

## EXACT BOUNDARY CONTROLLABILITY AND EXACT BOUNDARY SYNCHRONIZATION FOR A COUPLED SYSTEM OF WAVE EQUATIONS WITH COUPLED ROBIN BOUNDARY CONTROLS

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**Abstract.** In this paper, we consider the exact boundary controllability and the exact boundary synchronization (by groups) for a coupled system of wave equations with coupled Robin boundary controls. Owing to the difficulty coming from the lack of regularity of the solution, we confront a bigger challenge than that in the case with Dirichlet or Neumann boundary controls. In order to overcome this difficulty, we use the regularity results of solutions to the mixed problem with Neumann boundary conditions by Lasiecka and Triggiani [*J. Differ. Equ.* **94** (1991) 112–164] to get the regularity of solutions to the mixed problem with coupled Robin boundary conditions. Thus we show the exact boundary controllability of the system, and by a method of compact perturbation, we obtain the non-exact boundary controllability of the system with fewer boundary controls on some special domains. Based on this, we further study the exact boundary synchronization (by groups) for the same system, the determination of the exactly synchronizable state (by groups), as well as the necessity of the compatibility conditions of the coupling matrices.

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### 1. INTRODUCTION

Synchronization is a widespread natural phenomenon. It was first observed by Huygens in 1665 [4]. The theoretical research on synchronization from the mathematical point of view dates back to N. Wiener in 1950s (see Chap. 10 in [29], p. 199). Since 2012, Li and Rao started the research on the synchronization for coupled systems governed by PDEs, and they showed that the synchronization in this case could be realized in a finite

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time by means of proper boundary controls. Consequently, the study of synchronization becomes a part of research in control theory. Precisely speaking, Li and Rao considered the exact boundary synchronization for a coupled system of wave equations with Dirichlet boundary controls for any given space dimensions in the framework of weak solutions [11, 12, 15] and for the one-space-dimensional case in the framework of classical solutions [3, 20, 26]. Corresponding results were expanded to the exact boundary synchronization by  $p(\geq 1)$  groups [14, 17]. Moreover, Li and Rao proposed the concept of approximate boundary null controllability and approximate boundary synchronization in [13, 16] and further studied them.

Throughout this paper,  $\Omega \subset \mathbb{R}^n$  is a bounded domain with smooth boundary  $\Gamma$  or a parallelepiped. In the first situation, we assume that  $\Omega$  satisfies the usual multiplier geometrical condition [24]. Without loss of generality, assume that there exists an  $x_0 \in \mathbb{R}^n$ , such that by setting  $m = x - x_0$ , we have

$$(m, \nu) > 0, \quad \forall x \in \Gamma, \quad (1.1)$$

where  $\nu$  is the unit outward normal vector on the boundary, and  $(\cdot, \cdot)$  denotes the inner product in  $\mathbb{R}^n$ .

We define

$$\mathcal{H}_0 = \left\{ u : u \in L^2(\Omega), \int_{\Omega} u dx = 0 \right\}, \quad \mathcal{H}_1 = H^1(\Omega) \cap \mathcal{H}_0. \quad (1.2)$$

Inspired by the synchronization of the system with Dirichlet boundary controls, Li, Lu and Rao studied the null controllability and synchronization for the following coupled system of wave equations with Neumann boundary controls on a bounded domain  $\Omega \subset \mathbb{R}^n$  with smooth boundary:

$$\begin{cases} U'' - \Delta U + AU = 0 & \text{in } (0, +\infty) \times \Omega, \\ \partial_{\nu} U = DH & \text{on } (0, +\infty) \times \Gamma, \end{cases} \quad (1.3)$$

where, the coupling matrix  $A = (a_{ij})$  is of order  $N$ , the boundary control matrix  $D$  is an  $N \times M$  ( $M \leq N$ ) full column-rank matrix, namely,  $\text{rank}(D) = M$ , and both  $A$  and  $D$  have real constant elements,  $U = (u^{(1)}, \dots, u^{(N)})^T$  and  $H = (h^{(1)}, \dots, h^{(M)})^T$  denote the state variables and the boundary controls, respectively. The discussion on the control problem will become more flexible because of the introduction of the boundary control matrix  $D$ . Moreover,  $\partial_{\nu}$  denotes the outward normal derivative on the boundary.

**Remark 1.1.** Corresponding results on the exact boundary synchronization and the approximate boundary synchronization obtained in [9, 10, 18] were originally presented for the following system

$$\begin{cases} U'' - \Delta U + AU = 0 & \text{in } (0, +\infty) \times \Omega, \\ U = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ \partial_{\nu} U = DH & \text{on } (0, +\infty) \times \Gamma_1 \end{cases} \quad (1.4)$$

on a bounded domain  $\Omega \subset \mathbb{R}^n$  with smooth boundary  $\Gamma = \Gamma_1 \cup \Gamma_0$  with  $\bar{\Gamma}_1 \cap \bar{\Gamma}_0 = \emptyset$  and  $\text{mes}(\Gamma_1) \neq 0$ , where  $\text{mes}(\cdot)$  stands for the Lebesgue's surface measure on  $\Gamma$ . However, using the basic spaces defined by (1.2), all those results can be obtained for system (1.3).

We have

**Lemma 1.2.** *Assume that  $\Omega \subset \mathbb{R}^n$  is a smooth bounded domain. Assume furthermore that  $M = \text{rank}(D) = N$ . Then there exists a  $T > 0$ , for any given initial data  $(\widehat{U}_0, \widehat{U}_1) \in (\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$ , there exists a boundary control*

$H \in L^2(0, T; (L^2(\Gamma_1))^N)$ , such that the corresponding solution  $U = U(t, x)$  to system (1.3) satisfies

$$t \geq T : \quad U(t, x) \equiv 0, \quad x \in \Omega, \quad (1.5)$$

namely, system (1.3) is exactly null controllable at the time  $T$ .

**Remark 1.3.** By the method given in [18], the boundary control  $H$  can be chosen to continuously depend on the initial data:

$$\|H\|_{L^2(0, T, (L^2(\Gamma_1))^N)} \leq c \|(\widehat{U}_0, \widehat{U}_1)\|_{(\mathcal{H}_1)^N \times (\mathcal{H}_0)^N}, \quad (1.6)$$

here and hereafter,  $c$  is a positive constant independent of the initial data.

On the other hand, when there is a lack of boundary controls, we have

**Lemma 1.4.** *Assume that  $\Omega \subset \mathbb{R}^n$  is a smooth bounded domain. When  $M = \text{rank}(D) < N$ , no matter how large  $T > 0$  is, system (1.3) is not exactly controllable at the time  $T$  in the space  $(\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$ .*

The study on the synchronization will be more difficult with more complicated boundary conditions. In this paper we will consider a coupled system of wave equations with coupled Robin boundary controls as follows:

$$\begin{cases} U'' - \Delta U + AU = 0 & \text{in } (0, +\infty) \times \Omega, \\ \partial_\nu U + BU = DH & \text{on } (0, +\infty) \times \Gamma \end{cases} \quad (1.7)$$

with the corresponding initial condition

$$t = 0 : \quad U = \widehat{U}_0, \quad U' = \widehat{U}_1 \quad \text{in } \Omega, \quad (1.8)$$

where  $B = (b_{ij})$  is the boundary coupling matrix of order  $N$  with constant elements.

To study the exact boundary controllability and the exact boundary synchronization for a coupled system of wave equations with coupled Robin boundary controls, most of difficulties come from the complicated form of boundary conditions. To deeply study the non-exact boundary controllability and the necessity of the conditions of  $C_p$ -compatibility for the exact synchronization by  $p$ -groups, we have to further study the regularity of solutions to Robin problem, then these problems can be obtained on some special domains by a method of compact perturbation, based on the improved regularity results.

## 2. REGULARITY OF SOLUTIONS WITH NEUMANN BOUNDARY CONDITIONS

Similarly to the problem of wave equations with Neumann boundary conditions, a problem with Robin boundary conditions no longer enjoys the hidden regularity as in the case with Dirichlet boundary conditions. As a result, the solution to problem (1.7)–(1.8) with Robin boundary conditions is not smooth enough in general for the proof of the non-exact boundary controllability of the system. In order to overcome this difficulty, we should deeply study the regularity of solutions to wave equations with Neumann boundary conditions. For this purpose, we will review some existing results with Neumann boundary conditions.

Consider the following second order hyperbolic problem on a bounded domain  $\Omega \subset \mathbb{R}^n$  with boundary  $\Gamma$ :

$$\begin{cases} y_{tt} + A(x, \partial)y = f & \text{in } (0, T) \times \Omega, \\ \frac{\partial y}{\partial \nu_A} = g & \text{on } (0, T) \times \Gamma, \\ t = 0 : \quad y = y_0, \quad y_t = y_1 & \text{in } \Omega, \end{cases} \quad (2.1)$$

where

$$A(x, \partial) = - \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x) \frac{\partial}{\partial x_i} + c_0(x), \quad (2.2)$$

in which  $a_{ij}(x)$  with  $a_{ij}(x) = a_{ji}(x)$ ,  $b_i(x)$  and  $c_0(x)$  are smooth real coefficients, and the principal part of  $A(x, \partial)$  is supposed to be uniformly strong elliptic in  $\Omega$ :

$$\sum_{i,j=1}^n a_{ij}(x) \eta_i \eta_j \geq c \sum_{i=1}^n \eta_i^2 \quad (2.3)$$

for any given  $x \in \Omega$  and for any given  $\eta = (\eta_1, \dots, \eta_n) \in \mathbb{R}^n$ , where  $c > 0$  is a positive constant; moreover,  $\frac{\partial y}{\partial \nu_A}$  is the outward normal derivative associated with  $A$ :

$$\frac{\partial y}{\partial \nu_A} = \sum_{i=1}^N \sum_{j=1}^N a_{ij}(x) \frac{\partial y}{\partial x_i} \nu_j, \quad (2.4)$$

$\nu = (\nu_1, \dots, \nu_n)^T$  being the unit outward normal vector on the boundary  $\Gamma$ .

Define the operator  $\mathcal{A}$  by

$$\mathcal{A} = A(x, \partial), \quad \mathcal{D}(\mathcal{A}) = \left\{ y \in H^2(\Omega) : \frac{\partial y}{\partial \nu_A} = 0 \text{ on } \Gamma \right\}. \quad (2.5)$$

In [6], Lasiecka and Triggiani got the optimal regularity for the solution to problem (2.1) by means of the theory of cosine operator for  $\Omega \subset \mathbb{R}^n$  ( $n \geq 2$ ). On the other hand, when  $n = 1$ , better results can be obtained (see [7], Thm. 3.1–3.3 and Rem. 3.1). Moreover, more regularity results can be proved when the domain is a parallelepiped. For conciseness and clarity, we list only those results which are needed in this paper.

Let  $\epsilon > 0$  be an arbitrarily given small number. Here and hereafter, we always assume that  $\alpha, \beta$  are given, respectively, as follows:

$$\begin{cases} \alpha = 3/5 - \epsilon, \quad \beta = 3/5, & \Omega \text{ is a smooth bounded domain} \\ & \text{and } A(x, \partial) \text{ is defined by (2.2);} \\ \alpha = \beta = 3/4 - \epsilon, & \Omega \text{ is a parallelepiped and } A(x, \partial) = -\Delta. \end{cases} \quad (2.6)$$

**Lemma 2.1.** *Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain with boundary  $\Gamma$ . Assume that  $y_0 \equiv y_1 \equiv 0$  and  $f \equiv 0$ . For any given  $g \in L^2(0, T; L^2(\Gamma))$ , the unique solution  $y$  to problem (2.1) satisfies*

$$(y, y') \in C^0([0, T]; H^\alpha(\Omega) \times H^{\alpha-1}(\Omega)) \quad (2.7)$$

and

$$y|_\Sigma \in H^{2\alpha-1}(\Sigma) = L^2(0, T; H^{2\alpha-1}(\Gamma)) \cap H^{2\alpha-1}(0, T; L^2(\Gamma)), \quad (2.8)$$

where  $H^\alpha(\Omega)$  denotes the usual Sobolev space of order  $\alpha$  and  $\Sigma = (0, T) \times \Gamma$ .

**Lemma 2.2.** *Under the assumption on  $\Omega$  given in Lemma 2.1, assume that  $y_0 \equiv y_1 \equiv 0$  and  $g \equiv 0$ . For any given  $f \in L^2(0, T; L^2(\Omega))$ , the unique solution  $y$  to problem (2.1) satisfies*

$$(y, y') \in C^0([0, T]; H^1(\Omega) \times L^2(\Omega)) \quad (2.9)$$

and

$$y|_{\Sigma} \in H^{\beta}(\Sigma). \quad (2.10)$$

**Lemma 2.3.** *Under the assumption on  $\Omega$  given in Lemma 2.1, assume that  $f \equiv 0$  and  $g \equiv 0$ .*

(1) *If  $(y_0, y_1) \in H^1(\Omega) \times L^2(\Omega)$ , then the unique solution  $y$  to problem (2.1) satisfies*

$$(y, y') \in C^0([0, T]; H^1(\Omega) \times L^2(\Omega)) \quad (2.11)$$

and

$$y|_{\Sigma} \in H^{\beta}(\Sigma). \quad (2.12)$$

(2) *If  $(y_0, y_1) \in L^2(\Omega) \times (H^1(\Omega))'$ , where  $(H^1(\Omega))'$  denotes the dual space of  $H^1(\Omega)$  with respect to  $L^2(\Omega)$ , then the unique solution  $y$  to problem (2.1) satisfies*

$$(y, y') \in C^0([0, T]; L^2(\Omega) \times (H^1(\Omega))') \quad (2.13)$$

and

$$y|_{\Sigma} \in H^{\alpha-1}(\Sigma). \quad (2.14)$$

**Remark 2.4.** In the results mentioned above, the mappings from the given data to the solution are all continuous with respect to the corresponding topologies.

### 3. WELL-POSEDNESS OF A COUPLED SYSTEM OF WAVE EQUATIONS WITH COUPLED ROBIN BOUNDARY CONDITIONS

Let  $\Omega \subset \mathbb{R}^n$  be a smooth bounded domain or a parallelepiped as mentioned before. We now prove the well-posedness of problem (1.7) and (1.8).

Let  $\Phi = (\phi^{(1)}, \dots, \phi^{(N)})^T$ . We first consider the following adjoint system

$$\begin{cases} \Phi'' - \Delta\Phi + A^T\Phi = 0 & \text{in } (0, +\infty) \times \Omega, \\ \partial_{\nu}\Phi + B^T\Phi = 0 & \text{on } (0, +\infty) \times \Gamma \end{cases} \quad (3.1)$$

with the initial data

$$t = 0: \quad \Phi = \widehat{\Phi}_0, \quad \Phi' = \widehat{\Phi}_1 \quad \text{in } \Omega, \quad (3.2)$$

where  $A^T$  and  $B^T$  denote the transpose of  $A$  and  $B$ , respectively.

**Theorem 3.1.** *Assume that  $\Omega \subset \mathbb{R}^n$  is a smooth bounded domain or a parallelepiped. Assume furthermore that  $B$  is similar to a real symmetric matrix. Then for any given  $(\widehat{\Phi}_0, \widehat{\Phi}_1) \in (\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$ , the adjoint problem (3.1)–(3.2) admits a unique weak solution*

$$(\Phi, \Phi') \in C_{loc}^0([0, +\infty); (\mathcal{H}_1)^N \times (\mathcal{H}_0)^N) \quad (3.3)$$

in the sense of  $C_0$ -semigroup, where  $\mathcal{H}_1$  and  $\mathcal{H}_0$  are defined by (1.2).

*Proof.* Without loss of generality, we assume that  $B$  is a real symmetric matrix.

We formulate system (3.1) into the following variational form:

$$\int_{\Omega} \langle \Phi'', \widehat{\Phi} \rangle dx + \int_{\Omega} \langle \nabla \Phi, \nabla \widehat{\Phi} \rangle dx + \int_{\Gamma} \langle \Phi, B \widehat{\Phi} \rangle d\Gamma + \int_{\Omega} \langle \Phi, A \widehat{\Phi} \rangle dx = 0 \quad (3.4)$$

for any given test function  $\widehat{\Phi} \in (\mathcal{H}_1)^N$ , where  $(\cdot, \cdot)$  denotes the inner product of  $\mathbb{R}^N$ , while  $\langle \cdot, \cdot \rangle$  denotes the inner product of  $\mathbb{M}^{N \times N}(\mathbb{R})$ .

Recalling the following interpolation inequality ([25])

$$\int_{\Gamma} |\phi|^2 d\Gamma \leq c \|\phi\|_{H^1(\Omega)} \|\phi\|_{L^2(\Omega)}, \quad \forall \phi \in H^1(\Omega),$$

we have

$$\int_{\Gamma} \langle \Phi, B \Phi \rangle d\Gamma \leq \|B\| \int_{\Gamma} |\Phi|^2 d\Gamma \leq c \|B\| \|\Phi\|_{(\mathcal{H}_1)^N} \|\Phi\|_{(\mathcal{H}_0)^N},$$

then it is easy to see that

$$\int_{\Omega} \langle \nabla \Phi, \nabla \Phi \rangle dx + \int_{\Gamma} \langle \Phi, B \Phi \rangle d\Gamma + \lambda \|\Phi\|_{(\mathcal{H}_0)^N}^2 \geq c' \|\Phi\|_{(\mathcal{H}_1)^N}^2$$

for some suitable constants  $\lambda > 0$  and  $c' > 0$ . Moreover, the non-symmetric part in (3.4) satisfies

$$\int_{\Omega} \langle \Phi, A \widehat{\Phi} \rangle dx \leq \|A\| \|\Phi\|_{(\mathcal{H}_0)^N} \|\widehat{\Phi}\|_{(\mathcal{H}_0)^N}.$$

By Theorem 1.1 of Chapter 8 in [22] (p.151), the variational problem (3.4) with the initial data (3.2) admits a unique solution  $\Phi$  with the smoothness (3.3). The proof is complete.  $\square$

**Definition 3.2.**  $U$  is a weak solution to the mixed problem (1.7)–(1.8), if

$$U \in C_{loc}^0([0, +\infty); (\mathcal{H}_0)^N) \cap C_{loc}^1([0, +\infty); (\mathcal{H}_{-1})^N), \quad (3.5)$$

where  $\mathcal{H}_{-1}$  denotes the dual space of  $\mathcal{H}_1$  with the pivot space  $\mathcal{H}_0$ , such that for any given  $(\widehat{\Phi}_0, \widehat{\Phi}_1) \in (\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$  and for all given  $t \geq 0$ , we have

$$\begin{aligned} & \langle\langle (U'(t), -U(t)), (\Phi(t), \Phi'(t)) \rangle\rangle \\ &= \langle\langle (\widehat{U}_1, -\widehat{U}_0), (\widehat{\Phi}_0, \widehat{\Phi}_1) \rangle\rangle + \int_0^t \int_{\Gamma} \langle DH(\tau), \Phi(\tau) \rangle dx dt, \end{aligned} \quad (3.6)$$

in which  $\Phi(t)$  is the solution to the adjoint problem (3.1)–(3.2), and  $\langle\langle \cdot, \cdot \rangle\rangle$  denotes the duality between the spaces  $(\mathcal{H}_{-1})^N \times (\mathcal{H}_0)^N$  and  $(\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$ .

**Theorem 3.3.** Let  $\Omega \subset \mathbb{R}^n$  be a smooth bounded domain or a parallelepiped. Assume that  $B$  is similar to a real symmetric matrix. For any given  $H \in L_{loc}^2(0, +\infty; (L^2(\Gamma))^M)$  and  $(\widehat{U}_0, \widehat{U}_1) \in (\mathcal{H}_0)^N \times (\mathcal{H}_{-1})^N$ , problem (1.7)–(1.8) admits a unique weak solution  $U$ . Moreover, the mapping

$$(\widehat{U}_0, \widehat{U}_1, H) \rightarrow (U, U')$$

is continuous with respect to the corresponding topologies.

*Proof.* Let  $\Phi$  be the solution to the adjoint problem (3.1)–(3.2).

Define a linear functional as follows:

$$L_t(\widehat{\Phi}_0, \widehat{\Phi}_1) = \langle\langle \widehat{U}_1, -\widehat{U}_0, (\widehat{\Phi}_0, \widehat{\Phi}_1) \rangle\rangle + \int_0^t \int_{\Gamma} (DH(\tau), \Phi(\tau)) dx dt. \quad (3.7)$$

Clearly,  $L_t$  is bounded in  $(\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$ . Let  $S_t$  be the semigroup in  $(\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$ , corresponding to the adjoint problem (3.1)–(3.2).  $L_t \circ S_t^{-1}$  is bounded in  $(\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$ . Then, by Riesz–Fréchet representation theorem, for any given  $(\widehat{\Phi}_0, \widehat{\Phi}_1) \in (\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$ , there exists a unique  $(U'(t), -U(t)) \in (\mathcal{H}_{-1})^N \times (\mathcal{H}_0)^N$ , such that

$$L_t \circ S_t^{-1}(\Phi(t), \Phi'(t)) = \langle\langle (U'(t), -U(t)), (\Phi(t), \Phi'(t)) \rangle\rangle. \quad (3.8)$$

By

$$L_t \circ S_t^{-1}(\Phi(t), \Phi'(t)) = L_t(\widehat{\Phi}_0, \widehat{\Phi}_1) \quad (3.9)$$

for any given  $(\widehat{\Phi}_0, \widehat{\Phi}_1) \in (\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$ , (3.6) holds, then  $(U, U')$  is the unique weak solution to problem (1.7)–(1.8). Moreover, we have

$$\begin{aligned} \|(U'(t), -U(t))\|_{(\mathcal{H}_{-1})^N \times (\mathcal{H}_0)^N} &= \|L_t \circ S_t^{-1}\| \\ &\leq c(\|\widehat{U}_0, \widehat{U}_1\|_{(\mathcal{H}_0)^N \times (\mathcal{H}_{-1})^N} + \|H\|_{L^2(0, T; (L^2(\Gamma))^M)}) \end{aligned} \quad (3.10)$$

for all  $t \in [0, T]$ .

At last, by a classic argument of density, we obtain the regularity desired by (3.5).  $\square$

**Remark 3.4.** From now on, in order to guarantee the well-posedness of problem (1.7)–(1.8), we always assume that  $B$  is similar to a real symmetric matrix. This condition is also required for the well-posedness of weak solution even in one-space-dimensional case. However, the exact boundary controllability and the exact boundary synchronization of classical solutions in one-space-dimensional case were done without the symmetry of  $B$  in [8, 20].

#### 4. REGULARITY OF SOLUTIONS WITH COUPLED ROBIN BOUNDARY CONDITIONS

In this section, we will improve the regularity results for Robin problem by means of the regularity results for Neumann problem mentioned in Section 2.

**Theorem 4.1.** *When  $\Omega \subset \mathbb{R}^n$  is a smooth bounded domain or a parallelepiped, for any given  $H \in L^2(0, T; (L^2(\Gamma))^M)$  and any given  $(\widehat{U}_0, \widehat{U}_1) \in (\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$ , the weak solution  $U$  to problem (1.7)–(1.8) satisfies*

$$(U, U') \in C^0([0, T]; (H^\alpha(\Omega))^N \times (H^{\alpha-1}(\Omega))^N) \quad (4.1)$$

and

$$U|_{\Sigma} \in (H^{2\alpha-1}(\Sigma))^N, \quad (4.2)$$

where  $\Sigma = (0, T) \times \Gamma$ , and  $\alpha$  is defined by (2.6). Moreover, the linear mapping

$$(\widehat{U}_0, \widehat{U}_1, H) \rightarrow (U, U')$$

is continuous with respect to the corresponding topologies.

*Proof.* We first consider the case that  $\Omega$  is sufficiently smooth, for example, with  $C^3$  boundary. There exists a function  $h \in C^2(\overline{\Omega})$ , such that

$$\nabla h = \nu \quad \text{on } \Gamma, \quad (4.3)$$

where  $\nu$  is the unit outward normal vector on the boundary  $\Gamma$  ([24]).

Noting (2.11) and (2.12) in Lemma 2.3, it is easy to see that we need only to consider the case  $\widehat{U}_0 \equiv \widehat{U}_1 \equiv 0$ .

Let  $\lambda$  be an eigenvalue of  $B^T$  and let  $e$  be the corresponding eigenvector:

$$B^T e = \lambda e.$$

Defining

$$\phi = (e, U), \quad (4.4)$$

we have

$$\begin{cases} \phi'' - \Delta \phi = -(e, AU) & \text{in } (0, T) \times \Omega, \\ \partial_\nu \phi + \lambda \phi = (e, DH) & \text{on } (0, T) \times \Gamma, \\ t = 0 : \phi = 0, \phi' = 0 & \text{in } \Omega. \end{cases} \quad (4.5)$$

(Eq. (4.5) is actually valid in the weak sense, for simplicity of presentation, however, we write it in the classical sense here and hereafter.) Let

$$\psi = e^{\lambda h} \phi. \quad (4.6)$$

Problem (1.7)–(1.8) can be rewritten into the following problem with Neumann boundary conditions:

$$\begin{cases} \psi'' - \Delta \psi + b(\psi) = -e^{\lambda h} (e, AU) & \text{in } (0, T) \times \Omega, \\ \partial_\nu \psi = e^{\lambda h} (e, DH) & \text{on } (0, T) \times \Gamma, \\ t = 0 : \psi = 0, \psi' = 0 & \text{in } \Omega, \end{cases} \quad (4.7)$$

where  $b(\psi) = 2\lambda \nabla h \cdot \nabla \psi + \lambda(\Delta h - \lambda |\nabla h|^2) \psi$  is a first order linear form of  $\psi$  with smooth coefficients.

By Theorem 3.3,  $U \in C^0([0, T]; (\mathcal{H}_0)^N)$ . By (2.9) in Lemma 2.2, the solution  $\psi$  to the following problem with homogeneous Neumann boundary conditions:

$$\begin{cases} \psi'' - \Delta \psi + b(\psi) = -e^{\lambda h} (e, AU) & \text{in } (0, T) \times \Omega, \\ \partial_\nu \psi = 0 & \text{on } (0, T) \times \Gamma, \\ t = 0 : \psi = 0, \psi' = 0 & \text{in } \Omega \end{cases} \quad (4.8)$$

satisfies

$$(\psi, \psi') \in C^0([0, T]; H^1(\Omega) \times L^2(\Omega)). \quad (4.9)$$

Next, we consider the following problem with inhomogeneous Neumann boundary conditions but without internal force terms:

$$\begin{cases} \psi'' - \Delta\psi + b(\psi) = 0 & \text{in } (0, T) \times \Omega, \\ \partial_\nu\psi = e^{\lambda h}(e, DH) & \text{on } (0, T) \times \Gamma, \\ t = 0 : \quad \psi = 0, \quad \psi' = 0 & \text{in } \Omega. \end{cases} \quad (4.10)$$

By (2.7) and (2.8) in Lemma 2.1, we have

$$(\psi, \psi') \in C^0([0, T]; H^\alpha(\Omega) \times H^{\alpha-1}(\Omega)) \quad (4.11)$$

and

$$\psi|_\Sigma \in H^{2\alpha-1}(\Sigma) = H^{2\alpha-1}(0, T; L^2(\Gamma)) \cap L^2(0, T; H^{2\alpha-1}(\Gamma)), \quad (4.12)$$

where  $\alpha$  is given by the first formula of (2.6). Since this regularity result holds for all the eigenvectors of  $B^T$ , and all the eigenvectors of  $B^T$  constitute a set of basis in  $\mathbb{R}^N$ , we get the desired (4.1) and (4.2).

We next consider the case that  $\Omega$  is a parallelepiped. Although the boundary is only piecewise smooth, however, using direct eigenfunction expansions as for Theorem 6.1N.1 in [5], we can easily check that the regularity results (4.1)–(4.2) remain true with  $\alpha = 3/4 - \epsilon$ .  $\square$

## 5. EXACT BOUNDARY CONTROLLABILITY AND NON-EXACT BOUNDARY CONTROLLABILITY

In this section, we will study the exact boundary controllability and the non-exact boundary controllability for the coupled system (1.7) of wave equations with coupled Robin boundary controls. We will prove that, for a smooth bounded domain  $\Omega \subset \mathbb{R}^n$ , when the number of boundary controls is equal to  $N$ , the number of state variables, system (1.7) is exactly controllable for any given initial data  $(\widehat{U}_0, \widehat{U}_1) \in (\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$ , while, for a parallelepiped in  $\mathbb{R}^n$ , if  $M = \text{rank}(D) < N$ , namely, the number of controls needed is less than the number of variables, system (1.7) is not exactly controllable in  $(\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$ .

### 5.1. Exact boundary controllability

**Definition 5.1.** System (1.7) is exactly null controllable in the space  $(\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$ , if there exists a positive constant  $T > 0$ , such that for any given  $(\widehat{U}_1, \widehat{U}_0) \in (\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$ , there exists a boundary control  $H \in L^2(0, T; (L^2(\Gamma))^M)$ , such that problem (1.7)–(1.8) admits a unique weak solution  $U$  satisfying the final condition

$$t = T : \quad U = U' = 0. \quad (5.1)$$

**Theorem 5.2.** Assume that  $M = \text{rank}(D) = N$ . For a smooth bounded domain  $\Omega \subset \mathbb{R}^n$ , system (1.7) is exactly controllable at a certain time  $T > 0$ , and the boundary control continuously depends on the initial data:

$$\|H\|_{L^2(0, T; (L^2(\Gamma))^N)} \leq c \|(\widehat{U}_0, \widehat{U}_1)\|_{(\mathcal{H}_1)^N \times (\mathcal{H}_0)^N}, \quad (5.2)$$

where  $c > 0$  is a positive constant.

*Proof.* We first consider the corresponding problem (1.3) and (1.8). By Lemma 1.2 and Remark 1.3, for any given initial data  $(\widehat{U}_0, \widehat{U}_1) \in (\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$ , there exists a boundary control  $\widehat{H} \in L^2_{loc}(0, +\infty; (L^2(\Gamma))^N)$  with compact support in  $[0, T]$ , such that system (1.3) with Neumann boundary controls is exactly controllable at

the time  $T$ , and the boundary control  $\widehat{H}$  continuously depends on the initial data:

$$\|\widehat{H}\|_{L^2(0,T;(L^2(\Gamma))^N)} \leq c_1 \|(\widehat{U}_0, \widehat{U}_1)\|_{(\mathcal{H}_1)^N \times (\mathcal{H}_0)^N}, \quad (5.3)$$

where  $c_1 > 0$  is a positive constant.

Noting that  $M = \text{rank}(D) = N$ ,  $D$  is invertible and the boundary condition in system (1.3)

$$\partial_\nu U = D\widehat{H} \quad \text{on } (0, T) \times \Gamma \quad (5.4)$$

can be rewritten as

$$\partial_\nu U + BU = D(\widehat{H} + D^{-1}BU) \stackrel{\text{def.}}{=} DH \quad \text{on } (0, T) \times \Gamma. \quad (5.5)$$

Thus, problem (1.3) and (1.8) with (5.4) can be equivalently regarded as problem (1.7)–(1.8) with (5.5). In other words, the boundary control  $H$  given by

$$H = \widehat{H} + D^{-1}BU \quad \text{on } (0, T) \times \Gamma, \quad (5.6)$$

where  $U$  is the solution to problem (1.3) and (1.8) with (5.4), realizes the exact boundary controllability of system (1.7).

It remains to check that  $H$  given by (5.6) belongs to the control space  $L^2(0, T; (L^2(\Gamma))^N)$  with continuous dependence (5.2). By the regularity result given in Theorem 4.1 (in which we take  $B=0$ ), the trace  $U|_\Sigma \in (H^{2\alpha-1}(\Sigma))^N$ , where  $\alpha$  is defined by the first formula of (2.6). Since  $2\alpha - 1 > 0$ , we have  $H \in L^2(0, T; (L^2(\Gamma))^N)$ . Moreover, still by Theorem 4.1, we have

$$\|U\|_{L^2(0,T;(L^2(\Gamma))^N)} \leq c_2 (\|(\widehat{U}_0, \widehat{U}_1)\|_{(\mathcal{H}_1)^N \times (\mathcal{H}_0)^N} + \|\widehat{H}\|_{L^2(0,T;(L^2(\Gamma))^N)}), \quad (5.7)$$

where  $c_2 > 0$  is another positive constant. By the well-posedness theorem given in Theorem 3.3, it is easy to see that system (1.7) is exactly controllable by the boundary control function  $H$ . Moreover, noting (5.6), (5.2) follows from (5.3) and (5.7). The proof is complete.  $\square$

**Remark 5.3.** The parallelepiped domain is only piecewise smooth, however, the angles between the corners in a parallelepiped are all equal to  $\pi/2$ , then, by Grisvard's results in [2] (see also p. 534 of [1]), the Laplacian  $\Delta$  with Neumann boundary condition and  $L^2(\Omega)$  data has the  $H^2(\Omega)$ -regularity for a parallelepiped  $\Omega \subset \mathbb{R}^n$  with  $n \leq 3$ . Therefore, the exact controllability Theorem 5.1 is still valid at least in this case.

## 5.2. Non-exact boundary controllability

Differently from the case with Neumann boundary controls, the non-exact boundary controllability for the coupled system with coupled Robin boundary controls in a general domain is still an open problem. Fortunately, for some special domains, the solution to problem (1.7)–(1.8) may possess higher regularity. In particular, when  $\Omega$  is a parallelepiped, the optimal regularity of trace  $U|_\Sigma$  almost reaches  $(H^{\frac{1}{2}}(\Sigma))^N$ . This benefits a lot in the proof of the non-exact boundary controllability for the system with fewer boundary controls. We first give the following result of compactness, then we use it to prove the main Theorem 5.5 in this section that for a parallelepiped  $\Omega \subset \mathbb{R}^n$ , if  $M = \text{rank}(D) < N$ , then system (1.7) is not exactly null controllable.

**Lemma 5.4.** *Suppose that  $\Omega \subset \mathbb{R}^n$  is a parallelepiped. Let  $\mathcal{L}$  be a compact linear mapping from  $L^2(\Omega)$  to  $L^2(0, T; L^2(\Omega))$ , and let  $\mathcal{R}$  be a compact linear mapping from  $L^2(\Omega)$  to  $L^2(0, T; H^{1-\alpha}(\Gamma))$ , where  $\alpha$  is defined*

by (2.6). Then, for any given  $T > 0$ , there exists  $\theta \in L^2(\Omega)$ , such that the solution to the following problem:

$$\begin{cases} w'' - \Delta w = \mathcal{L}\theta & \text{in } (0, T) \times \Omega, \\ \partial_\nu w = \mathcal{R}\theta & \text{on } (0, T) \times \Gamma, \\ t = 0 : \quad w = 0, \quad w' = \theta & \text{in } \Omega \end{cases} \quad (5.8)$$

doesn't satisfy the final condition

$$w(T) = w'(T) = 0. \quad (5.9)$$

*Proof.* For any given  $\theta \in L^2(\Omega)$ , by Lemma 2.3, the following problem

$$\begin{cases} \phi'' - \Delta \phi = 0 & \text{in } (0, T) \times \Omega, \\ \partial_\nu \phi = 0 & \text{on } (0, T) \times \Gamma, \\ t = 0 : \quad \phi = \theta, \quad \phi' = 0 & \text{in } \Omega \end{cases} \quad (5.10)$$

admits a unique solution  $\phi$ . By (2.13) and (2.14) in Lemma 2.3, we have

$$\|\phi\|_{L^2(0,T;L^2(\Omega))} \leq c\|\theta\|_{L^2(\Omega)} \quad (5.11)$$

and

$$\|\phi\|_{L^2(0,T;H^{\alpha-1}(\Gamma))} \leq c\|\theta\|_{L^2(\Omega)}, \quad (5.12)$$

where  $\alpha$  is given by (2.6).

On the other hand, by Lemma 2.1 and Lemma 2.2, problem (5.8) admits a unique solution  $w$ . Assume by contradiction that (5.9) holds. Then taking the inner product with  $\phi$  on both sides of (5.8) and integrating by parts, it is easy to get

$$-\|\theta\|_{L^2(\Omega)}^2 = \int_0^T \int_\Omega \mathcal{L}\theta\phi dxdt + \int_0^T \int_\Gamma \mathcal{R}\theta\phi d\Gamma dt. \quad (5.13)$$

Noting (5.11)–(5.12), we then have

$$\|\theta\|_{L^2(\Omega)} \leq c(\|\mathcal{L}\theta\|_{L^2(0,T;L^2(\Omega))} + \|\mathcal{R}\theta\|_{L^2(0,T;H^{1-\alpha}(\Gamma))}) \quad (5.14)$$

for all  $\theta \in L^2(\Omega)$ , which contradicts the compactness of  $\mathcal{L}$  and  $\mathcal{R}$ .  $\square$

**Theorem 5.5.** *Assume that  $M = \text{rank}(D) < N$ . Assume furthermore that  $\Omega \subset \mathbb{R}^n$  is a parallelepiped. Then, no matter how large  $T > 0$  is, system (1.7) is not exactly null controllable in the space  $(\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$ .*

*Proof.* Assume that  $M = \text{rank}(D) < N$ . Then there exists an  $e \in \mathbb{R}^N$ , such that  $D^T e = 0$ . Take the special initial data

$$t = 0 : \quad U = 0, \quad U' = e\theta \quad (5.15)$$

for system (1.7). Assume by contradiction that the system is exactly controllable at the time  $T > 0$ . Then for any given  $\theta \in L^2(\Omega)$ , there exists a boundary control  $H \in L^2(0, T, (L^2(\Gamma))^M)$ , such that the corresponding solution satisfies

$$U(T) = U'(T) = 0. \quad (5.16)$$

Let

$$w = (e, U), \quad \mathcal{L}\theta = -(e, AU), \quad \mathcal{R}\theta = -(e, BU)|_{\Sigma}. \quad (5.17)$$

Noting that  $D^T e = 0$ , we see that  $w$  satisfies problem (5.8) and the final condition (5.9).

By Theorem 5.4, in order to prove Theorem 5.5, it suffices to show that the linear mapping  $\mathcal{L}$  is compact from  $L^2(\Omega)$  into  $L^2(0, T; L^2(\Omega))$ , and  $\mathcal{R}$  is compact from  $L^2(\Omega)$  into  $H^{1-\alpha}(\Sigma)$ , where  $\alpha$  is given by (2.6).

Since system (1.7) with special initial data (5.15) is exactly null controllable, the linear mapping  $\theta \rightarrow H$  is continuous from  $L^2(\Omega)$  into  $L^2(0, T; (L^2(\Gamma))^M)$ . By Theorem 4.1, the mapping  $(\theta, H) \rightarrow (U, U')$  is continuous from  $L^2(\Omega) \times L^2(0, T; (L^2(\Gamma))^M)$  into  $C^0([0, T]; (H^\alpha(\Omega))^N) \cap C^1([0, T]; (H^{\alpha-1}(\Omega))^N)$ . Besides, by Lions' compact embedding theorem (Theorem 5.1 in [23], p68), the following embedding

$$L^2(0, T; (H^\alpha(\Omega))^N) \cap H^1(0, T; (H^{\alpha-1}(\Omega))^N) \subset L^2(0, T; (L^2(\Omega))^N)$$

is compact, hence the linear mapping  $\mathcal{L}$  is compact from  $L^2(\Omega)$  into  $L^2(0, T; L^2(\Omega))$ .

On the other hand, by (4.2) in Theorem 4.1,  $H \rightarrow U|_{\Sigma}$  is continuous from  $L^2(0, T; (L^2(\Gamma))^M)$  into  $(H^{2\alpha-1}(\Sigma))^N$ , then,  $\mathcal{R} : \theta \rightarrow -(e, BU)|_{\Sigma}$  is a continuous mapping from  $L^2(\Omega)$  into  $H^{2\alpha-1}(\Sigma)$ . When  $\Omega$  is a parallelepiped,  $\alpha = 3/4 - \epsilon$ , then  $2\alpha - 1 > 1 - \alpha$ . Hence, by Simon's compact embedding result (Corollary 5 in [28], p. 86), the following embedding

$$H^{2\alpha-1}(\Sigma) = L^2(0, T; H^{2\alpha-1}(\Gamma)) \cap H^{2\alpha-1}(0, T; L^2(\Gamma)) \subset L^2(0, T; H^{1-\alpha}(\Gamma))$$

is compact, therefore the mapping  $\mathcal{R}$  is compact from  $L^2(\Omega)$  into  $L^2(0, T; H^{1-\alpha}(\Gamma))$ . The proof is complete.  $\square$

**Remark 5.6.** We obtain the non-exact boundary controllability for system (1.7) with coupled Robin boundary controls in a parallelepiped  $\Omega$  when there is a lack of boundary controls. The main idea is to use the compact perturbation theory which has a higher requirement on the regularity of the solution. The improved regularity (4.1)–(4.2) with  $\alpha = 3/4 - \epsilon$  for a parallelepiped domain of  $\mathbb{R}^n$  is a consequence of Lasiecka-Triggiani's sharp estimation for Neumann problem in [5].

On the other hand, the parallelepiped domain is only piecewise smooth. However, by Remark 5.3, the well-posedness Theorem 4.1 and the exact controllability Theorem 5.2 are still valid for a parallelepiped  $\Omega \subset \mathbb{R}^n$  with  $n \leq 3$ . Nevertheless, since Theorem 5.5 takes the assumption that the system is exactly controllable, so, it is valid for all parallelepiped  $\Omega \subset \mathbb{R}^n$  without any restriction on the dimension  $n$ .

In what follows, all the results on the synchronization will be established in a smooth bounded domain, while the parallelepiped domain will be only used to examine the necessity of compatibility conditions (see Sect. 7). Theorem 5.5 on the non-exact controllability can be regarded as a start in this direction. How to generalize this result to the general domain is still an open problem.

## 6. EXACT BOUNDARY SYNCHRONIZATION BY $p$ -GROUPS

Based on the results of the exact boundary controllability and the non-exact boundary controllability, we continue to study the exact boundary synchronization by  $p$ -groups for system (1.7) with coupled Robin boundary controls. Theorem 7.1 will show that in order to obtain the exact boundary synchronization by  $p$ -groups, we need at least  $(N - p)$  boundary controls.

Let  $p \geq 1$  be an integer and

$$0 = n_0 < n_1 < n_2 < \cdots < n_p = N \quad (6.1)$$

be integers such that  $n_r - n_{r-1} \geq 2$  for  $1 \leq r \leq p$ . We re-arrange the components of the state variable  $U$  into  $p$  groups:

$$(u^{(1)}, \dots, u^{(n_1)}), \quad (u^{(n_1+1)}, \dots, u^{(n_2)}), \dots, (u^{(n_{p-1}+1)}, \dots, u^{(n_p)}). \quad (6.2)$$

**Definition 6.1.** System (1.7) is exactly synchronizable by  $p$ -groups at the time  $T > 0$  in the space  $(\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$ , if for any given initial data  $(\widehat{U}_0, \widehat{U}_1) \in (\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$ , there exists a boundary control  $H \in (L^2_{loc}(0, +\infty; L^2(\Gamma)))^M$  with compact support in  $[0, T]$ , such that the corresponding solution  $U = U(t, x)$  to problem (1.7)–(1.8) satisfies

$$t \geq T : \quad u^{(i)} = u_r(t, x), \quad n_{r-1} + 1 \leq i \leq n_r, \quad 1 \leq r \leq p, \quad (6.3)$$

where,  $u = (u_1, \dots, u_p)^T$ , being unknown a priori, is called the corresponding exactly synchronizable state by  $p$ -groups.

**Remark 6.2.** In particular, when  $p = 1$ , system (1.7) is exactly synchronizable. The following theorems in this section also work in that case.

For the given division  $0 = n_0 < n_1 < n_2 < \dots < n_p = N$ , let  $S_r$  be an  $(n_r - n_{r-1} - 1) \times (n_r - n_{r-1})$  full row-rank matrix:

$$S_r = \begin{pmatrix} 1 & -1 & & & & \\ & 1 & -1 & & & \\ & & \ddots & \ddots & & \\ & & & 1 & -1 & \\ & & & & & \end{pmatrix}, \quad 1 \leq r \leq p, \quad (6.4)$$

and let  $C_p$  be the following  $(N - p) \times N$  matrix of synchronization by  $p$ -groups:

$$C_p = \begin{pmatrix} S_1 & & & & \\ & S_2 & & & \\ & & \ddots & & \\ & & & & S_p \end{pmatrix}. \quad (6.5)$$

Evidently, we have

$$\text{Ker}(C_p) = \text{Span}\{e_1, \dots, e_p\}, \quad (6.6)$$

where for  $1 \leq r \leq p$ ,

$$(e_r)_i = \begin{cases} 1, & n_{r-1} + 1 \leq i \leq n_r, \\ 0, & \text{others.} \end{cases}$$

Thus, the exact boundary synchronization by  $p$ -groups (6.3) can be equivalently written as

$$t \geq T : \quad C_p U \equiv 0 \quad (6.7)$$

or

$$t \geq T : \quad U = \sum_{r=1}^p u_r e_r. \quad (6.8)$$

**Theorem 6.3.** *Assume that  $\Omega \subset \mathbb{R}^n$  is a smooth bounded domain. Let  $C_p$  be the  $(N-p) \times N$  matrix of synchronization by  $p$ -groups defined by (6.4)–(6.5). Assume that both  $A$  and  $B$  satisfy the following conditions of  $C_p$ -compatibility:*

$$AKer(C_p) \subseteq Ker(C_p), \quad BKer(C_p) \subseteq Ker(C_p). \quad (6.9)$$

Then there exists a boundary control matrix  $D$  satisfying

$$M = \text{rank}(D) = \text{rank}(C_p D) = N - p, \quad (6.10)$$

such that system (1.7) is exactly synchronizable by  $p$ -groups, and the corresponding boundary control  $H$  possesses the following continuous dependence:

$$\|H\|_{L^2(0,T,(L^2(\Gamma))^{N-p})} \leq c \|C_p(\widehat{U}_0, \widehat{U}_1)\|_{(\mathcal{H}_1)^{N-p} \times (\mathcal{H}_0)^{N-p}}, \quad (6.11)$$

where  $c > 0$  is a positive constant.

*Proof.* Since both  $A$  and  $B$  satisfy the conditions of  $C_p$ -compatibility (6.9), by Lemma 3.3 in [21], there exist matrices  $\overline{A}_p$  and  $\overline{B}_p$  of order  $(N-p)$ , such that

$$C_p A = \overline{A}_p C_p, \quad C_p B = \overline{B}_p C_p. \quad (6.12)$$

Applying  $C_p$  to problem (1.7)–(1.8) and defining

$$W = C_p U, \quad \overline{D}_p = C_p D, \quad (6.13)$$

we have

$$\begin{cases} W'' - \Delta W + \overline{A}_p W = 0 & \text{in } (0, +\infty) \times \Omega, \\ \partial_\nu W + \overline{B}_p W = \overline{D}_p H & \text{on } (0, +\infty) \times \Gamma \end{cases} \quad (6.14)$$

with the initial data:

$$t = 0 : \quad W = C_p \widehat{U}_0, \quad W' = C_p \widehat{U}_1 \quad \text{in } \Omega. \quad (6.15)$$

Noting that  $C_p$  is a surjection from  $\mathbb{R}^N$  to  $\mathbb{R}^{N-p}$ , the exact boundary synchronization by  $p$ -groups for system (1.7) is equivalent to the exact boundary controllability for the reduced system (6.14), and the boundary control  $H$ , which realizes the exact boundary controllability for the reduced system (6.14), must be the boundary control which realizes the exact boundary synchronization by  $p$ -groups for system (1.7).

Let  $D$  be defined by

$$\text{Ker}(D^T) = \text{Span}\{e_1, \dots, e_p\} = \text{Ker}(C_p). \quad (6.16)$$

We have  $M = \text{rank}(D) = N - p$ , and

$$\text{Ker}(C_p) \cap \text{Im}(D) = \text{Ker}(C_p) \cap \{\text{Ker}(C_p)\}^\perp = \{0\}. \quad (6.17)$$

By Lemma 2.2 in [19], we get  $\text{rank}(C_p D) = \text{rank}(D) = M = N - p$ , thus  $\overline{D}_p$  is an invertible matrix of order  $(N - p)$ . By Theorem 5.2, the reduced system (6.14) is exactly null controllable, then system (1.7) is exactly synchronizable by  $p$ -groups. By (5.2), we get (6.11).  $\square$

**Remark 6.4.** Noting Theorem 5.2, it is easy to check from the above proof that as long as (7.1) holds, system (1.7) must be exactly synchronizable by  $p$ -groups under the assumptions of Theorem 6.3. Noticing (7.2), in fact, Theorem 6.3 gives a way to find the boundary control matrix  $D$  with minimum rank, but it is not the unique way.

The well-posedness of the reduced problem (6.14)–(6.15) is guaranteed by the following

**Lemma 6.5.** *If  $B$  is similar to a symmetric matrix, and  $B$  satisfies the condition of  $C_p$ -compatibility, then the reduced matrix  $\overline{B}_p$  of  $B$ , given by (6.12), is also similar to a symmetric matrix.*

*Proof.* Since  $B$  is similar to a symmetric matrix, there exists a symmetric matrix  $\widehat{B}$  and an invertible matrix  $P$  such that  $B = P\widehat{B}P^{-1}$ . By the second formula of (6.12), we have

$$C_p B P P^T C_p^T = \overline{B}_p C_p P P^T C_p^T. \quad (6.18)$$

Hence we get

$$\overline{B}_p = C_p P \widehat{B} P^T C_p^T (C_p P P^T C_p^T)^{-1}, \quad (6.19)$$

which is similar to the symmetric matrix

$$(C_p P P^T C_p^T)^{-\frac{1}{2}} C_p P \widehat{B} P^T C_p^T (C_p P P^T C_p^T)^{-\frac{1}{2}}. \quad (6.20)$$

The proof is complete.  $\square$

## 7. NECESSITY OF THE CONDITIONS OF $C_p$ -COMPATIBILITY

In this section, we will discuss the necessity of the conditions of  $C_p$ -compatibility. This problem is closely related to the number of applied boundary controls. The consideration will be based on Theorem 5.5, therefore, in this section  $\Omega$  is a parallelepiped.

### 7.1. Condition of $C_p$ -compatibility for the internal coupling matrix $A$

**Theorem 7.1.** *Let  $\Omega \subset \mathbb{R}^n$  be a parallelepiped. Assume that system (1.7) is exactly synchronizable by  $p$ -groups. Then we have*

$$\text{rank}(C_p D) = N - p. \quad (7.1)$$

*In particular, we have*

$$M = \text{rank}(D) \geq N - p. \quad (7.2)$$

*Proof.* If  $\text{Ker}(D^T) \cap \text{Im}(C_p^T) = \{0\}$ , by Lemma 2.2 in [19], we have

$$\text{rank}(C_p D) = \text{rank}(D^T C_p^T) = \text{rank}(C_p^T) = N - p. \quad (7.3)$$

Next, we prove that it is impossible to have  $\text{Ker}(D^T) \cap \text{Im}(C_p^T) \neq \{0\}$ . Otherwise, there exists a vector  $E \neq 0$ , such that

$$D^T C_p^T E = 0. \quad (7.4)$$

Since system (1.7) is exactly synchronizable by  $p$ -groups, taking the special initial data (5.15) for any given  $\theta \in L^2(\Omega)$ , the solution  $U$  to problem (1.7) and (5.15) satisfies (6.3) (or (6.7)) under boundary control  $H \in L^2(0, T, (L^2(\Gamma))^M)$ . Let

$$w = (E, C_p U), \quad \mathcal{L}\theta = -(E, C_p A U), \quad \mathcal{R}\theta = -(E, C_p B U). \quad (7.5)$$

We easily get

$$w(T) = w'(T) = 0. \quad (7.6)$$

Thus, we get again problem (5.8) for  $w$ . Besides, the exact boundary synchronization by  $p$ -groups for system (1.7) indicates that the final condition (5.9) holds. We then get a contradiction to Lemma 5.4.  $\square$

We have the following theorem on the condition of  $C_p$ -compatibility for the coupling matrix  $A$ :

**Theorem 7.2.** *Let  $\Omega \subset \mathbb{R}^n$  be a parallelepiped. Assume that  $M = \text{rank}(D) = N - p$ . If system (1.7) is exactly synchronizable by  $p$ -groups, then the coupling matrix  $A = (a_{ij})$  should satisfy the following condition of  $C_p$ -compatibility*

$$A \text{Ker}(C_p) \subseteq \text{Ker}(C_p). \quad (7.7)$$

*Proof.* It suffices to prove that

$$C_p A e_r = 0, \quad 1 \leq r \leq p. \quad (7.8)$$

By (6.8), taking the inner product with  $C_p$  on both sides of the equations in system (1.7), we get

$$t \geq T : \quad \sum_{r=1}^p C_p A e_r u_r = 0 \quad \text{in } \Omega. \quad (7.9)$$

If (7.8) fails, then there exist constant coefficients  $\alpha_r (1 \leq r \leq p)$ , not all equal to zero, such that

$$\sum_{r=1}^p \alpha_r u_r = 0 \quad \text{in } \Omega. \quad (7.10)$$

Let

$$c_{p+1} = \sum_{r=1}^p \frac{\alpha_r e_r^T}{\|e_r\|^2}. \quad (7.11)$$

Noting  $(e_r, e_s) = \|e_r\|^2 \delta_{rs}$ , we have

$$t \geq T : \quad c_{p+1} U = \sum_{r=1}^p \alpha_r u_r = 0 \quad \text{in } \Omega. \quad (7.12)$$

Let

$$\tilde{C}_{p-1} = \begin{pmatrix} C_p \\ c_{p+1} \end{pmatrix}. \quad (7.13)$$

We can easily get

$$t \geq T : \quad \tilde{C}_{p-1}U = 0 \quad \text{in } \Omega. \quad (7.14)$$

Noting (6.6) and (7.11), it is easy to see that  $c_{p+1}^T \notin \text{Im}(C_p^T)$ , then,  $\text{rank}(\tilde{C}_{p-1}) = N - p + 1$ . Since  $M = \text{rank}(D) = N - p$ , we have  $\text{Ker}(D^T) \cap \text{Im}(\tilde{C}_{p-1}^T) \neq \{0\}$ , then there exists a vector  $E \neq 0$ , such that

$$D^T \tilde{C}_{p-1}^T E = 0. \quad (7.15)$$

Since system (1.7) is exactly synchronizable by  $p$ -groups, taking the special initial data (5.15) for any given  $\theta \in L^2(\Omega)$ , the solution  $U$  to problem (1.7) and (5.15) satisfies (6.3) (or (6.7)) under boundary control  $H \in L^2(0, T, (L^2(\Gamma))^M)$ . Let

$$w = (E, \tilde{C}_{p-1}U), \quad \mathcal{L}\theta = -(E, \tilde{C}_{p-1}AU), \quad \mathcal{R}\theta = -(E, \tilde{C}_{p-1}BU). \quad (7.16)$$

We get again problem (5.8) for  $w$ . Noting (7.14), we have

$$t = T : \quad w(T) = 0. \quad (7.17)$$

Similarly, we have  $w'(T) = 0$ , then (5.9) holds. Noting that  $\Omega \subset \mathbb{R}^n$  is a parallelepiped, similarly to the proof of Theorem 7.1, we get a conclusion that contradicts Lemma 5.4.  $\square$

**Remark 7.3.** The condition of  $C_p$ -compatibility (7.7) is equivalent to the fact that there exist constants  $\alpha_{rs}$  ( $1 \leq r, s \leq p$ ) such that

$$Ae_r = \sum_{s=1}^p \alpha_{sr} e_s \quad 1 \leq r \leq p, \quad (7.18)$$

or  $A$  satisfies the following row-sum condition by blocks:

$$\sum_{j=n_{r-1}+1}^{n_r} a_{ij} = \alpha_{sr}, \quad n_{s-1} + 1 \leq i \leq n_s, \quad 1 \leq r, s \leq p. \quad (7.19)$$

In particular, when

$$\alpha_{sr} = 0, \quad 1 \leq r, s \leq p, \quad (7.20)$$

we say that  $A$  satisfies the zero-sum condition by blocks. In this case, we have

$$Ae_r = 0, \quad 1 \leq r \leq p. \quad (7.21)$$

## 7.2. Condition of $C_p$ -compatibility for the boundary coupling matrix $B$

Comparing with the internal coupling matrix  $A$ , the study on the necessity of the condition of  $C_p$ -compatibility for the boundary coupling matrix  $B$  is more complicated. It concerns the regularity of solution to the problem with coupled Robin boundary conditions.

Let

$$\varepsilon_i = (0, \dots, \overset{(i)}{1}, \dots, 0)^T, \quad 1 \leq i \leq N \quad (7.22)$$

be a set of classical orthogonal basis in  $\mathbb{R}^N$ , and let

$$V_r = \text{Span}\{\varepsilon_{n_{r-1}+1}, \dots, \varepsilon_{n_r}\}, \quad 1 \leq r \leq p. \quad (7.23)$$

Obviously, we have

$$e_r \in V_r, \quad 1 \leq r \leq p. \quad (7.24)$$

In what follows, we will discuss the necessity of the condition of  $C_p$ -compatibility for the boundary coupling matrix  $B$  under the assumption that  $Ae_r \in V_r$  and  $Be_r \in V_r$  ( $1 \leq r \leq p$ ).

**Theorem 7.4.** *Let  $\Omega \subset \mathbb{R}^n$  be a parallelepiped. Assume that  $M = \text{rank}(D) = N - p$  and*

$$Ae_r \in V_r, \quad Be_r \in V_r \quad (1 \leq r \leq p). \quad (7.25)$$

*If system (1.7) is exactly synchronizable by  $p$ -groups, then the boundary coupling matrix  $B$  should satisfy the following condition of  $C_p$ -compatibility:*

$$BKer(C_p) \subseteq Ker(C_p). \quad (7.26)$$

*Proof.* By (6.8), we have

$$\begin{cases} \sum_{r=1}^p (u_r'' e_r - \Delta u_r e_r + u_r A e_r) = 0 & \text{in } (T, +\infty) \times \Omega, \\ \sum_{r=1}^p (\partial_\nu u_r e_r + u_r B e_r) = 0 & \text{on } (T, +\infty) \times \Gamma. \end{cases} \quad (7.27)$$

Noting (7.24)–(7.25) and the fact that subspaces  $V_r$  ( $1 \leq r \leq p$ ) are orthogonal to each other, for  $1 \leq r \leq p$  we have

$$\begin{cases} u_r'' e_r - \Delta u_r e_r + u_r A e_r = 0 & \text{in } (T, +\infty) \times \Omega, \\ \partial_\nu u_r e_r + u_r B e_r = 0 & \text{on } (T, +\infty) \times \Gamma. \end{cases} \quad (7.28)$$

Taking the inner product with  $C_p$  on both sides of the boundary condition on  $\Gamma$  in (7.28), and noting (6.6), we get

$$u_r C_p B e_r \equiv 0 \quad \text{on } (T, +\infty) \times \Gamma, \quad 1 \leq r \leq p. \quad (7.29)$$

We claim that  $C_p B e_r = 0$  ( $r = 1, \dots, p$ ), which just mean that  $B$  satisfies the condition of  $C_p$ -compatibility (7.26). Otherwise, there exists an  $\bar{r}$  ( $1 \leq \bar{r} \leq p$ ) such that  $C_p B e_{\bar{r}} \neq 0$ , consequently, we have

$$u_{\bar{r}} \equiv 0 \quad \text{on } (T, +\infty) \times \Gamma. \quad (7.30)$$

Then, it follows from the boundary condition in system (7.28) that

$$\partial_\nu u_{\bar{r}} \equiv 0 \quad \text{on } (T, +\infty) \times \Gamma. \quad (7.31)$$

Hence, applying Holmgren's uniqueness theorem ([24], Thm. 8.2) to (7.28), we get

$$u_{\bar{r}} \equiv 0 \quad \text{in } (T, +\infty) \times \Omega, \quad (7.32)$$

then it is easy to check that

$$t \geq T : \quad e_{\bar{r}}^T U \equiv 0 \quad \text{in } \Omega. \quad (7.33)$$

Let

$$\tilde{C}_{p-1} = \begin{pmatrix} C_p \\ e_{\bar{r}}^T \end{pmatrix}. \quad (7.34)$$

We have (7.14). Since  $e_{\bar{r}}^T \notin \text{Im}(C_p^T)$ , it is easy to show that  $\text{rank}(\tilde{C}_{p-1}) = N - p + 1$ . On the other hand, since  $\text{rank}(\text{Ker}(D^T)) = p$ , we have  $\text{Ker}(D^T) \cap \text{Im}(\tilde{C}_{p-1}^T) \neq \{0\}$ , then, there exists a vector  $E \neq 0$ , such that

$$D^T \tilde{C}_{p-1}^T E = 0. \quad (7.35)$$

Again, since system (1.7) is exactly synchronizable by  $p$ -groups, taking the special initial data (5.15) for any given  $\theta \in L^2(\Omega)$ , the solution  $U$  to problem (1.7) and (5.15) satisfies (6.3) (or (6.7)) under boundary control  $H \in L^2(0, T, (L^2(\Gamma))^M)$ . Let

$$w = (E, \tilde{C}_{p-1} U), \quad \mathcal{L}\theta = -(E, \tilde{C}_{p-1} A U), \quad \mathcal{R}\theta = -(E, \tilde{C}_{p-1} B U). \quad (7.36)$$

We have  $w(T) = w'(T) \equiv 0$ , thus, we get again problem (5.8) for  $w$ , and (5.9) holds. Therefore, noting that  $\Omega$  is a parallelepiped, similarly to the proof of Theorem 7.1, we get a conclusion that contradicts Lemma 5.4.  $\square$

**Remark 7.5.** Noting that condition (7.25) obviously holds for  $p = 1$ , so, if system (1.7) is exactly synchronizable ( $p = 1$ ), the conditions of  $C_1$ -compatibility

$$A \text{Ker}(C_1) \subseteq \text{Ker}(C_1) \quad \text{and} \quad B \text{Ker}(C_1) \subseteq \text{Ker}(C_1) \quad (7.37)$$

are always satisfied for both  $A$  and  $B$ .

### 7.3. Conditions of $C_2$ -compatibility

In this section, we will study the necessity of the condition of  $C_2$ -compatibility (when  $p = 2$ ) for  $B$  for a specific example, where the restricted condition on  $B$  given in (7.25) can be removed.

**Theorem 7.6.** *Let  $\Omega \subset \mathbb{R}^n$  be a parallelepiped. Assume that the coupling matrix  $A$  satisfies the zero-sum condition by blocks (7.21). Assume furthermore that system (1.7) is exactly synchronizable by 2-groups with  $M = \text{rank}(D) = N - 2$ . Then the coupling matrix  $B$  necessarily satisfies the condition of  $C_2$ -compatibility:*

$$BKer(C_2) \subseteq Ker(C_2). \quad (7.38)$$

*Proof.* By the exact boundary synchronization by 2-groups of (1.7), we have

$$t \geq T : \quad U = e_1 u_1 + e_2 u_2 \quad \text{in } \Omega. \quad (7.39)$$

Noting (7.21), as  $t \geq T$  we have

$$AU = Ae_1 u_1 + Ae_2 u_2 = 0,$$

then it is easy to see that

$$\begin{cases} U'' - \Delta U = 0 & \text{in } (T, +\infty) \times \Omega, \\ \partial_\nu U + BU = 0 & \text{on } (T, +\infty) \times \Gamma. \end{cases} \quad (7.40)$$

Let  $P$  be a matrix such that  $\widehat{B} = PBP^{-1}$  is a real symmetric matrix. Denote

$$u = (u_1, u_2)^T. \quad (7.41)$$

Taking the inner product on both sides of (7.40) with  $P^T P e_i$  for  $i = 1, 2$ , we get

$$\begin{cases} Lu'' - L\Delta u = 0 & \text{in } (T, +\infty) \times \Omega, \\ L\partial_\nu u + \Lambda u = 0 & \text{on } (T, +\infty) \times \Gamma, \end{cases} \quad (7.42)$$

where the matrices  $L$  and  $\Lambda$  are given by

$$L = (Pe_i, Pe_j) \quad \text{and} \quad \Lambda = (\widehat{B}Pe_i, Pe_j), \quad 1 \leq i, j \leq 2, \quad (7.43)$$

respectively. Clearly,  $L$  is a symmetric and positive definite matrix and  $\Lambda$  is a symmetric matrix.

Taking the inner product on both sides of (7.42) with  $L^{-\frac{1}{2}}$  and denoting  $w = L^{\frac{1}{2}}u$ , we get

$$\begin{cases} w'' - \Delta w = 0 & \text{in } (T, +\infty) \times \Omega, \\ \partial_\nu w + \widehat{\Lambda}w = 0 & \text{on } (T, +\infty) \times \Gamma, \end{cases} \quad (7.44)$$

where  $\widehat{\Lambda} = L^{-\frac{1}{2}}\Lambda L^{-\frac{1}{2}}$  is also a symmetric matrix.

On the other hand, taking the inner product with  $C_2$  on both sides of the boundary condition on  $\Gamma$  in system (1.7) and noting (7.39), we get

$$t \geq T : \quad C_2 B e_1 u_1 + C_2 B e_2 u_2 \equiv 0 \quad \text{on } \Gamma. \quad (7.45)$$

We claim that  $C_2 B e_1 = C_2 B e_2 = 0$ , namely,  $B$  satisfies the condition of  $C_2$ -compatibility (7.38).

Otherwise, without loss of generality, we may assume that  $C_2 B e_1 \neq 0$ . Then it follows from (7.45) that there exists a non-zero vector  $D_2 \in \mathbb{R}^2$ , such that

$$t \geq T : \quad D_2^T u \equiv 0 \quad \text{on } \Gamma. \quad (7.46)$$

Denoting

$$\widehat{D}^T = D_2^T L^{-\frac{1}{2}},$$

we then have

$$t \geq T : \quad \widehat{D}^T w = 0 \quad \text{on } \Gamma, \quad (7.47)$$

in which  $w = L^{\frac{1}{2}}u$ . By the multiplier method, we can prove that the following Kalman's criterion

$$\text{rank}(\widehat{D}, \widehat{\Lambda}\widehat{D}) = 2 \quad (7.48)$$

is sufficient for the unique continuation of system (7.44) under the observation (7.47) on the infinite horizon  $[T, +\infty)$  (see [27], Thm. 3.22 and Rem. 3.12). Since  $M = \text{rank}(D) = N - 2$ , by Theorem 5.5, system (1.7) is not exactly null controllable. So, the rank condition (7.48) does not hold. Thus, there exists a vector  $E \neq 0$  in  $\mathbb{R}^2$ , such that

$$\widehat{\Lambda}^T E = \widehat{\Lambda}E = \mu E \quad \text{and} \quad \widehat{D}^T E = 0. \quad (7.49)$$

Noting (7.47) and the second formula of (7.49), we have both  $E$  and  $w|_{\Gamma} \in \text{Ker}(\widehat{D})$ . Since  $\dim \text{Ker}(\widehat{D}) = 1$ , there exists a constant  $\alpha$  such that  $w = \alpha E$  on  $\Gamma$ . Therefore, noting the first formula of (7.49), we have

$$\widehat{\Lambda}w = \widehat{\Lambda}\alpha E = \mu\alpha E = \mu w \quad \text{on } \Gamma. \quad (7.50)$$

Thus, (7.44) can be rewritten as

$$\begin{cases} w'' - \Delta w = 0 & \text{in } (T, +\infty) \times \Omega, \\ \partial_\nu w + \mu w = 0 & \text{on } (T, +\infty) \times \Gamma. \end{cases} \quad (7.51)$$

Let  $z = \widehat{D}^T w$ . Noting (7.47), it follows from (7.51) that

$$\begin{cases} z'' - \Delta z = 0 & \text{in } (T, +\infty) \times \Omega, \\ \partial_\nu z = z = 0 & \text{on } (T, +\infty) \times \Gamma. \end{cases} \quad (7.52)$$

Then, by Holmgren's uniqueness theorem, we have

$$t \geq T : \quad z = \widehat{D}^T w = D_2^T u \equiv 0 \quad \text{in } \Omega. \quad (7.53)$$

Let  $D_2^T = (\alpha_1, \alpha_2)$ . Define the following row vector

$$c_3 = \frac{\alpha_1 e_1^T}{\|e_1\|^2} + \frac{\alpha_2 e_2^T}{\|e_2\|^2}. \quad (7.54)$$

Noting  $(e_1, e_2) = 0$  and (7.53), we have

$$t \geq T : \quad c_3 U = \alpha_1 u_1 + \alpha_2 u_2 = D_2^T u \equiv 0 \quad \text{in } \Omega. \quad (7.55)$$

Let

$$\tilde{C}_1 = \begin{pmatrix} C_2 \\ c_3 \end{pmatrix}. \quad (7.56)$$

We get

$$t \geq T : \quad \tilde{C}_1 U = 0 \quad \text{in } \Omega. \quad (7.57)$$

Since  $c_3^T \notin \text{Im}(C_2^T)$ , it is easy to see that  $\text{rank}(\tilde{C}_1) = N - 1$ , and  $\text{Ker}(D^T) \cap \text{Im}(\tilde{C}_1^T) \neq \{0\}$ , thus there exists a vector  $\tilde{E} \neq 0$ , such that

$$D^T \tilde{C}_1^T \tilde{E} = 0. \quad (7.58)$$

Since system (1.7) is exactly synchronizable by 2-groups, taking the special initial data (5.15) for any given  $\theta \in L^2(\Omega)$ , the solution  $U$  to problem (1.7) and (5.15) satisfies (6.3) (or (6.7)) under boundary control  $H$ . Let

$$u = (\tilde{E}, \tilde{C}_1 U), \quad \mathcal{L}\theta = -(\tilde{E}, \tilde{C}_1 A U), \quad \mathcal{R}\theta = -(\tilde{E}, \tilde{C}_1 B U). \quad (7.59)$$

We get again  $w(T) = w'(T) \equiv 0$ , and problem (5.8) of  $w$  satisfies (5.9). Noting that  $\Omega$  is a parallelepiped, similarly to the proof of Theorem 7.1, we get a contradiction to Lemma 5.4.  $\square$

## 8. DETERMINATION OF THE EXACTLY SYNCHRONIZABLE STATE BY $p$ -GROUPS

In general, exactly synchronizable states by  $p$ -groups depend not only on initial data, but also on applied boundary controls. However, when the coupling matrices  $A$  and  $B$  satisfy certain algebraic conditions, the exactly synchronizable state by  $p$ -groups can be independent of applied boundary controls. In this section, we first discuss the case when the exactly synchronizable state by  $p$ -groups is independent of applied boundary controls, then we present the estimate on each exactly synchronizable state by  $p$ -groups in general situation.

**Theorem 8.1.** *Let  $\Omega \subset \mathbb{R}^n$  be a smooth bounded domain. Assume that both  $A$  and  $B$  satisfy the conditions of  $C_p$ -compatibility (6.9). Assume furthermore that  $A^T$  and  $B^T$  possess a common invariant subspace  $V$ , biorthogonal to  $\text{Ker}(C_p)$  (see [19], Def. 2.1). Then there exists a boundary control matrix  $D$  with  $M = \text{rank}(D) = \text{rank}(C_p D) = N - p$ , such that system (1.7) is exactly synchronizable by  $p$ -groups, and the exactly synchronizable state by  $p$ -groups  $u = (u_1, \dots, u_p)^T$  is independent of applied boundary controls.*

*Proof.* Define the boundary control matrix  $D$  by

$$\text{Ker}(D^T) = V. \quad (8.1)$$

Since  $V$  is biorthogonal to  $\text{Ker}(C_p)$ , by Lemma 2.5 in [19], we have

$$\text{Ker}(C_p) \cap \text{Im}(D) = \text{Ker}(C_p) \cap V^\perp = \{0\}, \quad (8.2)$$

then, by Lemma 2.2 in [19], we have

$$\text{rank}(C_p D) = \text{rank}(D) = M = N - p. \quad (8.3)$$

Therefore, by Theorem 6.3, system (1.7) is exactly synchronizable by  $p$ -groups. Let  $U$  be the solution to problem (1.7)–(1.8), which realizes the exact boundary synchronization by  $p$ -groups at time  $T > 0$  under such  $D$  and boundary control  $H$ .

By (8.2), noting  $\text{Ker}(C_p) = \text{Span}\{e_1, \dots, e_p\}$ , we may write

$$V = \text{Span}\{E_1, \dots, E_p\} \quad \text{with } (e_r, E_s) = \delta_{rs} (r, s = 1, \dots, p). \quad (8.4)$$

Since  $V$  is a common invariant subspace of  $A^T$  and  $B^T$ , there exist constants  $\alpha_{rs}$  and  $\beta_{rs} (r, s = 1, \dots, p)$  such that

$$A^T E_r = \sum_{s=1}^p \alpha_{rs} E_s, \quad B^T E_r = \sum_{s=1}^p \beta_{rs} E_s. \quad (8.5)$$

For  $r = 1, \dots, p$ , let

$$\phi_r = (E_r, U). \quad (8.6)$$

By system (1.7) and noting (8.1), for  $r = 1, \dots, p$  we have

$$\begin{cases} \phi_r'' - \Delta \phi_r + \sum_{s=1}^p \alpha_{rs} \phi_s = 0 & \text{in } (0, +\infty) \times \Omega, \\ \partial_\nu \phi_r + \sum_{s=1}^p \beta_{rs} \phi_s = 0 & \text{on } (0, +\infty) \times \Gamma, \\ t = 0: \quad \phi_r = (E_r, \widehat{U}_0), \quad \phi_r' = (E_r, \widehat{U}_1) & \text{in } \Omega. \end{cases} \quad (8.7)$$

On the other hand, for  $r = 1, \dots, p$  we have

$$t \geq T: \quad \phi_r = (E_r, U) = \sum_{s=1}^p (E_r, e_s) u_s = \sum_{s=1}^p \delta_{rs} u_s = u_r. \quad (8.8)$$

Thus, the exactly synchronizable state by  $p$ -groups  $u = (u_1, \dots, u_p)^T$  is entirely determined by the solution to problem (8.7), which is independent of applied boundary controls  $H$ .  $\square$

The following result gives the counterpart of Theorem 8.1.

**Theorem 8.2.** *Let  $\Omega \subset \mathbb{R}^n$  be a smooth bounded domain (say, with  $C^3$  boundary). Assume that both  $A$  and  $B$  satisfy the conditions of  $C_p$ -compatibility (6.9). Assume furthermore that system (1.7) is exactly synchronizable by  $p$ -groups. If there exists a subspace  $V = \text{Span}\{E_1, \dots, E_p\}$  of dimension  $p$ , such that the projection functions*

$$\phi_r = (E_r, U), \quad r = 1, \dots, p \quad (8.9)$$

*are independent of applied boundary controls  $H$ , where  $U$  is the solution to problem (1.7)–(1.8), which realizes the exact boundary synchronization by  $p$ -groups at time  $T$ , then  $V$  is a common invariant subspace of  $A^T$  and  $B^T$ ,  $V \subseteq \text{Ker}(D^T)$ , and biorthogonal to  $\text{Ker}(C_p)$ .*

*Proof.* Let  $(\widehat{U}_0, \widehat{U}_1) = (0, 0)$ . By Theorem 4.1, the linear mapping

$$F: H \rightarrow (U, U')$$

is continuous from  $L^2(0, T; (L^2(\Gamma))^M)$  to  $C^0([0, T]; (H^\alpha(\Omega))^N \times (H^{\alpha-1}(\Omega))^N)$ , where  $\alpha$  is defined by (2.6).

Let  $F'$  denote the Fréchet derivative of the application  $F$ . For any given  $\widehat{H} \in L^2(0, T; (L^2(\Gamma))^M)$ , we define

$$\widehat{U} = F'(0)\widehat{H}. \quad (8.10)$$

By linearity,  $\widehat{U}$  satisfies a system similar to that of  $U$ :

$$\begin{cases} \widehat{U}'' - \Delta\widehat{U} + A\widehat{U} = 0 & \text{in } (0, +\infty) \times \Omega, \\ \partial_\nu\widehat{U} + B\widehat{U} = D\widehat{H} & \text{on } (0, +\infty) \times \Gamma \\ t = 0: \quad \widehat{U} = \widehat{U}' = 0 & \text{in } \Omega. \end{cases} \quad (8.11)$$

Since the projection functions  $\phi_r = (E_r, U)$  ( $r = 1, \dots, p$ ) are independent of applied boundary controls  $H$ , we have

$$(E_r, \widehat{U}) \equiv 0, \quad \forall \widehat{H} \in L^2(0, T; (L^2(\Gamma))^M), \quad r = 1, \dots, p. \quad (8.12)$$

First, we prove that  $E_r \notin \text{Im}(C_p^T)$  for  $r = 1, \dots, p$ . Otherwise, there exist an  $\bar{r}$  and a vector  $R_{\bar{r}} \in \mathbb{R}^{N-p}$ , such that  $E_{\bar{r}} = C_p^T R_{\bar{r}}$ , then we have

$$0 = (E_{\bar{r}}, \widehat{U}) = (R_{\bar{r}}, C_p \widehat{U}), \quad \forall \widehat{H} \in L^2(0, T; (L^2(\Gamma))^M). \quad (8.13)$$

Since  $C_p \widehat{U}$  is the solution to the corresponding reduced problem (6.14)–(6.15), noting the equivalence between the exact boundary synchronization by  $p$ -groups for the original system and the exact boundary controllability for the reduced system, from the exact boundary synchronization by  $p$ -groups for system (1.7), we know that the reduced system (6.14) is exactly controllable, then the value of  $C_p \widehat{U}$  at the time  $T$  can be chosen arbitrarily, thus we get  $R_{\bar{r}} = 0$ , which contradicts  $E_{\bar{r}} \neq 0$ . Then, we have  $E_r \notin \text{Im}(C_p^T)$  ( $r = 1, \dots, p$ ). Thus  $V \cap \{\text{Ker}(C_p)\}^\perp = V \cap \text{Im}(C_p^T) = \{0\}$ . Hence by Lemma 4.2 and Lemma 4.3 in [27],  $V$  is bi-orthonormal to  $\text{Ker}(C_p)$ , and then  $(V, C_p^T)$  constitutes a set of basis in  $\mathbb{R}^N$ . Therefore, there exist constant coefficients  $\alpha_{rs}$  ( $r, s = 1, \dots, p$ ) and vectors  $P_r \in \mathbb{R}^{N-p}$  ( $r = 1, \dots, p$ ), such that

$$A^T E_r = \sum_{s=1}^p \alpha_{rs} E_s + C_p^T P_r, \quad r = 1, \dots, p. \quad (8.14)$$

Taking the inner product with  $E_r$  on both sides of the equations in (8.11) and noting (8.12), we get

$$0 = (A\widehat{U}, E_r) = (\widehat{U}, A^T E_r) = (\widehat{U}, C_p^T P_r) = (C_p \widehat{U}, P_r) \quad (8.15)$$

for  $r = 1, \dots, p$ . Similarly, by the exact boundary controllability for the reduced system (6.14), we get  $P_r = 0$  ( $r = 1, \dots, p$ ), thus we have

$$A^T E_r = \sum_{s=1}^p \alpha_{rs} E_s, \quad r = 1, \dots, p,$$

which means that  $V$  is an invariant subspace of  $A^T$ .

On the other hand, noting (8.12) and taking the inner product with  $E_r$  on both sides of the boundary condition on  $\Gamma$  in (8.11), we get

$$(E_r, B\widehat{U}) = (E_r, D\widehat{H}) \quad \text{on } \Gamma, \quad r = 1, \dots, p. \quad (8.16)$$

By Theorem 4.1, for  $r = 1, \dots, p$  we have

$$\begin{aligned} & \| (E_r, D\widehat{H}) \|_{H^{2\alpha-1}(\Sigma)} \\ &= \| (E_r, B\widehat{U}) \|_{H^{2\alpha-1}(\Sigma)} \leq c \| \widehat{H} \|_{L^2(0,T;L^2(\Gamma))^M}, \end{aligned} \quad (8.17)$$

where  $\alpha$  is given by (2.6).

We claim that  $D^T E_r = 0$  for  $r = 1, \dots, p$ . Otherwise, for  $r = 1, \dots, p$ , setting  $\widehat{H} = D^T E_r v$ , it follows from (8.17) that

$$\|v\|_{H^{2\alpha-1}(\Sigma)} \leq c \|v\|_{L^2(0,T;L^2(\Gamma))}. \quad (8.18)$$

Since  $2\alpha - 1 > 0$ , it contradicts the compactness of  $H^{2\alpha-1}(\Sigma) \hookrightarrow L^2(\Sigma)$ . Thus, by (8.16) we have

$$(E_r, B\widehat{U}) = 0 \quad \text{on } (0, T) \times \Gamma, \quad r = 1, \dots, p. \quad (8.19)$$

Similarly, there exist constants  $\beta_{rs}$  ( $r, s = 1, \dots, p$ ) and vectors  $Q_r \in \mathbb{R}^{N-p}$  ( $r = 1, \dots, p$ ), such that

$$B^T E_r = \sum_{s=1}^p \beta_{rs} E_s + C_p^T Q_r, \quad r = 1, \dots, p. \quad (8.20)$$

Substituting it into (8.19) and noting (8.12), we have

$$\sum_{s=1}^p \beta_{rs} (E_s, \widehat{U}) + (C_p^T Q_r, \widehat{U}) = (Q_r, C_p \widehat{U}) = 0, \quad r = 1, \dots, p. \quad (8.21)$$

By the exact boundary controllability for the reduced system (6.14), we get  $Q_r = 0$  ( $r = 1, \dots, p$ ), then we have

$$B^T E_r = \sum_{s=1}^p \beta_{rs} E_s, \quad r = 1, \dots, p, \quad (8.22)$$

which indicates that  $V$  is also an invariant subspace of  $B^T$ . The proof is complete.  $\square$

**Remark 8.3.** When  $\Omega \subset \mathbb{R}^n$  is a parallelepiped, Theorem 8.2 is still valid with the same proof.

When  $A$  and  $B$  do not satisfy all the conditions mentioned in Theorem 8.1, exactly synchronizable states by  $p$ -groups may depend on applied boundary controls. We have the following

**Theorem 8.4.** *Let  $\Omega \subset \mathbb{R}^n$  be a smooth bounded domain. Assume that both  $A$  and  $B$  satisfy the conditions of  $C_p$ -compatibility (6.9). Then there exists a boundary control matrix  $D$  such that system (1.7) is exactly synchronizable by  $p$ -groups, and each exactly synchronizable state by  $p$ -groups  $u = (u_1, \dots, u_p)^T$  satisfies the following estimate:*

$$\| (u, u')(T) - (\phi, \phi')(T) \|_{(H^{\alpha+1}(\Omega))^p \times (H^\alpha(\Omega))^p} \leq c \| C_p(\widehat{U}_0, \widehat{U}_1) \|_{(\mathcal{H}_1)^{N-p} \times (\mathcal{H}_0)^{N-p}}, \quad (8.23)$$

where  $\alpha$  is defined by the first formula of (2.6),  $c$  is a positive constant and  $\phi = (\phi_1, \dots, \phi_p)^T$  is the solution to the following problem ( $1 \leq r \leq p$ ):

$$\begin{cases} \phi_r'' - \Delta \phi_r + \sum_{s=1}^p \alpha_{rs} \phi_s = 0 & \text{in } (0, +\infty) \times \Omega, \\ \partial_\nu \phi_r + \sum_{s=1}^p \beta_{rs} \phi_s = 0 & \text{on } (0, +\infty) \times \Gamma, \\ t = 0: \quad \phi_r = (E_r, \widehat{U}_0), \quad \phi_r' = (E_r, \widehat{U}_1) & \text{in } \Omega, \end{cases} \quad (8.24)$$

in which

$$Ae_r = \sum_{s=1}^p \alpha_{sr} e_s, \quad Be_r = \sum_{s=1}^p \beta_{sr} e_s, \quad r = 1, \dots, p. \quad (8.25)$$

*Proof.* We first show that there exists a subspace  $V$  which is invariant for  $B^T$  and bi-orthonormal to  $\text{Ker}(C_p)$ .

Let  $B = P^{-1} \Lambda P$ , where  $P$  is an invertible matrix, and  $\Lambda$  be a symmetric matrix. Let  $V = \text{Span}\{E_1, \dots, E_p\}$  in which

$$E_r = P^T P e_r, \quad r = 1, \dots, p. \quad (8.26)$$

Noting (6.6) and the fact that  $\text{Ker}(C_p)$  is an invariant subspace of  $B$ , we get

$$B^T E_r = P^T P B e_r \subseteq P^T P \text{Ker}(C_p) \subseteq V \quad r = 1, \dots, p, \quad (8.27)$$

then  $V$  is invariant for  $B^T$ .

We next show that  $V^\perp \cap \text{Ker}(C_p) = \{0\}$ . Then, noting that  $\dim(V) = \dim \text{Ker}(C_p) = p$ , by Lemma 4.2 and Lemma 4.3 in [27],  $V$  is bi-orthonormal to  $\text{Ker}(C_p)$ . For this purpose, let  $a_1, \dots, a_p$  be coefficients such that

$$\sum_{r=1}^p a_r e_r \in V^\perp. \quad (8.28)$$

Then

$$\left( \sum_{r=1}^p a_r e_r, E_s \right) = \left( \sum_{r=1}^p a_r P e_r, P e_s \right) = 0, \quad s = 1, \dots, p. \quad (8.29)$$

It follows that

$$\left( \sum_{r=1}^p a_r P e_r, \sum_{s=1}^p a_s P e_s \right) = 0, \quad (8.30)$$

then  $a_1 = \dots = a_p = 0$ , namely,  $V^\perp \cap \text{Ker}(C_p) = \{0\}$ .

Denoting

$$B e_r = \sum_{s=1}^p \beta_{sr} e_s, \quad r = 1, \dots, p, \quad (8.31)$$

a direct calculation yields that

$$B^T E_r = \sum_{s=1}^p \beta_{rs} E_s, \quad r = 1, \dots, p. \quad (8.32)$$

Define the boundary control matrix  $D$  by

$$\text{Ker}(D^T) = V. \quad (8.33)$$

Noting (6.6), we have

$$\begin{aligned} & \text{Ker}(C_p) \cap \text{Im}(D) \\ &= \text{Ker}(C_p) \cap \{\text{Ker}(D^T)\}^\perp = \text{Ker}(C_p) \cap V^\perp = \{0\}, \end{aligned} \quad (8.34)$$

then, by Lemma 2.2 in [19], we have

$$\text{rank}(C_p D) = \text{rank}(D) = M = N - p. \quad (8.35)$$

Therefore, by Theorem 6.3, system (1.7) is exactly synchronizable by  $p$ -groups. Let  $U$  be the solution to problem (1.7)–(1.8), which realizes the exact boundary synchronization by  $p$ -groups at time  $T$  under such  $D$  and boundary control  $H$ .

Denoting  $\psi_r = (E_r, U)$  ( $r = 1, \dots, p$ ), we have

$$\begin{aligned} (E_r, AU) &= (A^T E_r, U) \\ &= \left( \sum_{s=1}^p \alpha_{rs} E_s + A^T E_r - \sum_{s=1}^p \alpha_{rs} E_s, U \right) \\ &= \sum_{s=1}^p \alpha_{rs} (E_s, U) + \left( A^T E_r - \sum_{s=1}^p \alpha_{rs} E_s, U \right) \\ &= \sum_{s=1}^p \alpha_{rs} \psi_s + \left( A^T E_r - \sum_{s=1}^p \alpha_{rs} E_s, U \right). \end{aligned} \quad (8.36)$$

By the assumption that  $V$  is bi-orthonormal to  $\text{Ker}(C_p)$ , without loss of generality, we may assume that

$$(E_r, e_s) = \delta_{rs} \quad (r, s = 1, \dots, p). \quad (8.37)$$

Then, for any given  $k = 1, \dots, p$ , by the first formula of (8.25), we get

$$\begin{aligned} & \left( A^T E_r - \sum_{s=1}^p \alpha_{rs} E_s, e_k \right) = (E_r, A e_k) - \sum_{s=1}^p \alpha_{rs} (E_s, e_k) \\ &= \sum_{s=1}^p \alpha_{sk} (E_r, e_s) - \alpha_{rk} = \alpha_{rk} - \alpha_{rk} = 0, \end{aligned}$$

hence

$$A^T E_r - \sum_{s=1}^p \alpha_{rs} E_s \in \{\text{Ker}(C_p)\}^\perp = \text{Im}(C_p^T), \quad r = 1, \dots, p. \quad (8.38)$$

Thus, there exist  $R_r \in \mathbb{R}^{N-p}$  ( $r = 1, \dots, p$ ), such that

$$A^T E_r - \sum_{s=1}^p \alpha_{rs} E_s = C_p^T R_r, \quad r = 1, \dots, p. \quad (8.39)$$

Taking the inner product on both sides of problem (1.7)–(1.8) with  $E_r$ , and noting (8.32)–(8.33), for  $r = 1, \dots, p$  we have

$$\begin{cases} \psi_r'' - \Delta \psi_r + \sum_{s=1}^p \alpha_{rs} \psi_s = -(R_r, C_p U) & \text{in } (0, +\infty) \times \Omega, \\ \partial_\nu \psi_r + \sum_{s=1}^p \beta_{rs} \psi_s = 0 & \text{on } (0, +\infty) \times \Gamma, \\ t = 0: \quad \psi_r = (E_r, \widehat{U}_0), \quad \psi_r' = (E_r, \widehat{U}_1) & \text{in } \Omega. \end{cases} \quad (8.40)$$

Then, by the classic semigroups theory, we have

$$\begin{aligned} & \|(\psi, \psi')(T) - (\phi, \phi')(T)\|_{(H^{\alpha+1}(\Omega))^p \times (H^\alpha(\Omega))^p} \leq c_1 \|(R_r, C_p U)\|_{L^2(0, T; H^\alpha(\Omega))} \\ & \leq c_2 \|C_p(\widehat{U}_0, \widehat{U}_1)\|_{(H^1(\Omega))^{N-p} \times (L^2(\Omega))^{N-p}}, \end{aligned} \quad (8.41)$$

where  $c_i$  for  $i = 1, 2$  are different positive constants,  $\alpha$  is given by the first formula of (2.6), and the second inequality follows from (5.2) and Theorem 4.1 since  $C_p U$  is the solution to the reduced problem (6.14)–(6.15).

On the other hand, noting (8.37), it is easy to see that

$$t \geq T: \quad \psi_r = (E_r, U) = \sum_{s=1}^p (E_r, e_s) u_s = u_r, \quad r = 1, \dots, p. \quad (8.42)$$

Substituting it into (8.41), we get (8.23).  $\square$

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