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STOCHASTIC MAXIMUM PRINCIPLE, DYNAMIC PROGRAMMING PRINCIPLE, AND THEIR RELATIONSHIP FOR FULLY COUPLED FORWARD-BACKWARD STOCHASTIC CONTROLLED SYSTEMS*

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Abstract. Within the framework of viscosity solution, we study the relationship between the maximum principle (MP) from M. Hu, S. Ji and X. Xue [*SIAM J. Control Optim.* **56** (2018) 4309–4335] and the dynamic programming principle (DPP) from M. Hu, S. Ji and X. Xue [*SIAM J. Control Optim.* **57** (2019) 3911–3938] for a fully coupled forward–backward stochastic controlled system (FBSCS) with a nonconvex control domain. For a fully coupled FBSCS, both the corresponding MP and the corresponding Hamilton–Jacobi–Bellman (HJB) equation combine an algebra equation respectively. With the help of a new decoupling technique, we obtain the desirable estimates for the fully coupled forward–backward variational equations and establish the relationship. Furthermore, for the smooth case, we discover the connection between the derivatives of the solution to the algebra equation and some terms in the first-order and second-order adjoint equations. Finally, we study the local case under the monotonicity conditions as from J. Li and Q. Wei [*SIAM J. Control Optim.* **52** (2014) 1622–1662] and Z. Wu [*Syst. Sci. Math. Sci.* **11** (1998) 249–259], and obtain the relationship between the MP from Z. Wu [*Syst. Sci. Math. Sci.* **11** (1998) 249–259] and the DPP from J. Li and Q. Wei [*SIAM J. Control Optim.* **52** (2014) 1622–1662].

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1. Introduction

It is well-known that Pontryagin's maximum principle (MP) and Bellman's dynamic programming principle (DPP) are two of the most important approaches in solving optimal control problems and there exist close relationship between them. The relation between the MP and the DPP will help us understand the MP and

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the DPP in a more profound way and is studied in many literatures (see [16, 26] and the references therein). For stochastic optimal control problems, the classical results on the relationship between the MP and the DPP were studied by Bensoussan [2]. Within the framework of viscosity solution, Zhou [27, 28] obtained the relation between these two approaches.

In this paper, we study the relationship between the MP and the DPP for a stochastic optimal control problem where the system is governed by the following controlled fully coupled forward–backward stochastic differential equation (FBSDE):

$$\begin{cases} dX(t)= & b(t,X(t),Y(t),Z(t),u(t))dt+\sigma(t,X(t),Y(t),Z(t),u(t))dB(t), \\ dY(t)= & -g(t,X(t),Y(t),Z(t),u(t))dt+Z(t)dB(t), \\ X(0)= & x_0,\ Y(T)=\phi(X(T)),\ t\in[0,T], \end{cases} \tag{1.1}$$

and the cost functional is defined by the solution to the backward stochastic differential equation (BSDE) of (1.1) at time 0, *i.e.*,

$$J(u(\cdot))=Y(0). \tag{1.2}$$

When the coefficients of the forward stochastic differential equation (SDE) are independent of the terms $Y(\cdot)$ and $Z(\cdot)$, we call the controlled system with the cost functional $Y(0)$ as a decoupled forward–backward stochastic controlled system (FBSCS). FBSCSs can be used to describe some important problems in mathematical finance and stochastic control theory. For example, the portfolios of a large investor, stochastic differential games, the generalized stochastic recursive utilities of consumers and principal-agent problems may involve in solving optimal controls for fully coupled FBSCSs [1, 5, 18].

Peng [23] first established a local stochastic maximum principle for the decoupled FBSCS. Then the local stochastic maximum principles for other various problems were studied in Dokuchaev and Zhou [4], Ji and Zhou [12], Øksendal and Sulem [17] and Haadem, Øksendal and Proske [6] and the references therein. When the control domain is nonconvex, the global stochastic maximum principle for a decoupled FBSCS has not been obtained for a long time since Peng [24] proposed it as an open problem. For this open problem, Hu [7] studied this decoupled FBSCS and obtained the first-order and second-order variational equations for BSDE which leads to a novel global maximum principle. Recently, Hu *et al.* [8] obtained a global stochastic maximum principle for the fully coupled FBSCS. In contrast with the progresses in deriving stochastic maximum principles, Peng [22] deduced the DPP and introduced the generalized Hamilton–Jacobi–Bellman (HJB) equation for a decoupled FBSCS. Under monotonicity conditions, Li and Wei [13] built the DPP and proved that the value function is a viscosity solution to the generalized HJB equation for a fully coupled FBSCS. Then, by establishing the DPP and various properties of the value function, Hue *et al.* [9] studied the existence and uniqueness of viscosity solutions to the generalized HJB equation for a fully coupled FBSCS.

As for the relationship between the MP and the DPP for the decoupled FBSCS, Nie *et al.* [16] studied the general case with the help of the first-order and second-order adjoint equations which are introduced in Hu [7]. Up to our knowledge, there are few works about the connection between the MP and the DPP for fully coupled FBSCSs. Especially, there is no research results in the case that the diffusion coefficient of the forward SDE depends on the term Z.

Inspired by the above works, in this paper, we investigate the connection between the MP and the DPP for fully coupled FBSCSs with a nonconvex control domain. We obtain that the connection between the adjoint process (p,P) in the maximum principle in [8] and the first-order and second-order sub- (resp. super-) jets of the value function W in [9] in the x-variable is, $\mathbb{P}$-$a.s.$

$$\begin{cases} \{p(s)\}\times[P(s),\infty)\subseteq D_x^{2,+}W(s,\bar{X}^{t,x;\bar{u}}(s)), \\ D_x^{2,-}W(s,\bar{X}^{t,x;\bar{u}}(s))\subseteq\{p(s)\}\times(-\infty,P(s)],\ \forall s\in[t,T], \end{cases} \tag{1.3}$$

and the connection between the function $\mathcal{H}_1$ and the right sub- (resp. super-) jets of W in the t-variable is

$$\begin{cases} [\mathcal{H}_1(s,\bar{X}^{t,x;\bar{u}}(s),\bar{Y}^{t,x;\bar{u}}(s),\bar{Z}^{t,x;\bar{u}}(s)),\infty)\subseteq D_{t+}^{1,+}W(s,X^{t,x;\bar{u}}(s)), \\ D_{t+}^{1,-}W(s,X^{t,x;\bar{u}}(s))\subseteq(-\infty,\mathcal{H}_1(s,\bar{X}^{t,x;\bar{u}}(s),\bar{Y}^{t,x;\bar{u}}(s),\bar{Z}^{t,x;\bar{u}}(s))],\ \mathbb{P}\text{-a.s..} \end{cases} \tag{1.4}$$

Comparing with the results in [16], the difficulties of proving the above relations come from the fully coupled property of our controlled system. Note that due to the fully coupled property, the MP in [8] includes an algebra equation which leads to the adjoint process (p,P) in [9] becomes much more complex than that in [7], and the value function W in [9] should satisfy the HJB equation combined with an algebra equation. When we establish the relation (1.3), we need to perturb the initial state x which leads to the fully coupled variational equation (3.5). The key step in obtaining (1.3) is to estimate the remainder terms $\varepsilon_i(\cdot)$ $(i=1,2,3)$ in (3.5). But the approach in [16] does not work. The reason is that for decoupled case, one can first estimate the remainder terms of the forward equation, and then estimate the remainder terms of the backward equation by standard estimates as in [16]. But for the fully coupled case, $\hat{Z}$ will appear in the remainder terms of the forward equation which yields that we cannot estimate the remainder terms in the forward equation firstly. To overcome this difficulty, by utilizing the relationship between $(\hat{Y}(\cdot),\hat{Z}(\cdot))$ and $\hat{X}(\cdot)$ (see (3.8) and (3.13)), we propose a new decoupling technique and estimate the remainder terms of the forward and backward equations simultaneously. Then, we obtain the desirable estimates which make the establishment of the relation (1.3) possible. The idea to prove the relation (1.4) is similar.

When the value function W is supposed to be smooth, we discover two novel connections:
(i) the relation between the algebra equation for the MP and the algebra equation for the HJB equation, *i.e.*, $\Delta(\cdot)$ in (2.16) and $V(\cdot)$ in (2.5), for $s\in[t,T]$,

$$\begin{aligned}\Delta(s)= \quad & V(s,\bar{X}^{t,x;\bar{u}}(s),W(s,\bar{X}^{t,x;\bar{u}}(s)),W_x(s,\bar{X}^{t,x;\bar{u}}(s)),u) \\ & -V(s,\bar{X}^{t,x;\bar{u}}(s),W(s,\bar{X}^{t,x;\bar{u}}(s)),W_x(s,\bar{X}^{t,x;\bar{u}}(s)),\bar{u}(s)),\end{aligned}$$

(ii) the relation between the derivatives of the solution V to the algebra equation (2.5) and the terms $K_1(\cdot)$, $K_2(\cdot)$ in the adjoint equations (2.11) and (2.13), for $s\in[t,T]$,

$$\begin{aligned} \frac{\partial V}{\partial x}(s,x,W(s,x),W_x(s,x),\bar{u}(s))\Big|_{x=\bar{X}^{t,x;\bar{u}}(s)} &= K_1(s), \\ \frac{\partial^2 V}{\partial x^2}(s,x,W(s,x),W_x(s,x),\bar{u}(s))\Big|_{x=\bar{X}^{t,x;\bar{u}}(s)} &= \tilde{K}_2(s), \end{aligned} \tag{1.5}$$

where $K_1(\cdot)$ (resp. $\tilde{K}_2(\cdot)$) is defined in (2.12) (resp. (4.11)).
From the point of view of the MP, $K_1(\cdot)$ is the coefficient of $X_1(\cdot)$ in the first-order variational equation of $Z(\cdot)$ in Lemma 3.13 in [8] and it measures the sensitivity of the variable $Z(\cdot)$ to the variable $X(\cdot)$ under the optimal state. From the viewpoint of the DPP and the HJB equation,

$$V(s,\bar{X}^{t,x;\bar{u}}(s),W(s,\bar{X}^{t,x;\bar{u}}(s)),W_x(s,\bar{X}^{t,x;\bar{u}}(s)),\bar{u}(s))=\bar{Z}^{t,x;\bar{u}}(s).$$

Thus, the connection (1.5) is naturally established. In fact, when σ is independent of y and z,

$$
\begin{aligned}
&K_1(s) = p(s)\,\sigma_x(s) + q(s)\,,\ s \in [t,T];\\
&\frac{\partial V}{\partial x}\left(s, x, W(s,x), W_x(s,x), \bar{u}(s)\right)\Big|_{x=\bar{X}^{t,x;\bar{u}}(s)}\\
&= W_x\left(s, \bar{X}^{t,x;\bar{u}}(s)\right)\sigma_x(s) + W_{xx}\left(s, \bar{X}^{t,x;\bar{u}}(s)\right)\sigma(s)\,,
\end{aligned}
$$

which can be directly deduced by the connection between the MP and the DPP for decoupled FBSCSs. The connection between $\tilde{K}_2(\cdot)$ and $\frac{\partial^2 V}{\partial x^2}(\cdot)$ can be analyzed similarly. Besides the smooth case, we also study other special cases. When the diffusion term σ of the forward stochastic differential equation in (2.1) is linear in z, we relax the assumption that $q(\cdot)$ is bounded. For the so-called local case in which the control domain is convex and compact, the relations in Theorem 3.1 are still hold under our Assumptions 2.1, 2.8 and 2.10 since our control domain is only supposed to be a nonempty and compact set. Then, we study the local case under the monotonicity conditions as in [13, 25] and obtain the relationship between the MP in [25] and the DPP in [13] for the fully coupled FBSCS.

The rest of the paper is organized as follows. In Section 2, we give the preliminaries. The connections between the value function and the adjoint processes within the framework of viscosity solution are given in Section 3. In the last section, we study some special cases.

2. Preliminaries

Let $T>0$ be fixed, and $U \subset \mathbb{R}^k$ be nonempty and compact. Given $t \in [0,T)$, denote by $\mathcal{U}^w[t,T]$ the set of all 5-tuples $(\Omega, \mathcal{F}, \mathbb{P}, B(\cdot); u(\cdot))$ satisfying the following:

- **(i)** $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space;
- **(ii)** $B(r) = (B_1(r), B_2(r), ... B_d(r))^{\intercal}_{r\geq t}$ is a d-dimensional standard Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with $B(t)=0$ a.s. Set $\mathbb{F}^t := \{\mathcal{F}_s^t\}_{s\geq t}$ and $\mathbb{F} = \mathbb{F}^0$, where $\mathcal{F}_s^t$ is the $\mathbb{P}$-augmentation of the natural filtration of $\sigma\{B(r) : t \leq r \leq s\}$;
- **(iii)** $u(\cdot) : [t,T] \times \Omega \to U$ is an $\mathbb{F}^t$-adapted process on $(\Omega, \mathcal{F}, \mathbb{P})$.

When there is no confusion, we also use $u(\cdot) \in \mathcal{U}^w[t,T]$. Denote by $\mathbb{R}^n$ the n-dimensional real Euclidean space and $\mathbb{R}^{k\times n}$ the set of $k \times n$ real matrices. Let $\langle \cdot, \cdot \rangle$ (resp. $\|\cdot\|$) denote the usual scalar product (resp. usual norm) of $\mathbb{R}^n$ and $\mathbb{R}^{k\times n}$. The scalar product (resp. norm) of $M=(m_{ij})$, $N=(n_{ij}) \in \mathbb{R}^{k\times n}$ is denoted by $\langle M, N\rangle = \mathrm{tr}\{MN^{\intercal}\}$ (resp. $\|M\| = \sqrt{MM^{\intercal}}$), where the superscript $^{\intercal}$ denotes the transpose of vectors or matrices.

For each given $p \geq 1$, we introduce the following spaces. $L^p(\mathcal{F}_T^t;\mathbb{R}^n)$: the space of $\mathcal{F}_T^t$-measurable $\mathbb{R}^n$-valued random vectors η such that $||\eta||_p := \left(\mathbb{E}\left[|\eta|^p\right]\right)^{\frac{1}{p}} < \infty$; $L^\infty(\mathcal{F}_T^t;\mathbb{R}^n)$: the space of $\mathcal{F}_T^t$-measurable $\mathbb{R}^n$-valued random vectors η such that $||\eta||_\infty = \mathrm{ess\ sup}_{\omega\in\Omega}|\eta(\omega)| < \infty$; $L^p_{\mathbb{F}^t}(t,T;\mathbb{R}^n)$: the space of $\mathbb{F}^t$-adapted $\mathbb{R}^n$-valued stochastic processes on $[t,T]$ such that $\mathbb{E}\left[\int_t^T |f(r)|^p dr\right] < \infty$; $L^\infty_{\mathbb{F}^t}(t,T;\mathbb{R}^n)$: the space of $\mathbb{F}^t$-adapted $\mathbb{R}^n$-valued stochastic processes on $[t,T]$ such that $||f(\cdot)||_\infty = \mathrm{ess\ sup}_{(r,\omega)\in[t,T]\times\Omega}|f(r,\omega)| < \infty$; $L^{p,q}_{\mathbb{F}^t}(t,T;\mathbb{R}^n)$: the space of $\mathbb{F}^t$-adapted $\mathbb{R}^n$-valued stochastic processes on $[t,T]$ such that $||f(\cdot)||_{p,q} = \left\{\mathbb{E}\left[\left(\int_t^T |f(r)|^p dr\right)^{\frac{q}{p}}\right]\right\}^{\frac{1}{q}} < \infty$; $L^p_{\mathbb{F}^t}(\Omega; C([t,T],\mathbb{R}^n))$: the space of $\mathbb{F}^t$-adapted $\mathbb{R}^n$-valued continuous stochastic processes on $[t,T]$ such that $\mathbb{E}\left[\sup_{t\leq r\leq T} |f(r)|^p\right] < \infty$.

To simplify the presentation, we only consider the case $d=1$. The results for $d>1$ are similar. For each fixed $(t,x)\in[0,T]\times\mathbb{R}$ and $u(\cdot)\in\mathcal{U}^w[t,T]$, consider the following controlled fully coupled FBSDE: for $s\in[t,T]$,

$$\begin{cases} dX^{t,x;u}(s)= & b(s,X^{t,x;u}(s),Y^{t,x;u}(s),Z^{t,x;u}(s),u(s))ds \\ & +\sigma(s,X^{t,x;u}(s),Y^{t,x;u}(s),Z^{t,x;u}(s),u(s))dB(s), \\ dY^{t,x;u}(s)= & -g(s,X^{t,x;u}(s),Y^{t,x;u}(s),Z^{t,x;u}(s),u(s))ds+Z^{t,x;u}(s)dB(s), \\ X^{t,x;u}(t)= & x,\ Y^{t,x;u}(T)=\phi(X^{t,x;u}(T)), \end{cases} \tag{2.1}$$

where $b:[t,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times U\to\mathbb{R}$, $\sigma:[t,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times U\to\mathbb{R}$, $g:[t,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times U\to\mathbb{R}$, $\phi:\mathbb{R}\to\mathbb{R}$.

Assumption 2.1. (i) b,σ,g,ϕ are continuous with respect to s,x,y,z,u, and there exist constants $L_i>0$, $i=1,2,3$ such that

$$|b(s,x_1,y_1,z_1,u)-b(s,x_2,y_2,z_2,u)|\le L_1|x_1-x_2|+L_2(|y_1-y_2|+|z_1-z_2|),$$

$$|\sigma(s,x_1,y_1,z_1,u)-\sigma(s,x_2,y_2,z_2,u)|\le L_1|x_1-x_2|+L_2|y_1-y_2|+L_3|z_1-z_2|,$$

$$|g(s,x_1,y_1,z_1,u)-g(s,x_2,y_2,z_2,u)|\le L_1(|x_1-x_2|+|y_1-y_2|+|z_1-z_2|),$$

$$|\phi(x_1)-\phi(x_2)|\le L_1|x_1-x_2|,$$

for all $s\in[0,T], x_i,y_i,z_i\in\mathbb{R},\ i=1,2,\ u\in U$.

(ii) For any $2\le\beta\le 8$, $\Lambda_\beta:=C_\beta 2^{\beta+1}(1+T^\beta)c_1^\beta<1$, where $c_1=\max\{L_2,L_3\}$, C_β is defined in Lemma 5.1 in [8] and can be found in Appendix.

Remark 2.2. Since U is compact, from the above assumption (i) we obtain that

$$|\psi(s,x,y,z,u)|\le L(1+|x|+|y|+|z|),$$

where $L>0$ is a constant and $\psi=b,\ \sigma,\ g$ and ϕ.

Remark 2.3. Note that $\beta=2$ is sufficient to guarantee the DPP. But, for the MP we need $2\le\beta\le 8$.

Given $u(\cdot)\in\mathcal{U}^w[t,T]$, by Theorem 2.2 in [8], the equation (2.1) has a unique solution $(X^{t,x;u}(\cdot),Y^{t,x;u}(\cdot),$ $Z^{t,x;u}(\cdot))\in L^\beta_{\mathbb{F}^t}(\Omega;C([t,T],\mathbb{R}))\times L^\beta_{\mathbb{F}^t}(\Omega;C([t,T],\mathbb{R}))\times L^{2,\beta}_{\mathbb{F}^t}(t,T;\mathbb{R})$ for $\beta\in[2,8]$. For the existence and uniqueness of solutions of FBSDEs, the readers may refer to ([10, 11, 14, 15, 21]). For each given $(t,x)\in[0,\,T]\times\mathbb{R}$, define the cost functional

$$J(t,x;u(\cdot))=Y^{t,x;u}(t). \tag{2.2}$$

Remark 2.4. Since the coefficients are deterministic and $u(\cdot)$ is an $\mathbb{F}^t$-adapted process, $J(t,x;u(\cdot))$ is deterministic.

For each given $(t,x)\in[0,T]\times\mathbb{R}$, define the value function

$$W(t,x)=\inf_{u(\cdot)\in\mathcal{U}^w[t,T]}J(t,x;u(\cdot)). \tag{2.3}$$

We introduce the following generalized HJB equation combined with an algebra equation for $W(\cdot,\cdot)$:

$$\begin{cases} \inf_{u\in U}\{G(t,x,W(t,x),W_x(t,x),W_{xx}(t,x),u)\}+W_t(t,x)=0, \\ W(T,x)=\phi(x), \end{cases} \tag{2.4}$$

where

$$\begin{aligned} &G(t,x,v,p,A,u) \\ &= pb(t,x,v,V(t,x,v,p,u),u)+\tfrac{1}{2}A(\sigma(t,x,v,V(t,x,v,p,u),u))^2 \\ &\quad +g(t,x,v,V(t,x,v,p,u),u), \\ &V(t,x,v,p,u)=p\sigma(t,x,v,V(t,x,v,p,u),u), \\ &\forall(t,x,v,p,A,u)\in[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times U. \end{aligned} \tag{2.5}$$

Now, we introduce the following definition of viscosity solution (see [3]).

Definition 2.5. (i) A real-valued continuous function $W(\cdot,\cdot)\in C([0,T]\times\mathbb{R})$ is called a viscosity subsolution (resp. supersolution) of (2.4) if $W(T,x)\le\phi(x)$ (resp. $W(T,x)\ge\phi(x)$) for all $x\in\mathbb{R}$ and if for all $f\in C_b^{2,3}([0,T]\times\mathbb{R})$ such that $W(t,x)=f(t,x)$ and $W-f$ attains a local maximum (resp. minimum) at $(t,x)\in[0,T)\times\mathbb{R}$, we have

$$\begin{cases} \inf_{u\in U}\{G(t,x,f(t,x),f_x(t,x),f_{xx}(t,x),u)\}+f_t(t,x)\ge 0 \\ (\text{resp. } \inf_{u\in U}\{G(t,x,f(t,x),f_x(t,x),f_{xx}(t,x),u)\}+f_t(t,x)\le 0). \end{cases}$$

(ii) A real-valued continuous function $W(\cdot,\cdot)\in C([0,T]\times\mathbb{R})$ is called a viscosity solution to (2.4), if it is both a viscosity subsolution and viscosity supersolution.

Remark 2.6. The viscosity solution to (2.4) can be equivalently defined by sub-jets and super-jets (see [3]).

Proposition 2.7. *(see [9]) Let Assumption 2.1 hold. Then, for each $t\in[0,T]$ and $x,x'\in\mathbb{R}$,*

$$|W(t,x)-W(t,x')|\le C|x-x'| \text{ and } |W(t,x)|\le C(1+|x|),$$

where $C>0$ depends on L_1, L_2, L_3, and T. Moreover, $W(\cdot,\cdot)$ satisfies the DPP. If L_3 is small enough, then $W(\cdot,\cdot)$ is the viscosity solution to (2.4).

Let $\bar{u}(\cdot)\in\mathcal{U}^w[t,T]$ be optimal, then $W(t,x)=J(t,x;\bar{u}(\cdot))$. The corresponding solution $(\bar{X}^{t,x;\bar{u}}(\cdot),\bar{Y}^{t,x;\bar{u}}(\cdot),\bar{Z}^{t,x;\bar{u}}(\cdot))$ to equation (2.1) is called optimal trajectory. To derive the MP, we give the following assumptions.

Assumption 2.8. For $\psi=b$, σ, g, and ϕ, we suppose
(i) ψ_x, ψ_y, ψ_z are bounded and continuous in (x,y,z,u); there exists a constant $L>0$ such that

$$|\sigma(t,0,0,z,u)-\sigma(t,0,0,z,u')|\le L(1+|u|+|u'|).$$

(ii) ψ_{xx}, ψ_{xy}, ψ_{yy} , ψ_{xz}, ψ_{yz}, ψ_{zz} are bounded and continuous in (x,y,z,u).

Remark 2.9. It is clear that L_1 in Assumption 2.1 is $\max\{||b_x||_\infty,||\sigma_x||_\infty,||g_x||_\infty,\ ||g_y||_\infty,||g_z||_\infty,||\phi_x||_\infty\}$, $L_2=\max\{||b_y||_\infty,||b_z||_\infty,||\sigma_y||_\infty\}$ and $L_3=||\sigma_z||_\infty$.

For $\beta_0 > 0$, set

$$F(y) = L_1 + \left(L_2 + L_1 + \beta_0^{-1} L_1 L_2\right)|y| \\ + \left[L_2 + \beta_0^{-1}(L_1 L_2 + L_2^2)\right] y^2 + \beta_0^{-1} L_2^2 |y|^3,\ y \in \mathbb{R}.$$

Let $s(\cdot)$ be the maximal solution to the following ordinary differential equation (ODE):

$$s(t) = L_1 + \int_t^T F(s(r))dr,\ t \in [0,T]. \tag{2.6}$$

Let $l(\cdot)$ be the minimal solution to the following ODE:

$$l(t) = -L_1 - \int_t^T F(l(r))dr,\ t \in [0,T]. \tag{2.7}$$

Moreover, set

$$t_1 = T - \int_{-\infty}^{-L_1} \frac{1}{F(y)}dy, \quad t_2 = T - \int_{L_1}^{\infty} \frac{1}{F(y)}dy,\ t^* = t_1 \vee t_2. \tag{2.8}$$

Assumption 2.10. There exists a positive constant $\beta_0 \in (0,1)$ such that

$$t^* < 0,\ [s(0) \vee (-l(0))]L_3 \le 1 - \beta_0. \tag{2.9}$$

Remark 2.11. When L_2 and L_3 are small enough, we derive that Assumption 2.10 holds in [8].

We introduce the following notations: for $\psi = b, \sigma, g$ and $\kappa = x, y, z$,

$$\begin{aligned} \psi(s) = &\quad \psi(s, \bar{X}^{t,x;\bar{u}}(s), \bar{Y}^{t,x;\bar{u}}(s), \bar{Z}^{t,x;\bar{u}}(s), \bar{u}(s)), \\ \psi_\kappa(s) = &\quad \psi_\kappa(s, \bar{X}^{t,x;\bar{u}}(s), \bar{Y}^{t,x;\bar{u}}(s), \bar{Z}^{t,x;\bar{u}}(s), \bar{u}(s)), \\ D\psi(s) = &\quad D\psi(s, \bar{X}^{t,x;\bar{u}}(s), \bar{Y}^{t,x;\bar{u}}(s), \bar{Z}^{t,x;\bar{u}}(s), \bar{u}(s)), \\ D^2\psi(s) = &\quad D^2\psi(s, \bar{X}^{t,x;\bar{u}}(s), \bar{Y}^{t,x;\bar{u}}(s), \bar{Z}^{t,x;\bar{u}}(s), \bar{u}(s)), \end{aligned} \tag{2.10}$$

where $D\psi$ is the gradient of ψ with respect to x, y, z, and $D^2\psi$ is the Hessian matrix of ψ with respect to x, y, z.

The first-order adjoint equation is

$$\left\{ \begin{aligned} dp(s) = &\quad -\{g_x(s) + g_y(s)p(s) + g_z(s)K_1(s) + b_x(s)p(s) \\ &\quad + b_y(s)p^2(s) + b_z(s)K_1(s)p(s) + \sigma_x(s)q(s) \\ &\quad + \sigma_y(s)p(s)q(s) + \sigma_z(s)K_1(s)q(s)\}\,ds + q(s)dB(s), \\ p(T) = &\quad \phi_x(\bar{X}(T)), \end{aligned} \right. \tag{2.11}$$

where

$$K_1(s) = (1 - p(s)\sigma_z(s))^{-1}\left[\sigma_x(s)p(s) + \sigma_y(s)p^2(s) + q(s)\right]. \tag{2.12}$$

The second-order adjoint equation is

$$
\begin{cases}
-dP(s)= & \left\{P(s)\left[(D\sigma(s)^{\intercal}(1,p(s),K_1(s))^{\intercal})^2 \ +2Db(s)^{\intercal}(1,p(s),K_1(s))^{\intercal}+H_y(s)\right]\right. \\
& +2Q(s)D\sigma(s)^{\intercal}(1,p(s),K_1(s))^{\intercal}+(1,p(s),K_1(s))\,D^2H(s)\,(1,p(s),K_1(s))^{\intercal} \\
& \left.+H_z(s)K_2(s)\right\}ds-Q(s)dB(s), \\
P(T)= & \phi_{xx}(\bar{X}(T)),
\end{cases}
\tag{2.13}
$$

where

$$
H(s,x,y,z,u,p,q)=\quad g(s,x,y,z,u)+pb(s,x,y,z,u)+q\sigma(s,x,y,z,u),
$$

$$
\begin{aligned}
K_2(s)= \quad & (1-p(s)\sigma_z(s))^{-1}\left\{p(s)\sigma_y(s)\ +2\left[\sigma_x(s)+\sigma_y(s)p(s)+\sigma_z(s)K_1(s)\right]\right\}P(s) \\
& +(1-p(s)\sigma_z(s))^{-1}\left\{Q(s)\ +p(s)\,(1,p(s),K_1(s))\,D^2\sigma(s)\,(1,p(s),K_1(s))^{\intercal}\right\},
\end{aligned}
\tag{2.14}
$$

$H(s)$, $DH(s)$ and $D^2H(s)$ are defined similarly in (2.10). It should be pointed out that under Assumptions 2.1, 2.8 and 2.10, there is a solution $(p(\cdot),q(\cdot))\in L^{\infty}_{\mathbb{F}}(0,T;\mathbb{R})\times L^{2,2}_{\mathbb{F}}(0,T;\mathbb{R})$ $(resp.(P(\cdot),Q(\cdot))\in L^{\infty}_{\mathbb{F}}(0,T;\mathbb{R})\times L^{2,2}_{\mathbb{F}}(0,T;\mathbb{R})$) to equation (2.11) (resp. (2.13))(see [8]). Define

$$
\begin{aligned}
\mathcal{H}(s,x,y,z,u,p,q,P) \quad &= pb(s,x,y,z+\Delta(s),u)+q\sigma(s,x,y,z+\Delta(s),u) \\
& \quad +g(s,x,y,z+\Delta(s),u)+\tfrac{1}{2}P(\sigma(s,x,y,z+\Delta(s),u)-\sigma(s))^2
\end{aligned}
\tag{2.15}
$$

where $\Delta(\cdot)$ is the solution to the following algebra equation

$$
\begin{aligned}
\Delta(s)= \quad & p(s)\left[\sigma(s,\bar{X}^{t,x;\bar{u}}(s),\bar{Y}^{t,x;\bar{u}}(s),\bar{Z}^{t,x;\bar{u}}(s)+\Delta(s),u)\right. \\
& \left.-\sigma(s,\bar{X}^{t,x;\bar{u}}(s),\bar{Y}^{t,x;\bar{u}}(s),\bar{Z}^{t,x;\bar{u}}(s),\bar{u}(s))\right],\ s\in[t,T].
\end{aligned}
\tag{2.16}
$$

Then, we have the following maximum principle.

Theorem 2.12. (See [8], Thm. 3.18) *Suppose that Assumptions 2.1, 2.8 and 2.10 hold, and $q(\cdot)$ in (2.11) is bounded. Then the following stochastic maximum principle holds:*

$$
\begin{aligned}
&\mathcal{H}(s,\bar{X}^{t,x;\bar{u}}(s),\bar{Y}^{t,x;\bar{u}}(s),\bar{Z}^{t,x;\bar{u}}(s),u,p(s),q(s),P(s)) \\
&\geq \mathcal{H}(s,\bar{X}^{t,x;\bar{u}}(s),\bar{Y}^{t,x;\bar{u}}(s),\bar{Z}^{t,x;\bar{u}}(s),\bar{u}(s),p(s),q(s),P(s)),\ \forall u\in U \text{ a.e., a.s..}
\end{aligned}
\tag{2.17}
$$

Remark 2.13. In the above theorem, if $\sigma(t,x,y,z,u)=A(t)z+\sigma_1(t,x,y,u)$, then we do not need the assumption that $q(\cdot)$ is bounded.

3. Main results

In the following, the constant $C>0$ will change from line to line for simplicity. All the obtained results can be extended to the multidimensional case for $X(\cdot)$ and $Z(\cdot)$.

3.1. Differentials in spatial variable

In this section, we investigate the relationship between the MP and the DPP. We first recall the notions of second-order super- and sub-jets in the spatial variable x see [3]). For $w \in C([0,T]\times\mathbb{R})$ and $(t,\hat{x}) \in [0,T]\times\mathbb{R}$, define

$$\begin{cases} D_x^{2,+}w(t,\hat{x}) := \{(p,P) \in \mathbb{R}\times\mathbb{R} : w(t,x) \leq w(t,\hat{x}) + \langle p, x-\hat{x}\rangle \\ \qquad +\frac{1}{2}(x-\hat{x})P(x-\hat{x}) + o\left(|x-\hat{x}|^2\right),\ \text{as } x \to \hat{x}\}, \\ D_x^{2,-}w(t,\hat{x}) := \{(p,P) \in \mathbb{R}\times\mathbb{R} : w(t,x) \geq w(t,\hat{x}) + \langle p, x-\hat{x}\rangle \\ \qquad +\frac{1}{2}(x-\hat{x})P(x-\hat{x}) + o\left(|x-\hat{x}|^2\right),\ \text{as } x \to \hat{x}\}. \end{cases}$$

Theorem 3.1. *Suppose Assumptions 2.1, 2.8 and 2.10 hold. Let $\bar{u}(\cdot)$ be optimal for problem (2.3), and let $(p(\cdot),q(\cdot)) \in L_{\mathbb{F}}^{\infty}(0,T;\mathbb{R})\times L_{\mathbb{F}}^{2,2}(0,T;\mathbb{R})$ and $(P(\cdot),Q(\cdot)) \in L_{\mathbb{F}}^{\infty}(0,T;\mathbb{R})\times L_{\mathbb{F}}^{2,2}(0,T;\mathbb{R})$ be the solution to equation (2.11) and (2.13) respectively. Furthermore, suppose that $q(\cdot)$ is bounded. Then for any $s \in [t,T]$,*

$$\begin{cases} \{p(s)\}\times[P(s),\infty) \subseteq D_x^{2,+}W(s,\bar{X}^{t,x;\bar{u}}(s)), \\ D_x^{2,-}W(s,\bar{X}^{t,x;\bar{u}}(s)) \subseteq \{p(s)\}\times(-\infty,P(s)],\ \mathbb{P}\text{-a.s.}. \end{cases} \tag{3.1}$$

Proof. The proof is divided into five steps.

Step 1: Variational equations.

For each fixed $s \in [t,T]$ and $x' \in \mathbb{R}$, denote by $(X^{s,x';\bar{u}}(\cdot), Y^{s,x';\bar{u}}(\cdot), Z^{s,x';\bar{u}}(\cdot))$ the solution to the following FBSDE:

$$\begin{cases} dX^{s,x';\bar{u}}(r) = & b(r,X^{s,x';\bar{u}}(r),Y^{s,x';\bar{u}}(r),Z^{s,x';\bar{u}}(r),\bar{u}(s))dr \\ & +\sigma(r,X^{s,x';\bar{u}}(r),Y^{s,x';\bar{u}}(r),Z^{s,x';\bar{u}}(r),\bar{u}(s))dB(r), \\ dY^{s,x';\bar{u}}(r) = & -g(r,X^{s,x';\bar{u}}(r),Y^{s,x';\bar{u}}(r),Z^{s,x';\bar{u}}(r),\bar{u}(s))dr + Z^{s,x';\bar{u}}(r)dB(r), \\ X^{s,x';\bar{u}}(s) = & x',\ Y^{s,x';\bar{u}}(T) = \phi(X^{s,x';\bar{u}}(T)),\quad r \in [s,T]. \end{cases} \tag{3.2}$$

Set $\bar{\Theta}(r) := \left(\bar{X}^{t,x;\bar{u}}(r), \bar{Y}^{t,x;\bar{u}}(r), \bar{Z}^{t,x;\bar{u}}(r)\right)$,

$$\begin{aligned} \hat{X}(r) := &\ X^{s,x';\bar{u}}(r) - \bar{X}^{t,x;\bar{u}}(r), \hat{Y}(r) := \ Y^{s,x';\bar{u}}(r) - \bar{Y}^{t,x;\bar{u}}(r), \\ \hat{Z}(r) := &\ Z^{s,x';\bar{u}}(r) - \bar{Z}^{t,x;\bar{u}}(r), \hat{\Theta}(r) := \ \left(\hat{X}(r),\hat{Y}(r),\hat{Z}(r)\right). \end{aligned} \tag{3.3}$$

By Theorem 2.2 in [8], for each $\beta \in [2,8]$, we have $\mathbb{P}$-a.s.

$$\mathbb{E}\left[\sup_{r\in[s,T]}\left(|\hat{X}(r)|^\beta + |\hat{Y}(r)|^\beta\right) + \left(\int_s^T |\hat{Z}(r)|^2 dr\right)^{\frac{\beta}{2}} \middle| \mathcal{F}_s^t\right] \leq C\left|x' - \bar{X}^{t,x;\bar{u}}(s)\right|^\beta. \tag{3.4}$$

It is easy to check that $\left(\hat{X}(\cdot),\hat{Y}(\cdot),\hat{Z}(\cdot)\right)$ satisfies the following FBSDE:

$$\left\{\begin{array}{rl} d\hat{X}(r)= & \left[\hat{\Theta}(r)Db(r)+\varepsilon_1(r)\right]dr+\left[\hat{\Theta}(r)D\sigma(r)+\varepsilon_2(r)\right]dB(r),\\ \hat{X}(s)= & x'-\bar{X}^{t,x;\bar{u}}(s),\\ d\hat{Y}(r)= & -\left[\hat{\Theta}(r)Dg(r)+\varepsilon_3(r)\right]dr+\hat{Z}(r)dB(r),\\ \hat{Y}(T)= & \phi_x(\bar{X}^{t,x;\bar{u}}(T))\hat{X}(T)+\varepsilon_4(T),\ r\in[s,T], \end{array}\right. \tag{3.5}$$

where

$$\begin{array}{rl} \varepsilon_1(r)= & \left(\tilde{b}_x(r)-b_x(r)\right)\hat{X}(r)+\left(\tilde{b}_y(r)-b_y(r)\right)\hat{Y}(r)\\ & +\left(\tilde{b}_z(r)-b_z(r)\right)\hat{Z}(r),\\ \varepsilon_2(r)= & \left(\tilde{\sigma}_x(r)-\sigma_x(r)\right)\hat{X}(r)+\left(\tilde{\sigma}_y(r)-\sigma_y(r)\right)\hat{Y}(r)\\ & +\left(\tilde{\sigma}_z(r)-\sigma_z(r)\right)\hat{Z}(r),\\ \varepsilon_3(r)= & \left(\tilde{g}_x(r)-g_x(r)\right)\hat{X}(r)+\left(\tilde{g}_y(r)-g_y(r)\right)\hat{Y}(r)\\ & +\left(\tilde{g}_z(r)-g_z(r)\right)\hat{Z}(r),\\ \varepsilon_4(T)= & [\tilde{\phi}_x(T)-\phi_x(\bar{X}^{t,x;\bar{u}}(T))]\hat{X}(T),\\ \tilde{\psi}_\kappa(r)= & \int_0^1\psi_\kappa(r,\bar{\Theta}(r)+\lambda\hat{\Theta}(r),\bar{u}(r))d\lambda\\ & \text{for } \psi=b,\sigma,g,\phi \text{ and } \kappa=x,y,z. \end{array} \tag{3.6}$$

Step 2: Estimates of the remainder terms of FBSDE.

By Assumption 2.8, we derive that, for $i=1,2,3$,

$$|\varepsilon_i(r)|\leq C\left(|\hat{X}(r)|^2+|\hat{Y}(r)|^2+|\hat{Z}(r)|^2\right)$$

and

$$|\varepsilon_4(T)|\leq C|\hat{X}(T)|^2.$$

Then, by (3.4), we obtain that for each $\beta\in[2,4]$

$$\begin{array}{rl} \mathbb{E}\left[\left(\int_s^T|\varepsilon_i(r)|dr\right)^\beta\middle|\mathcal{F}_s^t\right] & \leq C\left|x'-\bar{X}^{t,x;\bar{u}}(s)\right|^{2\beta},\ \text{i=1,2,3},\\ \mathbb{E}\left[|\varepsilon_4(T)|^\beta\middle|\mathcal{F}_s^t\right] & \leq C\left|x'-\bar{X}^{t,x;\bar{u}}(s)\right|^{2\beta}. \end{array} \tag{3.7}$$

Step 3: Relationship between $\hat{X}(\cdot)$ and $(\hat{Y}(\cdot),\hat{Z}(\cdot))$.
By Theorem A.3 in [8], we get

$$\begin{array}{rl} \hat{Y}(r)= & p(r)\hat{X}(r)+\varphi(r),\\ \hat{Z}(r)= & K_1(r)\hat{X}(r)+(1-p(r)\sigma_z(r))^{-1}[p(r)\sigma_y(r)\varphi(r)+p(r)\varepsilon_2(r)+\nu(r)], \end{array} \tag{3.8}$$

where $p(\cdot)$ is the solution to first-order adjoint equation (2.11), and $(\varphi(\cdot),\nu(\cdot))$ is the solution to the following linear BSDE:

$$\left\{\begin{array}{rl} d\varphi(r)= & -\left[A(r)\varphi(r)+C(r)\nu(r)+p(r)\varepsilon_1(r)+q(r)\varepsilon_2(r)\right.\\ & \left.+\varepsilon_3(r)+H_z(r)(1-p(r)\sigma_z(r))^{-1}p(r)\varepsilon_2(r)\right]dr\\ & +\nu(r)dB(r),\\ \varphi(T)= & \varepsilon_4(T),\end{array}\right. \tag{3.9}$$

$$\begin{aligned} A(r)= &\ p(r)b_y(r)+q(r)\sigma_y(r)+g_y(r)+(1-p(r)\sigma_z(r))^{-1}\sigma_y(r)p(r)H_z(r),\\ C(r)= &\ (1-p(r)\sigma_z(r))^{-1}H_z(r),\\ H_z(r)= &\ p(r)b_z(r)+q(r)\sigma_z(r)+g_z(r).\end{aligned}$$

Thus, we can write $\hat{\Theta}(r)$ as

$$\hat{\Theta}(r)=(1,p(r),K_1(r))\hat{X}(r)+\hat{L}(r), \tag{3.10}$$

where $\hat{L}(r):=(0,\varphi(r),(1-p(r)\sigma_z(r))^{-1}[p(r)\sigma_y(r)\varphi(r)+p(r)\varepsilon_2(r)+\nu(r)])$.

It follows from Theorem 3.6 in [8] that

$$|p(r)|\leq s(0)\vee(-l(0)) \text{ for } r\in[s,T]. \tag{3.11}$$

By relations (3.6) and (3.8), we get

$$\begin{aligned}\varepsilon_2(r)= &\ (\tilde{\sigma}_x(r)-\sigma_x(r))\hat{X}(r)+(\tilde{\sigma}_y(r)-\sigma_y(r))\cdot\left(p(r)\hat{X}(r)+\varphi(r)\right)\\ &+(\tilde{\sigma}_z(r)-\sigma_z(r))\cdot\Big[K_1(r)\hat{X}(r)+(1-p(r)\sigma_z)^{-1}\\ &\cdot(p(r)\sigma_y(r)\varphi(r)+p(r)\varepsilon_2(r)+\nu(r))\Big].\end{aligned} \tag{3.12}$$

Thus, we have

$$\begin{aligned}\varepsilon_2(r)= &\ \tfrac{1-p(r)\sigma_z(r)}{1-p(r)\tilde{\sigma}_z(r)}\Big\{\Big[(\tilde{\sigma}_x(r)-\sigma_x(r))\hat{X}(r)+(\tilde{\sigma}_y(r)-\sigma_y(r))\left(p(r)\hat{X}(r)+\varphi(r)\right)\Big]\\ &+(\tilde{\sigma}_z(r)-\sigma_z(r))\cdot\Big[K_1(r)\hat{X}(r)+(1-p(r)\sigma_z(r))^{-1}(p(r)\sigma_y(r)\varphi(r)+\nu(r))\Big]\Big\}\end{aligned} \tag{3.13}$$

By Assumption 2.10 and (3.11), one has $\left|\frac{1-p(r)\sigma_z(r)}{1-p(r)\tilde{\sigma}_z(r)}\right|\leq\beta_0^{-1}$.

Step 4: Variation of φ.

By (3.7) and the estimate of BSDE for (3.9), we obtain that, for each $\beta\in[2,4]$,

$$\begin{aligned}&\mathbb{E}\left[\sup_{r\in[s,T]}|\varphi(r)|^\beta+\left(\int_s^T|\nu(r)|^2dr\right)^{\frac{\beta}{2}}\middle|\mathcal{F}_s^t\right]\\ &\leq C\mathbb{E}\left[|\varepsilon_4(T)|^\beta+\left(\int_s^T(|\varepsilon_1(r)|+|\varepsilon_2(r)|+|\varepsilon_3(r)|)dr\right)^\beta\middle|\mathcal{F}_s^t\right]\\ &\leq C\left|x'-\bar{X}^{t,x;\bar{u}}(s)\right|^{2\beta},\ \mathbb{P}\text{-a.s..}\end{aligned} \tag{3.14}$$

In the following, we want to prove

$$\varphi(s) - \frac{1}{2}P(r)(\hat{X}(r))^2 = o(\left|x' - \bar{X}^{t,x;\bar{u}}(s)\right|^2),\ \mathbb{P}\text{-a.s.}. \tag{3.15}$$

Define

$$\tilde{\varphi}(r) = \frac{1}{2}P(r)(\hat{X}(r))^2; \tag{3.16}$$

$$\tilde{\nu}(r) = P(r)\hat{X}(r)(\hat{\Theta}(r)D\sigma(r) + \varepsilon_2(r)) + \frac{1}{2}Q(r)\hat{X}(r)^2. \tag{3.17}$$

Applying Itô's formula to $\frac{1}{2}P(r)\hat{X}(r)^2$, by (3.10), we obtain that $(\tilde{\varphi}(r), \tilde{\nu}(r))$ satisfies the following BSDE:

$$\begin{cases} d\tilde{\varphi}(r) = & \Big\{P(r)\Big[\Big(\hat{L}(r)Db(r) + \varepsilon_1(r)\Big)\hat{X}(r) \ + \frac{1}{2}(\hat{L}(r)D\sigma(r) + \varepsilon_2(r))^2 \\ & +(1, p(r), K_1(r))D\sigma(r)\hat{X}(r)(\hat{L}(r)D\sigma(r) + \varepsilon_2(r)) - \frac{1}{2}H_y(r)\hat{X}(r)^2\Big] \\ & -\frac{1}{2}\left[(1, p(r), K_1(r))D^2H(s)(1, p(r), K_1(r))^{\intercal} \ + H_z(s)K_2(s)\right]\hat{X}(r)^2 \\ & +Q(r)(\hat{L}(r)D\sigma(r) + \varepsilon_2(r))\hat{X}(r)\Big\}\, dr + \tilde{\nu}(r)dB(r), \\ \tilde{\varphi}(T) = & \frac{1}{2}\phi_{xx}(\bar{X}^{t,x;\bar{u}}(T))\left(\hat{X}(T)\right)^2. \end{cases} \tag{3.18}$$

Set

$$\hat{\varphi}(r) = \varphi(r) - \tilde{\varphi}(r),\ \hat{\nu}(r) = \nu\,(r) - \tilde{\nu}(r).$$

Replace $\varepsilon_1(r)$ by $\frac{1}{2}\hat{\Theta}(r)D^2b(r)\hat{\Theta}(r)^{\intercal} + \varepsilon_5(r)$, $\varepsilon_2(r)$ by $\frac{1}{2}\hat{\Theta}(r)D^2\sigma(r)\hat{\Theta}(r)^{\intercal} + \varepsilon_6(r)$, $\varepsilon_3(r)$ by $\frac{1}{2}\hat{\Theta}(r)D^2g(r)\hat{\Theta}(r)^{\intercal} + \varepsilon_7(r)$ and $\varepsilon_4(T)$ by $\frac{1}{2}\phi_{xx}(\bar{X}^{t,x;\bar{u}}(T))\left(\hat{X}(T)\right)^2 + \varepsilon_8(T)$ in (3.9), where

$$\begin{aligned} \varepsilon_5(r) &= \hat{\Theta}(r)\int_0^1\int_0^1 \lambda\left[D^2b(r, \bar{\Theta}^{t,x;\bar{u}}(r) + \theta\lambda\hat{\Theta}(r), \bar{u}(r)) - D^2b(r)\right] d\lambda d\theta\hat{\Theta}(r)^{\intercal}, \\ \varepsilon_6(r) &= \hat{\Theta}(r)\int_0^1\int_0^1 \lambda\left[D^2\sigma(r, \bar{\Theta}^{t,x;\bar{u}}(r) + \theta\lambda\hat{\Theta}(r), \bar{u}(r)) - D^2\sigma(r)\right] d\lambda d\theta\hat{\Theta}(r)^{\intercal}, \\ \varepsilon_7(r) &= \hat{\Theta}(r)\int_0^1\int_0^1 \lambda\left[D^2g(r, \bar{\Theta}^{t,x;\bar{u}}(r) + \theta\lambda\hat{\Theta}(r), \bar{u}(r)) - D^2g(r)\right] d\lambda d\theta\hat{\Theta}(r)^{\intercal}, \\ \varepsilon_8(T) &= \int_0^1\int_0^1 \lambda\left[\phi_{xx}(\bar{X}^{t,x;\bar{u}}(r) + \theta\lambda\hat{X}(T)) - \phi_{xx}(\bar{X}^{t,x;\bar{u}}(T))\right] d\lambda d\theta\left(\hat{X}(T)\right)^2. \end{aligned}$$

By (2.14), one can verify that $(\hat{\varphi}(\cdot), \hat{\nu}(\cdot))$ satisfies the following linear BSDE

$$\begin{cases} d\hat{\varphi}(r) = & -\left[A(r)\hat{\varphi}(r) + C(r)\hat{\nu}(r) + I(r)\right] dr + \hat{\nu}(r)dB\,(r)\,, \\ \hat{\varphi}(T) = & \varepsilon_8(T), \end{cases} \tag{3.19}$$

where

$$
\begin{aligned}
\mathrm{I}(r)= \quad & C(r)P(r)(\hat{L}(r)D\sigma(r)+\varepsilon_2(r))\hat{X}(r)+p(r)\varepsilon_5(r)+\varepsilon_7(r)\\
&+p(r)(1,p(r),K_1(r))D^2b(r)\hat{L}(r)^{\intercal}\hat{X}(r)+\tfrac{1}{2}p(r)\hat{L}(r)D^2b(r)\hat{L}(r)^{\intercal}\\
&+\left[q(r)+H_z(r)(1-p(r)\sigma_z(r))^{-1}p(r)\right]\\
&\quad\cdot\left[\tfrac{1}{2}\hat{L}(r)D^2\sigma(r)\hat{L}\,(r)^{\intercal}+\varepsilon_6(r)\ +(1,p(r),K_1(r))D^2\sigma(r)\hat{L}\,(r)^{\intercal}\,\hat{X}(r)\right]\\
&+(1,p(r),K_1(r))D^2g(r)\hat{L}(r)^{\intercal}\hat{X}(r)+\tfrac{1}{2}\hat{L}(r)D^2g(r)\hat{L}(r)^{\intercal}+P(r)\\
&\cdot\left[\left(\hat{L}(r)Db(r)+\varepsilon_1(r)\right)\hat{X}(r)\ +\tfrac{1}{2}(\hat{L}(r)D\sigma(r)+\varepsilon_2(r))^2\right.\\
&\left.+(1,p(r),K_1(r))D\sigma(r)\left(\hat{L}(r)D\sigma(r)+\varepsilon_2(r)\right)\hat{X}(r)\right]\\
&+Q(r)\left(\hat{L}(r)D\sigma(r)+\varepsilon_2(r)\right)\hat{X}(r).
\end{aligned}
\tag{3.20}
$$

Note that $A(\cdot)$ and $C(\cdot)$ are bounded. Then by the standard estimate of BSDE, we obtain that

$$
|\hat{\varphi}(s)|^2\le C\mathbb{E}\left[|\varepsilon_8(T)|^2+\left(\int_s^T|\mathrm{I}(r)|dr\right)^2\middle|\mathcal{F}_s^t\right]. \tag{3.21}
$$

Since $q(\cdot)$ is bounded, one can verify that $P(\cdot)$ is bounded. By (3.6), (3.10) and (3.13), it is easy to check that

$$
\begin{aligned}
|\mathrm{I}\,(r)|\le \quad & C\left[(1+|Q\,(r)|)\left(\rho\,(r)\left|\hat{X}(r)\right|^2+\left|\hat{X}(r)\varphi\,(r)\right|\right.\right.\\
&\left.\left.+\left|\hat{X}(r)\nu(r)\right|\right)+|\varphi\,(r)|^2+|\nu\,(r)|^2\right],
\end{aligned}
\tag{3.22}
$$

where C is a constant and

$$
\begin{aligned}
\rho\,(r)=&\textstyle\sum_{i=1}^3\int_0^1\int_0^1\lambda\left|D^2\psi_i(r,\bar{\Theta}(r)+\theta\lambda\hat{\Theta}(r),\bar{u}(r))-D^2\psi_i(r)\right|d\lambda d\theta\\
&+\textstyle\sum_{i=1}^2\int_0^1\left|D\psi_i(r,\bar{\Theta}(r)+\theta\hat{\Theta}(r),\bar{u}(r))-D\psi_i(r)\right|d\theta
\end{aligned}
\tag{3.23}
$$

for $\psi_1=b$, $\psi_2=\sigma$, $\psi_3=g$.

Next, we estimate term by term.

$$
\begin{aligned}
&\mathbb{E}\left[|\varepsilon_8(T)|^2\middle|\mathcal{F}_s^t\right]\\
&\le\left\{\mathbb{E}\left[\left|\hat{X}(T)\right|^8\middle|\mathcal{F}_s^t\right]\right\}^{\frac{1}{2}}\\
&\quad\cdot\left\{\mathbb{E}\left[\left|\textstyle\int_0^1\int_0^1\lambda[\phi_{xx}(\bar{X}^{t,x;\bar{u}}(r)+\theta\lambda\hat{X}(T))\ -\phi_{xx}(\bar{X}^{t,x;\bar{u}}(T))]d\lambda d\theta\right|^4\middle|\mathcal{F}_s^t\right]\right\}^{\frac{1}{2}}\\
&=o\left(\left|x'-\bar{X}^{t,x;\bar{u}}(s)\right|^4\right);
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\left[\left(\int_s^T (1+|Q(r)|)\,\rho(r)\,|\hat{X}(r)|^2 dr\right)^2 \middle| \mathcal{F}_s^t\right] \\
&\leq C\mathbb{E}\left[\sup_{s\leq r\leq T}\left|\hat{X}(r)\right|^4 \int_s^T (1+|Q(r)|)^2\,dr \ \cdot \int_s^T \rho(r)^2\,dr \middle| \mathcal{F}_s^t\right] \\
&= o\left(\left|x'-\bar{X}^{t,x;\bar{u}}(s)\right|^4\right);
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\left[\left(\int_s^T (1+|Q(r)|)\left|\hat{X}(r)\nu(r)\right| dr\right)^2 \middle| \mathcal{F}_s^t\right] \\
&\leq C\mathbb{E}\left[\sup_{s\leq r\leq T}\left|\hat{X}(r)\right|^2 \int_s^T (1+|Q(r)|)^2\,dr \ \cdot \int_s^T |\nu(r)|^2\,dr \middle| \mathcal{F}_s^t\right] \\
&\leq C\left\{\mathbb{E}\left[\sup_{s\leq r\leq T}\left|\hat{X}(r)\right|^4 \middle| \mathcal{F}_s^t\right]\right\}^{\frac{1}{2}} \cdot \left\{\mathbb{E}\left[\left(\int_s^T |\nu(r)|^2\,dr\right)^4 \middle| \mathcal{F}_s^t\right]\right\}^{\frac{1}{4}} \\
&\quad \cdot \left\{\mathbb{E}\left[\left(\int_s^T (1+|Q(r)|)^2\,dr\right)^4 \middle| \mathcal{F}_s^t\right]\right\}^{\frac{1}{4}} \\
&= o\left(\left|x'-\bar{X}^{t,x;\bar{u}}(s)\right|^4\right);
\end{aligned}
\tag{3.24}
$$

$$
\mathbb{E}\left[\left(\int_s^T |\varphi(r)|^2 + |\nu(r)|^2\,dr\right)^2\right] = o(|x'-\bar{X}^{t,x;\bar{u}}(s)|^4).
$$

The estimate for

$$
\mathbb{E}\left[\left(\int_s^T (1+|Q(r)|)\left|\hat{X}(r)\varphi(r)\right| dr\right)^2 \middle| \mathcal{F}_s^t\right]
$$

is similar to (3.24). Thus, we obtain

$$
|\hat{\varphi}(s)| = o(\left|x'-\bar{X}^{t,x;\bar{u}}(s)\right|^2),\ \mathbb{P}\text{-a.s.}.
$$

Step 5: Completion of the proof.

Since the set of all rationals $x'\in\mathbb{R}$ is countable, we can find a subset $\Omega_0\subseteq\Omega$ with $\mathbb{P}(\Omega_0)=1$ such that for any $\omega_0\in\Omega_0$,

$$
\begin{cases}
W(s,\bar{X}^{t,x;\bar{u}}(s,\omega_0)) = \bar{Y}^{t,x;\bar{u}}(s,\omega_0),\ (3.4),\ (3.7),\ (3.8),\\
(3.14),\ (3.15) \text{ are satisfied for any rational } x',\\
(\Omega,\mathcal{F},\mathbb{P}(\cdot|\mathcal{F}_s^t)(\omega_0), B(\cdot)-B(s); u(\cdot))|_{[s,T]} \in \mathcal{U}^w[s,T],\\
\text{and } \sup_{s\leq r\leq T}[|p(r,\omega_0)|+|P(r,\omega_0)|] < \infty.
\end{cases}
$$

The first relation of the above is obtained by the DPP (see [9]). Let $\omega_0\in\Omega_0$ be fixed, and then for any rational number x',

$$
|\hat{\varphi}(s,\omega_0)| = o(\left|x'-\bar{X}^{t,x;\bar{u}}(s,\omega_0)\right|^2),\ \text{for all } s\in[t,T]. \tag{3.25}
$$

By the definition of $\hat{\varphi}(s)$, we get for each $s \in [t,T]$,

$$
\begin{aligned}
&Y^{s,x';\bar{u}}(s,\omega_0) - \bar{Y}^{t,x;\bar{u}}(s,\omega_0)\\
&= p(s,\omega_0)\hat{X}(s,\omega_0) + \tfrac{1}{2}P(s,\omega_0)\hat{X}(s,\omega_0)^2 + o(\left|x' - \bar{X}^{t,x;\bar{u}}(s,\omega_0)\right|^2)\\
&= p(s,\omega_0)(x' - \bar{X}^{t,x;\bar{u}}(s,\omega_0)) + \tfrac{1}{2}P(s,\omega_0)\left(x' - \bar{X}^{t,x;\bar{u}}(s,\omega_0)\right)^2\\
&\quad + o(\left|x' - \bar{X}^{t,x;\bar{u}}(s,\omega_0)\right|^2).
\end{aligned}
$$

Thus, for each $s \in [t,T]$,

$$
\begin{aligned}
&W(s,x') - W(s,\bar{X}^{t,x;\bar{u}}(s,\omega_0))\\
&\leq Y^{s,x';\bar{u}}(s,\omega_0) - \bar{Y}^{t,x;\bar{u}}(s,\omega_0)\\
&= p(s,\omega_0)(x' - \bar{X}^{t,x;\bar{u}}(s,\omega_0)) + \tfrac{1}{2}P(s,\omega_0)\left(x' - \bar{X}^{t,x;\bar{u}}(s,\omega_0)\right)^2\\
&\quad + o(\left|x' - \bar{X}^{t,x;\bar{u}}(s,\omega_0)\right|^2).
\end{aligned}
\tag{3.26}
$$

Note that the term $o(\left|x' - \bar{X}^{t,x;\bar{u}}(s,\omega_0)\right|^2)$ in the above depends only on the size of $\left|x' - \bar{X}^{t,x;\bar{u}}(s,\omega_0)\right|$ and it is independent of x'. Therefore, by the continuity of $W(s,\cdot)$, we can easily obtain that (3.26) holds for all $x' \in \mathbb{R}$. By the definition of super-jets, we have

$$
\{p(s)\} \times [P(s),\infty) \subseteq D_x^{2,+}W(s,X^{t,x;\bar{u}}(s)).
$$

Now we prove that

$$
D_x^{2,-}W(s,X^{t,x;\bar{u}}(s)) \subseteq \{p(s)\} \times (-\infty,P(s)].
$$

Fix an $\omega \in \Omega$ such that (3.26) holds for all $x' \in \mathbb{R}$. For any

$$
(\hat{p},\hat{P}) \in D_x^{2,-}V(s,\bar{X}^{t,x;\bar{u}}(s)),
$$

by definition of sub-jets, we deduce

$$
\begin{aligned}
0 \;&\leq \liminf_{x' \to \bar{X}^{t,x;\bar{u}}(s)} \left\{ \frac{W(s,x') - W(s,\bar{X}^{t,x;\bar{u}}(s)) - \hat{p}\left(x' - \bar{X}^{t,x;\bar{u}}(s)\right)}{|x' - \bar{X}^{t,x;\bar{u}}(s)|^2} - \frac{\frac{1}{2}\hat{P}\left(x' - \bar{X}^{t,x;\bar{u}}(s)\right)^2}{|x' - \bar{X}^{t,x;\bar{u}}(s)|^2} \right\}\\
&\leq \liminf_{x' \to \bar{X}^{t,x;\bar{u}}(s)} \left\{ \frac{(p(s) - \hat{p})\left(x' - \bar{X}^{t,x;\bar{u}}(s)\right)}{|x' - \bar{X}^{t,x;\bar{u}}(s)|^2} + \frac{\frac{1}{2}(P(s) - \hat{P})\left(x' - \bar{X}^{t,x;\bar{u}}(s)\right)^2}{|x' - \bar{X}^{t,x;\bar{u}}(s)|^2} \right\}.
\end{aligned}
$$

Then it is necessary that

$$
\hat{p} = p(s), \hat{P} \leq P(s),\ \forall s \in [t,T],\ \mathbb{P}\text{-a.s.}.
$$

This completes the proof. $\square$

Example 3.2. Consider the following controlled FBSDE ($n = 1$) for $s \in [t, T]$:

$$\begin{cases} dX^{t,x;u}(s) & = [X^{t,x;u}(s) + cZ^{t,x;u}(s)]\, ds \\ & \quad + [cZ^{t,x;u}(s) + X^{t,x;u}(s)u(s)]\, dB(s) \\ dY^{t,x;u}(s) & = -\left[X^{t,x;u}(s)u(s) - Z^{t,x;u}(s)\right] ds \\ & \quad + Z^{t,x;u}(s)dB(s) \\ X^{t,x;u}(s) & = x, \ Y^{t,x;u}(T) = X^{t,x;u}(T), \end{cases}$$

where c is a positive number such that Assumption 2.1 holds. The control domain is $U = [0, 1] \cup [2, 4]$ and the cost functional is defined by $Y^{t,x;u}(t)$. The corresponding generalized HJB equation becomes

$$\begin{array}{l} W_t(t,x) + \inf_{u \in U} \left\{ W_x(t,x) \left(x + \frac{cW_x(t,x)xu}{1 - cW_x(t,x)} \right) + xu \right. \\ \left. - \frac{W_x(t,x)xu}{1 - cW_x(t,x)} + \frac{1}{2} W_{xx}(t,x) \left(xu + \frac{cW_x(t,x)xu}{1 - cW_x(t,x)} \right)^2 \right\} = 0. \end{array} \tag{3.27}$$

Set $W(t, x) = a(t)x$ where $a(t)$ satisfies

$$\dot{a}(t)x + \inf_{u \in U} \left\{ a(t) \left(x + \frac{ca(t)xu}{1 - ca(t)} \right) + xu - \frac{a(t)xu}{1 - ca(t)} \right\} = 0;$$

that is

$$\dot{a}(t)x + a(t)x + \inf_{u \in U} \left\{ (1 - a(t))\, xu \right\} = 0.$$

If $(1 - a(t))x > 0$, then $\bar{u}(t) = 0$ which leads to $a(t) = e^{T-t}$; if $(1 - a(t))x \leq 0$, then $\bar{u}(t) = 3$ which leads to $\dot{a}(t) + a(t) + 3\,(1 - a(t)) = 0$ and $a(t) = \frac{3}{2} - \frac{1}{2} e^{2(t-T)}$. Thus,

$$W(t,x) = \begin{cases} e^{T-t}x, & \text{if } x < 0, \\ \left[\frac{3}{2} - \frac{1}{2} e^{2(t-T)} \right] x, & \text{if } x \geq 0, \end{cases} \tag{3.28}$$

which satisfies the Lipschitz and linear growth condition. Noting that

$$D_x^{2,+} W(t,0) = \left(\frac{3}{2} - \frac{1}{2} e^{2(t-T)}, e^{T-t} \right) \times \mathbb{R} \cup \left\{ \frac{3}{2} - \frac{1}{2} e^{2(t-T)}, e^{T-t} \right\} \times [0, \infty),$$

it is easy to verify that $W(t, x)$ is the viscosity solution to (3.27). Applying the approach (polished functions) in Theorem 4.4 and Theorem 4.12 in [9], we can prove $Y^{t,x;u}(t) \geq W(t, x)$ for any admissible control $u(\cdot)$. By Theorem 4.4 in [9], we know that $W(t, x)$ is the value function. Moreover, it is obviously that $\bar{u} = 0$ (resp. $\bar{u} = 3$) is an optimal control when $x < 0$ (resp. $x > 0$), and any admissible control is optimal when $x = 0$.

The first-order and the second-order adjoint equations become

$$\begin{cases} dp(s) = & -\left\{ \bar{u}(s) + \frac{\bar{u}(s)p(s) + q(s)}{1 - cp(s)} \left(cp(s) + cq(s) - 1 \right) \right. \\ & \left. + p(s) + \bar{u}(s)q(s) \right\} ds + q(s)dB(s), \\ p(T) = & 1, \end{cases}$$

$$\begin{cases} dP(s) = & -\Big\{P(s)\Big[\Big(\bar{u}(s)+c\frac{p(s)+q(s)}{1-cp(s)}\Big)^2+2+2c\frac{p(s)+q(s)}{1-cp(s)}\Big] \\ & +2Q(s)\Big[\bar{u}(s)+c\frac{p(s)+q(s)}{1-cp(s)}\Big]+(cp(s)+cq(s)-1) \\ & \cdot\frac{\left[2\left(\bar{u}(s)+c\frac{\bar{u}(s)p(s)+q(s)}{1-cp(s)}\right)P(s)+Q(s)\right]}{1-cp(s)}\Big\}ds+Q(s)dB(s), \\ P(T) = & 0, \end{cases}$$

If $x<0$, then $\bar{X}^{t,x;\bar{u}}(\cdot)<0$, $W(s,\bar{X}^{t,x;\bar{u}}(s))=e^{T-s}\bar{X}^{t,x;\bar{u}}(s)$ and $D_x^{2,+}W(s,\bar{X}^{t,x;\bar{u}}(s))=\left\{e^{T-s}\right\}\times[0,\infty)$. The solutions of the first-order and second-order adjoint equations for the optimal control $\bar{u}(\cdot)=0$ are $(p(s),q(s))=(e^{T-s},0)$, $(P(s),Q(s))=(0,0)$. Thus, the relation (3.1) holds.

If $x>0$, then $\bar{X}^{t,x;\bar{u}}(\cdot)>0$,

$$\bar{Y}^{t,x;\bar{u}}(s)=\left[\frac{3}{2}-\frac{1}{2}e^{2(s-T)}\right]\bar{X}^{t,x;\bar{u}}(s),$$

and

$$\bar{Z}^{t,x;\bar{u}}(s)=3\left[1-\left(\frac{3}{2}-\frac{1}{2}e^{2(s-T)}\right)c\right]^{-1}\left(\frac{3}{2}-\frac{1}{2}e^{2(s-T)}\right)\bar{X}^{t,x;\bar{u}}(s).$$

Then $W(s,\bar{X}^{t,x;\bar{u}}(s))=\left[\frac{3}{2}-\frac{1}{2}e^{2(s-T)}\right]\bar{X}^{t,x;\bar{u}}(s)$ and $D_x^{2,+}W(s,\bar{X}^{t,x;\bar{u}}(s))=\left\{\frac{3}{2}-\frac{1}{2}e^{2(s-T)}\right\}\times[0,\infty)$. The solutions of the first-order and second-order adjoint equations for the optimal control $\bar{u}(\cdot)=3$ are $(p(s),q(s))=\left(\left[\frac{3}{2}-\frac{1}{2}e^{2(s-T)}\right],0\right)$, $(P(s),Q(s))=(0,0)$. Thus, the relation (3.1) holds.

If $x=0$, then $\bar{X}^{t,x;\bar{u}}(\cdot)=0$. From (3.28), then $D_x^{2,-}W(s,\bar{X}^{t,x;\bar{u}}(s))=\emptyset$ and $D_x^{2,+}W(s,\bar{X}^{t,x;\bar{u}}(s))=(\frac{3}{2}-\frac{1}{2}e^{2(t-T)},e^{T-t})\times\mathbb{R}\cup\left\{\frac{3}{2}-\frac{1}{2}e^{2(t-T)},e^{T-t}\right\}\times[0,\infty)$. It can be verified that $\bar{X}^{t,x;\bar{u}}(s)=\bar{Y}^{t,x;\bar{u}}(s)=\bar{Z}^{t,x;\bar{u}}(s)\equiv 0$ is the optimal trajectory for any $u(\cdot)\in\mathcal{U}^w[t,T]$. Set $p_1(s)=\frac{3}{2}-\frac{1}{2}e^{2(s-T)}$, $p_2(s)=e^{T-s}$. Then

$$\begin{cases} dp_1(s)= & -\left[3-2p_1(s)\right]ds, \\ dp_2(s)= & -p_2(s)ds, \end{cases}$$

By the comparison Theorem, we have the solutions of the first-order and second-order adjoint equations satisfy $p_1(s)\le p(s)\le p_2(s)$, $P(s)=0$. Thus, the relation (3.1) holds.

3.2. Differential in time variable

Let us recall the notions of right super-and sub-jets in the time variable t. For $w\in C([0,T]\times\mathbb{R})$ and $(\hat{t},\hat{x})\in[0,T)\times\mathbb{R}$, define

$$\begin{cases} D_{t+}^{1,+}w(\hat{t},\hat{x}) & := & \{q\in\mathbb{R}: w(t,\hat{x})\le w(\hat{t},\hat{x})+q(t-\hat{t})+o\left(|t-\hat{t}|\right) \text{ as } t\downarrow\hat{t}\}, \\ D_{t+}^{1,-}w(\hat{t},\hat{x}) & := & \{q\in\mathbb{R}: w(t,\hat{x})\ge w(\hat{t},\hat{x})+q(t-\hat{t})+o\left(|t-\hat{t}|\right) \text{ as } t\downarrow\hat{t}\}. \end{cases}$$

Theorem 3.3. *Suppose the same assumptions as in Theorem 3.1 hold. Then, for each $s\in[t,T]$,*

$$\begin{cases} [\mathcal{H}_1(s,\bar{X}^{t,x;\bar{u}}(s),\bar{Y}^{t,x;\bar{u}}(s),\bar{Z}^{t,x;\bar{u}}(s)),\infty)\subseteq D_{t+}^{1,+}W(s,X^{t,x;\bar{u}}(s)), \\ D_{t+}^{1,-}W(s,X^{t,x;\bar{u}}(s))\subseteq(-\infty,\mathcal{H}_1(s,\bar{X}^{t,x;\bar{u}}(s),\bar{Y}^{t,x;\bar{u}}(s),\bar{Z}^{t,x;\bar{u}}(s))],\ \mathbb{P}\text{-a.s.}, \end{cases} \tag{3.29}$$

where

$$\mathcal{H}_1(s,\bar{X}^{t,x;\bar{u}}(s),\bar{Y}^{t,x;\bar{u}}(s),\bar{Z}^{t,x;\bar{u}}(s))$$
$$=-\mathcal{H}(s,\bar{X}^{t,x;\bar{u}}(s),\bar{Y}^{t,x;\bar{u}}(s),\bar{Z}^{t,x;\bar{u}}(s),\bar{u}(s),p(s),q(s),P(s))+\tfrac{1}{2}P(s)\sigma(s)^2.$$

Proof.
Step 1: Variations and estimations for FBSDE.
For each $s\in(t,T)$, take $\tau\in(s,T]$. Denote by

$$\Theta^{\tau,\bar{X}^{t,x;\bar{u}}(s);\bar{u}}(\cdot)=(X^{\tau,\bar{X}^{t,x;\bar{u}}(s);\bar{u}}(\cdot),Y^{\tau,\bar{X}^{t,x;\bar{u}}(s);\bar{u}}(\cdot),Z^{\tau,\bar{X}^{t,x;\bar{u}}(s);\bar{u}}(\cdot))$$

the solution to the following FBSDE on $[\tau,T]$:

$$\begin{cases} X^{\tau,X^{t,x;\bar{u}}(s);\bar{u}}(r)= & \bar{X}^{t,x;\bar{u}}(s)+\int_\tau^r b(\alpha,\Theta^{\tau,\bar{X}^{t,x;\bar{u}}(s);\bar{u}}(\alpha),\bar{u}(\alpha))d\alpha \\ & \quad +\int_\tau^r \sigma(\alpha,\Theta^{\tau,\bar{X}^{t,x;\bar{u}}(s);\bar{u}}(\alpha),\bar{u}(\alpha))dB(\alpha), \\ Y^{\tau,X^{t,x;\bar{u}}(s);\bar{u}}(r)= & \phi(X^{\tau,X^{t,x;\bar{u}}(s);\bar{u}}(T))+\int_r^T g(\alpha,\Theta^{\tau,\bar{X}^{t,x;\bar{u}}(s);\bar{u}}(\alpha),\bar{u}(\alpha))d\alpha \\ & \quad -\int_r^T Z^{\tau,\bar{X}^{t,x;\bar{u}}(s);\bar{u}}(\alpha)dB(\alpha). \end{cases}$$

For $r\in[\tau,T]$, set

$$\begin{aligned} \hat{\xi}_\tau(r)= &\quad X^{\tau,\bar{X}^{t,x;\bar{u}}(s);\bar{u}}(r)-\bar{X}^{t,x;\bar{u}}(r), \\ \hat{\eta}_\tau(r)= &\quad Y^{\tau,\bar{X}^{t,x;\bar{u}}(s);\bar{u}}(r)-\bar{Y}^{t,x;\bar{u}}(r), \\ \hat{\zeta}_\tau(r)= &\quad Z^{\tau,\bar{X}^{t,x;\bar{u}}(s);\bar{u}}(r)-\bar{Z}^{t,x;\bar{u}}(r), \\ \hat{\Theta}_\tau(r)= &\quad (\hat{\xi}_\tau(r),\hat{\eta}_\tau(r),\hat{\zeta}_\tau(r)). \end{aligned}$$

Then, by Theorem 2.2 in [8], we have that for each $\beta\in[2,8]$

$$\mathbb{E}\left[\sup_{r\in[\tau,T]}\left(|\hat{\xi}_\tau(r)|^\beta+|\hat{\eta}_\tau(r)|^\beta\right)+\left(\int_\tau^T|\hat{\zeta}_\tau(r)|^2dr\right)^{\frac{\beta}{2}}\middle|\mathcal{F}_\tau^t\right]\le C\left|\bar{X}^{t,x;\bar{u}}(\tau)-\bar{X}^{t,x;\bar{u}}(s)\right|^\beta,\ \mathbb{P}\text{-a.s.} \tag{3.30}$$

Note that

$$\bar{X}^{t,x;\bar{u}}(\tau)-\bar{X}^{t,x;\bar{u}}(s)=\int_s^\tau b(r)dr+\int_s^\tau\sigma(r)dB(r).$$

Taking conditional expectation $\mathbb{E}\left[\cdot|\mathcal{F}_s^t\right]$ on both sides of (3.30), we obtain for a.e. $s\in[t,T]$,

$$\mathbb{E}\left[\sup_{r\in[\tau,T]}\left(|\hat{\xi}_\tau(r)|^\beta+|\hat{\eta}_\tau(r)|^\beta\right)+\left(\int_\tau^T|\hat{\zeta}_\tau(r)|^2dr\right)^{\frac{\beta}{2}}\middle|\mathcal{F}_s^t\right]\le O(|\tau-s|^{\frac{\beta}{2}}),\ \mathbb{P}\text{-a.s.}, \tag{3.31}$$

as $\tau \downarrow s$. We rewrite $\hat{\xi}_\tau(\cdot)$, $\hat{\eta}_\tau(\cdot)$ and $\hat{\zeta}_\tau(\cdot)$ as

$$\left\{\begin{array}{rl} d\hat{\xi}_\tau(r)= & \left[\hat{\Theta}_\tau(r)Db(r)+\varepsilon_{\tau 1}(r)\right]dr+\left[\hat{\Theta}_\tau(r)D\sigma(r)+\varepsilon_{\tau 2}(r)\right]dB(r),\\ \hat{\xi}_\tau(\tau)= & -\int_s^\tau b(r)dr-\int_s^\tau \sigma(r)dB(r),\\ d\hat{\eta}_\tau(r)= & -\left[\hat{\Theta}_\tau(r)Dg(r)+\varepsilon_{\tau 3}(r)\right]dr+\hat{\zeta}_\tau(r)dB(r),\\ \hat{\eta}_\tau(T)= & \phi_x(\bar{X}^{t,x;\bar{u}}(T))\hat{\xi}_\tau(T)+\varepsilon_{\tau 4}(T),\ r\in[\tau,T], \end{array}\right. \tag{3.32}$$

where

$$\begin{array}{rl} \varepsilon_{\tau 1}(r)= & \left(\tilde{b}_x(r)-b_x(r)\right)\hat{\xi}_\tau(r)+\left(\tilde{b}_y(r)-b_y(r)\right)\hat{\eta}_\tau(r)+\left(\tilde{b}_z(r)-b_z(r)\right)\hat{\zeta}_\tau(r),\\ \varepsilon_{\tau 2}(r)= & \left(\tilde{\sigma}_x(r)-\sigma_x(r)\right)\hat{\xi}_\tau(r)+\left(\tilde{\sigma}_y(r)-\sigma_y(r)\right)\hat{\eta}_\tau(r)+\left(\tilde{\sigma}_z(r)-\sigma_z(r)\right)\hat{\zeta}_\tau(r),\\ \varepsilon_{\tau 3}(r)= & \left(\tilde{g}_x(r)-g_x(r)\right)\hat{\xi}_\tau(r)+\left(\tilde{g}_y(r)-g_y(r)\right)\hat{\eta}_\tau(r)+\left(\tilde{g}_z(r)-g_z(r)\right)\hat{\zeta}_\tau(r),\\ \varepsilon_{\tau 4}(T)= & [\tilde{\phi}_x(\bar{X}^{t,x;\bar{u}}(T))-\phi_x(\bar{X}^{t,x;\bar{u}}(T))]\hat{\xi}_\tau(T),\\ \tilde{\psi}_\kappa(r)= & \int_0^1 \psi_\kappa(r,\bar{\Theta}^{t,x;\bar{u}}(r)+\lambda\hat{\Theta}(r),\bar{u}(r))d\lambda\\ & \text{for } \psi=b,\sigma,g,\phi \text{ and } \kappa=x,y,z. \end{array}$$

Similar to the proof in Theorem 3.1, we obtain

$$Y^{\tau,X^{t,x;\bar{u}}(s);\bar{u}}(\tau)-\bar{Y}^{t,x;\bar{u}}(\tau)=p(\tau)\hat{\xi}_\tau(\tau)+\frac{1}{2}P(\tau)\hat{\xi}_\tau(\tau)^2+o(|\hat{\xi}_\tau(\tau)|^2),\ \mathbb{P}\text{-a.s.},$$

which implies for a.e. $s\in[t,T]$,

$$\mathbb{E}\left[Y^{\tau,X^{t,x;\bar{u}}(s);\bar{u}}(\tau)-\bar{Y}^{t,x;\bar{u}}(\tau)\Big|\mathcal{F}_s^t\right]=\mathbb{E}\left[p(\tau)\hat{\xi}_\tau(\tau)+\tfrac{1}{2}P(\tau)\hat{\xi}_\tau(\tau)^2|\mathcal{F}_s^t\right]+o\left(|\tau-s|\right),$$

$\mathbb{P}$-a.s. as $\tau\downarrow s$.

Step 2: Completion of the proof.
By Proposition 3.4 in [9], we get

$$W(\tau,X^{t,x;\bar{u}}(s))\le\mathbb{E}\left[Y^{\tau,X^{t,x;\bar{u}}(s);\bar{u}}(\tau)|\mathcal{F}_s^t\right],\ \mathbb{P}\text{-a.s.}. \tag{3.33}$$

Similar to Theorem 3.1, we can find a subset $\Omega_0\subseteq\Omega$ with $\mathbb{P}(\Omega_0)=1$ such that for any $\omega_0\in\Omega_0$,

$$\left\{\begin{array}{l} W(s,\bar{X}^{t,x;\bar{u}}(s,\omega_0))=\bar{Y}^{t,x;\bar{u}}(s,\omega_0),\ (3.31),\ (3.33)\text{ are}\\ \text{satisfied for any rational } \ \tau>s,\\ (\Omega,\mathcal{F},\mathbb{P}(\cdot|\mathcal{F}_s^t)(\omega_0),B(\cdot)-B(s);u(\cdot))|_{[s,T]}\in\mathcal{U}^w[s,T],\\ \text{and } \sup\limits_{s\le r\le T}[|p(r,\omega_0)|+|P(r,\omega_0)|]<\infty. \end{array}\right.$$

The first relation of the above is a directly application of the DPP (see [9], Thm. 3.5). Let $\omega_0 \in \Omega_0$ be fixed. Then, for any rational number $\tau > s$ and for a.e. $s \in [t, T)$

$$\begin{aligned} & W(\tau, \bar{X}^{t,x;\bar{u}}(s,\omega_0)) - W(s, \bar{X}^{t,x;\bar{u}}(s,\omega_0)) \\ & \leq \mathbb{E}\left[Y^{\tau, X^{t,x;\bar{u}}(s);\bar{u}}(\tau) - \bar{Y}^{t,x;\bar{u}}(s)|\mathcal{F}_s^t\right](\omega_0) \\ & = \mathbb{E}\left[Y^{\tau, X^{t,x;\bar{u}}(s);\bar{u}}(\tau) - \bar{Y}^{t,x;\bar{u}}(\tau) + \bar{Y}^{t,x;\bar{u}}(\tau) - \bar{Y}^{t,x;\bar{u}}(s)|\mathcal{F}_s^t\right](\omega_0) \\ & = \mathbb{E}\left[p(\tau)\hat{\xi}_\tau(\tau) + \tfrac{1}{2}P(\tau)\hat{\xi}_\tau(\tau)^2 - \int_s^\tau g(r)dr|\mathcal{F}_s^t\right](\omega_0) + o(|\tau - s|), \end{aligned} \tag{3.34}$$

as $\tau \downarrow s$. Next we estimate the terms on the right-hand side of (3.34).

$$\begin{aligned} & \mathbb{E}\left[p(\tau)\hat{\xi}_\tau(\tau)|\mathcal{F}_s^t\right](\omega_0) \\ & = \mathbb{E}\left[p(s)\hat{\xi}_\tau(\tau) + (p(\tau) - p\,(s))\hat{\xi}_\tau(\tau)|\mathcal{F}_s^t\right](\omega_0) \\ & = \mathbb{E}\left[-p(s)\int_s^\tau b(r)dr - \int_s^\tau q(r)\sigma(r)dr|\mathcal{F}_s^t\right](\omega_0) + o(|\tau - s|), \end{aligned} \tag{3.35}$$

where the last equality is due to the Itô's formula for $(p(\tau) - p\,(s))\hat{\xi}_\tau(\tau)$. Similarly,

$$\mathbb{E}\left[\frac{1}{2}P(\tau)\hat{\xi}_\tau(\tau)^2|\mathcal{F}_s^t\right](\omega_0) = \mathbb{E}\left[\frac{1}{2}P(s)\int_s^\tau \sigma(r)^2 dr|\mathcal{F}_s^t\right](\omega_0) + o(|\tau - s|). \tag{3.36}$$

Thus, by (3.34)-(3.36) and the continuity of W, we obtain

$$\begin{aligned} & W(\tau, \bar{X}^{t,x;\bar{u}}(s)) - W(s, \bar{X}^{t,x;\bar{u}}(s)) \\ & \leq \mathbb{E}\left[-p(s)\int_s^\tau b(r)dr - \int_s^\tau q(r)\sigma(r)dr - \int_s^\tau g(r)dr\ + \tfrac{1}{2}P(s)\int_s^\tau \sigma(r)^2 dr|\mathcal{F}_s^t\right] + o(|\tau - s|) \\ & = (\tau - s)\mathcal{H}_1(s, \bar{X}^{t,x;\bar{u}}(s), \bar{Y}^{t,x;\bar{u}}(s), \bar{Z}^{t,x;\bar{u}}(s)) + o(|\tau - s|), \end{aligned}$$

which implies

$$[\mathcal{H}_1(s, \bar{X}^{t,x;\bar{u}}(s), \bar{Y}^{t,x;\bar{u}}(s), \bar{Z}^{t,x;\bar{u}}(s)), \infty) \subseteq D_{t+}^{1,+}W(s, X^{t,x;\bar{u}}(s))$$

by the definition of super-jets. For any $\hat{q} \in D_{t+}^{1,-}W(s, \bar{X}^{t,x;\bar{u}}(s))$, by definition of sub-jets, we have

$$\begin{aligned} 0 \quad & \leq \liminf_{\tau \downarrow s}\left\{\tfrac{V(\tau, \bar{X}^{t,x;\bar{u}}(s)) - V(s, \bar{X}^{t,x;\bar{u}}(s)) - \hat{q}(\tau - s)}{\tau - s}\right\} \\ & \leq \liminf_{\tau \downarrow s}\left\{\mathcal{H}_1(s, \bar{X}^{t,x;\bar{u}}(s), \bar{Y}^{t,x;\bar{u}}(s), \bar{Z}^{t,x;\bar{u}}(s)) - \hat{q}\right\}. \end{aligned}$$

Thus, for any $s \in [t, T)$,

$$\hat{q} \leq \mathcal{H}_1(s, \bar{X}^{t,x;\bar{u}}(s), \bar{Y}^{t,x;\bar{u}}(s), \bar{Z}^{t,x;\bar{u}}(s)),\ \mathbb{P}\text{-a.s.}.$$

This completes the proof. □

Example 3.4. (Continued) Consider the problem in Example 3.2 again. For $x < 0$,

$$D_{t+}^{1,+}W(s, \bar{X}^{t,x;\bar{u}}(s)) = [-e^{T-s}\bar{X}^{t,x;\bar{u}}(s), \infty)$$

and

$$D_{t+}^{1,-}W(s,\bar{X}^{t,x;\bar{u}}(s)) = (-\infty, -e^{T-s}\bar{X}^{t,x;\bar{u}}(s)].$$

Since the optimal control $\bar{u} = 0$, $p(s) = e^{T-s}$, $q(s) = P(s) = Q(s) = 0$,

$$\begin{aligned}
&\mathcal{H}_1(s,\bar{X}^{t,x;\bar{u}}(s),\bar{Y}^{t,x;\bar{u}}(s),\bar{Z}^{t,x;\bar{u}}(s)) \\
&= -\mathcal{H}(s,\bar{X}^{t,x;\bar{u}}(s),\bar{Y}^{t,x;\bar{u}}(s),\bar{Z}^{t,x;\bar{u}}(s),\bar{u}(s),p(s),q(s),P(s)) \\
&= -\left[p(s)\left(\bar{X}^{t,x;u}(s)+\bar{Z}^{t,x;u}(s)\right)+\bar{Z}^{t,x;u}(s)\right] \\
&= -e^{T-s}X^{t,x;\bar{u}}(s).
\end{aligned}$$

Thus, the relation (3.29) holds.

For $x > 0$,

$$D_{t+}^{1,+}W(s,\bar{X}^{t,x;\bar{u}}(s)) = \left[-e^{2(s-T)}\bar{X}^{t,x;\bar{u}}(s), \infty\right)$$

and

$$D_{t+}^{1,-}W(s,\bar{X}^{t,x;\bar{u}}(s)) = \left(-\infty, e^{2(s-T)}\bar{X}^{t,x;\bar{u}}(s)\right].$$

Since $\bar{u} = 3$, $p(s) = \frac{3}{2} - \frac{1}{2}e^{2(s-T)}$, $q(s) = P(s) = Q(s) = 0$,

$$\begin{aligned}
&\mathcal{H}_1(s,\bar{X}^{t,x;\bar{u}}(s),\bar{Y}^{t,x;\bar{u}}(s),\bar{Z}^{t,x;\bar{u}}(s)) \\
&= -\mathcal{H}(s,\bar{X}^{t,x;\bar{u}}(s),\bar{Y}^{t,x;\bar{u}}(s),\bar{Z}^{t,x;\bar{u}}(s),\bar{u}(s),p(s),q(s),P(s)) \\
&= -\left[p(s)\left(\bar{X}^{t,x;\bar{u}}(s)+\bar{Z}^{t,x;\bar{u}}(s)\right)+\bar{X}^{t,x;\bar{u}}(s)\bar{u}(s)-\bar{Z}^{t,x;\bar{u}}(s)\right] \\
&= -e^{2(s-T)}\bar{X}^{t,x;\bar{u}}(s).
\end{aligned}$$

Thus, the relation (3.29) holds.

For $x = 0$,

$$D_{t+}^{1,+}W(s,\bar{X}^{t,x;\bar{u}}(s)) = [0,\infty)$$

and

$$D_{t+}^{1,-}W(s,\bar{X}^{t,x;\bar{u}}(s)) = (-\infty, 0].$$

For any $\bar{u}(\cdot)$ we have $\bar{X}^{t,x;\bar{u}}(\cdot) = \bar{Y}^{t,x;\bar{u}}(s) = \bar{Z}^{t,x;\bar{u}}(s) \equiv 0 = 0$ and which implies

$$\mathcal{H}_1(s,\bar{X}^{t,x;\bar{u}}(s),\bar{Y}^{t,x;\bar{u}}(s),\bar{Z}^{t,x;\bar{u}}(s)) \equiv 0.$$

Thus, the relation (3.29) holds.

4. Special cases

In this section, we study three special cases. In the first case, the value function W is supposed to be smooth. In the second case, the diffusion term σ of the forward stochastic differential equation in (2.1) is linear in z. Finally, we study the case in which the control domain is convex and compact.

4.1. The smooth case

In this section, we assume that the value function W is smooth and obtain the relationship between the derivatives of W and the adjoint processes. Note that the HJB equation includes an algebra equation (2.4). It is worth pointing out that we discover two novel connections: (i) the relation between the derivatives of $V(\cdot)$ and the terms $K_1(\cdot)$, $K_2(\cdot)$ in the adjoint equations; (ii) the relation between the algebra equation $\Delta(\cdot)$ for the MP and the algebra equation $V(\cdot)$ for the HJB equation. We first give the following stochastic verification theorem.

Theorem 4.1. *Suppose Assumption 2.1 holds. Let $w(\cdot,\cdot)$ belong to $C_b^{1,2}([0,T]\times\mathbb{R})$ and be a solution to the HJB equation (2.4). If $||\sigma||_\infty<\infty$ and $||w_x||_\infty||\sigma_z||_\infty<1$, then*

$$w(t,x)\le J(t,x;u(\cdot)),\ \forall u(\cdot)\in\mathcal{U}^w[t,T],(t,x)\in[0,T]\times\mathbb{R}.$$

Furthermore, if $\bar{u}(\cdot)\in\mathcal{U}^w[t,T]$ such that

$$G(s,X^{t,x;\bar{u}}(s),w(s,X^{t,x;\bar{u}}(s)),w_x(s,X^{t,x;\bar{u}}(s),w_{xx}(s,X^{t,x;\bar{u}}(s)),\bar{u}(s))+w_s(s,X^{t,x;\bar{u}}(s))=0,$$

where $(X^{t,x;\bar{u}}(\cdot),Y^{t,x;\bar{u}}(\cdot),Z^{t,x;\bar{u}}(\cdot))$ is the solution to FBSDE (2.1) corresponding to $\bar{u}(\cdot)$, then $\bar{u}(\cdot)$ is an optimal control.

Proof. The proof is same to Theorem 4.11 in [9], thus we omit it. □

Now we study the relationship between the derivatives of the value function W and the adjoint processes.

Theorem 4.2. *Let Assumptions 2.1, 2.8, and 2.10 hold. Suppose that $\bar{u}(\cdot)\in\mathcal{U}^w[t,T]$ is an optimal control, and $(\bar{X}^{t,x;\bar{u}}(\cdot),\bar{Y}^{t,x;\bar{u}}(\cdot),\bar{Z}^{t,x;\bar{u}}(\cdot))$ is the corresponding optimal state. Let $(p(\cdot),q(\cdot))$ be the solution to (2.11). If the value function $W(\cdot,\cdot)\in C^{1,2}([t,T]\times\mathbb{R})$, then for each $s\in[t,T]$*

$$\bar{Y}^{t,x;\bar{u}}(s)=W(s,\bar{X}^{t,x;\bar{u}}(s)),$$

$$\bar{Z}^{t,x;\bar{u}}(s)=V(s,\bar{X}^{t,x;\bar{u}}(s),W(s,\bar{X}^{t,x;\bar{u}}(s)),W_x(s,\bar{X}^{t,x;\bar{u}}(s)),\bar{u}(s))$$

and

$$\begin{aligned}&-W_s(s,\bar{X}^{t,x;\bar{u}}(s))\\&=G\left(s,\bar{X}^{t,x;\bar{u}}(s),W(s,\bar{X}^{t,x;\bar{u}}(s)),W_x(s,\bar{X}^{t,x;\bar{u}}(s)),\right.\\&\qquad\left.W_{xx}(s,\bar{X}^{t,x;\bar{u}}(s)),\bar{u}(s)\right)\\&=\min_{u\in U}G\left(s,\bar{X}^{t,x;\bar{u}}(s),W(s,\bar{X}^{t,x;\bar{u}}(s)),W_x(s,\bar{X}^{t,x;\bar{u}}(s)),\right.\\&\qquad\left.W_{xx}(s,\bar{X}^{t,x;\bar{u}}(s)),u\right).\end{aligned}$$

Moreover, if $W(\cdot,\cdot) \in C^{1,3}([t,T]\times\mathbb{R})$ *and* $W_{sx}(\cdot,\cdot)$, $W_{sxx}(\cdot,\cdot)$ *are continuous, then, for* $s\in[t,T]$,

$$\begin{aligned} p(s) = & \quad W_x(s,\bar{X}^{t,x;\bar{u}}(s)), \\ q(s) = & \quad W_{xx}(s,\bar{X}^{t,x;\bar{u}}(s))\cdot\sigma(s,\bar{X}^{t,x;\bar{u}}(s),\bar{Y}^{t,x;\bar{u}}(s),\bar{Z}^{t,x;\bar{u}}(s),\bar{u}(s)). \end{aligned}$$

Furthermore, if $W(\cdot,\cdot)\in C^{1,4}([t,T]\times\mathbb{R})$ *and* $W_{sxx}(\cdot,\cdot)$ *is continuous, then*

$$P(s) \geq W_{xx}(s,\bar{X}^{t,x;\bar{u}}(s)),\ s\in[t,T],$$

where $(P(\cdot),Q(\cdot))$ *satisfies (2.13).*

Proof. By the DPP (see [9], Thm. 3.5), we get $\bar{Y}^{t,x;\bar{u}}(s) = W(s,\bar{X}^{t,x;\bar{u}}(s))$ $s\in[t,T]$. Applying Itô's formula to $W(s,\bar{X}^{t,x;\bar{u}}(s))$, we can get

$$\begin{aligned} &\bar{Y}^{t,x;\bar{u}}(s) = W(s,\bar{X}^{t,x;\bar{u}}(s)), \\ &\bar{Z}^{t,x;\bar{u}}(s) = V(s,\bar{X}^{t,x;\bar{u}}(s),W(s,\bar{X}^{t,x;\bar{u}}(s)),W_x(s,\bar{X}^{t,x;\bar{u}}(s)),\bar{u}(s)), \\ &G(s,\bar{X}^{t,x;\bar{u}}(s),W(s,\bar{X}^{t,x;\bar{u}}(s)),W_x(s,\bar{X}^{t,x;\bar{u}}(s)),W_{xx}(s,\bar{X}^{t,x;\bar{u}}(s)),\bar{u}(s))+W_s(s,\bar{X}^{t,x;\bar{u}}(s))=0. \end{aligned} \tag{4.1}$$

Since W satisfies the HJB equation (2.4), we obtain that, for each $u\in U$,

$$G\left(s,\bar{X}^{t,x;\bar{u}}(s),W(s,\bar{X}^{t,x;\bar{u}}(s)),W_x(s,\bar{X}^{t,x;\bar{u}}(s)),\ W_{xx}(s,\bar{X}^{t,x;\bar{u}}(s)),u\right)+W_s(s,\bar{X}^{t,x;\bar{u}}(s))\geq 0.$$

Thus, we deduce

$$\begin{aligned} &G(s,\bar{X}^{t,x;\bar{u}}(s),W(s,\bar{X}^{t,x;\bar{u}}(s)),W_x(s,\bar{X}^{t,x;\bar{u}}(s)),W_{xx}(s,\bar{X}^{t,x;\bar{u}}(s)),\bar{u}(s)) \\ &=\min_{u\in U} G\left(s,\bar{X}^{t,x;\bar{u}}(s),W(s,\bar{X}^{t,x;\bar{u}}(s)),W_x(s,\bar{X}^{t,x;\bar{u}}(s)),\ W_{xx}(s,\bar{X}^{t,x;\bar{u}}(s)),u\right). \end{aligned} \tag{4.2}$$

If $W(\cdot,\cdot)\in C^{1,3}([t,T]\times\mathbb{R})$ and $W_{sx}(\cdot,\cdot)$ is continuous, then, by applying Itô's formula to $W_x(s,\bar{X}^{t,x;\bar{u}}(s))$, we get

$$\begin{aligned} dW_x(s,\bar{X}^{t,x;\bar{u}}(s)) = & \quad \left\{W_{sx}(s,\bar{X}^{t,x;\bar{u}}(s))+W_{xx}(s,\bar{X}^{t,x;\bar{u}}(s))b(s)\ +\tfrac{1}{2}W_{xxx}(s,\bar{X}^{t,x;\bar{u}}(s))(\sigma(s))^2\right\}ds \\ & +W_{xx}(s,\bar{X}^{t,x;\bar{u}}(s))\sigma(s)dB(s). \end{aligned} \tag{4.3}$$

Note that W satisfies the HJB equation (2.4). Then we obtain

$$W_s(s,x)+G(s,x,W(s,x),W_x(s,x),W_{xx}(s,x),\bar{u}(s))\geq 0. \tag{4.4}$$

Combining (4.1) and (4.4), we conclude that the function

$$W_s(s,\cdot)+G(s,\cdot,W(s,\cdot),W_x(s,\cdot),W_{xx}(s,\cdot),\bar{u}(s))$$

achieves its minimum at $x=\bar{X}^{t,x;\bar{u}}(s)$. Thus

$$0=\frac{\partial}{\partial x}\{W_s(s,x)\ +G(s,x,W(s,x),W_x(s,x),W_{xx}(s,x),\bar{u}(s))\}|_{x=\bar{X}^{t,x;\bar{u}}(s)}\,. \tag{4.5}$$

By the implicit function theorem, we deduce

$$\begin{aligned}
&\tfrac{\partial V}{\partial x}(s,x,W(s,x),W_x(s,x),\bar{u}(s))\big|_{x=\bar{X}^{t,x;\bar{u}}(s)}\\
&=\left(1-W_x(s,\bar{X}^{t,x;\bar{u}}(s))\sigma_z(s)\right)^{-1}[W_{xx}(s,\bar{X}^{t,x;\bar{u}}(s))\sigma(s)\\
&\quad+W_x(s,\bar{X}^{t,x;\bar{u}}(s))\sigma_x(s)+\sigma_y(s)(W_x(s,\bar{X}^{t,x;\bar{u}}(s)))^2].
\end{aligned}\tag{4.6}$$

Thus, we can easily get

$$\begin{aligned}
&\tfrac{\partial}{\partial x}\{W_s(s,x)+G(s,x,W(s,x),W_x(s,x),W_{xx}(s,x),\bar{u}(s))\}|_{x=\bar{X}^{t,x;\bar{u}}(s)}\\
&=W_{sx}(s,\bar{X}^{t,x;\bar{u}}(s))+W_{xx}(s,\bar{X}^{t,x;\bar{u}}(s))b(s)+\tfrac{1}{2}W_{xxx}(s,\bar{X}^{t,x;\bar{u}}(s))\sigma(s)^2+W_x(s,\bar{X}^{t,x;\bar{u}}(s))\\
&\quad\cdot[b_x(s)+b_y(s)W_x(s,\bar{X}^{t,x;\bar{u}}(s))+b_z(s)V_x(s)]+W_{xx}(s,\bar{X}^{t,x;\bar{u}}(s))\sigma(s)\\
&\quad\cdot\left[\sigma_x(s)+\sigma_y(s)W_x(s,\bar{X}^{t,x;\bar{u}}(s))+\sigma_z(s)V_x(s)\right]+g_x(s)+g_y(s)W_x(s,\bar{X}^{t,x;\bar{u}}(s))+g_z(s)V_x(s),
\end{aligned}\tag{4.7}$$

where

$$V_x(s)=\frac{\partial V}{\partial x}(s,x,W(s,x),W_x(s,x),\bar{u}(s))\bigg|_{x=\bar{X}^{t,x;\bar{u}}(s)}.$$

Combining (4.3), (4.5) and (4.7), it is easy to check that

$$\left(W_x(s,\bar{X}^{t,x;\bar{u}}(s)),W_{xx}(s,\bar{X}^{t,x;\bar{u}}(s))\sigma(s)\right)$$

satisfies the adjoint equation (2.11), which implies

$$p(s)=W_x(s,\bar{X}^{t,x;\bar{u}}(s)),\ \ q(s)=W_{xx}(s,\bar{X}^{t,x;\bar{u}}(s))\sigma(s).$$

If $W(\cdot,\cdot)\in C^{1,4}([t,T]\times\mathbb{R})$ and $W_{sxx}(\cdot,\cdot)$ is continuous, then, applying Itô's formula to $W_{xx}(s,\bar{X}^{t,x;\bar{u}}(s))$, we obtain

$$\begin{aligned}
dW_{xx}(s,\bar{X}^{t,x;\bar{u}}(s))=\ &\left\{W_{sxx}(s,\bar{X}^{t,x;\bar{u}}(s))+W_{xxx}(s,\bar{X}^{t,x;\bar{u}}(s))b(s)\ +\tfrac{1}{2}W_{xxxx}(s,\bar{X}^{t,x;\bar{u}}(s))(\sigma(s))^2\right\}ds\\
&+W_{xxx}(s,\bar{X}^{t,x;\bar{u}}(s))\sigma(s)dB(s).
\end{aligned}\tag{4.8}$$

Since the function

$$W_s(s,\cdot)+G(s,\cdot,W(s,\cdot),W_x(s,\cdot),W_{xx}(s,\cdot),\bar{u}(s))$$

achieves its minimum at $x=\bar{X}^{t,x;\bar{u}}(s)$, we have

$$\frac{\partial^2}{\partial x^2}\{W_s(s,x)+G(s,x,W(s,x),W_x(s,x),W_{xx}(s,x),\bar{u}(s))\}|_{x=\bar{X}^{t,x;\bar{u}}(s)}\geq 0.\tag{4.9}$$

For $s\in[t,T]$, set

$$\tilde{P}(s)=W_{xx}(s,\bar{X}^{t,x;\bar{u}}(s))$$

and

$$\tilde{Q}(s) = W_{xxx}(s, \bar{X}^{t,x;\bar{u}}(s))\sigma(s).$$

In order to prove $P(s) \geq \tilde{P}(s)$, by comparison theorem of BSDE for equations (2.13) and (4.8), we only need to check

$$\begin{aligned}
&\tilde{P}(s)\left[(D\sigma(s)^{\intercal}(1,p(s),K_1(s)))^2 \ +2Db(s)^{\intercal}(1,p(s),K_1(s))^{\intercal} + H_y(s)\right] \\
&+2\tilde{Q}(s)D\sigma(s)^{\intercal}(1,p(s),K_1(s))^{\intercal} + (1,p(s),K_1(s))D^2H(s)(1,p(s),K_1(s))^{\intercal} \\
&+H_z(s)\tilde{K}_2(s) + W_{sxx}(s,\bar{X}^{t,x;\bar{u}}(s)) + W_{xxx}(s,\bar{X}^{t,x;\bar{u}}(s))b(s) \\
&+\tfrac{1}{2}W_{xxxx}(s,\bar{X}^{t,x;\bar{u}}(s))(\sigma(s))^2 \geq 0,
\end{aligned} \tag{4.10}$$

where

$$\begin{aligned}
\tilde{K}_2(s) = \ & (1-p(s)\sigma_z(s))^{-1}\left\{p(s)\sigma_y(s) + 2\left[\sigma_x(s) \ +\sigma_y(s)p(s) + \sigma_z(s)K_1(s)\right]\right\}\tilde{P}(s) \\
& +(1-p(s)\sigma_z(s))^{-1}\left\{\tilde{Q}(s) \ +p(s)(1,p(s),K_1(s))D^2\sigma(s)(1,p(s),K_1(s))^{\intercal}\right\}.
\end{aligned} \tag{4.11}$$

By (4.9), one can verify that the inequality (4.10) holds. □

From the proof in the above theorem, we can obtain the following corollary.

Corollary 4.3. *Under the same assumptions as in Theorem 4.2, we have the following relations:*
(i)

$$\begin{aligned}
\Delta(s) = \ & V(s,\bar{X}^{t,x;\bar{u}}(s), W(s,\bar{X}^{t,x;\bar{u}}(s)), W_x(s,\bar{X}^{t,x;\bar{u}}(s)), u) \\
& -V(s,\bar{X}^{t,x;\bar{u}}(s), W(s,\bar{X}^{t,x;\bar{u}}(s)), W_x(s,\bar{X}^{t,x;\bar{u}}(s)), \bar{u}(s)),
\end{aligned}$$

where $\Delta(\cdot)$ in (2.16) and $V(\cdot)$ in (2.5);
(ii)

$$\begin{aligned}
\frac{\partial V}{\partial x}(s,x,W(s,x),W_x(s,x),\bar{u}(s))\Big|_{x=\bar{X}^{t,x;\bar{u}}(s)} &= K_1(s), \\
\frac{\partial^2 V}{\partial x^2}(s,x,W(s,x),W_x(s,x),\bar{u}(s))\Big|_{x=\bar{X}^{t,x;\bar{u}}(s)} &= \tilde{K}_2(s),
\end{aligned}$$

where $\tilde{K}_2(s)$ is defined in (4.11).

Remark 4.4. It is worth to pointing out that $\tilde{K}_2(\cdot)$ and $K_2(\cdot)$ are closely related. If we replace $P(\cdot)$ (resp. $Q(\cdot)$) by $W_{xx}(\cdot,\bar{X}^{t,x;\bar{u}}(\cdot))$ (resp. $W_{xxx}(\cdot,\bar{X}^{t,x;\bar{u}}(\cdot))\sigma(\cdot)$) in $K_2(\cdot)$, then we have $\tilde{K}_2(\cdot)$.

If the value function is smooth enough, we can use the DPP to derive the MP in the following theorem.

Theorem 4.5. *Let Assumptions 2.1, 2.8 and 2.10 hold. Suppose that $\bar{u}(\cdot) \in \mathcal{U}^w[t,T]$ is an optimal control, and $(\bar{X}^{t,x;\bar{u}}(\cdot), \bar{Y}^{t,x;\bar{u}}(\cdot), \bar{Z}^{t,x;\bar{u}}(\cdot))$ is the corresponding optimal state. Let $(p(\cdot),q(\cdot))$ and $(P(\cdot),Q(\cdot))$ be the solutions to (2.11) and (2.13) respectively. If $W(\cdot,\cdot) \in C^{1,4}([t,T]\times\mathbb{R})$ and $W_{sx}(\cdot,\cdot)$, $W_{sxx}(\cdot,\cdot)$ are continuous, then*

$$\begin{aligned}
&\mathcal{H}(s,\bar{X}^{t,x;\bar{u}}(s),\bar{Y}^{t,x;\bar{u}}(s),\bar{Z}^{t,x;\bar{u}}(s),u,p(s),q(s),P(s)) \\
&\geq \mathcal{H}(s,\bar{X}^{t,x;\bar{u}}(s),\bar{Y}^{t,x;\bar{u}}(s),\bar{Z}^{t,x;\bar{u}}(s),\bar{u}(s),p(s),q(s),P(s)), \ \forall u \in U \text{ a.e., a.s..}
\end{aligned} \tag{4.12}$$

Proof. By (4.2) in Theorem 4.2, we have $\forall u \in U$ a.e., a.s.

$$
\begin{aligned}
&G(s, \bar{X}^{t,x;\bar{u}}(s), W(s, \bar{X}^{t,x;\bar{u}}(s)), W_x(s, \bar{X}^{t,x;\bar{u}}(s)), W_{xx}(s, \bar{X}^{t,x;\bar{u}}(s)), \bar{u}(s)) \\
&\leq G(s, \bar{X}^{t,x;\bar{u}}(s), W(s, \bar{X}^{t,x;\bar{u}}(s)), W_x(s, \bar{X}^{t,x;\bar{u}}(s)), W_{xx}(s, \bar{X}^{t,x;\bar{u}}(s)), u).
\end{aligned}
\tag{4.13}
$$

Since

$$
\begin{aligned}
\bar{Y}^{t,x;\bar{u}}(s) = &\quad W(s, \bar{X}^{t,x;\bar{u}}(s)), \\
\bar{Z}^{t,x;\bar{u}}(s) = &\quad W_x(s, \bar{X}^{t,x;\bar{u}}(s))\sigma(s), \\
p(s) = &\quad W_x(s, \bar{X}^{t,x;\bar{u}}(s)), \\
q(s) = &\quad W_{xx}(s, \bar{X}^{t,x;\bar{u}}(s))\sigma(s),
\end{aligned}
$$

we can obtain

$$
V(s, \bar{X}^{t,x;\bar{u}}(s), W(s, \bar{X}^{t,x;\bar{u}}(s)), W_x(s, \bar{X}^{t,x;\bar{u}}(s)), u) = \bar{Z}^{t,x;\bar{u}}(s) + \Delta(s) \tag{4.14}
$$

by the definition of $\Delta(s)$ in equation (2.16). Combining (4.13) and (4.14), we deduce that

$$
\begin{aligned}
&\mathcal{H}(s, \bar{X}^{t,x;\bar{u}}(s), \bar{Y}^{t,x;\bar{u}}(s), \bar{Z}^{t,x;\bar{u}}(s), u, p(s), q(s), P(s)) \\
&-\mathcal{H}(s, \bar{X}^{t,x;\bar{u}}(s), \bar{Y}^{t,x;\bar{u}}(s), \bar{Z}^{t,x;\bar{u}}(s), \bar{u}(s), p(s), q(s), P(s)) \\
&\geq \frac{1}{2}\left(P(s) - W_{xx}(s, \bar{X}^{t,x;\bar{u}}(s))\right) \cdot \left(\sigma(s, \bar{X}^{t,x;\bar{u}}(s), \bar{Y}^{t,x;\bar{u}}(s), \bar{Z}^{t,x;\bar{u}}(s), u) - \sigma(s)\right)^2.
\end{aligned}
$$

Noting that $P(s) \geq W_{xx}(s, \bar{X}^{t,x;\bar{u}}(s))$, then we obtain (4.12). □

4.2. The case that σ is linear in z

In this section, we consider the case that $\sigma(t,x,y,z,u) = \tilde{A}(t)z + \sigma_1(t,x,y,u)$. Under this case, we do not need the assumption that $q(\cdot)$ is bounded.

Assumption 4.6. $\sigma(t,x,y,x,u) = \tilde{A}(t)z + \sigma_1(t,x,y,u)$, $||\tilde{A}(\cdot)||_\infty$ is small enough.

Theorem 4.7. *Suppose Assumptions 2.1, 2.8, 2.10 and 4.6 hold. Let $\bar{u}(\cdot)$ be optimal for our problem (2.3), and let $(p(\cdot), q(\cdot)) \in L^\infty_{\mathbb{F}}(0,T;\mathbb{R}) \times L^{2,2}_{\mathbb{F}}(0,T;\mathbb{R})$ and $(P(\cdot), Q(\cdot)) \in L^2_{\mathbb{F}}(\Omega; C([0,T],\mathbb{R})) \times L^{2,2}_{\mathbb{F}}(0,T;\mathbb{R})$ be the solution to equation (2.11) and (2.13) respectively. Then*

$$
\begin{cases}
\{p(s)\} \times [P(s), \infty) \subseteq D_x^{2,+} W(s, \bar{X}^{t,x;\bar{u}}(s)), \\
D_x^{2,-} W(s, \bar{X}^{t,x;\bar{u}}(s)) \subseteq \{p(s)\} \times (-\infty, P(s)].
\end{cases}
$$

Proof. In this case

$$
\begin{aligned}
&\varepsilon_2(r) = (\tilde{\sigma}_x(r) - \sigma_x(r))\hat{X}(r) \\
&+ (\tilde{\sigma}_y(r) - \sigma_y(r))\left(p(r)\hat{X}(r) + \nu(r)\right).
\end{aligned}
\tag{4.15}
$$

It is easy to verify that

$$\begin{aligned} &|A(r)| \leq C(1+|q(r)|), \\ &|C(r)| \leq C\left(1+||\tilde{A}(\cdot)||_\infty |q(r)|\right), \end{aligned} \tag{4.16}$$

where C is a positive constant. By Theorem A.2 in [8], there exists a $\delta > 0$ such that for each $\lambda_1 < \delta$,

$$\begin{aligned} &\mathbb{E}\left[\exp\left(\lambda_1 \int_s^T |q(r)|^2 dr\right)\Big| \mathcal{F}_s^t\right] \leq C, \\ &\mathbb{E}\left[\sup_{s\leq\alpha\leq T} \exp\left(\lambda_1 \int_s^\alpha q(r)\, dB(r)\right)\Big| \mathcal{F}_s^t\right] \leq C. \end{aligned} \tag{4.17}$$

Set, for $r \in [s,T]$,

$$\begin{aligned} &\Gamma_1(r) = \exp\left(\int_s^r A(\alpha)d\alpha\right), \\ &\Gamma_2(r) = \exp\left(-\tfrac{1}{2}\int_s^r |C(\alpha)|^2 d\alpha + \int_s^r C(\alpha)dB(\alpha)\right). \end{aligned}$$

When $||\tilde{A}(\cdot)||_\infty$ is small enough, by (4.16) and (4.17), we can find a large enough constant $\lambda > 0$ such that

$$\mathbb{E}\left[\sup_{r\in[t,T]} |\Gamma_1(r)\Gamma_2(r)|^\lambda \bigg| \mathcal{F}_s^t\right] \leq C. \tag{4.18}$$

Set

$$\begin{aligned} &\varphi^1(r) = \varphi(r)\Gamma_1(r)\Gamma_2(r), \\ &\nu^1(r) = \nu(r)\Gamma_1(r)\Gamma_2(r) + C(r)\Gamma_1(r)\Gamma_2(r)\varphi(r), \end{aligned}$$

where $(\varphi(\cdot),\nu(\cdot))$ is the solution to BSDE (3.9). We obtain that $\left(\varphi^1(\cdot), \nu^1(\cdot)\right)$ satisfies the following BSDE by applying Itô's formula to $\varphi(\cdot)\Gamma_1(\cdot)\Gamma_2(\cdot)$,

$$\begin{cases} d\varphi^1(r) = & -\mathrm{II}(r)\, dr + \nu^1(r)\, dB(r), \\ \varphi^1(T) = & \varepsilon_4(T)\Gamma_1(T)\Gamma_2(T), \end{cases} \tag{4.19}$$

where

$$\begin{aligned} \mathrm{II}(r) = \; & \Gamma_1(r)\Gamma_2(r)[p(r)\varepsilon_1(r) + q(r)\varepsilon_2(r) + \varepsilon_3(r) \\ & + H_z(r)\left(1-p(r)\tilde{A}(r)\right)^{-1} p(r)\varepsilon_2(r)]. \end{aligned} \tag{4.20}$$

For each $\beta \in [2,3]$, by the estimate of BSDE, we have

$$\begin{aligned} &\mathbb{E}\left[|\varepsilon_4(T)\Gamma_1(T)\Gamma_2(T)|^\beta \big| \mathcal{F}_s^t\right] \\ &\leq \left\{\mathbb{E}\left[|\varepsilon_4(T)|^4 \big| \mathcal{F}_s^t\right]\right\}^{\frac{\beta}{4}} \left\{\mathbb{E}\left[|\Gamma_1(T)\Gamma_2(T)|^{\frac{4\beta}{4-\beta}} \Big| \mathcal{F}_s^t\right]\right\}^{\frac{4-\beta}{4}} \\ &\leq C\left|x' - \bar{X}^{t,x;\bar{u}}(s)\right|^{2\beta}, \end{aligned} \tag{4.21}$$

$$
\begin{aligned}
&\mathbb{E}\left[\left(\int_s^T |\text{II}(r)|dr\right)^{\beta}\middle|\mathcal{F}_s^t\right]\\
&\leq C\left\{\mathbb{E}\left[\left(\int_s^T (|\varepsilon_1(r)|+(1+|q(r)|)|\varepsilon_2(r)| \quad +|\varepsilon_3(r)|)dr\right)^{\frac{7}{2}}\middle|\mathcal{F}_s^t\right]\right\}^{\frac{2\beta}{7}}\\
&\quad\cdot\left\{\mathbb{E}\left[|\sup_{r\in[t,T]}|\Gamma_1(r)\Gamma_2(r)|^{\frac{7\beta}{7-2\beta}}\middle|\mathcal{F}_s^t\right]\right\}^{\frac{7-2\beta}{7}}\\
&\leq C\left|x'-\bar{X}^{t,x;\bar{u}}(s)\right|^{2\beta},
\end{aligned}
\tag{4.22}
$$

then

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{r\in[s,T]}|\varphi^1(r)|^{\beta}+\left(\int_s^T|\nu^1(r)|^2dr\right)^{\frac{\beta}{2}}\middle|\mathcal{F}_s^t\right]\\
&\leq C\mathbb{E}\left[|\varepsilon_4(T)\Gamma_1(T)\Gamma_2(T)|^{\beta}+\left(\int_s^T|\text{II}(r)|dr\right)^{\beta}\middle|\mathcal{F}_s^t\right]\\
&\leq\left|x'-\bar{X}^{t,x;\bar{u}}(s)\right|^{2\beta}.
\end{aligned}
\tag{4.23}
$$

Combining (4.18) (4.19) and (4.23), we obtain that, for each $\beta\in[2,\frac{5}{2}]$,

$$
\mathbb{E}\left[\sup_{r\in[s,T]}|\varphi(r)|^{\beta}+\left(\int_s^T|\nu(r)|^2dr\right)^{\frac{\beta}{2}}\middle|\mathcal{F}_s^t\right]\leq C\left|x'-\bar{X}^{t,x;\bar{u}}(s)\right|^{2\beta}.
\tag{4.24}
$$

Similar to the above analysis, there exists a large enough $\lambda>0$ such that for

$$
\mathbb{E}\left[\sup_{r\in[s,T]}|P(r)|^{\lambda}+\left(\int_s^T|Q(r)|^2dr\right)^{\frac{\lambda}{2}}\middle|\mathcal{F}_s^t\right]\leq C.
\tag{4.25}
$$

Applying Itô's formula to $\hat{\varphi}(r)\Gamma_1(r)\Gamma_2(r)$, where $(\hat{\varphi}(\cdot),\hat{\nu}(\cdot))$ is the equation (3.19) in Step 4 in the proof of Theorem 3.1, we get

$$
\hat{\varphi}(s)=\mathbb{E}\left[\Gamma_1(T)\Gamma_2(T)\varepsilon_8(T)+\int_s^T\Gamma_1(r)\Gamma_2(r)I(r)dr\middle|\mathcal{F}_s^t\right].
\tag{4.26}
$$

By (4.18) and (4.26), we deduce that

$$
|\hat{\varphi}(s)|\leq C\left\{\mathbb{E}\left[|\varepsilon_8(T)|^{\frac{9}{8}}+\left(\int_s^T|\text{I}(r)|dr\right)^{\frac{9}{8}}\middle|\mathcal{F}_s^t\right]\right\}^{\frac{8}{9}}.
\tag{4.27}
$$

The estimate for $\varepsilon_8(T)$ is the same as Theorem 3.1. Since we relax the assumption of $q(\cdot)$ being bounded, the corresponding $\mathrm{I}(r)$ in (3.20) has the following upper bound

$$
\begin{aligned}
&\left|\mathrm{I}\left(r\right)\right| \\
&\leq C\left[1+\left(1+\left|q\left(r\right)\right|\right)\left|P\left(r\right)\right|+\left|Q\left(r\right)\right|+\left|q\left(r\right)\right|\right] \\
&\cdot\left(\left|\varphi\left(r\right)\right|+\left|\nu\left(r\right)\right|+\rho\left(r\right)\left|\hat{X}\left(r\right)\right|\right)\left|\hat{X}\left(r\right)\right| \\
&+C(1+|q(r)|^2)\rho(r)|\hat{X}(r)|^2+C\left(1+\left|P\left(r\right)\right|\right)\left|\nu\left(r\right)\right|^2 \\
&+C\left(1+\left|q\left(r\right)\right|+\left|P\left(r\right)\right|\right)\varphi^2\left(r\right),
\end{aligned}
$$

where $\rho(r)$ is the same as (3.23). We estimate the following terms:

$$
\begin{aligned}
&\mathbb{E}\left[\left(\int_s^T\left|q\left(r\right)\right|P\left(r\right)\rho\left(r\right)\left|\hat{X}\left(r\right)\right|^2dr\right)^{\frac{9}{8}}\middle|\mathcal{F}_s^t\right] \\
&\leq\mathbb{E}\left[\sup_{r\in[t,T]}\left|\hat{X}\left(r\right)\right|^{\frac{9}{4}}\sup_{r\in[t,T]}\left|P\left(r\right)\right|^{\frac{9}{8}}\left(\int_s^T\left|q\left(r\right)\right|^2dr\right)^{\frac{9}{16}}\left(\int_s^T\left|\rho\left(r\right)\right|^2dr\right)^{\frac{9}{16}}\middle|\mathcal{F}_s^t\right] \\
&\leq\left\{\mathbb{E}\left[\sup_{r\in[t,T]}\left|\hat{X}\left(r\right)\right|^8\middle|\mathcal{F}_s^t\right]\right\}^{\frac{9}{32}}\cdot\left\{\mathbb{E}\left[\sup_{r\in[t,T]}\left|P\left(r\right)\right|^{\frac{36}{23}}\left(\int_s^T\left|q\left(r\right)\right|^2dr\right)^{\frac{18}{23}}\right.\right. \\
&\qquad\left.\left.\cdot\left(\int_s^T\left|\rho\left(r\right)\right|^2dr\right)^{\frac{18}{23}}\middle|\mathcal{F}_s^t\right]\right\}^{\frac{23}{32}} \\
&=o\left(\left|x-\bar{X}^{t,x;\bar{u}}\left(s\right)\right|^{\frac{9}{4}}\right);
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\left[\left(\int_s^T\left|q\left(r\right)\right|P\left(r\right)\left|\nu\left(r\right)\right|\left|\hat{X}\left(r\right)\right|dr\right)^{\frac{9}{8}}\middle|\mathcal{F}_s^t\right] \\
&\leq\mathbb{E}\left[\sup_{r\in[t,T]}\left|\hat{X}\left(r\right)\right|^{\frac{9}{8}}\sup_{r\in[t,T]}\left|P\left(r\right)\right|^{\frac{9}{8}}\left(\int_s^T\left|q\left(r\right)\right|^2dr\right)^{\frac{9}{16}}\left(\int_s^T\left|\nu\left(r\right)\right|^2dr\right)^{\frac{9}{16}}\middle|\mathcal{F}_s^t\right] \\
&\leq\left\{\mathbb{E}\left[\sup_{r\in[t,T]}\left|\hat{X}\left(r\right)\right|^{\frac{45}{22}}\sup_{r\in[t,T]}\left|P\left(r\right)\right|^{\frac{45}{22}}\cdot\left(\int_s^T\left|q\left(r\right)\right|^2dr\right)^{\frac{45}{44}}\middle|\mathcal{F}_s^t\right]\right\}^{\frac{11}{20}} \\
&\qquad\cdot\left\{\mathbb{E}\left[\left(\int_s^T\left|\nu\left(r\right)\right|^2dr\right)^{\frac{5}{4}}\middle|\mathcal{F}_s^t\right]\right\}^{\frac{9}{20}} \\
&=o\left(\left|x-\bar{X}^{t,x;\bar{u}}\left(s\right)\right|^{\frac{9}{4}}\right);
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\left[\left(\int_s^T\left|q\left(r\right)\right|^2\rho\left(r\right)\left|\hat{X}\left(r\right)\right|^2dr\right)^{\frac{9}{8}}\middle|\mathcal{F}_s^t\right] \\
&\leq\mathbb{E}\left[\sup_{r\in[t,T]}\left|\hat{X}\left(r\right)\right|^{\frac{9}{4}}\left(\int_s^T\left|q\left(r\right)\right|^2\rho\left(r\right)dr\right)^{\frac{9}{8}}\middle|\mathcal{F}_s^t\right] \\
&\leq\left\{\mathbb{E}\left[\sup_{r\in[t,T]}\left|\hat{X}\left(r\right)\right|^8\middle|\mathcal{F}_s^t\right]\right\}^{\frac{9}{32}}\cdot\left\{\mathbb{E}\left[\left(\int_s^T\left|q\left(r\right)\right|^2\rho\left(r\right)dr\right)^{\frac{36}{23}}\middle|\mathcal{F}_s^t\right]\right\}^{\frac{23}{32}} \\
&=o\left(\left|x-\bar{X}^{t,x;\bar{u}}\left(s\right)\right|^{\frac{9}{4}}\right);
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\left[\left(\int_s^T |q(r)|\,|\varphi(r)|^2\,dr\right)^{\frac{9}{8}}\middle|\mathcal{F}_s^t\right] \\
&\leq C\mathbb{E}\left[\sup_{r\in[t,T]}|\varphi(r)|^{\frac{9}{4}}\left(\int_s^T |q(r)|^2\,dr\right)^{\frac{9}{16}}\middle|\mathcal{F}_s^t\right] \\
&\leq \left\{\mathbb{E}\left[\sup_{r\in[t,T]}|\varphi(r)|^{\frac{5}{2}}\middle|\mathcal{F}_s^t\right]\right\}^{\frac{9}{10}}\cdot\left\{\mathbb{E}\left[\left(\int_s^T |q(r)|^2\,dr\right)^{\frac{45}{8}}\middle|\mathcal{F}_s^t\right]\right\}^{\frac{1}{10}} \\
&= o\left(\left|x-\bar{X}^{t,x;\bar{u}}(s)\right|^{\frac{9}{4}}\right);
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\left[\left(\int_s^T |P(r)|\,|\nu(r)|^2\,dr\right)^{\frac{9}{8}}\middle|\mathcal{F}_s^t\right] \\
&\leq \mathbb{E}\left[\sup_{r\in[t,T]}|P(r)|^{\frac{9}{8}}\left(\int_s^T |\nu(r)|^2\,dr\right)^{\frac{9}{8}}\middle|\mathcal{F}_s^t\right] \\
&\leq \left\{\mathbb{E}\left[\sup_{r\in[t,T]}|P(r)|^{\frac{45}{4}}\middle|\mathcal{F}_s^t\right]\right\}^{\frac{1}{10}}\cdot\left\{\mathbb{E}\left[\left(\int_s^T |\nu(r)|^2\,dr\right)^{\frac{5}{4}}\middle|\mathcal{F}_s^t\right]\right\}^{\frac{9}{10}} \\
&= o\left(\left|x-\bar{X}^{t,x;\bar{u}}(s)\right|^{\frac{9}{4}}\right),
\end{aligned}
$$

and the others are similar. The proof is completed. □

Theorem 4.8. *Suppose the same assumptions as in Theorem 4.7 hold. Then, for each $s \in [t,T]$,*

$$
\begin{cases}
[\mathcal{H}_1(s,\bar{X}^{t,x;\bar{u}}(s),\bar{Y}^{t,x;\bar{u}}(s),\bar{Z}^{t,x;\bar{u}}(s)),\infty) \subseteq D_{t+}^{1,+}W(s,X^{t,x;\bar{u}}(s)), \\
D_{t+}^{1,-}W(s,X^{t,x;\bar{u}}(s)) \subseteq (-\infty,\mathcal{H}_1(s,\bar{X}^{t,x;\bar{u}}(s),\bar{Y}^{t,x;\bar{u}}(s),\bar{Z}^{t,x;\bar{u}}(s))],
\end{cases}
$$

where

$$
\begin{aligned}
&\mathcal{H}_1(s,\bar{X}^{t,x;\bar{u}}(s),\bar{Y}^{t,x;\bar{u}}(s),\bar{Z}^{t,x;\bar{u}}(s)) \\
&= -\mathcal{H}(s,\bar{X}^{t,x;\bar{u}}(s),\bar{Y}^{t,x;\bar{u}}(s),\bar{Z}^{t,x;\bar{u}}(s),\bar{u}(s),p(s),q(s),P(s)) + \tfrac{1}{2}P(s)\sigma(s)^2.
\end{aligned}
$$

Proof. The proof is the same as in Theorem 3.3 by using the estimates in the proof of Theorem 4.7. □

4.3. The local case

In this case, the control domain is assumed to be a convex and compact set. Note that in the above theorems, our control domain is only supposed to be a nonempty and compact set. Then, for the local case we can still obtain the relations in Theorem 3.1 under our Assumptions 2.1, 2.8 and 2.10. In this section, we study the MP by convex variational method and its relationship with the DPP. For the convex variational method, we suppose that b, σ and g are continuously differentiable with respect to u, and we only need to consider the first-order variational equation. So, every assumptions that guarantee the existence and uniqueness of FBSDE (2.1) can be used in this case. Here we use the following monotonicity conditions as in [13, 25].

Define

$$
\Pi(s,x,y,z,u) = (-g,b,\sigma)^{\intercal}(s,x,y,z,u).
$$

Assumption 4.9. There exist three nonnegative constants β_1, β_2, β_3 such that $\beta_1+\beta_2>0$, $\beta_2+\beta_3>0$ and $\forall s\in[0,T]$, $\forall x$, x', y, y', z, $z'\in\mathbb{R}$, $\forall u\in U$,

$$\begin{aligned}&\langle \Pi(s,x,y,z,u)-\Pi(s,x',y',z',u),(x-x',y-y',z-z')^T\rangle\\&\leq -\beta_1|x-x'|^2-\beta_2(|y-y'|^2+|z-z'|^2),\end{aligned}$$

$$(\phi(x)-\phi(x'))(x-x')\geq \beta_3\left|x-x'\right|^2.$$

The adjoint equation in this case is the following linear FBSDE:

$$\begin{cases} dh(s)= & [g_y(s)h(s)+b_y(s)m(s)+\sigma_y(s)n(s)]\,ds\\ & +[g_z(s)h(s)+b_z(s)m(s)+\sigma_z(s)n(s)]\,dB(s),\\ h(t)= & 1,\\ dm(s)= & -[g_x(s)h(s)+b_x(s)m(s)+\sigma_x(s)n(s)]\,ds\\ & +n(s)dB(s),\ \ s\in[t,T],\\ m(T)= & \phi_x(\bar{x}(T))h(T).\end{cases} \tag{4.28}$$

Define the following Hamiltonian function:

$$H'(s,x,y,z,u,h,m,n)=mb(s,x,y,z,u)+n\sigma(s,x,y,z,u)+hg(s,x,y,z,u).$$

Suppose Assumptions 2.1 (i) and 4.9 hold. Let $\bar{u}(\cdot)\in\mathcal{U}^w[t,T]$ be optimal for problem (2.3) and $(h(\cdot),m(\cdot),n(\cdot))$ be the solution to FBSDE (4.28). Then Wu [25] obtained the following MP:

$$\begin{aligned}\langle H'_u(s,\bar{X}^{t,x;\bar{u}}(s),\bar{Y}^{t,x;\bar{u}}(s),\bar{Z}^{t,x;\bar{u}}(s),\bar{u}(s),h(s),m(s),n(s)),u-\bar{u}(s)\rangle\geq 0,\\ \forall u\in U\ \text{a.e.}s\in[t,T],\ \mathbb{P}\text{-a.s..}\end{aligned}\tag{4.29}$$

Theorem 4.10. *Suppose Assumptions 2.1 (i) and 4.9 hold. Let $\bar{u}(\cdot)$ be optimal for our problem (2.3) and $(h(\cdot),m(\cdot),n(\cdot))$ be the solution to FBSDE (4.28). If L_3 is small enough, then*

$$D_x^{1,-}W(s,\bar{X}^{t,x;\bar{u}}(s))\subseteq\left\{m(s)h^{-1}(s)\right\}\subseteq D_x^{1,+}W(s,\bar{X}^{t,x;\bar{u}}(s)),\ \forall s\in[t,T],\ \mathbb{P}\text{- a.s..}$$

Proof. We use notations (3.3), (3.6) and equations (3.2), (3.5) in Step 1 in the proof of Theorem 3.1. By the estimate of FBSDE (see [13]), we obtain

$$\mathbb{E}\left[\sup_{r\in[s,T]}\left(|\hat{X}(r)|^2+|\hat{Y}(r)|^2\right)+\int_s^T|\hat{Z}(r)|^2dr\middle|\mathcal{F}_s^t\right]\leq C\left|x'-\bar{X}^{t,x;\bar{u}}(s)\right|^2,\mathbb{P}\text{-a.s..}$$

Applying Itô's formula to $h(s)\hat{Y}(s)-m(s)\hat{X}(s)$, we get

$$\begin{aligned}&h(s)\hat{Y}(s)-m(s)\hat{X}(s)\\&=\mathbb{E}\left[h(T)\varepsilon_4(T)\ +\textstyle\int_s^T(m(r)\varepsilon_1(r)+n(r)\varepsilon_2(r)+h(r)\varepsilon_3(r))dr\middle|\mathcal{F}_s^t\right].\end{aligned}$$

Then, we want to prove $h(s)\hat{Y}(s)-m(s)\hat{X}(s)=o\left(\left|x'-\bar{X}^{t,x;\bar{u}}(s)\right|\right)$, and estimate the terms in the right hand as follows

$$\begin{aligned}
&\mathbb{E}\left[|h(T)\varepsilon_4(T)|\middle|\mathcal{F}_s^t\right]\\
&\le\left\{\mathbb{E}\left[|\hat{X}(T)|^2\middle|\mathcal{F}_s^t\right]\right\}^{1/2}\cdot\left\{\mathbb{E}\left[|h(T)(\tilde{\phi}_x(T)-\phi_x(T))|^2\middle|\mathcal{F}_s^t\right]\right\}^{1/2}\\
&=o\left(\left|x'-\bar{X}^{t,x;\bar{u}}(s)\right|\right);
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}\left[\int_s^T|n(r)\left(\tilde{\sigma}_z(r)-\sigma_z(r)\right)\hat{Z}(r)|dr\middle|\mathcal{F}_s^t\right]\\
&\le\left\{\mathbb{E}\left[\int_s^T|n(r)\left(\tilde{\sigma}_z(r)-\sigma_z(r)\right)|^2dr\middle|\mathcal{F}_s^t\right]\right\}^{1/2}\cdot\left\{\mathbb{E}\left[\int_s^T|\hat{Z}(r)|^2dr\middle|\mathcal{F}_s^t\right]\right\}^{1/2}\\
&=o\left(\left|x'-\bar{X}^{t,x;\bar{u}}(s)\right|\right).
\end{aligned}$$

The estimates for the other terms are similar. Similar to Step 5 in the proof of Theorem 3.1, we can find a subset $\Omega_0\subseteq\Omega$ with $\mathbb{P}(\Omega_0)=1$ such that for any $\omega_0\in\Omega_0$,

$$h(s,\omega_0)\hat{Y}(s,\omega_0)-m(s,\omega_0)\hat{X}(s,\omega_0)=o\left(\left|x'-\bar{X}^{t,x;\bar{u}}(s,\omega_0)\right|\right)\ \text{ for all } s\in[t,T].$$

By the DPP in [9, 13], we obtain

$$\begin{aligned}
&W(s,x')-W(s,\bar{X}^{t,x;\bar{u}}(s))\\
&\le Y^{s,x';\bar{u}}(s)-\bar{Y}^{t,x;\bar{u}}(s)\\
&=\hat{Y}(s).
\end{aligned}$$

When L_3 small enough, by Theorem A.4 in [8], we can obtain $h(s)>0$ and $p(s)=m(s)h(s)^{-1}$. Thus

$$\hat{Y}(s)=\quad m(s)h(s)^{-1}\left(X^{s,x';\bar{u}}(s)-\bar{X}^{t,x;\bar{u}}(s)\right)+o(|x'-\bar{X}^{t,x;\bar{u}}(s)|).$$

Since x' is arbitrary, from the definition of super-jet, we get

$$m(s)h(s)^{-1}\in D_x^{1,+}W(s,\bar{X}^{t,x;\bar{u}}(s)).$$

Now we prove

$$D_x^{1,-}W(s,\bar{X}^{t,x;\bar{u}}(s))\subseteq\left\{m(s)h(s)^{-1}\right\}.$$

If $D_x^{1,-}W(s,\bar{X}^{t,x;\bar{u}}(s))$ is not empty, then taking any $\xi\in D_x^{1,-}V(s,\bar{X}^{t,x;\bar{u}}(s))$, by definition of sub-jets, we have

$$\begin{aligned}
0\quad&\le\liminf_{x'\to\bar{X}^{t,x;\bar{u}}(s)}\left\{\frac{W(s,x')-W(s,\bar{X}^{t,x;\bar{u}}(s))-\xi\left(x'-\bar{X}^{t,x;\bar{u}}(s)\right)}{|x'-\bar{X}^{t,x;\bar{u}}(s)|}\right\}\\
&\le\liminf_{x'\to\bar{X}^{t,x;\bar{u}}(s)}\left\{\frac{\left(m(s)h(s)^{-1}-\xi\right)\left(x'-\bar{X}^{t,x;\bar{u}}(s)\right)}{|x'-\bar{X}^{t,x;\bar{u}}(s)|}\right\}.
\end{aligned}$$

Thus, we conclude that

$$\xi = m(s)h(s)^{-1}, \ \forall s \in [t,T], \ \ \mathbb{P}\text{-a.s.}.$$

The proof is completed. □

Theorem 4.11. *Suppose Assumptions 2.1 (i) and 4.9 hold. Let $\bar{u}(\cdot)$ be optimal for problem (2.3) and $(h(\cdot), m(\cdot), n(\cdot))$ be the solution to FBSDE (4.28). If L_3 is small enough and the value function $W(\cdot,\cdot) \in C^{1,2}([t,T]\times\mathbb{R})$, then, for $s \in [t,T]$,*

$$\begin{aligned} &\bar{Y}^{t,x;\bar{u}}(s) = W(s, \bar{X}^{t,x;\bar{u}}(s)), \\ &\bar{Z}^{t,x;\bar{u}}(s) = V(s, \bar{X}^{t,x;\bar{u}}(s), W(s, \bar{X}^{t,x;\bar{u}}(s)), W_x(s, \bar{X}^{t,x;\bar{u}}(s)), \bar{u}(s)), \end{aligned} \tag{4.30}$$

and for any $s \in [t,T]$,

$$\begin{aligned} &-W_s(s, \bar{X}^{t,x;\bar{u}}(s)) \\ &= G(s, \bar{X}^{t,x;\bar{u}}(s), W(s, \bar{X}^{t,x;\bar{u}}(s)), W_x(s, \bar{X}^{t,x;\bar{u}}(s)), \\ &\quad W_{xx}(s, \bar{X}^{t,x;\bar{u}}(s)), \bar{u}(s)) \\ &= \min_{u\in U} G\left(s, \bar{X}^{t,x;\bar{u}}(s), W(s, \bar{X}^{t,x;\bar{u}}(s)), W_x(s, \bar{X}^{t,x;\bar{u}}(s)),\right. \\ &\quad \left. W_{xx}(s, \bar{X}^{t,x;\bar{u}}(s)), u\right). \end{aligned} \tag{4.31}$$

Moreover, if $W(\cdot,\cdot) \in C^{1,3}([t,T]\times\mathbb{R})$ and $W_{sx}(\cdot,\cdot)$ is continuous, then, for $s \in [t,T]$,

$$\begin{aligned} m(s) = \ & W_x(s, \bar{X}^{t,x;\bar{u}}(s))h(s), \\ n(s) = \ & g_z(s)W_x(s, \bar{X}^{t,x;\bar{u}}) + W_{xx}(s, \bar{X}^{t,x;\bar{u}})\sigma(s)h(s) \\ & + \left(1 - W_x(s, \bar{X}^{t,x;\bar{u}}(s))\sigma_z(s)\right)^{-1} b_z(s)\cdot (W_x(s, \bar{X}^{t,x;\bar{u}}(s)))^2, \end{aligned} \tag{4.32}$$

and $\forall u \in U$ a.e. $s \in [t,T]$, $\mathbb{P}$-a.s.

$$\langle H_u'(s, \bar{X}^{t,x;\bar{u}}(s), \bar{Y}^{t,x;\bar{u}}(s), \bar{Z}^{t,x;\bar{u}}(s), \bar{u}(s), h(s), m(s), n(s)), u - \bar{u}(s)\rangle \geq 0. \tag{4.33}$$

Proof. The proof for (4.30) and (4.31) is the same as in Theorem 4.2. Applying Itô's formula to $W_x(s, \bar{X}^{t,x;\bar{u}}(s))h(s)$, one can check that $(h(\cdot), m(\cdot), n(\cdot))$ with $(m(\cdot), n(\cdot))$ given in (4.32) solve FBSDE (4.28). By (4.31), we have

$$\begin{aligned} &G(s, \bar{X}^{t,x;\bar{u}}(s), W(s, \bar{X}^{t,x;\bar{u}}(s)), W_x(s, \bar{X}^{t,x;\bar{u}}(s)), W_{xx}(s, \bar{X}^{t,x;\bar{u}}(s)), \bar{u}(s)) \\ &\leq G\left(s, \bar{X}^{t,x;\bar{u}}(s), W(s, \bar{X}^{t,x;\bar{u}}(s)), W_x(s, \bar{X}^{t,x;\bar{u}}(s)),\ W_{xx}(s, \bar{X}^{t,x;\bar{u}}(s)), u\right) \\ &\quad \forall u \in U \text{ a.e., a.s..} \end{aligned}$$

Thus, we obtain

$$\begin{aligned} &\left\langle \tfrac{\partial}{\partial u} G\left(s, \bar{X}^{t,x;\bar{u}}(s), W(s, \bar{X}^{t,x;\bar{u}}(s)), W_x(s, \bar{X}^{t,x;\bar{u}}(s)),\quad W_{xx}(s, \bar{X}^{t,x;\bar{u}}(s)), u\right)\Big|_{u=\bar{u}(s)}, u - \bar{u}(s)\right\rangle \geq 0, \\ &\quad \forall u \in U \text{ a.e., a.s.,} \end{aligned}$$

which implies

$$\begin{aligned}&\left\langle \left\{W_x(s,\bar{X}^{t,x;\bar{u}}(s))\left[b_z\left(s\right)V_u(s)+b_u(s)\right]\ +W_{xx}(s,\bar{X}^{t,x;\bar{u}}(s))\sigma(s)\left[\sigma_z\left(s\right)V_u(s)+\sigma_u(s)\right]\right.\right.\\&\left.\left.+g_z\left(s\right)V_u(s)\right\},u-u\left(s\right)\right\rangle\geq 0,\forall u\in U\ \text{a.e., a.s.,}\end{aligned}\tag{4.34}$$

where

$$V_u(s)=\quad \frac{\partial V}{\partial u}(s,\bar{X}^{t,x;\bar{u}}(s),W(s,\bar{X}^{t,x;\bar{u}}(s)),W_x(s,\bar{X}^{t,x;\bar{u}}(s)),u)\big|_{u=\bar{u}(s)}.$$

By implicit function theorem, we deduce that

$$\begin{aligned}&\tfrac{\partial V}{\partial u}(s,\bar{X}^{t,x;\bar{u}}(s),W(s,\bar{X}^{t,x;\bar{u}}(s)),W_x(s,\bar{X}^{t,x;\bar{u}}(s)),u)\big|_{u=\bar{u}(s)}\\&=\left(1-W_x(s,\bar{X}^{t,x;\bar{u}}(s))\sigma_z\left(s\right)\right)^{-1}W_x(s,\bar{X}^{t,x;\bar{u}}(s))\sigma_u(s).\end{aligned}\tag{4.35}$$

Combing (4.32), (4.34) and (4.35), we obtain the desired result (4.33). □

Remark 4.12. From Theorems 4.2 and 4.11, we can obtain the following relationship between $(p\left(\cdot\right),q\left(\cdot\right))$ and $(h\left(\cdot\right),m(\cdot),n\left(\cdot\right))$:

$$\begin{aligned}&m(s)=p(s)h(s);\\&n(s)=\left(1-p(s)\sigma_z(s)\right)^{-1}\left[b_z(s)p(s)^2+p(s)g_z(s)+q(s)\right]h(s).\end{aligned}$$

Appendix

The following Lemma is a combination of Theorem 3.17 and Theorem 5.17 in [20].

For each fixed $p>1$ and adapted stochastic process $(y(\cdot),z(\cdot))$, consider the following system

$$\begin{cases}dX(t)= & b(t,X(t),y(t),z(t))dt+\sigma(t,X(t),y(t),z(t))dB(t),\\ dY(t)= & -g(t,X(t),Y(t),Z(t))dt+Z(t)dB(t),\\ X(0)= & x_0,\ \ Y(t)=\phi(X(T)),\end{cases}\tag{A.1}$$

where b, σ, g, ϕ are the same in equation (2.1). If the coefficients satisfy

(i) $b(t,0,y(\cdot),z(\cdot))$, $\sigma(t,0,y(\cdot),z(\cdot))$, $g(t,0,0,0)$ are adapted processes and

$$\begin{aligned}\mathbb{E}\Bigg\{&|\phi(0)|^\beta+\left(\int_0^T\left[|b(t,0,y(t),z(t))|+|g(t,0,0,0)|\right]dt\right)^\beta\\&+\left(\int_0^T|\sigma(t,0,y(t),z(t))|^2dt\right)^{\frac{\beta}{2}}\Bigg\}<\infty,\end{aligned}$$

(ii)

$$\begin{aligned}|\psi(t,x_1,y,z)-\psi(t,x_2,y,z)|&\leq L_1|x_1-x_2|,\quad\text{for}\quad\psi=b,\sigma;\\|g(t,x_1,y_1,z_1)-g(t,x_2,y_2,z_2)|&\leq L_1(|x_1-x_2|+|y_1-y_2|+|z_1-z_2|),\end{aligned}$$

then equation (A.1) has a unique solution $(X(\cdot),Y(\cdot),Z(\cdot)) \in L^{\beta}_{\mathcal{F}}(\Omega;C([0,T],\mathbb{R})) \times L^{\beta}_{\mathcal{F}}(\Omega;C([0,T],\mathbb{R})) \times L^{2,\beta}_{\mathcal{F}}([0,T];\mathbb{R})$ and there exists a constant C_β which only depends on L_1, β, T such that

$$\begin{aligned}
&\mathbb{E}\left\{\sup_{t\in[0,T]}\left[|X(t)|^\beta+|Y(t)|^\beta\right]+\left(\int_0^T|Z(t)|^2dt\right)^{\frac{\beta}{2}}\right\}\\
&\leq C_\beta\mathbb{E}\left\{\left[\int_0^T\left(|b(t,0,y(t),z(t))|+|g(t,0,0,0)|\right)dt\right]^\beta+\left(\int_0^T|\sigma(t,0,y(t),z(t))|^2dt\right)^{\frac{\beta}{2}}\right.\\
&\qquad\left.+|\phi(0)|^\beta+|x_0|^\beta\right\}.
\end{aligned}$$

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