

TOWARDS A MATHEMATICAL THEORY OF BEHAVIORAL SWARMS*

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Abstract. This paper presents a unified mathematical theory of swarms where the dynamics of social behaviors interacts with the mechanical dynamics of self-propelled particles. The term *behavioral swarms* is introduced to characterize the specific object of the theory which is subsequently followed by applications. As concrete examples for our unified approach, we show that several Cucker-Smale type models with internal variables fall down to our framework. The second part of the paper shows how the modeling can be developed, beyond the Cucker-Smale approach. This will be illustrated with the aid of numerical simulations in swarms whose movement strategy is sensitive to individual social behaviors. Finally, the presentation looks ahead to research perspectives.

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1. PLAN OF THE PAPER

The celebrated paper by Cucker and Smale [24] has boosted a huge literature on the modeling, qualitative analysis, and computational applications of the mathematical theory of swarms. Namely, it describes, within a pseudo-Newtonian framework, the collective behaviors of interacting mechanical self-propelled particles. Interactions are generally nonlocal and nonlinearly additive. These produce accelerations, where inertia is hidden in the interaction parameters rather than being explicitly taken into account. The mathematical literature in the field has been reviewed and critically analyzed in [3], specifically treated in Sections 5 and 6 which are devoted to the theory of swarms. This topic has been related to the dynamics of crowds as well as to computational models to treat kinetic and multi-agent systems.

After Cucker-Smale's (C-S)'s seminal work, the (C-S) model has been extensively studied in literature, for example, we refer to [22] and references therein for mathematical aspects of the model. Among them, we briefly discuss some recent works incorporating social behaviors together with mechanical aspect of the model. Original C-S model describes the temporal evolution of the mechanical variables (positions and momentum of C-S particles), whereas in the modeling for the collective dynamics of biological and social complex systems,

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one needs to take into account of internal variables such as temperature, spin and excitation [30, 33, 35] *etc.*, see Section 3.

Recently, in [32], the authors found that the one-dimensional C-S model can be converted into the first-order nonlinear consensus model with monotone coupling function on the real line. Then, in the analogy with the corresponding result for the Kuramoto model, they showed that for a given initial data and coupling strength, they provided an algorithm determining the number of asymptotic clusters and their asymptotic group velocities which can be called “the complete cluster predictability” of the one-dimensional C-S model. Unfortunately, there is no multi-dimensional counterpart for this complete cluster predictability. The clustering dynamics of the C-S model under the attractive-repulsive couplings was also discussed in [26, 28, 38].

Swarms theory often interacts with the kinetic theory approaches which might be based on Fokker-Plank methods [45] or stochastic evolutionary games within the framework of the so-called kinetic theory of active particles [10]. An example of a kinetic theory approach to swarm modeling is given in [15].

A recent literature has shown a growing interest on the applications of swarm theory to model social and economical problems [1, 7, 8, 12], while the swarm literature on predator-preys system [16] suggests to introduce behavioral dynamics. Analogous requirements are induced in the control problems of sparse agents [4, 5, 18, 27, 41, 44, 46].

Our paper accounts for all aforementioned hints and proposes a development of a mathematical theory of swarms where the dynamics of social behaviors, including emotional states, interacts with the mechanical dynamics of self-propelled particles, where the term *behavioral swarms* is introduced to denote the specific class of dynamical systems under consideration. The theory proposed in this paper is subsequently followed by applications. In more detail, the contents are delivered as follows.

Section 2 presents the main results of our paper, namely the aforementioned theory, where a general framework includes the interaction between mechanics and social behaviors, hence towards a theory of behavioral swarms. The theory refers to both second order and hybrid first-second order models. It leads to general frameworks deemed to provide the conceptual basis for the derivation of specific models.

Section 3 shows how C-S type models with internal variables can be referred the general structure derived in Section 2 accounting for the heterogeneity features accounted for by our approach. Models refer to the hybrid structures, where the dynamics of the social variable is described by first order equations, while accelerations are included for the mechanical variable.

Section 4 shows, by selecting a number of simple cases studied how models which go beyond the C-S approach can be derived. The derivation refers to the mathematical structures which include accelerations for all dependent mechanical variables accounting for the role of the social dynamics over the motion by a first order model. Some simulations are presented to enlighten how the activity variables modify the trajectories of the motion.

Section 5 presents some research perspectives which are brought to the attention of the reader looking ahead to different fields of applications, for instance the dynamics of crowds with virus contagion and control problems for sparse and stochastic systems.

2. TOWARDS A MATHEMATICAL THEORY OF BEHAVIORAL SWARMS

Let us consider the dynamics of a heterogeneous swarm of N interacting self-propelled particles, labeled by the subscript $i = 1, \dots, N$, where heterogeneity is also induced by the ability of each particle to express a scalar social state which will be called *activity*. The individual state of the i th particle can be defined by position \mathbf{x}_i , velocity \mathbf{v}_i and activity u_i , while the speed of the activity variable, namely the time derivative of u_i is denoted by z_i . First, we consider the dynamics of one population only, subsequently the generalization to several interacting populations is treated.

The velocity \mathbf{v}_i can be represented in polar coordinates as follows:

$$\mathbf{v}_i = \{v_i, \boldsymbol{\omega}_i\}, \quad v_i \in [0, 1], \quad \boldsymbol{\omega}_i = \frac{\mathbf{v}_i}{\|\mathbf{v}_i\|}, \quad (2.1)$$

where the unit vector ω_i can be represented by the angular variables referred to a cartesian system and where the cartesian components of the position \mathbf{x}_i are referred to the characteristic length ℓ of the system, while v_i has been referred to the limit velocity v_M which can be reached by the fastest particle. In particular, if the system is localized in a bounded domain Σ , the positive constant ℓ is the diameter of the circle containing Σ , while if the system moves in an unbounded domain, ℓ is simply referred to the domain Σ_0 containing the particles at $t = 0$.

Since self-propelled particles express a social function, the term *active particle* or, in short, *a-particle* will be used to denote them, while the individual a-particle is denoted by *i-particle*. The state of each i-particle is called *micro-state*, while the sets of all positions, velocities, and activities of the whole system are denoted by \mathbf{x} , \mathbf{v} and \mathbf{u} , respectively.

In general, dimensionless variables are used dividing the linear component of the position by a characteristic length ℓ related to the geometry of the system, and the speed by the highest speed reachable by the specific active particles object of the modeling approach. The objective consists in referring all macroscopic states, namely geometrical, mechanical, and social, to the a domain of the order of unity. Accordingly, the activity variable u_i takes values in $[0, 1]$, where the limit values $u_i = 0$ and $u_i = 1$ represent, respectively, the lowest and highest values of the activity.

Bearing all the above in mind, let us consider the derivation of a general mathematical structure suitable to provide the conceptual framework for the derivation of models. This new conceptual approach should account, at least, for the following features:

1. Each particle is able to develop a specific strategy which is heterogeneously distributed.
2. Interactions can be nonlocal and nonlinearly additive.
3. A decisional hierarchy can be used by assuming that interactions first modify the activity and subsequently the motion which depends also on the activity.
4. The approach is such that firstly mathematical structures are derived to provide the conceptual framework for the derivation of models, and subsequently this structure is implemented by models interactions which lead to specific models.
5. Each a-particle has sensitivity domain and interacts with all particles within the said domain.

The following quantities can be introduced to model, still at a formal level, interactions:

η_i models the interaction rate of individual based interactions between *i*-particle with all particles in the sensitivity domain.

Ω_i is the vision-based sensitivity domain of the *i*-particle.

φ_i denotes the action, which occurs with the rate η_i , over the activity variable over the *i*-particle by all particles in Ω_i .

ψ_i denotes the acceleration, which occurs with the rate η_i , over the mechanical variable by all particles in Ω_i .

• *Second order mechanics and activity*: In general, all above quantities depend on all variables \mathbf{x} , \mathbf{v} , \mathbf{u} , \mathbf{z} to be selected within the sensitivity domain. The formal structure of the framework is as follows:

$$\begin{cases} \frac{d\mathbf{x}_i}{dt} = \mathbf{v}_i, \\ \frac{d\mathbf{v}_i}{dt} = \sum_{j \in \Omega_i} \eta_i(\mathbf{x}_i, \mathbf{v}_i, \mathbf{u}_i, \mathbf{x}_j, \mathbf{v}_j, \mathbf{u}_j) \psi_i(\mathbf{x}_i, \mathbf{v}_i, \mathbf{u}_i, \mathbf{x}_j, \mathbf{v}_j, \mathbf{u}_j), \\ \frac{du_i}{dt} = z_i, \\ \frac{dz_i}{dt} = \sum_{j \in \Omega_i} \eta_i(\mathbf{x}_i, \mathbf{v}_i, \mathbf{u}_i, \mathbf{x}_j, \mathbf{v}_j, \mathbf{u}_j) \varphi_i(\mathbf{x}_i, \mathbf{v}_i, \mathbf{u}_i, \mathbf{x}_j, \mathbf{v}_j, \mathbf{u}_j), \end{cases} \quad (2.2)$$

where the notation $j \in \Omega_i$ indicates that summation refers to all *j*-particles in the domain Ω_i .

The structure (2.2) consists of a second order framework as it involves acceleration terms for all variables, first order frameworks can be obtained by simplified models leading to velocity, rather than acceleration, for both activity and mechanical variables. However, the modeling approach can consider hybrid frameworks which are first order for the activity variable and second order for the mechanical variables. An example is reported, at a formal level, in the following, where the dynamics of the activity variable is of first order, while that of the mechanical variable is classically second order.

- *Second order mechanics and first order activity:*

$$\begin{cases} \frac{d\mathbf{x}_i}{dt} = \mathbf{v}_i, \\ \frac{d\mathbf{v}_i}{dt} = \sum_{j \in \Omega_i} \eta_i(\mathbf{x}_i, \mathbf{v}_i, \mathbf{u}_i, \mathbf{x}_j, \mathbf{v}_j, \mathbf{u}_j) \psi_i(\mathbf{x}_i, \mathbf{v}_i, \mathbf{u}_i, \mathbf{x}_j, \mathbf{v}_j, \mathbf{u}_j), \\ \frac{du_i}{dt} = \sum_{j \in \Omega_i} \eta_i(\mathbf{x}_i, \mathbf{v}_i, \mathbf{u}_i, \mathbf{x}_j, \mathbf{v}_j, \mathbf{u}_j) \mu_i(\mathbf{x}_i, \mathbf{v}_i, \mathbf{u}_i, \mathbf{x}_j, \mathbf{v}_j, \mathbf{u}_j), \end{cases} \quad (2.3)$$

with obvious meaning of the terms μ_i and where all components of the dependent variables, which play a role in the dynamics, have been explicitly indicated. By analogous calculations, we can consider a formal structure for first order models both on the mechanical and the activity variable, however we do not consider this specific case, as it appears to be too artificial with respect to rules of mechanics.

- *Second order mechanics and activity for a mixture of functional subsystems:* It can be shown how the formal structures (2.2) and (2.3) can be generalized to study the dynamics of a-particles grouped into n populations which can be called *functional subsystems*, in short FSS, labeled by the subscripts $k = 1, \dots, n$. Formal calculations and obvious generalization of notation yield:

$$\begin{cases} \frac{d\mathbf{x}_{ik}}{dt} = \mathbf{v}_{ik}, \\ \frac{d\mathbf{v}_{ik}}{dt} = \sum_{jq \in \Omega_{ik}} \eta_{ik}(\mathbf{x}_{ik}, \mathbf{v}_{ik}, \mathbf{u}_{ik}, \mathbf{z}_{ik}, \mathbf{x}_{jq}, \mathbf{v}_{jq}, \mathbf{u}_{jq}, \mathbf{z}_{jq}) \\ \quad \times \psi_{ik}(\mathbf{x}_{ik}, \mathbf{v}_{ik}, \mathbf{u}_{ik}, \mathbf{z}_{ik}, \mathbf{x}_{jq}, \mathbf{v}_{jq}, \mathbf{u}_{jq}, \mathbf{z}_{jq}), \\ \frac{du_{ik}}{dt} = z_{ik}, \\ \frac{dz_{ik}}{dt} = \sum_{jq \in \Omega_{ik}} \eta_{ik}(\mathbf{x}_{ik}, \mathbf{v}_{ik}, \mathbf{u}_{ik}, \mathbf{z}_{ik}, \mathbf{x}_{jq}, \mathbf{v}_{jq}, \mathbf{u}_{jq}, \mathbf{z}_{jq}) \\ \quad \times \varphi_{ik}(\mathbf{x}_{ik}, \mathbf{v}_{ik}, \mathbf{u}_{ik}, \mathbf{z}_{ik}, \mathbf{x}_{jq}, \mathbf{v}_{jq}, \mathbf{u}_{jq}, \mathbf{z}_{jq}). \end{cases} \quad (2.4)$$

The mathematical structure (2.4) can be simplified under the hypothesis that the sensibility domain of the ik -particle contains only the a-particles which belong to the same functional subsystem as the ik -particle,

namely the k th functional subsystem. The mathematical structure can be particularized in this case as follows:

$$\left\{ \begin{array}{l} \frac{d\mathbf{x}_{ik}}{dt} = \mathbf{v}_{ik}, \\ \frac{d\mathbf{v}_{ik}}{dt} = \sum_{j \in \Omega_{ik}} \eta_{ik}(\mathbf{x}_{ik}, \mathbf{v}_{ik}, \mathbf{u}_{ik}, \mathbf{z}_{ik}, \mathbf{x}_{jk}, \mathbf{v}_{jk}, \mathbf{u}_{jk}, \mathbf{z}_{jk}) \\ \quad \times \psi_{ik}(\mathbf{x}_{ik}, \mathbf{v}_{ik}, \mathbf{u}_{ik}, \mathbf{z}_{ik}, \mathbf{x}_{jk}, \mathbf{v}_{jk}, \mathbf{u}_{jk}, \mathbf{z}_{jk}), \\ \frac{d\mathbf{u}_{ik}}{dt} = \mathbf{z}_{ik}, \\ \frac{d\mathbf{z}_{ik}}{dt} = \sum_{j \in \Omega_{ik}} \eta_{ik}(\mathbf{x}_{ik}, \mathbf{v}_{ik}, \mathbf{u}_{ik}, \mathbf{z}_{ik}, \mathbf{x}_{jk}, \mathbf{v}_{jk}, \mathbf{u}_{jk}, \mathbf{z}_{jk}) \\ \quad \times \varphi_{ik}(\mathbf{x}_{ik}, \mathbf{v}_{ik}, \mathbf{u}_{ik}, \mathbf{z}_{ik}, \mathbf{x}_{jk}, \mathbf{v}_{jk}, \mathbf{u}_{jk}, \mathbf{z}_{jk}). \end{array} \right. \quad (2.5)$$

Analogous calculations yield:

- *Second order mechanics and first order activity for a mixture of functional subsystems:*

$$\left\{ \begin{array}{l} \frac{d\mathbf{x}_{ik}}{dt} = \mathbf{v}_{ik}, \\ \frac{d\mathbf{v}_{ik}}{dt} = \sum_{jq \in \Omega_{ik}} \eta_{ik}(\mathbf{x}_{ik}, \mathbf{v}_{ik}, \mathbf{u}_{ik}, \mathbf{x}_{jq}, \mathbf{v}_{jq}, \mathbf{u}_{jq}) \\ \quad \times \psi_{ik}(\mathbf{x}_{ik}, \mathbf{v}_{ik}, \mathbf{u}_{ik}, \mathbf{x}_{jq}, \mathbf{v}_{jq}, \mathbf{u}_{jq}), \\ \frac{d\mathbf{u}_{ik}}{dt} = \sum_{jq \in \Omega_{ik}} \eta_{ik}(\mathbf{x}_{ik}, \mathbf{v}_{ik}, \mathbf{u}_{ik}, \mathbf{x}_{jq}, \mathbf{v}_{jq}, \mathbf{u}_{jq}) \\ \quad \times \mu_{ik}(\mathbf{x}_{ik}, \mathbf{v}_{ik}, \mathbf{u}_{ik}, \mathbf{x}_{jq}, \mathbf{v}_{jq}, \mathbf{u}_{jq}), \end{array} \right. \quad (2.6)$$

where all components of the dependent variables, which play a role in the dynamics, appear in the argument of the interaction terms.

A detailed characterization of these mathematical structures, say the interaction terms $\Omega_{ik}, \eta_{ik}, \varphi_{ik}, \mu_{ik}$ and ψ_{ik} , can be developed only for well defined case studies. However, some common features of the modeling approach can be given by a qualitative description to be formalized for each case study, as we shall see in the next sections.

- **Hierarchy:** The study of human crowds [14] has suggested a hierarchy in the decision making by which walkers develop their walking strategies, namely by interactions which firstly induce modification of the social state, subsequently walkers modify their walking direction, and finally they adapt the speed to the local new flow direction. Introducing a hierarchy is an important step of the modeling approach. Arguably, an analogous sequence of dynamical choice can be developed for a broad variety of case studies.
- **Sensitivity domain:** This quantity is defined by a cone with vertex in \mathbf{x}_i , vertex angle Θ , and with axis along the velocity. Ω_i is finite being truncated at a distance R which might be related to the critical finite number of active particles which have a sensitive influence. The sensitivity domain might be modified by visibility problem.
- **Interaction rate:** The modeling of the interaction rates can be referred to a distance between the interacting entities by a metric suitable to account both for the distance between the interacting entities and that of their statistical distribution.
- **Social action:** The social action depends on the interaction of each particle with those in its sensitivity domain. It is specific of each system to be modeled.

- **Mechanical action:** Mechanical actions follow the rules of classical mechanics, but the parameters leading to accelerations depend on the social state by models to be properly defined.

3. CUCKER-SMALE TYPE MODELS WITH INTERNAL VARIABLES

In this section, we provide several C-S type models which fit into our proposed general framework (2.3). Our paper, as aforementioned, aims at going beyond this pioneer model, hence this section can be viewed as a preliminary step towards further developments.

Bearing in mind this introductory remark, we first present the original model for *mechanical particles* and, subsequently, we introduce several Cucker-Smale type models with internal variables related to (2.3).

3.1. The Cucker-Smale model

Consider an ensemble of mechanical Cucker-Smale particles whose states are represented by position and velocity variables. More precisely, let \mathbf{x}_i and \mathbf{v}_i be the position and velocity of the i th C-S particle in the free Euclidean space \mathbb{R}^d , respectively. Then, the temporal evolution of mechanical variables $(\mathbf{x}_i, \mathbf{v}_i)$ is governed by the Newton-like system:

$$\begin{cases} \frac{d\mathbf{x}_i}{dt} = \mathbf{v}_i, & t > 0, \quad 1 \leq i \leq N, \\ \frac{d\mathbf{v}_i}{dt} = \frac{\kappa}{N} \sum_{j=1}^N \phi(\|\mathbf{x}_j - \mathbf{x}_i\|) (\mathbf{v}_j - \mathbf{v}_i), \end{cases} \quad (3.1)$$

where $\|\cdot\|$ denotes the standard ℓ^2 -norm in \mathbb{R}^d , κ represents the strength of coupling and the communication weight $\phi : [0, \infty) \rightarrow \mathbb{R}_+$ is bounded, Lipschitz continuous and monotonically decreasing:

$$\begin{aligned} 0 < \phi(r) \leq \phi(0), \quad [\phi]_{Lip} := \sup_{x \neq y} \frac{|\phi(x) - \phi(y)|}{|x - y|} < \infty, \\ (\phi(r) - \phi(s))(r - s) \leq 0, \quad r, s \geq 0. \end{aligned} \quad (3.2)$$

System (3.1) does not conserve the speed of particles. Thus, as a slight variation of system (3.1), we introduce the Cucker-Smale model with unit speed. For this, we replace the velocity coupling term $\mathbf{v}_j - \mathbf{v}_i$ by $\mathbf{v}_j - (\mathbf{v}_j \cdot \mathbf{v}_i)\mathbf{v}_i$ which results in

$$\begin{cases} \frac{d\mathbf{x}_i}{dt} = \mathbf{v}_i, & t > 0, \quad 1 \leq i \leq N, \\ \frac{d\mathbf{v}_i}{dt} = \frac{\kappa}{N} \sum_{j=1}^N \phi(\|\mathbf{x}_j - \mathbf{x}_i\|) (\mathbf{v}_j - (\mathbf{v}_j \cdot \mathbf{v}_i)\mathbf{v}_i). \end{cases} \quad (3.3)$$

Then, it is easy to see that system (3.3) preserves the unit speed of particles:

$$\|\mathbf{v}_i(t)\| = \|\mathbf{v}_i^0\| = 1, \quad t > 0, \quad i = 1, \dots, N.$$

In the following four subsections, we provide four active C-S particles with internal variables such as excitation level, temperature, spin and phase which takes the form:

$$\begin{cases} \frac{d\mathbf{x}_i}{dt} = \mathbf{v}_i, & i = 1, \dots, N, \\ \frac{d\mathbf{v}_i}{dt} = \frac{\kappa_1}{N} \sum_{j=1}^N \psi_1(\mathbf{x}_i, \mathbf{x}_j) C_1(\mathbf{v}_i, \mathbf{v}_j, u_i, u_j), \\ \frac{du_i}{dt} = \frac{\kappa_2}{N} \sum_{j=1}^N \psi_2(\mathbf{x}_i, \mathbf{x}_j) C_2(\mathbf{v}_i, \mathbf{v}_j, u_i, u_j). \end{cases} \quad (3.4)$$

Note that the choice

$$\kappa_2 = 0, \quad \kappa_1 = \kappa, \quad \psi_1(\mathbf{x}_i, \mathbf{x}_j) = \phi(\|\mathbf{v}_j - \mathbf{v}_i\|), \quad C_1(\mathbf{v}_i, \mathbf{v}_j, u_i, u_j) = \mathbf{v}_j - \mathbf{v}_i$$

corresponds to the C-S model (3.1).

3.2. Thermodynamic Cucker-Smale model

In this subsection, we introduce the C-S model [31, 35] for thermodynamic particles. Consider an ensemble of C-S particles with temperatures as an internal variable. For this, we denote $u_i \in \mathbb{R}$ to be the temperature of the i th C-S particle. Then, the thermodynamic Cucker-Smale model reads as follows.

$$\begin{cases} \frac{d\mathbf{x}_i}{dt} = \mathbf{v}_i, \\ \frac{d\mathbf{v}_i}{dt} = \frac{\kappa_1}{N} \sum_{j=1}^N \phi_1(\|\mathbf{x}_j - \mathbf{x}_i\|) \left(\frac{\mathbf{v}_j - \mathbf{v}_c}{u_j} - \frac{\mathbf{v}_i - \mathbf{v}_c}{u_i} \right), \\ \frac{d}{dt} \left(u_i + \frac{1}{2} \|\mathbf{v}_i\|^2 \right) = \frac{\kappa_2}{N} \sum_{j=1}^N \phi_2(\|\mathbf{x}_j - \mathbf{x}_i\|) \left(\frac{1}{u_i} - \frac{1}{u_j} \right) \\ \quad + \frac{\kappa_1}{N} \sum_{j=1}^N \phi_1(\|\mathbf{x}_j - \mathbf{x}_i\|) \left(\frac{\mathbf{v}_j - \mathbf{v}_c}{u_j} - \frac{\mathbf{v}_i - \mathbf{v}_c}{u_i} \right) \cdot \mathbf{v}_c, \end{cases} \quad (3.5)$$

where $\mathbf{v}_c := \frac{1}{N} \sum_{i=1}^N \mathbf{v}_i$, and κ_1 and κ_2 denote positive coupling strengths.

Note that system (3.5) is Galilean invariant, so without loss of generality, we can choose the center of mass as origin of reference frame $\mathbf{v}_c = 0$. In this setting, system (3.5) becomes

$$\begin{cases} \frac{d\mathbf{x}_i}{dt} = \mathbf{v}_i, \\ \frac{d\mathbf{v}_i}{dt} = \frac{\kappa_1}{N} \sum_{j=1}^N \phi_1(\|\mathbf{x}_j - \mathbf{x}_i\|) \left(\frac{\mathbf{v}_j}{u_j} - \frac{\mathbf{v}_i}{u_i} \right), \\ \frac{d}{dt} \left(u_i + \frac{1}{2} \|\mathbf{v}_i\|^2 \right) = \frac{\kappa_2}{N} \sum_{j=1}^N \phi_2(\|\mathbf{x}_j - \mathbf{x}_i\|) \left(\frac{1}{u_i} - \frac{1}{u_j} \right). \end{cases} \quad (3.6)$$

The model (3.5) takes into account the mutual interactions not only of “mechanical” type but also of thermodynamic type, that is, the “temperature effect” due to the presence of different “internal energy”. In the case

in which all particles have the same temperature $u_i = u_\infty$, the first two equations in (3.5) becomes the C-S model (3.1) with $\kappa = \frac{\kappa_1}{Nu_\infty}$.

On the other hand, under the small diffusion velocity assumption $|\mathbf{v}_i - \mathbf{v}_c| \ll 1$ and by taking the barycenter observer such that $\mathbf{v}_c = 0$, we can further simplify the temperature equation in system (3.6) and derive an approximate model:

$$\begin{cases} \frac{d\mathbf{x}_i}{dt} = \mathbf{v}_i, \\ \frac{d\mathbf{v}_i}{dt} = \frac{\kappa_1}{N} \sum_{j=1}^N \phi_1(\|\mathbf{x}_j - \mathbf{x}_i\|) \left(\frac{\mathbf{v}_j}{u_j} - \frac{\mathbf{v}_i}{u_i} \right), \\ \frac{du_i}{dt} = \frac{\kappa_2}{N} \sum_{j=1}^N \phi_2(\|\mathbf{x}_j - \mathbf{x}_i\|) \left(\frac{1}{u_i} - \frac{1}{u_j} \right). \end{cases} \quad (3.7)$$

3.3. Thermodynamic Kuramoto model

In this subsection, we present the thermodynamic Kuramoto model [34] which can be derived from the two-dimensional thermodynamic Cucker-Smale model (3.6) with constant communication weight and uniform modulus ratio $\frac{v_i}{u_i}$ between velocity and temperature:

$$\psi_{ij} : \text{constant}, \quad \left| \frac{\mathbf{v}_i}{u_i} \right| = \left| \frac{\mathbf{v}_j}{u_j} \right|, \quad i, j = 1, \dots, N.$$

Now, we introduce a heading angle (or phase) θ_i such that

$$\frac{\mathbf{v}_i}{u_i} = \frac{\eta}{u_*} e^{i\theta_i}, \quad \text{or equivalently} \quad \mathbf{v}_i = \eta \frac{u_i}{u_*} e^{i\theta_i}, \quad (3.8)$$

where η is a constant parameter with speed dimension, and u_* is a constant reference temperature.

We substitute the ansatz (3.8) into (3.6)₂ and compare the imaginary part of the resulting relation to get

$$u_i \dot{\theta}_i = \frac{\kappa_1}{N} \sum_{j=1}^N \psi_{ij} \sin(\theta_j - \theta_i).$$

Once all the temperatures u_i are strictly positive, then the above relation can be rewritten as

$$\dot{\theta}_i = \frac{\kappa_1}{N} \sum_{j=1}^N \frac{\psi_{ij}}{u_i} \sin(\theta_j - \theta_i). \quad (3.9)$$

Like the Kuramoto model [42], we may add a natural frequency ν_i in the R.H.S of (3.9) to model the heterogeneity of particles to obtain

$$\dot{\theta}_i = \nu_i + \frac{\kappa_1}{N} \sum_{j=1}^N \frac{\psi_{ij}}{u_i} \sin(\theta_j - \theta_i). \quad (3.10)$$

For the evolution of temperature u_i , we first use relation (3.8) to get

$$\frac{1}{2} |\dot{\mathbf{v}}_i|^2 = \frac{\eta^2}{2u_*^2} |u_i|^2, \quad \text{i.e.,} \quad \frac{d}{dt} |\mathbf{v}_i|^2 = \frac{2\eta^2}{u_*^2} u_i \dot{u}_i. \quad (3.11)$$

Then, we use (3.6)₃ and (3.11) to find the dynamics of u_i :

$$\dot{u}_i = \frac{\kappa_2 u_*^2}{N(u_*^2 + \eta^2 u_i)} \sum_{j=1}^N \zeta_{ij} \left(\frac{1}{u_i} - \frac{1}{u_j} \right). \quad (3.12)$$

Finally, we combine (3.10) and (3.12) to derive the thermodynamic Kuramoto model:

$$\begin{cases} \dot{\theta}_i = \nu_i + \frac{\kappa_1}{N} \sum_{j=1}^N \frac{\psi_{ij}}{u_i} \sin(\theta_j - \theta_i), \\ \dot{u}_i = \frac{\kappa_2}{N} \sum_{j=1}^N \frac{\zeta_{ij} u_*^2}{(u_*^2 + \eta^2 u_i)} \left(\frac{1}{u_i} - \frac{1}{u_j} \right), \end{cases} \quad (3.13)$$

where elements in network topologies (ψ_{ij}) and (ζ_{ij}) are assumed to be positive and symmetric:

$$\psi_{ij} = \psi_{ji} > 0, \quad \zeta_{ij} = \zeta_{ji} > 0, \quad 1 \leq i, j \leq N. \quad (3.14)$$

We refer to [34] for a detailed emergent dynamics of (3.13).

3.4. Spinning Cucker-Smale particles

In this subsection, we present the C-S model [30] for the spinning particles. In this case, the internal variable \mathbf{u}_i corresponds to the spin of the i th particle. Then, the dynamics of the state $(\mathbf{x}_i, \mathbf{v}_i, \mathbf{u}_i)$ is governed by the inertial spin model:

$$\begin{cases} \frac{d\mathbf{x}_i}{dt} = \mathbf{v}_i, \\ \frac{d\mathbf{v}_i}{dt} = \frac{1}{\chi} \mathbf{u}_i \times \mathbf{v}_i, \\ \frac{d\mathbf{u}_i}{dt} = \mathbf{v}_i \times \left[\frac{\kappa}{N} \sum_{j=1}^N p_{ij} (\mathbf{v}_j - \mathbf{v}_i) - \gamma \frac{d\mathbf{v}_i}{dt} \right], \end{cases} \quad (3.15)$$

where $\chi > 0$ is the generalized moment of inertia. Here, p_{ij} represents a bounded communication weight or an interaction rate between i th and j th particles, and γ , and κ represent strengths of damping, and coupling, respectively. The friction term $-\gamma\dot{\mathbf{v}}$ in spin dynamics induces a rotational dissipation so that the total spin tends to zero exponentially fast. Moreover, in a formal zero inertia limit $\chi \rightarrow 0$, system (3.15) with metric dependent weight $p_{ij} = \psi(\|\mathbf{x}_i - \mathbf{x}_j\|)$ reduces to the C-S type model with a unit speed constraint (3.3):

$$\begin{cases} \frac{d\mathbf{x}_i}{dt} = \mathbf{v}_i, \\ \frac{d\mathbf{v}_i}{dt} = \frac{\bar{\kappa}}{N} \sum_{j=1}^N \psi(\|\mathbf{x}_i - \mathbf{x}_j\|) (\mathbf{v}_j - (\mathbf{v}_j \cdot \mathbf{v}_i) \mathbf{v}_i), \quad \bar{\kappa} := \frac{\kappa}{\gamma}. \end{cases} \quad (3.16)$$

4. BEYOND CUCKER-SMALE MODELS: CASE STUDIES

This section shows how models for behavioral swarms can be derived referring to the mathematical structures proposed in Section 2. The presentation refers to specific case studies which are followed by numerical simulations. Models account for the influence over the motion of the heterogeneously distributed activity variable

corresponding to a specific social state. The activity variable is modified by interactions. A plane dynamics is studied just to simplify notation, however models can be rapidly transferred to three space dimensions.

In more details, we consider a system constituted by one FS of n interacting particles which heterogeneously share a social state. This internal variable modifies the trajectories of their motion. The models are somehow inspired to human crowd dynamics as interactions account, at least partially, for a dynamics which appears in crowds [3, 6, 14]. The derivation accounts for the so-called topological interactions introduced in [9], see also [15, 16]. The presentation is delivered in the next subsections, where the first one deals with modeling, the second one presents some simulations related to the case studies selected in the first subsection.

4.1. Modeling case studies

The derivation of models is here developed with the aim of showing, by a number of simple case studies, how the dynamics of the activity variable modifies the patterns of the flow dynamics. Specific models can be derived within the following framework:

1. The state of the system is given by the whole set of directions θ_i , rotation speeds σ_i and activities u_i of all a-particles with $i = 1, \dots, n$, while each a-particle in the swarm moves with the same speed $v = v_0 = 1$. The activity u_i is supposed to correspond to the level of stress with $u_i \in [0, 1]$, where $u_i = 0$ defines the lack of stress, whereas $u_i = 1$ corresponds to the maximal admissible level.
2. All a-particles have a visibility angle, hence a visibility domain $\Omega_i = [\theta_i - \Theta, \theta_i + \Theta]$ with the radius R , each a-particle interacts only with other particles in Ω_i or even with a fixed “small” number of entities within a sensitivity domain $\Omega_i^s \subseteq \Omega_i$, where the radius R^s is finite and depends on the number of particles selected for the interaction.
3. Each a-particle has an individual movement direction ν_i , but all of them share a preferred direction ν which might depend on time. All particles have a trend towards ν , but each of them is subject to an attraction towards the mean velocity direction ω_i of the particles in the sensitivity domain. This attraction depends also on the social state of the particle which is heterogeneously distributed over the swarm and hence in Ω_i .
4. The decision process by which an a-particles develops its movement strategy follows a decisional sequence by which an a-particle : firstly modifies the activity and subsequently the direction of motion.
5. The dynamics of the social state depends on the specific features of the system under consideration. A simple case corresponds to a consensus dynamics with respect to the a-particles in the sensitivity domain.

Let us now transfer these simple rules into interactions models, to be inserted into the general structures proposed in Section 2, referring specifically to equation (2.3) which needs some technical modifications (simplifications) to account for the assumptions in the above items.

The assumption that the same speed is shared by all particles defines the velocity of the i-particle as follows:

$$\mathbf{v}_i = \cos \theta_i \mathbf{i} + \sin \theta_i \mathbf{j} \quad \Rightarrow \quad \frac{d\mathbf{v}_i}{dt} = (-\sin \theta_i \mathbf{i} + \cos \theta_i \mathbf{j}) \sigma_i, \quad \sigma_i = \frac{d\theta_i}{dt}, \quad (4.1)$$

where \mathbf{i} and \mathbf{j} denote the unit vectors of an orthogonal frame, and $\theta_i \in [0, 2\pi)$ denotes the velocity direction.

Hence the mechanical state of each particle is defined by flight direction θ_i and rotational speed σ_i , namely the time derivative of θ_i . In addition, the following physical quantities can be defined:

The interaction rate $\eta_{ij} \cong \eta_0$ is supposed to be given a constant quantity subsequently weighted by the distance between the interacting a-particles:

$$g = g(\mathbf{x}_j, \mathbf{x}_i; \alpha) = \exp \left\{ -\alpha \cdot (\|\mathbf{x}_j - \mathbf{x}_i\|) \right\}, \quad (4.2)$$

where α is a positive defined constant parameter.

The attraction direction ω_i^s over the i th a-particle from the a-particles in Ω_i decays with the distance between the interacting entities:

$$\omega_i^s = \omega_i^s(\mathbf{x}_i, \theta_i) = \sum_{j \in \Omega_i} g(\mathbf{x}_j, \mathbf{x}_i; \alpha) \theta_j. \quad (4.3)$$

The direction ω_i , which effectively attracts the movement direction of the i-particle, is defined by a convex combination of θ^ν and ω_i^s weighted by the activity

$$\omega_i = \omega_i(\mathbf{x}_i, \theta_i, u_i; \nu) = u_i \theta^\nu + (1 - u_i) \omega_i^s(\mathbf{x}_i, \theta_i), \quad (4.4)$$

where θ^ν is the angle related to a commonly preferred direction.

The dynamics by a consensus of the i-particle to the j-particles in Ω_i depends on the parameter β as follows:

$$\frac{du_i}{dt} = \beta \sum_{j \in \Omega_i} \exp \{ -\alpha (\|\mathbf{x}_j - \mathbf{x}_i\|) \} (u_j - u_i). \quad (4.5)$$

When we go beyond the consensus dynamics, different models of interaction can be considered keeping, however, the decay with distance as in (4.5).

4.1.1. First order models

Let us now consider first order models, where the alignment dynamic is described by a rotational speed, depending on the parameter γ , on the angular distance between the i-particle and that of the j-particles in Ω_i :

$$\frac{d\theta_i}{dt} = \gamma (\omega_i(\mathbf{x}_i, \theta_i, u_i) - \theta_i). \quad (4.6)$$

Let us specialize the components of \mathbf{x} by $\mathbf{x} = \{x, y\}$ and transfer the above assumptions into the mathematical structure (2.3) to yield:

$$\begin{cases} \frac{du_i}{dt} = \beta \sum_{j \in \Omega_i} \exp \{ -\alpha (\|\mathbf{x}_j - \mathbf{x}_i\|) \} (u_j - u_i), \\ \frac{dx_i}{dt} = \cos \theta_i, \\ \frac{dy_i}{dt} = \sin \theta_i, \\ \frac{d\theta_i}{dt} = \gamma (u_i \theta^\nu + (1 - u_i) \sum_{j \in \Omega_i} \exp \{ -\alpha (\|\mathbf{x}_j - \mathbf{x}_i\|) \} \theta_j - \theta_i). \end{cases} \quad (4.7)$$

The following simplifications or generalizations can be considered:

• **The case of a constant activity:** If u_i attains the same value for all particles $u_i = u$, then the first equation is automatically satisfied, while the rest of system can be rewritten as follows:

$$\begin{cases} \frac{dx_i}{dt} = \cos \theta_i, \\ \frac{dy_i}{dt} = \sin \theta_i, \\ \frac{d\theta_i}{dt} = \gamma \left(u \theta^\nu + (1-u) \sum_{j \in \Omega_i} \exp \{ -\alpha (\|\mathbf{x}_j - \mathbf{x}_i\|) \} \theta_j - \theta_i \right), \end{cases} \quad (4.8)$$

where the very special case $u = 1$ simplifies the alignment equation as follows:

$$\frac{d\theta_i}{dt} = \gamma (\theta_i^\nu - \theta_i), \quad (4.9)$$

which is independent of the space variable. If $\varepsilon = 1 - u$, where ε is small with respect to zero, then equation (4.8) can be treated as a perturbation, by the small parameter ε , of the model corresponding to $u = 1$.

• **Attraction to high values of u :** An additional possible modification of model (4.7) is that only high levels of u_j are attractive, namely

$$\begin{cases} \frac{du_i}{dt} = \beta \sum_{j \in \Omega_i} \exp \{ -\alpha (\|\mathbf{x}_j - \mathbf{x}_i\|) \} (u_j - u_i) & \text{if } u_j > u_i, \\ \frac{du_i}{dt} = 0 & \text{if } u_j \leq u_i. \end{cases}$$

The model can be particularized in this case as follows:

$$\begin{cases} \frac{du_i}{dt} = \beta \sum_{j \in \Omega_i} \exp \{ -\alpha (\|\mathbf{x}_j - \mathbf{x}_i\|) \} (u_j - u_i) \delta_{\{u_j > u_i\}}, \\ \frac{dx_i}{dt} = \cos \theta_i, \\ \frac{dy_i}{dt} = \sin \theta_i, \\ \frac{d\theta_i}{dt} = \gamma \left(u_i \theta_i^\nu + (1-u_i) \sum_{j \in \Omega_i} \exp \{ -\alpha (\|\mathbf{x}_j - \mathbf{x}_i\|) \} \theta_j - \theta_i \right), \end{cases} \quad (4.10)$$

where δ stand for the Kronecker delta symbol:

$$\delta_{\{u_j > u_i\}} = 1 \quad \text{if } u_j > u_i \quad \text{and} \quad \delta_{\{u_j > u_i\}} = 0 \quad \text{if } u_j \leq u_i.$$

4.1.2. Second order models

Second order models can be developed by introducing a rotational speed σ_i as the time derivative of θ_i and modeling the acceleration as follows:

$$\frac{d\sigma_i}{dt} = \gamma_a (\omega_i(\mathbf{x}_i, \theta_i, u_i) - \theta_i) - \gamma_b \frac{d\theta_i}{dt}, \quad (4.11)$$

corresponding to an acceleration induced by the chased and flight directions reduced by a viscous action, where γ_a and γ_b are a positive defined parameters. This phenomenological assumption yields:

$$\left\{ \begin{array}{l} \frac{du_i}{dt} = \beta \sum_{j \in \Omega_i} \exp \{ -\alpha (\|\mathbf{x}_j - \mathbf{x}_i\|) \} (u_j - u_i), \\ \frac{dx_i}{dt} = \cos \theta_i, \\ \frac{dy_i}{dt} = \sin \theta_i, \\ \frac{d\theta_i}{dt} = \sigma_i, \\ \frac{d\sigma_i}{dt} = \gamma_a (u_i \theta^\nu + (1 - u_i) \sum_{j \in \Omega_i} \exp \{ -\alpha (\|\mathbf{x}_j - \mathbf{x}_i\|) \} \theta_j - \theta_i) - \gamma_b \frac{d\theta_i}{dt}, \end{array} \right. \quad (4.12)$$

while this model can be simplified under the assumption of a constant activity.

4.1.3. Topological interactions

Following [9], interactions involve only a fixed number $m < n$ of i-particles within the sensitivity domain $\Omega_i^s \subseteq \Omega_i$. The space dependence of the activity can be neglected in the modeling of interactions as m is of a smaller order with respect to n , do that R^s is also small with respect to the visibility radius R . In this case, the model simply reads as follows:

$$\left\{ \begin{array}{l} \frac{du_i}{dt} = \beta \sum_{j \in \Omega_i^s} (u_j - u_i), \\ \frac{dx_i}{dt} = \cos \theta_i, \\ \frac{dy_i}{dt} = \sin \theta_i, \\ \frac{d\theta_i}{dt} = \gamma (u_i \theta^\nu + (1 - u_i) \sum_{j \in \Omega_i^s} (\theta_j - \theta_i)), \end{array} \right. \quad (4.13)$$

while analogous calculations can be applied to the structures corresponding to the case of a constant activity.

4.2. Simulations

Some numerical simulations, related to the case studies treated in the preceding subsection, are presented in the following with the aim of showing how case studies of social dynamics modify the patterns of the collective motion. The focus is on the influence of the activity variable over the alignment of particles and on how it contributes to the flocking behaviors.

4.2.1. On the role of constant activity

Let us refer to first order models (4.8), sample simulations are developed to enlighten how the activity variable, which acts as a parameter as it has a constant value u , can affect the collective dynamics. Two specific cases are presented in the following.

- **Fixed preferred direction:** Simulations correspond to $n = 100$ a-particles and to a fixed values of the preferred direction, specifically the value $\theta^\nu = \frac{\pi}{3}$ is selected. Initially, the active particles are randomly

distributed in a square region of dimension $[0, 1] \times [0, 1]$ while the velocity direction of each a-particle has a random distribution in

$$\Omega = \left[\theta^\nu - \frac{\pi}{3}, \theta^\nu + \frac{\pi}{3} \right].$$

Simulations, developed with the following parameters $\alpha = 1$, $\gamma = 0.1$ and $\Theta = \frac{\pi}{4}$, are shown in Figure 1 which reports the flow patterns for $u = 0.25, 0.5, 0.75$ corresponding to $t = 0, 4, 8, 12$. It can be observed that high values of u contribute to flocking and alignment toward the shared preferred direction, while decreasing values of u reduce the alignment.

• **Rotating preferred direction:** We consider the case of a rotating direction of the preferred velocity. The study is analogous to that treated in the above case study, but with the difference that the preferred direction shared by all particles rotates anticlockwise with constant angular velocity. Therefore, the model is the same given in equation (4.8) with $\nu(t) = (\cos(a_0 t), \sin(a_0 t))$, where a_0 is the angular velocity, hence the angle related to commonly preferred direction is given by:

$$\theta^\nu(t) = a_0 t.$$

Similar to the case of a constant velocity direction simulations refer to three different values of $u = 0.25, 0.5, 0.75$ for $n = 100$ a-particles which at $t = 0$ are randomly distributed in a unit square, while the initial velocity direction has a random distribution in Ω . Other parameters are $\alpha = 1$, $\gamma = 0.1$, $\Theta = \frac{\pi}{4}$ and $a_0 = 0.02$.

Simulations are shown in Figure 2 which enlightens how different values of the activity can have an influence on the alignment dynamics of the active particles. In fact, it appears that decreasing values of u reduce the alignment of the particles, while high values of u contribute to flocking. However, active particles are subject to an anticlockwise rotation. It is shown that increasing values of u significantly affect the rotation of the particles and increase the radius of their circular trajectory.

4.2.2. On the role of variable activity

Consider now the model (4.7) where, compared to (4.8), the dynamics of each u_i is delivered by a consensus towards a mean activity, see equation (4.5).

Two cases are studied:

- I: Each u_i is delivered by a consensus towards a mean activity (model (4.7)).
- II: Attraction to high values of activity variable (model (4.10)).

We consider initial data for the activity variable randomly distributed in the interval $[0, 1]$ and we chose $\beta = 0.1$. The other parameters are the same as in the case of a fixed preferred direction and a constant activity. Figure 3 reports the time dynamic of active particles for $t = 0, 4, 8, 12$ in cases I and II. In both cases, a flocking behavior is observed and is more relevant in case II which shows that attraction to high values of u enhances flocking.

5. RESEARCH PERSPECTIVES

A new class of behavioral swarm models have been derived in this paper for large systems of active particles, where the micro-state of the interacting entities includes, in addition to mechanical variables, typically position and velocity, also additional micro-scale variables which model the heterogeneous behavior by which the said entities develop their overall dynamics. These variables, in analogy with the kinetic theory of active particles [10], have been called *activity*. Firstly, the mathematical structures underlying the aforementioned

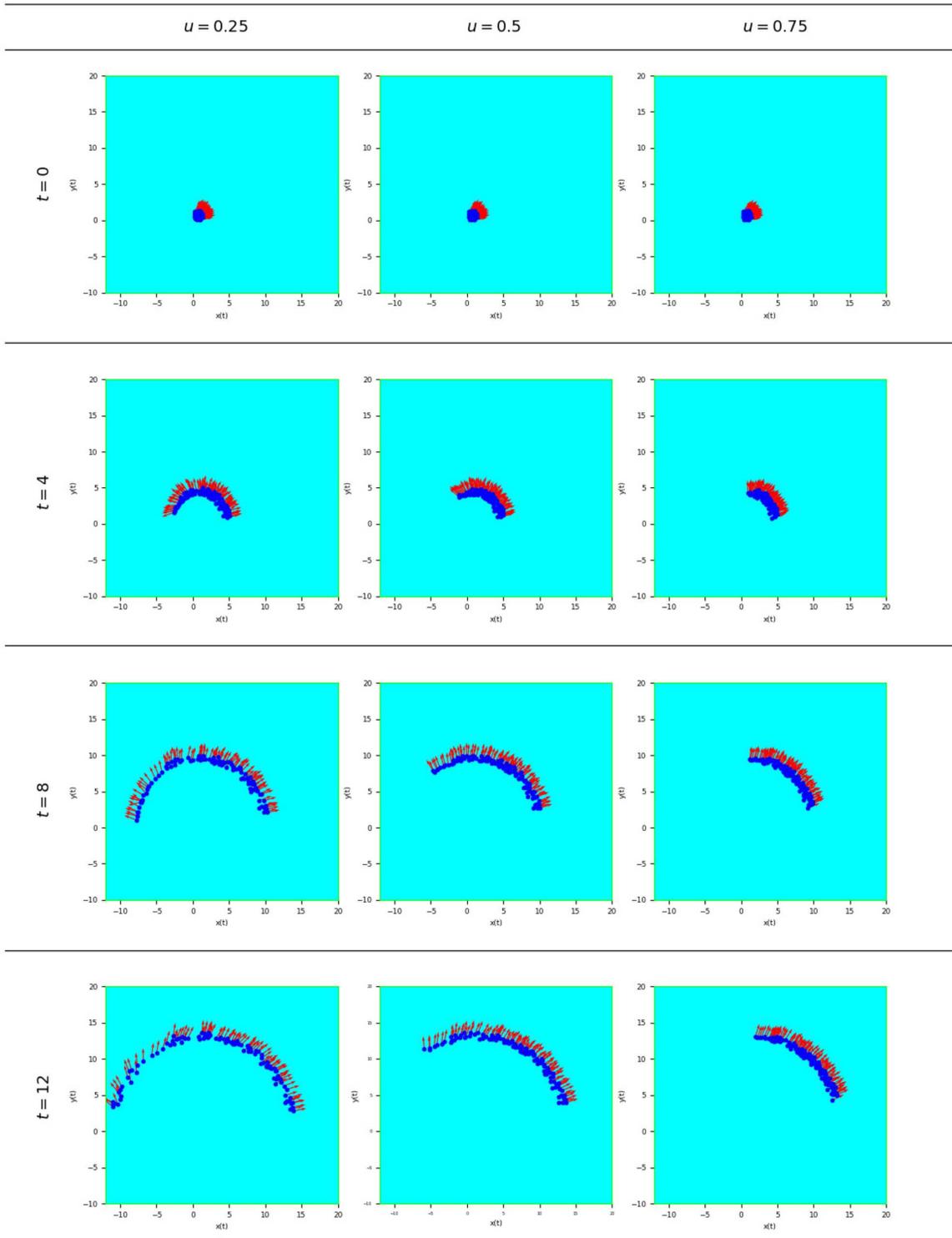


FIGURE 1. Temporal dynamics of particles for different values of $u = 0.25$, $u = 0.5$ and $u = 0.75$ corresponding to $t = 0, 4, 8, 12$ with a fixed preferred direction.

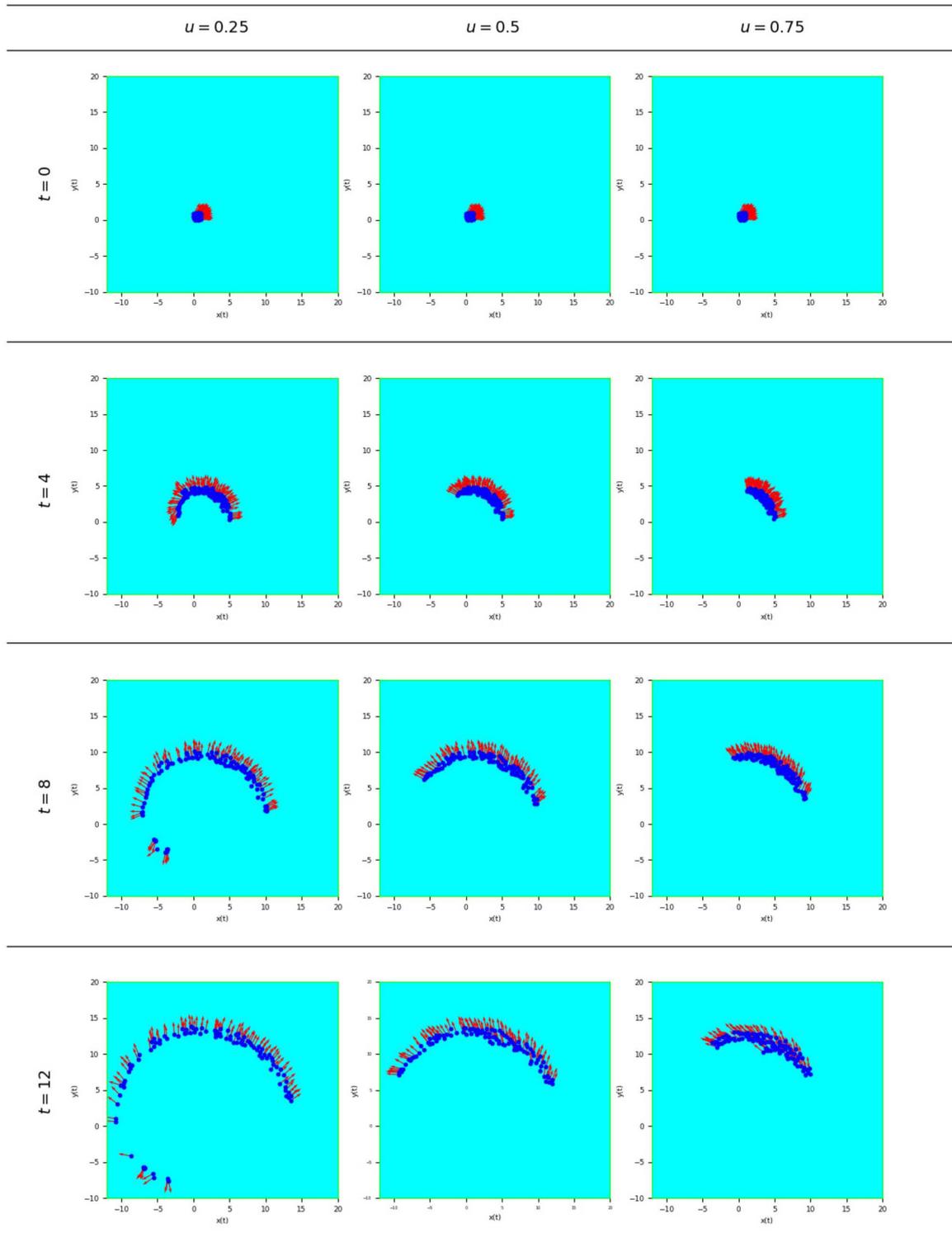


FIGURE 2. Temporal dynamics of particles for different values of $u = 0.25$, $u = 0.5$ and $u = 0.75$ corresponding to $t = 0, 4, 8, 12$ with a rotating preferred direction.

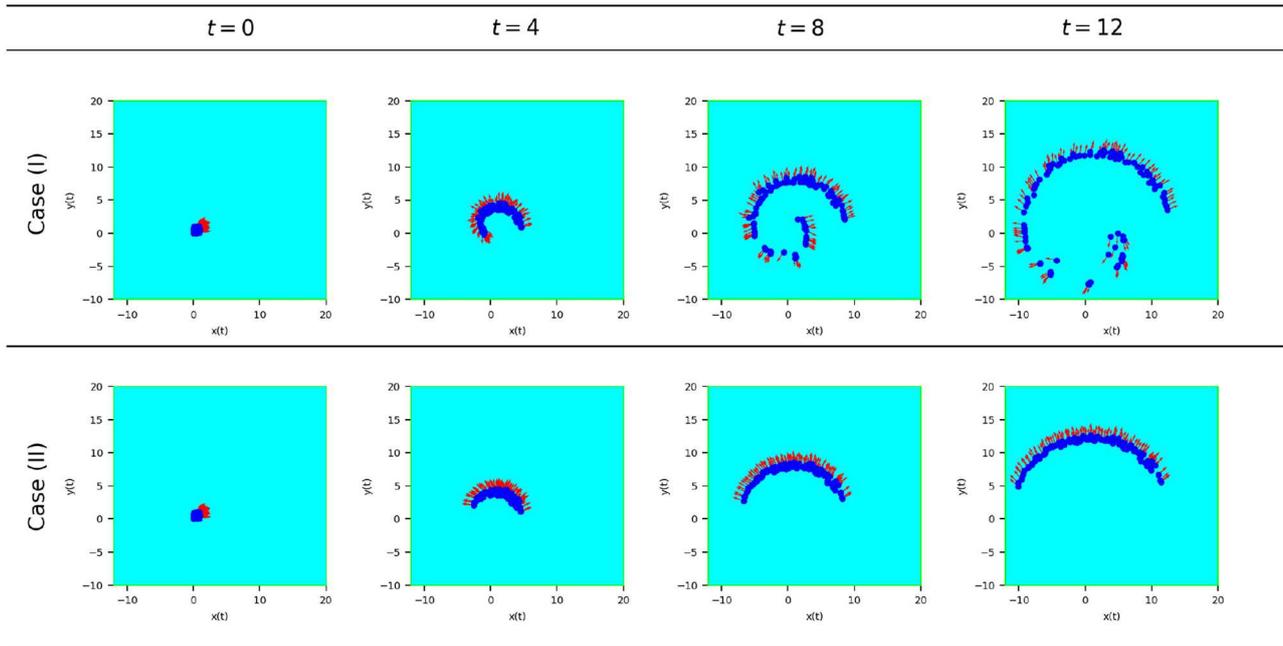


FIGURE 3. Time dynamics of particles for different values of $t = 0, 4, 8, 12$. Fixed preferred direction and variable activity.

behavioral dynamics have been derived, subsequently, a variety of sample models have been presented. Simulations have enlightened how the dynamics might show large deviations corresponding to large deviations of the activity variables.

The rationale towards the modeling approach presents some analogy with that of the kinetic theory of active particles [10], the kinetic approach requires a large number of interacting active particles to justify the continuity assumptions of the distribution function over the micro-state of the said particles. On the other hand, the derivation of swarm models does not require this specific assumptions.

This approach has already generated some models of interest in economy concerning the role of idiosyncratic learning of firms as an engine for the derivation of market sharing [13] or the modeling of price dynamics in the complex interaction between buyers and sellers [12]. These specific models refer to a dynamics in space homogeneity, while the theory proposed in our paper develops a general study of time-space dynamics with pattern formation.

Research perspectives should be mainly focused on behavioral systems, where the dynamics is over time and space. Five key problems, which appear to us worth to be chased in a possible research quest, are brought to the attention of the interested reader referring to systems where the number of interacting particles is finite.

- *Derivation of a more general framework* suitable to include the main key features of living systems specifically proliferative-destructive events as well as Darwinist mutations and selection which are typical of the immune competition [11, 17, 25, 50] in almost all biological systems, specifically in cancer dynamics [49].
- *Applications to the modeling specific systems such as crowds in crisis situations.* For instance, evacuation under stress conditions [14] or under contagion risk [11, 39, 40].
- *Learning dynamics* within a finite number of interacting individuals [21] can be derived within the framework of the behavioral swarms theory. Models of collective learning can contribute to describe the dynamics of opinion formation [19, 43].

- *Implementation of control actions* to lead the dynamics towards optimal behaviors, while this specific modeling should be followed by development of mathematical control tools for sparse and stochastic systems [18, 27, 41, 46].
- *Study of analytic problems within a multiscale vision* by suitable asymptotic time-space dynamics and continuity assumptions, firstly kinetic type models are derived from the underlying description and subsequently hydrodynamical models are derived from the underlying kinetic description [20].

It is plain that these key problems have been generated by the authors' bias which looks ahead to the perspectives to develop the approach introduced in our paper toward a mathematical theory of dynamical systems which aims at modeling the complex behavior of living systems.

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