

SOME ISOPERIMETRIC INEQUALITIES WITH RESPECT TO MONOMIAL WEIGHTS

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Abstract. We solve a class of isoperimetric problems on \mathbb{R}_+^2 with respect to monomial weights. Let α and β be real numbers such that $0 \leq \alpha < \beta + 1$, $\beta \leq 2\alpha$. We show that, among all smooth sets Ω in \mathbb{R}_+^2 with fixed weighted measure $\iint_{\Omega} y^{\beta} dx dy$, the weighted perimeter $\int_{\partial\Omega} y^{\alpha} ds$ achieves its minimum for a smooth set which is symmetric w.r.t. to the y -axis, and is explicitly given. Our results also imply an estimate of a weighted Cheeger constant and a bound for eigenvalues of some nonlinear problems.

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1. INTRODUCTION

The last two decades have seen a growing interest in weighted isoperimetric problems in the Euclidean space. In many cases, volume and perimeter in those problems carried the same weight. Such a setting corresponds to *manifolds with density*, see for instance [5, 7, 8, 12, 15–18, 21, 36, 37, 41–43] and the references therein.

More recently, also problems with *different* weight functions for perimeter and volume were addressed, see for example [2–4, 6, 23–26, 33, 38, 39, 45, 46] and the references therein.

The minimizing sets for the isoperimetric problems are radial in all of the above-mentioned articles. There is only a sparse literature in the Euclidean setting with non-radial isoperimetric sets, see [1, 20, 27].

We wish to mention that isoperimetric problems w.r.t. non-Euclidean metrics, in particular in the framework of the Grushin space, have received a lot of attention recently, too, see for instance [28–30, 40] and references therein.

In this paper we deal with an isoperimetric problem in the upper half-plane, where the minimizers are *not* half balls, centered at the origin.

More precisely, we study the following problem:

$$\text{Minimize } \int_{\partial\Omega} y^{\alpha} ds \text{ among all smooth sets } \Omega \subset \mathbb{R}_+^2 \text{ satisfying } \iint_{\Omega} y^{\beta} dx dy = 1$$

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or equivalently

$$(P) \quad \inf \left\{ \frac{\int_{\partial\Omega} y^\alpha ds}{\left[\iint_{\Omega} y^\beta dx dy \right]^{(\alpha+1)/(\beta+2)}} : 0 < \iint_{\Omega} y^\beta dx dy < +\infty \right\} =: \mu(\alpha, \beta).$$

Our main result, proved in Section 2, is the following:

Theorem 1.1. *Assume that*

$$0 \leq \alpha < \beta + 1 \tag{1.1}$$

and

$$\beta \leq 2\alpha. \tag{1.2}$$

Then problem (P) has a minimizer which is given by

$$\Omega^* := \{(x, y) : |x| < f(y), 0 < y < 1\}, \quad \text{where} \tag{1.3}$$

$$f(y) := \int_y^1 \frac{t^{\beta-\alpha+1}}{\sqrt{1-t^{2(\beta-\alpha+1)}}} dt, \quad (0 < y < 1). \tag{1.4}$$

Moreover, we have

$$\mu(\alpha, \beta) = \gamma^{\frac{\alpha+1}{\beta+2}-1} \cdot \left[\frac{(\beta+1)(\beta+2)}{\alpha+1} \right]^{\frac{\alpha+1}{\beta+2}} \cdot \left[B \left(\frac{\alpha+1}{2\gamma}, \frac{1}{2} \right) \right]^{\frac{\gamma}{\beta+2}}, \tag{1.5}$$

where $\gamma := \beta + 1 - \alpha$ and B denotes the Beta function. In particular,

$$\mu(\alpha, 2\alpha) = \sqrt{\frac{2\pi(2\alpha+1)}{\alpha+1}}. \tag{1.6}$$

Remark 1.2. (a) First observe that Ω^* is the half-circle when $\alpha = \beta$. Therefore Theorem 1.1 includes the result obtained by Maderna and Salsa in [37] (see also [11, 15]).

(b) Let $B_1^+ := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1, y > 0\}$. It is elementary to verify that,

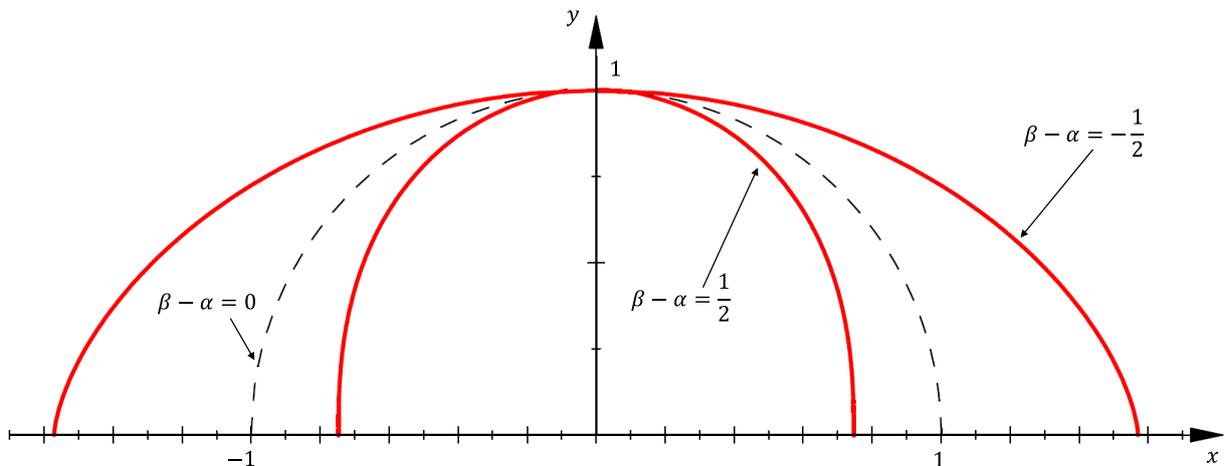
1. if $\beta - \alpha < 0$ then $\Omega \supseteq B_1^+$,
2. if $\beta - \alpha > 0$ then $\Omega \subseteq B_1^+$,
3. if $\beta - \alpha = 0$ then $\Omega \equiv B_1^+$; see Figure 1.

Theorem 1.1 also allows to obtain a *Faber-Krahn*-type inequality for the so-called *weighted Cheeger constant*, and in turn a lower bound for the first eigenvalue for a degenerate elliptic operators. For similar results see also [9, 13, 14, 19, 34, 44, 47].

We emphasize that our main result generalizes the isoperimetric inequality in the Grushin plane proved in [40] (see also [28–30]) which corresponds to Theorem 1.1 with $\alpha = 0$ and $\beta \in (-1, 0]$ (see Rem. 2.15 in Sect. 2).

In a forthcoming paper we will study the N -dimensional case, which has to be treated with a different approach.

Our paper is organized as follows. We state and prove the isoperimetric inequality, Theorem 1.1., in Section 2. We will also point out some relations to recent other literature, at the end of the section, see Remark 2.15. Further, we give two applications of Theorem 1.1., in Section 3. Namely, Theorem 3.1. gives an estimate for

FIGURE 1. Isoperimetric sets for different values of α and β .

a weighted Cheeger constant while Theorem 3.2. provides a lower bound for the first eigenvalue of a class of non-linear degenerate weighted eigenvalue problems.

2. ISOPERIMETRIC INEQUALITY IN THE UPPER HALF PLANE

Let $\mathbb{R}_+^2 := \{(x, y) \in \mathbb{R}^2 : y > 0\}$. Throughout this paper, we assume that $\alpha, \beta \in \mathbb{R}$ and

$$\beta + 1 > 0 \text{ and } \alpha \geq 0. \quad (2.1)$$

If $\Omega \subset \mathbb{R}_+^2$ is measurable, we set

$$\Omega(y) := \{x \in \mathbb{R} : (x, y) \in \Omega\}, \quad (y \in \mathbb{R}_+), \quad (2.2)$$

$$\Omega' := \{y \in \mathbb{R}_+ : \Omega(y) \neq \emptyset\}. \quad (2.3)$$

Further, we define the *weighted area* of Ω by

$$A_\beta(\Omega) := \iint_{\Omega} y^\beta \, dx dy,$$

and the *weighted relative perimeter* of Ω as

$$P_\alpha(\Omega) := \sup \left\{ \iint_{\mathbb{R}_+^2} \operatorname{div}(y^\alpha \mathbf{v}) \, dx dy : \mathbf{v} \in C_0^1(\mathbb{R}_+^2, \mathbb{R}^2), |\mathbf{v}| \leq 1 \text{ in } \Omega \right\}. \quad (2.4)$$

It is well-known that, if Ω is an open, rectifiable set, then the following equality holds

$$P_\alpha(\Omega) = \int_{\partial\Omega \cap \mathbb{R}_+^2} y^\alpha \, d\mathcal{H}_1. \quad (2.5)$$

(\mathcal{H}_1 denotes 1-dimensional Hausdorff-measure.)

Remark 2.1. The following properties of the perimeter are well-known:

Let Ω be measurable with $0 < A_\beta(\Omega) < +\infty$ and $P_\alpha(\Omega) < +\infty$.

Then there exists a sequence of open, rectifiable sets $\{\Omega_n\}$ with $\lim_{n \rightarrow \infty} A_\beta(\Omega \Delta \Omega_n) = 0$ and

$$P_\alpha(\Omega) = \lim_{n \rightarrow \infty} P_\alpha(\Omega_n). \quad (2.6)$$

Further, we have

$$P_\alpha(\Omega) \leq \liminf_{n \rightarrow \infty} P_\alpha(\Omega_n) \quad (2.7)$$

for any sequence of open, rectifiable sets $\{\Omega_n\}$ satisfying $\lim_{n \rightarrow \infty} A_\beta(\Omega \Delta \Omega_n) = 0$.

We define the ratio

$$\mathcal{R}_{\alpha,\beta}(\Omega) := \frac{P_\alpha(\Omega)}{[A_\beta(\Omega)]^{(\alpha+1)/(\beta+2)}}, \quad (0 < A_\beta(\Omega) < +\infty).$$

Remark 2.2. We have $\mathcal{R}_{\alpha,\beta}(t\Omega) = \mathcal{R}_{\alpha,\beta}(\Omega)$ for every $t > 0$.

We study the following isoperimetric problem:

$$(\mathbf{P}) \quad \inf \{ \mathcal{R}_{\alpha,\beta}(\Omega) : 0 < A_\beta(\Omega) < +\infty \} =: \mu(\alpha, \beta).$$

Our first aim is to reduce the class of admissible sets in the isoperimetric problem (\mathbf{P}) .

Throughout our proofs let C denote a generic constant which may vary from line but does not depend on the other parameters.

The first two Lemmata give necessary conditions for a minimizer to exist.

Lemma 2.3. *If $\alpha > \beta + 1$, then*

$$\mu(\alpha, \beta) = 0, \quad (2.8)$$

and (\mathbf{P}) has no minimizer.

Proof. Let $\Omega(t) := (0, t) \times (0, 1)$, ($t > 0$). Then

$$\begin{aligned} \int_{\partial\Omega(t) \cap \mathbb{R}_+^2} y^\alpha \, ds &= t + 2 \int_0^t y^\alpha \, dy = t + \frac{2}{\alpha + 1}, \\ \iint_{\Omega(t)} y^\beta \, dx dy &= \frac{t}{\beta + 1}. \end{aligned}$$

Hence

$$\mathcal{R}_{\alpha,\beta}(\Omega(t)) = \frac{t + (2/(\alpha + 1))}{[t/(\beta + 1)]^{(\alpha+1)/(\beta+2)}} \longrightarrow 0, \text{ as } t \rightarrow +\infty,$$

and the assertion follows. \square

Lemma 2.4. *Assume $\alpha < \beta + 1$. Further, let $\Omega \subset \mathbb{R}_+^2$ be a nonempty, open and rectifiable set, which is not simply connected. Then there exists a nonempty, open and rectifiable set $U \subset \mathbb{R}_+^2$ which is simply connected, such that*

$$\mathcal{R}_{\alpha,\beta}(U) < \mathcal{R}_{\alpha,\beta}(\Omega). \quad (2.9)$$

Proof. (i) First assume that Ω is connected. Let G be the unbounded component of $\mathbb{R}^2 \setminus \overline{\Omega}$ and set $U := \mathbb{R}^2 \setminus \overline{G}$. Then U is simply connected with $\Omega \subset U \subset \mathbb{R}_+^2$ and $\partial U \subset \partial\Omega$, so that (2.9) follows.

(ii) Next, let $\Omega = \cup_{k=1}^m \Omega_k$, with mutually disjoint, nonempty, open, connected and rectifiable sets Ω_k , ($k = 1, \dots, m$, $m \geq 2$). We set $\mathcal{R}_{\alpha,\beta}(\Omega) =: \lambda$. Let us assume that $\mathcal{R}_{\alpha,\beta}(\Omega_k) \geq \lambda$ for every $k \in \{1, \dots, m\}$. Then we have, since $(\alpha + 1)/(\beta + 2) < 1$,

$$\begin{aligned} P_\alpha(\Omega) &= \sum_{k=1}^m P_\alpha(\Omega_k) \geq \lambda \sum_{k=1}^m [A_\beta(\Omega_k)]^{(\alpha+1)/(\beta+2)} \\ &> \lambda \left[\sum_{k=1}^m A_\beta(\Omega_k) \right]^{(\alpha+1)/(\beta+2)} = \lambda [A_\beta(\Omega)]^{(\alpha+1)/(\beta+2)} \\ &= P_\alpha(\Omega), \end{aligned}$$

a contradiction. Hence there exists a number $k_0 \in \{1, \dots, m\}$ with $\mathcal{R}_{\alpha,\beta}(\Omega_{k_0}) < \lambda$. Then, repeating the argument of part (i), with Ω_{k_0} in place of Ω , we again arrive at (2.9). \square

Lemma 2.5. *There holds*

$$\mu(\beta + 1, \beta) = \beta + 1, \quad (2.10)$$

but (P) has no open rectifiable minimizer.

Proof. With $\Omega(t)$ as in the proof of Lemma 2.3, we calculate $\mathcal{R}_{\beta+1,\beta}(\Omega(t)) > \beta + 1$ and

$$\lim_{t \rightarrow \infty} \mathcal{R}_{\beta+1,\beta}(\Omega(t)) = \beta + 1. \quad (2.11)$$

Let $\Omega \subset \mathbb{R}_+^2$ be open, rectifiable and simply connected. Then $\partial\Omega$ is a closed Jordan curve \mathcal{C} with counter-clockwise representation

$$\mathcal{C} = \{(\xi(t), \eta(t)) : 0 \leq t \leq a\}, \quad (a \in 0, +\infty),$$

where $\xi, \eta \in C[0, a] \cap C^{0,1}(0, a)$, $\xi(0) = \xi(a)$, $\eta(0) = \eta(a)$, and $(\xi')^2 + (\eta')^2 > 0$ on $[0, a]$. Using Green's Theorem we evaluate

$$\begin{aligned} A_\beta(\Omega) &= - \int_{\mathcal{C}} \frac{y^{\beta+1}}{\beta+1} dx = - \int_0^a \frac{[\eta(t)]^{\beta+1}}{\beta+1} \xi'(t) dt \\ &\leq \frac{1}{\beta+1} \int_0^a [\eta(t)]^{\beta+1} \sqrt{(\xi')^2 + (\eta')^2} dt = \frac{1}{\beta+1} P_{\beta+1}(\Omega). \end{aligned} \quad (2.12)$$

Equality in (2.12) can hold only if $\eta' \equiv 0$ and $\xi' \leq 0$ on $\partial\Omega \cap \mathbb{R}_+^2$, that is, if $\partial\Omega \cap \mathbb{R}_+^2$ is a single straight segment which is parallel to the x -axis. But this is impossible. Hence we find that

$$P_{\beta+1}(\Omega) > (\beta + 1)A_\beta(\Omega). \quad (2.13)$$

To show the assertion in the general case, we proceed similarly as in the proof of Lemma 2.4:

Assume first that Ω is connected and define the sets G and U as in the last proof. Using (2.13), with U in place of Ω , we obtain

$$\begin{aligned} P_{\beta+1}(\Omega) &\geq P_{\beta+1}(U) \\ &> (\beta+1)A_\beta(U) \geq (\beta+1)A_\beta(\Omega). \end{aligned} \quad (2.14)$$

Finally, let Ω be open and rectifiable. Then $\Omega = \cup_{k=1}^m \Omega_k$, with mutually disjoint, connected sets Ω_k , ($k = 1, \dots, m$). Then (2.14) yields

$$\begin{aligned} P_{\beta+1}(\Omega) &= \sum_{k=1}^m P_{\beta+1}(\Omega_k) \\ &> (\beta+1) \sum_{k=1}^m A_\beta(\Omega_k) = (\beta+1)A_\beta(\Omega). \end{aligned} \quad (2.15)$$

Now the assertion follows from (2.15) and (2.11). \square

Lemma 2.6. *Let $2\alpha < \beta$. Then (2.8) holds and (\mathbf{P}) has no minimizer.*

Proof. Let $z(t) := (0, t)$, ($t \geq 2$). Then we have for all $t \geq 2$,

$$\begin{aligned} \int_{\partial B_1(z(t))} y^\alpha ds &\leq Ct^\alpha \quad \text{and} \\ \iint_{B_1(z(t))} y^\beta dx dy &\geq Ct^\beta. \end{aligned}$$

This implies

$$\mathcal{R}_{\alpha,\beta}(B_1(z(t))) \leq Ct^{\alpha-\beta(1+\alpha)/(2+\beta)} \longrightarrow 0, \quad \text{as } t \rightarrow +\infty, \quad (2.16)$$

and the assertion follows. \square

Next we recall the definition of the Steiner symmetrization w.r.t. the x -variable. If Ω is measurable, we set

$$S(\Omega) := \{(x, y) : x \in S(\Omega(y)), y \in \Omega'\}$$

where $\Omega(y)$, Ω' are defined in (2.2) and

$$S(\Omega(y)) := \begin{cases} (-\frac{1}{2}\mathcal{L}^1(\Omega(y)), +\frac{1}{2}\mathcal{L}^1(\Omega(y))) & \text{if } 0 < \mathcal{L}^1(\Omega(y)) < +\infty \\ \emptyset & \text{if } \mathcal{L}^1(\Omega(y)) = 0 \\ \mathbb{R} & \text{if } \mathcal{L}^1(\Omega(y)) = +\infty \end{cases}.$$

Note that $S(\Omega)(y)$ is a symmetric interval with $\mathcal{L}^1(S(\Omega)(y)) = \mathcal{L}^1(\Omega(y))$.

Since the weight functions in the functionals P_α and A_β do not depend on x , we have the following well-known properties, see ([32], Prop. 3).

Lemma 2.7. *Let $\Omega \subset \mathbb{R}_+^2$ be measurable. Then*

$$P_\alpha(\Omega) \geq P_\alpha(S(\Omega)) \quad \text{and} \quad (2.17)$$

$$A_\beta(\Omega) = A_\beta(S(\Omega)). \quad (2.18)$$

For nonempty open sets Ω with $\Omega = S(\Omega)$ we set $\Omega_t := \{(x, y) \in \Omega : y > t\}$ and

$$y^+ := \inf\{t \geq 0 : A_\beta(\Omega_t) = 0\}, \quad (2.19)$$

$$y^- := \sup\{t \geq 0 : A_\beta(\Omega \setminus \Omega_t) = 0\}. \quad (2.20)$$

Remark 2.8. Assume that $\Omega \subset \mathbb{R}_+^2$ is a bounded, open and rectifiable set with $0 < A_\beta(\Omega) < +\infty$ and $\Omega = S(\Omega)$. Then it has the following representation,

$$\begin{aligned} \Omega &= \{(x, y) : |x| < f(y), y^- < y < y^+\}, \quad \text{where} \\ f &: (y^-, y^+) \rightarrow (0, +\infty) \text{ is lower semi-continuous.} \end{aligned} \quad (2.21)$$

Lemma 2.9. Let Ω be a nonempty, bounded, open and rectifiable set with $\Omega = S(\Omega)$. Then we have

$$P_\alpha(\Omega) \geq \frac{2}{\alpha + 1}((y^+)^{\alpha+1} - (y^-)^{\alpha+1}) \quad \text{and} \quad (2.22)$$

$$P_\alpha(\Omega) \geq 2y^\alpha f(y) \quad \forall y \in (y^-, y^+), \quad (2.23)$$

where f is given by (2.21).

Proof. Assume first that Ω is represented by (2.21) where

$$f \in C^1[y^-, y^+] \quad \text{and} \quad f(y^-) = f(y^+) = 0. \quad (2.24)$$

Then we have for every $y \in (y^-, y^+)$,

$$\begin{aligned} P_\alpha(\Omega) &= 2 \int_{y^-}^{y^+} t^\alpha \sqrt{1 + (f'(t))^2} dt \geq 2 \int_y^{y^+} t^\alpha \sqrt{1 + (f'(t))^2} dt \\ &\geq 2y^\alpha \int_y^{y^+} |f'(t)| dt \geq 2y^\alpha f(y). \end{aligned}$$

Furthermore, there holds

$$\begin{aligned} P_\alpha(\Omega) &= 2 \int_{y^-}^{y^+} t^\alpha \sqrt{1 + (f'(t))^2} dt \geq 2 \int_{y^-}^{y^+} t^\alpha dt \\ &= \frac{2}{\alpha + 1}((y^+)^{\alpha+1} - (y^-)^{\alpha+1}). \end{aligned}$$

In the general case the assertions follow from these calculations by approximation with sets Ω of the type given by (2.21), (2.24). \square

Lemma 2.10. Assume that

$$\alpha < \beta + 1, \quad \text{and} \quad (2.25)$$

$$\beta < 2\alpha, \quad (2.26)$$

and let Ω be a bounded, open and rectifiable set with $A_\beta(\Omega) = 1$, $\Omega = S(\Omega)$ and $P_\alpha(\Omega) < \mu(\alpha, \beta) + 1$. Then there exist positive numbers C_1 and C_2 which depend only on α and β such that

$$C_1 \geq y^+ \quad \text{and} \quad y^+ - y^- \geq C_2. \quad (2.27)$$

Proof. By (2.22) and (2.23) we have

$$\frac{1}{2}(\mu(\alpha, \beta) + 1)(\alpha + 1) \geq (y^+)^{\alpha+1} - (y^-)^{\alpha+1} \quad \text{and} \quad (2.28)$$

$$\frac{\mu(\alpha, \beta) + 1}{2} \geq y^\alpha f(y) \quad \forall y \in (y^-, y^+). \quad (2.29)$$

It follows that

$$\begin{aligned} 1 = A_\beta(\Omega) &= 2 \int_{y^-}^{y^+} y^\beta f(y) dy \\ &\leq (\mu(\alpha, \beta) + 1) \int_{y^-}^{y^+} y^{\beta-\alpha} dy = \frac{\mu(\alpha, \beta) + 1}{\beta + 1 - \alpha} ((y^+)^{\beta+1-\alpha} - (y^-)^{\beta+1-\alpha}). \end{aligned} \quad (2.30)$$

Setting $z := \frac{y^-}{y^+} (\in [0, 1))$, we obtain from (2.28) and (2.30),

$$\begin{aligned} (y^+)^{\alpha+1} &\leq \frac{(\mu(\alpha, \beta) + 1)(\alpha + 1)}{2(1 - z^{\alpha+1})} \quad \text{and} \\ (y^+)^{\beta+1-\alpha} &\geq \frac{\beta + 1 - \alpha}{(\mu(\alpha, \beta) + 1)(1 - z^{\beta+1-\alpha})}, \end{aligned} \quad (2.31)$$

which implies that

$$f(z) := \frac{(1 - z^{\alpha+1})^{(\beta+1-\alpha)/(\alpha+1)}}{1 - z^{\beta+1-\alpha}} \leq C,$$

with a constant C which depends only on α and β . By (2.26) we have that

$$\lim_{z \rightarrow 1^-} f(z) = +\infty.$$

Hence it follows that

$$z = \frac{y^-}{y^+} \leq 1 - \delta \quad \text{for some } \delta \in (0, 1). \quad (2.32)$$

Using (2.31) and (2.32) this leads to (2.27). \square

Lemma 2.11. *Assume (2.25) and (2.26). Then problem (P) has a minimizer Ω^* which is symmetric w.r.t. the y -axis.*

Proof. We proceed in 4 steps.

Step 1: *A minimizing sequence:*

Let $\{\Omega_n\}$ a minimizing sequence, that is, $\lim_{n \rightarrow \infty} \mathcal{R}_{\alpha, \beta}(\Omega_n) = \mu(\alpha, \beta)$. In view of the Remarks 2.1 and 2.2 and the Lemmata 2.4 and 2.9 we may assume that Ω_n is open, simply connected and rectifiable with $\Omega_n = S(\Omega_n)$,

$A_\beta(\Omega_n) = 1$ and $P_\alpha(\Omega_n) \leq \mu(\alpha, \beta) + \frac{1}{n}$, ($n \in \mathbb{N}$).

Step 2: *Parametrization of $\partial\Omega_n$:*

Denote $\mathcal{C}_n := \overline{\partial\Omega_n \cap \{(x, y) : x > 0, y > 0\}}$. By a standard approximation procedure we may assume that \mathcal{C}_n is a simple smooth curve with

$$\mathcal{C}_n = \{(x_n(s), y_n(s)) : s \in [0, L_n]\}, \quad (2.33)$$

where s denotes the usual arclength parameter, $L_n \in (0, +\infty)$, $x_n, y_n \in C[0, L_n] \cap C^{0,1}(0, L_n)$, $x_n(s) > 0$ and $y_n(s) > 0$ for every $s \in (0, L_n)$, ($n \in \mathbb{N}$). We orientate \mathcal{C}_n in such a way that the mapping $s \mapsto y_n(s)$ is nonincreasing and $x_n(0) = 0$. Setting $y_n(0) =: y_n^+$ and $y_n(L_n) =: y_n^-$, we have by Lemma 2.9,

$$C_1 \geq y_n^+ \quad \text{and} \quad y_n^+ - y_n^- \geq C_2, \quad (2.34)$$

where C_1 and C_2 do not depend on n . Note that $x_n(L_n) = 0$ in case that $y_n^- > 0$. Further, Lemma 2.9, (2.23) shows that

$$\mu(\alpha, \beta) + \frac{1}{n} \geq 2y_n(s)^\alpha x_n(s), \quad \forall s \in [0, L_n]. \quad (2.35)$$

For our purposes it will be convenient to work with another parametrization of \mathcal{C}_n : We set

$$\begin{aligned} \varphi_n(s) &:= \frac{2}{P_\alpha(\Omega_n)} \int_0^s y_n^\alpha(t) dt, \quad \text{and} \\ X_n(\varphi_n(s)) &:= x_n(s), \quad Y_n(\varphi_n(s)) := y_n(s), \quad (s \in [0, L_n]). \end{aligned}$$

Then $X_n, Y_n \in C[0, 1] \cap C^{0,1}(0, 1)$, and we evaluate

$$1 = A_\beta(\Omega_n) = \frac{2}{\beta + 1} \int_0^1 Y_n^{\beta+1}(\sigma) X_n'(\sigma) d\sigma = -2 \int_0^1 Y_n^\beta(\sigma) Y_n'(\sigma) X_n(\sigma) d\sigma, \quad (2.36)$$

$$P_\alpha(\Omega_n) = 2 \int_0^1 Y_n^\alpha(\sigma) \sqrt{(X_n'(\sigma))^2 + (Y_n'(\sigma))^2} d\sigma, \quad (2.37)$$

$$\frac{d}{d\sigma} X_n(\sigma) = \frac{P_\alpha(\Omega_n)}{2} (y_n(s))^{-\alpha} x_n'(s), \quad (2.38)$$

$$\frac{d}{d\sigma} Y_n(\sigma) = \frac{P_\alpha(\Omega_n)}{2} (y_n(s))^{-\alpha} y_n'(s), \quad (\sigma = \varphi_n(s)). \quad (2.39)$$

Step 3: *Limit of the minimizing sequence:*

Since $(x_n'(s))^2 + (y_n'(s))^2 \equiv 1$, we obtain from (2.34) and (2.39) that the family $\{Y_n^{\alpha+1}\}$ is equibounded and uniformly Lipschitz continuous on $(0, 1)$. Hence there is a function $Y \in C[0, 1]$ with $Y^{\alpha+1} \in C^{0,1}(0, 1)$ such that, up to a subsequence,

$$Y_n^{\alpha+1} \longrightarrow Y^{\alpha+1} \quad \text{uniformly on } [0, 1]. \quad (2.40)$$

Moreover, setting $Y(0) =: y^+$, $Y(1) =: y^-$, the bounds (2.27) are in place and $\sigma \mapsto Y(\sigma)$ is nonincreasing.

Let

$$\sigma_0 := \sup\{\sigma \in (0, 1] : Y(\sigma) > 0\}.$$

Then from (2.35) and (2.38) we obtain that the families $\{X_n\}$ and $\{dX_n/d\sigma\}$ are equibounded on every closed subset of $[0, \sigma_0)$. Hence there exists a function $X \in C[0, \sigma_0)$ which is locally Lipschitz continuous on $[0, \sigma_0)$, such that, up to a subsequence,

$$X_n \rightarrow X \quad \text{uniformly on closed subsets of } [0, \sigma_0). \quad (2.41)$$

Moreover, from (2.35) and (2.38) we find that

$$\mu(\alpha, \beta) \geq 2(Y(\sigma))^\alpha X(\sigma) \quad \text{and} \quad (2.42)$$

$$\mu(\alpha, \beta) + 1 \geq 2(Y(\sigma))^\alpha X'(\sigma), \quad (\sigma \in [0, \sigma_0)). \quad (2.43)$$

Let Ω be the set in \mathbb{R}_+^2 with $\Omega = S(\Omega)$ such that $\partial\Omega \cap \{(x, y) : x > 0, y > 0\}$ is represented by the pair of functions

$$X(\sigma), Y(\sigma) : \quad \sigma \in [0, \sigma_0).$$

In view of (2.43) we have that

$$A_\beta(\Omega) = \frac{2}{\beta + 1} \int_0^{\sigma_0} Y^{\beta+1} X' \, d\sigma. \quad (2.44)$$

Step 4: A minimizing set:

We prove that

$$A_\beta(\Omega) = 1, \quad P_\alpha(\Omega) = \mu(\alpha, \beta). \quad (2.45)$$

In order to prove the first equality, since $A_\beta(\Omega_n) = 1$, we prove that

$$\lim_{n \rightarrow +\infty} A_\beta(\Omega_n) = A_\beta(\Omega). \quad (2.46)$$

Fix some $\delta \in (0, \sigma_0)$. Then (2.38), (2.39), (2.43), (2.40) and (2.41) yield

$$\begin{aligned} & \left| \int_0^{\sigma_0-\delta} Y_n^{\beta+1} X'_n \, d\sigma - \int_0^{\sigma_0-\delta} Y^{\beta+1} X' \, d\sigma \right| \\ & \leq \left| \int_0^{\sigma_0-\delta} Y_n^{\beta+1} (X'_n - X') \, d\sigma \right| + \int_0^{\sigma_0-\delta} |Y_n^{\beta+1} - Y^{\beta+1}| |X'| \, d\sigma \\ & \leq Y_n^{\beta+1} |X_n - X| \Big|_0^{\sigma_0-\delta} + (\beta + 1) \int_0^{\sigma_0-\delta} Y_n^\beta |Y'_n| |X_n - X| \, d\sigma \\ & \quad + \int_0^{\sigma_0-\delta} |Y_n^{\beta+1} - Y^{\beta+1}| |X'| \, d\sigma \quad \longrightarrow 0, \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (2.47)$$

Further, (2.38) and (2.43) give

$$\lim_{t \rightarrow 0} \int_{\sigma_0-t}^{\sigma_0} (Y_n)^{\beta+1} X'_n \, d\sigma = 0, \quad \text{uniformly for all } n \in \mathbb{N}, \text{ and} \quad (2.48)$$

$$\lim_{t \rightarrow 0} \int_{\sigma_0-t}^{\sigma_0} Y^{\beta+1} X' \, d\sigma = 0. \quad (2.49)$$

Now (2.48), (2.49), (2.47) and (2.44) yield (2.46) and therefore the first of the equalities in (2.45).

Now we prove the second inequality in (2.45). With δ as above we also have

$$\begin{aligned} P_\alpha(\Omega_n) &= 2 \int_0^1 Y_n^\alpha \sqrt{(X'_n)^2 + (Y'_n)^2} \, d\sigma \\ &\geq \int_0^{\sigma_0-\delta} (Y_n^\alpha - Y^\alpha) \sqrt{(X'_n)^2 + (Y'_n)^2} \, d\sigma + \int_0^{\sigma_0-\delta} Y^\alpha \sqrt{(X'_n)^2 + (Y'_n)^2} \, d\sigma \\ &=: I_{n,\delta}^1 + I_{n,\delta}^2. \end{aligned}$$

In view of (2.40) and (2.41) it follows that $\lim_{n \rightarrow \infty} I_{n,\delta}^1 = 0$.

On the other hand, we have

$$\liminf_{n \rightarrow \infty} I_{n,\delta}^2 \geq \int_0^{\sigma_0-\delta} Y^\alpha \sqrt{(X')^2 + (Y')^2} \, d\sigma.$$

Define

$$S := \{\sigma \in (0, 1) : X(\sigma) > 0, Y(\sigma) > 0\}.$$

Letting $\delta \rightarrow 0$ we obtain

$$\begin{aligned} \mu(\alpha, \beta) &\geq \liminf_{n \rightarrow \infty} P_\alpha(\Omega_n) \geq \int_0^{\sigma_0} Y^\alpha \sqrt{(X')^2 + (Y')^2} \, d\sigma \\ &\geq \int_S Y^\alpha \sqrt{(X')^2 + (Y')^2} \, d\sigma \\ &= P_\alpha(\Omega) \geq \mu(\alpha, \beta). \end{aligned}$$

Hence Ω is a minimizing set.

Note that Ω must be simply connected in view of Lemma 2.4, which implies that there is a number $\sigma_1 \in [0, \sigma_0)$ such that

$$S = (\sigma_1, \sigma_0). \quad (2.50)$$

This finishes the proof of Lemma 2.11. □

Next we obtain differential equations for the functions X and Y in the proof in Lemma 2.11.

Lemma 2.12. *Assume (2.25) and (2.26). Then the minimizer Ω obtained in Lemma 2.8 is bounded, and its boundary given parametrically by*

$$\begin{aligned} & \partial\Omega \cap \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\} \\ & = \{(x(s), y(s)) : s \in (0, L)\}, \quad (s : \text{arclength}), \end{aligned} \quad (2.51)$$

where the functions $x, y \in C^2[0, L]$ satisfy the following equations:

$$-(y^\alpha y')' + \alpha y^{\alpha-1} = \lambda(\beta + 1)y^\beta x', \quad (2.52)$$

$$-(y^\alpha x')' = -\lambda(y^{\beta+1})', \quad (2.53)$$

together with the boundary conditions

$$x(0) = 0, \quad y(0) = y^+, \quad y'(0) = 0, \quad (2.54)$$

$$y(L) = y^-, \quad \text{and either} \quad (2.55)$$

$$(i) \quad y^- > 0, \quad y'(L) = 0, \quad \text{or} \quad (2.56)$$

$$(ii) \quad y^- = 0, \quad \lim_{s \nearrow L} y^\alpha(s)x'(s) = 0,$$

for some numbers $\lambda > 0$ and $0 \leq y^- < y^+$. Finally, the curve (2.51) is strictly convex.

Proof. We proceed in 4 steps.

Step 1: Euler equations:

After a rescaling the parameter σ , we see that the functions $X(\sigma)$ and $Y(\sigma)$ in the previous proof annihilate the first variation of the functional

$$\int_0^1 Y^\alpha \sqrt{(X')^2 + (Y')^2} \, d\sigma$$

under the constraint

$$\int_0^1 Y^{\beta+1} X' \, d\sigma = \text{const. } (> 0).$$

Hence X and Y satisfy the Euler equations

$$-\frac{d}{d\sigma} \left(\frac{Y^\alpha Y'}{\sqrt{(X')^2 + (Y')^2}} \right) + \alpha Y^{\alpha-1} \sqrt{(X')^2 + (Y')^2} = \lambda(\beta + 1)Y^\beta X', \quad (2.57)$$

$$-\frac{d}{d\sigma} \left(\frac{Y^\alpha X'}{\sqrt{(X')^2 + (Y')^2}} \right) = -\lambda \frac{d}{d\sigma} (Y^{\beta+1}), \quad (2.58)$$

where λ is a Lagrangian multiplier, and

$$X(\sigma) > 0, \quad Y(\sigma) > 0. \quad (2.59)$$

In addition, the following boundary conditions are satisfied:

$$X(0) = 0, \quad Y(0) =: y^+ > 0, \quad Y'(0) = 0, \quad (2.60)$$

$$\text{if } Y(1) = 0 \text{ and } \lim_{\sigma \rightarrow 1} X(\sigma) \text{ exists, then } \lim_{\sigma \rightarrow 1} Y^\alpha(\sigma)X'(\sigma) = 0, \quad (2.61)$$

$$\text{if } Y(1) =: y^- > 0, \text{ then } X(1) = 0 \text{ and } X'(1) = 0. \quad (2.62)$$

Step 2: Boundedness:

It will be more convenient to rewrite the above conditions in terms of the arclength parameter s : Set

$$s \equiv \psi(\sigma) := \int_0^\sigma \sqrt{(X'(t))^2 + (Y'(t))^2} dt,$$

$$x(s) := X(\psi(s)), \quad y(s) := Y(\psi(s)), \quad (s \in (0, L)),$$

where $L \in (0, +\infty]$. Then we have $(x'(s))^2 + (y'(s))^2 = 1$ so that (2.57), (2.58) yield the system of equations (2.52), (2.53).

Now we prove that x, y are smooth functions. Firstly let us prove that $x, y \in C^1(]0, L[)$. Indeed, by the previous step x, y are Lipschitz continuous functions, *i.e.* $x, y \in W_{loc}^{1,\infty}(]0, L[)$. By Lemma 8.1, p. 204 of [10], there exists a constant d such that

$$y^\alpha x' - \lambda y^{\beta+1} = d, \quad \text{a.e. in }]0, L[,$$

or equivalently, since $y(\sigma) > 0$ in $]0, L[$,

$$x' = \lambda y^{\beta+1-\alpha} + d y^{-\alpha}, \quad \text{a.e. in }]0, L[. \quad (2.63)$$

Since $y^\alpha \in C(]0, L[)$, we deduce that $x' \in C(]0, L[)$, that is

$$x \in C^1(]0, L[).$$

Now by (2.53)

$$-(y^\alpha y')' = \lambda(\beta + 1)y^\beta x' - \alpha y^{\alpha-1} \in C(]0, L[), \quad (2.64)$$

and, since $y^{\alpha+1}, y$ are locally Lipschitz continuous functions by the previous step, we conclude that

$$y^\alpha y' \in W_{loc}^{1,\infty}(]0, L[).$$

By Remark 6, p. 204 of [10], we conclude that

$$y^\alpha y' \in C_{loc}^1(]0, L[).$$

Since $y^\alpha \in C(]0, L[)$, we deduce $y' \in C(]0, L[)$ or, equivalently

$$y \in C^1(]0, L[).$$

By (2.63) this implies that $x' \in C^1(]0, L[)$, *i.e.* $x \in C^2(]0, L[)$.

Moreover by (2.64), $-(y^\alpha y')' \in C^1(]0, L[)$ since $y^\alpha \in C^1(]0, L[)$, this means that $y' \in C^1(]0, L[)$. In a similar way we deduce that also $y \in C^2(]0, L[)$. By repeating this argument we conclude that x, y are smooth functions on $]0, L[$.

Integrating (2.53) we obtain

$$y^\alpha x' = \lambda y^{\beta+1} + d, \quad (2.65)$$

for some $d \in \mathbb{R}$.

Assume first that $x(s)$ is unbounded. Then $L = +\infty$, and in view of (2.42) we have that $\lim_{s \rightarrow \infty} y(s) = 0$. If $\alpha = 0$, then this would imply $P_0(\Omega) = +\infty$, which is impossible. Hence we may restrict ourselves to the case $\alpha > 0$.

There is a sequence $s_n \rightarrow +\infty$ such that $\lim_{n \rightarrow \infty} x'(s_n) = 1$. Using $s = s_n$ in (2.65) and passing to the limit $n \rightarrow \infty$ gives $d = 0$. Plugging this into (2.52), we find

$$-(y^\alpha y')' + \alpha y^{\alpha-1} = \lambda^2(\beta+1)y^{2\beta+1-\alpha}. \quad (2.66)$$

Multiplying (2.66) with $y^\alpha y'$ and integrating, we obtain

$$-y^{2\alpha}(y')^2 + \alpha y^{2\alpha} = \lambda^2(\beta+1)y^{2\beta+2} + D,$$

or equivalently,

$$-(y')^2 + \alpha = \lambda^2(\beta+1)y^{2\beta+2-2\alpha} + Dy^{-2\alpha}, \quad (2.67)$$

for some $D \in \mathbb{R}$. Using $s = s_n$ in (2.67) and taking into account that $\lim_{n \rightarrow +\infty} y'(s_n) = 0$, $\beta+1-\alpha > 0$ and $\alpha > 0$, we arrive again at a contradiction. Hence $x(s)$ is bounded, and we deduce $L < +\infty$.

Finally we can apply ([10], Thm. 8.2, p. 204) and we deduce the existence of continuous representatives on $[0, L]$ \bar{x} and \bar{y} of x and y respectively such that

$$\bar{x}, \bar{y} \in C^2([0, L]) \quad \text{and} \quad x = \bar{x}, y = \bar{y} \quad \text{a.e. in } [0, L].$$

We identify x and y with their continuous representative \bar{x} , \bar{y} .

Therefore we deduce that $x, y \in C^2[0, L]$ and the boundary conditions (2.54)–(2.56).

Step 3: λ is positive:

Multiplying (2.52) with y and integrating from $s = 0$ to $s = L$ gives

$$-\int_0^L (y^\alpha y')' y \, ds + \alpha \int_0^L y^\alpha \, ds = \lambda(\beta+1) \int_0^L y^{\beta+1} x' \, ds.$$

Using integration by parts this yields

$$-y^{\alpha+1} y' \Big|_0^L + \int_0^L y^\alpha ((y')^2 + \alpha) \, ds = \lambda(\beta+1) y^{\beta+1} x \Big|_0^L + \lambda(\beta+1)^2 \int_0^L y^\beta (-y') x \, ds.$$

The first two boundary terms in this identity vanish due to the boundary conditions (2.54)–(2.56) and the two integrals are positive since $y' \leq 0$ and $y' \neq 0$. It follows that

$$\lambda > 0. \quad (2.68)$$

Now, considering equation (2.65) at $s = L$ and taking into account the boundary conditions (2.56), we find

$$0 \geq \lambda(y^-)^{\beta+1} + d,$$

which implies that

$$d \leq 0. \quad (2.69)$$

Step 4: Strict convexity:

From (2.52) and (2.53) we obtain

$$\begin{aligned} x'' &= \lambda(\beta + 1)y^{\beta-\alpha}y' - \frac{\alpha}{y}x'y', \\ y'' &= -\lambda(\beta + 1)y^{\beta-\alpha}x' + \frac{\alpha}{y}(x')^2. \end{aligned}$$

Hence, using (2.65), we find for the curvature $\kappa(s)$ of the curve $(x(s), y(s))$, $(s \in (0, L))$,

$$\begin{aligned} \kappa &= \frac{-x'y'' + y'x''}{((x')^2 + (y')^2)^{3/2}} = -x'y'' + y'x'' \\ &= -\frac{\alpha}{y}x' + \lambda(\beta + 1)y^{\beta-\alpha} = \lambda y^{\beta-\alpha}(\beta + 1 - \alpha) - \alpha dy^{-1-\alpha}. \end{aligned} \tag{2.70}$$

The last expression is positive by (2.68) and (2.69), which means that Ω is strictly convex. The Lemma is proved. \square

Lemma 2.13. *Assume (2.25) and (2.26), and let $\partial\Omega$ be given by (2.51)–(2.56). Then $y^- = 0$.*

Proof. Supposing that $y^- > 0$, we will argue by contradiction. We proceed in 3 steps.

Step 1: Another parametrization of $\partial\Omega$:

Let

$$x_0 := \sup\{x : (x, y) \in \partial\Omega\}.$$

Since Ω is strictly convex, there are functions $u_1, u_2 \in C^2[0, x_0] \cap C[0, x_0]$ such that

$$\partial\Omega \cap \{(x, y) : x > 0, y > 0\} = \{(x, y) : u_1(x) < y < u_2(x), 0 < x < x_0\}, \tag{2.71}$$

$$u_1(0) = y^-, \quad u_2(0) = y^+, \tag{2.72}$$

$$u_1(x_0) = u_2(x_0) =: y_0, \tag{2.73}$$

$$u_1'(0) = u_2'(0) = 0. \tag{2.74}$$

Furthermore, the Euler equations (2.52), (2.53) lead to

$$\frac{\alpha u_1^{\alpha-1}}{\sqrt{1 + (u_1')^2}} - \frac{u_1^\alpha u_1''}{(1 + (u_1')^2)^{3/2}} = -\lambda(\beta + 1)u_1^\beta, \tag{2.75}$$

$$\frac{\alpha u_2^{\alpha-1}}{\sqrt{1 + (u_2')^2}} - \frac{u_2^\alpha u_2''}{(1 + (u_2')^2)^{3/2}} = \lambda(\beta + 1)u_2^\beta. \tag{2.76}$$

Using (2.73) and the fact that

$$\lim_{x \rightarrow x_0} u_1'(x) = +\infty \quad \text{and} \tag{2.77}$$

$$\lim_{x \rightarrow x_0} u_2'(x) = -\infty, \tag{2.78}$$

lead to

$$\frac{u_1^\alpha}{\sqrt{1+(u_1')^2}} = \lambda(y_0^{\beta+1} - u_1^{\beta+1}), \quad (2.79)$$

$$\frac{u_2^\alpha}{\sqrt{1+(u_2')^2}} = \lambda(u_2^{\beta+1} - y_0^{\beta+1}). \quad (2.80)$$

Finally, the boundary conditions (2.72)–(2.74) lead to the following formulas:

$$y_0^{\beta+1} = \frac{(y^-)^{\beta+1}(y^+)^\alpha + (y^+)^{\beta+1}(y^-)^\alpha}{(y^-)^\alpha + (y^+)^\alpha}, \quad (2.81)$$

$$\lambda = \frac{(y^-)^\alpha + (y^+)^\alpha}{(y^+)^{\beta+1} - (y^-)^{\beta+1}}. \quad (2.82)$$

Step 2: Curvature:

In the following, we will refer to points $(x, u_1(x))$ as points of the ‘lower curve’ and to points $(x, u_2(x))$ as points of the ‘upper curve’, $(x \in (0, x_0))$.

The signed curvature κ (see (2.70)) can be expressed in terms of the functions u_1 and u_2 . More precisely, we have

$$\kappa = \begin{cases} \frac{u_1''}{(1+(u_1')^2)^{3/2}} & \text{on the lower curve} \\ -\frac{u_2''}{(1+(u_2')^2)^{3/2}} & \text{on the upper curve} \end{cases}. \quad (2.83)$$

Accordingly, we will write

$$\begin{aligned} \kappa_1(x) &:= \frac{u_1''}{(1+(u_1')^2)^{3/2}}, \\ \kappa_2(x) &:= -\frac{u_2''}{(1+(u_2')^2)^{3/2}}, \quad (x \in (0, x_0)). \end{aligned}$$

Finally, let $s_0 \in (0, L)$ be taken such that $y(s_0) = y_0$ and $x(s_0) = x_0$. Then formula (2.65) taken at $s = s_0$ leads to

$$d = -\lambda y_0^{\beta+1}. \quad (2.84)$$

Plugging this into (2.70) we find

$$\kappa(s) = \lambda \left((\beta + 1 - \alpha)y^{\beta-\alpha} + \alpha y^{-1-\alpha}y_0^{\beta+1} \right), \quad (s \in [0, L]). \quad (2.85)$$

Differentiating (2.85) we evaluate

$$\kappa'(s) = -\lambda y^{-2-\alpha}y' \left(\alpha(\alpha + 1)y_0^{\beta+1} - (\beta + 1 - \alpha)(\beta - \alpha)y^{\beta+1} \right). \quad (2.86)$$

Since $y'(s) < 0$ for $s \in (0, L)$, this in particular implies

$$\kappa'(s) > 0 \quad \forall s \in [0, L], \quad \text{if } \beta \leq \alpha. \quad (2.87)$$

Step 4:

We claim that

$$\kappa_1(x) > \kappa_2(x) \quad \forall x \in (0, x_0). \quad (2.88)$$

First observe that (2.88) immediately follows from (2.87) if $\alpha \geq \beta$. Thus it remains to consider the case

$$0 < \alpha < \beta < 2\alpha. \quad (2.89)$$

From (2.86) and the fact that $y(s) < y_0$ for $s \in [s_0, L]$ we find that

$$\begin{aligned} \kappa'(s) &\geq -\lambda y' y^{-2-\alpha} y_0^{\beta+1} [\alpha(\alpha+1) - (\beta+1-\alpha)(\beta-\alpha)] \\ &= -\lambda y' y^{-2-\alpha} y_0^{\beta+1} (2\alpha-\beta)(\beta+1) \\ &> 0 \quad \forall s \in [s_0, L]. \end{aligned} \quad (2.90)$$

This means that

$$\kappa_1(x) > \kappa_2(x) \quad \forall x \in (x_0 - \varepsilon, x_0), \quad (2.91)$$

for some (small) $\varepsilon > 0$. Now assume that (2.88) does not hold. By (2.91) there exists a number $x_1 \in (0, x_0)$ such that

$$\kappa_1(x) > \kappa_2(x) \quad \forall x \in (x_1, x_0), \quad \text{and} \quad (2.92)$$

$$\kappa_1(x_1) = \kappa_2(x_1). \quad (2.93)$$

We claim that (2.92) implies that

$$u'_1(x_1) < -u'_2(x_1). \quad (2.94)$$

To prove (2.94), observe first that

$$\lim_{x \rightarrow x_0} u'_1(x) = \lim_{x \rightarrow x_0} (-u'_2(x)) = +\infty.$$

Then, integrating (2.92) over (x_1, x_0) leads to

$$\begin{aligned} 1 - \frac{u'_1(x_1)}{\sqrt{1 + (u'_1(x_1))^2}} &= \int_{x_1}^{x_0} \left(\frac{u'_1(x)}{\sqrt{1 + (u'_1(x))^2}} \right)' dx \\ &= \int_{x_1}^{x_0} \frac{u''_1(x)}{[1 + (u'_1(x))^2]^{3/2}} dx = \int_{x_1}^{x_0} \kappa_1(x) dx \\ &> \int_{x_1}^{x_0} \kappa_2(x) dx = \int_{x_1}^{x_0} \frac{-u''_2(x)}{[1 + (u'_2(x))^2]^{3/2}} dx \\ &= \int_{x_1}^{x_0} \left(\frac{-u'_2(x)}{\sqrt{1 + (u'_2(x))^2}} \right)' dx = 1 + \frac{u'_2(x_1)}{\sqrt{1 + (u'_2(x_1))^2}}, \end{aligned}$$

which implies (2.94). Now, from (2.94) we deduce

$$\frac{1}{\sqrt{1 + (u_1'(x_1))^2}} > \frac{1}{\sqrt{1 + (u_2'(x_1))^2}}.$$

Together with (2.79) and (2.80) we obtain from this

$$\frac{y_0^{\beta+1} - (u_1(x_1))^{\beta+1}}{(u_1(x_1))^\alpha} > \frac{(u_2(x_1))^{\beta+1} - y_0^{\beta+1}}{(u_2(x_1))^\alpha},$$

or, equivalently

$$(u_1(x_1))^{\beta+1-\alpha} + (u_2(x_1))^{\beta+1-\alpha} < y_0^{\beta+1} ((u_1(x_1))^{-\alpha} + (u_2(x_1))^{-\alpha}). \quad (2.95)$$

Furthermore, multiplying (2.75) by $(u_1)^{-\alpha}$, respectively (2.76) by $(u_2)^{-\alpha}$, adding both equations and taking into account (2.93) leads to

$$\frac{\alpha}{u_1 \sqrt{1 + (u_1')^2}} + \frac{\alpha}{u_2 \sqrt{1 + (u_2')^2}} = \lambda(\beta + 1) ((u_2)^{\beta-\alpha} - (u_1)^{\beta-\alpha}) \quad \text{at } x = x_1.$$

Using once more (2.79) and (2.80) then gives

$$\frac{\alpha(y_0^{\beta+1} - (u_1)^{\beta+1})}{(u_1)^{\alpha+1}} + \frac{\alpha((u_2)^{\beta+1} - y_0^{\beta+1})}{(u_2)^{\alpha+1}} = (\beta + 1) ((u_2)^{\beta-\alpha} - (u_1)^{\beta-\alpha}),$$

or equivalently,

$$\alpha y_0^{\beta+1} ((u_1)^{-\alpha-1} - (u_2)^{-\alpha-1}) = (\beta + 1 - \alpha) ((u_2)^{\beta-\alpha} - (u_1)^{\beta-\alpha}) \quad \text{at } x = x_1.$$

From this and (2.95) we then obtain

$$\begin{aligned} & (\beta + 1 - \alpha) ((u_2)^{\beta-\alpha} - (u_1)^{\beta-\alpha}) \\ & > \alpha ((u_1)^{-\alpha-1} - (u_2)^{-\alpha-1}) \frac{(u_2)^{\beta+1-\alpha} + (u_1)^{\beta+1-\alpha}}{(u_1)^{-\alpha} + (u_2)^{-\alpha}} \quad \text{at } x = x_1. \end{aligned} \quad (2.96)$$

Setting

$$z := \frac{u_1(x_1)}{u_2(x_1)} \in (0, 1),$$

this leads to

$$(\beta + 1 - \alpha) (z - z^{\beta+1-\alpha} + z^{\alpha+1} - z^{\beta+1}) > \alpha (1 - z^{\alpha+1} + z^{\beta+1-\alpha} - z^{\beta+2}). \quad (2.97)$$

But this contradicts Lemma A (Appendix). This finishes the proof of (2.88).

Step 5: Since $\lim_{x \rightarrow x_0} u_1'(x) = -\lim_{x \rightarrow x_0} u_2'(x) = +\infty$, (2.88) implies that

$$u_1'(0) < -u_2'(0) = 0,$$

which contradicts the boundary conditions (2.74). Hence we must have that $y^- = 0$. \square

Now we are in a position to give a

Proof of Theorem 1.1: We split into two cases.

Case (i): Assume that

$$\beta < 2\alpha. \quad (2.98)$$

By Lemma 2.13 we have $y^- = 0$. Now, since $x'(L) = 0$, equation (2.53) at $s = L$ yields

$$d = 0, \quad (2.99)$$

so that

$$x' = \lambda y^{\beta+1-\alpha}. \quad (2.100)$$

In view of Remark 2.2 we may rescale Ω in such a way that $y^+ = 1$. Then (2.100) at $s = 0$ gives $\lambda = 1$. Since $y'(s) < 0$ for $s \in (0, L)$, we have $s = g(y)$ with a decreasing function $g \in C^1(0, 1)$. Writing

$$f(y) := x(g(y)),$$

we obtain

$$-\frac{f'(y)}{\sqrt{1+(f'(y))^2}} = y^{\beta+1-\alpha}, \quad (2.101)$$

and integrating this leads to (1.3) and (1.4).

Case (ii) Now assume that

$$\beta = 2\alpha. \quad (2.102)$$

Since the case $\alpha = 0$ is trivial, we may assume $\alpha > 0$. Let us fix such α .

First observe that for every smooth domain $U \subset \mathbb{R}_+^2$, the mapping

$$\beta \mapsto \mathcal{R}_{\alpha,\beta}(U), \quad (-1 < \beta \leq 2\alpha), \quad (2.103)$$

is continuous. Furthermore, from Case (i) we see that the mapping

$$\beta \mapsto \mu(\alpha, \beta), \quad (-1 < \beta < 2\alpha),$$

is continuous, and the limit

$$Z := \lim_{\beta \rightarrow 2\alpha^-} \mu(\alpha, \beta)$$

exists.

Now let Ω^* be the domain that is given by formulas (1.3), (1.4), with $\beta = 2\alpha$. Then we also have

$$Z = \mathcal{R}_{\alpha,2\alpha}(\Omega^*),$$

which implies that $Z \geq \mu(\alpha, 2\alpha)$.

Assume that

$$Z > \mu(\alpha, 2\alpha).$$

Then there is a smooth set $\Omega' \subset \mathbb{R}_+^2$ such that also

$$Z > \mathcal{R}_{\alpha, 2\alpha}(\Omega').$$

But by (2.103) this implies that

$$\mathcal{R}_{\alpha, \beta}(\Omega') < \mu(\alpha, \beta),$$

when $\beta < 2\alpha$ and $|\beta - 2\alpha|$ is small, which is impossible. Hence we have that

$$Z = \mu(\alpha, 2\alpha) = \mathcal{R}_{\alpha, 2\alpha}(\Omega^*).$$

This finishes the proof of the Theorem.

Finally we evaluate $\mu(\alpha, \beta)$. Put $\gamma := \beta + 1 - \alpha (> 0)$. With the Beta function B and the function f given by (1.4) we have

$$\begin{aligned} P_\alpha(\Omega^*) &= 2 \int_0^1 y^\alpha \sqrt{1 + (f'(y))^2} dy = 2 \int_0^1 \frac{y^\alpha}{\sqrt{1 - y^{2\gamma}}} dy \\ &= \frac{1}{\gamma} \int_0^1 \frac{z^{\frac{\alpha+1}{2\gamma} - 1} dz}{\sqrt{1 - z}} = \frac{1}{\gamma} \cdot B\left(\frac{\alpha+1}{2\gamma}, \frac{1}{2}\right), \end{aligned}$$

and

$$\begin{aligned} A_\beta(\Omega^*) &= 2 \int_0^1 y^\beta f(y) dy = -\frac{2}{\beta+1} \int_0^1 y^{\beta+1} f'(y) dy \\ &= \frac{2}{\beta+1} \int_0^1 \frac{y^{2\gamma}}{\sqrt{1 - y^{2\gamma}}} dy = \frac{1}{\gamma(\beta+1)} \int_0^1 \frac{z^{\frac{\alpha+1}{2\gamma}} dz}{\sqrt{1 - z}} \\ &= \frac{1}{\gamma(\beta+1)} B\left(1 + \frac{\alpha+1}{2\gamma}, \frac{1}{2}\right). \end{aligned}$$

Using the identity

$$B(a+1, b) = B(a, b) \cdot \frac{a}{a+b}, \quad (\operatorname{Re}(a) > 0, \operatorname{Re}(b) > 0),$$

we obtain

$$\begin{aligned} \mu(\alpha, \beta) &= \mathcal{R}_{\alpha, \beta}(\Omega^*) = \frac{P_\alpha(\Omega^*)}{[A_\beta(\Omega^*)]^{\frac{\alpha+1}{\beta+2}}} \\ &= \frac{1}{\gamma} \cdot B\left(\frac{\alpha+1}{2\gamma}, \frac{1}{2}\right) \cdot \left[\frac{1}{\gamma(\beta)} \cdot B\left(1 + \frac{\alpha+1}{2\gamma}, \frac{1}{2}\right) \right]^{-\frac{\alpha+1}{\beta+2}} \\ &= \gamma^{\frac{\alpha+1}{\beta+2} - 1} \cdot \left[\frac{(\beta+1)(\beta+2)}{\alpha+1} \right]^{\frac{\alpha+1}{\beta+2}} \cdot \left[B\left(\frac{\alpha+1}{2\gamma}, \frac{1}{2}\right) \right]^{\frac{\gamma}{\beta+2}}, \end{aligned}$$

which is (1.5). In case of $\beta = 2\alpha$ this leads to

$$\mu(\alpha, 2\alpha) = (\alpha + 1)^{-\frac{1}{2}} \cdot [2(2\alpha + 1)]^{\frac{1}{2}} \cdot \left[B\left(\frac{1}{2}, \frac{1}{2}\right) \right]^{\frac{1}{2}} = \sqrt{\frac{2\pi(2\alpha + 1)}{\alpha + 1}}. \quad \square$$

Remark 2.14. It is also well-known that the isoperimetric inequality is equivalent to the following functional inequality, (see [1], Lemma 3.5).

$$\iint_{\mathbb{R}_+^2} |\nabla u| y^\alpha \, dx dy \geq \mu(\alpha, \beta) \left(\iint_{\mathbb{R}_+^2} |u|^{\frac{\beta+2}{\alpha+1}} y^\beta \, dx dy \right)^{\frac{\alpha+1}{\beta+2}}, \quad \forall u \in C_0^\infty(\mathbb{R}^2). \quad (2.104)$$

Remark 2.15. In [40] (see also [28–30]) the authors obtain a sharp isoperimetric inequality in the Grushin plane, and they also compute the corresponding isoperimetric sets. Their proof consists of three steps. First, performing a certain transformation in one of the coordinates, the problem becomes equivalent to the task of minimizing the (unweighted) Euclidean perimeter of sets $U \subset \mathbb{R}^2$ having fixed weighted measure $\iint_U |y|^\beta \, dx \, dy$, where $\beta \in (-1, 0]$. Then, using two consecutive Steiner symmetrizations in the x - and the y -variable, the perimeter of an admissible set U decreases while the weighted measure increases. This means that the problem can be further reduced to sets which are Steiner symmetric in both variables. In turn, the boundary of the minimizer must be a polar graph which is then found in the last step of the proof.

In view of the symmetry of the minimizer w.r.t. the x -axis, this isoperimetric result can also be read off from Theorem 1.1 when $\alpha = 0$ and $\beta \in (-1, 0]$. Note however, that in the study of our problem **(P)** extra difficulties appeared because rearrangement arguments in the y -variable failed in the general case.

3. APPLICATIONS

In this section we firstly show that our isoperimetric inequality implies a sharp estimate of the so-called *weighted Cheeger constant*.

Then we deduce an estimate of the first eigenvalue to a degenerate elliptic Dirichlet boundary values problem. We begin by introducing some function spaces that will be used in the sequel.

Let Ω be an open subset of \mathbb{R}_+^2 and $p \in [1, +\infty)$.

By $L^p(\Omega; y^\beta)$ we denote the weighted Hölder space of measurable functions $u : \Omega \rightarrow \mathbb{R}$ such that

$$\|u\|_{L^p(\Omega; y^\beta)} := \left(\iint_{\Omega} |u|^p y^\beta \, dx dy \right)^{1/p} < +\infty.$$

Then let $W^{1,p}(\Omega; y^\alpha, y^\beta)$ be the weighted Sobolev space of all functions $u \in L^p(\Omega; y^\beta)$ possessing weak first partial derivatives which belong to $L^p(\Omega; y^\beta)$. A norm in $W^{1,p}(\Omega; y^\alpha, y^\beta)$ is given by

$$\|u\|_{W^{1,p}(\Omega; y^\alpha, y^\beta)} := \|\nabla u\|_{L^p(\Omega; y^\alpha)} + \|u\|_{L^p(\Omega; y^\beta)}.$$

For any function $u \in L^1(\Omega; y^\beta)$ we write

$$|Du|(\Omega; y^\alpha) := \sup \left\{ \iint_{\Omega} u \operatorname{div}(\mathbf{v} y^\alpha) \, dx dy : \mathbf{v} \in C_0^\infty(\Omega, \mathbb{R}^2), |\mathbf{v}| \leq 1 \right\}.$$

Then let $BV(\Omega; y^\alpha, y^\beta)$ be the weighted BV-space of all functions $u \in L^1(\Omega; y^\beta)$ such that $|Du|(\Omega; y^\alpha) < +\infty$. A norm on $BV(\Omega; y^\alpha, y^\beta)$ is given by

$$\|u\|_{BV(\Omega; y^\alpha, y^\beta)} := |Du|(\Omega; y^\alpha) + \|u\|_{L^1(\Omega; y^\beta)}.$$

Let us explicitly remark that for an open bounded set $\Omega \subset \mathbb{R}_+^2$ the following equality holds

$$P_\alpha(\Omega) = |D\chi_\Omega|(\mathbb{R}_+^2; y^\alpha).$$

Finally let X be the set of all the functions $w \in C^1(\overline{\Omega})$ that vanish in a neighborhood of $\partial\Omega \cap \mathbb{R}_+^2$. Then $V^p(\Omega; y^\alpha, y^\beta)$ will denote the closure of X in the norm of $W^{1,p}(\Omega; y^\alpha, y^\beta)$.

Finally we denote by Ω^\star the set $t\Omega^\star$, for $t > 0$, such that $A_\beta(\Omega) = A_\beta(\Omega^\star)$.

3.1. Weighted Cheeger sets

We define the weighted Cheeger constant of an open bounded set $\Omega \subset \mathbb{R}_+^2$ as

$$h_{\alpha,\beta}(\Omega) = \inf \left\{ \frac{P_\alpha(E)}{A_\beta(E)} : E \subset \Omega, 0 < A_\beta(E) < +\infty \right\}. \quad (3.1)$$

(see also [34, 47])

We firstly prove that the existence of an ammissible set which realizes the minimum in (3.1) (see also [47]).

Lemma 3.1. *Assume $0 \leq \alpha < \beta + 1$ and $\beta \leq 2\alpha$. For any open bounded set $\Omega \subset \subset \mathbb{R}_+^2$, there exists at least one set $M \subseteq \Omega$, the so-called weighted Cheeger set, such that*

$$h_{\alpha,\beta}(\Omega) = \frac{P_\alpha(M)}{A_\beta(M)}. \quad (3.2)$$

Proof. Since Ω is open, $h_{\alpha,\beta}(\Omega)$ is finite: Indeed, it is easy to verify that for any ball B with $B \subset \subset \Omega$, the ratio $\frac{P_\alpha(B)}{A_\beta(B)}$ is finite.

Let $\{E_k\}$ be a minimizing sequence for (3.1). Since Ω is bounded, we have

$$A_\beta(E_k) = \iint_{E_k} y^\beta \, dx dy \leq A_\beta(\Omega) = \iint_{\Omega} y^\beta \, dx dy < +\infty.$$

Now fix $\varepsilon > 0$. There exists an index \bar{k} such that

$$\left| h_{\alpha,\beta}(\Omega) - \frac{P_\alpha(E_k)}{A_\beta(E_k)} \right| < \varepsilon, \quad \forall k > \bar{k}.$$

Since Ω is bounded, for all $k > \bar{k}$, we get

$$P_\alpha(E_k) < (\varepsilon + h_{\alpha,\beta}(\Omega))A_\beta(E_k) \leq (\varepsilon + h_{\alpha,\beta}(\Omega))A_\beta(\Omega) \equiv C$$

This implies

$$|D\chi_{E_k}|(\mathbb{R}_+^2, y^\alpha) = P_\alpha(E_k) \leq C \quad \forall k > \bar{k}.$$

Hence $\{E_k\}$ is an equibounded family in weighted $BV(\Omega; y^\alpha, y^\beta)$ -norm. Thus by Lemma B (Appendix), up to subsequences, $\{\chi(E_k)\}$ converges in the weighted $L^1(\Omega; y^\beta)$ -norm and pointwise a.e. to a function u . Moreover there exists a subset $M \subseteq \Omega$ such that $u = \chi_M$.

Since $\{E_k\}$ is a minimizing sequence, by lower semicontinuity of perimeter P_α and Lebesgue dominated convergence theorem, we get

$$h_{\alpha,\beta}(\Omega) = \lim_{k \rightarrow +\infty} \frac{P_\alpha(E_k)}{A_\beta(E_k)} \geq \frac{P_\alpha(M)}{A_\beta(M)}. \quad (3.3)$$

It remains to prove that M is an admissible set, that is we need to prove that

$$A_\beta(M) > 0. \quad (3.4)$$

Assume by contradiction that $A_\beta(M) = 0$. This implies that $\lim_{k \rightarrow +\infty} A_\beta(E_k) = 0$.

Now for a fixed $\eta > 0$ consider the set

$$E_{k,\eta} = \{(x, y) \in E_k : y > \eta\}.$$

Now the following inequality holds true

$$A_\beta(E_{k,\eta}) < \delta^\beta A_0(E_{k,\eta}), \quad P_\alpha(E_{k,\eta}) < \eta^\alpha P_0(E_{k,\eta}),$$

where $\delta = \eta$, if $\beta \leq 0$ and $\delta = R$ for a suitable $R > 0$ such that a ball B_R of radius R contains Ω , if $\beta > 0$. Denote by $B_{r_k,\eta}$ a ball of radius r_k having the same Lebesgue measure $A_0(E_{k,\eta})$ of $E_{k,\eta}$. By the classical isoperimetric inequality, we get

$$\frac{P_\alpha(E_{k,\eta})}{A_\beta(E_{k,\eta})} \geq \frac{\eta^\alpha P_\alpha(E_{k,\eta})}{\delta^\beta A_0(B_{r_k,\eta})} = \frac{\eta^\alpha P_\alpha(B_{r_k,\eta})}{\delta^\beta A_0(B_{r_k,\eta})} = \frac{2\eta^\alpha}{\delta^\beta r_k} \rightarrow +\infty, \quad \text{as } k \text{ goes to } +\infty.$$

This yields a contradiction. Therefore (3.4) holds true and the conclusion follows. \square

We recall that Ω^\star is the set $t\Omega^\star$, for $t > 0$, such that $A_\beta(\Omega) = A_\beta(\Omega^\star)$.

Once we have proved the existence of a weighted Cheeger set, we can obtain the following result.

Theorem 3.2. *Assume $0 \leq \alpha < \beta + 1$ and $\beta \leq 2\alpha$ and $\alpha < \beta + 1$, and let Ω be a bounded open subset \mathbb{R}_+^2 . Then the following estimate holds true*

$$h_{\alpha,\beta}(\Omega) \geq h_{\alpha,\beta}(\Omega^\star) = \frac{P_\alpha(\Omega^\star)}{A_\beta(\Omega^\star)}. \quad (3.5)$$

Proof. Let E be a nonempty subset of Ω with $P_\alpha(E) < +\infty$. By our isoperimetric inequality Theorem 1.1 and since $E^\star \subset \Omega^\star$ with $A_\beta(E) = A_\beta(E^\star)$, we have that

$$\frac{P_\alpha(E)}{A_\beta(E)} \geq \frac{P_\alpha(E^\star)}{A_\beta(E^\star)} \geq h_{\alpha,\beta}(\Omega^\star). \quad (3.6)$$

It remains to prove the equality in (3.5). Let F be a nonempty subset of Ω^\star . Then we have $F^\star \subset \Omega^\star$ and

$$\frac{P_\alpha(F)}{A_\beta(F)} \geq \frac{P_\alpha(F^\star)}{A_\beta(F^\star)} = t^{\beta+1-\alpha} \frac{P_\alpha(tF^\star)}{A_\beta(tF^\star)} \quad (3.7)$$

for all $t > 0$. Since $F^\star \subset \Omega^\star$, there exists $t \geq 1$ such that $tF^\star = \Omega^\star$. Therefore

$$\frac{P_\alpha(F)}{A_\beta(F)} \geq t^{\beta+1-\alpha} \frac{P_\alpha(\Omega^\star)}{A_\beta(\Omega^\star)}$$

which proves the equality in (3.5). \square

Remark 3.3. Theorem 3.1 could be stated as an estimate of the first eigenvalue of the weighted 1-laplacian.

3.2. A nonlinear eigenvalue problem

Let $\Omega \subset \mathbb{R}_+^2$ be a bounded domain and let $p \in (1, +\infty)$. We consider the following weighted eigenvalue problem

$$\begin{cases} -\operatorname{div} \left(y^{p\gamma_1} |\nabla u|^{p-2} \nabla u \right) = \lambda y^{\frac{p\gamma_2}{p-1}} |u|^{p-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \cap \mathbb{R}_+^2, \end{cases} \quad (3.8)$$

where

$$\gamma_1 = \alpha - \frac{p-1}{p}\beta \quad \text{and} \quad \gamma_2 = \frac{p-1}{p}\beta.$$

By a solution to problem (3.8) we mean a function $u \in V^p \left(\Omega; y^{p\gamma_1}, y^{\frac{p\gamma_2}{p-1}} \right)$ such that

$$\iint_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \psi y^{p\gamma_1} dx dy = \lambda \iint_{\Omega} |u|^{p-2} u \psi y^{\frac{p\gamma_2}{p-1}} dx dy$$

for all function $\psi \in C^1(\overline{\Omega})$ such that $\psi = 0$ on $\partial\Omega \cap \mathbb{R}_+^2$.

Let us denote by T the range of values of α and β for which the isoperimetric inequality holds true. We have that

$$(\alpha, \beta) \in T = \{\alpha \geq 0\} \cap \{\beta > \alpha - 1\} \cap \{\beta \leq 2\alpha\} \quad (3.9)$$

if and only if

$$(\gamma_1, \gamma_2) \in U := \{\gamma_1 + \gamma_2 \geq 0\} \cap \left\{ \frac{p\gamma_2}{p-1} > \gamma_1 + \gamma_2 - 1 \right\} \cap \left\{ \frac{p\gamma_2}{p-1} \leq 2(\gamma_1 + \gamma_2) \right\}. \quad (3.10)$$

Furthermore the smallest eigenvalue of problem (3.8), $\lambda_{1,p}^{\gamma_1, \gamma_2}(\Omega)$, has the following variational characterization

$$\lambda_{1,p}^{\gamma_1, \gamma_2}(\Omega) = \min \left\{ \frac{\iint_{\Omega} |\nabla u|^p y^{p\gamma_1} dx dy}{\iint_{\Omega} u^p y^{\frac{p\gamma_2}{p-1}} dx dy} \text{ with } u \in V^p \left(\Omega; y^{p\gamma_1}, y^{\frac{p\gamma_2}{p-1}} \right) \setminus \{0\} \right\}.$$

Indeed, see *e.g.* Theorem 8.9 in [31], for any $(\gamma_1, \gamma_2) \in U$, the following compact embedding holds true

$$V^p \left(\Omega; y^{p\gamma_1}, y^{\frac{p\gamma_2}{p-1}} \right) \hookrightarrow L^p \left(\Omega; y^{\frac{p\gamma_2}{p-1}} \right).$$

By adapting the arguments used in [22, 35], we obtain the following result

Theorem 3.4. *Let $(\gamma_1, \gamma_2) \in U$, then the following estimate holds true*

$$\lambda_{1,p}^{\gamma_1, \gamma_2}(\Omega) \geq \frac{1}{p^p} [h_{\alpha, \beta}(\Omega^\star)]^p = \frac{1}{p^p} \left[\frac{P_\alpha(\Omega^\star)}{A_\beta(\Omega^\star)} \right]^p.$$

Remark 3.5. If $\alpha = \beta > 0$ and $p = 2$, a Faber-Krahn type inequality for $\lambda_{1,p}^{\gamma_1, \gamma_2}(\Omega)$ holds true (see [37]). Indeed in this case, we have $\alpha = 2\gamma_1 = 2\gamma_2 = \beta$.

Proof of Theorem 3.4 We claim that

$$\lambda_{1,p}^{\gamma_1, \gamma_2}(\Omega) \geq \frac{[h_{\alpha, \beta}(\Omega)]^p}{p^p}, \quad (3.11)$$

where $h_{\alpha, \beta}(\Omega)$ is defined in (3.1). Let u be an eigenfunction corresponding to $\lambda_{1,p}^{\gamma_1, \gamma_2}(\Omega)$. Hölder's inequality gives

$$\begin{aligned} \iint_{\Omega} |\nabla u| u^{p-1} y^\alpha dx dy &= \iint_{\Omega} |\nabla u| u^{p-1} y^{\gamma_1} y^{\gamma_2} dx dy \\ &\leq \left(\iint_{\Omega} |\nabla u|^p y^{p\gamma_1} dx dy \right)^{\frac{1}{p}} \left(\iint_{\Omega} u^p y^{\frac{\gamma_2 p}{p-1}} dx dy \right)^{\frac{p-1}{p}}, \end{aligned}$$

and therefore

$$\iint_{\Omega} |\nabla u|^p y^{p\gamma_1} dx \geq \frac{\left(\iint_{\Omega} |\nabla u| u^{p-1} y^\alpha dx dy \right)^p}{\left(\iint_{\Omega} u^p y^\beta dx dy \right)^{p-1}} = \frac{\left(\iint_{\Omega} |\nabla (u^p)| y^\alpha dx dy \right)^p}{p^p \left(\iint_{\Omega} u^p y^\beta dx dy \right)^{p-1}}.$$

Let $f := u^p$, the previous inequality gives

$$\frac{\iint_{\Omega} |\nabla u|^p y^{p\gamma_1} dx dy}{\iint_{\Omega} u^p y^\beta dx dy} \geq \frac{\left(\iint_{\Omega} |\nabla f| y^\alpha dx dy \right)^p}{p^p \left(\iint_{\Omega} f y^\beta dx dy \right)^p}.$$

On the other hand Coarea formula yields

$$\begin{aligned}
& \frac{\left(\iint_{\Omega} |\nabla f| y^{\alpha} dx dy\right)^p}{p^p \left(\iint_{\Omega} f y^{\beta} dx dy\right)^p} = \frac{1}{p^p \left(\iint_{\Omega} f y^{\beta} dx dy\right)^p} \left(\int_0^{\max f} P_{\alpha}(\{f(x) > t\}) dt\right)^p \\
& \geq \frac{1}{p^p \left(\iint_{\Omega} f y^{\beta} dx dy\right)^p} \left(\int_0^{\max f} \frac{P_{\alpha}(\{f(x) > t\}) dt}{A_{\beta}(\{f(x) > t\})} A_{\beta}(\{f(x) > t\}) dt\right)^p \\
& \geq \frac{1}{p^p \left(\iint_{\Omega} f y^{\beta} dx dy\right)^p} \left(\int_0^{\max f} h_{\alpha, \beta}(\{f(x) > t\}) A_{\beta}(\{f(x) > t\}) dt\right)^p \\
& \geq \frac{[h_{\alpha, \beta}(\Omega)]^p}{p^p \left(\iint_{\Omega} f y^{\beta} dx dy\right)^p} \left(\int_0^{\max f} A_{\beta}(\{f(x) > t\}) dt\right)^p = \frac{[h_{\alpha, \beta}(\Omega)]^p}{p^p}.
\end{aligned}$$

The claim is hence proved. It immediately implies Theorem 3.4, thanks to (3.5).

APPENDIX A.

We prove two technical results.

Lemma A: *Let $\alpha, \beta \in \mathbb{R}$ with $0 < \alpha \leq \beta \leq 2\alpha$. Then*

$$\begin{aligned}
& \alpha(1 - z^{\alpha+1} + z^{\beta+1-\alpha} - z^{\beta+2}) + (\beta + 1 - \alpha)(-z + z^{\beta+1-\alpha} - z^{\alpha+1} + z^{\beta+1}) \\
& > 0 \quad \forall z \in (0, 1).
\end{aligned} \tag{A.1}$$

Proof. We fix $\alpha > 0$ and define

$$g(z, \beta) := \alpha(1 - z^{\alpha+1} + z^{\beta+1-\alpha} - z^{\beta+2}) + (\beta + 1 - \alpha)(-z + z^{\beta+1-\alpha} - z^{\alpha+1} + z^{\beta+1}),$$

and

$$\begin{aligned}
h(z) & := g(z, 2\alpha) = \alpha(1 - z^{2\alpha+2}) + (\alpha + 1)(-z + z^{2\alpha+1}), \\
& (z \in [0, 1], \beta \in [\alpha, 2\alpha]).
\end{aligned}$$

Then

$$g(z, \alpha) = \alpha(1 - z^{\alpha+1})(1 + z) > 0, \quad (\text{A.2})$$

$$h(0) = \alpha, \quad h(1) = 0, \quad (\text{A.3})$$

$$\begin{aligned} h'(z) &= -2\alpha(\alpha + 1)z^{2\alpha+1} + (\alpha + 1)[-1 + (2\alpha + 1)z^{2\alpha}] \quad \text{and} \\ h''(z) &= 2\alpha(\alpha + 1)(2\alpha + 1)z^{2\alpha+1}(1 - z) > 0. \end{aligned} \quad (\text{A.4})$$

Hence $h'(1) = 0$ which together with (A.3) and (A.4) implies that

$$h(z) = g(z, \alpha) > 0 \quad \forall z \in (0, 1). \quad (\text{A.5})$$

Furthermore we have

$$\begin{aligned} \frac{\partial g}{\partial \beta}(z, \beta) &= z^{\beta+1-\alpha} [\alpha(1 - z^{1+\alpha}) + (\beta + 1 - \alpha)(1 + z^\alpha)] \ln z \\ &\quad - (1 - z^\beta)(z + z^{\alpha+1}) \\ &< 0 \quad \text{if } z \in (0, 1) \text{ and } \beta \in [\alpha, 2\beta]. \end{aligned}$$

Together with (A.2) and (A.4) this implies

$$g(z, \alpha) > 0 \quad \text{if } z \in (0, 1) \text{ and } \beta \in [\alpha, 2\alpha],$$

which is (A.1). □

Lemma B: Let $\{u_n\} \subset BV(\Omega; y^\alpha, y^\beta)$ be a bounded sequence. Then there exists a subsequence that converges in $L^1(\Omega; y^\beta)$ and a.e. in Ω to some function u .

Proof. Put $\gamma = \frac{\beta+2}{\alpha+1} (> 1)$ and let $\Omega_\varepsilon = \Omega \cap \{(x, y) : y > \varepsilon\}$ for any $\varepsilon > 0$.

Let $k \in \mathbb{N}$. By a classical compactness result in the unweighted case, there exists a function $u^k \in L^1(\Omega_{2^{-k}}; y^\beta)$ and an increasing sequence of integers $\{a(k, m)\}_{m \geq 1}$ such that

$$u_{a(k, m)} \rightarrow u^k \quad \text{in } L^1(\Omega_{2^{-k}}; y^\beta) \text{ and a.e. in } \Omega. \quad (\text{A.6})$$

By choosing $\{a(k+1, m)\}$ to be a subsequence of $\{a(k, m)\}$, ($k \in \mathbb{N}$), we can achieve that $u^k = u^{k+1}$ in $\Omega_{2^{-k}}$, $k \in \mathbb{N}$.

Now put

$$u(x) = \begin{cases} u^1(x) & \text{if } x \in \Omega_{2^{-1}} \\ u^k(x) & \text{if } x \in \Omega_{2^{-k}} \setminus \bar{\Omega}_{2^{-k+1}}, k = 2, 3, \dots \end{cases}$$

In view of our isoperimetric inequality, the sequence u_n is equibounded in $L^\gamma(\Omega; y^\beta)$. We have the following estimate:

$$\begin{aligned} \|u\|_{L^1(\Omega_{2^{-k}}; y^\beta)} &= \sum_{j=0}^k \int_{\Omega_{2^{-j}} \setminus \bar{\Omega}_{2^{-j+1}}} |u| y^\beta \, dx dy \\ &\leq \sum_{j=0}^k \left(\int_{\Omega_{2^{-j}} \setminus \bar{\Omega}_{2^{-j+1}}} |u|^\gamma y^\beta \, dx dy \right)^{\frac{1}{\gamma}} \left(\int_{\Omega_{2^{-j}} \setminus \bar{\Omega}_{2^{-j+1}}} y^\beta \, dx dy \right)^{1 - \frac{1}{\gamma}} \end{aligned}$$

$$\leq \sum_{j=0}^k C_1 \left[y^{\beta+1} \left| \frac{2^{-j+1}}{2^{-j}} \right| \right]^{1-\frac{1}{\gamma}} \leq C_2 \sum_{j=0}^{\infty} 2^{(-j+1)(\beta+1)(1-\frac{1}{\gamma})} < +\infty,$$

with constants that do not depend on k . This implies that $u \in L^1(\Omega; y^\beta)$.

Let $\varepsilon > 0$. Choose k large enough such that

$$\int_{\Omega \setminus \overline{\Omega}_{2^{-k}}} y^\beta dx dy < \varepsilon$$

and then m large enough such that

$$\int_{\Omega_{2^{-k}}} |u - u_{a(k,m)}| y^\beta dx dy < \varepsilon.$$

Then we obtain

$$\begin{aligned} \int_{\Omega} |u - u_{a(k,m)}| y^\beta dx dy &< \varepsilon + \int_{\Omega \setminus \overline{\Omega}_{2^{-k}}} |u - u_{a(k,m)}| y^\beta dx dy \\ &\leq \varepsilon + \left(\int_{\Omega} |u - u_{a(k,m)}| y^\beta dx dy \right)^{\frac{1}{\gamma}} \left(\int_{\Omega \setminus \overline{\Omega}_{2^{-k}}} y^\beta dx dy \right)^{1-\frac{1}{\gamma}} \\ &\leq \varepsilon + C \varepsilon^{(1-\frac{1}{\gamma})(\beta+1)}, \end{aligned}$$

where C does not depend on k . From this the assertion follows. \square

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