

OPTIMAL ENERGY DECAY RATES FOR ABSTRACT SECOND ORDER EVOLUTION EQUATIONS WITH NON-AUTONOMOUS DAMPING*

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Abstract. We consider an abstract second order non-autonomous evolution equation in a Hilbert space H : $u'' + Au + \gamma(t)u' + f(u) = 0$, where A is a self-adjoint and nonnegative operator on H , f is a conservative H -valued function with polynomial growth (not necessarily to be monotone), and $\gamma(t)u'$ is a time-dependent damping term. How exactly the decay of the energy is affected by the damping coefficient $\gamma(t)$ and the exponent associated with the nonlinear term f ? There seems to be little development on the study of such problems, with regard to *non-autonomous* equations, even for strongly positive operator A . By an idea of asymptotic rate-sharpening (among others), we obtain the optimal decay rate of the energy of the non-autonomous evolution equation in terms of $\gamma(t)$ and f . As a byproduct, we show the optimality of the energy decay rates obtained previously in the literature when f is a monotone operator.

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1. INTRODUCTION

In view of the importance in mathematical theory and applications in physics, engineering, mechanics, biology and others, various nonautonomous equations including the non-autonomous evolution equations in abstract (infinite-dimensional) spaces, have been studied by many researchers and a lot of good results on this issue have been established (*cf.*, *e.g.* [1, 3, 8–17, 24–26, 28, 30, 32, 34, 35] and references therein).

In this paper, we are concerned with an abstract second order non-autonomous evolution equation in a real Hilbert space H , with time dependent damping, as follows

$$\begin{cases} u''(t) + Au(t) + \gamma(t)u'(t) + f(u(t)) = 0, & \forall t \geq 0, \\ u(0) = u_0, \quad u'(0) = u_1, \end{cases} \quad (1.1)$$

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where A is a nonnegative self-adjoint linear operator on H , the nonlinear term $f : D(A^{1/2}) \rightarrow H$ is assumed to be conservative with polynomial growth, and $\gamma u'$ is the time dependent damping with

$$R_1(1+t)^{-\alpha_1} \leq \gamma(t) \leq R_2(1+t)^{-\alpha_2}, \quad \forall t \geq 0, \quad (1.2)$$

where R_1 and R_2 are two positive constants, and $0 \leq \alpha_2 \leq \alpha_1 < 1$.

For the study of the asymptotic behaviors and decay rates for the damped autonomous evolution equations, there have been many developments. We refer the reader to, *e.g.*, [2, 4, 6, 12, 13, 15, 17–21, 27, 31, 35] and references therein. Especially, when A is strongly positive, the asymptotic behaviors for the autonomous case of (1.1) have been extensively studied too, *cf.* [10, 22, 23, 29, 30, 32, 33]. In this paper, we assume A to just have a nontrivial kernel; for the case of $\gamma(t)$ being constant, optimal decay rates have been obtained, and fast and slow solutions are classified subtly (see [18–20]).

It was shown in [7] that any bounded solution of (1.1) converges weakly in H_1 to a stationary point of the potential energy, when f is a general monotone operator. Following the work, it was proved in [31] that any solution energy $E(t)$ satisfies that for every $\bar{\alpha} < \alpha_1$,

$$E(t) = o\left(\frac{1}{t^{1+\bar{\alpha}}}\right) \quad \text{as } t \rightarrow +\infty, \quad (1.3)$$

under the condition that $\gamma(t)$ is decreasing,

$$\gamma(t) \geq c(1+t)^{-\alpha_1}, \quad t \geq 0 \quad (1.4)$$

(with a constant $c > 0$), and

$$\gamma'(t) \leq -\alpha_1 \frac{\gamma(t)}{1+t}, \quad t \geq t_0 \geq 0 \quad \text{a.e.} \quad (1.5)$$

We note that the two inequalities (1.4) and (1.5) together imply that

$$\gamma(t) \leq C(1+t)^{-\alpha_1}, \quad t \geq 0$$

(with a constant $C > 0$). For

$$\gamma(t) = (1+t)^{-\alpha} \quad (\alpha \in [0, 1]), \quad (1.6)$$

the estimate (1.3) was later improved in [4] as

$$E(t) = o\left(\frac{1}{t^{1+\alpha}}\right) \quad \text{as } t \rightarrow +\infty. \quad (1.7)$$

However, what is the optimal decay rate of the energy of the non-autonomous equation (1.1), *i.e.*, how exactly the decay of the energy of (1.1) is affected by the damping coefficient $\gamma(t)$ and the exponent associated with the nonlinear term f ? This problem is still unsolved. To the best of our knowledge, there has been little development so far on the study of such problems, with regard to non-autonomous equations, even for strongly positive operator A ; see Section 1.3 of [30] about an optimality result for one dimensional wave equation damped by a time-dependent boundary feedback, whose proof relies on d'Alembert's formula.

In this paper, we devote ourselves to studying the problem.

Let $\gamma(t)$ be as in (1.6), and $f(u) = |u|^p u$ ($p > 0$). By Theorem 2.4 of [28], the energy has the upper estimate

$$E(t) \leq C(1+t)^{-(1-\alpha)(p+2)/p}. \quad (1.8)$$

On the other hand, to get a lower bound we may exploit the hyperbolic version of the Dirichlet quotient

$$G(t) := \frac{\|u'(t)\|^2 + \|A^{1/2}u(t)\|^2}{\|u(t)\|^{2p+2}}, \quad (1.9)$$

introduced in [18]. Estimating the time-derivative of $\hat{G}(t)$, a small perturbation of G with γ as a product factor (see (4.9), Sect. 4), with the aid of either (1.7) or (1.8) we might probably obtain a lower estimate

$$E(t) \geq c(1+t)^{-(1+\alpha)(p+2)/p}$$

for a nonempty open set of initial data, but under the restriction

$$\alpha < p/(p+4) \quad \text{or} \quad \alpha < 1/3.$$

Obviously, there is a gap between the exponents $-(1-\alpha)(p+2)/p$ and $-(1+\alpha)(p+2)/p$, when $\alpha > 0$. Actually, it is still unknown what is the best possible upper estimate.

By an idea of asymptotic rate-sharpening, we obtain the optimal decay rate of the energy of this abstract second order non-autonomous evolution equation in terms of $\gamma(t)$ and f . Moreover, as a byproduct, we show that, when f is a monotone operator, some energy decay rates given previously in the literature are optimal.

More explicitly, we obtain upper estimates for the solutions and their energies of (1.1) in a general setting, and single out slow solutions (decaying exactly at the rates corresponding to the upper estimates) with a nonempty open set of initial data. When $\gamma(t)$ is a constant, the estimates recover those given in Theorems 2.2 and 2.3 of [18]. Moreover, specialized to the case of (1.6), the exponents of the upper and lower bounds coincide and read $-(1+\alpha)(p+2)/p$ for all $\alpha \in [0, 1)$. In addition, since

$$-(1+\alpha)(p+2)/p \rightarrow -(1+\alpha) \quad \text{as } p \rightarrow +\infty, \quad (1.10)$$

we see that the decay rate in (1.7) is the best possible for a general monotone f (as considered in [4, 7, 31]).

The outline of this paper is the following. In Section 2 we state assumptions and present our main theorems. Section 3 is devoted to formulating several preliminary results. We employ these results in Section 4 to prove the main theorems. Finally, in Section 5 we give two examples of our theorems being applied to Neumann or Dirichlet problems for nonautonomous, semilinear wave equations.

2. MAIN THEOREMS

We denote by $\langle v, w \rangle$ the inner product of two vectors v, w in H , and by $\|v\|$ the H -norm of v , and we define

$$H_1 = D(A^{1/2}) \quad \text{with norm} \quad \|v\|_{H_1} := \left(\|v\|^2 + \|A^{1/2}v\|^2 \right)^{1/2}.$$

The following is the basic assumptions on γ and f .

Assumption (H1)

- (i) $\gamma \in W_{\text{loc}}^{1,\infty}(R^+)$ satisfies (1.2);
- (ii) $f = \nabla F : H_1 \rightarrow H$ is a locally Lipschitz continuous gradient operator of some nonnegative functional F on H_1 , with $F(0) = 0$.

One knows

$$F(v) = \int_0^1 \langle f(tv), v \rangle dt, \quad \forall v \in H_1.$$

A function $u \in C([0, +\infty); H_1) \cap C^1([0, +\infty); H)$ is called a *mild solution* of problem (1.1), if it satisfies the integral equation

$$u(t) = S'(t)u_0 + S(t)u_1 - \int_0^t S(t-\tau)[\gamma(\tau)u'(\tau) + f(u)]d\tau, \quad t \geq 0.$$

Here $S(\cdot) : [0, +\infty) \rightarrow \mathcal{L}(H)$ (the space of bounded linear operators on H) is a solution operator for the linear equation

$$u''(t) + Au(t) = 0, \quad t \geq 0,$$

with $S(0) = 0$ and $S'(0) = I$ (the identity; the derivative being in the sense of strong topology).

In particular, u is called a *strong solution* if $u \in C^1([0, +\infty); H_1) \cap C^2([0, +\infty); H)$ and (1.1) holds.

The following result concerning the wellposedness of (1.1) is from Proposition 2.3 of [28].

Proposition 2.1. *Assume (H1). Then for every $(u_0, u_1) \in H_1 \times H$, problem (1.1) admits a unique mild solution u , which depends continuously on the initial data. In particular, u is a strong solution if $(u_0, u_1) \in D(A) \times H_1$. Moreover, defining the energy*

$$E(t) := \frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\|A^{1/2}u(t)\|^2 + F(u(t)), \quad t \geq 0, \quad (2.1)$$

one has

$$\frac{dE(t)}{dt} = -\gamma(t)\|u'(t)\|^2, \quad t \geq 0 \quad (2.2)$$

(for strong solutions).

For energy decay, more conditions are required.

Assumption (H2)

- (1) For $v \in H_1$,

$$\langle f(v), v \rangle \geq cF(v) \geq 0,$$

with some constant $c > 0$;

- (2) for $v \in H_1$,

$$\|v\|^{p+2} \leq M_0(\|A^{1/2}v\|) \left(\|A^{1/2}v\|^2 + F(v) \right),$$

with constant $p > 0$ and some positive function M_0 on R_+ that is bounded on bounded sets;

- (3) for $t \geq 0$,

$$|\gamma'(t)| \leq C\gamma^2(t) \text{ with some constant } C > 0, \text{ or } \alpha_1 = \alpha_2;$$

- (4) for $t \geq 0$,

$$|\gamma'(t)| \leq C\gamma(t)(1+t)^{-1} \text{ with some constant } C > 0 \text{ or } \gamma'(t) \leq 0, \text{ in the case } \alpha_1 > 1/2;$$

(5) $2\alpha_1 < 1 + \alpha_2$, in the case $\alpha_1 > 1/2$.

Now, we present our first main result.

Theorem 2.2. *Let Assumptions (H1) and (H2) hold. Let $(u_0, u_1) \in H_1 \times H$, and let $u(t)$ be the unique global mild solution of problem (1.1). Then*

$$E(t) \leq M_1(E(0)) \left(1 + \int_0^t \gamma(\tau)^{-1} d\tau\right)^{-\frac{p+2}{p}}, \quad \forall t \geq 0, \quad (2.3)$$

$$\|u(t)\| \leq M_2(E(0)) \left(1 + \int_0^t \gamma(\tau)^{-1} d\tau\right)^{-\frac{1}{p}}, \quad \forall t \geq 0,$$

for some positive functions M_1 and M_2 on R_+ that are bounded on bounded sets.

For the existence of slow solutions, we need additional conditions.

Assumption (H3)

(1) $\ker A \neq \{0\}$, and there exists $\zeta > 0$ such that

$$\|A^{1/2}u\|^2 \geq \zeta \|u\|^2, \quad \forall u \in H_1 \cap (\ker A)^\perp;$$

(2) there exist positive number ξ and M such that

$$\|f(u)\| \leq M \left(\|u\|^{1+p} + \|A^{1/2}u\|^{1+p} \right), \quad \forall u \in H_1 \text{ with } \|u\|_{H_1} \leq \xi;$$

(3) $\gamma(t)\gamma(s)^{-1} \leq C_0$, $\forall t \geq s \geq 0$, with some constant $C_0 > 0$.

The following is our second main result.

Theorem 2.3. *Let Assumptions (H1)–(H3) hold. Then there exists a nonempty open set $S \subset H_1 \times H$ such that for every $(u_0, u_1) \in S$, the unique global solution of problem (1.1) satisfies*

$$\|u(t)\| \geq c_0 \left(1 + \int_0^t \gamma(\tau)^{-1} d\tau\right)^{-\frac{1}{p}}, \quad \forall t \geq 0,$$

$$E(t) \geq c_0 \left(1 + \int_0^t \gamma(\tau)^{-1} d\tau\right)^{-\frac{p+2}{p}}, \quad \forall t \geq 0, \quad (2.4)$$

for some positive function c_0 depending on $\|u_0\|$, $\|A^{1/2}u_0\|$ and $\|u_1\|$.

Remark 2.4. (1) When γ is decreasing and $\alpha_1 = \alpha_2 = \alpha$, we know that (H2)(3)–(5) and (H3)(3) are satisfied, and the upper estimate of the energy in (2.3) reads

$$E(t) \leq M_1(E(0)) (1+t)^{-(1+\alpha)(p+2)/p}, \quad \forall t \geq 0,$$

and the lower estimate in (2.4) reads

$$E(t) \geq c_0 (1+t)^{-(1+\alpha)(p+2)/p}, \quad \forall t \geq 0.$$

- (2) The inequality $|\gamma'(t)| \leq C\gamma^2(t)$ in (H2)(3) is weaker than $|\gamma'(t)| \leq C\gamma(t)(1+t)^{-1}$ in (H2)(4).
- (3) The inequality $2\alpha_1 < 1 + \alpha_2$ (in (H2)(5)) holds automatically, when $\alpha_1 \in [0, 1/2)$.
- (4) Condition (H3)(2) is stronger than the corresponding one in [18, (2.14)]:

$$\|f(u)\| \leq M \left(\|u\|^{1+p} + \|A^{1/2}u\|^{1+\beta} \right), \quad \forall u \in H_1 \text{ with } \|u\|_{H_1} \leq \xi, \quad (2.5)$$

where M, ξ, β are positive constants. It remains open whether condition (H3)(2) in Theorem 2.3 can be improved as (2.5).

3. PRELIMINARY RESULTS AND PROOFS

Throughout this section, $\bar{C}, \tilde{C}, C_1, C_2, \dots, C_{17}$ denote positive constants depending on the values of $E(0)$ and being bounded on bounded sets of the values, and they may be different at different positions.

From (2.2),

$$\frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\|A^{1/2}u(t)\|^2 \leq E(t) \leq E(0),$$

for strong solutions (and so for mild solution as well, by a density argument). Thus, from (H2)(2) we have

$$\|u\|^{p+2} \leq \bar{C}(E(0))E(t), \quad \forall u \in H_1, t \geq 0, \quad (3.1)$$

for some positive function \bar{C} on R_+ that is bounded on bounded sets.

Now, we establish an important lemma with respect to a perturbed energy functional $E_{\varepsilon, \eta}(t)$ (see below), which is useful in the proof of Theorem 2.2. The functional with $\eta(t) = 1$ was employed in [18] to establish the upper estimate of the energy when $\gamma(t) = 1$.

Lemma 3.1. *Assume (H1) and (H2)(1). Let $u(t)$ be a strong solution of (1.1), and let $\eta \in W_{\text{loc}}^{1, \infty}(R^+)$ be a positive function, $\varepsilon \in (0, 1)$, and*

$$r := \frac{p}{p+2}.$$

Define

$$E_{\varepsilon, \eta}(t) := E(t) + \varepsilon\eta(t)[E(t)]^r \langle u'(t), u(t) \rangle.$$

Then

$$E'_{\varepsilon, \eta}(t) \leq -\frac{1}{2}\gamma(t)\|u'(t)\|^2 + \varepsilon\tilde{C}(E(0))V_1(t)\eta(t)\|u'(t)\|^2 + \varepsilon[\varepsilon\tilde{C}(E(0))V_2(t) - \tilde{c}]\eta(t)[E(t)]^{r+1}, \quad t \geq 0,$$

with some $\tilde{c} \in (0, 1)$ and some positive function \tilde{C} that are bounded on bounded sets. Here,

$$V_1(t) := [E(t)]^{\frac{p}{2p+4}}\gamma(t) + [E(t)]^r,$$

$$V_2(t) := \gamma(t)\eta(t) \left[1 + \left(\frac{\eta'(t)}{\eta(t)\gamma(t)} \right)^2 \right]. \quad (3.2)$$

Proof. The time-derivative of $E_{\varepsilon,\eta}(t)$ is

$$\begin{aligned} E'_{\varepsilon,\eta}(t) &= -\gamma(t)\|u'\|^2 - \varepsilon r \eta(t)[E(t)]^{r-1} \gamma(t) \langle u', u \rangle \|u'\|^2 + \varepsilon \eta(t)[E(t)]^r \|u'\|^2 \\ &\quad - \varepsilon \eta(t)[E(t)]^r \gamma(t) \langle u', u \rangle - \varepsilon \eta(t)[E(t)]^r \left(\|A^{1/2}u(t)\|^2 + \langle f(u), u \rangle \right) \\ &\quad + \varepsilon \eta'(t)[E(t)]^r \langle u', u \rangle \\ &=: I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \end{aligned}$$

We first estimate I_2 . By (3.1) and the Cauchy-Schwarz inequality we obtain

$$r[E(t)]^{r-1} \langle u', u \rangle \leq r[E(t)]^{r-1} \|u'\| \|u\| \leq C_1 [E(t)]^{r-1+\frac{1}{2}+\frac{1}{p+2}} = C_1 [E(t)]^{\frac{p}{2p+4}},$$

with a constant $C_1 > 0$. Hence

$$I_2 \leq C_1 \varepsilon \eta(t) [E(t)]^{\frac{p}{2p+4}} \gamma(t) \|u'\|^2. \quad (3.3)$$

For I_4 , we use Young's inequality, as well as the Cauchy-Schwarz inequality and (3.1) to infer that

$$\begin{aligned} \varepsilon \eta(t) [E(t)]^r \gamma(t) \langle u', u \rangle &\leq \frac{1}{4} \gamma(t) \|u'\|^2 + \gamma(t) \varepsilon^2 \eta^2(t) [E(t)]^{2r} \|u\|^2 \\ &\leq \frac{1}{4} \gamma(t) \|u'\|^2 + C_2 \varepsilon^2 \gamma(t) \eta^2(t) [E(t)]^{2r+\frac{2}{p+2}}. \end{aligned}$$

Since $2r + \frac{2}{p+2} = r + 1$, this means that

$$I_4 \leq \frac{1}{4} \gamma(t) \|u'\|^2 + C_2 \varepsilon^2 \gamma(t) \eta^2(t) [E(t)]^{r+1}. \quad (3.4)$$

Similarly, we have

$$I_6 \leq \frac{1}{4} \gamma(t) \|u'\|^2 + C_3 \varepsilon^2 \gamma(t)^{-1} [\eta'(t)]^2 [E(t)]^{r+1}. \quad (3.5)$$

Moreover, in view of (H2)(1) and (2.1) we deduce that

$$\begin{aligned} [E(t)]^r \left(\|A^{1/2}u(t)\|^2 + \langle f(u), u \rangle \right) &\geq c_1 [E(t)]^r \left(\frac{1}{2} \|A^{1/2}u(t)\|^2 + F(u) \right) \\ &= c_1 [E(t)]^r \left(E(t) - \frac{1}{2} \|u'\|^2 \right) \\ &= c_1 [E(t)]^{r+1} - \frac{c_1}{2} [E(t)]^r \|u'\|^2, \end{aligned}$$

with some constant $c_1 > 0$, so that

$$I_5 \leq -c_1 \varepsilon \eta(t) [E(t)]^{r+1} + \frac{c_1}{2} \varepsilon [E(t)]^r \eta(t) \|u'\|^2. \quad (3.6)$$

Finally, combining (3.3)–(3.6) together we have

$$\begin{aligned} E'_{\varepsilon,\eta}(t) &\leq -\frac{1}{2}\gamma(t)\|u'\|^2 + C_1\varepsilon\eta(t)[E(t)]^{\frac{p}{2p+4}}\gamma(t)\|u'\|^2 + \frac{C_1}{2}\varepsilon\eta(t)[E(t)]^r\|u'\|^2 \\ &\quad + C_2\varepsilon^2\gamma(t)\eta^2(t)[E(t)]^{r+1} - c_1\varepsilon\eta(t)[E(t)]^{r+1} + C_3\varepsilon^2\gamma(t)^{-1}[\eta'(t)]^2[E(t)]^{r+1}. \end{aligned}$$

This gives the estimate of $E'_{\varepsilon,\eta}(t)$ as required. \square

Next, we derive a preliminary upper estimate of energy, by an application of Lemma 3.1.

Proposition 3.2. *Let Assumptions (H1), (H2)(1), and (H2)(3) hold. Then*

$$E(t) \leq M_1(E(0)) \left(1 + \int_0^t \gamma(\tau) d\tau\right)^{-\frac{p+2}{p}}, \quad \forall t \geq 0,$$

for some positive function M_1 on R_+ that is bounded on bounded sets.

Proof. It suffices to show the estimate for strong solutions, by a density argument.

First we consider the case $\alpha_1 = \alpha_2 = \alpha$ and take $\eta(t) = R_1(1+t)^{-\alpha}$. By (1.2) we see that

$$\left| \frac{\eta'(t)}{\eta(t)\gamma(t)} \right| \leq \frac{\alpha}{R_1}(1+t)^{\alpha-1}.$$

Thus, for V_1 and V_2 in (3.2) we have

$$V_1(t) \leq C_4, \quad V_2(t) \leq C_4, \quad t \geq 0,$$

with a constant $C_4 > 0$. Accordingly,

$$E'_{\varepsilon,\gamma}(t) \leq -\gamma(t) \left(\frac{1}{2} - \varepsilon\tilde{C}C_4 \right) \|u'\|^2 + \varepsilon R_1(\varepsilon\tilde{C}C_4 - \tilde{c})(1+t)^{-\alpha}[E(t)]^{1+r} \quad (3.7)$$

by virtue of Lemma 3.1. On the other hand, using the Cauchy-Schwarz inequality, (1.2) and (3.1), we obtain

$$\begin{aligned} \varepsilon R_1(1+t)^{-\alpha}[E(t)]^r |\langle u', u \rangle| &\leq \varepsilon R_1[E(t)]^r \|u'\| \|u\| \\ &\leq C_5\varepsilon[E(t)]^{r+\frac{1}{2}+\frac{1}{p+2}} = C_5\varepsilon[E(t)]^{1+\frac{p}{2(p+2)}} \\ &\leq C_6\varepsilon E(t). \end{aligned} \quad (3.8)$$

Take $\varepsilon = \tilde{c}(2\tilde{C}C_4 + 2C_6 + \tilde{c})^{-1}$. Then $\varepsilon \in (0, 1)$, and

$$\varepsilon\tilde{C}C_4 \leq c/2 < 1/2, \quad C_6\varepsilon \leq 1/2.$$

Thus, it follows from (3.8) and (3.7) that

$$\frac{1}{2}E(t) \leq E_{\varepsilon,\gamma}(t) \leq 2E(t), \quad \forall t \geq 0,$$

and

$$E'_{\varepsilon,\gamma}(t) \leq -\frac{1}{2}\tilde{c}R_1\varepsilon(1+t)^{-\alpha}[E(t)]^{1+r} \leq -\left(\frac{1}{2}\right)^{2+r} \tilde{c}R_1\varepsilon(1+t)^{-\alpha}[E_{\varepsilon,\gamma}(t)]^{1+r}.$$

Therefore,

$$E(t) \leq 2E_{\varepsilon, \gamma}(t) \leq C_7 \left(1 + \int_0^t (1 + \tau)^{-\alpha} d\tau\right)^{-\frac{1}{\tau}} \leq C_7 \left(1 + \int_0^t \gamma(\tau) d\tau\right)^{-\frac{1}{\tau}}. \quad (3.9)$$

When $\alpha_1 \neq \alpha_2$, by (H2)(3) we derive

$$|\gamma'(t)| \leq C_0 \gamma^2(t), \quad t \geq 0.$$

Therefore, taking $\eta(t) = \gamma(t)$, we see that

$$\left| \frac{\eta'(t)}{\eta(t)\gamma(t)} \right| = \left| \frac{\gamma'(t)}{\gamma(t)^2} \right|$$

is also bounded. Then, arguing similarly as in the paragraph above, and using (1.2) whenever in need, we still obtain the energy estimate as in (3.9).

The proof is complete. \square

Finally, we make an improvement over the upper estimate of energy in Proposition 3.2, which will play a key role for the case $\alpha_1 > 1/2$, by following the way as in Section 2 of [31] (see also [7], Sect. 3.1) with effective adaptations for the present situation.

Proposition 3.3. *Suppose $1/2 < \alpha_1 < 1$. Let Assumptions (H1) and (H2)(1)-(4) hold. Then*

$$E(t) \leq M_2(E(0)) \left(\frac{1}{1+t} \right)^{(\alpha_1 + \alpha_2) \cdot \frac{p+2}{p}}, \quad \forall t \geq 0,$$

for some positive function M_2 that are bounded on bounded sets.

Proof. First we write

$$k_0 := (1 - \alpha_1) \cdot \frac{p+2}{p}, \quad k_* := (\alpha_1 + \alpha_2) \frac{p+2}{p}.$$

It is clear that $k_0 < k_*$. Applying Proposition 3.2 enables us to obtain

$$E(t) \leq C_8 \left(\frac{1}{1 + \int_0^t \gamma(\tau) d\tau} \right)^{\frac{p+2}{p}} \leq C_8 (1+t)^{-k_0}, \quad \forall t \geq 0, \quad (3.10)$$

by (1.2).

Consider the function $q(t) := \frac{1}{2} \|u(t)\|^2$. Using (H2)(1) and (1.1), we have

$$\begin{aligned} q''(t) + \gamma(t)q'(t) &= \|u'\|^2 - \|A^{1/2}u\|^2 - \langle f(u), u \rangle \\ &\leq \|u'\|^2 - \min\{2, c\} \left(\frac{1}{2} \|A^{1/2}u\|^2 + F(u) \right) \\ &\leq 2\|u'\|^2 - \min\{2, c\}E(t). \end{aligned}$$

Therefore,

$$\min\{2, c\} \int_0^T \lambda_k(t)E(t)dt \leq 2 \int_0^T \lambda_k(t)\|u'(t)\|^2 dt - \int_0^T \lambda_k(t)q''(t)dt - \int_0^T \lambda_k(t)\gamma(t)q'(t)dt,$$

where

$$T > 0, \quad \lambda_k(t) := (1+t)^k \text{ with } k \in [k_0 - 1, k_*].$$

Integrating by parts yields that

$$\begin{aligned} \min\{2, c\} \int_0^T \lambda_k(t)E(t)dt &\leq 2 \int_0^T \lambda_k\|u'(t)\|^2 dt - \lambda_k(T)q'(T) + [\lambda'_k - (\gamma\lambda_k)](T)q(T) \\ &\quad + \int_0^T [(\lambda_k\gamma)' - \lambda''_k](t)q(t)dt + q'(0) + (\gamma(0) - k)q(0). \end{aligned}$$

Notice

$$\lambda'_k(t) = k(1+t)^{k-1} \quad \text{and} \quad k \leq k_* < \frac{2p+4}{p}.$$

We see

$$\lambda'_k(T) - (\gamma\lambda_k)(T) \leq -\frac{1}{2}(\gamma\lambda_k)(T), \quad \forall T \geq T_0,$$

where

$$T_0 := \left(\frac{4p+8}{pR_1} \right)^{1/(1-\alpha_1)}.$$

Also,

$$|q'(t)| \leq \|u'(t)\|\|u(t)\| \leq 2\sqrt{E(t)}\sqrt{q(t)}.$$

Therefore,

$$\begin{aligned} & -\lambda_k(T)q'(T) + [\lambda'_k - (\gamma\lambda_k)](T)q(T) \\ & \leq \left[2(\lambda_k\sqrt{E})(T) \right] \sqrt{q(T)} - \left[\frac{1}{2}(\lambda_k\gamma)(T) \right] \left(\sqrt{q(T)} \right)^2 \\ & \leq \frac{\left[2(\lambda_k\sqrt{E})(T) \right]^2}{2(\lambda_k\gamma)(T)} \quad \left(\text{the maximum of the above formal function of } \sqrt{q(T)} \right) \\ & \leq \frac{2}{R_1} \lambda_{k+\alpha_1}(T)E(T) \quad (\text{by (1.2)}), \quad \forall T \geq T_0. \end{aligned}$$

Accordingly, we infer, for $T \geq T_0$,

$$\int_0^T \lambda_k(t)E(t)dt \leq \bar{C}_0 \int_0^T \lambda_k(t)\|u'(t)\|^2 dt + \bar{C}_0 \lambda_{k+\alpha_1}(T)E(T) + \bar{C}_0 \int_0^T [(\lambda_k\gamma)' - \lambda''_k](t)q(t)dt + C_9$$

$$:= \bar{C}_0(J_1 + J_2 + J_3) + C_9, \quad \forall T \geq T_0, \quad (3.11)$$

with constants $\bar{C}_0, C_9 > 0$.

From (1.2) and (3.10) it follows that

$$\begin{aligned} J_1 &= \int_0^T \frac{\lambda_k}{\gamma(t)} \gamma(t) \|u'(t)\|^2 dt \leq \frac{1}{R_1} \int_0^T \lambda_{k+\alpha_1}(-E') dt \\ &\leq -\frac{1}{R_1} \lambda_{k+\alpha_1}(T) E(T) + \frac{E(0)}{R_1} + \frac{k+\alpha_1}{R_1} \int_0^T \lambda_{k+\alpha_1-1}(t) E(t) dt \\ &\leq \frac{E(0)}{R_1} + \frac{C_8(k+\alpha_1)}{R_1} \int_0^T \lambda_{k+\alpha_1-1-k_0}(t) dt. \end{aligned} \quad (3.12)$$

Also using (3.10) gives

$$J_2 \leq \frac{2C_8}{R_1} \lambda_{k+\alpha_1-k_0}(T). \quad (3.13)$$

As for J_3 , we observe

$$(\lambda_k \gamma)'(t) = \lambda'_k(t) \gamma(t) + \lambda_k(t) \gamma'(t) \leq (k+C) R_2 (1+t)^{k-1} (1+t)^{-\alpha_2},$$

by (1.2) and (H2)(4),

$$\lambda''_k(t) = k(k-1)(1+t)^{k-2},$$

and

$$q(t) = \frac{1}{2} \|u(t)\|^2 \leq C_{10} [E(t)]^{\frac{2}{p+2}} \leq C_{11} (1+t)^{-\frac{2k_0}{p+2}},$$

by (3.10) again. Accordingly,

$$J_3 \leq C_{12} \int_0^T (1+t)^{k-\alpha_2-1-\frac{2k_0}{p+2}} dt. \quad (3.14)$$

Take $k = k_0 - \alpha_1 - \epsilon$, where ϵ is a fixed positive number satisfying $1 - \alpha_1 - \epsilon > 0$. Noticing

$$\tilde{k} - \alpha_1 \leq \frac{2\tilde{k}}{p+2} + \alpha_2, \quad \text{whenever } \tilde{k} \in [k_0, k_*],$$

we see that

$$\begin{cases} k < k_0 - \alpha_1, \\ k < \frac{2k_0}{p+2} + \alpha_2. \end{cases}$$

Accordingly, combining (3.11) with (3.12)–(3.14) yields that

$$\int_0^{+\infty} \lambda_k E(t) dt \leq C_{13}.$$

Since $E(t)$ is decreasing, we see

$$E(t) \int_{t/2}^t \lambda_k(s) ds \leq \int_{t/2}^t \lambda_k E(s) ds.$$

Hence,

$$E(t) \leq C_{14} \left(\frac{1}{1+t} \right)^{k+1} = C_{14} \left(\frac{1}{1+t} \right)^{k_1}, \quad t \geq 0, \quad (3.15)$$

where

$$k_1 := k_0 + 1 - \alpha_1 - \epsilon.$$

If $k_1 \geq k_*$, then we have obtained the required energy estimate. If not, then

$$E(t) \leq C_{15} \left(\frac{1}{1+t} \right)^{k_2}, \quad t \geq 0,$$

where

$$k_2 := k_1 + 1 - \alpha_1 - \epsilon = k_0 + 2(1 - \alpha_1 - \epsilon);$$

this is because (3.12)–(3.14), with k_0 replaced by k_1 , are satisfied, by using (3.15) (instead of (3.10)). Thus, proceeding like this and denoting by n the positive integer satisfying

$$\begin{cases} k_0 + n(1 - \alpha_1 - \epsilon) < k_*, \\ k_n := k_0 + (n + 1)(1 - \alpha_1 - \epsilon) \geq k_*, \end{cases}$$

we obtain

$$E(t) \leq C_{16} \left(\frac{1}{1+t} \right)^{k_n} \leq C_{17} \left(\frac{1}{1+t} \right)^{k_*}, \quad t \geq 0.$$

This ends the proof. □

4. PROOFS OF THE THEOREMS

Throughout the section, $\bar{C}_1, \bar{C}_2, \dots, \bar{C}_{15}$ denote positive constants depending on the values of $E(0)$ and being bounded on bounded sets of the values, and they may be different at different positions.

4.1. Proof of Theorem 2.2

It suffices to deal with the strong solution case.

We divide the proof into three steps.

Step 1. Obtain a better estimate of energy for case $\alpha_1 \leq 1/2$.

Take $\eta(t) = 1$ in Lemma 3.1. Then $V_2(t) = \gamma(t)$ is bounded. For $V_1(t)$ we exploit Proposition 3.2 to deduce that

$$E(t)^r \leq \bar{C}_1 \left(1 + \int_0^t \gamma(\tau) d\tau\right)^{-\frac{p+2}{p} \cdot r} \leq \bar{C}_1 \left(\frac{1}{1+t}\right)^{1-\alpha_1} \leq \bar{C}_1 \left(\frac{1}{1+t}\right)^{\alpha_1} \leq \bar{C}_1 \gamma(t),$$

since $\alpha_1 \leq 1/2$. Accordingly,

$$V_1(t) \leq \bar{C}_2 \gamma(t).$$

Thus, similarly as in the proof of Proposition 3.2, we can let ε small enough such that

$$\frac{1}{2}E(t) \leq E_{\varepsilon,1}(t) \leq 2E(t), \quad \forall t \geq 0,$$

and

$$E'_{\varepsilon,1}(t) \leq -\frac{1}{2}\tilde{c}\varepsilon[E(t)]^{1+r}.$$

This gives that

$$E(t) \leq \bar{C}_3(1+t)^{-\frac{1}{r}} = \bar{C}_3(1+t)^{-\frac{p+2}{p}}. \quad (4.1)$$

Step 2. Obtain a better estimate of energy for case $\alpha_1 > 1/2$.

Take

$$\eta(t) = (1+t)^{2\alpha_1-1}$$

in Lemma 3.1. Making use of Proposition 3.3 we infer that

$$\eta(t)[E(t)]^{\frac{p}{2p+4}} \leq \bar{C}_4(1+t)^{2\alpha_1-1}(1+t)^{-\frac{\alpha_1+\alpha_2}{2}} \leq \bar{C}_4(1+t)^{2\alpha_1-1}(1+t)^{-\alpha_2} \leq \bar{C}_4, \quad (4.2)$$

by (H2)(5). This yields that

$$\varepsilon\eta(t)[E(t)]^r |\langle u', u \rangle| \leq \varepsilon\eta(t)[E(t)]^r \|u'\| \|u\| \leq \bar{C}_5 \varepsilon \eta(t)[E(t)]^{1+\frac{p}{2p+4}} \leq \bar{C}_5 \varepsilon E(t),$$

which implies that

$$\frac{1}{2}E(t) \leq E_{\varepsilon,\eta}(t) \leq 2E(t), \quad \forall t \geq 0, \quad (4.3)$$

whenever ε is small enough.

Moreover, using Proposition 3.3 again, we obtain

$$\eta(t)[E(t)]^r \leq \bar{C}_6(1+t)^{2\alpha_1-1}(1+t)^{-\alpha_1-\alpha_2} = \bar{C}_6(1+t)^{2\alpha_1-1-\alpha_2}(1+t)^{-\alpha_1} \leq \bar{C}_6 \gamma(t),$$

due to (H2)(5) and (1.2). Furthermore, we have

$$\gamma(t)\eta(t) \leq R_2(1+t)^{-\alpha_2}(1+t)^{2\alpha_1-1} \leq R_2,$$

and

$$\eta'(t) = (2\alpha_1 - 1)(1+t)^{2\alpha_1-2} \leq \bar{C}_7 \gamma(t) \eta(t),$$

by (1.2). These estimates combined with (4.2) indicate that $V_2(t)$ is bounded, and

$$V_1(t) \eta(t) \leq \bar{C}_8 \gamma(t).$$

Therefore, there exists a sufficiently small ε such that (4.3) is satisfied, and

$$E'_{\varepsilon, \eta}(t) \leq -\frac{1}{2} \tilde{c} \varepsilon \eta(t) [E(t)]^{1+r}.$$

Consequently,

$$E_{\varepsilon, \eta}(t) \leq \bar{C}_9 \left(1 + \int_0^t \eta(\tau) d\tau \right)^{-1/r}.$$

So

$$E(t) \leq \bar{C}_{10} (1+t)^{-2\alpha_1/r} = \bar{C}_{10} (1+t)^{-2\alpha_1 \cdot \frac{p+2}{p}}. \quad (4.4)$$

Step 3. Achieve the desired estimate of energy.

We take

$$\eta(t) = \eta_0(t) := \begin{cases} (1+t)^\alpha, & \text{if } \alpha_1 = \alpha_2 = \alpha, \\ \gamma(t)^{-1}, & \text{if } \alpha_1 \neq \alpha_2 \end{cases}$$

in Lemma 3.1. The boundedness of $V_2(t)$ is easy to see by (H2)(3).

Employing (4.1) and (4.4), we obtain

$$[E(t)]^r \leq \bar{C}_{11} (1+t)^{-2\alpha_1} \leq \bar{C}_{11} \gamma(t)^2$$

by (1.2). This yields that

$$[E(t)]^{\frac{p}{2p+4}} = [E(t)]^{r/2} \leq \bar{C}_{12} \gamma(t).$$

Hence

$$V_1(t) \eta_0(t) = \left([E(t)]^{\frac{p}{2p+4}} \gamma(t) + [E(t)]^r \right) \eta_0(t) \leq \bar{C}_{13} \gamma(t).$$

Moreover,

$$\varepsilon \eta_0(t) [E(t)]^r |\langle u', u \rangle| \leq \bar{C}_{14} \varepsilon \eta_0(t) [E(t)]^{1+\frac{p}{2(p+2)}} \leq \bar{C}_{14} \varepsilon E(t).$$

Accordingly, choosing ε small enough we deduce that

$$\frac{1}{2} E(t) \leq E_{\varepsilon, \eta_0}(t) \leq 2E(t), \quad \forall t \geq 0,$$

$$E'_{\varepsilon, \eta_0}(t) \leq -\frac{1}{2} \tilde{c} \varepsilon \eta_0(t) [E(t)]^{1+r}.$$

Therefore,

$$E(t) \leq \bar{C}_{15} \left(1 + \int_0^t \eta_0(\tau) d\tau \right)^{-1/r} \leq \bar{C}_{15} \left(1 + \int_0^t \gamma(\tau)^{-1} d\tau \right)^{-1/r}.$$

4.2. Proof of Theorem 2.3

We follow the strategy in Section 3.4 of [18] (for the case of $\gamma(t) = 1$) to prove the theorem. It is worth noting that exploitations of the (already obtained) fine upper bound estimates will play an important role.

Let $u(t)$ be a mild solution of (1.1) with $u_0 \neq 0$. Set

$$\gamma_1 = \frac{R_2^2(C_0^2 + 1)}{\|u_0\|^{2p+2}} \left(\|u_1\|^2 + \|A^{1/2}u_0\|^2 \right) + \frac{129C_0^2M^2}{\delta^2}, \quad (4.5)$$

where R_2 is as in (1.2), C_0 as in (H3)(3), M as in (H3)(2), and $\delta \in (0, 1/2)$ is a constant that will be determined later; set

$$G(t) := \frac{\|u'(t)\|^2 + \|A^{1/2}u(t)\|^2}{2\|u(t)\|^{2p+2}}. \quad (4.6)$$

We will show the existence of a nonempty open set $S \subset (H_1 \setminus \{0\}) \times H$ such that for $(u_0, u_1) \in S$, the solution $u(t)$ of problem (1.1) satisfies the property: given $T > 0$, one has

$$u(T) \neq 0 \quad \text{and} \quad G(T) < \frac{2\gamma_1}{\gamma^2(T)}, \quad (4.7)$$

whenever

$$u(t) \neq 0 \quad \text{and} \quad G(t) \leq \frac{2\gamma_1}{\gamma^2(t)} \quad \text{for all } t \in [0, T]. \quad (4.8)$$

Next, we assume (4.8). We will find such a set S , and with it prove (4.7) (for strong solutions), by seven steps.

Step 1. Construct a small perturbation of $G(t)$.

Denoting by Q the orthogonal projection from H to $(\ker A)^\perp$, we set

$$\hat{G}(t) = G(t) + \delta \gamma(t) \frac{\langle u'(t), Qu(t) \rangle}{\|u(t)\|^{2p+2}}, \quad t \in [0, T], \quad (4.9)$$

for the case of $\alpha_1 \neq \alpha_2$; replace $\gamma(t)$ by $(1+t)^{-\alpha}$ in (4.9) for the case of $\alpha_1 = \alpha_2 = \alpha$. Below we only address the former case (the latter case can be dealt with similarly). We have

$$\begin{aligned} \hat{G}'(t) &= -\gamma(t) \frac{\|u'\|^2}{\|u\|^{2p+2}} - \delta \gamma(t) \frac{\|A^{1/2}u\|^2}{\|u\|^{2p+2}} + \delta \gamma(t) \frac{\|Qu'\|^2 - \gamma(t) \langle u', Qu \rangle}{\|u(t)\|^{2p+2}} \\ &\quad - \frac{\langle f(u), u' + \delta \gamma(t) Qu \rangle}{\|u\|^{2p+2}} - (2p+2) \frac{\langle u', u \rangle}{\|u\|^2} \cdot \hat{G}(t) + \delta \gamma'(t) \frac{\langle u', Qu \rangle}{\|u\|^{2p+2}} \\ &=: K_1 + K_2 + K_3 + K_4 + K_5 + K_6, \quad t \in [0, T]. \end{aligned} \quad (4.10)$$

Step 2. Estimate K_3 and K_6 .

It follows from (1.2) and (H3)(1) that

$$\gamma(t)\langle u', Qu \rangle \leq \frac{R_2}{\sqrt{\zeta}} \|u'\| \sqrt{\zeta} \|Qu\| \leq \frac{R_2}{\sqrt{\zeta}} \|u'\| \|A^{1/2}u\| \leq \frac{2R_2^2}{\zeta} \|u'\|^2 + \frac{1}{8} \|A^{1/2}u\|^2,$$

noting that $\|A^{1/2}Qu\| = \|A^{1/2}u\|$. Hence, we have

$$K_3 \leq \tilde{D}_1 \delta \gamma(t) \frac{\|u'\|^2}{\|u\|^{2p+2}} + \frac{1}{8} \delta \gamma(t) \frac{\|A^{1/2}u\|^2}{\|u\|^{2p+2}} \leq \tilde{D}_1 \delta \gamma(t) \frac{\|u'\|^2}{\|u\|^{2p+2}} + \frac{1}{4} \delta \gamma(t) G(t), \quad t \in [0, T], \quad (4.11)$$

where $\tilde{D}_1 := (2R_2^2 + \zeta)/\zeta$.

As for K_6 , we use (H2)(3) to get

$$|\gamma'(t)| \leq C\gamma^2(t) \leq CR_2\gamma(t), \quad \forall t \geq 0.$$

Also,

$$|\langle u', Qu \rangle| \leq \frac{1}{\sqrt{\zeta}} \|u'\| \sqrt{\zeta} \|Qu\| \leq \frac{1}{\sqrt{\zeta}} \|u'\| \|A^{1/2}u\| \leq \frac{2CR_2}{\zeta} \|u'\|^2 + \frac{1}{8CR_2} \|A^{1/2}u\|^2.$$

Accordingly,

$$K_6 \leq \tilde{D}_2 \delta \gamma(t) \frac{\|u'\|^2}{\|u\|^{2p+2}} + \frac{\delta \gamma(t)}{4} G(t), \quad t \in [0, T], \quad (4.12)$$

where $\tilde{D}_2 := 2C^2R_2^2/\zeta$.

Step 3. Estimate K_4 and K_5 in the case $\alpha < 1/2$.

From now on, we assume

$$E(0) \leq 1, \text{ and } \left(\sup_{s \in [0,1]} \bar{C}(s) + 2 \right) [E(0)]^{\frac{2}{p+2}} \leq \xi^2, \quad (4.13)$$

where \bar{C} and ξ are, respectively, as in (3.1) and (H3)(2). Then

$$\begin{aligned} \|u(t)\|_{H_1}^2 &= \|u(t)\|^2 + \|A^{1/2}u(t)\|^2 \leq \left(\sup_{s \in [0,1]} \bar{C}(s) \right) [E(t)]^{\frac{2}{p+2}} + 2E(t) \\ &\leq \left(\sup_{s \in [0,1]} \bar{C}(s) + 2 \right) [E(0)]^{\frac{2}{p+2}} \leq \xi^2. \end{aligned}$$

Thus, in view of (2.1), (4.1), (3.1), (1.2) and (H3)(2), we deduce that

$$\begin{aligned} \frac{\|f(u)\|}{\|u\|^{p+1}} &\leq M \left(\frac{\|u\|^{p+1} + \|A^{1/2}u\|^{p+1}}{\|u\|^{p+1}} \right) \\ &\leq M \left(1 + \frac{\|A^{1/2}u\|}{\|u\|^{p+1}} \cdot [2E(t)]^{p/2} \right) \\ &\leq M \left(1 + \sqrt{2G(t)} \cdot [2E(t)]^{p/4} [2E(0)]^{p/4} \right) \end{aligned}$$

$$\begin{aligned} &\leq M \left(1 + D_1 \sqrt{G(t)} \cdot (1+t)^{-\frac{p+2}{4}} [E(0)]^{p/4} \right) \\ &\leq M \left(1 + D_2 [E(0)]^{p/4} \gamma(t) \sqrt{G(t)} \right), \quad t \in [0, T], \end{aligned}$$

noting

$$\frac{p+2}{4} > \frac{1}{2} > \alpha_1.$$

Here, D_1, D_2 are positive constants independent of initial data (because of $E(0) \leq 1$). Also, we have

$$\delta\gamma(t)\|Qu\| \leq \frac{\delta R_2}{\sqrt{\zeta}} \|A^{1/2}u\|,$$

by (H3)(1). Hence,

$$\frac{\|u_t\| + \delta\gamma(t)\|Qu\|}{\|u\|^{p+1}} \leq \frac{\|u_t\| + R_2/\sqrt{\zeta}\|A^{1/2}u\|}{\|u\|^{p+1}} \leq D_3 \sqrt{G(t)}, \quad t \in [0, T],$$

where $D_3 := 2(1 + R_2/\sqrt{\zeta})$. Thus,

$$K_4 \leq \frac{f(u)}{\|u\|^{p+1}} \cdot \frac{\|u_t\| + \delta\gamma(t)\|Qu\|}{\|u\|^{p+1}} \leq D_3 M \sqrt{G(t)} + M D_2 D_3 [E(0)]^{p/4} \gamma(t) G(t).$$

So

$$K_4 \leq \frac{4M^2}{\delta\gamma(t)} + \frac{\delta\gamma(t)}{4} G(t) + \tilde{D}_3 [E(0)]^{p/4} \gamma(t) G(t), \quad t \in [0, T]. \quad (4.14)$$

where \tilde{D}_3 is a positive constant independent of initial data.

For K_5 , we note that

$$\exists \theta > 0 \text{ such that } \alpha_1 + \theta = 1/2,$$

since $\alpha_1 < 1/2$. From the definition of $G(t)$, (3.1), (4.8) and (4.1), it follows that for $t \in [0, T]$,

$$\begin{aligned} \left| \frac{\langle u', u \rangle}{\|u\|^2} \right| &\leq \frac{\|u'\|}{\|u\|^{1+p}} \cdot \|u\|^p \\ &\leq \sqrt{2G(t)} \cdot \|u\|^{2p\alpha_1} \|u\|^{2p\theta} \\ &\leq \frac{\sqrt{\gamma_1}}{\gamma(t)} \cdot D_4 (1+t)^{-2\alpha_1} [E(0)]^{\frac{2p\theta}{p+2}} \\ &\leq D_5 \sqrt{\gamma_1} \gamma(t) [E(0)]^{\frac{2p\theta}{p+2}}, \end{aligned}$$

with positive constants D_4, D_5 independent of initial data. Therefore

$$K_5 \leq \tilde{D}_4 [E(0)]^{\frac{p\theta'}{p+2}} \sqrt{\gamma_1} \gamma(t) \cdot \hat{G}(t), \quad t \in [0, T], \quad (4.15)$$

for some $\theta' > 0$, and some positive constant \tilde{D}_4 independent of initial data.

Step 4. Estimate K_4 and K_5 in the case $\alpha \geq 1/2$.

In view of (2.1), (4.4) and (H3)(2), we obtain

$$\begin{aligned} \frac{\|f(u)\|}{\|u\|^{p+1}} &\leq M \left(1 + \frac{\|A^{1/2}u\|}{\|u\|^{p+1}} \cdot [2E(t)]^{p/4} [2E(0)]^{p/4} \right) \\ &\leq M \left(1 + D_6 \sqrt{G(t)} \cdot (1+t)^{-\alpha_1 \frac{p+2}{2}} [E(0)]^{p/4} \right) \\ &\leq M \left(1 + D_7 E(0) \right)^{p/4} \gamma(t) \sqrt{G(t)}, \quad t \in [0, T], \end{aligned}$$

with positive constants D_6, D_7 independent of initial data. This means that the estimate (4.14) for K_4 holds too in this case.

By virtue of Theorem 2.2, we get

$$\|u(t)\| \leq D_8 \left(1 + \int_0^t \gamma(\tau)^{-1} d\tau \right)^{-\frac{1}{p}} \leq D_9 (1+t)^{-\frac{1+\alpha_2}{p}},$$

where D_8, D_9 are positive constants independent of initial data. In addition,

$$\exists \theta_1 \in (0, 1) \text{ such that } 2\alpha_1 = \theta_1(1 + \alpha_2)$$

by (H2)(5). Hence

$$\begin{aligned} \left| \frac{\langle u', u \rangle}{\|u\|^2} \right| &\leq \sqrt{2G(t)} \cdot \|u\|^{p\theta_1} \|u\|^{(1-\theta_1)p} \\ &\leq \frac{\sqrt{\gamma_1}}{\gamma(t)} \cdot D_{10} (1+t)^{-\theta_1(1+\alpha_2)} \cdot [E(0)]^{(1-\theta_1)\frac{p}{p+2}} \\ &= \frac{\sqrt{\gamma_1}}{\gamma(t)} \cdot D_{10} (1+t)^{-2\alpha_1} [E(0)]^{(1-\theta_1)\frac{p}{p+2}} \\ &\leq D_{11} \sqrt{\gamma_1} \gamma(t) [E(0)]^{\frac{(1-\theta_1)p}{p+2}}, \quad t \in [0, T], \end{aligned}$$

where D_{10}, D_{11} are positive constants independent of initial data. So the estimate (4.15) for K_5 holds too in this case.

Step 5. Estimate \hat{G}' .

Plugging (4.11)–(4.15) into (4.10), we obtain

$$\begin{aligned} \hat{G}'(t) &\leq - \left(1 - \tilde{D}_1 \delta - \tilde{D}_2 \delta \right) \gamma(t) \frac{\|u'\|^2}{\|u\|^{2p+2}} - \delta \gamma(t) \frac{\|A^{1/2}u\|^2}{\|u\|^{2p+2}} + \frac{4M^2}{\delta \gamma(t)} \\ &\quad + \left(3\delta/4 + \tilde{D}_3 [E(0)]^{p/4} \right) \gamma(t) G(t) + \tilde{D}_4 \sqrt{\gamma_1} [E(0)]^{\frac{p\theta_1'}{p+2}} \cdot \gamma(t) \hat{G}(t), \quad t \in [0, T]. \end{aligned}$$

From (H3)(1) and (1.2), we know that

$$\gamma(t) \frac{\langle u', Qu \rangle}{\|u\|^{2p+2}} \leq 2 (R_2^2 \zeta^{-1} + 1/4) G(t), \quad t \in [0, T].$$

Now, we choose $\delta \in (0, 1/2)$ such that

$$\begin{cases} (R_2^2 \zeta^{-1} + 1/4) \delta \leq 1/10, \\ (\tilde{D}_1 + \tilde{D}_2) \delta \leq 1/2; \end{cases}$$

then, choose a set S of initial data satisfying (4.13) and

$$\begin{cases} \tilde{D}_3 [E(0)]^{p/4} \leq \delta/32, \\ \tilde{D}_4 \sqrt{\gamma_1} [E(0)]^{\frac{p\theta'}{p+2}} \leq \delta/32. \end{cases} \quad (4.16)$$

We here emphasize that \tilde{D}_3, \tilde{D}_4 are independent of all the initial data satisfying $E(0) \leq 1$. Thus,

$$\frac{5}{6}G(t) \leq \hat{G}(t) \leq 2G(t), \quad t \in [0, T],$$

and

$$\begin{aligned} \hat{G}'(t) &\leq -\delta\gamma(t) \left(\frac{\|u'\|^2}{\|u\|^{2p+2}} + \frac{\|A^{1/2}u\|^2}{\|u\|^{2p+2}} \right) + \frac{4M^2}{\delta\gamma(t)} + \frac{25}{32}\delta\gamma(t)G(t) + \frac{1}{32}\delta\gamma(t)\hat{G}(t) \\ &\leq -\delta\gamma(t)\hat{G}(t) + \frac{4M^2}{\delta\gamma(t)} + \frac{15}{16}\delta\gamma(t)\hat{G}(t) + \frac{1}{32}\delta\gamma(t)\hat{G}(t), \quad t \in [0, T], \end{aligned}$$

by noting

$$\begin{aligned} & - \left(1 - \tilde{D}_1\delta - \tilde{D}_2\delta \right) \gamma(t) \frac{\|u'\|^2}{\|u\|^{2p+2}} - \delta\gamma(t) \frac{\|A^{1/2}u\|^2}{\|u\|^{2p+2}} \\ & \leq - (1 - 1/2)\gamma(t) \frac{\|u'\|^2}{\|u\|^{2p+2}} - \delta\gamma(t) \frac{\|A^{1/2}u\|^2}{\|u\|^{2p+2}} \\ & \leq - \min\{1/2, \delta\}\gamma(t) \left(\frac{\|u'\|^2}{\|u\|^{2p+2}} + \frac{\|A^{1/2}u\|^2}{\|u\|^{2p+2}} \right), \end{aligned}$$

and $\delta \in (0, 1/2)$. Hence,

$$\hat{G}'(t) \leq -\frac{1}{32}\delta\gamma(t)\hat{G}(t) + \frac{4M^2}{\delta\gamma(t)}, \quad t \in [0, T]. \quad (4.17)$$

Step 6. Estimate \hat{G} .

Integrating (4.17) and setting

$$H(t) := \exp\left(\frac{\delta}{32} \int_0^t \gamma(\tau) d\tau\right), \quad t \geq 0,$$

we deduce that

$$\hat{G}(t) \leq \left(\hat{G}(0) + \frac{4M^2}{\delta} \int_0^t \gamma(s)^{-1} H(s) ds \right) H(t)^{-1}, \quad t \in [0, T].$$

By (H3)(3), we have

$$\gamma^{-1}(s) \leq C_0^2 \frac{\gamma(s)}{\gamma^2(t)}, \quad \forall t \geq s > 0.$$

Then

$$\hat{G}(t) \leq \hat{G}(0)H(t)^{-1} + \frac{4C_0^2 M^2}{\delta \gamma^2(t)} \frac{\int_0^t \gamma(s)H(s)ds}{H(t)}, \quad t \in [0, T].$$

From

$$H'(t) = \frac{\delta}{32} \gamma(t) \exp\left(\frac{\delta}{32} \int_0^t \gamma(\tau) d\tau\right) = \frac{\delta}{32} \gamma(t) H(t),$$

it follows that

$$\int_0^t \gamma(s)H(s)ds = \frac{32}{\delta} (H(t) - H(0)) \leq \frac{32}{\delta} H(t).$$

Therefore, noting $H(t) \geq 1$ for $t \in [0, T]$, we derive

$$\hat{G}(t) \leq \hat{G}(0) + \frac{128C_0^2 M^2}{\delta^2 \gamma^2(t)}, \quad t \in [0, T]. \quad (4.18)$$

Step 7. Obtain (4.7).

Using (4.8), we have

$$\left| \frac{d}{dt} \|u(t)\|^2 \right| = 2|\langle u'(t), u(t) \rangle| \leq 2\sqrt{2G(t)} \cdot \|u\|^{2+p} \leq \frac{4\sqrt{\gamma_1}}{\gamma(t)} (\|u(t)\|^2)^{1+\frac{p}{2}}, \quad t \in [0, T].$$

So we obtain

$$\|u(t)\|^2 \geq \left(2p\sqrt{\gamma_1} \int_0^t \gamma(s)^{-1} ds + \|u_0\|^{-p} \right)^{-\frac{2}{p}}, \quad t \in [0, T].$$

Thus, we see that $u(T) \neq 0$.

Letting $t \rightarrow T^-$ in (4.18) and using (H3)(3), we get

$$\hat{G}(T) \leq \left(\frac{1}{C_0^2} \cdot C_0^2 \hat{G}(0) + \frac{128C_0^2 M^2}{\delta^2 \gamma^2(T)} \right) \leq \left(\frac{\gamma^2(0)}{\gamma^2(T)} \cdot C_0^2 \hat{G}(0) + \frac{128C_0^2 M^2}{\delta^2 \gamma^2(T)} \right).$$

So

$$G(T) \leq 2\hat{G}(T) \leq \frac{2}{\gamma^2(T)} \left(2R_2^2 C_0^2 \cdot G(0) + \frac{128C_0^2 M^2}{\delta^2} \right) < \frac{2\gamma_1}{\gamma^2(T)}.$$

Therefore, (4.7) is satisfied.

Consequently, for any $(u_0, u_1) \in S$ (given by (4.13) and (4.16)) and any $T > 0$, we have

$$u(T) \neq 0 \quad \text{and} \quad G(T) < \frac{2\gamma_1}{\gamma^2(T)},$$

provided $u(t)$ is a strong solution, and

$$u(t) \neq 0 \quad \text{and} \quad G(t) \leq \frac{2\gamma_1}{\gamma^2(t)}, \quad \text{for all } t \in [0, T]. \quad (4.19)$$

This implies that

$$\sup\{T > 0 : (4.19) \text{ holds}\} = +\infty,$$

due to the continuity of G and the fact that (4.19) is indeed satisfied for some $T > 0$, as can be seen from the estimate

$$\gamma^2(0)G(0) \leq R_2^2 G(0) < \gamma_1/2.$$

Therefore, we obtain

$$u(t) \neq 0 \quad \text{and} \quad G(t) \leq \frac{2\gamma_1}{\gamma^2(t)}, \quad \text{for all } t \geq 0.$$

Accordingly,

$$\left| \frac{d}{dt} \|u(t)\|^2 \right| \leq \frac{4\sqrt{\gamma_1}}{\gamma(t)} (\|u(t)\|^2)^{1+\frac{p}{2}}, \quad t \geq 0,$$

for strong solutions (and so for mild solution as well). Hence,

$$\|u(t)\| \geq \left(2p\sqrt{\gamma_1} \int_0^t \gamma^{-1}(s) ds + \|u_0\|^{-p} \right)^{-\frac{1}{p}}, \quad \text{for all } t \geq 0.$$

This combined with (3.1) gives (2.4). Thus, we complete the proof.

5. APPLICATIONS

In this section, $\Omega \subset R^n$ is a bounded domain with smooth boundary $\partial\Omega$, ν is the unit outward normal on $\partial\Omega$, and $\gamma(t)$ is a decreasing function on R^+ satisfying (1.2) with $\alpha_1 = \alpha_2 = \alpha$.

Example 5.1. We consider the following nonautonomous wave equation with the Neumann boundary condition:

$$\begin{cases} u_{tt}(t, x) - \Delta u(t, x) + \gamma(t)u_t(t, x) + \left(\int_{\Omega} |u(t, x)|^2 dx \right)^{p/2} u(t, x) = 0, & \text{in } [0, +\infty) \times \Omega, \\ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), & x \in \Omega, \\ \frac{\partial u(t, x)}{\partial \nu} = 0, & \text{on } \partial\Omega \times (0, \infty), \end{cases} \quad (5.1)$$

where $p \geq 1$.

Take $H = L^2(\Omega)$, and define operator A by

$$Av(x) = -\Delta v(x), \quad x \in \Omega \text{ a.e.}$$

with

$$v \in D(A) := \left\{ u \in H^2(\Omega); \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \right\}.$$

It is known that A is a self-adjoint nonnegative operator on H , $H_1 := D(A^{1/2}) = H^1(\Omega)$, and (H3)(1) holds.

Moreover, we set

$$f(v)(x) = \left(\int_{\Omega} |v(x)|^2 dx \right)^{p/2} v(x) = \|v\|^p v(x), \quad v \in H_1.$$

Then

$$\|f(v)\| = \|v\|^{1+p}, \quad \forall v \in H_1,$$

and so (H3)(2) is satisfied. Besides (H1)(ii) and (H2)(1)-(2) are also satisfied (*cf.* [28], Exam. 4.2 for details). Therefore, applying Theorems 2.2 and 2.3 (seeing also Rem. 2.4 (1)), we obtain the following conclusions regarding the mild solution $u(t)$ and the energy

$$E_u(t) := \frac{1}{2} \int_{\Omega} (|u_t|^2 + |\nabla u|^2) dx + \frac{1}{p+2} \left(\int_{\Omega} |u(t, x)|^2 dx \right)^{\frac{p+2}{2}}$$

of the problem (5.1).

(i) For some positive function M_1 on R_+ that are bounded on bounded sets,

$$E(t), \|u(t)\|^{p+2} \leq M_1(E(0)) (1+t)^{-(1+\alpha)(p+2)/p}, \quad \forall t \geq 0;$$

(ii) there exists a nonempty open set $S \subset H_1 \times H$ such that for some positive function c_0 depending on $\|u_0\|$, $\|A^{1/2}u_0\|$ and $\|u_1\|$,

$$E(t), \|u(t)\|^{p+2} \geq c_0 (1+t)^{-(1+\alpha)(p+2)/p}, \quad \forall t \geq 0,$$

whenever $(u_0, u_1) \in S$,

Example 5.2. Consider the Dirichlet problem for a nonautonomous wave equation

$$\begin{cases} u_{tt}(t, x) - \Delta u(t, x) - \lambda_1 u(t, x) + \gamma(t)u_t + |u(t, x)|^p u(t, x) = 0, \\ \quad \text{in } [0, +\infty) \times \Omega, \\ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \Omega, \\ u(t, x) = 0, \quad \text{on } \partial\Omega \times (0, \infty), \end{cases} \quad (5.2)$$

where λ_1 is the first eigenvalue of the negative Dirichlet-Laplacian on Ω , $p > 0$ and

$$p \leq 2/(n-2) \quad \text{if } n > 2. \quad (5.3)$$

Take $H = L^2(\Omega)$, and define operator A by

$$Av(x) = -\Delta v(x) - \lambda_1 v(x), \quad x \in \Omega \quad \text{a.e.}$$

with $v \in D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Then we know that A is self-adjoint and nonnegative, (H3)(1) is satisfied, and $H_1 := D(A^{1/2}) = H_0^1(\Omega)$. Set

$$f(v)(x) = |v(x)|^p v(x), \quad v \in H_1.$$

Using (5.3) gives

$$\|f(v)\| = \|v\|_{L^{2p+2}}^{1+p} \leq c_2 \|v\|_{H_1}^{1+p}, \quad \forall v \in H_1,$$

with a constant $c_2 > 0$. So (H3)(2) holds. Moreover, (H1)(ii) and (H2)(1)-(2) hold too (cf. [18, the proof of Thm. 4.1] for details). Accordingly, Theorems 2.2 and 2.3 are applicable to problem (5.2), and so for the mild solution $u(t)$ and the energy

$$E_u(t) := \frac{1}{2} \int_{\Omega} (|u_t|^2 + |\nabla u|^2 - \lambda_1 |u|^2) dx + \frac{1}{p+2} \left(\int_{\Omega} |u(t, x)|^{p+2} dx \right)$$

of the problem (5.2), we have the same conclusions as in Example 5.1.

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