j^2 < +\infty. \end{aligned}$$

This, together witgh Assumption 5.1 and $\widehat{v}(x,t) = \widetilde{v}(x,t) - xd(t)$, gives $\sup_{t\geq 0} \|\widehat{v}(\cdot,t)\|_{L^2(0,1)} < +\infty$. Now, suppose $\lim_{t\to\infty} {}_0^C D_t^\alpha d(t) = 0$. We claim that $\lim_{t\to\infty} \|\widehat{v}(\cdot,t)\|_{L^2(0,1)} = 0$. From the proof of the boundedness of $\sup_{t\geq 0} \|\widetilde{v}(\cdot,t)\|_{L^2(0,1)}^2$ and

$$\left| \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\mu_j(t-s)^\alpha) {}_0^C D_t^\alpha d(s) \mathrm{d}s \right| \leq \int_0^t \frac{M_1(t-s)^{\alpha-1}}{1+\mu_0^2(t-s)^{2\alpha}} |{}_0^C D_t^\alpha d(s)| \mathrm{d}s, \ j \in \mathbb{N},$$

it suffices to show that

$$\lim_{t\to\infty} \int_0^t \frac{M_1(t-s)^{\alpha-1}}{1+\mu_0^2(t-s)^{2\alpha}} |{}_0^C D_t^\alpha d(s)| \mathrm{d}s = 0. \tag{5.10}$$

Indeed, for any given $\varepsilon > 0$, since $\lim_{t\to\infty} {}_0^C D_t^\alpha d(t) = 0$, we can take $T > 0$ sufficiently large so that $|{}_0^C D_t^\alpha d(t)| \leq \varepsilon$ for all $t \geq T$. By $\lim_{t\to\infty} \max_{s\in[0,T]} \frac{M_1 T(t-s)^{\alpha-1}}{1+\mu_0^2(t-s)^{2\alpha}} = 0$, there exists $T' > T$ such that $|\max_{s\in[0,T]} \frac{M_1 T(t-s)^{\alpha-1}}{1+\mu_0^2(t-s)^{2\alpha}}| \leq \varepsilon$. Thus,

$$\begin{aligned}
\int_0^t \frac{M_1(t-s)^{\alpha-1}}{1+\mu_0^2(t-s)^{2\alpha}} |D_t^\alpha d(s)| \mathrm{d}s &\leq \varepsilon \|D_t^\alpha d(\cdot)\|_{L^\infty(0,\infty)} + \varepsilon \int_T^t \frac{M_1(t-s)^{\alpha-1}}{1+\mu_0^2(t-s)^{2\alpha}} \mathrm{d}s \\
&\leq \varepsilon \|D_t^\alpha d(\cdot)\|_{L^\infty(0,\infty)} + \varepsilon \int_0^\infty \frac{M_1 s^{\alpha-1}}{1+\mu_0^2 s^{2\alpha}} \mathrm{d}s,
\end{aligned}$$

which implies that (5.10) holds. Since $\widehat{v}(x,t) = \widetilde{v}(x,t) - xd(t)$, it has $\sup_{t\geq 0} \|\widehat{v}(\cdot,t)\|_{L^2(0,1)} < +\infty$. Next, suppose that $(\widehat{v}_0, \widehat{v}_1) \in [H^2(0,1)]^2$. We claim that $\sup_{t\geq 0} \|(\widehat{v}(\cdot,t), \widehat{v}_t(\cdot,t))\|_{L^2(0,1)} < +\infty$. Actually, from (5.8),

$$\begin{aligned}
\dot{\phi}_j(t) &= b_j t^{-1} E_{\alpha,0}(\mu_j t^\alpha) + c_j E_{\alpha,1}(\mu_j t^\alpha) \\
&\quad + a_j \int_0^t (t-s)^{\alpha-2} E_{\alpha,\alpha-1}(\mu_j(t-s)^\alpha) {}_0^C D_t^\alpha d(s) \mathrm{d}s.
\end{aligned} \tag{5.11}$$

By Lemma 2.2, (4.42), (4.43) and (5.11), we estimate $\|\widetilde{v}_t(\cdot,t)\|_{L^2(0,1)}^2$ to yield

$$\begin{aligned}
\|\widetilde{v}_t(\cdot,t)\|_{L^2(0,1)}^2 &= \sum_{j=0}^\infty \dot{\phi}_j^2(t) \leq 3 \sum_{j=0}^\infty \left[ b_j t^{-1} E_{\alpha,0}(\mu_j t^\alpha) + c_j E_{\alpha,1}(\mu_j t^\alpha) \right. \\
&\quad \left. + a_j^2 \left( \int_0^t (t-s)^{\alpha-2} E_{\alpha,\alpha-1}(\mu_j(t-s)^\alpha) {}_0^C D_t^\alpha d(s) \mathrm{d}s \right)^2 \right] \\
&\leq 3 \sum_{j=0}^\infty \left[ b_j^2 \frac{M_2^2(-\mu_j)^{2/\alpha}}{(1-\mu_j t^\alpha)^2} + c_j^2 \frac{M_2^2}{(1-\mu_j t^\alpha)^2} \right. \\
&\quad \left. + a_j^2 M_2^2 \|D_t^\alpha d(\cdot)\|_{L^\infty(0,\infty)}^2 \left( \int_0^t \frac{(t-s)^{\alpha-2}}{1+\mu_j^2(t-s)^{2\alpha}} \mathrm{d}s \right)^2 \right] \\
&\leq 3 \frac{M_2^2(1+t^2)}{(1-\mu_0 t^\alpha)^2} \sum_{j=0}^\infty [\mu_j^2 b_j^2 + c_j^2] \\
&\quad + M_2^2 \|D_t^\alpha d(\cdot)\|_{L^\infty(0,\infty)}^2 \left( \int_0^t \frac{s^{\alpha-2}}{1+\mu_0^2 s^{2\alpha}} \mathrm{d}s \right)^2 \sum_{j=0}^\infty a_j^2
\end{aligned}$$

for some $M_2 > 0$. Since $(\widehat{v}_0, \widehat{v}_1) \in [H^2(0,1)]^2$, it has $(\widetilde{v}_0, \widetilde{v}_1) \in [H^2(0,1)]^2$, which gives $\sum_{j=0}^{\infty}[\mu_j^2 b_j^2 + c_j^2] < +\infty$. From $\alpha \in (1,2)$, it is easy to verify that $\int_0^{\infty} \frac{s^{\alpha-2}}{1+\mu_0^2 s^{2\alpha}} \mathrm{d}s < +\infty$. Hence, $\sup_{t \geq 0} \|\widetilde{v}_t(\cdot, t)\|_{L^2(0,1)} < +\infty$, which yields $\sup_{t \geq 0} \|\widehat{v}_t(\cdot, t)\|_{L^2(0,1)} < +\infty$. This ends the proof of the lemma. $\qquad\square$

**Remark 5.3.** The condition $\lim_{t \to \infty} {}_0^C D_t^{\alpha} d(t) = 0$ in Lemma 5.2 does not imply that $d(t)$ tends to zero as time $t$ goes to infinity. For example, $d(t) = t^{\beta}$ with $\beta$ satisfying $1 < \beta < \alpha < 2$. A straight computation gives ${}_0^C D_t^{\alpha} d(t) = \frac{\Gamma(2-\alpha)\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha}$. Clearly, $\lim_{t \to \infty} {}_0^C D_t^{\alpha} d(t) = 0$ but $\lim_{t \to \infty} d(t) = +\infty$. From the proof of Lemma 5.2, it is concluded that in order to ensure $\lim_{t \to \infty} \|\widehat{v}(\cdot, t)\|_{L^2(0,1)} = 0$, the condition $\lim_{t \to \infty} d(t) = 0$ cannot be removed.

**Remark 5.4.** The second statement that $\lim_{t \to \infty} \|\widehat{v}(\cdot, t)\|_{L^2(0,1)} = 0$ of Lemma 5.2 is invalid when $\alpha = 2$. Actually, system (5.4) with $\alpha = 2$ becomes

$$\begin{cases} \widehat{v}_{tt}(x,t) = \widehat{v}_{xx}(x,t), & x \in (0,1), \ t \geq 0, \\ \widehat{v}(0,t) = 0, \quad \widehat{v}_x(1,t) = -d(t), \ t \geq 0, \\ \widehat{v}(x,0) = \widehat{v}_0(x), \ \widehat{v}_t(x,0) = \widehat{v}_1(x), \ 0 \leq x \leq 1, \end{cases} \qquad (5.12)$$

which is the classical wave equation with Neumann boundary control input $-d(t)$. Let $d(t) \equiv 0$. Then, from the boundary condition $\widehat{v}(0,t) = 0, \widehat{v}_x(1,t) = 0$, it is easy to verify that system (5.12) is a conservation system, *i.e.*, $\|\widehat{v}_t(\cdot, t)\|_{L^2(0,1)}^2 + \|\widehat{v}_x(\cdot, t)\|_{L^2(0,1)}^2 = \|\widehat{v}_t(\cdot, 0)\|_{L^2(0,1)}^2 + \|\widehat{v}_x(\cdot, 0)\|_{L^2(0,1)}^2$ for all $t \geq 0$. Moreover, from Remark 3.3, the solution of (5.12) with $d(t) \equiv 0$ is given by $\widehat{v}(x,t) = \sum_{j=0}^{\infty} \left[ a_j \cos\left(j + \frac{1}{2}\right)\pi t + b_j((j + \frac{1}{2})\pi)^{-1}\sin\left(j + \frac{1}{2}\right)\pi t \right] e_j(x)$ where $e_j(x) = \sqrt{2}\sin\left(j + \frac{1}{2}\right)\pi x$, $a_j = \int_0^1 e_j(x)\widehat{v}_0(x)\mathrm{d}x$ and $b_j = \int_0^1 e_j(x)\widehat{v}_1(x)\mathrm{d}x$. Note that there exist $i_0, j_0$ such that $a_{i_0}^2 + b_{j_0}^2 \neq 0$ whence $(\widehat{v}_0, \widehat{v}_1) \neq 0$, we obtain $\|\widehat{v}(\cdot, t)\|_{L^2(0,1)}^2 \geq a_{i_0}^2 \cos^2\left(i_0 + \frac{1}{2}\right)\pi t + b_{j_0}^2((j_0 + \frac{1}{2})\pi)^{-2}\sin^2\left(j_0 + \frac{1}{2}\right)\pi t$, which implies $\lim_{t \to \infty} \|\widehat{v}(\cdot, t)\|_{L^2(0,1)} \neq 0$.

Secondly, the "$z$"-part of system (5.3) is to estimate the disturbance $d(t)$. Actually, let $\widetilde{z}(x,t) = z(x,t) - \widehat{v}(x,t)$, then, we can verify that $\widetilde{z}(x,t)$ is governed by

$$\begin{cases} {}_0^C D_t^{\alpha} \widetilde{z}(x,t) = \widetilde{z}_{xx}(x,t), & x \in (0,1), \ t \geq 0, \\ \widetilde{z}(0,t) = 0, \quad \widetilde{z}(1,t) = 0, \ t \geq 0, \\ \widetilde{z}(x,0) = \widetilde{z}_0(x), \ \widetilde{z}_t(x,0) = \widetilde{z}_1(x), \ 0 \leq x \leq 1. \end{cases} \qquad (5.13)$$

**Lemma 5.5.** *For any initial value $(\widetilde{z}_0, \widetilde{z}_1) \in [L^2(0,1)]^2$, system (5.13) admits a unique solution $\widetilde{z} \in C(0,\infty; L^2(0,1))$, and there exists a constant $M > 0$ such that*

$$\|\widetilde{z}(\cdot, t)\|_{[L^2(0,1)]^2}^2 \leq \frac{M}{1 + t^{2\alpha-2}} \|(\widetilde{z}_0, \widetilde{z}_1)\|_{[L^2(0,1)]^2}^2.$$

*When $(w_0, w_1) \in [H^2(0,1)]^2$, there exists a constant $M' > 0$ such that*

$$\|(\widetilde{z}(\cdot, t), \widetilde{z}_t(\cdot, t))\|_{[L^2(0,1)]^2}^2 \leq \frac{M'}{1 + t^{2\alpha-2}} \|(\widetilde{z}_0, \widetilde{z}_1)\|_{[H^2(0,1)]^2}^2.$$

*Proof.* Since the proof is similar to the proof of Lemma 3.1, we omit the details. $\qquad\square$

By (3.25), we propose the following control law

$$U(t) = k(1,1)u(1,t) + \int_0^1 k_x(1,y)u(y,t)\mathrm{d}y - (-z_x(1,t)). \qquad (5.14)$$

The purpose of this control law is essentially just cancelling the disturbance by its estimate. The closed-loop system is governed by

$$
\begin{cases}
{}^{C}_{0}D_t^\alpha u(x,t) = u_{xx}(x,t) + \lambda(x)u(x,t), \\
u(0,t) = 0, \quad u_x(1,t) = k(1,1)u(1,t) + \displaystyle\int_0^1 k_x(1,y)u(y,t)\mathrm{d}y + z_x(1,t) + d(t), \\
{}^{C}_{0}D_t^\alpha v(x,t) = v_{xx}(x,t) + \lambda(x)u(x,t), \\
v(0,t) = 0, crrv_x(1,t) = k(1,1)u(1,t) + \displaystyle\int_0^1 k_x(1,y)u(y,t)\mathrm{d}y + z_x(1,t), \\
{}^{C}_{0}D_t^\alpha z(x,t) = z_{xx}(x,t), \\
z(0,t) = 0, \quad z(1,t) = v(1,t) - u(1,t)
\end{cases}
\tag{5.15}
$$

**Theorem 5.6.** *Suppose that Assumption 5.1 holds. System (1.1) can be stabilized by the controller (5.14) and the disturbance estmator (5.3), which gives the closed-loop system (5.15). Moreover, for any initial value $Z_0 := (u_0, v_0, z_0, u_1, v_1, z_1) \in [L^2(0,1)]^6$, there exists a unique solution $(u,v,z) \in C(0,\infty; [L^2(0,1)]^3)$ to (5.15), and there exist two positive constants $M > 0$ such that*

$$
\|u(\cdot,t)\|_{L^2(0,1)}^2 \le \frac{M}{1+t^{2\alpha-2}}\|Z_0\|_{[L^2(0,1)]^6}^2,
\tag{5.16}
$$

*and $\sup_{t\ge 0}\|(v(\cdot,t), z(\cdot,t))\|_{[L^2(0,1)]^2} < +\infty$. Moreover, when $(u_0, v_0, z_0, u_1, v_1, z_1) \in [H^2(0,1)]^6$, there exists a constant $\overline{M}' > 0$ such that*

$$
\|(u(\cdot,t), u_t(\cdot,t))\|_{[L^2(0,1)]^2}^2 \le \frac{M'\|Z_0\|_{[H^2(0,1)]^6}^2}{1+t^{2\alpha-2}},
\tag{5.17}
$$

*and $\sup_{t\ge 0}\|(v(\cdot,t), v_t(\cdot,t), z(\cdot,t), z_t(\cdot,t))\|_{[L^2(0,1)]^4} < +\infty$.*

*Proof.* Using the variables $\widehat{v}(x,t) = v(x,t) - u(x,t)$, $\widetilde{z}(x,t) = z(x,t) - \widehat{v}(x,t)$ and $p(x,t) = w(x,t) - \widetilde{z}(x,t)$ by the transformation (3.1), an equivalent system of (5.15) can be obtained as follows:

$$
\begin{cases}
{}^{C}_{0}D_t^\alpha p(x,t) = p_{xx}(x,t), \quad x \in (0,1), \ t \ge 0, \\
p(0,t) = 0, \quad p_x(1,t) = 0, \ t \ge 0, \\
{}^{C}_{0}D_t^\alpha \widehat{v}(x,t) = \widehat{v}_{xx}(x,t), \quad x \in (0,1), \ t \ge 0, \\
\widehat{v}(0,t) = 0, \quad \widehat{v}_x(1,t) = -d(t), \ t \ge 0, \\
{}^{C}_{0}D_t^\alpha \widetilde{z}(x,t) = \widetilde{z}_{xx}(x,t), \quad x \in (0,1), \ t \ge 0, \\
\widetilde{z}(0,t) = 0, \quad \widetilde{z}(1,t) = 0, \ t \ge 0.
\end{cases}
\tag{5.18}
$$

By Lemma 5.2, $\sup_{t\ge 0}\|\widehat{v}(\cdot,t)\|_{L^2(0,1)} < +\infty$. From Lemmas 3.1 and 5.5, there exists a constant $C > 0$ such that

$$
\|p(\cdot,t)\|_{L^2(0,1)}^2 \le \frac{C}{1+t^{2\alpha-2}}\|(p_0, p_1)\|_{[L^2(0,1)]^2}^2
\tag{5.19}
$$

and

$$
\|\widetilde{z}(\cdot,t)\|_{L^2(0,1)}^2 \le \frac{C}{1+t^{2\alpha-2}}\|(\widetilde{z}_0, \widetilde{z}_1)\|_{[L^2(0,1)]^2}^2.
\tag{5.20}
$$

Noting the invertible transformation

$$
\begin{pmatrix} u(x,t) \\ v(x,t) \\ z(x,t) \end{pmatrix} = \begin{pmatrix} (I+\mathbb{P}_1)^{-1} & 0 & (I+\mathbb{P}_1)^{-1} \\ (I+\mathbb{P}_1)^{-1} & -I & (I+\mathbb{P}_1)^{-1} \\ 0 & I & I \end{pmatrix} \begin{pmatrix} p(x,t) \\ \widehat{v}(x,t) \\ \widetilde{z}(x,t) \end{pmatrix},
\tag{5.21}
$$

it follows from (5.19) and (5.20) that (5.16) holds and $\sup_{t \geq 0} \|(v(\cdot, t), z(\cdot, t))\|_{[L^2(0,1)]^2} < +\infty$. Next, suppose that $(u_0, v_0, z_0, u_1, v_1, z_1) \in [H^2(0, 1)]^6$. Since

$$\begin{pmatrix} u_t(x,t) \\ v_t(x,t) \\ z_t(x,t) \end{pmatrix} = \begin{pmatrix} (I + \mathbb{P}_1)^{-1} & 0 & (I + \mathbb{P}_1)^{-1} \\ (I + \mathbb{P}_1)^{-1} & -I & (I + \mathbb{P}_1)^{-1} \\ 0 & I & I \end{pmatrix} \begin{pmatrix} p_t(x,t) \\ \widehat{v}_t(x,t) \\ \widetilde{z}_t(x,t) \end{pmatrix},$$

by Lemmas 3.1, 5.2 and 5.5 again, and by (5.21), we know that (5.17) and $\sup_{t \geq 0} \|(v(\cdot, t), v_t(\cdot, t), z(\cdot, t), z_t(\cdot, t))\|_{[L^2(0,1)]^4} < +\infty$ hold. This completes the proof of the therem.                    □

**Remark 5.7.** As discussed in Remark 3.5, from (5.16) and (5.17), we can see that the state $(u, u_t)$ of system (5.15) is Mittag-Leffler stable.

**Remark 5.8.** From the proof of Theorem 5.6 and Lemma 5.2, it is seen that if the bondedness of disturbance estimator is not required, then, $d \in L^\infty(0, \infty)$ in Assumption 5.1 is not necessary. For instance, take $d(t) = a + bt$ with some constants $a, b$ with $b \neq 0$, ${}_0^C D_t^\alpha d(t) = 0$ for all $t \geq 0$. Clearly, $d(t) \to \infty$ as $t \to \infty$ and $d \notin L^\infty(0, \infty)$.

**Remark 5.9.** As dicussed in Remarks 3.3 and 4.4, the result of Theorem 5.6 is not valid for $\alpha = 2$ in (5.15).

At the end of this section, we point out that Theorem 5.6 provides a disturbance estimator for system considered in [21] which reads as

$$\begin{cases} {}_0^C D_t^\alpha u(x,t) = u_{xx}(x,t), \quad x \in (0,1), \ t \geq 0, \\ u(0,t) = 0, \quad u_x(1,t) = U(t) + d(t), \ t \geq 0, \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ 0 \leq x \leq 1. \end{cases} \tag{5.22}$$

The disturbance estimator for (5.22) is given by

$$\begin{cases} {}_0^C D_t^\alpha v(x,t) = v_{xx}(x,t), \quad x \in (0,1), \ t \geq 0, \\ v(0,t) = 0, \quad v_x(1,t) = U(t), \ t \geq 0, \\ {}_0^C D_t^\alpha z(x,t) = z_{xx}(x,t), \quad x \in (0,1), \ t \geq 0, \\ z(0,t) = 0, \\ z(1,t) = v(1,t) - u(1,t), \ t \geq 0, \end{cases}$$

which is completely determined by output $u(1, t)$ and input $U(t)$ and thus is an output-based disturbance estimator. By Theorem 5.6, the control $U(t)$ can be designed as $U(t) = z_x(1, t)$ that is based only on the the disturbance estimator. Since the disturbance estimator depends only on $u(1, t)$ and $U(t)$, the control $U(t) = z_x(1, t)$ is an output feedback controller. Moreover, Theorem 5.6 also provides a mathematical proof for the stability of the closed-loop system of system (5.22). Hence, the Problem II arised in [21] is completely solved.

## 6. Numerical simulation

In this section, we present some numerical simulations results for the closed-loop system (5.15). For numerical computations, we take fractional order $\alpha = 1.5$ and $\alpha = 2$. The disturbance is taken as $d(t) = \sin(t)$, and the initial values are $u(x,0) = 2\sin x$, $u_t(x,0) = (x+1)\sin x$, $v(x,0) = 2x^2 + 2\sin x$, $v_t(x,0) = 3x(x-1) + (x+1)\sin x$, $z(x,0) = \cos x - 2x + 2x^2$, $z_t(x,0) = 2x\cos x + 3x(x-1)$, $\lambda(x) = 4$. The numerical algorithm is based on the combination of the $L1$ scheme [31] in time and the second order centred difference scheme [27] in space. We take the spacial step as $dx = 0.001$ and the time step as $dt = 0.01$.

Figure 1 displays the state $(u(\cdot, t), v(\cdot, t), z(\cdot, t))$ of system (5.15) with $\alpha = 1.5$. It can be observed that the control law (5.14) is very effective to make the "$u$-part" of system (5.15) convergent to zero and make the "$(v, z)$-part" of system (5.15) being bounded. Figure 2 shows the state $(u(\cdot, t), v(\cdot, t), z(\cdot, t))$ of system (5.15) with $\alpha = 2$. It is seen that the "$(u, v, z)$-part" of system (5.15) with $\alpha = 2$ is bounded, where the "$u$-part" of

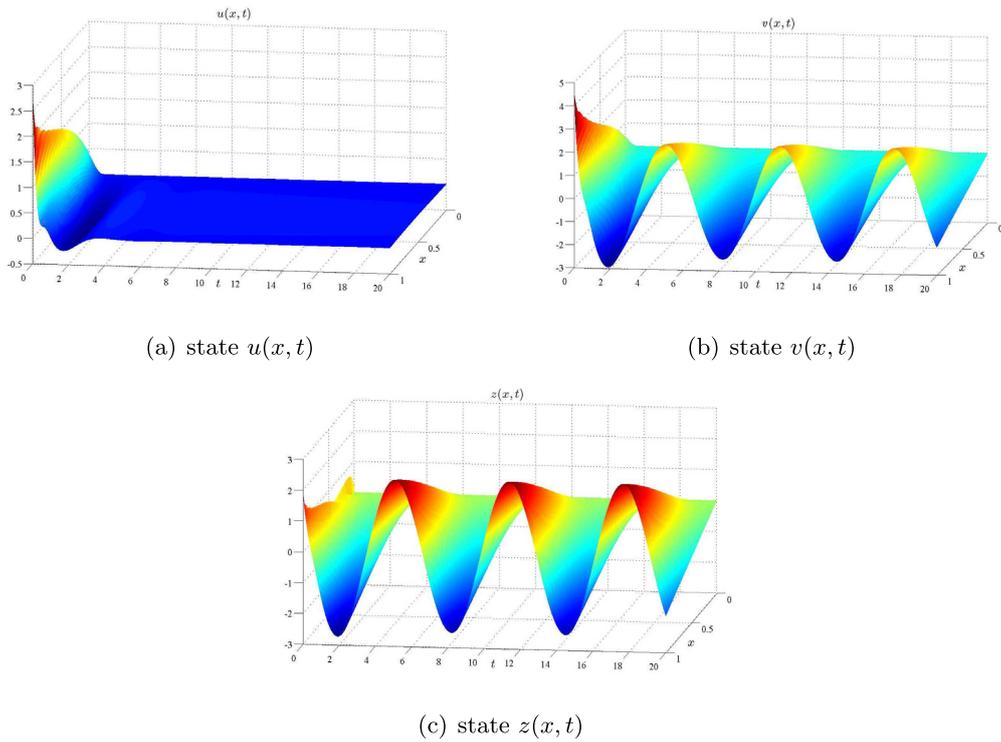(a) state $u(x,t)$

(b) state $v(x,t)$

(c) state $z(x,t)$

FIGURE 1. Simulation for system (5.15) with $\alpha = 1.5$.



(a) state $u(x,t)$

(b) state $v(x,t)$

(c) state $z(x,t)$

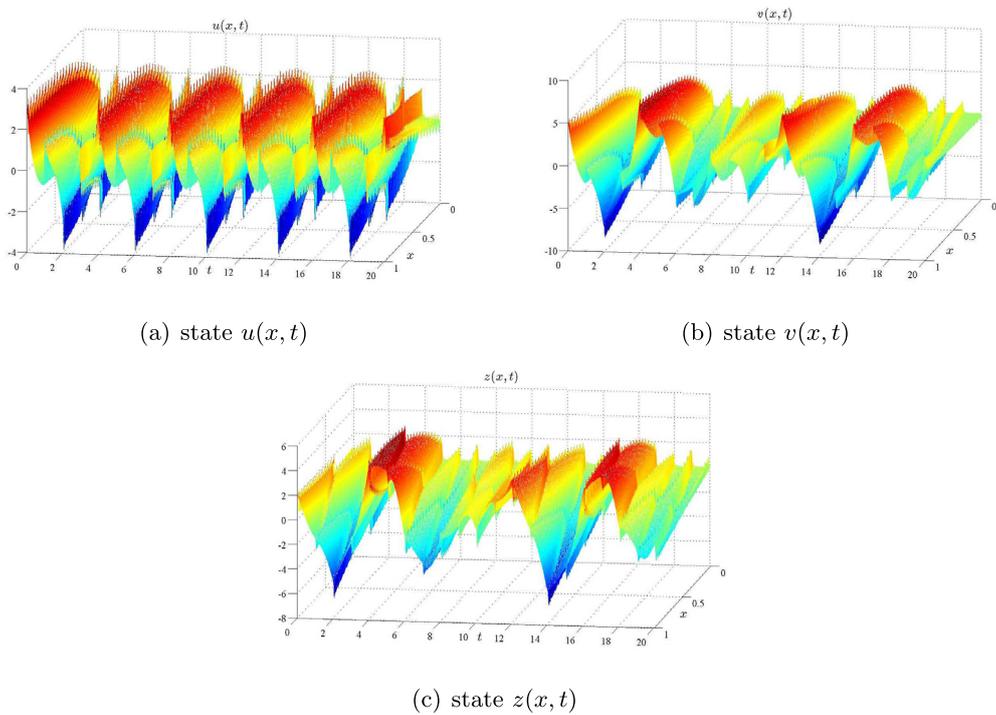FIGURE 2. Simulation for system (5.15) with $\alpha = 2$.

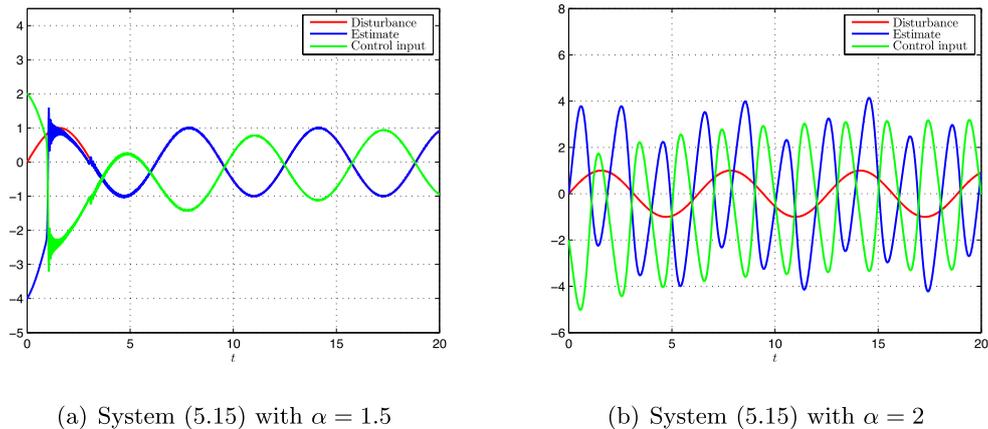(a) System (5.15) with $\alpha = 1.5$        (b) System (5.15) with $\alpha = 2$

FIGURE 3. Disturbance, estimate and control input.

system (5.15) does not converge to zero. This is exactly as that pointed out in Remark 5.9: The control law (5.14) does not work when $\alpha = 2$. Figure 3a and b demonstrates the disturbance $d(t)$, its estimate $-z_x(1,t)$ and the control input $U(t)$ for system (5.15) with $\alpha = 1.5$ and for system (5.15) with $\alpha = 2$, respectively. It is seen that for $\alpha = 1.5$ the disturbance is fast estimated and the control approximates the value $-d(t)$ as the time $t$ is sufficiently large while the disturbance cannot be estimated for $\alpha = 2$. This is similarly as that pointed out in Remark 3.3: The solution of the $\widetilde{z}$-part of system (5.18) with $\alpha = 2$ is not convergent to zero.

## 7. CONCLUDING REMARKS

In this paper, the boundary stabilization of an unstable time fractional diffusion-wave equation system with or without boundary external disturbance is considered. For the system free from boundary external disturbance, we achieve the Mittag-Leffler stability *via* state feedback and asymptotic stability *via* output feedback, by using the backstepping method and Riesz basis approach to the analytic solution of the closed-loop system. For the system subject to boundary external disturbance, we construct an infinite dimensional disturbance estimator consisting of two subsystems to estimate the disturbance and compensate the disturbance in the feedback loop. It is shown that the $u$-part of the resulting closed-loop system is asymptotically sable, while the $(v, z)$-part is bounded. As a result, the paper solves completely two long-standing mathematical control problems raised in [21]. As future works, designing an output feedback controller for system (5.1) by the active disturbance rejection control or sliding mode control method would be an interesting problem. It is worth pointing that the difficulty in designing output feedback for system (5.1) is because the system (5.1) has the term $\lambda(x)u(x,t)$.

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