

RIESZ BASIS PROPERTY AND EXPONENTIAL STABILITY FOR ONE-DIMENSIONAL THERMOELASTIC SYSTEM WITH VARIABLE COEFFICIENTS*

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Abstract. In this paper, we study Riesz basis property and stability for a nonuniform thermoelastic system with Dirichlet-Dirichlet boundary condition, where the heat subsystem is considered as a control to the whole coupled system. By means of the matrix operator pencil method, we obtain the asymptotic expressions of the eigenpairs, which are exactly coincident to the constant coefficients case. We then show that there exists a sequence of generalized eigenfunctions of the system, which forms a Riesz basis for the state space and the spectrum determined growth condition is therefore proved. As a consequence, the exponential stability of the system is concluded.

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1. INTRODUCTION

It is known that the derivation of the classical heat equation presumes the conducting body to be rigid and hence ignores the interaction between thermal effects and mechanical effects. Since many engineering processes are controlled by the temperature, the generalization of the heat equation which incorporates the effect of thermo mechanical coupling and visa versa has been studied by many authors in the last a few decades. A typical phenomenon for thermoelasticity is that the heat dissipation alone is used as a control to stabilize exponentially the thermoelastic system through in-domain and boundary weak connections. A simple one-dimensional linear thermoelasticity for a homogeneous and isotropic body was derived ([3], p. 3) by incorporating the effect of thermo mechanical coupling and the effect of inertia, which is described by

$$\begin{cases} u_{tt}(x, t) - u_{xx}(x, t) + r\theta_x(x, t) = 0, & 0 \leq x \leq 1, t > 0, \\ \theta_t(x, t) + ru_{xt}(x, t) - k\theta_{xx}(x, t) = 0, & 0 \leq x \leq 1, t > 0, \\ u(0, t) = u(1, t) = \theta(0, t) = \theta(1, t) = 0, & t > 0, \end{cases} \quad (1.1)$$

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where $u(x, t)$ represents the displacement of the string vibration, $\theta(x, t)$ represents the absolute temperature, and $k > 0$ represents the thermal conductivity. The coupling constant $r > 0$ which is a measure of the mechanical thermal coupling present in the system is generally much smaller than one. The exponential stability of system (1.1) was first obtained in [17] by frequency domain multiplier method. In [4], the distribution of eigenvalues of system (1.1) was studied, and the Riesz basis property of the system was obtained in [5]. There are some computation errors in [4, 5], which were corrected in a recent monograph ([9], Sect. 3.6). By taking the memory effect into account, the second equation of (1.1) was replaced by

$$\theta_t(x, t) + ru_{xt}(x, t) - (k * \theta_{xx})(x, t) = 0$$

in [25], where the sign $*$ represents the convolution product, and the exponential stability was obtained as well. The exponential stability and analyticity of abstract linear thermoelastic system were studied in [16]. An interesting fact presented in [6] is that the first real eigenvalue of system (1.1) associated with heat equation must be greater than the first eigenvalue of the pure heat equation, guessed from physical point of view in [20]. This explains clearly the interaction of the mechanical-thermal coupling. The heat makes coupled system exponentially stable and conversely, the vibration of the string increases the temperature of the object.

In this paper, we consider system (1.1) with variable coefficients described by

$$\begin{cases} u_{tt}(x, t) - (a(x)u_x(x, t))_x + r\theta_x(x, t) = 0, & 0 \leq x \leq 1, t > 0, \\ \theta_t(x, t) + ru_{xt}(x, t) - (k(x)\theta_x(x, t))_x = 0, & 0 \leq x \leq 1, t > 0, \\ u(0, t) = u(1, t) = \theta(0, t) = \theta(1, t) = 0, & t > 0, \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), \theta(x, 0) = \theta_0(x), & 0 \leq x \leq 1, \end{cases} \quad (1.2)$$

where $a(x) > 0$ represents the velocity of the wave subsystem and $k(x) > 0$ represents the thermal conductivity, both are varying with spatial variable. The total energy of system (1.2) is given by ([3], p.4):

$$E(t) = \frac{1}{2} \int_0^1 [u_t^2(x, t) + a(x)u_x^2(x, t) + \theta^2(x, t)] dx. \quad (1.3)$$

Formally, the derivative of $E(t)$ with respect to time t satisfies

$$\dot{E}(t) = - \int_0^1 k(x)\theta_x^2(x, t) dx, \quad (1.4)$$

which shows that $E(t)$ is non-increasing with time. However, the right-hand side of (1.4) depends only on the gradient of temperature which is considered as controller and does not depend on the elastic subsystem explicitly. Such a weak coupling gives rise to a serious problem for the stability of system (1.2). Although the frequency domain multiplier method developed in [17] might be applicable to obtain the exponential stability of the system, some other profound problems like the spectrum-determined growth condition cannot be answered by the multiplier method. To tackle this problem, we adopt the Riesz basis approach. First, throughout the paper, we always assume that

$$a(x), k(x) \in C^2[0, 1]. \quad (1.5)$$

Our goal of this paper is to show that the variable coefficients thermoelastic system (1.2) is a Riesz spectral system in the energy state space. This follows from the following several facts: (a) the system is well-posed and the system operator has compact resolvent; (b) the spectrum of the system consists of two branches: one branch has an asymptote which is parallel to the imaginary axis from the left side and the other distributes along the

real axis like 1-D heat equation; (c) the generalized eigenfunctions of the system form a Riesz basis for the state space and therefore the spectrum determined growth condition holds. The latter implies the exponential stability of the system.

The difficulty in spectral analysis for the current system is caused by the variable coefficients. A classical method of finding the asymptotic eigenvalues was introduced in Section 4 of [18] which consists of two steps. The first step is to transform the “dominant term” of the eigenfunction into a uniform “dominant term” by space scaling and state transformation, where no variable coefficient is involved any more, and the second step is to approximate the eigenfunctions of the system by those of the uniform “dominant equation”. This approach has been successfully used in dealing with the Euler-Bernoulli beam equations with variable coefficients in [7, 12, 24]. However, this approach is not applicable to the thermoelastic system investigated in this paper because the “dominant term” of the eigenfunction cannot be transformed into a uniform form. In [21], the eigenvalue problem of a Timoshenko beam was written as $Y' = \lambda A(x)Y$ from which the polynomial operator pencil method can be applied. While, the eigenvalue problem of the thermoelastic system is equivalent to $Y' = A(\lambda, x)Y$ where λ and x in $A(\lambda, x)$ are inseparable. This results in failure of the method used in [21] to deal with the thermoelastic system. The essential reason behind is that the thermoelastic system is a coupled system consisting of a wave subsystem and heat subsystem, and the eigenvalues of the two subsystems are of different orders. We therefore first transform the eigenvalue problem of (1.2) into a first order matrix differential equation and find out the explicit asymptotic expression of the matrix fundamental solutions by introducing an invertible matrix function and using the asymptotic technique for the first order matrix differential equation, referred to as the matrix operator pencil method (see, *e.g.*, [22, 23]). This method has been used to spectral analysis for system of coupled partial differential equations in [13, 26]. However, there is a remarkable difference between [13, 26] and the present one in the process of investigation, that is, the invertible function matrix defined in [26] either associates with the eigenvalues only or the spatial variables only [13], while in present paper, the invertible matrix function is associated with both eigenvalues and the spatial variable. This is caused by the fact that the eigenvalues in those systems discussed in [13, 26] are of the same order, which is not true for thermoelastic systems (1.1) and (1.2).

The remaining part of this paper is organized as follows. In the next section, Section 2, we transform system (1.2) into an evolution equation in the energy Hilbert space and the well-posedness of the system is concluded. Asymptotic estimation of the eigenvalues is presented in Section 3. Based on the asymptotic eigenvalues, Section 4 is devoted to calculating the asymptotic expression of the corresponding eigenfunctions. The Riesz basis generation of the system is developed in Section 5, which leads to the spectrum-determined growth condition. As a consequence, the system is shown to be exponentially stable, which covers the main result of [17] on exponential stability of constant coefficients thermoelastic system as a special case.

2. WELL-POSEDNESS OF THE SYSTEM

We consider system (1.2) in the state Hilbert space $\mathcal{H} = H_0^1(0, 1) \times L^2(0, 1) \times L^2(0, 1)$ with the inner product defined by

$$\langle X_1, X_2 \rangle = \int_0^1 [a(x)f_1'(x)\overline{f_2'(x)} + g_1(x)\overline{g_2(x)} + h_1(x)\overline{h_2(x)}] dx$$

for all $X_i = (f_i, g_i, h_i) \in \mathcal{H}, i = 1, 2$. Define the system operator $A : D(A) \subset \mathcal{H} \rightarrow \mathcal{H}$ by

$$\begin{cases} A(f, g, h)^\top = (g(x), (a(x)f'(x))' - \gamma h'(x), -\gamma g'(x) + (k(x)h'(x))')^\top, \forall (f, g, h)^\top \in D(A), \\ D(A) = \{(f, g, h)^\top \in (H^2 \cup H_0^1) \times H_0^1 \times (H_0^1 \cup H^2)\}. \end{cases} \quad (2.1)$$

If we set $X = (u(\cdot, t), u_t(\cdot, t), \theta(\cdot, t))^\top$, then system (1.2) can be formulated into an abstract evolution problem in \mathcal{H} :

$$\begin{cases} \frac{d}{dt}X(t) = AX(t), t > 0, \\ X(0) = (u_0, u_1, \theta_0)^\top. \end{cases} \quad (2.2)$$

Theorem 2.1. *Let A be defined by (2.1). Then, A is dissipative in \mathcal{H} . Moreover, A^{-1} exists and is compact on \mathcal{H} . Therefore, A generates a C_0 -semigroup of contractions on \mathcal{H} and $\sigma(A)$, the spectrum set of A , consists of isolated eigenvalues only.*

Proof. For $X = (f, g, h)^\top \in D(A)$, we compute

$$\begin{aligned} \langle AX, X \rangle &= \langle (g(x), (a(x)f'(x))' - \gamma h'(x), -\gamma g'(x) + (k(x)h'(x))')^\top, (f(x), g(x), h(x))^\top \rangle \\ &= \int_0^1 a(x)g'(x)\overline{f'(x)} + (a(x)f'(x))'\overline{g(x)} - \gamma h'(x)\overline{g(x)} - \gamma g'(x)\overline{h(x)} + (k(x)h'(x))'\overline{h(x)} dx \\ &= \int_0^1 g'(x)\overline{a(x)f'(x)} - a(x)f'(x)\overline{g'(x)} + \gamma h(x)\overline{g'(x)} - \gamma g'(x)\overline{h(x)} - k(x)h'(x)\overline{h'(x)} dx \\ &\quad + a(x)f'(x)\overline{g(x)}|_0^1 - \gamma h(x)\overline{g(x)}|_0^1 + k(x)h'(x)\overline{h(x)}|_0^1 \\ &= \int_0^1 g'(x)\overline{a(x)f'(x)} - a(x)f'(x)\overline{g'(x)} + \gamma h(x)\overline{g'(x)} - \gamma g'(x)\overline{h(x)} - k(x)|h'(x)|^2 dx. \end{aligned}$$

On account of $k(x) > 0$, it holds

$$\operatorname{Re}\langle AX, X \rangle = - \int_0^1 k(x)|h'(x)|^2 dx \leq 0.$$

This shows that A is dissipative. For any $(\phi, \psi, \omega)^\top \in \mathcal{H}$, we seek $(f, g, h)^\top \in D(A)$ such that

$$A(f, g, h)^\top = (\phi, \psi, \omega)^\top,$$

which yields

$$\begin{cases} g(x) = \phi(x), \\ (a(x)f'(x))' - \gamma h'(x) = \psi(x), \\ -\gamma g'(x) + (k(x)h'(x))' = \omega(x), \\ f(0) = f(1) = g(0) = g(1) = h(0) = h(1) = 0. \end{cases} \quad (2.3)$$

Substituting the first equation of (2.3) into the third one, we obtain

$$(k(x)h'(x))' = \omega(x) + \gamma\phi'(x).$$

By $h(0) = h(1) = 0$, a direct calculation gives

$$\begin{cases} h(x) = \int_0^x \int_0^\xi \frac{\omega(\eta)}{k(\xi)} d\eta d\xi + \int_0^x \frac{1}{k(\xi)} (\gamma\phi(\xi) - \gamma\phi(0) + k(0)h'(0)) d\xi, \\ h'(0) = -\frac{1}{\int_0^1 \frac{k(0)}{k(\xi)} d\xi} \left[\int_0^1 \int_0^\xi \frac{\omega(\eta)}{k(\xi)} d\eta d\xi + \int_0^1 \frac{1}{k(\xi)} (\gamma\phi(\xi) - \gamma\phi(0)) d\xi \right]. \end{cases} \quad (2.4)$$

To obtain $f(x)$, we solve

$$\begin{cases} (a(x)f'(x))' = \psi(x) + \gamma h'(x), \\ f(0) = f(1) = 0, \end{cases}$$

to obtain

$$\begin{cases} f(x) = \int_0^x \int_0^\xi \frac{\psi(\eta)}{a(\xi)} d\eta d\xi + \int_0^x \frac{1}{a(\xi)} (\gamma h(\xi) - \gamma h(0) + a(0)f'(0)) d\xi, \\ f'(0) = -\frac{1}{\int_0^1 \frac{a(0)}{a(\xi)} d\xi} \left[\int_0^1 \int_0^\xi \frac{\psi(\eta)}{a(\xi)} d\eta d\xi + \int_0^1 \frac{1}{a(\xi)} (\gamma h(\xi) - \gamma h(0)) d\xi \right], \end{cases}$$

where $h(x)$ is given by (2.4). We thus obtain a unique solution $(f, g, h)^\top$ such that $A(f, g, h)^\top = (\phi, \psi, \omega)^\top$, which in turn implies that A^{-1} exists and is bounded. Moreover, by the Sobolev embedding theorem ([1], Thm. 4.12, p. 85), A^{-1} is compact on \mathcal{H} and thus $\sigma(A)$ consists of isolated eigenvalues ([14], Thm. 6.26, p. 185). Therefore, A generates a C_0 -semigroup of contractions on \mathcal{H} according to the Lumer-Phillips theorem ([19], Thm. 4.3, p. 14). \square

Corollary 2.2. *Let A be defined by (2.1) and let $T(t)$ be the C_0 -semigroup generated by A on \mathcal{H} . Then, $T(t)$ is asymptotically stable on \mathcal{H} , i.e.*

$$\lim_{t \rightarrow \infty} \|T(t)X\| = 0, \quad \forall X \in \mathcal{H}.$$

Proof. From Theorem 2.1, it suffices to show that there is no eigenvalue on the imaginary axis ([15], Thm. 3.26, p. 130). Assume that $\lambda = i\tau \in \sigma(A)$ with $0 \neq \tau \in \mathbb{R}$ and $X = (f, g, h)^\top \in D(A)$ is the corresponding eigenfunction satisfying $AX = i\tau X$. Then, $(f(\cdot), g(\cdot), h(\cdot))$ satisfies the following differential equations:

$$\begin{cases} g(x) = i\tau f(x), & 0 < x < 1, \\ (a(x)f'(x))' - \gamma h'(x) = i\tau g(x), & 0 < x < 1, \\ -\gamma g'(x) + (k(x)h'(x))' = i\tau h'(x), & 0 < x < 1, \\ f(0) = f(1) = g(0) = g(1) = h(0) = h(1) = 0. \end{cases} \quad (2.5)$$

Since

$$\operatorname{Re}\langle i\tau X, X \rangle = \operatorname{Re}\langle AX, X \rangle = -\int_0^1 k(x)|h'(x)|^2 dx = 0,$$

it has $h'(\cdot) \equiv 0$ which together with $h(0) = 0$ gives $h(\cdot) \equiv 0$. From the third equation of (2.5), we have $g'(\cdot) \equiv 0$ which together with $g(0) = 0$ leads to $g(\cdot) \equiv 0$. Hence, $(a(\cdot)f'(\cdot))' \equiv 0$ which together with $f(0) = f(1) = 0$ gives $f(\cdot) \equiv 0$. We have thus proved that (2.5) admits only null solution which contradicts that $(f, g, h)^\top$ is an eigenfunction. Therefore, there is no eigenvalue of A located on the imaginary axis, proving the theorem. \square

3. SPECTRAL ANALYSIS

In this section, we consider the eigenvalue problem for system operator A . For any $(f, g, h)^\top \in D(A)$ and $\operatorname{Re}\lambda < 0$, let $A(f, g, h)^\top = \lambda(f, g, h)^\top$. Then, $g(x) = \lambda f(x)$, and $f(\cdot)$ and $h(\cdot)$ satisfy:

$$\begin{cases} (a(x)f'(x))' - \gamma h'(x) = \lambda^2 f(x), \\ (k(x)h'(x))' - \gamma \lambda f'(x) = \lambda h(x), \\ f(i) = h(i) = 0, i = 0, 1. \end{cases} \quad (3.1)$$

A direct calculation transforms (3.1) into a coupled system of second order ordinary differential equations:

$$\begin{cases} f''(x) = -\frac{a'(x)}{a(x)}f'(x) + \frac{\lambda^2}{a(x)}f(x) + \frac{\gamma}{a(x)}h'(x), \\ h''(x) = \frac{\gamma\lambda}{k(x)}f'(x) - \frac{k'(x)}{k(x)}h'(x) + \frac{\lambda}{k(x)}h(x). \end{cases} \quad (3.2)$$

In order to solve these equations, we shall use matrix operator pencil method which is a standard technique for the asymptotic expression of eigenpairs. Setting $\lambda = \rho^2$ in (3.1), let

$$\Phi = (f, f', h, h')^\top.$$

Then, (3.2) becomes a first order matrix differential equation:

$$\begin{cases} T^D(x, \rho)\Phi(x) := \Phi'(x) - M(x, \rho)\Phi(x) = 0, \\ T^R(x, \rho)\Phi(x) := W^0(\rho)\Phi(0) + W^1(\rho)\Phi(1) = 0, \end{cases} \quad (3.3)$$

where

$$M(x, \rho) := \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{\rho^4}{a(x)} & -\frac{a'(x)}{a(x)} & 0 & \frac{\gamma}{a(x)} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{\gamma\rho^2}{k(x)} & \frac{\rho^2}{k(x)} & -\frac{k'(x)}{k(x)} \end{pmatrix}, \quad (3.4)$$

and

$$W^0(\rho) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad W^1(\rho) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (3.5)$$

In the sequel, we diagonalize the leading term for ρ in (3.4). Define a matrix $P(x, \rho)$ by

$$P(x, \rho) := \begin{pmatrix} 1 & 1 & 1 & 1 \\ -\frac{\rho}{\sqrt{k(x)}} & \frac{\rho}{\sqrt{k(x)}} & -\frac{\rho^2}{\sqrt{a(x)}} & \frac{\rho^2}{\sqrt{a(x)}} \\ \frac{\rho^3\sqrt{k(x)}}{\gamma} & -\frac{\rho^3\sqrt{k(x)}}{\gamma} & -\frac{\gamma\sqrt{a(x)}}{k(x)} & \frac{\gamma\sqrt{a(x)}}{k(x)} \\ -\frac{\rho^4}{\gamma} & -\frac{\rho^4}{\gamma} & \frac{\gamma\rho^2}{k(x)} & \frac{\gamma\rho^2}{k(x)} \end{pmatrix}. \quad (3.6)$$

It is easy to check that for $\text{Re}\lambda < 0$ and $\lambda = \rho^2$, the matrix $P(x, \rho)$ is invertible when $\lambda \neq -\frac{\gamma^2}{k(x)}$. A direct computation gives that

$$P^{-1}(x, \rho) = \begin{pmatrix} \frac{\gamma^2}{2\gamma^2+2\rho^2k(x)} & -\frac{\gamma^2 a(x)\sqrt{k(x)}}{2k(x)^2\rho^5+2\gamma^2 a(x)\rho} & \frac{\gamma\rho k(x)\sqrt{k(x)}}{2k(x)^2\rho^4+2\gamma^2 a(x)} & -\frac{\gamma k(x)}{2k(x)\rho^4+2\gamma^2\rho^2} \\ \frac{\gamma^2}{2\gamma^2+2\rho^2k(x)} & \frac{\gamma^2 a(x)\sqrt{k(x)}}{2k(x)^2\rho^5+2\gamma^2 a(x)\rho} & -\frac{\gamma\rho k(x)\sqrt{k(x)}}{2k(x)^2\rho^4+2\gamma^2 a(x)} & -\frac{\gamma k(x)}{2k(x)\rho^4+2\gamma^2\rho^2} \\ \frac{\rho^2 k(x)}{2\gamma^2+2\rho^2k(x)} & \frac{\rho^2\sqrt{a(x)k(x)^2}}{2k(x)^2\rho^4+2\gamma^2 a(x)} & -\frac{\gamma\sqrt{a(x)k(x)}}{2k(x)^2\rho^4+2\gamma^2 a(x)} & \frac{\gamma k(x)}{2k(x)\rho^4+2\gamma^2\rho^2} \\ \frac{\rho^2 k(x)}{2\gamma^2+2\rho^2k(x)} & \frac{\rho^2\sqrt{a(x)k(x)^2}}{2k(x)^2\rho^4+2\gamma^2 a(x)} & \frac{\gamma\sqrt{a(x)k(x)}}{2k(x)^2\rho^4+2\gamma^2 a(x)} & \frac{\gamma k(x)}{2k(x)\rho^4+2\gamma^2\rho^2} \end{pmatrix}, \quad (3.7)$$

where $\lambda = \rho^2 \neq -\frac{\gamma^2}{k(x)}$ and $\text{Re}\lambda < 0$. Transform $\Phi(x)$ through

$$\Psi(x, \rho) = P^{-1}(x, \rho)\Phi(x). \quad (3.8)$$

Taking the derivative of (3.8) with respect to x and substituting $\Phi'(x)$ given by the first equation of (3.3), we obtain:

$$\begin{aligned} \frac{\partial\Psi(x, \rho)}{\partial x} &= P^{-1}(x, \rho) \left[M(x, \rho) + \frac{\partial P(x, \rho)}{\partial x} P^{-1}(x, \rho) \right] P(x, \rho)\Psi(x, \rho) \\ &= \left[\widehat{M}(x, \rho) - P^{-1}(x, \rho) \frac{\partial P(x, \rho)}{\partial x} \right] \Psi(x, \rho), \end{aligned} \quad (3.9)$$

where $\widehat{M}(x, \rho) = P^{-1}(x, \rho)M(x, \rho)P(x, \rho)$. By using power series expansion, we find that $\widehat{M}(x, \rho)$ and $P^{-1}(x, \rho)\frac{\partial P(x, \rho)}{\partial x}$ have the following asymptotic expression as $|\rho| \rightarrow \infty$:

$$\widehat{M}(x, \rho) = M_1(x)\rho^2 + M_2(x)\rho + M_3(x) + M_5(x)\frac{1}{\rho} + O(\rho^{-2}), \quad (3.10)$$

and

$$P^{-1}(x, \rho)\frac{\partial P(x, \rho)}{\partial x} = M_4(x) + M_6(x)\frac{1}{\rho} + O(\rho^{-2}), \quad (3.11)$$

where

$$M_1(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{a(x)}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{a(x)}} \end{pmatrix}, \quad M_2(x) = \begin{pmatrix} -\frac{1}{\sqrt{k(x)}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{k(x)}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.12)$$

$$M_3(x) = \begin{pmatrix} -\frac{k'(x)}{2k(x)} & -\frac{k'(x)}{2k(x)} & 0 & 0 \\ -\frac{k'(x)}{2k(x)} & -\frac{k'(x)}{2k(x)} & 0 & 0 \\ \frac{\sqrt{a(x)+k'(x)}}{2k(x)} & \frac{\sqrt{a(x)+k'(x)}}{2k(x)} & -\frac{\gamma^2}{2\sqrt{a(x)k(x)}} - \frac{a'(x)}{2a(x)} & \frac{a'(x)}{2a(x)} - \frac{\gamma^2}{2\sqrt{a(x)k(x)}} \\ \frac{k'(x)-\sqrt{a(x)}}{2k(x)} & \frac{k'(x)-\sqrt{a(x)}}{2k(x)} & \frac{\gamma^2}{2\sqrt{a(x)k(x)}} + \frac{a'(x)}{2a(x)} & \frac{\gamma^2}{2\sqrt{a(x)k(x)}} - \frac{a'(x)}{2a(x)} \end{pmatrix}, \quad (3.13)$$

$$M_5(x) = \begin{pmatrix} \frac{\gamma^2}{2k(x)\sqrt{k(x)}} & -\frac{\gamma^2}{2k(x)\sqrt{k(x)}} & 0 & 0 \\ \frac{\gamma^2}{2k(x)\sqrt{k(x)}} & -\frac{\gamma^2}{2k(x)\sqrt{k(x)}} & 0 & 0 \\ -\frac{a'(x)}{2\sqrt{a(x)}\sqrt{k(x)}} - \frac{\gamma^2}{2k(x)\sqrt{k(x)}} & \frac{a'(x)}{2\sqrt{a(x)}\sqrt{k(x)}} + \frac{\gamma^2}{2k(x)\sqrt{k(x)}} & 0 & 0 \\ \frac{a'(x)}{2\sqrt{a(x)}\sqrt{k(x)}} - \frac{\gamma^2}{2k(x)\sqrt{k(x)}} & \frac{\gamma^2}{2k(x)\sqrt{k(x)}} - \frac{a'(x)}{2\sqrt{a(x)}\sqrt{k(x)}} & 0 & 0 \end{pmatrix}, \quad (3.14)$$

and

$$M_4(x) = \begin{pmatrix} \frac{k'(x)}{4k(x)} & -\frac{k'(x)}{4k(x)} & 0 & 0 \\ -\frac{k'(x)}{4k(x)} & \frac{k'(x)}{4k(x)} & 0 & 0 \\ 0 & 0 & -\frac{a'(x)}{4a(x)} & \frac{a'(x)}{4a(x)} \\ 0 & 0 & \frac{a'(x)}{4a(x)} & -\frac{a'(x)}{4a(x)} \end{pmatrix}, \quad M_6(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{\sqrt{a(x)}k'(x)}{2k(x)\sqrt{k(x)}} & \frac{\sqrt{a(x)}k'(x)}{2k(x)\sqrt{k(x)}} & 0 & 0 \\ \frac{\sqrt{a(x)}k'(x)}{2k(x)\sqrt{k(x)}} & -\frac{\sqrt{a(x)}k'(x)}{2k(x)\sqrt{k(x)}} & 0 & 0 \end{pmatrix}. \quad (3.15)$$

Thus,

$$\frac{\partial \Psi(x, \rho)}{\partial x} = [M_1(x)\rho^2 + M_2(x)\rho + M_3(x) - M_4(x) + (M_5(x) - M_6(x))\frac{1}{\rho} + O(\rho^{-2})]\Psi(x, \rho). \quad (3.16)$$

On the basis of these transformations, we are now in a position to find a asymptotic expression for the fundamental matrix solution of system (3.9).

Theorem 3.1. *Let A be defined by (2.1). Then, $\lambda = -\frac{\gamma^2}{k(x)} \notin \sigma(A)$ for any $x \in [0, 1]$. For $\text{Re} \lambda < 0$, let $\widehat{M}(x, \rho)$ and $P^{-1}(x, \rho)\frac{\partial P(x, \rho)}{\partial x}$ be given by (3.10) and (3.11) respectively, and for $x \in [0, 1]$, set*

$$E(x, \rho) := \text{diag}[F_1(x, \rho), F_2(x, \rho), F_3(x, \rho), F_4(x, \rho)], \quad (3.17)$$

where

$$\begin{cases} F_1(x, \rho) = e^{-\rho \int_0^x \frac{1}{\sqrt{k(\xi)}} d\xi}, & F_2(x, \rho) = e^{\rho \int_0^x \frac{1}{\sqrt{k(\xi)}} d\xi}, \\ F_3(x, \rho) = e^{-\rho^2 \int_0^x \frac{1}{\sqrt{a(\xi)}} d\xi}, & F_4(x, \rho) = e^{\rho^2 \int_0^x \frac{1}{\sqrt{a(\xi)}} d\xi}. \end{cases} \quad (3.18)$$

Then, there exists a fundamental matrix solution $\widehat{\Psi}(x, \rho)$ for system (3.9), which satisfies

$$\frac{\partial \widehat{\Psi}(x, \rho)}{\partial x} = [\widehat{M}(x, \rho) - P^{-1}(x, \rho)\frac{\partial P(x, \rho)}{\partial x}]\widehat{\Psi}(x, \rho), \quad (3.19)$$

such that for large enough $|\rho|$,

$$\widehat{\Psi}(x, \rho) = \left(\widehat{\Psi}_0(x) + \frac{\widehat{\Psi}_1(x)}{\rho} + \frac{\widehat{\Psi}_2(x)}{\rho^2} + \dots \right) E(x, \rho), \quad (3.20)$$

where

$$\widehat{\Psi}_0(x) := \text{diag}[q_1(x), q_1(x), q_2(x)Q_1(x), q_2(x)Q_2(x)], \quad (3.21)$$

and all entries of $\hat{\Psi}_i(x)$ are uniformly bounded in $[0, 1]$, $i = 0, 1, 2, \dots$. Here,

$$q_1(x) = \left(\frac{k(0)}{k(x)}\right)^{3/4}, q_2(x) = \left(\frac{a(0)}{a(x)}\right)^{1/4}, Q_1(x) = e^{-\int_0^x \frac{\gamma^2}{2\sqrt{a(\xi)k(\xi)}} d\xi}, Q_2(x) = e^{\int_0^x \frac{\gamma^2}{2\sqrt{a(\xi)k(\xi)}} d\xi} \quad (3.22)$$

are uniformly bounded in $[0, 1]$.

Proof. We first assume that $\lambda = \rho^2 \neq -\frac{\gamma^2}{k(x)}$ for any $x \in [0, 1]$. Substituting (3.12) into (3.10), we obtain

$$M_1(x)\rho^2 + M_2(x)\rho = \text{diag} \left\{ -\frac{\rho}{\sqrt{k(x)}}, \frac{\rho}{\sqrt{k(x)}}, -\frac{\rho^2}{\sqrt{a(x)}}, \frac{\rho^2}{\sqrt{a(x)}} \right\},$$

which is a diagonal matrix. It follows that $E(\cdot, \rho)$ given by (3.17) is a fundamental matrix solution to

$$\frac{\partial E(x, \rho)}{\partial x} = (M_1(x)\rho^2 + M_2(x)\rho) E(x, \rho),$$

in which the right-hand side involves the higher order terms of ρ from the right side of (3.19). Now we look for a fundamental matrix solution of (3.19) given by (3.20). Taking the derivative of both sides of (3.20) with respect to x leads to

$$\begin{aligned} \frac{\partial \hat{\Psi}(x, \rho)}{\partial x} &= \left(\hat{\Psi}'_0(x) + \frac{\hat{\Psi}'_1(x)}{\rho} + \frac{\hat{\Psi}'_2(x)}{\rho^2} + \dots \right) E(x, \rho) \\ &+ \left(\hat{\Psi}_0(x) + \frac{\hat{\Psi}_1(x)}{\rho} + \frac{\hat{\Psi}_2(x)}{\rho^2} + \dots \right) (M_1(x)\rho^2 + M_2(x)\rho) E(x, \rho). \end{aligned}$$

Comparing it with the right-hand side of (3.19):

$$\begin{aligned} &\left(M_1(x)\rho^2 + M_2(x)\rho + M_3(x) - M_4(x) + (M_5(x) - M_6(x))\frac{1}{\rho} + O(\rho^{-2}) \right) \\ &\quad \left(\hat{\Psi}_0(x) + \frac{\hat{\Psi}_1(x)}{\rho} + \frac{\hat{\Psi}_2(x)}{\rho^2} + \dots \right) E(x, \rho), \end{aligned}$$

and letting each coefficient with same power of ρ of both hands be equal, we arrive at

$$\hat{\Psi}_0(x)M_1(x) = M_1(x)\hat{\Psi}_0(x), \quad (3.23)$$

$$\hat{\Psi}_0(x)M_2(x) + \hat{\Psi}_1(x)M_1(x) = M_2(x)\hat{\Psi}_0(x) + M_1(x)\hat{\Psi}_1(x), \quad (3.24)$$

$$\hat{\Psi}_1(x)M_2(x) + \hat{\Psi}_2(x)M_1(x) + \hat{\Psi}'_0(x) = (M_3(x) - M_4(x))\hat{\Psi}_0(x) + M_2(x)\hat{\Psi}_1(x) + M_1(x)\hat{\Psi}_2(x), \quad (3.25)$$

$$\begin{aligned} &\hat{\Psi}_2(x)M_2(x) + \hat{\Psi}_3(x)M_1(x) + \hat{\Psi}'_1(x) \\ &= (M_5(x) - M_6(x))\hat{\Psi}_0(x) + (M_3(x) - M_4(x))\hat{\Psi}_1(x) \\ &\quad + M_2(x)\hat{\Psi}_2(x) + M_1(x)\hat{\Psi}_3(x), \end{aligned} \quad (3.26)$$

⋮

According to the argument in [22], we conclude that there is an asymptotic fundamental matrix solution $\hat{\Psi}(\cdot, \rho)$ for system (3.19). The proof will be accomplished if the leading order term $\hat{\Psi}_0(\cdot)$ is given by (3.21). Indeed, $\hat{\Psi}_0(\cdot)$ can be determined by (3.23), (3.24) and (3.25), and then $\hat{\Psi}_1(\cdot)$ can be obtained by (3.24), (3.25), (3.26) and the $\hat{\Psi}_0(\cdot)$ obtained above. Similarly, $\hat{\Psi}_i(\cdot)$, $i = 2, 3, \dots$ in (3.20) can also be deduced if the coefficients of ρ^{-i} , $i = 2, 3, \dots$ in (3.10) and (3.11) are given.

Now we find the leading term $\hat{\Psi}_0(\cdot)$. Let us denote $c_{ij}(\cdot)$ as the (i, j) -entry of the matrix $\hat{\Psi}_0(\cdot)$ with $i, j = 1, 2, \dots, 6$. Substituting $M_1(\cdot)$ given by (3.12) into (3.23), we can obtain that the entries $c_{ij}(\cdot)$ of $\hat{\Psi}_0(\cdot)$ satisfy

$$\begin{cases} c_{ij}(x) = 0 & \text{if } 1 \leq i \leq 2, 3 \leq j \leq 4, \\ c_{ij}(x) = 0 & \text{if } 3 \leq i \leq 4, 1 \leq j \leq 4, i \neq j. \end{cases}$$

Substituting them and $M_1(\cdot)$ and $M_2(\cdot)$ given by (3.12) into (3.24), we have $c_{12}(\cdot) = c_{21}(\cdot) = 0$ immediately. Hence, $\hat{\Psi}_0(\cdot) = \text{diag}\{c_{11}(\cdot), c_{22}(\cdot), c_{33}(\cdot), c_{44}(\cdot)\}$ can be found by substituting into (3.25) as

$$\begin{cases} c'_{11}(x) = -\frac{3k'(x)}{4k(x)}c_{11}(x), \\ c'_{22}(x) = -\frac{3k'(x)}{4k(x)}c_{22}(x), \\ c'_{33}(x) = -\frac{2\gamma^2\sqrt{a(x)+k(x)}a'(x)}{4a(x)k(x)}c_{33}(x), \\ c'_{44}(x) = \frac{2\gamma^2\sqrt{a(x)-k(x)}a'(x)}{4a(x)k(x)}c_{44}(x). \end{cases}$$

The (3.21) then follows from $\hat{\Psi}_0(0) = I$. This completes the proof of the theorem. What left is to check whether $\lambda = -\frac{\gamma^2}{k(x)}$ is an eigenvalue of A . For any fixed $x_0 \in [0, 1]$, we solve $AX = \lambda X$ with $X = (f, g, h)^\top$ and $\lambda = -\frac{\gamma^2}{k(x_0)}$ to obtain

$$\begin{cases} g(x) = -\frac{\gamma^2}{k(x_0)}f(x) \\ (a(x)f'(x))' - \gamma h'(x) = -\frac{\gamma^2}{k(x_0)}g(x), \\ -\gamma g'(x) + (k(x)h'(x))' = -\frac{\gamma^2}{k(x_0)}h(x). \end{cases} \quad (3.27)$$

Setting $\Phi = (f, f', h, h')^\top$, the (3.27) equals a first order differential equation system:

$$\Phi'(x) = M(x)\Phi(x), \quad (3.28)$$

where

$$M(x) := \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{\gamma^4}{k^2(x_0)a(x)} & -\frac{a'(x)}{a(x)} & 0 & \frac{\gamma}{a(x)} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{\gamma^3}{k(x_0)k(x)} & -\frac{\gamma^2}{k(x_0)k(x)} & -\frac{k'(x)}{k(x)} \end{pmatrix}.$$

It is easy to check that $M(x)$ is continuous in $[0, 1]$. Hence, we obtain the existence and uniqueness of the solution for (3.28) under the boundary conditions $f(0) = f(1) = h(0) = h(1)$. Meanwhile, as $f(\cdot) = h(\cdot) \equiv 0$ is a solution of (3.28), system (3.27) has only null solution which contradicts that $\lambda = -\frac{\gamma^2}{k(x_0)}$ is an eigenvalue. \square

Note that in the process of proving Theorem 3.1, $\hat{\Psi}_1(\cdot)$ is of the following form:

$$\hat{\Psi}_1(x) = \begin{pmatrix} d_{11}(x) & d_{12}(x) & 0 & 0 \\ d_{21}(x) & d_{22}(x) & 0 & 0 \\ 0 & 0 & d_{33}(x) & 0 \\ 0 & 0 & 0 & d_{44}(x) \end{pmatrix}, \quad (3.29)$$

and the explicit expressions of the nonzero terms can be obtained by (3.26), which are not listed here. By virtue of the transformation for $\hat{\Psi}(\cdot, \rho)$ in (3.8), we obtain immediately the relationship between systems (3.9) and (3.3) as the following Corollary 3.2.

Corollary 3.2. *For $\operatorname{Re} \lambda < 0$, let $\hat{\Psi}(\cdot, \rho)$ given by (3.20) be a fundamental matrix solution of system (3.9). Then,*

$$\hat{\Phi}(x, \rho) := P(x, \rho) \hat{\Psi}(x, \rho) \quad (3.30)$$

is a fundamental matrix solution for the system (3.3).

Next, we estimate asymptotically the eigenvalues of system (2.2). By the fundamental matrix obtained in Theorem 3.1, $\lambda = \rho^2 \in \sigma(A)$ if and only if ρ satisfies

$$\Delta(\rho) = \det \left(T^R \hat{\Phi}(x, \rho) \right) = 0, \quad \rho \in \mathbb{C}, \quad (3.31)$$

where the operator T^R is defined in (3.3) and $\hat{\Phi}(x, \rho)$ is any fundamental matrix of $T^D(x, \rho)\Phi(x) = 0$, which is similar to [22].

By Theorem 3.1, any eigenvalue $\lambda = \rho^2 \neq -\frac{\gamma^2}{k(x)}$ for any $x \in [0, 1]$. Substituting (3.20) and (3.30) into (3.31), the boundary condition in (3.3) implies that

$$T^R \hat{\Phi}(x, \rho) = W^0(\rho)P(0, \rho)\hat{\Psi}(0, \rho) + W^1(\rho)P(1, \rho)\hat{\Psi}(1, \rho), \quad (3.32)$$

where $W^0(\rho)$ and $W^1(\rho)$ are given by (3.5). Using (3.5) and (3.6), a direct computation gives

$$W^0(\rho)P(0, \rho) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ \rho^3 p_1 & -\rho^3 p_1 & -p_2 & p_2 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$W^1(\rho)P(1, \rho) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ \rho^3 p_3 & -\rho^3 p_3 & -p_4 & p_4 \end{pmatrix},$$

where

$$p_1 = \frac{\sqrt{k(0)}}{\gamma}, p_2 = \frac{\sqrt{a(0)}\gamma}{k(0)}, p_3 = \frac{\sqrt{k(1)}}{\gamma}, p_4 = \frac{\sqrt{a(1)}\gamma}{k(1)}. \quad (3.33)$$

For notational simplicity, set

$$[a]_1 := a + O(\rho^{-1}).$$

Since $\widehat{\Psi}_0(0) = I$, and $E(0, \rho) = I$, a direct computation gives

$$W^0(\rho)P(0, \rho)\widehat{\Psi}(0, \rho) = \begin{pmatrix} [1]_1 & [1]_1 & [1]_1 & [1]_1 \\ 0 & 0 & 0 & 0 \\ \rho^3[p_1]_1 & -\rho^3[p_1]_1 & -[p_2]_1 & [p_2]_1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$W^1(\rho)P(1, \rho)\widehat{\Psi}(1, \rho) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ F_1[p_5]_1 & F_2[p_5]_1 & F_3Q_1[p_6]_1 & F_4Q_2[p_6]_1 \\ 0 & 0 & 0 & 0 \\ \rho^3F_1[p_3p_5]_1 & -\rho^3F_2[p_3p_5]_1 & -F_3Q_1[p_4p_6]_1 & F_4Q_2[p_4p_6]_1 \end{pmatrix},$$

where

$$\begin{cases} p_5 = \left(\frac{k(0)}{k(1)}\right)^{3/4}, p_6 = \left(\frac{a(0)}{a(1)}\right)^{1/4}, F_1 = e^{-\rho \int_0^1 \frac{1}{\sqrt{k(\xi)}} d\xi}, F_2 = e^{\rho \int_0^1 \frac{1}{\sqrt{k(\xi)}} d\xi}, \\ F_3 = e^{-\rho^2 \int_0^1 \frac{1}{\sqrt{a(\xi)}} d\xi}, F_4 = e^{\rho^2 \int_0^1 \frac{1}{\sqrt{a(\xi)}} d\xi}, Q_1 = e^{-\int_0^1 \frac{\gamma^2}{2\sqrt{a(\xi)k(\xi)}} d\xi}, Q_2 = e^{\int_0^1 \frac{\gamma^2}{2\sqrt{a(\xi)k(\xi)}} d\xi}. \end{cases} \quad (3.34)$$

Therefore,

$$\Delta(\rho) = \det(T^R\widehat{\Phi}(x, \rho)) = \begin{vmatrix} [1]_1 & [1]_1 & [1]_1 & [1]_1 \\ F_1[p_5]_1 & F_2[p_5]_1 & F_3Q_1[p_6]_1 & F_4Q_2[p_6]_1 \\ \rho^3[p_1]_1 & -\rho^3[p_1]_1 & -[p_2]_1 & [p_2]_1 \\ \rho^3F_1[p_3p_5]_1 & -\rho^3F_2[p_3p_5]_1 & -F_3Q_1[p_4p_6]_1 & F_4Q_2[p_4p_6]_1 \end{vmatrix}. \quad (3.35)$$

This leads to the following Lemma 3.3.

Lemma 3.3. *Let A be defined by (2.1). For any $\rho \in \mathbb{C}$, $\lambda = \rho^2 \in \sigma(A)$ if and only if ρ satisfies*

$$\Delta(\rho) = \det(T^R\widehat{\Phi}(x, \rho)) = 0,$$

where $\Delta(\rho)$ is given by (3.35).

Theorem 3.4. *Let A be defined by (2.1). Then, the eigenvalues of A contain at least two families:*

$$\{\lambda_{1n}, \overline{\lambda_{1n}}, n \in \mathbb{N}\} \cup \{\lambda_{2n}, n \in \mathbb{N}\} \subset \sigma_p(A),$$

where λ_{1n} and λ_{2n} have the following asymptotic expansions:

$$\begin{cases} \lambda_{1n} = -\mu_1\mu_2 + i\mu_2n\pi + O\left(\frac{1}{n}\right), \\ \lambda_{2n} = -\mu_3^2n^2\pi^2 + O(1), \end{cases} \quad (3.36)$$

with $\mu_1 = \int_0^1 \frac{\gamma^2}{2\sqrt{a(\xi)k(\xi)}} d\xi$, $\mu_2 = \frac{1}{\left(\int_0^1 \frac{1}{\sqrt{a(\xi)}} d\xi\right)}$, $\mu_3 = \frac{1}{\left(\int_0^1 \frac{1}{\sqrt{k(\xi)}} d\xi\right)}$ and n being the positive integers.

Proof. Since all eigenvalues are symmetric to the real axis, we only need to consider those λ which lie in the second quadrant of the complex plane:

$$\lambda := \rho^2, \rho \in \mathcal{S} := \left\{ \rho \in \mathbb{C} \mid \frac{\pi}{4} \leq \arg \rho \leq \frac{\pi}{2} \right\}.$$

It is easy to check that for any $\rho \in \mathcal{S}$,

$$\operatorname{Re}(-\rho) \leq 0, \operatorname{Re}(\rho^2) \leq 0. \quad (3.37)$$

Since the positive functions $a(\cdot)$ and $k(\cdot)$ are bounded, a direct computation shows that $p_i (i = 1, 2, \dots, 6)$ and Q_1, Q_2 are constants and $|F_1|, |F_4|$ are not larger than 1, according to the expressions of $p_i (i = 1, 2, \dots, 6)$, $Q_j (j = 1, 2)$ and $F_k (k = 1, 2, \dots, 4)$ given by (3.33) and (3.34).

By multiplying some factors, we make each entry of the determinant $\Delta(\rho)$ be bounded as $|\rho| \rightarrow \infty$:

$$\frac{\rho^6}{F_1 F_4} \Delta(\rho) = \begin{vmatrix} [1]_1 & F_1 [1]_1 & F_4 [1]_1 & [1]_1 \\ F_1 [p_5]_1 & [p_5]_1 & Q_1 [p_6]_1 & F_4 Q_2 [p_6]_1 \\ [p_1]_1 & -F_1 [p_1]_1 & -\frac{1}{\rho^3} F_4 [p_2]_1 & \frac{1}{\rho^3} [p_2]_1 \\ F_1 [p_3 p_5]_1 & -[p_3 p_5]_1 & -\frac{1}{\rho^3} Q_1 [p_4 p_6]_1 & \frac{1}{\rho^3} F_4 Q_2 [p_4 p_6]_1 \end{vmatrix}. \quad (3.38)$$

Denote $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ with

$$\mathcal{S}_1 = \{ \rho \in \mathbb{C} \mid \pi/4 \leq \arg \rho \leq 3\pi/8 \}, \quad \mathcal{S}_2 = \{ \rho \in \mathbb{C} \mid 3\pi/8 \leq \arg \rho \leq \pi/2 \}. \quad (3.39)$$

When $\rho \in \mathcal{S}_1$, we have

$$\operatorname{Re}(-\rho) = -|\rho| \cos(\arg \rho) \leq -|\rho| \cos(3\pi/8) < 0.$$

Thus, there exists a positive constant C_1 such that

$$|F_1| = O(e^{-C_1|\rho|}), \quad |F_4| = O(1) \text{ as } |\rho| \rightarrow \infty. \quad (3.40)$$

It is easily seen that

$$\frac{\rho^6}{F_1 F_4} \Delta(\rho) = \begin{vmatrix} [1]_1 & 0 & F_4 [1]_1 & [1]_1 \\ 0 & [p_5]_1 & Q_1 [p_6]_1 & F_4 Q_2 [p_6]_1 \\ [p_1]_1 & 0 & 0 & 0 \\ 0 & -[p_3 p_5]_1 & 0 & 0 \end{vmatrix} + O(\rho^{-3}).$$

From this, we see that $\Delta(\rho) = 0$ if and only if

$$\begin{vmatrix} [p_1]_1 & 0 \\ 0 & -[p_3 p_5]_1 \end{vmatrix} \begin{vmatrix} F_4 [1]_1 & [1]_1 \\ Q_1 [p_6]_1 & F_4 Q_2 [p_6]_1 \end{vmatrix} = O(\rho^{-3}),$$

which is equivalent to

$$(Q_1 - F_4^2 Q_2) [p_1 p_3 p_5 p_6]_1 = O(\rho^{-3}).$$

Since Q_1 , Q_2 and F_4 are bounded, there holds

$$F_4^2 = e^{2\rho^2 \int_0^1 \frac{1}{\sqrt{a(\xi)}} d\xi} = \frac{Q_1}{Q_2} + O(\rho^{-1}) = e^{-\int_0^1 \frac{\gamma^2}{\sqrt{a(\xi)k(\xi)}} d\xi} + O(\rho^{-1}). \quad (3.41)$$

Since the solutions of the equation

$$e^{2\rho^2 \int_0^1 \frac{1}{\sqrt{a(\xi)}} d\xi} = e^{-\int_0^1 \frac{\gamma^2}{\sqrt{a(\xi)k(\xi)}} d\xi}$$

are given by

$$\tilde{\lambda}_{1n} = \tilde{\rho}_{1n}^2 = -\mu_1\mu_2 + i\mu_2n\pi, \quad n \in \mathbb{Z},$$

it follows from Rouché's theorem that the solutions to (3.41) are in the form

$$\lambda_{1n} = \rho_{1n}^2 = -\mu_1\mu_2 + i\mu_2n\pi + O(n^{-1}), \quad n \in \mathbb{Z} \text{ and } |n| \rightarrow \infty, \quad (3.42)$$

and hence

$$\rho_{1n} = (-1)^{1/4} \sqrt{\mu_2 n \pi} + \frac{(-1)^{3/4} \mu_1 \sqrt{\mu_2}}{2\sqrt{n\pi}} + O(n^{-3/2}). \quad (3.43)$$

Similarly, when $\rho \in \mathcal{S}_2$, it is easy to verify that

$$\operatorname{Re}(\rho^2) = |\rho|^2 \cos(\arg(\rho^2)) \leq |\rho|^2 \cos(3\pi/4) < 0,$$

Thus, there exists a positive constant C_2 , such that

$$|F_1| = O(1), \quad |F_4| = O(e^{-C_2|\rho|^2}) \text{ as } |\rho| \rightarrow \infty. \quad (3.44)$$

Now

$$\frac{\rho^6}{F_1 F_4} \Delta(\rho) = \begin{vmatrix} [1]_1 & F_1[1]_1 & 0 & [1]_1 \\ F_1[p_5]_1 & [p_5]_1 & Q_1[p_6]_1 & 0 \\ [p_1]_1 & -F_1[p_1]_1 & 0 & 0 \\ F_1[p_3 p_5]_1 & -[p_3 p_5]_1 & 0 & 0 \end{vmatrix} + O(\rho^{-3}).$$

Hence $\Delta(\rho) = 0$ if and only if

$$\begin{vmatrix} [p_1]_1 & -F_1[p_1]_1 \\ F_1[p_3 p_5]_1 & -[p_3 p_5]_1 \end{vmatrix} \begin{vmatrix} 0 & [1]_1 \\ Q_1[p_6]_1 & 0 \end{vmatrix} = O(\rho^{-3}), \quad (3.45)$$

which is equivalent to

$$(F_1^2 - 1)Q_1[p_1 p_3 p_5 p_6]_1 = O(\rho^{-3}).$$

Therefore, F_1 satisfies

$$F_1^2 = e^{-2\rho \int_0^1 \frac{1}{\sqrt{k(\xi)}} d\xi} = 1 + O(\rho^{-1})$$

which gives

$$\rho_{2n} = i\mu_3 n\pi + O(n^{-1}), \quad n \in \mathbb{Z}. \quad (3.46)$$

Therefore, the second branch of eigenvalues $\lambda_{2n} = \rho_{2n}^2$ has the asymptotic expression given by (3.36). The proof is complete. \square

Notice that the asymptotic expressions (3.36) are exactly the same as those obtained in Section 3.6 of [9], for the eigenvalues of the thermoelastic system with constant coefficients, which given by

$$\begin{cases} \lambda_{1n} = -\frac{\gamma^2}{2k} + in\pi - 2\gamma^2 k^{-1} \sqrt{\frac{k}{n\pi}} \frac{1-i}{\sqrt{2}} + O\left(\frac{1}{n}\right), \\ \lambda_{2n} = -kn^2\pi^2 + (4 + k^{-1})\gamma^2 + O\left(\frac{1}{n}\right). \end{cases}$$

However, we do not know whether λ_{2n} are real until Corollary 5.6 in Section 5.

4. ASYMPTOTIC BEHAVIOR OF EIGENFUNCTIONS

In this section, we shall consider the asymptotic expression of eigenfunctions which are solutions of (3.2) with respect to eigenvalues given in Theorem 3.4.

Theorem 4.1. *Let A be defined by (2.1) and $\{\lambda_{1n}, \overline{\lambda_{1n}}, n \in \mathbb{N}\} \cup \{\lambda_{2n}, n \in \mathbb{N}\}$ be a subset of $\sigma_p(A)$. Let $\lambda_{1n} = \rho_{1n}^2$ and $\lambda_{2n} = \rho_{2n}^2$ with ρ_{1n} and ρ_{2n} being given by (3.43) and (3.46) respectively. Then, there are two families of asymptotic eigenfunctions of A :*

- (1) *One family $\{X_{1n}(x) = (\lambda_{1n}^{-1} f_{1n}(x), f_{1n}(x), \lambda_{1n}^{-1} h_{1n}(x))^\top\}$, where X_{1n} is the eigenfunction of A corresponding to the eigenvalue λ_{1n} , has the following asymptotic expression:*

$$(\lambda_{1n}^{-1} f'_{1n}(x), f_{1n}(x), \lambda_{1n}^{-1} h_{1n}(x))^\top = \left(\frac{1}{\sqrt{a(x)}} \cos(a_n(x)), i \sin(a_n(x)), 0 \right)^\top + O(n^{-3/2}) \quad (4.1)$$

as $n \rightarrow \infty$, where

$$a_n(x) = \mu_2 n\pi \int_0^x \frac{1}{\sqrt{a(\xi)}} d\xi + i \left(\mu_1 \mu_2 \int_0^x \frac{1}{\sqrt{a(\xi)}} d\xi - \int_0^x \frac{\gamma^2}{2\sqrt{a(\xi)}k(\xi)} d\xi \right) + O(n^{-1}). \quad (4.2)$$

- (2) *The other family $\{X_{2n}(x) = (\lambda_{2n}^{-1} f_{2n}(x), f_{2n}(x), \lambda_{2n}^{-1} h_{2n}(x))^\top\}$, where X_{2n} is the eigenfunction of A corresponding to the eigenvalue λ_{2n} , has the following asymptotic expression:*

$$(\lambda_{2n}^{-1} f'_{2n}(x), f_{2n}(x), \lambda_{2n}^{-1} h_{2n}(x))^\top = \left(0, \sin \left(\mu_3 n\pi \int_0^x \frac{d\xi}{\sqrt{k(\xi)}} + O(n^{-1}) \right) \right)^\top + O(n^{-1}) \quad (4.3)$$

as $n \rightarrow \infty$.

Proof. Suppose that $\Phi(x) = (f(x), f'(x), h(x), h'(x))^\top$ is the solution of (3.3) with respect to the eigenvalue $\lambda = \rho^2$, we can obtain the i -th component of $\Phi(x)$ by taking the determinants of the matrices which are replaced one of the row of $T^R \widehat{\Phi}(x, \rho)$ in (3.32) with the i -th row of the fundamental matrix $\widehat{\Phi}(x, \rho)$ given by (3.30) so that the determinants are non zeros. To obtain the eigenfunctions which are solutions of the eigen system (3.1),

it is sufficient to calculate $f(x)$, $f'(x)$ and $h(x)$, namely the first three components of $\widehat{\Phi}(x)$. From (3.30), we obtain the fundamental matrix solution of system (3.9) as

$$\widehat{\Phi}(x, \rho) = (B_1, B_2)$$

with

$$B_1 = \begin{pmatrix} F_1(x, \rho)q_1(x)[1]_1 & F_2(x, \rho)q_1(x)[1]_1 \\ -\frac{\rho}{\sqrt{k(x)}}F_1(x, \rho)q_1(x)[1]_1 & \frac{\rho}{\sqrt{k(x)}}F_2(x, \rho)q_1(x)[1]_1 \\ \rho^3 F_1(x, \rho)p_1(x)q_1(x)[1]_1 & -\rho^3 F_2(x, \rho)p_1(x)q_1(x)[1]_1 \\ -\frac{\rho^4}{\gamma}F_1(x, \rho)q_1(x)[1]_1 & -\frac{\rho^4}{\gamma}F_2(x, \rho)q_1(x)[1]_1 \end{pmatrix},$$

and

$$B_2 = \begin{pmatrix} F_3(x, \rho)Q_1(x)q_2(x)[1]_1 & F_4(x, \rho)Q_2(x)q_2(x)[1]_1 \\ -\frac{\rho^2}{\sqrt{a(x)}}F_3(x, \rho)Q_1(x)q_2(x)[1]_1 & \frac{\rho^2}{\sqrt{a(x)}}F_4(x, \rho)Q_2(x)q_2(x)[1]_1 \\ -F_3(x, \rho)Q_1(x)p_2(x)q_2(x)[1]_1 & F_4(x, \rho)Q_2(x)p_2(x)q_2(x)[1]_1 \\ \frac{\gamma\rho^2}{k(x)}F_3(x, \rho)Q_1(x)q_2(x)[1]_1 & \frac{\gamma\rho^2}{k(x)}F_4(x, \rho)Q_2(x)q_2(x)[1]_1 \end{pmatrix},$$

where $p_1(x) = \frac{\sqrt{k(x)}}{\gamma}$, $p_2(x) = \frac{\gamma\sqrt{a(x)}}{k(x)}$, and other functions are defined by (3.18) and (3.22) respectively.

In the sequel, we calculate the eigenfunctions associated with eigenvalues given by (3.36). Replacing the second row of $T^R\widehat{\Phi}(x, \rho)$ by the first row of $\widehat{\Phi}(x, \rho)$, we obtain

$$f(x, \rho) = \begin{pmatrix} [1]_1 & [1]_1 & [1]_1 & [1]_1 \\ F_1(x, \rho)q_1(x)[1]_1 & F_2(x, \rho)q_1(x)[1]_1 & F_3(x, \rho)Q_1(x)q_2(x)[1]_1 & F_4(x, \rho)Q_2(x)q_2(x)[1]_1 \\ \rho^3[p_1]_1 & -\rho^3[p_1]_1 & -[p_2]_1 & [p_2]_1 \\ \rho^3 F_1[p_3p_5]_1 & -\rho^3 F_2[p_3p_5]_1 & -F_3Q_1[p_4p_6]_1 & F_4Q_2[p_4p_6]_1 \end{pmatrix}. \quad (4.4)$$

From the expression of $\rho = \rho_{1n}$ given by (3.43), we have

$$\begin{cases} F_1(x, \rho_{1n}) = e^{-\sqrt{i\mu_2 n\pi} \int_0^x \frac{1}{\sqrt{k(\xi)}} d\xi + O(n^{-1/2})}, & F_1 F_2(x, \rho_{1n}) = e^{-\sqrt{i\mu_2 n\pi} \int_x^1 \frac{1}{\sqrt{k(\xi)}} d\xi + O(n^{-1/2})}, \\ F_4(x, \rho_{1n}) = e^{i\mu_2 n\pi \int_0^x \frac{1}{\sqrt{a(\xi)}} d\xi + O(1)}, & F_4 F_3(x, \rho_{1n}) = e^{i\mu_2 n\pi \int_x^1 \frac{1}{\sqrt{a(\xi)}} d\xi + O(1)}, \end{cases} \quad (4.5)$$

and the estimations $\|F_1(x, \rho_{1n})\| = O_x(n^{-1/4})$ and $\|F_1 F_2(x, \rho_{1n})\| = O_x(n^{-1/4})$, where $O_x(n^{-1/4})$ means that $\|O_x(n^{-1/4})\|_{L^2[0,1]} = O(n^{-1/4})$.

Substituting $\lambda_{1n} = \rho_{1n}^2$ with ρ_{1n} given by (3.43), it is seen that $\widehat{f}_{1n}(x) = f(x, \rho_{1n})$ satisfies:

$$\begin{aligned} & \frac{F_1 F_4}{\rho_{1n}^6} \widehat{f}_{1n}(x) = \frac{F_1 F_4}{\rho_{1n}^6} f(x, \rho_{1n}) \\ = & \begin{pmatrix} [1]_1 & F_1[1]_1 & F_4[1]_1 & [1]_1 \\ F_1(x, \rho_{1n})q_1(x)[1]_1 & F_1 F_2(x, \rho_{1n})q_1(x)[1]_1 & F_4 F_3(x, \rho_{1n})Q_1(x)q_2(x)[1]_1 & F_4(x, \rho_{1n})Q_2(x)q_2(x)[1]_1 \\ [p_1]_1 & -F_1[p_1]_1 & -\frac{1}{\rho_{1n}^3} F_4[p_2]_1 & \frac{1}{\rho_{1n}^3} [p_2]_1 \\ F_1[p_3p_5]_1 & -[p_3p_5]_1 & -\frac{1}{\rho_{1n}^3} Q_1[p_4p_6]_1 & \frac{1}{\rho_{1n}^3} F_4 Q_2[p_4p_6]_1 \end{pmatrix}. \end{aligned}$$

By the estimations (3.40) and (4.5), it follows from the boundedness of p_i ($i = 1, \dots, 6$), $Q_j, Q_j(x)$ ($j = 1, 2$) that

$$\begin{aligned} & \frac{F_1 F_4}{\rho_{1n}^6} \widehat{f}_{1n}(x) = \frac{F_1 F_4}{\rho_{1n}^6} f(x, \rho_{1n}) \\ & = \left| \begin{array}{cc|cc} [p_1]_1 & 0 & F_4[1]_1 & [1]_1 \\ 0 & -[p_3 p_5]_1 & F_4 F_3(x, \rho_{1n}) Q_1(x) q_2(x) [1]_1 & F_4(x, \rho_{1n}) Q_2(x) q_2(x) [1]_1 \end{array} \right| + O(\rho_{1n}^{-3}) \\ & = -F_4 q_2(x) (F_4(x, \rho_{1n}) Q_2(x) - F_3(x, \rho_{1n}) Q_1(x)) [p_1 p_3 p_5]_1 + O(n^{-3/2}). \end{aligned}$$

Substitute $F_i(x, \rho_{1n}), i = 3, 4$ and $Q_j(x), j = 1, 2$ given by (4.5) and (3.22) to obtain

$$\frac{F_1 F_4}{\rho_{1n}^6} \widehat{f}_{1n}(x) = \frac{F_1 F_4}{\rho_{1n}^6} f(x, \rho_{1n}) = -2i F_4 q_2(x) \sin(a_n(x)) [p_1 p_3 p_5]_1 + O(n^{-3/2}), \quad (4.6)$$

where $a_n(x) = \mu_2 n \pi \int_0^x \frac{1}{\sqrt{a(\xi)}} d\xi + i \left(\mu_1 \mu_2 \int_0^x \frac{1}{\sqrt{a(\xi)}} d\xi - \int_0^x \frac{\gamma^2}{2\sqrt{a(\xi)k(\xi)}} d\xi \right) + O(n^{-1})$.

From the expression of $\rho = \rho_{2n}$ given by (3.46), we have

$$\left\{ \begin{array}{l} F_1(x, \rho_{2n}) = e^{-i\mu_3 n \pi \int_0^x \frac{1}{\sqrt{k(\xi)}} d\xi + O(n^{-1})}, F_1 F_2(x, \rho_{2n}) = e^{-i\mu_3 n \pi \int_x^1 \frac{1}{\sqrt{k(\xi)}} d\xi + O(n^{-1})}, \\ F_4(x, \rho_{2n}) = e^{-\mu_3^2 n^2 \pi^2 \int_0^x \frac{1}{\sqrt{a(\xi)}} d\xi + O(1)}, F_4 F_3(x, \rho_{2n}) = e^{-\mu_3^2 n^2 \pi^2 \int_x^1 \frac{1}{\sqrt{a(\xi)}} d\xi + O(1)}, \end{array} \right. \quad (4.7)$$

and the estimations $\|F_4(x, \rho_{2n})\| = O_x(n^{-1})$ and $\|F_4 F_3(x, \rho_{2n})\| = O_x(n^{-1})$, where $O_x(n^{-1})$ means that $\|O_x(n^{-1})\|_{L^2[0,1]} = O(n^{-1})$.

Similarly, we can obtain $\widehat{f}_{2n}(x) = f(x, \rho_{2n})$ corresponding to $\lambda_{2n} = \rho_{2n}^2$ as follows:

$$\begin{aligned} & \frac{F_1 F_4}{\rho_{2n}^4} \widehat{f}_{2n}(x) = \frac{F_1 F_4}{\rho_{2n}^4} f(x, \rho_{2n}) \\ & = \rho_{2n}^2 \left| \begin{array}{cccc|cc} & [1]_1 & & & & \\ F_1(x, \rho_{2n}) q_1(x) [1]_1 & & F_1[1]_1 & & F_4[1]_1 & \\ & [p_1]_1 & -F_1[p_1]_1 & & -\frac{1}{\rho_{2n}^3} F_4[p_2]_1 & \\ F_1[p_3 p_5]_1 & & -[p_3 p_5]_1 & & -\frac{1}{\rho_{2n}^3} Q_1[p_4 p_6]_1 & \\ & & & & \frac{1}{\rho_{2n}^3} F_4 Q_2[p_4 p_6]_1 & \end{array} \right|. \end{aligned}$$

By (3.44), (4.7) and the boundedness of p_i ($i = 1, \dots, 6$) $Q_j, Q_j(x)$ ($j = 1, 2$),

$$\begin{aligned} & \frac{F_1 F_4}{\rho_{2n}^4} \widehat{f}_{2n}(x) = \frac{F_1 F_4}{\rho_{2n}^4} f(x, \rho_{2n}) \\ & = \rho_{2n}^2 \left| \begin{array}{cc|cc} [p_1]_1 & -F_1[p_1]_1 & 0 & [1]_1 \\ F_1[p_3 p_5]_1 & -[p_3 p_5]_1 & F_4 F_3(x, \rho_{2n}) Q_1(x) q_2(x) [1]_1 & F_4(x, \rho_{2n}) Q_2(x) q_2(x) [1]_1 \end{array} \right| + O(\rho_{2n}^{-1}). \end{aligned}$$

By (3.45) and the estimation $\|F_4 F_3(x, \rho_{2n})\| = O_x(n^{-1})$, we conclude that

$$\frac{F_1 F_4}{\rho_{2n}^4} \widehat{f}_{2n}(x) = \frac{F_1 F_4}{\rho_{2n}^4} f(x, \rho_{2n}) = O(\rho_{2n}^{-1}) O_x(n^{-1}) + O(n^{-1}) = O(n^{-1}). \quad (4.8)$$

In the following, we look for $f'(x)$, the second component of $\Phi(x)$. Replacing the second row of $T^R \widehat{\Phi}(x, \rho)$ by the second row of $\widehat{\Phi}(x, \rho)$, we obtain

$$f'(x, \rho) = \left| \begin{array}{cccc|cc} & [1]_1 & & & & \\ -\frac{\rho F_1(x, \rho) q_1(x)}{\sqrt{k(x)}} [1]_1 & \frac{\rho F_2(x, \rho) q_1(x)}{\sqrt{k(x)}} [1]_1 & -\frac{\rho^2 F_3(x, \rho) Q_1(x) q_2(x)}{\sqrt{a(x)}} [1]_1 & \frac{\rho^2 F_4(x, \rho) Q_2(x) q_2(x)}{\sqrt{a(x)}} [1]_1 & & \\ \rho^3 [p_1]_1 & -\rho^3 [p_1]_1 & -[p_2]_1 & [p_2]_1 & & \\ \rho^3 F_1[p_3 p_5]_1 & -\rho^3 F_2[p_3 p_5]_1 & -F_3 Q_1[p_4 p_6]_1 & F_4 Q_2[p_4 p_6]_1 & & \end{array} \right|. \quad (4.9)$$

Substituting $\rho = \rho_{1n}$ into (4.9), we obtain $\lambda_{1n}^{-1} \widehat{f'_{1n}}(x) = \lambda_{1n}^{-1} f'(x, \rho_{1n})$ corresponding to eigenvalue $\lambda_{1n} = \rho_{1n}^2$, which satisfies

$$\begin{aligned} & \frac{F_1 F_4}{\rho_{1n}^6} \lambda_{1n}^{-1} \widehat{f'_{1n}}(x) = \frac{F_1 F_4}{\rho_{1n}^6} \lambda_{1n}^{-1} f'(x, \rho_{1n}) \\ = & \begin{vmatrix} [1]_1 & F_1 [1]_1 & F_4 [1]_1 & [1]_1 \\ -\frac{F_1(x, \rho_{1n}) q_1(x)}{\rho_{1n} \sqrt{k(x)}} [1]_1 & \frac{F_1 F_2(x, \rho_{1n}) q_1(x)}{\rho_{1n} \sqrt{k(x)}} [1]_1 & -\frac{F_4 F_3(x, \rho_{1n}) Q_1(x) q_2(x)}{\sqrt{a(x)}} [1]_1 & \frac{F_4(x, \rho_{1n}) Q_2(x) q_2(x)}{\sqrt{a(x)}} [1]_1 \\ [p_1]_1 & -F_1 [p_1]_1 & -\frac{F_4}{\rho_{1n}^3} [p_2]_1 & \frac{1}{\rho_{1n}^3} [p_2]_1 \\ F_1 [p_3 p_5]_1 & -[p_3 p_5]_1 & -\frac{Q_1}{\rho_{1n}^3} [p_4 p_6]_1 & \frac{F_4 Q_2}{\rho_{1n}^3} [p_4 p_6]_1 \end{vmatrix}. \end{aligned}$$

By (3.40), (4.5) and the boundedness of p_i ($i = 1, \dots, 6$), $Q_j, Q_j(x)$ ($j = 1, 2$), we have

$$\begin{aligned} & \frac{F_1 F_4}{\rho_{1n}^6} \lambda_{1n}^{-1} \widehat{f'_{1n}}(x) = \frac{F_1 F_4}{\rho_{1n}^6} \lambda_{1n}^{-1} f'(x, \rho_{1n}) \\ = & \begin{vmatrix} [p_1]_1 & 0 \\ 0 & -[p_3 p_5]_1 \end{vmatrix} \begin{vmatrix} -\frac{F_4 F_3(x, \rho_{1n}) Q_1(x) q_2(x)}{\sqrt{a(x)}} [1]_1 & \frac{F_4(x, \rho_{1n}) Q_2(x) q_2(x)}{\sqrt{a(x)}} [1]_1 \end{vmatrix} + O(\rho_{1n}^{-3}) \\ = & -\frac{F_4 q_2(x)}{\sqrt{a(x)}} (F_4(x, \rho_{1n}) Q_2(x) + F_3(x, \rho_{1n}) Q_1(x)) [p_1 p_3 p_5]_1 + O(\rho_{1n}^{-3}). \end{aligned}$$

Substitute $F_i(x, \rho_{1n}), i = 3, 4$ and $Q_j(x), j = 1, 2$ given by (4.7) and (3.22) to obtain

$$\frac{F_1 F_4}{\rho_{1n}^6} \lambda_{1n}^{-1} \widehat{f'_{1n}}(x) = \frac{F_1 F_4}{\rho_{1n}^6} \lambda_{1n}^{-1} f'(x, \rho_{1n}) = -2 \frac{F_4 q_2(x)}{\sqrt{a(x)}} \cos(a_n(x)) [p_1 p_3 p_5]_1 + O(n^{-3/2}). \quad (4.10)$$

Similarly, we substitute $\rho = \rho_{2n}$ into (4.9) and obtain $\lambda_{2n}^{-1} \widehat{f'_{2n}}(x) = \lambda_{2n}^{-1} f'(x, \rho_{2n})$ corresponding to eigenvalue $\lambda_{2n} = \rho_{2n}^2$, which satisfies

$$\begin{aligned} & \frac{F_1 F_4}{\rho_{2n}^4} \lambda_{2n}^{-1} \widehat{f'_{2n}}(x) = \frac{F_1 F_4}{\rho_{2n}^4} \lambda_{2n}^{-1} f'(x, \rho_{2n}) \\ = & \rho_{2n}^2 \begin{vmatrix} [1]_1 & F_1 [1]_1 & F_4 [1]_1 & [1]_1 \\ -\frac{F_1(x, \rho_{2n}) q_1(x)}{\rho_{2n} \sqrt{k(x)}} [1]_1 & \frac{F_1 F_2(x, \rho_{2n}) q_1(x)}{\rho_{2n} \sqrt{k(x)}} [1]_1 & -\frac{F_4 F_3(x, \rho_{2n}) Q_1(x) q_2(x)}{\sqrt{a(x)}} [1]_1 & \frac{F_4(x, \rho_{2n}) Q_2(x) q_2(x)}{\sqrt{a(x)}} [1]_1 \\ [p_1]_1 & -F_1 [p_1]_1 & -\frac{F_4}{\rho_{2n}^3} [p_2]_1 & \frac{1}{\rho_{2n}^3} [p_2]_1 \\ F_1 [p_3 p_5]_1 & -[p_3 p_5]_1 & -\frac{Q_1}{\rho_{2n}^3} [p_4 p_6]_1 & \frac{F_4 Q_2}{\rho_{2n}^3} [p_4 p_6]_1 \end{vmatrix}. \end{aligned}$$

By (3.44), (4.7) and the boundedness of p_i ($i = 1, \dots, 6$), $Q_j, Q_j(x)$ ($j = 1, 2$), we have

$$\begin{aligned} & \frac{F_1 F_4}{\rho_{2n}^4} \lambda_{2n}^{-1} \widehat{f'_{2n}}(x) = \frac{F_1 F_4}{\rho_{2n}^4} \lambda_{2n}^{-1} f'(x, \rho_{2n}) \\ = & \rho_{2n}^2 \begin{vmatrix} [p_1]_1 & -F_1 [p_1]_1 \\ F_1 [p_3 p_5]_1 & -[p_3 p_5]_1 \end{vmatrix} \begin{vmatrix} 0 & \frac{F_4(x, \rho_{2n}) Q_2(x) q_2(x)}{\sqrt{a(x)}} [1]_1 \\ -\frac{F_4 F_3(x, \rho_{2n}) Q_1(x) q_2(x)}{\sqrt{a(x)}} [1]_1 & \frac{F_4(x, \rho_{2n}) Q_2(x) q_2(x)}{\sqrt{a(x)}} [1]_1 \end{vmatrix} + O(\rho_{2n}^{-1}). \end{aligned}$$

By (3.45) and the estimation $\|F_4 F_3(x, \rho_{2n})\| = O_x(n^{-1})$, we conclude that

$$\frac{F_1 F_4}{\rho_{2n}^4} \lambda_{2n}^{-1} \widehat{f'_{2n}}(x) = \frac{F_1 F_4}{\rho_{2n}^4} \lambda_{2n}^{-1} f(x, \rho_{2n}) = O(\rho_{2n}^{-1}) O_x(n^{-1}) + O(n^{-1}) = O(n^{-1}). \quad (4.11)$$

Finally, we look for the third component of $\Phi(x)$. Replacing the third row of $T^R \hat{\Phi}(x, \rho)$ by the second row of $\hat{\Phi}(x, \rho)$, we obtain

$$h(x, \rho) = \left| \hat{B}_1, \hat{B}_2 \right| \quad (4.12)$$

with

$$\hat{B}_1 = \begin{pmatrix} [1]_1 & [1]_1 \\ F_1[p_5]_1 & F_2[p_5]_1 \\ \rho^3[p_1]_1 & -\rho^3[p_1]_1 \\ \rho^3 F_1(x, \rho)p_1(x)q_1(x)[1]_1 & -\rho^3 F_2(x, \rho)p_1(x)q_1(x)[1]_1 \end{pmatrix}$$

and

$$\hat{B}_2 = \begin{pmatrix} [1]_1 & [1]_1 \\ F_3 Q_1[p_6]_1 & F_4 Q_2[p_6]_1 \\ -[p_2]_1 & [p_2]_1 \\ -F_3(x, \rho)Q_1(x)p_2(x)q_2(x)[1]_1 & F_4(x, \rho)Q_2(x)p_2(x)q_2(x)[1]_1 \end{pmatrix}.$$

For notational simplicity, denote

$$\tilde{B}_1(x, \rho) = \begin{pmatrix} [1]_1 & F_1[1]_1 \\ F_1[p_5]_1 & [p_5]_1 \\ [p_1]_1 & -F_1[p_1]_1 \\ F_1(x, \rho)p_1(x)q_1(x)[1]_1 & -F_1 F_2(x, \rho)p_1(x)q_1(x)[1]_1 \end{pmatrix} \quad (4.13)$$

and

$$\tilde{B}_2(x, \rho) = \begin{pmatrix} F_4[1]_1 & [1]_1 \\ Q_1[p_6]_1 & F_4 Q_2[p_6]_1 \\ -\frac{F_4}{\rho^3}[p_2]_1 & \frac{1}{\rho^3}[p_2]_1 \\ -\frac{F_4 F_3(x, \rho)Q_1(x)p_2(x)q_2(x)}{\rho^3}[1]_1 & \frac{F_4(x, \rho)Q_2(x)p_2(x)q_2(x)}{\rho^3}[1]_1 \end{pmatrix} \quad (4.14)$$

as two matrix functions of (x, ρ) .

Substituting $\rho = \rho_{1n}$ into (4.12), we obtain $\lambda_{1n}^{-1} \widehat{h}_{1n}(x) = \lambda_{1n}^{-1} h(x, \rho_{1n})$ corresponding to eigenvalues $\lambda_{1n} = \rho_{1n}^2$, which satisfies

$$\frac{F_1 F_4}{\rho_{1n}^6} \lambda_{1n}^{-1} \widehat{h}_{1n}(x) = \frac{F_1 F_4}{\rho_{1n}^6} \lambda_{1n}^{-1} h(x, \rho_{1n}) = \frac{1}{\rho_{1n}^2} \left| \tilde{B}_1(x, \rho_{1n}), \tilde{B}_2(x, \rho_{1n}) \right|.$$

By (3.40), (4.5) and the boundedness of p_i ($i = 1, \dots, 6$) $Q_j, Q_j(x)$ ($j = 1, 2$), we have

$$\begin{aligned} & \frac{F_1 F_4}{\rho_{1n}^6} \lambda_{1n}^{-1} \widehat{h}_{1n}(x) = \frac{F_1 F_4}{\rho_{1n}^6} \lambda_{1n}^{-1} h(x, \rho_{1n}) \\ &= \frac{1}{\rho_{1n}^2} \left| \begin{array}{cc} [p_1]_1 & 0 \\ F_1(x, \rho_{1n})p_1(x)q_1(x)[1]_1 & -F_1 F_2(x, \rho_{1n})p_1(x)q_1(x)[1]_1 \end{array} \right| \left| \begin{array}{cc} F_4[1]_1 & [1]_1 \\ Q_1[p_6]_1 & F_4 Q_2[p_6]_1 \end{array} \right| + O(\rho_{1n}^{-3}) \\ &= -\frac{[p_1 p_6]_1}{\rho_{1n}^2} p_1(x)q_1(x) F_1 F_2(x, \rho_{1n}) (F_4^2 Q_2 - Q_1) + O(\rho_{1n}^{-3}). \end{aligned}$$

By (3.41) and the estimation $\|F_1 F_2(x, \rho_{1n})\| = O_x(n^{-1/4})$, we conclude that

$$\frac{F_1 F_4}{\rho_{1n}^4} \lambda_{1n}^{-1} \widehat{h_{1n}}(x) = \frac{F_1 F_4}{\rho_{1n}^6} \lambda_{1n}^{-1} h(x, \rho_{1n}) = O(\rho_{1n}^{-3}) O_x(n^{-1/4}) + O(\rho_{1n}^{-3}) = O(n^{-3/2}). \quad (4.15)$$

Similarly, we substitute $\rho = \rho_{2n}$ into (4.12) to obtain $\lambda_{2n}^{-1} \widehat{h_{2n}}(x) = \lambda_{2n}^{-1} h(x, \rho_{2n})$ corresponding to eigenvalues $\lambda_{2n} = \rho_{2n}^2$, which satisfies:

$$\frac{F_1 F_4}{\rho_{2n}^4} \lambda_{2n}^{-1} \widehat{h_{2n}}(x) = \frac{F_1 F_4}{\rho_{2n}^4} \lambda_{2n}^{-1} h(x, \rho_{2n}) = \left| \tilde{B}_1(x, \rho_{2n}), \tilde{B}_1(x, \rho_{2n}) \right|. \quad (4.16)$$

By (3.44), (4.7) and the boundedness of p_i ($i = 1, \dots, 6$) $Q_j, Q_j(x)$ ($j = 1, 2$), we have

$$\begin{aligned} & \frac{F_1 F_4}{\rho_{2n}^4} \lambda_{2n}^{-1} \widehat{h_{2n}}(x) = \frac{F_1 F_4}{\rho_{2n}^4} \lambda_{2n}^{-1} h(x, \rho_{2n}) \\ &= \left| \begin{array}{cc} [p_1]_1 & -F_1 [p_1]_1 \\ F_1(x, \rho) p_1(x) q_1(x) [1]_1 & -F_1 F_2(x, \rho) p_1(x) q_1(x) [1]_1 \end{array} \right| \left| \begin{array}{cc} 0 & [1]_1 \\ Q_1 [p_6]_1 & 0 \end{array} \right| + O(\rho_{2n}^{-3}) \\ &= p_1 p_6 F_1 Q_1 p_1(x) q_1(x) (F_2(x, \rho_{2n}) - F_1(x, \rho_{2n})) + O(\rho_{2n}^{-1}). \end{aligned}$$

Substitute $F_i(x, \rho_{2n}), i = 1, 2$ given by (4.7) to obtain

$$\begin{aligned} \frac{F_1 F_4}{\rho_{2n}^4} \lambda_{2n}^{-1} \widehat{h_{2n}}(x) &= \frac{F_1 F_4}{\rho_{2n}^4} \lambda_{2n}^{-1} h(x, \rho_{2n}) \\ &= 2i p_1 p_6 F_1 Q_1 p_1(x) q_1(x) \sin \left(\mu_3 n \pi \int_0^1 \frac{d\xi}{\sqrt{k(\xi)}} + O(n^{-1}) \right) + O(n^{-1}). \end{aligned} \quad (4.17)$$

By $\hat{\Psi}_1(x)$ given by (3.29) which is the coefficient matrix of ρ^{-1} in the fundamental matrix solution $\hat{\Psi}(x, \rho)$ given by (3.20), a direct computation gives that the first two items of the asymptotic expression of factor $[p_1 p_3 p_5]_1$ in (4.6) and (4.10) are the same. Combining with the expressions of $\lambda_{1n}^{-1} \widehat{f_{1n}}(x), \widehat{f_{1n}}(x), \lambda^{-1} \widehat{h_{1n}}(x)$ given by (4.6), (4.10), (4.15), respectively, and setting

$$X_{1n} = \begin{pmatrix} \lambda_{1n}^{-1} \widehat{f_{1n}}(x) \\ \widehat{f_{1n}}(x) \\ \lambda^{-1} \widehat{h_{1n}}(x) \end{pmatrix} = \frac{F_1 F_4}{-2 F_4 q_2(x) \rho^6 [p_1 p_3 p_5]_1} \begin{pmatrix} \lambda_{1n}^{-1} \widehat{f_{1n}}(x) \\ \widehat{f_{1n}}(x) \\ \lambda^{-1} \widehat{h_{1n}}(x) \end{pmatrix},$$

we obtain the first family of eigenfucions X_{1n} corresponding to λ_{1n} , which satisfies (4.1). Similarly, combining with the expressions of $\lambda_{2n}^{-1} \widehat{f_{2n}}(x), \widehat{f_{2n}}(x), \lambda^{-1} \widehat{h_{2n}}(x)$ given by (4.8), (4.11), (4.17), respectively, and setting

$$X_{2n} = \begin{pmatrix} \lambda_{2n}^{-1} \widehat{f_{2n}}(x) \\ \widehat{f_{2n}}(x) \\ \lambda^{-1} \widehat{h_{2n}}(x) \end{pmatrix} = \frac{F_4}{2i p_1 p_6 Q_1 p_1(x) q_1(x) \rho^4} \begin{pmatrix} \lambda_{2n}^{-1} \widehat{f_{2n}}(x) \\ \widehat{f_{2n}}(x) \\ \lambda^{-1} \widehat{h_{2n}}(x) \end{pmatrix},$$

we get further the second family of eigenfunctions X_{2n} corresponding to λ_{2n} , which satisfies (4.3). \square

5. RIESZ BASIS PROPERTY AND EXPONENTIAL STABILITY

In this section, we shall use the asymptotic expressions of eigenpairs obtained in Section 3 and Section 4 to prove that there exists a sequence of generalized eigenfunctions of operator A , which forms a Riesz basis for \mathcal{H} . Furthermore, the exponential stability of the system can be determined by its spectrum distribution.

Before continuing, we recall some notations. For a closed operator \mathcal{A} in a Hilbert space \mathbb{H} , a nonzero $X \in \mathbb{H}$ is called a generalized eigenvector of \mathcal{A} , corresponding to an eigenvalue λ of \mathcal{A} , if there is an integer $m > 0$ such that $(\lambda I - \mathcal{A})^m X = 0$. If $m = 1$, X is just an eigenvector. A sequence $\{X_n\}_{n=1}^{\infty} \subset \mathbb{H}$ is called a Riesz basis for \mathbb{H} if there exists an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ in \mathbb{H} and a linear bounded invertible operator T such that

$$TX_n = e_n, \quad n = 1, 2, \dots.$$

To prove the Riesz basis property of our system, we recall two associated results. The following Lemma 5.1 comes from [2].

Lemma 5.1. *Let $\{\lambda_n \in \mathbb{C}, n \in \mathbb{Z}_+\}$ be a sequence satisfying both of the following conditions:*

- (1) $\sup_n |\operatorname{Im}(\lambda_n)| \leq M$;
- (2) $\sup_n |\operatorname{Re}(\lambda_n) - n\pi| < \pi/4$;

where M is a positive constant. Then, the sine system $\{\sin \lambda_n x, n \in \mathbb{Z}_+\}$ and the cosine system $\{1, \cos \lambda_n x, n \in \mathbb{Z}_+\}$ are two Riesz bases for $L^2(0, 1)$.

The succeeding Lemma 5.2 comes from [10].

Lemma 5.2. *Let \mathcal{A} be a densely defined closed linear operator in a Hilbert space \mathcal{H} with isolated eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ and $\sigma_r(\mathcal{A}) = \emptyset$. Let $\{\phi_i\}_{i=1}^{\infty}$ be a Riesz basis for \mathcal{H} . Suppose that there is an integer $N \geq 1$ and a sequence of generalized eigenvectors $\{\psi_i\}_{i=N}^{\infty}$ of \mathcal{A} such that*

$$\sum_{i=N}^{\infty} \|\psi_i - \phi_i\|^2 < \infty. \quad (5.1)$$

Then, there exists $M(> N)$ number of generalized eigenvectors $\{\psi_{i_0}\}_{i=1}^M$ of \mathcal{A} such that $\{\psi_{i_0}\}_{i=1}^M \cup \{\psi_i\}_{i=M+1}^{\infty}$ forms a Riesz basis for \mathcal{H} .

To prove the Riesz basis property for the operator A defined by (2.1), we define a linear subspace of $L^2(0, 1)$ by

$$L_0^2(0, 1) = \left\{ g \in L^2(0, 1) \mid \int_0^1 b(x)g(x)dx = 0 \right\}, \quad b(x) = \frac{1}{\sqrt{a(x)}}, \quad (5.2)$$

With the inner product in $L^2(0, 1)$ defined by

$$\langle f, g \rangle = \int_0^1 b(x)f(x)\overline{g(x)}dx,$$

there holds an orthogonal decomposition of $L^2(0, 1)$:

$$L^2(0, 1) = \{1\} \oplus L_0^2(0, 1).$$

Then, for any $f \in L^2(0, 1)$, a simple computation shows that there exists a unique direct sum decomposition

$$f(x) = C + g(x), \quad g \in L_0^2(0, 1), \quad C = \frac{\int_0^1 b(x)f(x)dx}{\int_0^1 b(x)dx}, \quad (5.3)$$

where $b(x)$ is given in (5.2).

The following Lemma 5.3 is trivial from decomposition (5.3).

Lemma 5.3. *Let $\{\lambda_n\}$ be the sequence defined by Lemma 5.1. Decompose*

$$\cos \lambda_n x = C_n + g_n(x), \quad g_n \in L_0^2(0, 1), \quad C_n = \mathcal{O}(n^{-1}). \quad (5.4)$$

Then, $\{g_n(x)\}$ forms a Riesz basis for $L_0^2(0, 1)$.

Proof. The conclusion is trivial by (5.3) and we only need to notice that

$$C_n = \frac{\int_0^1 b(x) \cos \lambda_n x dx}{\int_0^1 b(x) dx} = \frac{-\int_0^1 b'(x) \sin \lambda_n x dx + b(1) \sin \lambda_n}{\lambda_n \int_0^1 a(x) dx} = \mathcal{O}(n^{-1}).$$

□

Lemma 5.4. *Let λ_n satisfy the conditions of Lemma 5.1 and C_n be defined in Lemma 5.3. Then, the vector family $\{Y_n = (\cos \lambda_n x - C_n, i \sin \lambda_n x)^\top\}_{n=1}^\infty$ forms a Riesz basis for the space $L_0^2(0, 1) \times L^2(0, 1)$.*

Proof. Based on Lemma 5.1 and Lemma 5.3, it is easy to check that

$$\left\{ \left(\begin{array}{c} 0 \\ \sin \lambda_n x \end{array} \right), \left(\begin{array}{c} \cos \lambda_n x - C_n \\ 0 \end{array} \right) \right\}_{n=1}^\infty$$

constitutes a Riesz basis for $L_0^2(0, 1) \times L^2(0, 1)$. Hence, the sequence

$$\left\{ \left(\begin{array}{c} \cos \lambda_n x - C_n \\ \sin \lambda_n x \end{array} \right), \left(\begin{array}{c} \cos \lambda_n x - C_n \\ -\sin \lambda_n x \end{array} \right) \right\}_{n \in \mathbb{N}}$$

also forms a Riesz basis for $L_0^2(0, 1) \times L^2(0, 1)$. We introduce an invertible bounded operator $T_1 = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$ which produces

$$\left(\begin{array}{c} \cos \lambda_n x - C_n \\ -i \sin \lambda_n x \end{array} \right) = T_1 \left(\begin{array}{c} \cos \lambda_n x - C_n \\ \sin \lambda_n x \end{array} \right), \quad \left(\begin{array}{c} \cos \lambda_n x - C_n \\ i \sin \lambda_n x \end{array} \right) = T_1 \left(\begin{array}{c} \cos \lambda_n x - C_n \\ -\sin \lambda_n x \end{array} \right).$$

The conclusion then follows straightforwardly. □

Theorem 5.5. *Let A be defined by (2.1), and let X_{1n}, X_{2n} be the two families of eigenfunctions given by (4.1) and (4.3) respectively. Then, $\{X_{1n}, X_{2n}, n \in \mathbb{Z}_+\}$ forms a Riesz basis for \mathcal{H} the state space of the system (1.2).*

Proof. By the bounded invertible mapping:

$$T_2(f, g, h)^\top = (\sqrt{a(x)}f', g, h),$$

the state space \mathcal{H} is mapped onto $L_0^2(0, 1) \times (L^2(0, 1))^2$. The above map is also needed for constant case, which corrects an error for (3.271) of [9]. To complete the proof, we only need to prove that $\{T_2 X_{1n}, T_2 X_{2n}, n \in \mathbb{Z}_+\}$ forms a Riesz basis for $L_0^2(0, 1) \times (L^2(0, 1))^2$. Set

$$z_1 = z_1(x) = \mu_2 \int_0^x \frac{1}{\sqrt{a(\xi)}} d\xi = \frac{\int_0^x \frac{1}{\sqrt{a(\xi)}} d\xi}{\int_0^1 \frac{1}{\sqrt{a(\xi)}} d\xi}$$

and

$$z_2 = z_2(x) = \mu_3 \int_0^x \frac{1}{\sqrt{k(\xi)}} d\xi = \frac{\int_0^x \frac{1}{\sqrt{k(\xi)}} d\xi}{\int_0^1 \frac{1}{\sqrt{k(\xi)}} d\xi}.$$

By (4.1), (4.3) and the space scaling above, the sequences $\{T_2 X_{1n}, n \in \mathbb{Z}_+\}$ and $\{T_2 X_{2n}, n \in \mathbb{Z}_+\}$ are transformed into two equivalent sequences in $L_0^2(0, 1) \times (L^2(0, 1))^2$:

$$Y_{1n} = (\cos(n\pi z_1 + ib_n(z_1)) + O(n^{-1}), \sin(n\pi z_1 + ib_n(z_1)) + O(n^{-1}), 0)^\top + O(n^{-3/2}) \text{ as } n \rightarrow \infty, \quad (5.5)$$

and

$$Y_{2n} = (0, 0, \sin(n\pi z_2 + O(n^{-1})))^\top + O(n^{-1}) \text{ as } n \rightarrow \infty, \quad (5.6)$$

where $b_n(z_1) = \mu_1 z_1 - \int_0^{z_1^{-1}} \frac{1}{2\sqrt{a(\xi)k(\xi)}} d\xi$ with z_1^{-1} being the inverse of $z_1(x)$.

As for $a(\cdot), k(\cdot) \in C^2[0, 1]$, it is easy to check that $\sup_n |b_n(z_1)|$ is bounded and both of the real parts in (5.5) and (5.6), $n\pi z_1 + O(n^{-1})$ and $n\pi z_2 + O(n^{-1})$, satisfy the second condition of Lemma 5.1. By Lemma 5.4, we conclude that

$$\left\{ (\cos(n\pi z_1 + ib_n(z_1)) + O(n^{-1})) - C_n, i \sin(n\pi z_1 + ib_n(z_1)) + O(n^{-1}) \right\}^\top, n \in \mathbb{Z}_+$$

forms a Riesz basis for $L_0^2(0, 1) \times L^2(0, 1)$ with associated $C_n = \mathcal{O}(n^{-1})$ in terms of Lemma 5.3, and

$$\{\sin(n\pi z_2 + O(n^{-1})), n \in \mathbb{Z}_+\}$$

forms a Riesz basis for $L^2(0, 1)$.

Denote $\widehat{Y}_{1n} = \{(\cos(n\pi z_1 + ib_n(z_1)) + O(n^{-1})) - C_n, i \sin(n\pi z_1 + ib_n(z_1)) + O(n^{-1}), 0\}^\top, n \in \mathbb{Z}_+$ and $\widehat{Y}_{2n} = \{(0, 0, \sin(n\pi z_2 + O(n^{-1})))^\top, n \in \mathbb{Z}_+\}$. It is obviously that $\{\widehat{Y}_{1n}, \widehat{Y}_{2n}, n \in \mathbb{Z}_+\}$ forms a Riesz basis for $L_0^2(0, 1) \times (L^2(0, 1))^2$. By the expressions of Y_{1n} and Y_{2n} given by (5.5) and (5.6) respectively, there exists an $N > 0$ such that

$$\sum_{n \geq N} \left[\|Y_{1n} - \widehat{Y}_{1n}\|^2 + \|Y_{2n} - \widehat{Y}_{2n}\|^2 \right] \leq \sum_{n \geq N} \mathcal{O}(n^{-2}) < \infty.$$

The proof is completed by applying Lemma 5.2. \square

It is noted that since the eigenvalues are symmetric about the real axis, theoretically, $\{\overline{\lambda_{2n}}\}$ are also eigenvalues of A . However, since the two families of asymptotic eigenfunctions of operator A , which correspond to the two branches of eigenvalues $\{\lambda_{1n}, \overline{\lambda_{1n}}, n \in \mathbb{N}\} \cup \{\lambda_{2n}, n \in \mathbb{N}\}$ given by Theorem 3.4, are used in forming a Riesz basis for \mathcal{H} only, all λ_{2n} with sufficiently large n must be real. We therefore obtain the following Corollary 5.6.

Corollary 5.6. *Let A be defined by (2.1). Then, the eigenvalues of A consist of two families:*

$$\sigma(A) = \sigma_p(A) = \{\lambda_{1n}, \overline{\lambda_{1n}}, n \in \mathbb{N}\} \cup \{\lambda_{2n}, n \in \mathbb{N}\},$$

where λ_{1n} and λ_{2n} are given by (3.36) in Theorem 3.4, and λ_{2n} are real for all sufficiently large n .

As a consequence, we obtain the following Theorem 5.7 which covers main result of [17] on exponential stability as a special case.

Theorem 5.7. *Let A be defined by (2.1). Then, the spectrum-determined growth condition holds for e^{At} : $\omega(A) = S(A)$ where $\omega(A) = \inf\{\omega \mid \text{there exists an } M \text{ such that } \|e^{At}\| \leq Me^{\omega t}\}$ is the growth bound of the C_0 -semigroup, and $S(A) = \sup\{\operatorname{Re}(\lambda) \mid \lambda \in \sigma(A)\}$ is the spectral bound of A . Furthermore, the C_0 -semigroup e^{At} is exponentially stable:*

$$\|e^{At}\| \leq Me^{-\omega t},$$

for some $M, \omega > 0$.

Proof. The spectrum-determined growth condition is a direct consequence of Theorem 5.5. By the expressions of λ_{1n} and λ_{2n} given by (3.36), we have

$$\lim_{n \rightarrow \infty} \operatorname{Re}(\lambda_{1n}) = -\mu_1\mu_2, \text{ and } \lim_{n \rightarrow \infty} \operatorname{Re}(\lambda_{2n}) = -\infty.$$

By Corollary 2.2, there exists a positive constant $\omega > 0$ such that

$$S(A) = \sup\{\operatorname{Re}(\lambda) \mid \lambda \in \sigma(A)\} < -\omega.$$

The exponential stability then follows from the spectrum-determined growth condition, which is similar to [8, 10, 11]. \square

REFERENCES

- [1] R.A. Adams and J.J.F. Fournier, Sobolev Spaces. Academic Press, Amsterdam, second edition (2003).
- [2] B.T. Bilalov, Bases of exponentials, cosines, and sines formed by eigenfunctions of differential operators. *Differ. Equ.* **39** (2003) 652–657.
- [3] A. Day, Heat Conduction within Linear Thermoelasticity. Springer-Verlag, New York (1985).
- [4] B.Z. Guo and S.P. Yung, Asymptotic behavior of the eigenfrequency of a one-dimensional linear thermoelastic system. *J. Math. Anal. Appl.* **213** (1997) 406–421.
- [5] B.Z. Guo, Further results for a one-dimensional linear thermoelastic equation with Dirichlet-Dirichlet boundary conditions. *ANZIAM J.* **43** (2002) 449–462.
- [6] B.Z. Guo and J.C. Chen, The first real eigenvalue of a one-dimensional linear thermoelastic system. *Comput. Math. Appl.* **38** (1999) 249–256.
- [7] B.Z. Guo, J.M. Wang and G.D. Zhang, Spectral analysis of a wave equation with Kelvin-Voigt damping. *ZAMM Z. Angew. Math. Mech.* **90** (2010) 323–342.
- [8] B.Z. Guo, Riesz basis approach to the stabilization of a flexible beam with a tip mass. *SIAM J. Control Optim.* **39** (2001) 1736–1747.
- [9] B.Z. Guo and J.M. Wang, Control of Wave and Beam PDEs – The Riesz Basis Approach. Springer-Verlag, Cham (2019).
- [10] B.Z. Guo and G.D. Zhang, On spectrum and Riesz basis property for one-dimensional wave equation with Boltzmann damping. *ESAIM: COCV* **18** (2012) 889–913.
- [11] B.Z. Guo and K.Y. Chan, Riesz basis generation, eigenvalues distribution, and exponential stability for a Euler-Bernoulli beam with joint feedback control. *Rev. Mat. Complut.* **14** (2001) 205–229.
- [12] B.Z. Guo, Riesz basis property and exponential stability of controlled Euler-Bernoulli beam equations with variable coefficients. *SIAM J. Control Optim.* **40** (2002) 1905–1923.
- [13] Z.J. Han and G.Q. Xu, Spectral analysis and stability of thermoelastic Bresse system with second sound and boundary viscoelastic damping. *Math. Methods Appl. Sci.* **38** (2015) 94–112.
- [14] T. Kato, Perturbation Theory for Linear Operators. Springer-Verlag, Berlin, second edition (1976).
- [15] Z.H. Luo, B.Z. Guo and O. Morgül, Stability and Stabilization of Infinite-Dimensional Systems with Applications. Springer-Verlag, London (1999).
- [16] K.S. Liu and Z.Y. Liu, Exponential stability and analyticity of abstract linear thermoelastic systems. *Z. Angew. Math. Phys.* **48** (1997) 885–904.
- [17] Z.Y. Liu and S.M. Zheng, Exponential stability of the semigroup associated with a thermoelastic system. *Quart. Appl. Math.* **51** (1993) 535–545.

- [18] M.A. Naimark, *Linear Differential Operators, Part I: Elementary Theory of Linear Differential Operators*. Frederick Ungar Publishing Co., New York (1967).
- [19] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer-Verlag, New York (1983).
- [20] M. Renardy, On the type of certain C_0 -semigroups. *Comm. Partial Differ. Equ.* **18** (1993) 1299–1307.
- [21] M.A. Shubov, Asymptotic and spectral analysis of the spatially nonhomogeneous Timoshenko beam model. *Math. Nachr.* **241** (2002) 125–162.
- [22] C. Tretter, Spectral problems for systems of differential equations $y' + A_0y = \lambda A_1y$ with λ -polynomial boundary conditions. *Math. Nachr.* **214** (2000) 129–172.
- [23] C. Tretter, Boundary eigenvalue problems for differential equations $N\eta = \lambda P\eta$ and λ -polynomial boundary conditions. *J. Differ. Equ.* **170** (2001) 408–471.
- [24] J.M. Wang, G.Q. Xu and S.P. Yung, Riesz basis property, exponential stability of variable coefficient Euler-Bernoulli beams with indefinite damping. *IMA J. Appl. Math.* **70** (2005) 459–477.
- [25] J.M. Wang and B.Z. Guo, On dynamic behavior of a hyperbolic system derived from a thermoelastic equation with memory type. *J. Franklin Inst.* **344** (2007) 75–96.
- [26] J.M. Wang, G.Q. Xu and S.P. Yung, Exponential stabilization of laminated beams with structural damping and boundary feedback controls. *SIAM J. Control Optim.* **44** (2005) 1575–1597.

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