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STRONG RATES OF CONVERGENCE FOR A SPACE-TIME DISCRETIZATION OF THE BACKWARD STOCHASTIC HEAT EQUATION, AND OF A LINEAR-QUADRATIC CONTROL PROBLEM FOR THE STOCHASTIC HEAT EQUATION*

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Abstract. We verify strong rates of convergence for a time-implicit, finite-element based space-time discretization of the backward stochastic heat equation, and the forward-backward stochastic heat equation from stochastic optimal control. The fully discrete version of the forward-backward stochastic heat equation is then used within a gradient descent algorithm to approximately solve the linear-quadratic control problem for the stochastic heat equation driven by additive noise. This work is thus giving a theoretical foundation for the computational findings in Dunst and Prohl, *SIAM J. Sci. Comput.* **38** (2016) A2725–A2755.

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1. INTRODUCTION

Let $D \subset \mathbb{R}^d$ be a bounded domain with C^2 boundary, $T > 0$, and a (deterministic) function $\widetilde{X} \equiv \{\widetilde{X}(t);\, t \in [0,T]\} \in C([0,T];\mathbb{H}_0^1 \cap \mathbb{H}^2)$ be given. Our goal is to numerically approximate the $\mathbb{L}^2$-valued, $\mathbb{F}$-adapted control process $U^* \equiv \{U^*(t);\, t \in [0,T]\}$ on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ that minimizes the functional $(\alpha \geq 0)$

$$\mathcal{J}(X,U) = \frac{1}{2}\mathbb{E}\Big[\int_0^T \big(\|X(t) - \widetilde{X}(t)\|_{\mathbb{L}^2}^2 + \|U(t)\|_{\mathbb{L}^2}^2\big)\,\mathrm{d}t + \alpha\|X(T) - \widetilde{X}(T)\|_{\mathbb{L}^2}^2\Big] \tag{1.1}$$

subject to the (controlled forward) stochastic heat equation (**SPDE**, for short)

$$\begin{cases} \mathrm{d}X(t) = \big[\Delta X(t) + U(t)\big]\,\mathrm{d}t + \sigma(t)\mathrm{d}W(t) \quad \forall\, t \in [0,T]\,, \\ X(0) = X_0\,, \end{cases} \tag{1.2}$$

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which is supplemented by homogeneous Dirichlet boundary conditions. Here $W \equiv \{W(t);\, t \in [0,T]\}$ is an $\mathbb{R}^m$-valued Wiener process, and $X_0 \in \mathbb{H}_0^1 \cap \mathbb{H}^2$, and let $\sigma \equiv \{(\sigma_1(t), \sigma_2(t), \ldots, \sigma_m(t));\, t \in [0,T]\}$, with $\sigma_i \in L^\infty_{\mathbb{F}}\big(0,T; L^2(\Omega; \mathbb{H}_0^1 \cap \mathbb{H}^2)\big)$ for $i = 1, 2, \ldots, m$. For every $U \in L^2_{\mathbb{F}}\big(\Omega; L^2(0,T;\mathbb{L}^2)\big)$, there exists a unique $\mathbb{H}_0^1$-valued mild/weak solution $X \equiv X(U) \in L^2_{\mathbb{F}}\big(\Omega; C([0,T];\mathbb{L}^2) \cap L^2(0,T;\mathbb{H}_0^1)\big)$ in (1.2) (see *e.g.* [26]), and also a unique minimizer $(X^*, U^*) \in L^2_{\mathbb{F}}\big(\Omega; C([0,T];\mathbb{L}^2) \cap L^2(0,T;\mathbb{H}_0^1) \times L^2(0,T;\mathbb{L}^2)\big)$ of the stochastic optimal control problem: 'minimize (1.1) subject to (1.2)' can be deduced—which we below refer to as **SLQ** (see *e.g.* [5]).

We consider problem **SLQ** as a prototype example of a (linear-quadratic) stochastic optimal control problem involving a stochastic PDE, for which corresponding numerical analyses so far are rare in the existing literature; see *e.g.* [15, 35]. This is in contrast to the deterministic counterpart problem **LQ** which involves a linear PDE, where optimal rates of convergence are available for (finite-element based) space-time discretization of related optimality conditions (see *e.g.* [20, 27–29, 32]), which may then be used as part of a gradient descent algorithm with step size control [21] to approximate the minimizing tuple (X^*, U^*), which here consists of deterministic state and control functions. If compared to problem **LQ**, problem **SLQ** owns some distinctive characters and additional difficulties caused by the driving Wiener process in the SPDE (1.2), which make the generalization of the numerical results for the deterministic control problem to **SLQ** a non-trivial task. For example, a crucial difficulty consists in solving the adjoint equation in the context of **SLQ**, which here is a backward stochastic PDE (**BSPDE**, for short) of the form

$$\left\{\begin{array}{l} \mathrm{d}Y(t) = \big[-\Delta Y(t) + [X(t) - \widetilde{X}(t)]\big]\,\mathrm{d}t + Z(t)\,\mathrm{d}W(t) \quad \forall\, t \in [0,T]\,, \\ Y(T) = -\alpha\big(X(T) - \widetilde{X}(T)\big)\,, \end{array}\right. \tag{1.3}$$

having a unique solution tuple $(Y, Z) \in L^2_{\mathbb{F}}\big(\Omega; C([0,T];\mathbb{H}_0^1) \cap L^2(0,T;\mathbb{H}_0^1 \cap \mathbb{H}^2)\big) \times L^2_{\mathbb{F}}\big(\Omega; L^2(0,T;\mathbb{H}_0^1)\big)$; *cf.* [14]. The adjoint variable Y is then related to the optimal control by Pontryagin's maximum principle, which in the case of problem **SLQ** is

$$0 = U^*(t) - Y(t) \qquad \forall\, t \in (0,T)\,. \tag{1.4}$$

The combination of equations (1.2), (1.3), and (1.4) then uniquely determines the optimal process tuple (X^*, U^*) of problem **SLQ**.

Both, the numerical discretization and related analysis of **SLQ** governed by **SPDE**s are non-trivial. In this work, we apply Pontryagin's maximum principle to discretize problem **SLQ**. There exists related works to approximate optimal control problems driven by SDE systems; see, *e.g.*, [1, 2, 19]; however, to the best of the authors' knowledge, works on discretization of **SLQ** with **SPDE**s constraints so-far are rare. In [15], a scheme to numerically solve (a slightly different) problem is given, which combines a proper selection of involved finite element spaces with an implicit Euler method for temporal discretization to inherit relevant stability properties of the limiting system, with a least squares Monte-Carlo method, in combination with a stochastic gradient method, which is based on the (discretization of) optimality system (1.2), (1.3), and (1.4). The scheme was implemented in [15] for a problem that uses a two-dimensional bounded domain D, and comprehensive computational studies are reported, among which evidence is given for

a) strong convergence order $\frac{1}{2}$ for the iterates of both, the temporal discretization of **BSPDE**, and **SLQ** (with the parameter $\tau > 0$),
b) optimal strong convergence order for the iterates of both, the spatial discretization of **BSPDE**, and **SLQ** (with the parameter $h > 0$), depending on the spatial $\mathbb{L}^2$- or $\mathbb{H}^1$-norm, in which the error is considered.

By evidence, a convergence rate as stated in a) may be expected from the literature on backward stochastic (ordinary) differential equations (BSDEs, for short), where the dimension of the involved state spaces is *fixed*; see *e.g.* [39], and the discussion below. In the present setting of **BSPDE** and **SLQ**, however, where solutions take values in (infinite dimensional) Hilbert spaces, the related dimensions in a (finite-element based) discretization depend on the inverse of the discretization parameter $h > 0$, and a naive transfer of results from [39] would result in error estimates which *couple* temporal and spatial discretization parameters—which severely restricts the wanted flexibility of spatio-temporal meshes to resolve relevant multiscale solution features. In this work, we use Malliavin calculus and variational analysis as technical tools to exploit inherent stability properties of time-implicit finite-element based discretizations to validate strong error estimates for (the space-time discretization of) both problems, where both discretization parameters enter *additively*, and no coupling terms arise. As such, this work is meant to give a first theoretical guidance on the design and analysis of numerical methods for problems **BSPDE** and **SLQ**, as well as an explanation for the computational findings in [15]—where *no* couplings of discretization parameters (τ, h) were observed, thus allowing for general quasiuniform triangulations of D for example, in the computational studies for (slightly different version of) **BSPDE** and **SLQ**; see also Section 4 below.

Conceptually, we start our work below with the numerical analysis of **BSPDE**, which is the crucial building block for the (numerical analysis of the) optimality conditions (1.2), (1.3), and (1.4) for **SLQ**, which is then considered afterwards. There is a rich body of literature on numerical schemes for BSDEs; see *e.g.* [3, 6, 9, 11, 16, 18, 22, 33, 37, 39, 41], and the references therein. In contrast, the convergence analysis of space-time discretizations of **BSPDE** is only a recent research subject, and available results are rare: we are only aware of the works [34, 35], where a first error analysis for an (abstract) time-space discretization based on the implicit Euler method for the above **BSPDE** (1.3) is proposed, where the error depends on the ratio of temporal discretization and Galerkin parameters; see our discussion above. In [15], the authors derive rates of convergence for a conforming finite element semi-discretization, and discuss its actual implementation. The proofs in [15] use simple variational arguments, resting on improved regularity properties of the variational solution, Itô's formula, and approximation results for the finite element method. However, the interplay of spatial and temporal discretization errors was left open in [15], in particular the relevant question regarding unconditional convergence rates (which were discussed above), which allow discretization parameters w.r.t. time and space to *independently* tend to zero, and *general* quasiuniform space-time meshes.

A stability and strong error analysis for the spatial semi-discretization (2.14) already exists in the literature (*cf.* Sect. 2.3 of a summary); however, the role of an additional *temporal* discretization of (2.14) was not theoretically accessible in [15]; see again our discussion above. In Section 3, we fill this gap by using Malliavin calculus for the (finite-element type) solutions (Y_h, Z_h) of (2.14) to derive estimates for its increments in relevant norms, such as the *uniform* estimate (see Lem. 3.2)

$$\sup_{h>0} \mathbb{E}\big[\|Z_h(t) - Z_h(s)\|^2_{\mathbb{L}^2}\big] \leq C|t-s|\,. \tag{1.5}$$

One key tool to validate this estimate for the solution (Y_h, Z_h) of (2.14) is the observation that Z_h is the Malliavin derivative of Y_h, and the Malliavin derivative of (Y_h, Z_h) also satisfies a **BSPDE** (see Lem. 2.2); another tool is the auxiliary equation (3.7), which allows to subsequently study of temporal discretization effects on the first, and the second solution component of the solutions (Y_h, Z_h) of (2.14). We use this again in Section 4 for the study the space-time discretization effects for **FBSPDE**; *cf.* (4.32).

The second goal in this work is addressed in Section 4, where strong error estimates for a space-time discretization (4.15)–(4.16) of the coupled forward-backward SPDE (1.2)-(1.3) (**FBSPDE** for short) are shown: its derivation starts (*cf.* Sect. 4.1) with the error analysis of a spatial semi-discretization (4.5)–(4.7) of the optimality system **FBSPDE** of optimization problem **SLQ**, where our main result are optimal error estimates in Theorem 4.1. Conceptionally, its proof first deduces estimates for the (discrete) optimal control (see (i) in Thm. 4.1) *via* the use of the (strictly convex, differentiable) reduced functional $\widehat{\mathcal{J}}_h$ in (4.8), in particular, the representation of its derivative in (4.9), and stability properties (uniformly in h) of the spatial discretizations of the forward, and the adjoint/backward problem (see Sect. 3) in the given context of finite elements. Once (i) of Theorem 4.1 is settled, the estimates (ii) and (iii) now essentially follow from Section 3. Section 4.2 then focuses on the error due to (additional) temporal discretization of (4.5)–(4.7) in the form (4.15)–(4.16), which—as for the **BSPDE** case—, is much more intricate: the key tools to validate the main result here (*i.e.*, Thm. 4.3) are a representation of the derivative of the reduced functional $\widehat{\mathcal{J}}_{h\tau}$ in (4.26) *via* (4.15), together with stability and convergence estimates from Section 3 to show (i) in Theorem 4.3. Statements (ii) and (iii) in Theorem 4.3 then follow from the results in Section 3 by separately considering the (space-time) discretization of the forward, and the adjoint/backward equation, and using Theorem 3.5, in particular. The results of Section 4 therefore establish optimal rates of convergence for the minimizer of the space-time discrete optimization problem $\mathbf{SLQ}_{h\tau}$ (see (4.13)–(4.14)) towards the minimizer of **SLQ**.

We remark that even for numerical methods of coupled forward-backward stochastic differential (ordinary) equations (SDEs, for short), the convergence analysis is non-trivial, and to obtain rates of convergence, usually assumptions such as small time intervals $[0,T]$ are needed (see *e.g.* [4]). In the present setting, the results in Section 4 extend available ones (*cf.* [15]) in the literature in several aspects: the obtained strong convergence rates for the used finite element based *space-time* discretization (4.15) hold for *arbitrary* times T—and is not only a semi-discretization in space where optimal rates are obtained in [15] for *small times* T *via* a contraction argument.

To solve a **BSPDE** computationally requires huge computational resources (see [15]), and it is even more computationally demanding (in terms of computational storage requirements and computational times) to solve the coupled **FBSPDE**. Consequently, an alternative numerical strategy to the space-time discretization (4.15)–(4.16) of **FBSPDE** which couples the forward and the backward part is needed in practice. While a simple fixed-point method on the level of optimality conditions to accomplish this goal is known to converge only for small times $T>0$ (*cf.* [4, 15]), the construction of a *decoupled* system should be based on the fully discretized problem $\mathbf{SLQ}_{h\tau}$ (4.13)–(4.14), where convergence of a gradient descent method exploits its character as a minimization problem, and allows for computational decouplings; we refer to Section 5 for details of this iterative scheme, and its convergence, which is the final goal in this work.

The rest of this paper is organized as follows. In Section 2, we introduce notations, and review relevant properties of the problems **BSPDE** (2.10) and **FBSPDE** considered in this work. In Section 3, we prove strong error estimates for a space-time discretization of **BSPDE**. By virtue of the obtained error estimates, in Section 4, we prove a convergence rate for a space-time discretization of **FBSPDE**, which is related to problem **SLQ**. Convergence of the related iterative gradient descent method towards the minimizer U^* of **SLQ** is shown in Section 5.

2. Preliminaries

2.1. Notation—involved processes and the finite element method

Let $\big(\mathbb{K},(\cdot,\cdot)_{\mathbb{K}}\big)$ be a separable Hilbert space. By $\|\cdot\|_{\mathbb{L}^2}$ resp. $(\cdot,\cdot)_{\mathbb{L}^2}$, we denote the norm resp. the scalar product in Lebesgue space $\mathbb{L}^2 := L^2(D)$. The norm in sobolev space $\mathbb{H}^1_0 := H^1_0(D)$, $\mathbb{H}^2 := H^2(D)$ is denoted by $\|\cdot\|_{\mathbb{H}^1_0}$, $\|\cdot\|_{\mathbb{H}^2}$ respectively. Let $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ be a complete filtered probability space, where $\mathbb{F}=\{\mathcal{F}_t\}_{t\in[0,T]}$ is the filtration generated by the $\mathbb{R}^m$-valued Wiener process W, which is augmented by all the $\mathbb{P}$-null sets. Below, we set $m=1$ for simplicity. The space of all $\mathbb{F}$-adapted processes $X:\Omega\times[0,T]\to\mathbb{K}$ satisfying $\mathbb{E}[\int_0^T\|X(t)\|^2_{\mathbb{K}}\,\mathrm{d}t]<\infty$ is denoted by $L^2_{\mathbb{F}}(\Omega;L^2(0,T;\mathbb{K}))$; the space of all $\mathbb{F}$-adapted processes $X:\Omega\times[0,T]\to\mathbb{K}$ satisfying $\mathbb{E}[\sup_{t\in[0,T]}\|X(t)\|^2_{\mathbb{K}}]<\infty$ is denoted by $L^2_{\mathbb{F}}\big(\Omega;C([0,T];\mathbb{K})\big)$.

We partition the bounded domain $D\subset\mathbb{R}^d$ *via* a regular triangulation $\mathcal{T}_h$ into elements K with maximum mesh size $h:=\max\{\mathrm{diam}(K):\,K\in\mathcal{T}_h\}$, and consider spaces

$$\mathbb{V}^1_h := \left\{\phi\in\mathbb{H}^1_0:\,\phi\big|_K\in\mathbb{P}_1(K)\quad\forall\,K\in\mathcal{T}_h\right\},\qquad \mathbb{V}^0_h := \left\{\phi\in\mathbb{L}^2:\,\phi\big|_K\in\mathbb{P}_0(K)\quad\forall\,K\in\mathcal{T}_h\right\},$$

where $\mathbb{P}_i(K)$ denotes the space of polynomials of degree i $(i=0,1)$. The $\mathbb{L}^2$-projection $\Pi^i_h:\mathbb{L}^2\to\mathbb{V}^i_h$ is defined by $(\Pi^i_h\xi-\xi,\phi_h)_{\mathbb{L}^2}=0$ for all $\phi_h\in\mathbb{V}^i_h$. We define the discrete Laplacean $\Delta_h:\mathbb{V}^1_h\to\mathbb{V}^1_h$ by $(-\Delta_h\xi_h,\phi_h)_{\mathbb{L}^2} = (\nabla\xi_h,\nabla\phi_h)_{\mathbb{L}^2}$ for all $\xi_h,\phi_h\in\mathbb{V}^1_h$.

We use approximation estimates for the projection Π^1_h, and an inverse estimate (*cf.* [8]) to conclude that

$$\|\Delta_h\Pi^1_h\xi\|_{\mathbb{L}^2}\le C\|\nabla^2\xi\|_{\mathbb{L}^2}\qquad\forall\,\xi\in\mathbb{H}^1_0\cap\mathbb{H}^2\,, \tag{2.1}$$

since

$$\begin{aligned}\|\Delta_h \Pi_h^1 \xi\|_{\mathbb{L}^2}^2 &= -\big(\nabla[\Pi_h^1\xi - \xi], \nabla\Delta_h\Pi_h^1\xi\big)_{\mathbb{L}^2} - (\nabla\xi, \nabla\Delta_h\Pi_h^1\xi)_{\mathbb{L}^2} \\ &\le Ch\|\nabla^2\xi\|_{\mathbb{L}^2}\|\nabla\Delta_h\Pi_h^1\xi\|_{\mathbb{L}^2} + (\Delta\xi, \Delta_h\Pi_h^1\xi)_{\mathbb{L}^2} \\ &\le C\big(\|\nabla^2\xi\|_{\mathbb{L}^2} + \|\Delta\xi\|_{\mathbb{L}^2}\big)\|\Delta_h\Pi_h^1\xi\|_{\mathbb{L}^2}\,.\end{aligned}$$

We denote by $I_\tau = \{t_n\}_{n=0}^N \subset [0,T]$ a time mesh with maximum step size $\tau := \max\{t_{n+1} - t_n : n = 0,1,\dots,N-1\}$, and $\Delta_n W = W(t_n) - W(t_{n-1})$ for all $n = 1,\dots,N$. For simplicity, we choose a uniform partition, *i.e.* $\tau = T/N$ and $\tau \le 1$. The results in this work still hold for general partitions.

2.2. The stochastic heat equation—strong convergence rates for a space-time discretization

To obtain a convergence rate for the space-time discretization of (1.2), we assume that $X_0 \in \mathbb{H}_0^1 \cap \mathbb{H}^2$, that $U \in L^2_{\mathbb{F}}\big(\Omega; L^2(0,T;\mathbb{H}_0^1)\big)$, and $\sigma \in L^\infty_{\mathbb{F}}\big(0,T; L^2(\Omega;\mathbb{H}_0^1\cap\mathbb{H}^2)\big)$. Under the above assumptions, **SPDE** (1.2) admits a unique strong solution $X \in L^2_{\mathbb{F}}\big(\Omega; C([0,T];\mathbb{H}_0^1) \cap L^2(0,T;\mathbb{H}_0^1\cap\mathbb{H}^2)\big)$, see *e.g.* [13, 26, 40], such that

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\|X(t)\|_{\mathbb{H}_0^1}^2 + \int_0^T \|X(t)\|_{\mathbb{H}^2}^2\,\mathrm{d}t\Big] \le C\mathbb{E}\Big[\|X_0\|_{\mathbb{H}_0^1}^2 + \int_0^T \|U(t)\|_{\mathbb{L}^2}^2\,\mathrm{d}t\Big]\,, \tag{2.2}$$

where $C \equiv C(D,T) > 0$, which satisfies the following weak form $\mathbb{P}$-a.s. for all $t \in [0,T]$

$$\begin{aligned}\big(X(t),\phi\big)_{\mathbb{L}^2} - (X_0,\phi)_{\mathbb{L}^2} + \int_0^t \Big[\big(\nabla X(s),\nabla\phi\big)_{\mathbb{L}^2} - \big(U(s),\phi\big)_{\mathbb{L}^2}\Big]\,\mathrm{d}s \\ = \int_0^t \big(\sigma(s),\phi\big)_{\mathbb{L}^2}\,\mathrm{d}W(s) \qquad \forall\,\phi\in\mathbb{H}_0^1\,.\end{aligned} \tag{2.3}$$

A finite element discretization of (2.3) then reads: For all $t\in[0,T]$, find $X_h \in L^2_{\mathbb{F}}\big(\Omega; C([0,T];\mathbb{V}_h^1)\big)$ such that $\mathbb{P}$-a.s. and for all times $t\in[0,T]$

$$\begin{aligned}\big(X_h(t),\phi_h\big)_{\mathbb{L}^2} - (X_0,\phi_h)_{\mathbb{L}^2} + \int_0^t \Big[\big(\nabla X_h(s),\nabla\phi_h\big)_{\mathbb{L}^2} - \big(U(s),\phi_h\big)_{\mathbb{L}^2}\Big]\,\mathrm{d}s \\ = \int_0^t \big(\sigma(s),\phi_h\big)_{\mathbb{L}^2}\,\mathrm{d}W(s) \qquad \forall\,\phi_h\in\mathbb{V}_h^1\,.\end{aligned} \tag{2.4}$$

Equation (2.4) may be recast into the following SDE system,

$$\begin{cases} dX_h(t) = \big[\Delta_h X_h(t) + \Pi_h^1 U(t)\big]\mathrm{d}t + \Pi_h^1\sigma(t)\,\mathrm{d}W(t) \qquad \forall\; t\in[0,T]\,, \\ X_h(0) = \Pi_h^1 X_0\,. \end{cases} \tag{2.5}$$

Thanks to this equivalence, we do not distinguish between **SPDE**$_h$ (2.4) and SDE (2.5) throughout this paper.

The derivation of a strong error estimate is standard, and uses the improved (spatial) regularity properties of the strong variational solution,

$$\sup_{t\in[0,T]}\mathbb{E}\big[\|X_h(t)-X(t)\|_{\mathbb{L}^2}^2\big] + \mathbb{E}\Big[\int_0^T \big\|\nabla\big[X_h(t)-X(t)\big]\big\|_{\mathbb{L}^2}^2\,\mathrm{d}t\Big] \le Ch^2\,. \tag{2.6}$$

We now consider a time-implicit discretization of (2.4) on a partition I_τ of $[0,T]$. The problem then reads: For every $0 \le n \le N-1$, find a solution $X_h^{n+1} \in L^2_{\mathcal{F}_{t_{n+1}}}(\Omega;\mathbb{V}_h^1)$ such that $\mathbb{P}$-a.s.

$$(X_h^{n+1} - X_h^n, \phi_h)_{\mathbb{L}^2} + \tau\big[(\nabla X_h^{n+1}, \nabla\phi_h)_{\mathbb{L}^2} - (U(t_n),\phi_h)_{\mathbb{L}^2}\big] = \big(\sigma(t_n),\phi_h\big)_{\mathbb{L}^2}\Delta_{n+1}W\,, \tag{2.7}$$

where $\Delta_{n+1}W := W(t_{n+1}) - W(t_n)$. The verification of the error estimate (see [36])

$$\max_{0\le n\le N}\mathbb{E}\big[\|X_h(t_n) - X_h^n\|^2_{\mathbb{L}^2}\big] + \tau\sum_{n=1}^N \mathbb{E}\big[\big\|\nabla\big[X_h(t_n) - X_h^n\big]\big\|^2_{\mathbb{L}^2}\big] \le C\tau \tag{2.8}$$

rests on stability properties of the implicit Euler, as well as the bound

$$\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} \mathbb{E}\big[\|X_h(t) - X_h(t_n)\|^2_{\mathbb{H}_0^1}\big]\,\mathrm{d}t \le C\tau\,, \tag{2.9}$$

which requires additional regularity properties of involved data, *i.e.*,

$$\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} \mathbb{E}\left[\|U(t) - U(t_n)\|^2_{\mathbb{L}^2} + \|\sigma(t) - \sigma(t_n)\|^2_{\mathbb{L}^2}\right]\,\mathrm{d}t \le C\tau\,,$$

and the $\mathbb{H}^1$-stability of the $\mathbb{L}^2$-projection Π_h^1; *cf.* [7, 10].

2.3. The backward stochastic heat equation—a finite element based spatial discretization

Let $Y_T \in L^2_{\mathcal{F}_T}\big(\Omega;\mathbb{H}_0^1\big)$ and $f \in L^2_{\mathbb{F}}\big(\Omega; L^2(0,T;\mathbb{L}^2)\big)$. A strong solution to the backward stochastic heat equation

$$\begin{cases} \mathrm{d}Y(t) = \big[-\Delta Y(t) + f(t)\big]\mathrm{d}t + Z(t)\mathrm{d}W(t) \quad \forall\, t\in[0,T]\,, \\ Y(T) = Y_T \end{cases} \tag{2.10}$$

is a pair of square integrable $\mathbb{F}$-adapted processes $(Y,Z) \in L^2_{\mathbb{F}}\big(\Omega; C([0,T];\mathbb{H}_0^1) \cap L^2_{\mathbb{F}}\big(0,T;\mathbb{H}_0^1\cap\mathbb{H}^2\big)\big) \times L^2_{\mathbb{F}}\big(0,T;\mathbb{H}_0^1\big)$, and there exists a constant $C \equiv C(D,T) > 0$ such that

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\|Y(t)\|^2_{\mathbb{H}_0^1}\Big] + \mathbb{E}\Big[\int_0^T \|Y(t)\|^2_{\mathbb{H}^2} + \|Z(t)\|^2_{\mathbb{H}_0^1}\,\mathrm{d}t\Big] \le C\Big[\mathbb{E}\|Y_T\|^2_{\mathbb{H}_0^1} + \mathbb{E}\int_0^T \|f(t)\|^2_{\mathbb{L}^2}\,\mathrm{d}t\Big]\,. \tag{2.11}$$

The existence of a strong solution to (2.10), as well as its uniqueness are shown in [14]. Obviously, (Y,Z) satisfies the following variational form $\mathbb{P}$-a.s. for all times $t\in[0,T]$

$$\begin{aligned} \big(Y_T,\phi\big)_{\mathbb{L}^2} - (Y(t),\phi)_{\mathbb{L}^2} - \int_t^T \big[\big(\nabla Y(s),\nabla\phi\big)_{\mathbb{L}^2} + \big(f(s),\phi\big)_{\mathbb{L}^2}\big]\,\mathrm{d}s \\ = \int_t^T \big(Z(s),\phi\big)_{\mathbb{L}^2}\,\mathrm{d}W(s) \qquad \forall\,\phi\in\mathbb{H}_0^1\,. \end{aligned} \tag{2.12}$$

We now consider a finite element discretization of the **BSPDE** (2.10). Let $Y_{T,h} \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{V}_h^1)$ be an approximation of Y_T. The problem $\textbf{BSPDE}_h$ then reads: Find $(Y_h,Z_h) \in L^2_{\mathbb{F}}\big(\Omega; C([0,T];\mathbb{V}_h^1)\big) \times L^2_{\mathbb{F}}\big(\Omega; L^2(0,T;\mathbb{V}_h^1)\big)$

such that $\mathbb{P}$-a.s. for all $t \in [0,T]$

$$
\begin{aligned}
(Y_{T,h}, \phi_h)_{\mathbb{L}^2} - \big(Y_h(t), \phi_h\big)_{\mathbb{L}^2} - \int_t^T \big[\big(\nabla Y_h(s), \nabla \phi_h\big)_{\mathbb{L}^2} + \big(f(s), \phi_h\big)_{\mathbb{L}^2}\big]\,\mathrm{d}s \\
= \int_t^T \big(Z_h(s), \phi_h\big)_{\mathbb{L}^2}\,\mathrm{d}W(s) \qquad \forall\, \phi_h \in \mathbb{V}_h^1 \,.
\end{aligned}
\tag{2.13}
$$

Equation (2.13) is equivalent to the following BSDE:

$$
\begin{cases}
\mathrm{d}Y_h(t) = \big[-\Delta_h Y_h(t) + \Pi_h^1 f(t)\big]\mathrm{d}t + Z_h(t)\,\mathrm{d}W(t) \quad \forall\, t \in [0,T]\,, \\
Y_h(T) = Y_{T,h}\,.
\end{cases}
\tag{2.14}
$$

Based on this equivalence, we do not distinguish $\mathbf{BSPDE}_h$ (2.13) and BSDE (2.14) throughout this paper. The existence and uniqueness of a solution tuple (Y_h, Z_h) *e.g.* follows from Theorem 2.1 of [17]. Moreover, there exists $C \equiv C(f,T) > 0$ such that

$$
\sup_{t\in[0,T]} \mathbb{E}\big[\|\nabla Y_h(t)\|_{\mathbb{L}^2}^2\big] + \mathbb{E}\Big[\int_0^T \|\Delta_h Y_h(t)\|_{\mathbb{L}^2}^2 + \|\nabla Z_h(t)\|_{\mathbb{L}^2}^2\,\mathrm{d}t\Big] \le C\mathbb{E}\Big[\|\nabla Y_{T,h}\|_{\mathbb{L}^2}^2 + \int_0^T \|f(t)\|_{\mathbb{L}^2}^2\,\mathrm{d}t\Big]\,; \tag{2.15}
$$

cf. Lemma 3.1 of [15].—The following result is taken from Theorem 3.2 of [15], whose proof exploits the bounds (2.11).

Theorem 2.1. *Let $Y_T \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{H}_0^1)$, $Y_{T,h} \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{V}_h^1)$. Let (Y,Z) be the solution to (2.12), and (Y_h,Z_h) solve (2.13). There exists $C \equiv C(Y_T,f,T) > 0$ such that*

$$
\begin{aligned}
\sup_{t\in[0,T]} \mathbb{E}\big[\|Y(t) - Y_h(t)\|_{\mathbb{L}^2}^2\big] + \mathbb{E}\Big[\int_0^T \big\|\nabla\big[Y(t) - Y_h(t)\big]\big\|_{\mathbb{L}^2}^2 + \|Z(t) - Z_h(t)\|_{\mathbb{L}^2}^2\,\mathrm{d}t\Big] \\
\le C\big(\mathbb{E}\big[\|Y_T - Y_{T,h}\|_{\mathbb{L}^2}^2\big] + h^2\big)\,.
\end{aligned}
$$

Choosing $Y_{T,h} = \Pi_h^1 Y_T$ thus leads to an error estimate for the spatial semi-discretization (2.14).

2.4. Temporal discretization of the backward stochastic heat equation—the role of the Malliavin derivative

The numerical analysis of a temporal discretization of (2.14) requires Malliavin calculus to bound temporal increments $\mathbb{E}\big[\|Z_h(t) - Z_h(s)\|_{\mathbb{L}^2}^2\big]$ *uniformly in* h in terms of $|t-s|$, where $s,t \in [0,T]$. We therefore recall the definition of the Malliavin derivative of processes, and the crucial connection between the Malliavin derivative of Y_h and Z_h from (2.14). For further details, we refer to [17, 31].

Let us recall that $\mathcal{F}_T = \sigma\{W(t); 0 \le t \le T\}$, and that $\mathbb{K}$ denotes a separable Hilbert space. We define the Itô isometry $W : L^2(0,T;\mathbb{R}) \to L^2_{\mathcal{F}_T}(\Omega;\mathbb{R})$ by

$$
W(g) = \int_0^T g(t)\,\mathrm{d}W(t)\,.
$$

For $\ell \in \mathbb{N}$, we denote by $C_p^\infty(\mathbb{R}^\ell)$ the space of all smooth functions $s : \mathbb{R}^\ell \to \mathbb{R}$ such that s and all of its partial derivatives have polynomial growth. Let $\mathcal{P}$ be the set of $\mathbb{R}$-valued random variables of the form

$$
F = s\big(W(h_1), \dots, W(h_\ell)\big) \tag{2.16}
$$

for some $s \in C_p^\infty(\mathbb{R}^\ell)$, $\ell \in \mathbb{N}$, and $g_1, \dots, g_\ell \in L^2(0,T;\mathbb{R})$. To any $F \in \mathcal{P}$ we define its $\mathbb{R}$-valued Malliavin derivative $DF := \{D_\theta F;\, 0 \le \theta \le T\}$ process *via*

$$D_\theta F = \sum_{i=1}^{\ell} \frac{\partial s}{\partial x_i}(W(g_1), \dots, W(g_\ell)) g_i(\theta)\,.$$

In general, we can define the k-th iterated derivative of F by $D^k F = D(D^{k-1}F)$, for any $k \in \mathbb{N}$.

Now we extend the derivative operator to $\mathbb{K}$-valued variables. For any $k \in \mathbb{N}$, and u in the set of $\mathbb{K}$-valued variables:

$$\mathcal{P}_{\mathbb{K}} = \Big\{ u = \sum_{j=1}^{n} F_j \phi_j : F_j \in \mathcal{P},\, \phi_j \in \mathbb{K},\, n \in \mathbb{N} \Big\}\,,$$

we can define the k-th iterated derivative of u by

$$D^k u = \sum_{j=1}^{n} D^k F_j \otimes \phi_j\,.$$

For $p \ge 1$, we define the norm $\|\cdot\|_{k,p}$ *via*

$$\|u\|_{k,p} := \Big(\mathbb{E}\Big[\|u\|_{\mathbb{K}}^p + \sum_{j=1}^{k} \big\| D^j u \big\|^p_{(L^2(0,T;\mathbb{R}))^{\otimes j} \otimes \mathbb{K}} \Big] \Big)^{\frac{1}{p}}\,.$$

Then $\mathbb{D}^{k,p}(\mathbb{K})$ is the completion of $\mathcal{P}_{\mathbb{K}}$ under the norm $\|\cdot\|_{k,p}$.

We may now express Z_h in BSDE (2.14) in terms of the Malliavin derivative of Y_h.

Lemma 2.2 ([17], Prop. 5.3)**.** *Suppose that $Y_{T,h} \in \mathbb{D}^{1,2}(\mathbb{L}^2)$, $f \in L^2_{\mathbb{F}}\big(\Omega; L^2(0,T;\mathbb{L}^2)\big)$, and*

$$\mathbb{E}\Big[\int_0^T \|D_\theta Y_{T,h}\|^2_{\mathbb{L}^2}\, \mathrm{d}\theta \Big] + \mathbb{E}\Big[\int_0^T \int_0^T \|D_\theta f(t)\|^2_{\mathbb{L}^2}\, \mathrm{d}t \mathrm{d}\theta \Big] < \infty\,.$$

Let (Y_h, Z_h) be the solution to BSDE (2.14). Then

$$(Y_h, Z_h) \in L^2_{\mathbb{F}}\big(\Omega; C\big([0,T]; \mathbb{D}^{1,2}(\mathbb{L}^2)\big) \times L^2\big(0,T; \mathbb{D}^{1,2}(\mathbb{L}^2)\big)\big)\,,$$

and its Malliavin derivative $(D_\theta Y_h, D_\theta Z_h)$ solves

$$\begin{cases} D_\theta Y_h(t) - D_\theta Y_h(T) + \displaystyle\int_t^T -\Delta_h D_\theta Y_h(s) + \Pi_h^1 D_\theta f(s)\, \mathrm{d}s \\ \qquad\qquad\qquad\qquad = -\displaystyle\int_t^T D_\theta Z_h(s)\, \mathrm{d}W(s) \qquad 0 \le \theta \le t \le T\,, \\ D_\theta Y_h(t) = D_\theta Z_h(t) = 0 \qquad 0 \le t < \theta \le T\,. \end{cases} \tag{2.17}$$

Moreover, $\{D_t Y_h(t) : 0 \le t \le T\}$ is a version of $\{Z_h(t) : 0 \le t \le T\}$.

3. Strong rates of convergence for a space-time discretization of the BSPDE (2.10)

In this section, we introduce the temporal discretization scheme (3.6) to approximate the solution (Y_h, Z_h) to the $\mathbf{BSPDE}_h$ (2.14) by a finite sequence $\{(Y_h^n, Z_h^n)\}_{n=0}^{N-1}$ on a mesh I_τ. The main results are Theorems 3.5 and 3.7 in Section 3.2. Their derivation crucially hinges on the time regularity of the solution (Y_h, Z_h) to (2.14), and a related *uniform* bound w.r.t. the mesh parameter $h > 0$, which is provided in the subsequent Section 3.1.

3.1. Uniform bounds for temporal increments of the solution (Y_h, Z_h) to (2.14)

We start with the derivation of uniform estimates for Y_h which control its temporal increments. We note again that all involved generic constants $C > 0$ do not depend on h.

Lemma 3.1. *Suppose that $Y_{T,h} \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{H}^1_0)$, $f \in L^2_{\mathbb{F}}\big(\Omega; L^2(0,T;\mathbb{H}^1_0)\big)$, I_τ is a temporal partition of $[0,T]$. Let (Y_h, Z_h) be the solution to (2.14). Then*

(i)

$$\sum_{n=0}^{N-1} \mathbb{E}\Big[\int_{t_n}^{t_{n+1}} \|Y_h(t) - Y_h(t_n)\|^2_{\mathbb{L}^2}\,\mathrm{d}t\Big] \le C\tau \mathbb{E}\Big[\|Y_{T,h}\|^2_{\mathbb{H}^1_0} + \int_0^T \|f(t)\|^2_{\mathbb{L}^2}\,\mathrm{d}t\Big].$$

(ii) *Assume further* $\sup_{h>0} \mathbb{E}\big[\|\Delta_h Y_{T,h}\|^2_{\mathbb{L}^2}\big] < \infty$. *Then*

$$\sum_{n=0}^{N-1} \mathbb{E}\Big[\int_{t_n}^{t_{n+1}} \|\nabla\left(Y_h(t) - Y_h(t_n)\right)\|^2_{\mathbb{L}^2}\,\mathrm{d}t\Big] \le C\tau \mathbb{E}\Big[\|\Delta_h Y_{T,h}\|^2_{\mathbb{L}^2} + \|\nabla Y_{T,h}\|^2_{\mathbb{H}^1_0} + \int_0^T \|f(t)\|^2_{\mathbb{H}^1_0}\,\mathrm{d}t\Big].$$

(iii) *Assume further* $\sup_{h>0} \mathbb{E}\big[\|\Delta_h Y_{T,h}\|^2_{\mathbb{H}^1_0}\big] < \infty$ *and* $f \in L^2_{\mathbb{F}}\big(\Omega; L^2(0,T;\mathbb{H}^1_0 \cap \mathbb{H}^2)\big)$. *Then*

$$\sum_{n=0}^{N-1} \mathbb{E}\Big[\int_{t_n}^{t_{n+1}} \|\Delta_h\left(Y_h(t) - Y_h(t_n)\right)\|^2_{\mathbb{L}^2}\,\mathrm{d}t\Big] \le C\tau \mathbb{E}\Big[\|\Delta_h Y_{T,h}\|^2_{\mathbb{H}^1_0} + \int_0^T \|f(t)\|^2_{\mathbb{H}^2}\,\mathrm{d}t\Big].$$

Here, the constant $C > 0$ only depends on $Y_{T,h}$, f and T.

Proof. We only prove (i). The other statements can be proved in a similar vein.

By BSDE (2.14), we get

$$\sum_{n=0}^{N-1} \mathbb{E}\Big[\int_{t_n}^{t_{n+1}} \|Y_h(t) - Y_h(t_n)\|^2_{\mathbb{L}^2}\,\mathrm{d}t\Big] \le C\tau \mathbb{E}\Big[\int_0^T \|\Delta_h Y_h(t)\|^2_{\mathbb{L}^2} + \|\Pi^1_h f(t)\|^2_{\mathbb{L}^2} + \|Z_h(t)\|^2_{\mathbb{L}^2}\,\mathrm{d}t\Big].$$

Applying Itô's formula for $\|Y_h\|^2_{\mathbb{L}^2}$ and $\|\nabla Y_h\|^2_{\mathbb{L}^2}$ in (2.14), we find that

$$\mathbb{E}\Big[\int_0^T \|Z_h(t)\|^2_{\mathbb{L}^2}\,\mathrm{d}t\Big] \le C\mathbb{E}\Big[\|Y_{T,h}\|^2_{\mathbb{L}^2} + \int_0^T \|\Pi^1_h f(t)\|^2_{\mathbb{L}^2}\,\mathrm{d}t\Big],$$
$$\mathbb{E}\Big[\int_0^T \|\Delta_h Y_h(t)\|^2_{\mathbb{L}^2}\,\mathrm{d}t\Big] \le C\mathbb{E}\Big[\|\nabla Y_{T,h}\|^2_{\mathbb{L}^2} + \int_0^T \|\Pi^1_h f(t)\|^2_{\mathbb{L}^2}\,\mathrm{d}t\Big].$$

Then (i) can be deduced by the above estimates. □

Lemma 3.2. *Suppose that $Y_{T,h} \in \mathbb{D}^{1,2}(\mathbb{H}^1_0)$, and $f \in L^2_{\mathbb{F}}\big(\Omega; L^2(0,T;\mathbb{L}^2)\big)$ satisfy*

$$\sup_{0\le t\le T} \mathbb{E}\big[\|D_t Y_{T,h}\|^2_{\mathbb{H}^1_0}\big] + \sup_{0\le\theta\le T}\sup_{0\le t\le T} \mathbb{E}\big[\|D_\theta D_t Y_{T,h}\|^2_{\mathbb{L}^2}\big] \le C\,,$$
$$\sup_{0\le t\le T} \mathbb{E}\Big[\int_t^T \|D_t f(\tau)\|^2_{\mathbb{L}^2}\,\mathrm{d}\tau\Big] + \sup_{0\le\theta\le T}\sup_{0\le t\le T} \mathbb{E}\Big[\int_{\theta\vee t}^T \|D_\theta D_t f(\tau)\|^2_{\mathbb{L}^2}\,\mathrm{d}\tau\Big] \le C\,,$$

and for any $s,t \in [0,T]$ with $s \le t$,

$$\mathbb{E}\big[\|\left(D_t - D_s\right) Y_{T,h}\|^2_{\mathbb{L}^2}\big] + \mathbb{E}\Big[\int_t^T \|\left(D_t - D_s\right) f(\tau)\|^2_{\mathbb{L}^2}\,\mathrm{d}\tau\Big] \le C|t-s|\,. \tag{3.1}$$

Then, it holds that

$$\mathbb{E}\big[\|Z_h(t) - Z_h(s)\|^2_{\mathbb{L}^2}\big] \le C|t-s|\,.$$

Proof. By Lemma 2.2, we know that $Z_h(t) = D_t Y_h(t)$ for all $0 \le t \le T$, and therefore, for $0 \le s \le t \le T$,

$$\frac{1}{2}\mathbb{E}\big[\|Z_h(t) - Z_h(s)\|^2_{\mathbb{L}^2}\big] \le \mathbb{E}\big[\|D_t Y_h(t) - D_s Y_h(t)\|^2_{\mathbb{L}^2}\big] + \mathbb{E}\big[\|D_s Y_h(t) - D_s Y_h(s)\|^2_{\mathbb{L}^2}\big]\,. \tag{3.2}$$

In what follows, we estimate the two terms on the right-hand side of (3.2) independently.

Step 1. Fix two $0 \le \theta_2 \le \theta_1 \le t \le T$ and define $\delta_\theta = D_{\theta_1} - D_{\theta_2}$. By (2.17), we have the BSDE

$$\delta_\theta Y_h(t) - \delta_\theta Y_h(T) + \int_t^T \big[-\Delta_h \delta_\theta Y_h(s) + \Pi^1_h \delta_\theta f(s)\big]\,\mathrm{d}s = -\int_t^T \delta_\theta Z_h(s)\,\mathrm{d}W(s) \ \ \forall\, t \in [\theta_1, T]\,. \tag{3.3}$$

Itô's formula and Poincaré's inequality lead to

$$\begin{aligned}\mathbb{E}\big[\|\delta_\theta Y_h(t)\|^2_{\mathbb{L}^2}\big] &+ \int_t^T \mathbb{E}\big[\|\nabla\delta_\theta Y_h(s)\|^2_{\mathbb{L}^2} + \|\delta_\theta Z_h(s)\|^2_{\mathbb{L}^2}\big]\,\mathrm{d}s\\ &\le \mathbb{E}\Big[\|\delta_\theta Y_h(T)\|^2_{\mathbb{L}^2} + \int_t^T \|\Pi^1_h \delta_\theta f(s)\|^2_{\mathbb{L}^2}\,\mathrm{d}s\Big]\,.\end{aligned}$$

Taking $\theta_2 = s$ and $\theta_1 = t$ and using (3.1) then lead to

$$\mathbb{E}\big[\|D_t Y_h(t) - D_s Y_h(t)\|^2_{\mathbb{L}^2}\big] \le \mathbb{E}\Big[\|\left(D_t - D_s\right) Y_{h,T}\|^2_{\mathbb{L}^2} + \int_t^T \|\left(D_t - D_s\right) f(\tau)\|^2_{\mathbb{L}^2}\,\mathrm{d}\tau\Big] \le C|t-s|\,. \tag{3.4}$$

Step 2. By (2.17), Itô's isometry together with Poincaré's inequality,

$$
\begin{aligned}
&\mathbb{E}\big[\|D_sY_h(t)-D_sY_h(s)\|_{\mathbb{L}^2}^2\big]\\
=&\mathbb{E}\Big[\Big\|\int_s^t\big[-\Delta_hD_sY_h(\tau)+\Pi_h^1D_sf(\tau)\big]\,\mathrm{d}\tau+\int_s^tD_sZ_h(\tau)\,\mathrm{d}W(\tau)\Big\|_{\mathbb{L}^2}^2\Big]\\
\le&2|t-s|\int_s^T\mathbb{E}\big[\|\Delta_hD_sY_h(\tau)\|_{\mathbb{L}^2}^2+\|\Pi_h^1D_sf(\tau)\|_{\mathbb{L}^2}^2\big]\,\mathrm{d}\tau+2\int_s^t\mathbb{E}\big[\|D_sZ_h(\tau)\|_{\mathbb{L}^2}^2\big]\,\mathrm{d}\tau\\
\le&C|t-s|\,\mathbb{E}\Big[\|\nabla D_sY_h(T)\|_{\mathbb{L}^2}^2+\int_s^T\|\Pi_h^1D_sf(\tau)\|_{\mathbb{L}^2}^2\,\mathrm{d}\tau\Big]\\
&+C|t-s|\sup_{0\le\theta\le T}\sup_{0\le t\le T}\mathbb{E}\Big[\|D_\theta D_tY_{T,h}\|_{\mathbb{L}^2}^2+\int_{\theta\vee t}^T\|D_\theta D_tf(\tau)\|_{\mathbb{L}^2}^2\,\mathrm{d}\tau\Big]\,.
\end{aligned}
\tag{3.5}
$$

Inserting (3.5) and (3.4) into (3.2) then settles the proof of the lemma. ☐

Remark 3.3. In Lemma 3.2, we derive the Malliavin differentiability of the solution to $\mathbf{BSPDE}_h$ (2.14), *i.e.*, the spatial discretization for **BSPDE** (2.10). Indeed, the Malliavin differentiability of solution to (2.10) can also be verified; see *e.g.* Proposition 3.2 of [12].

3.2. A time-implicit space-time discretization of the BSPDE (2.10)

We use a time implicit discretization on the mesh I_τ to approximate $\mathbf{BSPDE}_h$ (2.14); we refer to it as $\mathbf{BSPDE}_{h\tau}$, and the discretization reads as follows: For every $0\le n\le N-1$, find $\big(Y_h^n,Z_h^n\big)\in L^2_{\mathcal{F}_{t_n}}\big(\Omega;\mathbb{V}_h^1\times\mathbb{V}_h^1\big)$ such that

$$
\begin{cases}
[\mathbb{1}-\tau\Delta_h]Y_h^n=\mathbb{E}\big[Y_h^{n+1}\big|\mathcal{F}_{t_n}\big]-\tau\Pi_h^1f(t_n)\,,\\
Z_h^n=\dfrac{1}{\tau}\mathbb{E}\big[Y_h^{n+1}\Delta_{n+1}W\big|\mathcal{F}_{t_n}\big]\,,\\
Y_h^N=Y_{T,h}\,.
\end{cases}
\tag{3.6}
$$

We introduce an auxiliary BSDE for the convergence analysis of (3.6), which, in particular, uses a time-continuous diffusion term:

$$
\begin{cases}
\mathrm{d}\overline{Y}_h(t)=\big[-\Delta_hY_h^{\pi(t)}+\Pi_h^1f(\tau(t))\big]\,\mathrm{d}t+\overline{Z}_h(t)\,\mathrm{d}W(t)\quad\forall\,t\in[0,T)\,,\\
\overline{Y}_h(T)=Y_{T,h}\,,
\end{cases}
\tag{3.7}
$$

where $\pi(\cdot)$, and $\tau(\cdot)$ are defined as follows: For $n=0,1,\ldots,N-1$ and $t\in[t_n,t_{n+1})$, we set $\pi(t)=n$ and $\tau(t)=t_n$.

Lemma 3.4. *Let $\{(Y_h^n,Z_h^n)\}_{n=0}^{N-1}$ solve (3.6), and $(\overline{Y}_h,\overline{Z}_h)$ solve (3.7). For all $0\le n\le N-1$,*

$$
Y_h^n=\overline{Y}_h(t_n)\,,\qquad Z_h^n=\frac{1}{\tau}\mathbb{E}\Big[\int_{t_n}^{t_{n+1}}\overline{Z}_h(s)\,\mathrm{d}s\Big|\mathcal{F}_{t_n}\Big]\,.
$$

Proof. The first identity is immediate; the second follows from multiplication of (3.7) with the admissible $\int_{t_n}^{t_{n+1}}1\,\mathrm{d}W(s)$, and application of conditional expectation $\mathbb{E}[\cdot|\mathcal{F}_{t_n}]$. ☐

We may now prove a strong error estimate for the first component of (Y_h,Z_h) that solves (2.14); its proof uses the auxiliary BSDE (3.7), and the relation of its solution at time-grid points to iterates $\{(Y_h^n,Z_h^n)\}_{n=0}^{N-1}$ from (3.6) thanks to Lemma 3.4.

Theorem 3.5. *Suppose that $Y_{T,h} \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{H}^1_0)$, $\Delta_h Y_{T,h} \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{L}^2)$, $f \in L^2_{\mathbb{F}}\big(\Omega;L^2(0,T;\mathbb{H}^1_0)\big)$ as well as*

$$\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} \mathbb{E}\big[\|f(t)-f(t_n)\|^2_{\mathbb{L}^2}\big]\,\mathrm{d}t \le C\tau\,.$$

Let (Y_h,Z_h) solve (2.14), and $\{(Y_h^n,Z_h^n)\}_{n=0}^{N-1}$ solve (3.6). There exists a constant $C \equiv (Y_{T,h},f,T) > 0$ such that

$$\max_{0\le n\le N}\mathbb{E}\big[\|Y_h(t_n)-Y_h^n\|^2_{\mathbb{L}^2}\big] + \tau\sum_{n=0}^{N-1}\mathbb{E}\big[\|\nabla(Y_h(t_n)-Y_h^n)\|^2_{\mathbb{L}^2}\big] \le C\tau\,. \tag{3.8}$$

Proof. Consider $(\overline{Y}_h,\overline{Z}_h)$ from (3.7), and define $\{e_n\}_{n=0}^{N-1}$, where each $e_n = Y_h(t_n)-\overline{Y}_h(t_n)$ is a $\mathbb{V}^1_h$-valued random variable. Subtracting (3.7) from (2.14) yields $\mathbb{P}$-a.s.

$$\begin{aligned} e_n - e_{n+1} - \int_{t_n}^{t_{n+1}}\Delta_h e_n\,\mathrm{d}s &= \int_{t_n}^{t_{n+1}}\Delta_h\big[Y_h(s)-Y_h(t_n)\big] - \Pi_h^1[f(s)-f(t_n)]\,\mathrm{d}s \\ &\quad - \int_{t_n}^{t_{n+1}}\big[Z_h(s)-\overline{Z}_h(s)\big]\,\mathrm{d}W(s)\,. \end{aligned} \tag{3.9}$$

Fixing one realization $\omega\in\Omega$, testing with the admissible $e_n(\omega)\in\mathbb{V}^1_h$, using binomial formula, and then taking expectation, and Poincare's and Young's inequality lead to

$$\begin{aligned} &\frac{1}{2}\mathbb{E}\Big[\|e_n\|^2_{\mathbb{L}^2} - \|e_{n+1}\|^2_{\mathbb{L}^2} + \|e_n-e_{n+1}\|^2_{\mathbb{L}^2} + 2\int_{t_n}^{t_{n+1}}\|\nabla e_n\|^2_{\mathbb{L}^2}\,\mathrm{d}s\Big] \\ &\le \mathbb{E}\Big[\int_{t_n}^{t_{n+1}}\frac{1}{2}\big\|\nabla\big[Y_h(s)-Y_h(t_n)\big]\big\|^2_{\mathbb{L}^2} + \|f(s)-f(t_n)\|^2_{\mathbb{L}^2}\,\mathrm{d}s\Big] \\ &\quad + \frac{1}{2}\mathbb{E}\Big[\int_{t_n}^{t_{n+1}}\|\nabla e_n\|^2_{\mathbb{L}^2} + \|e_n-e_{n+1}\|^2_{\mathbb{L}^2}\,\mathrm{d}s\Big] + \frac{\tau}{2}\mathbb{E}\big[\|e_{n+1}\|_{\mathbb{L}^2}\big]^2\,. \end{aligned} \tag{3.10}$$

Subsequently, the discrete Gronwall inequality leads to

$$\max_{0\le n\le N}\mathbb{E}\big[\|e_n\|^2_{\mathbb{L}^2}\big] \le 2e^T\sum_{n=0}^{N-1}\mathbb{E}\Big[\int_{t_n}^{t_{n+1}}\big\|\nabla\big[Y_h(s)-Y_h(t_n)\big]\big\|^2_{\mathbb{L}^2} + \|f(s)-f(t_n)\|^2_{\mathbb{L}^2}\,\mathrm{d}s\Big]\,. \tag{3.11}$$

Then, summing up over all steps of (3.10) yields

$$\begin{aligned} &\sum_{n=0}^{N-1}\mathbb{E}\Big[\int_{t_n}^{t_{n+1}}\|\nabla e_n\|^2_{\mathbb{L}^2}\,\mathrm{d}s\Big] \\ &\le \tau\sum_{n=0}^{N-1}\mathbb{E}\big[\|e_{n+1}\|_{\mathbb{L}^2}\big]^2 + 2\sum_{n=0}^{N-1}\mathbb{E}\Big[\int_{t_n}^{t_{n+1}}\big\|\nabla\big[Y_h(s)-Y_h(t_n)\big]\big\|^2_{\mathbb{L}^2} + \|f(s)-f(t_n)\|^2_{\mathbb{L}^2}\,\mathrm{d}s\Big]\,. \end{aligned} \tag{3.12}$$

Then, (3.11), (3.12) together with Lemma 3.1 (ii), and Lemma 3.4 lead to the desired estimate. ∎

By Theorems 2.1, 3.5 and Lemma 3.1 (i), we thus get the following convergence rate for the approximation $\{Y_h^n\}_{n=0}^N$ of the first solution component Y to (2.12) *via* the space-time discretization scheme (3.6),

$$\max_{0\le n\le N} \mathbb{E}\big[\|Y(t_n)-Y_h^n\|_{\mathbb{L}^2}^2\big] + \sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} \mathbb{E}\big[\|\nabla\big(Y(t)-Y_h^n\big)\|_{\mathbb{L}^2}^2\big]\,\mathrm{d}t \le C\left(\tau+h^2\right) . \tag{3.13}$$

Remark 3.6. If the drift term of (2.10) is $-\Delta Y(t,x)+f\big(t,x,Y(t,x)\big)$, where f is a Lipschitz nonlinearity, we may apply a similar procedure to get the above convergence rate. However, the above strategy is not clear to be successful if Z appears as well in the drift term.

We now derive estimates for the approximation $\{Z_h^n\}_{n=0}^{N-1}$ of the second solution component Z to (2.12), which uses the characterization $Z_h(t)=D_tY_h(t)$, and (2.17).

Theorem 3.7. *Let (Y_h,Z_h) solve (2.13), where data satisfy the assumptions in Lemma 3.1 (ii), Lemma 3.2, as well as*

$$\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} \mathbb{E}\big[\|f(t)-f(t_n)\|_{\mathbb{L}^2}^2\big]\,\mathrm{d}t \le C\tau\,.$$

Let $\{(Y_h^n,Z_h^n)\}_{n=0}^{N-1}$ solve (3.6). There exists a constant $C\equiv(Y_T,f,T)>0$ such that

$$\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} \mathbb{E}\big[\|Z_h(t)-Z_h^n\|_{\mathbb{L}^2}^2\big]\,\mathrm{d}t \le C\tau\,.$$

The proof begins with an estimate for $Z_h-\overline{Z}_h$, which exploits time regularity properties of the solution (Y_h,Z_h) in stronger norms; *cf.* Lemma 3.1, (ii). Moreover, the following technical result is needed; see also [35].

Lemma 3.8. *For any $\varphi\in L^2_{\mathbb{F}}(0,T;\mathbb{K})$ and $0\le s<t\le T$, define*

$$\varphi_0=\frac{1}{t-s}\mathbb{E}\Big[\int_s^t\varphi(\tau)\,\mathrm{d}\tau\Big|\mathcal{F}_s\Big]\,.$$

For any $\xi\in L^2_{\mathcal{F}_s}(\Omega;\mathbb{K})$ there holds

$$\mathbb{E}\Big[\int_s^t\|\varphi(\tau)-\varphi_0\|_{\mathbb{K}}^2\,\mathrm{d}\tau\Big]\le\mathbb{E}\Big[\int_s^t\|\varphi(\tau)-\xi\|_{\mathbb{K}}^2\,\mathrm{d}\tau\Big]\,.$$

Proof. Let $\{\phi_i\}_{i=1}^\infty$ be an orthonormal basis of $\mathbb{K}$, and Π_n be the projection from $\mathbb{K}$ to $\mathrm{span}\{\phi_i: i=1,2,\ldots,n\}$. For any $n\in\mathbb{Z}$, one has

$$\begin{aligned}
&\mathbb{E}\Big[\int_s^t\|\varphi(\tau)-\xi\|_{\mathbb{K}}^2\,\mathrm{d}\tau\Big]\ge\mathbb{E}\Big[\int_s^t\|\Pi_n\big(\varphi(\tau)-\xi\big)\|_{\mathbb{K}}^2\,\mathrm{d}\tau\Big]\\
=&\mathbb{E}\Big[\int_s^t\|\Pi_n\big(\varphi(\tau)-\varphi_0\big)\|_{\mathbb{K}}^2+\|\Pi_n(\varphi_0-\xi)\|_{\mathbb{K}}^2+2\big(\Pi_n\big(\varphi(\tau)-\varphi_0\big),\Pi_n(\varphi_0-\xi)\big)_{\mathbb{K}}\,\mathrm{d}\tau\Big]\\
=&\mathbb{E}\Big[\int_s^t\|\Pi_n(\varphi(\tau)-\varphi_0)\|_{\mathbb{K}}^2+\|\Pi_n(\varphi_0-\xi)|_{\mathbb{K}}^2\,\mathrm{d}\tau\Big]\\
&\quad+2\mathbb{E}\Big[\mathbb{E}\Big[\Big(\int_s^t\Pi_n\varphi(\tau)\,\mathrm{d}\tau-\mathbb{E}\Big[\int_s^t\Pi_n\varphi(\tau)\,\mathrm{d}\tau\Big|\mathcal{F}_s\Big],\Pi_n(\varphi_0-\xi)\Big)_{\mathbb{K}}\Big|\mathcal{F}_s\Big]\Big]
\end{aligned}$$

Since, φ_0 and ξ are $\mathcal{F}_s$-measurable, the last term vanishes, *i.e.*,

$$\mathbb{E}\Big[\mathbb{E}\Big[\Big(\int_s^t \Pi_n\varphi(\tau)\,\mathrm{d}\tau - \mathbb{E}\Big[\int_s^t \Pi_n\varphi(\tau)\,\mathrm{d}\tau\Big|\mathcal{F}_s\Big], \Pi_n(\varphi_0-\xi)\Big)_{\mathbb{K}}\Big|\mathcal{F}_s\Big]\Big]$$
$$= \mathbb{E}\left[\Big(\mathbb{E}\Big[\int_s^t \Pi_n\varphi(\tau)\,\mathrm{d}\tau - \mathbb{E}\Big[\int_s^t \Pi_n\varphi(\tau)\,\mathrm{d}\tau\Big|\mathcal{F}_s\Big]\Big|\mathcal{F}_s\Big], \Pi_n(\varphi_0-\xi)\Big)_{\mathbb{K}}\right] = 0\,.$$

Therefore,

$$\mathbb{E}\Big[\int_s^t \|\varphi(\tau)-\xi\|_{\mathbb{K}}^2\,\mathrm{d}\tau\Big] \geq \mathbb{E}\Big[\int_s^t \|\Pi_n(\varphi(\tau)-\varphi_0)\|_{\mathbb{K}}^2\,\mathrm{d}\tau\Big] + \mathbb{E}\Big[\int_s^t \|\Pi_n(\varphi_0-\xi)|_{\mathbb{K}}^2\,\mathrm{d}\tau\Big]$$
$$\geq \mathbb{E}\Big[\int_s^t \|\Pi_n(\varphi(\tau)-\varphi_0)\|_{\mathbb{K}}^2\,\mathrm{d}\tau\Big]\,.$$

By letting $n\uparrow\infty$, we may therefore conclude

$$\mathbb{E}\Big[\int_s^t \|\varphi(\tau)-\varphi_0\|_{\mathbb{K}}^2\,\mathrm{d}\tau\Big] = \lim_{n\to\infty}\mathbb{E}\Big[\int_s^t \|\Pi_n\big(\varphi(\tau)-\varphi_0\big)\|_{\mathbb{K}}^2\,\mathrm{d}\tau\Big] \leq \mathbb{E}\Big[\int_s^t \|\varphi(\tau)-\xi\|_{\mathbb{K}}^2\,\mathrm{d}\tau\Big]\,,$$

which completes the proof. □

Proof of Theorem 3.7. Step 1. Claim: theres exists a constant C, which is independent of h, τ, such that

$$\mathbb{E}\Big[\int_0^T \|Z_h(s)-\overline{Z}_h(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s\Big] \leq C\tau\,. \tag{3.14}$$

We recall the definition of $\{e_n\}_{n=0}^{N-1}$ in the proof of Theorem 3.5, as well as equation (3.9), which we recast into the form

$$(\mathbb{1}-\tau\Delta_h)e_n + \int_{t_n}^{t_{n+1}} \big[\overline{Z}_h(s)-Z_h(s)\big]\,\mathrm{d}W(s)$$
$$= e_{n+1} + \int_{t_n}^{t_{n+1}} \big[\Delta_h(Y_h(s)-Y_h(t_n)) - \Pi_h^1(f(s)-f(t_n))\big]\,\mathrm{d}s\,.$$

Taking squares and afterwards expectations on both sides, by binomial formula, Itô isometry, and Young's inequality, we arrive at

$$\mathbb{E}\Big[\|(\mathbb{1}-\tau\Delta_h)e_n\|_{\mathbb{L}^2}^2 + \Big\|\int_{t_n}^{t_{n+1}} \overline{Z}_h(s)-Z_h(s)\,\mathrm{d}W(s)\Big\|_{\mathbb{L}^2}^2\Big]$$
$$= \mathbb{E}\Big[\|(\mathbb{1}-\tau\Delta_h)e_n\|_{\mathbb{L}^2}^2 + \int_{t_n}^{t_{n+1}} \|\overline{Z}_h(s)-Z_h(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s\Big]$$
$$\leq (1+\tau)\,\mathbb{E}\Big[\|e_{n+1}\|_{\mathbb{L}^2}^2 + \big(1+\frac{1}{4\tau}\big)\tau\int_{t_n}^{t_{n+1}} \big\|\Delta_h\big[Y_h(s)-Y_h(t_n)\big] - \Pi_h^1[f(s)-f(t_n)]\big\|_{\mathbb{L}^2}^2\,\mathrm{d}s\Big]\,.$$

Note that $\|(\mathbb{1}-\tau\Delta_h)e_n\|_{\mathbb{L}^2}^2 = \|e_n\|_{\mathbb{L}^2}^2 + 2\tau\|\nabla e_n\|_{\mathbb{L}^2}^2 + \tau^2\|\Delta_h e_n\|_{\mathbb{L}^2}^2$. Summation over $0 \le n \le N-1$ then leads to

$$
\begin{aligned}
&\mathbb{E}\Big[\|e_0\|_{\mathbb{L}^2}^2 + 2\tau\sum_{n=0}^{N-1}\|\nabla e_n\|_{\mathbb{L}^2}^2 + \int_0^T \|\overline{Z}_h(s) - Z_h(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s\Big] \\
&\quad\le \tau\sum_{n=0}^{N-1}\mathbb{E}\big[\|e_{n+1}\|_{\mathbb{L}^2}^2\big] + 2\sum_{n=0}^{N-1}\mathbb{E}\Big[\int_{t_n}^{t_{n+1}}\big\|\Delta_h\big[Y_h(s)-Y_h(t_n)\big] - \Pi_h^1[f(s)-f(t_n)]\big\|_{\mathbb{L}^2}^2\,\mathrm{d}s\Big].
\end{aligned}
$$

By the discrete version of Gronwall's inequality, and Lemma 3.1, (iii), the right-hand side is bounded by $C\tau$. Hence, (3.14) is proved.

Step 2. We use the triangle inequality, Lemma 3.4 and Lemma 3.8 with $\xi = Z_h(\tau(t))$, as well as Lemma 3.2, to deduce

$$
\begin{aligned}
&\sum_{n=0}^{N-1}\mathbb{E}\Big[\int_{t_n}^{t_{n+1}}\|Z_h(t) - Z_h^n\|_{\mathbb{L}^2}^2\,\mathrm{d}t\Big] \\
&\quad\le 2\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}\mathbb{E}\big[\|Z_h(t)-\overline{Z}_h(t)\|_{\mathbb{L}^2}^2 + \big\|\overline{Z}_h(t) - Z_h^n\big\|_{\mathbb{L}^2}^2\big]\,\mathrm{d}t \\
&\quad\le 2\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}\mathbb{E}\big[3\|Z_h(t)-\overline{Z}_h(t)\|_{\mathbb{L}^2}^2 + 2\|Z_h(t) - Z_h(t_n)\|_{\mathbb{L}^2}^2\big]\,\mathrm{d}t \\
&\quad\le C\tau\,.
\end{aligned}
$$

That completes the proof. ☐

4. Strong rates of convergence for a space-time discretization of SLQ

In this part, we discretize the original problem **SLQ** within two steps, starting with its semi-discretization in space (which is referred to as $\mathbf{SLQ}_h$), which is then followed by a discretization in space and time (which is referred to as $\mathbf{SLQ}_{h\tau}$). Our goal is to prove strong convergence rates in both cases. By *e.g.* [25], problem **SLQ** is uniquely solvable, and its solution (X^*, U^*) may be characterized by the following **FBSPDE** with the unique solution (X^*, Y, Z, U^*),

$$
\begin{cases}
\mathrm{d}X^*(t) = \big[\Delta X^*(t) + U^*(t)\big]\,\mathrm{d}t + \sigma(t)\,\mathrm{d}W(t) \qquad \forall\, t\in(0,T)\,,\\
\mathrm{d}Y(t) = \big[-\Delta Y(t) + [X^*(t) - \widetilde{X}(t)]\big]\,\mathrm{d}t + Z(t)\,\mathrm{d}W(t) \qquad \forall\, t\in(0,T)\,,\\
X^*(0) = X_0\,, \qquad Y(T) = -\alpha\big(X^*(T) - \widetilde{X}(T)\big)\,,
\end{cases}
\tag{4.1}
$$

with the condition

$$
U^* - Y = 0\,. \tag{4.2}
$$

We remark that by $(4.1)_1$, X^* may be written as $X^* = \mathcal{S}(U^*)$, where

$$
\mathcal{S} : L^2_{\mathbb{F}}\big(\Omega; L^2(0,T;\mathbb{L}^2)\big) \to L^2_{\mathbb{F}}\big(\Omega; C([0,T];\mathbb{H}^1_0)\cap L^2(0,T;\mathbb{H}^2)\big)
$$

is the bounded 'control-to-state' map. Moreover, we introduce the reduced functional

$$\widehat{\mathcal{J}} : L^2_{\mathbb{F}}\big(\Omega; L^2(0,T;\mathbb{L}^2)\big) \to \mathbb{R} \qquad \textit{via} \qquad \widehat{\mathcal{J}}(U) = \mathcal{J}\big(\mathcal{S}(U), U\big)\,,$$

where $\mathcal{J}$ is defined in (1.1). The first component of the solution to equation $(4.1)_2$ may be written as $Y = \mathcal{T}(X^*)$, where $\mathcal{T}$

$$\mathcal{T} : L^2_{\mathbb{F}}\big(\Omega; C([0,T];\mathbb{L}^2)\big) \to L^2_{\mathbb{F}}\big(\Omega; C([0,T];\mathbb{H}^1_0) \cap L^2(0,T;\mathbb{H}^1_0 \cap \mathbb{H}^2)\big)\,,$$

which is also bounded. For every $U \in L^2_{\mathbb{F}}\big(\Omega; L^2(0,T;\mathbb{L}^2)\big)$, the Fréchet derivative $D\widehat{\mathcal{J}}(U)$ is also a bounded operator on $L^2_{\mathbb{F}}\big(\Omega; L^2(0,T;\mathbb{L}^2)\big)$ and takes the form

$$D\widehat{\mathcal{J}}(U) = U - \mathcal{T}\big(\mathcal{S}(U)\big)\,. \tag{4.3}$$

4.1. Problem $\mathbf{SLQ}_h$: Semi-discretization in space

We begin with a spatial semi-discretization $\mathbf{SLQ}_h$ of the problem **SLQ** stated in the introduction, which reads: Find an optimal pair $(X^*_h, U^*_h) \in L^2_{\mathbb{F}}\big(\Omega; C([0,T];\mathbb{V}^1_h) \times L^2(0,T;\mathbb{V}^0_h)\big)$ that minimizes the functional ($\alpha \geq 0$)

$$\mathcal{J}(X_h, U_h) = \frac{1}{2}\mathbb{E}\Big[\int_0^T \|X_h(t) - \widetilde{X}(t)\|^2_{\mathbb{L}^2} + \|U_h(t)\|^2_{\mathbb{L}^2}\,\mathrm{d}t + \alpha\|X_h(T) - \widetilde{X}(T)\|^2_{\mathbb{L}^2}\Big] \tag{4.4}$$

subject to the equation

$$\begin{cases} \mathrm{d}X_h(t) = \big[\Delta_h X_h(t) + \Pi^1_h U_h(t)\big]\,\mathrm{d}t + \Pi^1_h \sigma(t)\,\mathrm{d}W(t) \quad \forall\, t \in [0,T]\,, \\ X_h(0) = \Pi^1_h X_0\,. \end{cases} \tag{4.5}$$

The existence of a unique optimal pair (X^*_h, U^*_h) follows from [38], as well as its characterization *via* Pontryagin's maximum principle, *i.e.*,

$$0 = U^*_h(t) - \Pi^0_h Y_h(t) \qquad \forall\, t \in (0,T)\,, \tag{4.6}$$

where the adjoint $(Y_h, Z_h) \in L^2_{\mathbb{F}}\big(\Omega; C([0,T];\mathbb{V}^1_h)\big) \times L^2_{\mathbb{F}}\big(\Omega; L^2(0,T;\mathbb{V}^1_h)\big)$ solves the $\mathbf{BSPDE}_h$

$$\begin{cases} \mathrm{d}Y_h(t) = \big[-\Delta_h Y_h(t) + [X^*_h(t) - \Pi^1_h \widetilde{X}(t)]\big]\,\mathrm{d}t + Z_h(t)\,\mathrm{d}W(t) \quad \forall\, t \in [0,T]\,, \\ Y_h(T) = -\alpha\big(X^*_h(T) - \Pi^1_h \widetilde{X}(T)\big)\,. \end{cases} \tag{4.7}$$

In [15], optimal error estimates have been obtained for (X^*_h, Y_h, Z_h) with the help of a fixed point argument—which crucially exploits $T > 0$ to be *sufficiently small*. The goal in this section is to derive corresponding estimates for (X^*_h, U^*_h, Y_h, Z_h) for *arbitrary* $T > 0$ *via* a variational argument which exploits properties of the reduced functional $\widehat{\mathcal{J}} \equiv \widehat{\mathcal{J}}(u)$ that is now defined: once an estimate for $\int_0^T \mathbb{E}\big[\|U^*(s) - U^*_h(s)\|^2_{\mathbb{L}^2}\big]\,\mathrm{d}s$ has been obtained, we use the convergence analysis from Section 3 to derive estimates for $X^* - X^*_h$, as well as $Y - Y_h$ and $Z - Z_h$.

By the unique solvability property of (4.5), we associate to this equation the bounded solution operator

$$\mathcal{S}_h : L^2_{\mathbb{F}}\big(\Omega; L^2(0,T;\mathbb{V}^0_h)\big) \to L^2_{\mathbb{F}}\big(\Omega; C([0,T];\mathbb{V}^1_h)\big)\,,$$

which allows to introduce the reduced functional

$$\widehat{\mathcal{J}}_h : L^2_{\mathbb{F}}\big(\Omega; L^2(0,T;\mathbb{V}^0_h)\big) \to \mathbb{R}\,, \qquad \textit{via} \qquad \widehat{\mathcal{J}}_h(U_h) = \mathcal{J}\big(\mathcal{S}_h(U_h), U_h\big)\,, \tag{4.8}$$

where $\mathcal{J}$ is defined in (1.1). The first solution component to equation (4.7) may be written as $Y_h = \mathcal{T}_h(X^*_h)$, where

$$\mathcal{T}_h : L^2_{\mathbb{F}}\big(\Omega; C([0,T];\mathbb{V}^1_h)\big) \to L^2_{\mathbb{F}}\big(\Omega; C([0,T];\mathbb{V}^1_h)\big)\,.$$

For every $U_h \in L^2_{\mathbb{F}}\big(\Omega; L^2(0,T;\mathbb{V}^0_h)\big)$, the Fréchet derivative $D\widehat{\mathcal{J}}_h(U_h)$ is a bounded operator (uniformly in h) on $L^2_{\mathbb{F}}\big(\Omega; L^2(0,T;\mathbb{V}^0_h)\big)$ at U_h, and has the form

$$D\widehat{\mathcal{J}}_h(U_h) = U_h - \Pi^0_h \mathcal{T}_h\big(\mathcal{S}_h(U_h)\big)\,. \tag{4.9}$$

Let $U_h \in L^2_{\mathbb{F}}\big(\Omega; L^2(0,T;\mathbb{V}^0_h)\big)$ be arbitrary; it is due to the quadratic structure of the reduced functional (4.8) that

$$\mathbb{E}\Big[\big(D^2\widehat{\mathcal{J}}_h(U_h)R_h, R_h\big)_{L^2(0,T;\mathbb{L}^2)}\Big] \geq \mathbb{E}\big[\|R_h\|^2_{L^2(0,T;\mathbb{L}^2)}\big] \qquad \forall\, R_h \in L^2_{\mathbb{F}}\big(\Omega; L^2(0,T;\mathbb{V}^0_h)\big)\,.$$

As a consequence, on putting $R_h = U^*_h - \Pi^0_h U^*$,

$$\begin{aligned}
\mathbb{E}\big[\|U^*_h - \Pi^0_h U^*\|^2_{L^2(0,T;\mathbb{L}^2)}\big] &\leq \mathbb{E}\Big[\big(D^2\widehat{\mathcal{J}}_h(U_h)(U^*_h - \Pi^0_h U^*), U^*_h - \Pi^0_h U^*\big)_{L^2(0,T;\mathbb{L}^2)}\Big] \\
&= \mathbb{E}\Big[\big(D\widehat{\mathcal{J}}_h(U^*_h), U^*_h - \Pi^0_h U^*\big)_{L^2(0,T;\mathbb{L}^2)} - \big(D\widehat{\mathcal{J}}_h(\Pi^0_h U^*), U^*_h - \Pi^0_h U^*\big)_{L^2(0,T;\mathbb{L}^2)}\Big]\,.
\end{aligned} \tag{4.10}$$

Note that $D\widehat{\mathcal{J}}_h(U^*_h) = 0$ by (4.6), as well as $D\widehat{\mathcal{J}}(U^*) = 0$ by (4.2), such that the last line equals

$$\begin{aligned}
&= \mathbb{E}\Big[\big(D\widehat{\mathcal{J}}(U^*), U^*_h - \Pi^0_h U^*\big)_{L^2(0,T;\mathbb{L}^2)} - \big(D\widehat{\mathcal{J}}(\Pi^0_h U^*), U^*_h - \Pi^0_h U^*\big)_{L^2(0,T;\mathbb{L}^2)}\Big] \\
&\quad + \mathbb{E}\Big[\big(D\widehat{\mathcal{J}}(\Pi^0_h U^*), U^*_h - \Pi^0_h U^*\big)_{L^2(0,T;\mathbb{L}^2)} - \big(D\widehat{\mathcal{J}}_h(\Pi^0_h U^*), U^*_h - \Pi^0_h U^*\big)_{L^2(0,T;\mathbb{L}^2)}\Big] \\
&=: I + II\,.
\end{aligned} \tag{4.11}$$

We use (4.3) to bound I as follows,

$$\begin{aligned}
I &= \mathbb{E}\Big[\Big(U^* - \Pi^0_h U^* + \mathcal{T}\big(\mathcal{S}(\Pi^0_h U^*)\big) - \mathcal{T}\big(\mathcal{S}(U^*)\big), U^*_h - \Pi^0_h U^*\Big)_{L^2(0,T;\mathbb{L}^2)}\Big] \\
&\leq \Big(\Big(\mathbb{E}\big[\|U^* - \Pi^0_h U^*\|^2_{L^2(0,T;\mathbb{L}^2)}\big]\Big)^{1/2} + I_a\Big)\Big(\mathbb{E}\big[\|U^*_h - \Pi^0_h U^*\|^2_{L^2(0,T;\mathbb{L}^2)}\big]\Big)^{1/2}\,,
\end{aligned}$$

where $I_a^2 = \mathbb{E}\big[\|\mathcal{T}\big(\mathcal{S}(U^*) - \mathcal{S}(\Pi^0_h U^*)\big)\|^2_{L^2(0,T;\mathbb{L}^2)}\big]$. By Poincaré's inequality, and a stability bound (see also (2.11)) for the backward stochastic heat equation (2.10), as well as for the stochastic heat equation (2.3) (see also (2.2)),

$$\begin{aligned}
I_a^2 &\leq C\mathbb{E}\big[\|\big(\mathcal{S}(U^*) - \mathcal{S}(\Pi^0_h U^*)\big)(T)\|^2_{\mathbb{L}^2} + \|\mathcal{S}(U^*) - \mathcal{S}(\Pi^0_h U^*)\|^2_{L^2(0,T;\mathbb{L}^2)}\big] \\
&\leq C\mathbb{E}\big[\|U^* - \Pi^0_h U^*\|^2_{L^2(0,T;\mathbb{L}^2)}\big]\,.
\end{aligned} \tag{4.12}$$

By optimality condition (4.2), and the regularity properties of the solution to **BSPDE** (2.10), we know that already $U^* \in L^2_{\mathbb{F}}\big(\Omega; C([0,T];\mathbb{H}^1_0) \cap L^2(0,T;\mathbb{H}^1_0 \cap \mathbb{H}^2)\big)$; as a consequence, the right-hand side of (4.12) may be bounded by Ch^2.

We use the representation (4.9) and properties of the projection Π^0_h to bound II *via*

$$
\begin{aligned}
II &= \mathbb{E}\Big[\Big(\mathcal{T}\big(\mathcal{S}(\Pi^0_h U^*)\big) - \Pi^0_h \mathcal{T}_h\big(\mathcal{S}_h(\Pi^0_h U^*)\big), U^*_h - \Pi^0_h U^*\Big)_{L^2(0,T;\mathbb{L}^2)}\Big] \\
&\leq II_a \times \Big(\mathbb{E}\big[\|U^*_h - \Pi^0_h U^*\|^2_{L^2(0,T;\mathbb{L}^2)}\big]\Big)^{1/2},
\end{aligned}
$$

where $II_a^2 := \mathbb{E}\big[\|\mathcal{T}\big(\mathcal{S}(\Pi^0_h U^*)\big) - \mathcal{T}_h\big(\mathcal{S}_h(\Pi^0_h U^*)\big)\|^2_{L^2(0,T;\mathbb{L}^2)}\big]$. We split II_a^2 into two terms

$$
\begin{aligned}
& II^2_{a,1} = \mathbb{E}\big[\|\mathcal{T}\big(\mathcal{S}(\Pi^0_h U^*)\big) - \mathcal{T}\big(\mathcal{S}_h(\Pi^0_h U^*)\big)\|^2_{L^2(0,T;\mathbb{L}^2)}\big] \\
\text{and} \qquad & II^2_{a,2} = \mathbb{E}\big[\|\mathcal{T}\big(\mathcal{S}_h(\Pi^0_h U^*)\big) - \mathcal{T}_h\big(\mathcal{S}_h(\Pi^0_h U^*)\big)\|^2_{L^2(0,T;\mathbb{L}^2)}\big].
\end{aligned}
$$

In order to bound $II^2_{a,1}$, we use stability properties for **BSPDE** (2.10), in combination with the error estimate (2.6) for (2.5) to conclude

$$
II^2_{a,1} \leq C\mathbb{E}\big[\big\|\mathcal{S}(\Pi^0_h U^*) - \mathcal{S}_h(\Pi^0_h U^*)\big\|^2_{L^2(0,T;\mathbb{L}^2)}\big] \leq Ch^2 .
$$

In order to bound $II^2_{a,2}$, we use the error estimate in Theorem 2.1 for **BSPDE** (2.10), in combination with stability properties of (2.5), and again the error estimate (2.6) for (2.5) to find

$$
II^2_{a,2} \leq C\Big(\mathbb{E}\big[\big\|\mathcal{S}(\Pi^0_h U^*)(T) - \mathcal{S}_h(\Pi^0_h U^*)(T)\big\|^2_{\mathbb{L}^2}\big] + h^2\Big) \leq Ch^2 .
$$

We now insert these estimates into (4.11) resp. (4.10) to obtain the bound

$$
\mathbb{E}\big[\|U^*_h - \Pi^0_h U^*\|^2_{L^2(0,T;\mathbb{L}^2)}\big] \leq Ch^2 .
$$

By arguing as below (4.12), this settles part (i) of the following

Theorem 4.1. *Let (X^*, Y, Z, U^*) be the solution to problem* **SLQ**, *and (X^*_h, Y_h, Z_h, U^*_h) be the solution to problem* $\mathbf{SLQ}_h$. *There exists $C \equiv C(X_0, T) > 0$ such that*

(i) $\displaystyle \mathbb{E}\Big[\int_0^T \|U^*(t) - U^*_h(t)\|^2_{\mathbb{L}^2}\,\mathrm{d}t\Big] \leq Ch^2$;

(ii) $\displaystyle \sup_{0\leq t\leq T} \mathbb{E}\big[\|X^*(t) - X^*_h(t)\|^2_{\mathbb{L}^2}\big] + \int_0^T \mathbb{E}\big[\|X^*(t) - X^*_h(t)\|^2_{\mathbb{H}^1_0}\big]\,\mathrm{d}t \leq Ch^2$;

(iii) $\displaystyle \sup_{0\leq t\leq T} \mathbb{E}\big[\|Y(t) - Y_h(t)\|^2_{\mathbb{L}^2}\big] + \int_0^T \mathbb{E}\big[\|Y(t) - Y_h(t)\|^2_{\mathbb{H}^1_0} + \|Z(t) - Z_h(t)\|^2_{\mathbb{L}^2}\big]\,\mathrm{d}t \leq Ch^2$.

Proof. Since $U^* \in L^2_{\mathbb{F}}(\Omega; L^2(0,T;\mathbb{H}^1_0))$, and (i), the first estimate of (ii) can be deduced as (2.6). Assertion (iii) now follows accordingly as Theorem 2.1, thanks to (ii). □

4.2. Problem $\mathbf{SLQ}_{h\tau}$: Discretization in space and time

In this part, we provide the temporal discretization of problem $\mathbf{SLQ}_h$ which was analyzed in Section 4.1. For this purpose, we use a mesh I_τ covering $[0,T]$, and consider processes $(X_{h\tau}, U_{h\tau}) \in \mathbb{X}_{h\tau} \times \mathbb{U}_{h\tau} \subset$

$L^2_{\mathbb{F}}\big(\Omega; L^2(0,T;\mathbb{V}^1_h)\big) \times L^2_{\mathbb{F}}\big(\Omega; L^2(0,T;\mathbb{V}^0_h)\big)$, where

$$\begin{aligned}
\mathbb{X}_{h\tau} &:= \big\{X \in L^2_{\mathbb{F}}\big(\Omega; L^2(0,T;\mathbb{V}^1_h)\big):\ X(t) = X(t_n),\ \forall t \in [t_n, t_{n+1}),\ n = 0,1,\ldots,N-1\big\}\,,\\
\mathbb{U}_{h\tau} &:= \big\{U \in L^2_{\mathbb{F}}\big(\Omega; L^2(0,T;\mathbb{V}^0_h)\big):\ U(t) = U(t_n),\ \forall t \in [t_n, t_{n+1}),\ n = 0,1,\ldots,N-1\big\}\,,
\end{aligned}$$

and for any $X \in \mathbb{X}_{h\tau}$, $U \in \mathbb{U}_{h\tau}$,

$$\|X\|_{\mathbb{X}_{h\tau}} := \Big(\tau \sum_{n=1}^{N} \mathbb{E}\big[\|X(t_n)\|^2_{\mathbb{L}^2}\big]\Big)^{1/2},\ \ \|U\|_{\mathbb{U}_{h\tau}} := \Big(\tau \sum_{n=0}^{N-1} \mathbb{E}\big[\|U(t_n)\|^2_{\mathbb{L}^2}\big]\Big)^{1/2}.$$

Problem $\mathbf{SLQ}_{h\tau}$ then reads as follows: Find an optimal pair $(X^*_{h\tau}, U^*_{h\tau}) \in \mathbb{X}_{h\tau} \times \mathbb{U}_{h\tau}$ which minimizes the cost functional

$$\mathcal{J}_\tau(X_{h\tau}, U_{h\tau}) = \frac{1}{2}\|X_{h\tau} - \widetilde{X}\|^2_{\mathbb{X}_{h\tau}} + \frac{1}{2}\|U_{h\tau}\|^2_{\mathbb{U}_{h\tau}} + \frac{\alpha}{2}\mathbb{E}\big[\|X_{h\tau}(T) - \widetilde{X}(T)\|^2_{\mathbb{L}^2}\big]\,, \tag{4.13}$$

subject to the difference equation

$$\begin{cases} X_{h\tau}(t_{n+1}) - X_{h\tau}(t_n) = \tau\big[\Delta_h X_{h\tau}(t_{n+1}) + \Pi^1_h U_{h\tau}(t_n)\big] + \Pi^1_h \sigma(t_n)\Delta_{n+1}W \quad n = 0,1,\cdots,N-1\,,\\ X_{h\tau}(0) = \Pi^1_h X_0\,, \end{cases} \tag{4.14}$$

where $\Delta_{n+1}W = W(t_{n+1}) - W(t_n)$. The following result states Pontryagin's maximum principle for problem $\mathbf{SLQ}_{h\tau}$, which is later used to verify convergence rates for the solution to problem $\mathbf{SLQ}_{h\tau}$ towards the solution to $\mathbf{SLQ}$.

Theorem 4.2. *Problem* $\mathbf{SLQ}_{h\tau}$ *admits a unique minimizer* $(X^*_{h\tau}, U^*_{h\tau}) \in \mathbb{X}_{h\tau} \times \mathbb{U}_{h\tau}$*, which is (part of) the unique solution*

$$(X^*_{h\tau}, Y_{h\tau}, U^*_{h\tau}) \in \big[\mathbb{X}_{h\tau}\big]^2 \times \mathbb{U}_{h\tau}$$

to the following forward-backward difference equation for $0 \le n \le N-1$*,*

$$\begin{cases} [\mathbb{1} - \tau\Delta_h]X^*_{h\tau}(t_{n+1}) = X^*_{h\tau}(t_n) + \tau\Pi^1_h U^*_{h\tau}(t_n) + \Pi^1_h\sigma(t_n)\Delta_{n+1}W\,,\\ [\mathbb{1} - \tau\Delta_h]Y_{h\tau}(t_n) = \mathbb{E}\Big[Y_{h\tau}(t_{n+1}) - \tau\big(X^*_{h\tau}(t_{n+1}) - \Pi^1_h\widetilde{X}(t_{n+1})\big)\big|\mathcal{F}_{t_n}\Big]\,,\\ X^*_{h\tau}(0) = \Pi^1_h X_0\,, \qquad Y_{h\tau}(T) = -\alpha\big(X^*_{h\tau}(T) - \Pi^1_h\widetilde{X}(T)\big)\,, \end{cases} \tag{4.15}$$

together with

$$U^*_{h\tau}(t_n) - \Pi^0_h Y_{h\tau}(t_n) = 0 \qquad n = 0,1,\ldots,N-1\,. \tag{4.16}$$

By (4.16), we can see that $U^*_{h\tau}$ is càdlàg, and then $U^*_{h\tau} \in \mathbb{U}_{h\tau}$. Inserting (4.16) into $(4.15)_1$ leads to a coupled problem for $\big(\{X^*_{h\tau}(t_{n+1})\}_{n=0}^{N-1}, \{Y_{h\tau}(t_n)\}_{n=0}^{N-1}\big)$, where $(4.15)_2$ is similar to (3.6). Note that no Z-component appears explicitly in $(4.15)_2$, where the conditional expectation is used to compute the Y-component. It is in particular due to the need to compute conditional expectations in $(4.15)_2$ that the optimality system (4.15)–(4.16) is still not amenable to an actual implementation, but serves as a key step towards a practical method which approximately solves $\mathbf{SLQ}_{h\tau}$—which is proposed and studied in Section 5.

Proof. We divide the proof into three steps.

Step 1. Let $A_0 := (\mathbb{1} - \tau\Delta_h)^{-1}$. For any $U_{h\tau} \in \mathbb{U}_{h\tau}$, by equation $(4.15)_1$, we have

$$X_{h\tau}(t_n) = A_0\big[X_{h\tau}(t_{n-1}) + \tau\Pi_h^1 U_{h\tau}(t_{n-1}) + \Pi_h^1\sigma(t_{n-1})\Delta_n W\big]\,. \tag{4.17}$$

Hence, by iteration we arrive at

$$\begin{aligned} X_{h\tau}(t_n) &= A_0^n X_{h\tau}(0) + \tau\sum_{j=0}^{n-1} A_0^{n-j}\Pi_h^1 U_{h\tau}(t_j) + \sum_{j=1}^{n} A_0^{n+1-j}\Pi_h^1\sigma(t_{j-1})\Delta_j W \\ &=: (\Gamma\Pi_h^1 X_0)(t_n) + (LU_{h\tau})(t_n) + f(t_n)\,. \end{aligned} \tag{4.18}$$

Here, $\Gamma : \mathbb{V}_h^1 \to \mathbb{X}_{h\tau}$ and $L : \mathbb{U}_{h\tau} \to \mathbb{X}_{h\tau}$ are bounded operators. Below, we use the abbreviations

$$\widehat{\Gamma}\Pi_h^1 X_0 := \Gamma\Pi_h^1 X_0(T)\,, \qquad \widehat{L}U_{h\tau} := (LU_{h\tau})(T)\,, \qquad \widehat{f} = f(T)\,. \tag{4.19}$$

Claim: For any $\xi \in \mathbb{X}_{h\tau}$, and any $\eta \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{V}_h^1)$,

$$L^*\xi = -\Pi_h^0 Y_0\,, \qquad \widehat{L}^*\eta = -\Pi_h^0 Y_1\,, \tag{4.20}$$

where (Y_0, Z_0) solves the following backward stochastic difference equation:

$$\begin{cases} Y_0(t_{n+1}) - Y_0(t_n) = \tau\big[-\Delta_h Y_0(t_n) + \xi(t_{n+1})\big] + \displaystyle\int_{t_n}^{t_{n+1}} Z_0(t)\,\mathrm{d}W(t) \quad n = 0,1,\ldots,N-1\,, \\ Y_0(t_N) = Y_0(T) = 0\,, \end{cases} \tag{4.21}$$

and (Y_1, Z_1) solves

$$\begin{cases} Y_1(t_{n+1}) - Y_1(t_n) = -\tau\Delta_h Y_1(t_n) + \displaystyle\int_{t_n}^{t_{n+1}} Z_1(t)\,\mathrm{d}W(t) \qquad n = 0,1,\ldots,N-1\,, \\ Y_1(T) = -\eta\,. \end{cases}$$

Proof of Claim: The existence and the uniqueness of solutions to (4.21) are obvious. Note that

$$Y_0(t_j) = \mathbb{E}\big[A_0 Y_0(t_{j+1}) - \tau A_0\xi(t_{j+1})\big|\mathcal{F}_{t_j}\big]. \tag{4.22}$$

With the similar procedure as that in (4.17), we conclude from (4.22) and $(4.21)_2$,

$$\begin{aligned} Y_0(t_j) &= \mathbb{E}\big[A_0^{N-j} Y_0(t_N)\big|\mathcal{F}_{t_j}\big] - \mathbb{E}\Big[\tau\sum_{k=j+1}^{N} A_0^{k-j}\xi(t_k)\big|\mathcal{F}_{t_j}\Big] \\ &= -\mathbb{E}\Big[\tau\sum_{k=j+1}^{N} A_0^{k-j}\xi(t_k)\big|\mathcal{F}_{t_j}\Big]\,. \end{aligned} \tag{4.23}$$

Let $U_{h\tau} \in \mathbb{U}_{h\tau}$ be arbitrary. By the definition of L, (4.23) and the fact that A_0 is self-adjoint, we can calculate that

$$\tau \sum_{n=1}^{N} \mathbb{E}\big[\big((LU_{h\tau})(t_n), \xi(t_n)\big)_{\mathbb{L}^2}\big] = \tau \sum_{n=1}^{N} \mathbb{E}\Big[\big(\tau \sum_{j=0}^{n-1} A_0^{n-j} \Pi_h^1 U_{h\tau}(t_j), \xi(t_n)\big)_{\mathbb{L}^2}\Big]$$
$$= \tau \sum_{j=0}^{N-1} \mathbb{E}\Big[\Big(\Pi_h^1 U_{h\tau}(t_j), \mathbb{E}\big[\tau \sum_{n=j+1}^{N} A_0^{n-j} \xi(t_n) \big| \mathcal{F}_{t_j}\big]\Big)_{\mathbb{L}^2}\Big] .$$

Since the second argument is $\mathbb{V}_h^1$-valued, we may skip the projection operator in the first argument, and may continue instead

$$= \tau \sum_{j=0}^{N-1} \mathbb{E}\Big[\Big(U_{h\tau}(t_j), \Pi_h^0 \mathbb{E}\big[\tau \sum_{k=j+1}^{N} A_0^{k-j} \xi(t_k) \big| \mathcal{F}_{t_j}\big]\Big)_{\mathbb{L}^2}\Big] .$$

Because of (4.23), the latter equals

$$= \tau \sum_{j=0}^{N-1} \mathbb{E}\big[\big(U_{h\tau}(t_j), -\Pi_h^0 Y_0(t_j)\big)_{\mathbb{L}^2}\big] ,$$

which is the first part of the claim.

The remaining part can be deduced from the fact that $Y_1(t_j) = -\mathbb{E}\big[A_0^{N-j}\eta \big| \mathcal{F}_{t_j}\big]$ for $j = 0, 1, \ldots, N-1$, and the following calculation:

$$\begin{aligned} \mathbb{E}\big[\big(\widehat{L}U_{h\tau}, \eta\big)_{\mathbb{L}^2}\big] &= \mathbb{E}\Big[\Big(\tau \sum_{j=0}^{N-1} A_0^{N-j} \Pi_h^1 U_{h\tau}(t_j), \eta\Big)_{\mathbb{L}^2}\Big] \\ &= \tau \sum_{j=0}^{N-1} \mathbb{E}\big[\big(\Pi_h^1 U_{h\tau}(t_j), \mathbb{E}\big[A_0^{N-j}\eta \big| \mathcal{F}_{t_j}\big]\big)_{\mathbb{L}^2}\big] \\ &= \tau \sum_{j=0}^{N-1} \mathbb{E}\big[\big(U_{h\tau}(t_j), -\Pi_h^0 Y_1(t_j)\big)_{\mathbb{L}^2}\big] \qquad \forall\, U_{h\tau} \in \mathbb{U}_{h\tau} . \end{aligned} \tag{4.24}$$

Step 2. By (4.18) and (4.19), we can rewrite $\mathcal{J}_\tau(X_{h\tau}, U_{h\tau})$ as follows:

$$\begin{aligned} &\mathcal{J}_\tau(X_{h\tau}, U_{h\tau}) \\ &\quad = \frac{1}{2}\Big[\|X_{h\tau} - \Pi_\tau \widetilde{X}\|^2_{L^2_{\mathbb{F}}(\Omega;L^2(0,T;\mathbb{L}^2))} + \|U_{h\tau}\|^2_{L^2_{\mathbb{F}}(\Omega;L^2(0,T;\mathbb{L}^2))} + \alpha\|X_{h\tau}(T) - \widetilde{X}(T)\|^2_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{L}^2)}\Big] \\ &\quad = \frac{1}{2}\Big[\big(\Gamma\Pi_h^1 X_0 + LU_{h\tau} + f - \Pi_\tau \widetilde{X}, \Gamma\Pi_h^1 X_0 + LU_{h\tau} + f - \Pi_\tau \widetilde{X}\big)_{L^2_{\mathbb{F}}(\Omega;L^2(0,T;\mathbb{L}^2))} \\ &\qquad\quad + (U_{h\tau}, U_{h\tau})_{L^2_{\mathbb{F}}(\Omega;L^2(0,T;\mathbb{L}^2))} \\ &\qquad\quad + \alpha\big(\widehat{\Gamma}\Pi_h^1 X_0 + \widehat{L}U_{h\tau} + \widehat{f} - \widetilde{X}(T), \widehat{\Gamma}\Pi_h^1 X_0 + \widehat{L}U_{h\tau} + \widehat{f} - \widetilde{X}(T)\big)_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{L}^2)}\Big] , \end{aligned}$$

where $\Pi_\tau \widetilde{X}(t) = \widetilde{X}(t_n)$, for $t \in [t_n, t_{n+1})$, $n = 0, 1, \ldots, N-1$. Rearranging terms then leads to

$$
\begin{aligned}
=&\frac{1}{2}\Big[\big(\big[\mathbb{1} + L^*L + \alpha \widehat{L}^*\widehat{L}\big]U_{h\tau}, U_{h\tau}\big)_{L^2_{\mathbb{F}}(\Omega;L^2(0,T;\mathbb{L}^2))} \\
&+ 2\big(\big[L^*\Gamma + \alpha\widehat{L}^*\widehat{\Gamma}\big]\Pi_h^1 X_0 + L^* f + \alpha \widehat{L}^*\widehat{f} - L^*\Pi_\tau\widetilde{X} - \alpha\widehat{L}^*\widetilde{X}(T), U_{h\tau}\big)_{L^2_{\mathbb{F}}(\Omega;L^2(0,T;\mathbb{L}^2))} \\
&+ \Big\{\big(\Gamma\Pi_h^1 X_0 + f - \Pi_\tau\widetilde{X}, \Gamma\Pi_h^1 X_0 + f - \Pi_\tau\widetilde{X}\big)_{L^2_{\mathbb{F}}(\Omega;L^2(0,T;\mathbb{L}^2))} \\
&+ \alpha\big(\widehat{\Gamma}\Pi_h^1 X_0 + \widehat{f} - \widetilde{X}(T), \widehat{\Gamma}\Pi_h^1 X_0 + \widehat{f} - \widetilde{X}(T)\big)_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{L}^2)}\Big\}\Big] \\
=:&\frac{1}{2}\Big[\big(\mathfrak{N}U_{h\tau}, U_{h\tau}\big)_{L^2_{\mathbb{F}}(\Omega;L^2(0,T;\mathbb{L}^2))} + 2\big(\mathfrak{H}(\Pi_h^1 X_0, f, \widetilde{X}), U_{h\tau}\big)_{L^2_{\mathbb{F}}(\Omega;L^2(0,T;\mathbb{L}^2))} + \mathfrak{M}(\Pi_h^1 X_0, f, \widetilde{X})\Big]\,.
\end{aligned}
$$

Since $\mathfrak{N} = \mathbb{1} + L^*L + \alpha\widehat{L}^*\widehat{L}$ is positive definite, there exists a unique $U_{h\tau}^* \in \mathbb{U}_{h\tau}$ such that

$$
\mathfrak{N}U_{h\tau}^* + \mathfrak{H}(\Pi_h^1 X_0, f, \widetilde{X}) = 0\,.
$$

Therefore, for any $U_{h\tau} \in \mathbb{U}_{h\tau}$ such that $U_{h\tau} \neq U_{h\tau}^*$,

$$
\begin{aligned}
&\mathcal{J}_\tau(X_{h\tau}, U_{h\tau}) - \mathcal{J}_\tau(X_{h\tau}^*, U_{h\tau}^*) \\
&\quad= \big(\mathfrak{N}U_{h\tau}^* + \mathfrak{H}(\Pi_h^1 X_0, f, \widetilde{X}), U_{h\tau} - U_{h\tau}^*\big)_{L^2_{\mathbb{F}}(\Omega;L^2(0,T;\mathbb{L}^2))} \\
&\qquad+\frac{1}{2}\big(\mathfrak{N}(U_{h\tau} - U_{h\tau}^*), U_{h\tau} - U_{h\tau}^*\big)_{L^2_{\mathbb{F}}(\Omega;L^2(0,T;\mathbb{L}^2))} \\
&\quad= \frac{1}{2}\big(\mathfrak{N}(U_{h\tau} - U_{h\tau}^*), U_{h\tau} - U_{h\tau}^*\big)_{L^2_{\mathbb{F}}(\Omega;L^2(0,T;\mathbb{L}^2))} \\
&\quad> 0\,,
\end{aligned}
$$

which means that $U_{h\tau}^*$ is the unique optimal control, and $(X_{h\tau}^*, U_{h\tau}^*)$ is the unique optimal pair.
Step 3. By the definition of $\mathfrak{N}, \mathfrak{H}, L^*, \widehat{L}^*$, and properties (4.20) and (4.18), we can get

$$
\begin{aligned}
0 &= \mathfrak{N}U_{h\tau}^* + \mathfrak{H}(\Pi_h^1 X_0, f, \widetilde{X}) \\
&= U_{h\tau}^* + L^*\big(\Gamma\Pi_h^1 X_0 + LU_{h\tau}^* + f - \widetilde{X}\big) + \alpha\widehat{L}^*\big(\widehat{\Gamma}\Pi_h^1 X_0 + \widehat{L}U_{h\tau}^* + \widehat{f} - \widetilde{X}(T)\big) \\
&= U_{h\tau}^* - \Pi_h^0\big[Y_0\big(\cdot; X_{h\tau}^* - \widetilde{X}\big) + Y_1\big(\cdot; \alpha(X_{h\tau}^*(T) - \widetilde{X}(T))\big)\big] \\
&= U_{h\tau}^* - \Pi_h^0 Y_{h\tau}\,,
\end{aligned}
$$

which is (4.16). This completes the proof. □

We are now ready to verify strong rates of convergence for the solution to $\mathbf{SLQ}_{h\tau}$; it is as in Section 4.1 that the reduced cost functional $\widehat{\mathcal{J}}_{h\tau} : \mathbb{U}_{h\tau} \to \mathbb{R}$ is used, which is defined *via*

$$
\widehat{\mathcal{J}}_{h\tau}(U_{h\tau}) = \mathcal{J}_\tau\big(\mathcal{S}_{h\tau}(U_{h\tau}), U_{h\tau}\big)\,,
$$

where $\mathcal{S}_{h\tau} : \mathbb{U}_{h\tau} \to \mathbb{X}_{h\tau}$ is the solution operator to the forward equation $(4.15)_1$. Moreover, we use the solution operator $\mathcal{T}_{h\tau} : \mathbb{X}_{h\tau} \to \mathbb{X}_{h\tau}$ for the first solution component of the backward equation $(4.15)_2$.

Theorem 4.3. *Suppose that* $X(0) \in \mathbb{H}_0^1 \cap \mathbb{H}^2$, *and*

$$
\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} \mathbb{E}\big[\,\|\sigma(t) - \sigma(t_n)\|_{\mathbb{H}_0^1}^2\,\big] + \big\|\widetilde{X}(t) - \widetilde{X}(t_n)\big\|_{\mathbb{L}^2}^2 + \big\|\widetilde{X}(t) - \widetilde{X}(t_{n+1})\big\|_{\mathbb{L}^2}^2 \,\mathrm{d}t \le C\tau\,. \tag{4.25}
$$

Let (X_h^*, Y_h, Z_h, U_h^*) be the solution to problem $\mathbf{SLQ}_h$, and $(X_{h\tau}^*, Y_{h\tau}, U_{h\tau}^*)$ be the solution to problem $\mathbf{SLQ}_{h\tau}$. There exists $C \equiv C(X_0, T) > 0$ such that

$$\text{(i)}\quad \sum_{k=0}^{N-1} \mathbb{E}\Big[\int_{t_k}^{t_{k+1}} \|U_h^*(t) - U_{h\tau}^*(t_k)\|_{\mathbb{L}^2}^2 \,\mathrm{d}t\Big] \le C\tau\,;$$

$$\text{(ii)}\quad \max_{0\le k\le N} \mathbb{E}\big[\|X_h^*(t_k) - X_{h\tau}^*(t_k)\|_{\mathbb{L}^2}^2\big] + \mathbb{E}\Big[\tau \sum_{k=1}^{N} \|X_h^*(t_k) - X_{h\tau}^*(t_k)\|_{\mathbb{H}_0^1}^2\Big] \le C\tau\,;$$

$$\text{(iii)}\quad \max_{0\le k\le N} \mathbb{E}\big[\|Y_h(t_k) - Y_{h\tau}(t_k)\|_{\mathbb{L}^2}^2\big] + \mathbb{E}\Big[\tau \sum_{k=0}^{N-1} \|Y_h(t_k) - Y_{h\tau}(t_k)\|_{\mathbb{H}_0^1}^2\Big] \le C\tau\,.$$

Proof. We divide the proof into three steps.
Step 1. We follow the argumentation in the proof of Theorem 4.1. For every $U_{h\tau},\ R_{h\tau} \in \mathbb{U}_{h\tau}$, the first Fréchet derivative $D\widehat{\mathcal{J}}_{h\tau}(U_{h\tau})$, and the second Fréchet derivative $D^2\widehat{\mathcal{J}}_{h\tau}(U_{h\tau})$ satisfy

$$\begin{aligned} &D\widehat{\mathcal{J}}_{h\tau}\left(U_{h\tau}\right) = U_{h\tau} - \Pi_h^0 \mathcal{T}_{h\tau}\big(\mathcal{S}_{h\tau}\left(U_{h\tau}\right)\big)\,,\\ &\mathbb{E}\big[\big(D^2\widehat{\mathcal{J}}_{h\tau}(U_{h\tau})R_{h\tau}, R_{h\tau}\big)_{L^2(0,T;\mathbb{L}^2)}\big] \ge \mathbb{E}\big[\|R_{h\tau}\|_{L^2(0,T;\mathbb{L}^2)}^2\big]\,. \end{aligned} \tag{4.26}$$

Define the (piecewise constant) operator $\Pi_\tau : L^2_{\mathbb{F}}\big(\Omega; C(0,T;\mathbb{V}_h^0)\big) \to \mathbb{U}_{h\tau}$ by

$$\Pi_\tau U_h(t) := U_h(t_n) \qquad \forall\, t \in [t_n, t_{n+1}) \qquad n = 0, 1, \ldots, N-1\,.$$

By putting $R_{h\tau} = U_{h\tau}^* - \Pi_\tau U_h^*$ in (4.26), and applying the fact $D\widehat{\mathcal{J}}_{h\tau}(U_{h\tau}^*) = D\widehat{\mathcal{J}}_h(U_h^*) = 0$, we see that

$$\begin{aligned} &\mathbb{E}\big[\|U_{h\tau}^* - \Pi_\tau U_h^*\|_{L^2(0,T;\mathbb{L}^2)}^2\big]\\ &\quad\le \mathbb{E}\Big[\big(D\widehat{\mathcal{J}}_{h\tau}(U_{h\tau}^*), U_{h\tau}^* - \Pi_\tau U_h^*\big)_{L^2(0,T;\mathbb{L}^2)} - \big(D\widehat{\mathcal{J}}_{h\tau}(\Pi_\tau U_h^*), U_{h\tau}^* - \Pi_\tau U_h^*\big)_{L^2(0,T;\mathbb{L}^2)}\Big]\\ &\quad= \mathbb{E}\Big[\big(D\widehat{\mathcal{J}}_h(U_h^*) - D\widehat{\mathcal{J}}_h(\Pi_\tau U_h^*), U_{h\tau}^* - \Pi_\tau U_h^*\big)_{L^2(0,T;\mathbb{L}^2)}\Big]\\ &\qquad+\mathbb{E}\Big[\big(D\widehat{\mathcal{J}}_h(\Pi_\tau U_h^*) - D\widehat{\mathcal{J}}_{h\tau}(\Pi_\tau U_h^*), U_{h\tau}^* - \Pi_\tau U_h^*\big)_{L^2(0,T;\mathbb{L}^2)}\Big]\,. \end{aligned} \tag{4.27}$$

Therefore,

$$\begin{aligned} &\mathbb{E}\big[\|U_{h\tau}^* - \Pi_\tau U_h^*\|_{L^2(0,T;\mathbb{L}^2)}^2\big]\\ &\quad\le 2\mathbb{E}\big[\|D\widehat{\mathcal{J}}_h(U_h^*) - D\widehat{\mathcal{J}}_h(\Pi_\tau U_h^*)\|_{L^2(0,T;\mathbb{L}^2)}^2\big] + 2\mathbb{E}\big[\|D\widehat{\mathcal{J}}_h(\Pi_\tau U_h^*) - D\widehat{\mathcal{J}}_{h\tau}(\Pi_\tau U_h^*)\|_{L^2(0,T;\mathbb{L}^2)}^2\big]\\ &\quad=: 2I' + 2II'\,. \end{aligned} \tag{4.28}$$

We use (4.9) and (4.6), and stability properties of the projection Π_h^0 to bound I' as follows,

$$\begin{aligned} I' &= \mathbb{E}\Big[\big\|U_h^* - \Pi_\tau U_h^* + \Pi_h^0\mathcal{T}_h\big(\mathcal{S}_h(\Pi_\tau U_h^*)\big) - \Pi_h^0\mathcal{T}_h\big(\mathcal{S}_h(U_h^*)\big)\big\|_{L^2(0,T;\mathbb{L}^2)}^2\Big]\\ &\le 2\mathbb{E}\Big[\|U_h^* - \Pi_\tau U_h^*\|_{L^2(0,T;\mathbb{L}^2)}^2 + \big\|\mathcal{T}_h\big(\mathcal{S}_h(\Pi_\tau U_h^*)\big) - \mathcal{T}_h\big(\mathcal{S}_h(U_h^*)\big)\big\|_{L^2(0,T;\mathbb{L}^2)}^2\Big]\,. \end{aligned} \tag{4.29}$$

By stability properties of solutions to **BSPDE**$_h$ (2.14), and the discretization (2.5) of **BSPDE**, we obtain

$$
\begin{aligned}
&\mathbb{E}\Big[\big\|\mathcal{T}_h\big(\mathcal{S}_h(\Pi_\tau U_h^*)\big) - \mathcal{T}_h\big(\mathcal{S}_h(U_h^*)\big)\big\|^2_{L^2(0,T;\mathbb{L}^2)}\Big] \\
&\quad \le C\mathbb{E}\Big[\big\|\big(\mathcal{S}_h(U_h^*) - \mathcal{S}_h(\Pi_\tau U_h^*)\big)(T)\big\|^2_{\mathbb{L}^2} + \|\mathcal{S}_h(U_h^*) - \mathcal{S}_h(\Pi_\tau U_h^*)\|^2_{L^2(0,T;\mathbb{L}^2)}\Big] \\
&\quad \le C\,\|U_h^* - \Pi_\tau U_h^*\|^2_{L^2_{\mathbb{F}}(\Omega;L^2(0,T;\mathbb{L}^2))}\,.
\end{aligned} \tag{4.30}
$$

By the optimality condition (4.6), estimate (2.15), and Theorem 4.1 (i) we have

$$
\begin{aligned}
&\|U_h^* - \Pi_\tau U_h^*\|^2_{L^2_{\mathbb{F}}(\Omega;L^2(0,T;\mathbb{L}^2))} \\
&\quad \le C\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}}\int_{t_k}^{t}\mathbb{E}\Big[\tau\big\|-\Delta_h Y_h(s) + \big(X_h^*(s) - \Pi_h^1\widetilde{X}(s)\big)\big\|^2_{\mathbb{L}^2} + \ \|Z_h(s)\|^2_{\mathbb{L}^2}\Big]\,\mathrm{d}s\mathrm{d}t \\
&\quad \le C\tau\int_0^T\mathbb{E}\Big[\big\| - \Delta_h Y_h(s) + \big(X_h^*(s) - \Pi_h^1\widetilde{X}(s)\big)\big\|^2_{\mathbb{L}^2} + \|Z_h(s)\|^2_{\mathbb{L}^2}\Big]\,\mathrm{d}s \\
&\quad \le C\tau\,.
\end{aligned} \tag{4.31}
$$

Next, we turn to II', for which we use the representations (4.9), (4.26) and the stability property of Π_h^0 to conclude

$$
\begin{aligned}
II' &= \mathbb{E}\big[\big\|\Pi_h^0\mathcal{T}_h\big(\mathcal{S}_h(\Pi_\tau U_h^*)\big) - \Pi_h^0\mathcal{T}_{h\tau}\big(\mathcal{S}_{h\tau}(\Pi_\tau U_h^*)\big)\big\|^2_{L^2(0,T;\mathbb{L}^2)}\big] \\
&\le 2\mathbb{E}\Big[\big\|\mathcal{T}_h\big(\mathcal{S}_h(\Pi_\tau U_h^*)\big) - \mathcal{T}_h\big(\mathcal{S}_{h\tau}(\Pi_\tau U_h^*)\big)\big\|^2_{L^2(0,T;\mathbb{L}^2)} \\
&\quad + \big\|\mathcal{T}_h\big(\mathcal{S}_{h\tau}(\Pi_\tau U_h^*)\big) - \mathcal{T}_{h\tau}\big(\mathcal{S}_{h\tau}(\Pi_\tau U_h^*)\big\|^2_{L^2(0,T;\mathbb{L}^2)}\Big] \\
&=: 2\left(II'_{a,1} + II'_{a,2}\right)\,.
\end{aligned}
$$

In order to bound $II'_{a,1}$, we use stability properties for **SPDE**$_h$ (2.5), **BSPDE**$_h$ (4.7), in combination with the error estimate (2.8) for (2.5) to conclude

$$
II'_{a,1} \le C\mathbb{E}\big[\|(\mathcal{S}_h(\Pi_\tau U_h^*) - \mathcal{S}_{h\tau}(\Pi_\tau U_h^*))\,(T)\|^2_{\mathbb{L}^2} + \|\mathcal{S}_h(\Pi_\tau U_h^*) - \mathcal{S}_{h\tau}(\Pi_\tau U_h^*)\|^2_{L^2(0,T;\mathbb{L}^2)}\big] \le C\tau\,.
$$

To bound $II'_{a,2}$, it is easy to see

$$
\begin{aligned}
II'_{a,2} &\le 2\sum_{n=0}^{N-1}\mathbb{E}\Big[\int_{t_n}^{t_{n+1}}\|Y_h(t;\mathcal{S}_{h\tau}(\Pi_\tau U_h^*)) - Y_h(t_n;\mathcal{S}_{h\tau}(\Pi_\tau U_h^*))\|^2_{\mathbb{L}^2}\ \mathrm{d}t\Big] \\
&\quad + 2T\max_{0\le n\le N}\mathbb{E}\big[\|Y_h(t_n;\mathcal{S}_{h\tau}(\Pi_\tau U_h^*)) - Y_h^n(\mathcal{S}_{h\tau}(\Pi_\tau U_h^*))\|^2_{\mathbb{L}^2}\big].
\end{aligned}
$$

By (4.9), Lemma 3.1 (i), stable property of $(4.15)_1$, we can get

$$
\begin{aligned}
&\sum_{n=0}^{N-1}\mathbb{E}\Big[\int_{t_n}^{t_{n+1}}\|Y_h(t;\mathcal{S}_{h\tau}(\Pi_\tau U_h^*))-Y_h(t_n;\mathcal{S}_{h\tau}(\Pi_\tau U_h^*))\|_{\mathbb{L}^2}^2\,\mathrm{d}t\Big]\\
&\quad\le C\tau\left(\mathbb{E}\big[\|\mathcal{S}_{h\tau}(\Pi_\tau U_h^*)(T)\|_{\mathbb{H}_0^1}^2\big]+\|\widetilde{X}(T)\|_{\mathbb{H}_0^1}^2+\|\mathcal{S}_{h\tau}(\Pi_\tau U_h^*)\|_{L^2_{\mathbb{F}}(\Omega;L^2(0,T;\mathbb{L}^2))}^2+\|\widetilde{X}\|_{L^2(0,T;\mathbb{L}^2)}^2\right)\\
&\quad\le C\tau\left(\|X_0\|_{\mathbb{H}_0^1}^2+\|\widetilde{X}(T)\|_{\mathbb{H}_0^1}^2+\|\sigma\|_{L^2_{\mathbb{F}}(\Omega;L^2(0,T;\mathbb{H}_0^1))}^2+\|Y_h\|_{L^2_{\mathbb{F}}(\Omega;L^2(0,T;\mathbb{H}_0^1))}^2+\|\widetilde{X}\|_{L^2(0,T;\mathbb{L}^2)}^2\right)\\
&\quad\le C\tau\,.
\end{aligned}
$$

Utilizing Theorem 3.5 for $\mathbf{BSPDE}_h$ (2.14) with $\Pi_h f=\mathcal{S}_{h\tau}(\Pi_\tau U_h^*)-\Pi_h^1\widetilde{X}$ and (4.25), we can find that

$$
\begin{aligned}
&\max_{0\le n\le N}\mathbb{E}\big[\|Y_h(t_n;\mathcal{S}_{h\tau}(\Pi_\tau U_h^*))-Y_h^n(\mathcal{S}_{h\tau}(\Pi_\tau U_h^*))\|_{\mathbb{L}^2}^2\big]\\
&\quad\le C\mathbb{E}\Big[\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}\big\|\nabla\big[Y_h(s;\mathcal{S}_{h\tau}(\Pi_\tau U_h^*))-Y_h(t_n;\mathcal{S}_{h\tau}(\Pi_\tau U_h^*))\big]\big\|_{\mathbb{L}^2}^2+\|\widetilde{X}(s)-\widetilde{X}(t_n)\|_{\mathbb{L}^2}^2\,\mathrm{d}s\Big]\\
&\quad\le C\tau\Big[\max_{0\le n\le N}\mathbb{E}\big[\big\|\nabla X_{h\tau}(t_n;\mathcal{S}_{h\tau}(\Pi_\tau U_h^*))\big\|_{\mathbb{L}^2}^2\big]+\mathbb{E}\int_0^T\big\|\nabla\Pi_h^1\widetilde{X}(t)\big\|_{\mathbb{L}^2}^2\,\mathrm{d}t\Big]\\
&\qquad+C\mathbb{E}\Big[\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}\|\widetilde{X}(s)-\widetilde{X}(t_n)\|_{\mathbb{L}^2}^2\,\mathrm{d}s\Big]\\
&\quad\le C\tau\,.
\end{aligned}
$$

Here, we apply the representation of $X_{h\tau}$ (4.18), the fact $\widetilde{X}\in L^2(0,T;\mathbb{H}_0^1)$, and condition (4.25).

Now we insert above estimates into (4.28) to obtain assertion (i).

Step 2. For all $k=0,1,\dots,N$, we define $e_X^k=X_h^*(t_k)-X_{h\tau}^*(t_k)$. Subtracting (4.14) from (4.5) leads to

$$
\begin{aligned}
e_X^{k+1}-e_X^k&=\tau\Delta_h e_X^{k+1}+\tau\Pi_h^1[U_h^*(t_k)-U_{h\tau}^*(t_k)]+\int_{t_k}^{t_{k+1}}\Pi_h^1[\sigma(s)-\sigma(t_k)]\,\mathrm{d}W(s)\\
&\quad+\int_{t_k}^{t_{k+1}}\big(\Delta_h[X_h^*(s)-X_h^*(t_{k+1})]+\Pi_h^1[U_h^*(s)-U_h^*(t_k)]\big)\,\mathrm{d}s\,.
\end{aligned}
$$

Testing with e_X^{k+1}, and using binomial formula, Poincaré's inequality, independence, and absorption lead to

$$
\begin{aligned}
&\frac12\mathbb{E}\big[\|e_X^{k+1}\|_{\mathbb{L}^2}^2-\|e_X^k\|_{\mathbb{L}^2}^2+\frac12\|e_X^{k+1}-e_X^k\|_{\mathbb{L}^2}^2\big]+\frac{\tau}{2}\mathbb{E}\big[\|\nabla e_X^{k+1}\|_{\mathbb{L}^2}^2\big]\\
&\quad\le C\tau\mathbb{E}\big[\|U_h^*(t_k)-U_{h\tau}^*(t_k)\|_{\mathbb{L}^2}^2\big]+C\mathbb{E}\Big[\Big\|\int_{t_k}^{t_{k+1}}\Pi_h^1[\sigma(s)-\sigma(t_k)]\,\mathrm{d}W(s)\Big\|_{\mathbb{L}^2}^2\Big]\\
&\qquad+C\mathbb{E}\Big[\int_{t_k}^{t_{k+1}}\|\nabla\big[X_h^*(s)-X_h^*(t_{k+1})\big]\|_{\mathbb{L}^2}^2\,\mathrm{d}s\Big]+C\mathbb{E}\Big[\int_{t_k}^{t_{k+1}}\|U_h^*(s)-U_h^*(t_k)\|_{\mathbb{L}^2}^2\,\mathrm{d}s\Big].
\end{aligned}
$$

By taking the sum over all $0 \le k \le n$ and $0 \le k \le N-1$, and noting that $e_X^0 = 0$, we find that

$$\begin{aligned}
&\max_{0\le n\le N} \mathbb{E}\big[\|e_X^n\|_{\mathbb{L}^2}^2\big] + \sum_{n=1}^{N} \tau \mathbb{E}\big[\|\nabla e_X^n\|_{\mathbb{L}^2}^2\big] \\
&\quad \le C\sum_{k=0}^{N-1} \mathbb{E}\Big[\tau\|U_h^*(t_k) - U_{h\tau}^*(t_k)\|_{\mathbb{L}^2}^2 + \int_{t_k}^{t_{k+1}} \Big(\|\sigma(s)-\sigma(t_k)\|_{\mathbb{L}^2}^2 \\
&\qquad + \|\nabla\left[X_h^*(s) - X_h^*(t_{k+1})\right]\|_{\mathbb{L}^2}^2 + \|U_h^*(s) - U_h^*(t_k)\|_{\mathbb{L}^2}^2\Big)\,\mathrm{d}s\Big].
\end{aligned}$$

By (4.28), the first term on the right-hand side is bounded by $C\tau$. We use Itô isometry for the second term, and Hölder regularity in time of σ to bound it equally. Adopting the method in (4.31), we can bound the third term by $C\tau\big(\|\Delta_h X_h^*(0)\|_{\mathbb{L}^2}^2 + \|\nabla \Pi_h^1 U_h^*(t)\|_{L^2_{\mathbb{F}}(\Omega;L^2(0,T;\mathbb{L}^2))}^2 + \|\sigma(t)\|_{L^2_{\mathbb{F}}(\Omega;L^2(0,T;\mathbb{H}^2))}^2\big)$. We use (4.31) to bound the last term by $C\tau$. That is assertion (ii).

Step 3. Firstly, we introduce an auxiliary BSDE

$$\begin{cases}
Y_\tau(t_{n+1}) - Y_\tau(t_n) = \tau\big[-\Delta_h Y_\tau(t_n) + [X_{h\tau}^*(t_{n+1}) - \Pi_h^1 \widetilde{X}(t_{n+1})]\big] \\
\qquad\qquad\qquad\qquad + \displaystyle\int_{t_n}^{t_{n+1}} Z_\tau(t)\,\mathrm{d}W(t) \quad n = 0,1,\ldots,N-1\,, \\
Y_\tau(T) = -\alpha\big(X_{h\tau}^*(T) - \Pi_h^1 \widetilde{X}(T)\big).
\end{cases} \tag{4.32}$$

It is easy to see that $Y_\tau = Y_{h\tau}$. Define $e_Y^n = Y_h(t_n) - Y_\tau(t_n)$, $n = 0,1,\ldots,N$. With the same argument as that in the proof of Theorem 3.5, we can deduce

$$\begin{aligned}
&\max_{0\le n\le N} \mathbb{E}\big[\|e_Y^n\|_{\mathbb{L}^2}^2\big] + \sum_{n=1}^{N} \tau \mathbb{E}\big[\|\nabla e_Y^n\|_{\mathbb{L}^2}^2\big] \\
\le& C\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}} \mathbb{E}\big[\|\nabla\big[Y_h(s) - Y_h(t_k)\big]\|_{\mathbb{L}^2}^2 + \|X_h^*(s) - X_{h\tau}^*(t_{k+1})\|_{\mathbb{L}^2}^2 + \|\widetilde{X}(s) - \widetilde{X}(t_{k+1})\|_{\mathbb{L}^2}^2\big]\,\mathrm{d}s\,.
\end{aligned}$$

Applying Lemma 3.1 (ii), the first integral term is bounded by

$$C\tau\Big[\|\Delta_h X_h(0)\|_{\mathbb{L}^2}^2 + \|\Delta_h \Pi_h^1 \widetilde{X}(T)\|_{\mathbb{L}^2}^2 + \int_0^T \mathbb{E}\big[\|\nabla X_h^*(t)\|_{\mathbb{L}^2}^2 + \|\nabla \Pi_h^1 \widetilde{X}_h^*(t)\|_{\mathbb{L}^2}^2 + \|\Pi_h^1 U_h^*(t)\|_{\mathbb{L}^2}^2\big]\,\mathrm{d}t\Big]\,.$$

It remains to estimate the second integral term, which is bounded by

$$\begin{aligned}
&C\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}} \mathbb{E}\big[\|X_h^*(s) - X_h^*(t_{k+1})\|_{\mathbb{L}^2}^2 + \|X_h^*(t_{k+1}) - X_{h\tau}^*(t_{k+1})\|_{\mathbb{L}^2}^2\big]\,\mathrm{d}s \\
&\le C\tau\Big[\|\nabla X_h(0)\|_{\mathbb{L}^2}^2 + \int_0^T \mathbb{E}\big[\|\Pi_h^1 U_h^*(t)\|_{\mathbb{L}^2}^2 + \|\nabla \Pi_h^1 \sigma(t)\|_{\mathbb{L}^2}^2\big]\,\mathrm{d}t\Big] \\
&\quad + C\max_{0\le k\le N} \mathbb{E}\big[\|X_h^*(t_k) - X_{h\tau}^*(t_k)\|_{\mathbb{L}^2}^2\big] \\
&\le C\tau\,.
\end{aligned}$$

Assertion (iii) now follows from the above three statements and conditions on $X_0, \sigma, \widetilde{X}$. □

Remark 4.4. In this work, we apply implicit Euler method to solve $\mathbf{BSPDE}_h$ (2.14) and (4.7). In the literature, some higher order schemes exist for the simulation of **BSDE**s of *fixed, finite* dimension; see, *e.g.*, [9, 37, 41]. These schemes are also candidates to solve problem **SLQ**, and this is a meaningful research topic.

Remark 4.5. Error analysis in this section can be trivially transferred to some generalized **SLQ** problems. For example, the cost functional and controlled equation could be

$$\begin{aligned}\mathcal{J}(X,U) =&\frac{1}{2}\mathbb{E}\Big[\int_0^T\int_D \big(\alpha_X(t,x)|X(t,x)-\widetilde{X}(t,x)|^2+\alpha_U(t,x)|U(t,x)|^2\big)\,\mathrm{d}x\mathrm{d}t\\&+\int_D \alpha(x)|X(T,x)-\widetilde{X}(T,x)|^2\,\mathrm{d}x\Big]\,,\end{aligned}\tag{4.33}$$

and

$$\begin{cases}\mathrm{d}X(t,x)=\big[-\mathcal{L}X(t,x)+U(t,x)\big]\,\mathrm{d}t+\sigma(t,x)\,\mathrm{d}W(t) & (t,x)\in[0,T]\times D\,,\\ X(t,x)=0 & (t,x)\in(0,T)\times\partial D,\\ X(0,x)=X_0(x) & x\in D\,,\end{cases}$$

where $\mathcal{L}X=-\sum_{i,j=1}^d\frac{\partial}{\partial x_i}\big(a_{ij}(x)\frac{\partial X}{\partial x_j}\big)+c(x)X$, (a_{ij}) is symmetric and uniformly elliptic, α_X, α_U, α, c are positive bounded functions, and $\alpha_U\geq\delta>0$. Besides, based on Pontryagin's maximum principle, the spatio-temporal discretization strategy for problem **SLQ** can also be utilized to general stochastic optimal control problems with stochastic parabolic equations constraints.

5. The gradient descent method to solve $\mathbf{SLQ}_{h\tau}$

By Theorem 4.2, solving minimization problem $\mathbf{SLQ}_{h\tau}$ is equivalent to solving the system of coupled forward-backward difference equations (4.15) and (4.16). We may exploit the variational character of problem $\mathbf{SLQ}_{h\tau}$ to construct a gradient descent method $\mathbf{SLQ}_{h\tau}^{\mathrm{grad}}$ where approximate iterates of the optimal control $U_{h\tau}^*$ in the Hilbert space $\mathbb{U}_{h\tau}$ are obtained; see also [23, 30].

Algorithm 5.1. **($\mathbf{SLQ}_{h\tau}^{\mathrm{grad}}$)** Let $U_{h\tau}^{(0)}\in\mathbb{U}_{h\tau}$, and fix $\kappa>0$. For any $\ell\in\mathbb{N}_0$, update $U_{h\tau}^{(\ell)}\in\mathbb{U}_{h\tau}$ as follows:

1. Compute $X_{h\tau}^{(\ell)}\in\mathbb{X}_{h\tau}$ by

$$\begin{cases}[\mathbb{1}-\tau\Delta_h]X_{h\tau}^{(\ell)}(t_{n+1})=X_{h\tau}^{(\ell)}(t_n)+\tau\Pi_h^1U_{h\tau}^{(\ell)}(t_n)+\Pi_h^1\sigma(t_n)\Delta_{n+1}W \quad n=0,1,\dots,N-1\,,\\ X_{h\tau}^{(\ell)}(0)=\Pi_h^1X_0\,.\end{cases}$$

2. Use $X_{h\tau}^{(\ell)}\in\mathbb{X}_{h\tau}$ to compute $Y_{h\tau}^{(\ell)}\in\mathbb{X}_{h\tau}$ *via*

$$\begin{cases}[\mathbb{1}-\tau\Delta_h]Y_{h\tau}^{(\ell)}(t_n)=\mathbb{E}\left[Y_{h\tau}^{(\ell)}(t_{n+1})-\tau\big(X_{h\tau}^{(\ell)}(t_{n+1})-\Pi_h^1\widetilde{X}(t_{n+1})\big)\big|\mathcal{F}_{t_n}\right]\\ \qquad\qquad n=0,1,\dots,N-1\,,\\ Y_{h\tau}^{(\ell)}(T)=-\alpha\big(X_{h\tau}^{(\ell)}(T)-\Pi_h^1\widetilde{X}(T)\big)\,.\end{cases}$$

3. Compute the update $U_{h\tau}^{(\ell+1)}\in\mathbb{U}_{h\tau}$ *via*

$$U_{h\tau}^{(\ell+1)}=U_{h\tau}^{(\ell)}-\frac{1}{\kappa}\big(U_{h\tau}^{(\ell)}-\Pi_h^0Y_{h\tau}^{(\ell)}\big)\,.$$

Note that Steps 1 and 2 are now decoupled: the first step requires to solve a space-time discretization (2.7) of **SPDE** (2.3), while the second requires to solve the space-time discretization $(3.6)_1$ of the **BSPDE** $(4.1)_2$. We refer to related works on how to approximate conditional expectations [3, 6, 18, 24, 33]; a similar method to $\mathbf{SLQ}_{h\tau}^{\text{grad}}$ to solve problem $\mathbf{SLQ}_{h\tau}$ has been proposed in [15]. For general optimal control problem, to reduce computation cost, one can adopt stochastic gradient descent methods; see *e.g.* [1].

Remark 5.2. In [24], based on variational method, a finite transposition method for solving **BSPDE**s like $(4.1)_2$ is proposed, which avoids the computation of conditional expectations *via* Monte-Carlo simulation; see [24] for further details.

We want to show convergence of $\mathbf{SLQ}_{h\tau}^{\text{grad}}$ for $\kappa > 0$ sufficiently large and $\ell \uparrow \infty$. For this purpose, we recall the notations $\mathcal{S}_{h\tau}, \mathcal{T}_{h\tau}, \widehat{\mathcal{J}}_{h\tau}$ introduced in Section 4.2. For this purpose, we first recall Lipschitz continuity of $D\widehat{\mathcal{J}}_{h\tau}$: since

$$D^2\widehat{\mathcal{J}}_{h\tau}(U_{h\tau}) = \big(\mathbb{1} + L^*L + \alpha\widehat{L}^*\widehat{L}\big)U_{h\tau}\,,$$

where operators L, $\widehat{L}$ are defined in (4.18), we find $K := \|\mathbb{1} + L^*L + \alpha\widehat{L}^*\widehat{L}\|_{\mathcal{L}(\mathbb{U}_{h\tau};\mathbb{U}_{h\tau})}$, such that

$$\|D\widehat{\mathcal{J}}_{h\tau}\left(U_{h\tau}^1\right) - D\widehat{\mathcal{J}}_{h\tau}\left(U_{h\tau}^2\right)\|_{\mathbb{U}_{h\tau}} \le K\left\|U_{h\tau}^1 - U_{h\tau}^2\right\|_{\mathbb{U}_{h\tau}}.$$

Indeed, noting that $\|(\mathbb{1} - \tau\Delta_h)^{-1}\|_{\mathcal{L}(\mathbb{V}_h^1;\mathbb{V}_h^1)} \le 1$, we conclude

$$\|LU_{h\tau}\|_{\mathbb{X}_{h\tau}}^2 = \sum_{n=1}^N \tau\mathbb{E}\big[\|LU_{h\tau}(t_n)\|_{\mathbb{L}^2}^2\big] = \sum_{n=1}^N \tau\mathbb{E}\Big[\|\tau\sum_{j=0}^{n-1}\big[(\mathbb{1} - \tau\Delta_h)^{-1}\big]^{n-j}\,\Pi_h^1 U_{h\tau}(t_j)\|_{\mathbb{L}^2}^2\Big] \le T^2\|U_{h\tau}\|_{\mathbb{U}_{h\tau}}^2\,,$$

and

$$\|\widehat{L}U_{h\tau}\|_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{L}^2)}^2 = \mathbb{E}\Big[\Big\|\tau\sum_{j=0}^{N-1}\big[(\mathbb{1} - \tau\Delta_h)^{-1}\big]^{N-j}\,\Pi_h^1 U_{h\tau}(t_j)\Big\|_{\mathbb{L}^2}^2\Big] \le T\|U_{h\tau}\|_{\mathbb{U}_{h\tau}}^2.$$

Hence

$$K = \|\mathbb{1} + L^*L + \alpha\widehat{L}^*\widehat{L}\|_{\mathcal{L}(\mathbb{U}_{h\tau};\mathbb{U}_{h\tau})} \le 1 + \alpha T + T^2.$$

Since $\mathbf{SLQ}_{h\tau}^{\text{grad}}$ is the gradient descent method for $\mathbf{SLQ}_{h\tau}$, we have the following result.

Theorem 5.3. *Suppose that $\kappa \ge K$. Let $\{U_{h\tau}^{(\ell)}\}_{\ell\in\mathbb{N}_0} \subset \mathbb{U}_{h\tau}$ be generated by $\mathbf{SLQ}_{h\tau}^{\text{grad}}$, and $U_{h\tau}^*$ solve $\mathbf{SLQ}_{h\tau}$. Then*

$$\begin{cases} \widehat{\mathcal{J}}_{h\tau}(U_{h\tau}^{(\ell)}) - \widehat{\mathcal{J}}_{h\tau}(U_{h\tau}^*) \le \dfrac{2\kappa\|U_{h\tau}^{(0)} - U_{h\tau}^*\|_{\mathbb{U}_{h\tau}}^2}{\ell}, \\ \|U_{h\tau}^{(\ell)} - U_{h\tau}^*\|_{\mathbb{U}_{h\tau}}^2 \le \Big(1 - \dfrac{1}{\kappa}\Big)^\ell \|U_{h\tau}^{(0)} - U_{h\tau}^*\|_{\mathbb{U}_{h\tau}}^2 \qquad \ell = 1, 2, \ldots. \end{cases}$$

Proof. We know that $D\widehat{\mathcal{J}}_{h\tau}$ is Lipschitz continuous with constant $K > 0$. Also, $\widehat{\mathcal{J}}_{h\tau}$ is strongly convex. Hence, the gradient descent method in abstract form is the following iteration (see Algorithm 5.1, Step 3.)

$$U_{h\tau}^{(\ell+1)} = U_{h\tau}^{(\ell)} - \frac{1}{\kappa}D\widehat{\mathcal{J}}_{h\tau}(U_{h\tau}^{(\ell)}), \quad \ell = 0, 1, 2\cdots. \tag{5.1}$$

By the proof of Theorem 4.2, we have obtained the following facts:

$$\begin{cases} D\widehat{\mathcal{J}}_{h\tau}(U_{h\tau}^{(\ell)}) = U_{h\tau}^{(\ell)} - \Pi_h^0 \mathcal{T}_{h\tau}\big(\mathcal{S}_{h\tau}(U_{h\tau}^{(\ell)})\big)\,, \\ \Pi_h^0 \mathcal{T}_{h\tau}\big(\mathcal{S}_{h\tau}(U_{h\tau}^{(\ell)})\big) = -L^*\big(\Gamma\Pi_h^1 X_0 + LU_{h\tau}^{(\ell)} + f - \widetilde{X}\big) - \alpha\widehat{L}^*\big(\widehat{\Gamma}\Pi_h^1 X_0 + \widehat{L}U_{h\tau}^{(\ell)} + \widehat{f} - \widetilde{X}(T)\big)\,, \end{cases}$$

where $L, \widehat{L}, \Gamma, \widehat{\Gamma}, \widehat{f}$ are defined in (4.18) and (4.19). *Via* (4.20), we have that $\Pi_h^0 \mathcal{T}_{h\tau}\big(\mathcal{S}_{h\tau}(U_{h\tau}^{(\ell)})\big)$ is just $Y_{h\tau}^{(\ell)}$, the solution of Step 2 in Algorithm 5.1. Therefore, (5.1) is consistent with the gradient descent method $\mathbf{SLQ}_{h\tau}^{\mathrm{grad}}$. The desired error estimates now follow by standard estimates for the gradient descent method (see *e.g.* [30], Thm. 1.2.4). ☐

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