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NULL CONTROLLABILITY OF A COUPLED DEGENERATE SYSTEM WITH THE FIRST AND ZERO ORDER TERMS BY A SINGLE CONTROL

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Abstract. This paper concerns the null controllability of a system of m linear degenerate parabolic equations with coupling terms of first and zero order, and only one control force localized in some arbitrary nonempty open subset ω of Ω. The key ingredient for proving the null controllability is to obtain the observability inequality for the corresponding adjoint system. Due to the degeneracy, we transfer to study an approximate nondegenerate adjoint system. In order to deal with the coupling first order terms, we first prove a new Carleman estimate for a degenerate parabolic equation in Sobolev spaces of negative order. Based on this Carleman estimate, we obtain a uniform Carleman estimate and then an observation inequality for this approximate adjoint system.

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1. Introduction

In this paper, we consider the null controllability of the following coupled degenerate parabolic system with the first and zero order terms:

$$\begin{cases} \partial_t u_1 = (x^{\alpha_1} u_{1,x})_x + \sum\limits_{i=1}^m b_{1i} u_{i,x} + \sum\limits_{i=1}^m a_{1i} u_i + f\mathbf{1}_\omega, & (x,t) \in \Omega_T, \\ \partial_t u_2 = (x^{\alpha_2} u_{2,x})_x + \sum\limits_{i=1}^m b_{2i} u_{i,x} + \sum\limits_{i=1}^m a_{2i} u_i, & (x,t) \in \Omega_T, \\ \cdots & \\ \partial_t u_m = (x^{\alpha_m} u_{m,x})_x + \sum\limits_{i=1}^m b_{mi} u_{i,x} + \sum\limits_{i=1}^m a_{mi} u_i, & (x,t) \in \Omega_T, \\ u_j(0,t) = u_j(1,t) = 0, \quad t \in (0,T), \quad u_j(x,0) = u_{j,0}(x), & x \in \Omega, \quad 1 \le j \le m, \end{cases} \tag{1.1}$$

where $\Omega = (0,1)$, $\Omega_T := \Omega \times (0,T)$, $m \ge 2$, $0 < \alpha_i < 1$, $a_{ij}, b_{ij} \in L^\infty(\Omega_T)$ and f is the control function, $\mathbf{1}_\omega$ is the characteristic function of the set $\omega = (x_1, x_2)$ with $0 < x_1 < x_2 < 1$. Here $B = (b_{ij})_{1\le i,j\le m}$ is an upper

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triangular matrix and $A = (a_{ij})_{1\le i,j\le m}$ is an upper triangular matrix plus a subdiagonal matrix, details see (A3) below.

The null controllability for nondegenerate coupled parabolic system has been extensively studied by many authors. For instance, González-Burgos and Pérez-Garca [15] proved a controllability result for some coupled parabolic systems with nonlinearities depending on the gradient of the state by a single control. Further, González-Burgos and Pérez-Garca [17] extended the controllability result to a general cascade system of m linear parabolic equations, also see Fernandez-Cara, Gonzalez-Burgos and de Teresa [14]. Additionally, Duprez and Lissy [12] considered a system of m linear parabolic equations with coupling terms of first and zero order, and $m-1$ controls localized in some arbitrary nonempty open subset ω of Ω. A necessary and sufficient condition to obtain the null or approximate controllability is obtained, a general result also see Fadili and Maniar [13].

As for the null controllability for degenerate coupled parabolic system, we refer to [1, 5, 7, 16, 17, 25]. In [2], Ait Benhassi, Ammar Khodja, Hajjaj and Maniar studied the null controllability of the following system of two degenerate parabolic equations coupling with the zero order terms

$$\begin{cases} u_t - (x^{\alpha_1}u_x)_x + b_{11}u + b_{12}v = h\mathbf{1}_\omega, & (x,t)\in\Omega_T,\\ v_t - (x^{\alpha_1}v_x)_x + b_{21}u + b_{22}v = 0, & (x,t)\in\Omega_T. \end{cases} \tag{1.2}$$

To deal with different degenerate powers α_1 and α_2, they proved a new Carleman estimate for degenerate parabolic equation with an equivalence between weight functions $e^{s\varphi_1}$ and $e^{s\varphi_2}$ and then obtained the null controllability by means of this Carleman estimate. Du and Xu [11] considered the following coupled degenerate parabolic system

$$\begin{cases} u_t - (x^{\alpha}u_x)_x + b_1u_x + b_2v_x + c_{11}u + c_{12}v = h\mathbf{1}_\omega, & (x,t)\in\Omega_T,\\ v_t - (x^{\alpha}v_x)_x + b_3v_x + c_{21}u + c_{22}v = 0, & (x,t)\in\Omega_T. \end{cases} \tag{1.3}$$

The null controllability for (1.3) with only one control was shown under the condition

$$|b_i(x,t)| \le Kx^{\frac{\alpha}{2}} \quad \text{for} \quad i=1,2,3 \tag{1.4}$$

with a positive constant K, which means that the coefficients b_i of the coupling first order terms go to zero at some polynomial rate as $x\to 0$. Thanks to (1.4), the coupling first order terms could be absorbed by the existing Carleman estimates for the degenerate equation, for example the ones in [4, 7]. Additionally, in (1.3) there is only one equation coupling with the first order term of another equation, rather than coupling with the first order terms each other. Letting $\alpha_i = \alpha$, Fadili and Maniar [13] studied the null controllability properties of the following coupled degenerate system with only zero order terms

$$\begin{cases} \partial_t u_1 = d_1\left(x^\alpha u_{1,x}\right)_x + \sum\limits_{i=1}^n a_{1i}u_i + b_1f_1\mathbf{1}_\omega, & (x,t)\in\Omega_T,\\ \partial_t u_2 = d_2\left(x^\alpha u_{2,x}\right)_x + \sum\limits_{i=1}^n a_{2i}u_i + b_2f_2\mathbf{1}_\omega, & (x,t)\in\Omega_T,\\ \cdots & \\ \partial_t u_m = d_m\left(x^\alpha u_{m,x}\right)_x + \sum\limits_{i=1}^n a_{mi}u_i + b_mf_m\mathbf{1}_\omega, & (x,t)\in\Omega_T,\\ \partial_t u_{m+1} = d_{m+1}\left(x^\alpha u_{m+1,x}\right)_x + \sum\limits_{i=1}^n a_{m+1,i}u_i, & (x,t)\in\Omega_T,\\ \cdots & \\ \partial_t u_n = d_n\left(x^\alpha u_{n,x}\right)_x + \sum\limits_{i=1}^n a_{n+1,i}u_i, & (x,t)\in\Omega_T. \end{cases} \tag{1.5}$$

They proved that the Kalman rank condition on the coupling and the control matrices $A = (a_{ij})_{1\le i,j\le n}$ and $B = \mathrm{diag}(b_1, \cdots, b_m, 0, \cdots, 0)$ characterizes the controllability properties of system (1.5) in a particular case: $D = (d, \cdots, d), d > 0$ and A, B are constant in time and space. Additionally, when $D = (d_1, \cdots, d_n), d_i > 0$ and A is cascade, they also obtained the controllability result. A general result also see Ait Benhassi, Fadili and Maniar [3].

The goal of this paper is to derive the null controllability for the coupled degenerate parabolic system with the first and zero order terms. We only assume that the coefficients of the coupling first order terms $b_{ij} \in L^\infty(\Omega_T)$, rather than $x^{-\alpha_j/2}b_{ij} \in L^\infty(\Omega_T)$, which leads to a different method introduced to deal with the coupling first order terms. To do this, we have to prove a new Carleman estimate for a degenerate equation in Sobolev space with negative order on the basis of the method introduced by Imanuvilov and Yamamoto [19]. Meanwhile we need the term $\int_{\Omega_T} s^3\theta^3 x^{-\gamma}|v_i|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t$ with $\gamma = \max\{\alpha_1, \cdots, \alpha_m\}$ included in our Carleman estimate to handle the dual operation in proving the Carleman estimate in $L^2(0, T; H^{-1}(\Omega))$. For this reason, we only obtain the null controllability result for $\alpha_j \in (0, \frac{1}{3})$. Moreover, due to degeneracy of (1.1), we transfer to study a uniform null controllability in ε for a nondegenerate approximate system. To do this, letting $0 < \varepsilon < 1$ and $u^\varepsilon_{0,j} \in H^2(\Omega) \cap H^1_0(\Omega)$ such that

$$v^\varepsilon_{j,0} \to v_{j,0}, \quad \text{strongly in } L^2(\Omega) \text{ for } 1 \le j \le m,$$

we first consider the null controllability for the following approximate system:

$$\begin{cases} \partial_t u^\varepsilon_1 = \big((x+\varepsilon)^{\alpha_1} u^\varepsilon_{1,x}\big)_x + \sum\limits_{i=1}^m b_{1i}u^\varepsilon_{i,x} + \sum\limits_{i=1}^m a_{1i}u^\varepsilon_i + f^\varepsilon \mathbf{1}_\omega, & (x,t) \in \Omega_T, \\ \partial_t u^\varepsilon_2 = \big((x+\varepsilon)^{\alpha_2} u^\varepsilon_{2,x}\big)_x + \sum\limits_{i=1}^m b_{2i}u^\varepsilon_{i,x} + \sum\limits_{i=1}^m a_{2i}u^\varepsilon_i, & (x,t) \in \Omega_T, \\ \cdots \\ \partial_t u^\varepsilon_m = \big((x+\varepsilon)^{\alpha_m} u^\varepsilon_{m,x}\big)_x + \sum\limits_{i=1}^m b_{mi}u^\varepsilon_{i,x} + \sum\limits_{i=1}^m a_{mi}u^\varepsilon_i, & (x,t) \in \Omega_T, \\ u^\varepsilon_j(0,t) = u^\varepsilon_j(1,t) = 0, \quad t \in (0,T), \quad u^\varepsilon_j(x,0) = u^\varepsilon_{j,0}(x), \quad x \in \Omega, \quad 1 \le j \le m. \end{cases} \tag{1.6}$$

It is well known that the key ingredient for proving the controllability result is to obtain the Carleman estimate for the corresponding adjoint system. Carleman estimate is a class of weighted energy estimates connected with the differential operator, which has various applications such as inverse problems [6, 20, 22–24, 26, 31, 32, 34], unique continuations [27, 28], the control theory [18, 19, 21, 29] and so on. As for Carleman estimate for degenerate equation, we refer to [4, 9, 10].

The main result of this paper is the null controllability for the coupled degenerate system (1.1) by only one control force. We state this result as Theorem 1.1 below in this section.

The following assumptions are for Theorem 1.1.

(A1) $\alpha_j \in (0, \frac{1}{3})$ for all $1 \le j \le m$;

(A2) $a_{ij}, b_{ij} \in L^\infty(\Omega_T)$ for $1 \le i, j \le m$, $(u_{1,0}, \cdots, u_{m,0})^T \in L^2(\Omega)^m$;

(A3) $A = (a_{ij})_{1\le i,j\le m}$ and $B = (b_{ij})_{1\le i,j\le m}$ have the structure

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ 0 & a_{32} & a_{33} & \cdots & a_{3m} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{m,m-1} & a_{mm} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ 0 & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{mm} \end{pmatrix} \tag{1.7}$$

and $a_{i,i-1}$ such that

$$|a_{i,i-1}(x,t)| \geq a_0 > 0, \quad (x,t) \in \omega_T, \quad 2 \leq i \leq m. \tag{1.8}$$

Remark 1.1. Condition (A3) is used to prove the null controllability by only one control. Generally, similar controllability result with one control does not hold for general complete matrices. Nevertheless, for general complete matrices A and B our method could still be applied to prove the following null controllability with m controls

$$\begin{cases} \partial_t u_1 = (x^{\alpha_1} u_{1,x})_x + \sum\limits_{i=1}^m b_{1i} u_{i,x} + \sum\limits_{i=1}^m a_{1i} u_i + f_1 \mathbf{1}_\omega, & (x,t) \in \Omega_T, \\ \partial_t u_2 = (x^{\alpha_2} u_{2,x})_x + \sum\limits_{i=1}^m b_{2i} u_{i,x} + \sum\limits_{i=1}^m a_{2i} u_i + f_2 \mathbf{1}_\omega, & (x,t) \in \Omega_T, \\ \cdots \\ \partial_t u_m = (x^{\alpha_m} u_{m,x})_x + \sum\limits_{i=1}^m b_{mi} u_{i,x} + \sum\limits_{i=1}^m a_{mi} u_i + f_m \mathbf{1}_\omega, & (x,t) \in \Omega_T, \\ u_j(0,t) = u_j(1,t) = 0, \quad t \in (0,T), \quad u_j(x,0) = u_{j,0}(x), & x \in \Omega, \quad 1 \leq j \leq m. \end{cases} \tag{1.9}$$

Moreover, our interest is limited to obtain a controllability result for the degenerate parabolic system under condition (A2) on the coefficients of the first order coupling terms, which has wider application range than the condition $x^{-\alpha_j/2} b_{ij} \in L^\infty(\Omega_T)$ in [11]. So here we only give a sufficient condition (A3) on the matrices A and B to guarantee our null controllability, which is introduced by González-Burgos and de Teresa to prove the null controllability for a nongenerate parabolic system with only one control force [17]. It would be very interesting to try to generalize (A3) to a sufficient and necessary condition for the null controllability results, such as the so-called Kalman rank condition for nongenerate parabolic system in [14], also for degenerate parabolic system in [13].

Theorem 1.1. *Let (A1)-(A3) be held. Then for any* $(u_{1,0}, \cdots, u_{m,0})^T \in L^2(\Omega)^m$*, there exists a control* $f \in L^2(\omega_T)$ *such that the corresponding solution* $(u_1, \cdots, u_m)^T$ *of system (1.1) satisfies*

$$u_j(x,T) = 0, \quad x \in \Omega, \quad 1 \leq j \leq m. \tag{1.10}$$

Moreover, f *satisfies the following estimate*

$$\|f\|_{L^2(\omega_T)} \leq C \sum_{j=1}^m \|u_{j,0}\|_{L^2(\Omega)}, \tag{1.11}$$

where C *is depending on* $m, \omega, \Omega, T, \alpha_j, a_{ij}, b_{ij}$.

The remainder of the paper is organized as follows. In next section, we prove the well-posedness of degenerate parabolic system with the couplings of the first and zero order. In Section 3, we show a Carleman estimate for the adjoint system of (1.1) with only one local integral. Next, we prove, in Section 4, an observability inequality and then the null controllability for the degenerate system (1.1), *i.e.* Theorem 1.1.

2. Well-posedness

In this section, we use an approximate argument to prove the well-posedness of the following coupled degenerate system

$$\begin{cases} \partial_t u_1 = (x^{\alpha_1} u_{1,x})_x + \sum\limits_{i=1}^m b_{1i} u_{i,x} + \sum\limits_{i=1}^m a_{1i} u_i + f_1, & (x,t) \in \Omega_T, \\ \partial_t u_2 = (x^{\alpha_2} u_{2,x})_x + \sum\limits_{i=1}^m b_{2i} u_{i,x} + \sum\limits_{i=1}^m a_{2i} u_i + f_2, & (x,t) \in \Omega_T, \\ \cdots \\ \partial_t u_m = (x^{\alpha_m} u_{m,x})_x + \sum\limits_{i=1}^m b_{mi} u_{i,x} + \sum\limits_{i=1}^m a_{mi} u_i + f_m, & (x,t) \in \Omega_T, \\ u_j(0,t) = u_j(1,t) = 0, \quad t \in (0,T), \quad u_j(x,0) = u_{j,0}(x), \quad x \in \Omega, \quad 1 \le j \le m. \end{cases} \tag{2.1}$$

Since system (2.1) is degenerate at $x = 0$, we have to prove the well-posedness in the following weighted spaces

$$H^1_\alpha(\Omega) := \left\{ z \in L^2(\Omega) \mid x^{\frac{\alpha}{2}} z_x \in L^2(\Omega) \quad \text{and} \quad z|_{\partial\Omega} = 0 \right\}$$

endowed with the norm

$$\|z\|_{H^1_\alpha(\Omega)} = \|z\|_{L^2(\Omega)} + \|x^{\frac{\alpha}{2}} z_x\|_{L^2(\Omega)}.$$

Throughout this paper, the solution of system (2.1) is in the following sense.

Definition. $(u_1, \cdots, u_m)^T$ is called the weak solution of (2.1) if $u_j \in L^2(0,T; H^1_{\alpha_j}(\Omega)) \cap L^\infty(0,T; L^2(\Omega))$, $\partial_t u_j \in L^2(0,T; H^{-1}(\Omega))$ for $1 \le j \le m$ and $u_j|_{t=0} = u_{j,0}$ and there hold

$$\begin{aligned} &\int_0^T \langle \partial_t u_j, \xi \rangle_{H^{-1}(\Omega), H^1(\Omega)} + \int_{\Omega_T} x^{\alpha_j} u_{j,x} \xi_x \mathrm{d}x\mathrm{d}t \\ =& \sum_{i=1}^m \int_{\Omega_T} b_{ji} u_{i,x} \xi \mathrm{d}x\mathrm{d}t + \sum_{i=1}^m \int_{\Omega_T} a_{ji} u_i \xi \mathrm{d}x\mathrm{d}t + \int_{\Omega_T} f_j \xi \mathrm{d}x\mathrm{d}t, \quad 1 \le j \le m \end{aligned}$$

for all $\xi \in L^2(0,T; H^1(\Omega))$, where $H^{-1}(\Omega)$ denotes the dual space of $H^1(\Omega)$.

We have the following well-posedness for system (2.1) for $\alpha_j \in (0,1)$.

Theorem 2.1. *Let (A2) be held, $\alpha_j \in (0,1)$, $f_j \in L^2(\Omega_T)$ for all $1 \le j \le m$. Then there exists a unique weak solution $(u_1, \cdots, u_m)^T$ of the problem (2.1) that verifies moreover*

$$\sum_{j=1}^m \left(\|u_j\|_{L^\infty(0,T;L^2(\Omega))} + \|u_j\|_{L^2(0,T;H^1_{\alpha_j}(\Omega))} \right) \le C \sum_{j=1}^m \left(\|u_{j,0}\|_{L^2(\Omega)} + \|f_j\|_{L^2(\Omega_T)} \right), \tag{2.2}$$

where C is depending on $m, \Omega, T, \alpha_j, a_{ij}, b_{ij}$.

Proof. We first consider the following nondegenerate approximate system

$$\begin{cases} \partial_t u_1^\varepsilon = \left((x+\varepsilon)^{\alpha_1} u_{1,x}^\varepsilon\right)_x + \sum\limits_{i=1}^m b_{1i} u_{i,x}^\varepsilon + \sum\limits_{i=1}^m a_{1i} u_i^\varepsilon + f_1, & (x,t) \in \Omega_T, \\ \partial_t u_2^\varepsilon = \left((x+\varepsilon)^{\alpha_2} u_{2,x}^\varepsilon\right)_x + \sum\limits_{i=1}^m b_{2i} u_{i,x}^\varepsilon + \sum\limits_{i=1}^m a_{2i} u_i^\varepsilon + f_2, & (x,t) \in \Omega_T, \\ \cdots \\ \partial_t u_m^\varepsilon = \left((x+\varepsilon)^{\alpha_m} u_{m,x}^\varepsilon\right)_x + \sum\limits_{i=1}^m b_{mi} u_{i,x}^\varepsilon + \sum\limits_{i=1}^m a_{mi} u_i^\varepsilon + f_m, & (x,t) \in \Omega_T, \\ u_j^\varepsilon(0,t) = u_j^\varepsilon(1,t) = 0, \quad t \in (0,T), \quad u_j^\varepsilon(x,0) = u_{j,0}^\varepsilon(x), & x \in \Omega, \quad 1 \le j \le m, \end{cases} \tag{2.3}$$

where $0 < \varepsilon < 1$ and $(u_{1,0}^\varepsilon, \cdots, u_{m,0}^\varepsilon) \in (H^2(\Omega) \cap H_0^1(\Omega))^m$ satisfies

$$u_{j,0}^\varepsilon \to u_{j,0} \quad \text{strongly in } L^2(\Omega) \text{ for } 1 \le j \le m.$$

By the standard theory of parabolic equation, we know that the problem (2.3) has a unique solution $u^\varepsilon := (u_1^\varepsilon, \cdots, u_m^\varepsilon)^T \in (L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega)) \cap L^\infty(0,T; L^2(\Omega)))^m$.

In order to prove that the limit of u^ε is a weak solution of problem (2.1), we need the following uniform estimate

$$\begin{aligned} &\mathrm{Sup}_{t\in(0,T)} \sum_{j=1}^m \int_\Omega |u_j^\varepsilon|^2(x,t)\mathrm{d}x + \sum_{j=1}^m \int_{\Omega_T} (x+\varepsilon)^{\alpha_j} |u_{j,x}^\varepsilon|^2 \mathrm{d}x\mathrm{d}t \\ \le& C \sum_{j=1}^m \left(\|u_{j,0}\|_{L^2(\Omega)}^2 + \|f_j\|_{L^2(\Omega_T)}^2 \right), \end{aligned} \tag{2.4}$$

where C is depending on $m, \Omega, T, \alpha_j, a_{ij}, b_{ij}$, but independent of ε.

To do this, we multiply the equation of u_j^ε by u_j^ε and integrate over Ω_t to obtain

$$\begin{aligned} &\frac{1}{2} \int_\Omega |u_j^\varepsilon|^2(x,t)\mathrm{d}x + \int_{\Omega_t} (x+\varepsilon)^{\alpha_j} |u_{j,x}^\varepsilon|^2 \mathrm{d}x\mathrm{d}t \\ =& \frac{1}{2} \int_\Omega |u_{j,0}^\varepsilon|^2 \mathrm{d}x + \sum_{i=1}^m \int_{\Omega_t} b_{ji} u_{i,x}^\varepsilon u_j^\varepsilon \mathrm{d}x\mathrm{d}t + \sum_{i=1}^m \int_{\Omega_t} a_{ji} u_i^\varepsilon u_j^\varepsilon \mathrm{d}x\mathrm{d}t + \int_{\Omega_t} f_j u_j^\varepsilon \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{2.5}$$

We use the method proposed by Wang and Du in [30] to estimate the second term on the right-hand side of (2.5). Let $\gamma = \max\{\alpha_1, \alpha_2, \cdots, \alpha_m\}$. By Young's inequality, we have

$$\begin{aligned} &\sum_{i=1}^m \int_{\Omega_t} b_{ji} u_{i,x}^\varepsilon u_j^\varepsilon \mathrm{d}x\mathrm{d}t \\ \le& \frac{1}{2m} \sum_{i=1}^m \int_{\Omega_t} (x+\varepsilon)^{\alpha_i} |u_{i,x}^\varepsilon|^2 \mathrm{d}x\mathrm{d}t + C \int_{\Omega_t} (x+\varepsilon)^{-\alpha_i} |u_j^\varepsilon|^2 \mathrm{d}x\mathrm{d}t \\ \le& \frac{1}{2m} \sum_{i=1}^m \int_{\Omega_t} (x+\varepsilon)^{\alpha_i} |u_{i,x}^\varepsilon|^2 \mathrm{d}x\mathrm{d}t + C \int_0^t \int_0^{1-\varepsilon} (x+\varepsilon)^{-\gamma} |u_j^\varepsilon|^2 \mathrm{d}x\mathrm{d}t \\ &+ C \int_0^t \int_{1-\varepsilon}^1 (x+\varepsilon)^{-\gamma} \frac{(x+\varepsilon)^{-\alpha_i}}{(x+\varepsilon)^{-\gamma}} |u_j^\varepsilon|^2 \mathrm{d}x\mathrm{d}t \\ \le& \frac{1}{2m} \sum_{i=1}^m \int_{\Omega_t} (x+\varepsilon)^{\alpha_i} |u_{i,x}^\varepsilon|^2 \mathrm{d}x\mathrm{d}t + C \int_{\Omega_t} (x+\varepsilon)^{-\gamma} |u_j^\varepsilon|^2 \mathrm{d}x\mathrm{d}t, \end{aligned} \tag{2.6}$$

where we have used $(x+\varepsilon)^{\gamma-\alpha_i} \le 2^{\gamma-\alpha_i}$ for $x \in (1-\varepsilon, 1)$, due to $0<\varepsilon<1$. On the other hand, for a sufficiently small parameter $\kappa > 0$ which will be specified below, we have

$$\begin{aligned}
&\int_{\Omega_t} (x+\varepsilon)^{-\gamma}|u_j^\varepsilon|^2 \mathrm{d}x\mathrm{d}t \\
\le& \int_0^t \int_0^\kappa (x+\varepsilon)^{-\gamma}|u_j^\varepsilon|^2 \mathrm{d}x\mathrm{d}t + \kappa^{-\gamma}\int_{\Omega_t} |u_j^\varepsilon|^2\mathrm{d}x\mathrm{d}t \\
\le& \int_0^t \int_0^\kappa (x+\varepsilon)^{-\gamma}\left|\int_0^x u_{j,x}^\varepsilon(\zeta,t)\mathrm{d}\zeta\right|^2 \mathrm{d}x\mathrm{d}t + \kappa^{-\gamma}\int_{\Omega_t} |u_j^\varepsilon|^2\mathrm{d}x\mathrm{d}t \\
\le& \frac{1}{1-\alpha_j}\int_0^t \int_0^\kappa (x+\varepsilon)^{1-\alpha_j-\gamma}\left(\int_0^x (\zeta+\varepsilon)^{\alpha_j}|u_{j,x}^\varepsilon|^2(\zeta,t)\mathrm{d}\zeta\right) \mathrm{d}x\mathrm{d}t + \kappa^{-\gamma}\int_{\Omega_t} |u_j^\varepsilon|^2\mathrm{d}x\mathrm{d}t \\
\le& C_{\kappa,\varepsilon}\int_{\Omega_t} (x+\varepsilon)^{\alpha_j}|u_{j,x}^\varepsilon|^2\mathrm{d}x\mathrm{d}t + \kappa^{-\gamma}\int_{\Omega_t} |u_j^\varepsilon|^2\mathrm{d}x\mathrm{d}t
\end{aligned} \tag{2.7}$$

with $C_{\kappa,\varepsilon} = \frac{(\kappa+\varepsilon)^{2-\alpha_j-\gamma}-\varepsilon^{2-\alpha_j-\gamma}}{(1-\alpha_j)(2-\alpha_j-\gamma)}$. For any $\varepsilon \in (0,1)$, we can choose $\kappa > 0$ independent of ε such that $C_{\kappa,\varepsilon} < \frac{1}{4C}$. Then by (2.6) and (2.7), we obtain

$$\begin{aligned}
&\sum_{i=1}^m \int_{\Omega_t} b_{ji}u_{i,x}^\varepsilon u_j^\varepsilon \mathrm{d}x\mathrm{d}t \\
\le& \frac{1}{2m}\sum_{i=1}^m \int_{\Omega_t} (x+\varepsilon)^{\alpha_i}|u_{i,x}^\varepsilon|^2\mathrm{d}x\mathrm{d}t + \frac{1}{4}\int_{\Omega_t} (x+\varepsilon)^{\alpha_j}|u_{j,x}^\varepsilon|^2\mathrm{d}x\mathrm{d}t + C\int_{\Omega_t} |u_j^\varepsilon|^2\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{2.8}$$

Summing (2.5) over j from 1 to m and substituting (2.8) into the resulting inequality, and absorbing the first and second terms on the right-hand side of (2.8) by the terms on the left-hand side of (2.5), we obtain

$$\begin{aligned}
&\sum_{j=1}^m \int_{\Omega} |u_j^\varepsilon|^2(x,t)\mathrm{d}x + \sum_{j=1}^m \int_{\Omega_t} (x+\varepsilon)^{\alpha_j}|u_{j,x}^\varepsilon|^2\mathrm{d}x\mathrm{d}t \\
\le& C\sum_{j=1}^m \left(\int_{\Omega} |u_{j,0}|^2\mathrm{d}x + \int_{\Omega_t} |f_j|^2\mathrm{d}x\mathrm{d}t + \int_{\Omega_t} |u_j^\varepsilon|^2\mathrm{d}x\mathrm{d}t\right), \quad t \in [0,T].
\end{aligned} \tag{2.9}$$

From Gronwall inequality, we deduce (2.4) and then obtain a weak solution $(u_1, \cdots, u_m)$ of problem (2.1) by a standard limit process as $\varepsilon \to 0$ [33]. Moreover, for any solution $(u_1, \cdots, u_m)^T$, by an argument similar to (2.4) (substituting u_j^ε by u_j), we obtain (2.2). The uniqueness of the solution of problem (2.1) is a direct consequence from (2.2). This completes the proof of Theorem 2.1. □

Remark 2.1. By Lemma 5.1 in Appendix and (2.4), we easily obtain

$$\begin{aligned}
&\sum_{j=1}^m \int_{\Omega_T} \left[(x+\varepsilon)^{-\gamma}|u_j^\varepsilon|^2 + (x+\varepsilon)^{\alpha_j}|u_{j,x}^\varepsilon|^2\right]\mathrm{d}x\mathrm{d}t \\
\le& \frac{4}{(\gamma-1)^2}\sum_{j=1}^m \int_{\Omega_T} \left[(x+\varepsilon)^{2-\gamma}|u_{j,x}^\varepsilon|^2 + (x+\varepsilon)^{\alpha_j}|u_{j,x}^\varepsilon|^2\right]\mathrm{d}x\mathrm{d}t \\
\le& C\sum_{j=1}^m \left(\|u_{j,0}\|_{L^2(\Omega)}^2 + \|f_i\|_{L^2(\Omega_T)}^2\right),
\end{aligned} \tag{2.10}$$

where $\gamma = \max\{\alpha_1, \alpha_2, \cdots, \alpha_m\}$ such that $2-\gamma > \alpha_j$ for all $1 \le j \le m$ and C is independent of ε. This uniform estimate ensures that

$$\sum_{j=1}^{m}\int_{\Omega_T}(x+\varepsilon)^{-\gamma}|u_j^\varepsilon|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t$$

included in our Carleman estimate is well-defined. Here the weight function φ is specified in the next section.

3. Carleman estimates

In this section we present and prove a Carleman estimate for the following adjoint system

$$\begin{cases} -\partial_t v_1^\varepsilon = \left((x+\varepsilon)^{\alpha_1} v_{1,x}^\varepsilon\right)_x - \sum\limits_{i=1}^{m}\left(b_{i1}v_i^\varepsilon\right)_x + \sum\limits_{i=1}^{m} a_{i1}v_i^\varepsilon, & (x,t)\in\Omega_T,\\ -\partial_t v_2^\varepsilon = \left((x+\varepsilon)^{\alpha_2} v_{2,x}^\varepsilon\right)_x - \sum\limits_{i=1}^{m}\left(b_{i2}v_i^\varepsilon\right)_x + \sum\limits_{i=1}^{m} a_{i2}v_i^\varepsilon, & (x,t)\in\Omega_T,\\ \cdots \\ -\partial_t v_m^\varepsilon = \left((x+\varepsilon)^{\alpha_m} v_{m,x}^\varepsilon\right)_x - \sum\limits_{i=1}^{m}\left(b_{im}v_i^\varepsilon\right)_x + \sum\limits_{i=1}^{m} a_{im}v_i^\varepsilon, & (x,t)\in\Omega_T,\\ v_j^\varepsilon(0,t) = v_j^\varepsilon(1,t) = 0, \quad t\in(0,T), \quad v_j^\varepsilon(x,T) = v_{j,T}^\varepsilon, & x\in\Omega, \quad 1\le j\le m, \end{cases} \tag{3.1}$$

where $v_{j,T}^\varepsilon \in H^2(\Omega)\cap H_0^1(\Omega)$ satisfies

$$v_{j,T}^\varepsilon \to v_{j,T}, \quad \text{strongly in } L^2(\Omega) \text{ for } 1\le j\le m.$$

In the proof of Carleman estimate for (3.1), the main difficulty is how to absorb the first order couplings. Since we only assume $b_{ij}\in L^\infty(\Omega_T)$ rather than $b_{ij}\in L^\infty(0,T;W^{1,\infty}(\Omega))$ and $x^{-\frac{\alpha_i}{2}}b_{ij}\in L^\infty(\Omega_T)$ as [11], we could not use the existing Carleman estimate for degenerate parabolic equation to absorb the coupling first order terms. For example, applying the Carleman estimate in [4] to each equation of (3.1) and adding up the resulting inequality, we obtain

$$\begin{aligned} &\sum_{j=1}^{m}\int_{\Omega_T} s^3\theta^3(x+\varepsilon)^{2-\alpha_i}|v_j^\varepsilon|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t + \sum_{j=1}^{m}\int_{\Omega_T} s\theta(x+\varepsilon)^{\alpha_j}|v_{j,x}^\varepsilon|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t\\ \le& C\sum_{i,j=1}^{m}\int_{\Omega_T}\left(|\left(b_{ij}v_i^\varepsilon\right)_x|^2 + |a_{ij}v_i^\varepsilon|^2\right)e^{2s\varphi}\mathrm{d}x\mathrm{d}t + C\sum_{j=1}^{m}\int_{\omega_T} s^3\theta^3|v_j^\varepsilon|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{3.2}$$

in which the terms including $(b_{ij}v_i^\varepsilon)_x$ on the right-hand side of (3.2) could not be absorbed by the terms on the left-hand side of (3.2) directly. Based on this reason, we have to prove first a new Carleman estimate for degenerate parabolic equation in Sobolev spaces of negative order. Moreover, we also need that there is the term $\int_{\Omega_T} s^3\theta^3(x+\varepsilon)^{-\alpha_i}|v_i|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t$ on the left-hand side of Carleman estimate to control the remainders after making a dual operation $\langle -\left(b_{ij}v_i^\varepsilon\right)_x, \hat{W}\rangle_{H^{-1}(\Omega),H^1(\Omega)} = \langle (x+\varepsilon)^{-\frac{\alpha_i}{2}}b_{ij}v_i^\varepsilon, (x+\varepsilon)^{\frac{\alpha_i}{2}}\hat{W}_x\rangle_{L^2(\Omega),L^2(\Omega)}$, where $\hat{W}$ is the solution of an adjoint null controllability problem (3.12) below.

3.1. Carleman estimate for a degenerate equation in Sobolev Spaces of negative order

In this subsection, we show a new Carleman estimate for the following degenerate parabolic equation with the terms on right-hand side in $L^2(0,T;H^{-1}(\Omega))$:

$$-w_t - ((x+\varepsilon)^\alpha w_x)_x = B_0 + (B_1)_x, \quad (x,t)\in\Omega_T, \tag{3.3}$$

where w satisfies

$$w(0,t) = w(1,t) = 0, \quad t \in (0,T). \tag{3.4}$$

To do this, we introduce some weight functions. For $\omega = (x_1, x_2)$, we choose $\omega^{(i)} := (x_1^{(i)}, x_2^{(i)})(i = 1, 2, 3)$ such that $\omega^{(3)} \Subset \omega^{(2)} \Subset \omega^{(1)} \Subset \omega$. Let $\chi \in C^2(\overline{\Omega})$ be a cut-off function such that $0 \le \chi(x) \le 1$ for $x \in \Omega$, $\chi(x) \equiv 1$ for $x \in (0, x_1^{(3)})$ and $\chi(x) \equiv 0$ for $x \in (x_2^{(3)}, 1)$. Let

$$\eta_1(x) = (x+\varepsilon)^{1+\beta}$$

with $\beta \in \big(0, \frac{2}{9} - \frac{2\alpha}{3}\big)$ and $\eta_2 \in C^2(\overline{\Omega})$ satisfy

$$\eta_2(x) > 0, \quad x \in \Omega, \quad \eta|_{\partial\Omega} = 0 \quad \text{and} \quad |\eta_{2,x}(x)| > 0, \quad x \in \overline{\Omega \setminus \omega^{(2)}}$$

and

$$\eta_1(x) = \eta_2(x), \quad x \in \omega^{(3)}.$$

Let us define

$$\theta(t) = \frac{1}{t^4(T-t)^4}, \quad \psi_i(x) = e^{\lambda\eta_i(x)} - 2e^{\lambda\|\eta_i\|_{C(\overline{\Omega})}}, \quad \phi_i(x) = e^{\lambda\eta_i(x)}, \quad i = 1, 2,$$

with a positive parameter λ. Now let us define the weight function

$$\varphi(x,t) = \chi(x)\varphi_1(x,t) + (1-\chi(x))\varphi_2(x,t),$$

where

$$\varphi_i(x,t) = \psi_i(x)\theta(t), \quad i = 1, 2.$$

Obviously, for the weight function φ, we have

$$\varphi(x,t) = \begin{cases} \varphi_1(x,t), & (x,t) \in (0, x_1^{(3)}) \times (0,T), \\ \varphi_1(x,t) = \varphi_2(x,t), & (x,t) \in (x_1^{(3)}, x_2^{(3)}) \times (0,T), \\ \varphi_2(x,t), & (x,t) \in (x_2^{(3)}, 1) \times (0,T) \end{cases} \tag{3.5}$$

and

$$-2e^{C\lambda}\theta \le \varphi \le -e^{C\lambda}\theta, \quad |\varphi_t| \le e^{C\lambda}\theta^2, \quad |\varphi_{tt}| \le e^{C\lambda}\theta^3, \quad |\varphi_x| \le C\lambda e^{C\lambda}\theta, \tag{3.6}$$

where C is depending on $\omega, \Omega, T, x_1, x_2, \alpha, \beta$, but independent of ε.

Remark 3.1. The choice of β is used to guarantee $-\gamma > 2\alpha + 3\beta - 1$ for any $\gamma \in \big[\alpha, \frac{1}{3}\big)$, which is necessary to overcome the difficulty from the dual operation $\langle -(b_{ij}v_i^\varepsilon)_x, \hat{W}\rangle_{H^{-1}(\Omega),H^1(\Omega)} = \langle (x+\varepsilon)^{-\frac{\alpha_i}{2}} b_{ij}v_i^\varepsilon, (x+\varepsilon)^{\frac{\alpha_i}{2}}\hat{W}_x\rangle_{L^2(\Omega),L^2(\Omega)}$ in proving the Carleman estimate in Sobolev Spaces of negative order. In order to control the term of $(x+\varepsilon)^{-\alpha_i} b_{ij}v_i^\varepsilon$ with $b_{ij} \in L^\infty(\Omega_T)$, we have to append $\int_{\Omega_T} (x+\varepsilon)^{-\gamma}|v_i^\varepsilon|^2 e^{2s\varphi} \mathrm{d}x\mathrm{d}t$ in our Carleman estimate. Moreover, $-\gamma > 2\alpha + 3\beta - 1$ and $\beta > 0$ only hold for $\alpha \in (0, \frac{1}{3})$. Therefore, the constraint of $\alpha \in (0, \frac{1}{3})$ is essentially caused by $b_{ij} \in L^\infty(\Omega_T)$.

Our main result in this subsection is as follows.

Theorem 3.1. *Let $\alpha \in \left(0, \frac{1}{3}\right)$, $B_0 \in L^2(\Omega_T), (B_1)_x \in L^2(0,T;H^{-1}(\Omega))$ and $d \in \mathbb{N}$ such that $d \geq 3$. Then for any $\varepsilon \in (0,1)$ and any $\gamma \in \left[\alpha, \frac{1}{3}\right)$, there exist positive constants $\lambda_1 = \lambda_1(\omega,\Omega,T,\alpha,\beta)$, $s_1 = s_1(\omega,\Omega,T,\alpha,\beta,\lambda)$ and $C = C(\omega,\Omega,T,\alpha,\beta,\lambda)$ such that*

$$\begin{aligned}\mathcal{I}^{\varepsilon}_{\gamma,\alpha,d}(w) \leq & C\int_{\Omega_T} s^{d-3}\theta^{d-3}|B_0|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t + C\int_{\Omega_T} s^{d-\frac{1}{2}}\theta^{d-\frac{1}{2}}(x+\varepsilon)^{-\gamma}|B_1|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t \\ & + C\int_{\omega_T^{(1)}} s^d\theta^d|w|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t \end{aligned} \tag{3.7}$$

for all $\lambda \geq \lambda_1$, $s \geq s_1$ and all $w \in L^2(0,T;H^1_\alpha(\Omega))$ satisfying (3.3)-(3.4), where

$$\mathcal{I}^{\varepsilon}_{\gamma,\alpha,d}(w) = \int_{\Omega_T} s^d\theta^d(x+\varepsilon)^{-\gamma}|w|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t + \int_{\Omega_T} s^{d-2}\theta^{d-2}(x+\varepsilon)^{\alpha}|w_x|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t. \tag{3.8}$$

Remark 3.2. Notice that here C is independent of ε. Therefore (3.7) is a uniformly Carleman estimate in ε.
Remark 3.3. Carleman estimate (3.7) will be applied to v_j^ε to prove Theorem 3.3 with $B_0 = \sum_{i=1}^m a_{ij}v_i^\varepsilon \in L^2(\Omega_T)$ and $(B_1)_x = -\sum_{i=1}^m (b_{ij}v_i^\varepsilon)_x \in L^2(0,T;H^{-1}(\Omega))$, due to $(v_1^\varepsilon,\cdots,v_m^\varepsilon)^T \in \left(L^2(0,T;H^1_\alpha(\Omega))\right)^m$ for each $\varepsilon > 0$. Such a regularity is the same as the one in our well-posedness in Section 2. In other words, the conditions on B_0, B_1 and w are unified in the equation of v_j^ε.

In order to prove Theorem 3.1, we need a new Carleman estimate for the following approximate version of a degenerate operator

$$\mathcal{L}[y] := -y_t - ((x+\varepsilon)^\alpha y_x)_x, \quad (x,t) \in \Omega_T, \tag{3.9}$$

where y satisfies

$$y(0,t) = y(1,t) = 0, \quad t \in (0,T). \tag{3.10}$$

Lemma 3.2. *Let $\alpha \in \left(0, \frac{1}{3}\right)$ and $d \in \mathbb{N}$ such that $d \geq 3$. Then for any $\varepsilon \in (0,1)$ and any $\gamma \in \left[\alpha, \frac{1}{3}\right)$, there exist positive constants $\lambda_2 = \lambda_2(\omega,\Omega,T,\alpha,\beta)$, $s_2 = s_2(\omega,\Omega,T,\alpha,\beta,\lambda)$ and $C = C(\omega,\Omega,T,\alpha,\beta,\lambda)$ such that*

$$\mathcal{I}^{\varepsilon}_{\gamma,\alpha,d}(y) \leq C\int_{\Omega_T} s^{d-3}\theta^{d-3}|\mathcal{L}[y]|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t + C\int_{\omega_T^{(1)}} s^d\theta^d|y|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t \tag{3.11}$$

for all $\lambda \geq \lambda_2$, $s \geq s_2$ and all $y \in L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))$ satisfying (3.9)-(3.10).

For proving this lemma, we borrow some ideas used to deal with the null controllability of a 2×2 weakly degenerate parabolic systems without the first order coupling terms in [2]. The main difference between our proof and the one in [2] is that we need a new integral term including $(x+\varepsilon)^{-\gamma}|y|^2$ on the left-hand side of (3.11) to prove the Carleman estimate (3.7). Moreover, we need the same weight functions on the left and the right-hand side of (3.7) to absorb the coupling terms. To do this, we applied the method in [7] or [4] to split the proof into degenerate part and nondegenerate part. The proof of this lemma is detailed in Appendix and omitted here.

Now we prove Theorem 3.1, which is based on a duality argument introduced by Imanuvilov and Yamamoto [19].

Proof of Theorem 3.1. Let us consider the following null controllability problem

$$\begin{cases} \hat{W}_t - \Big((x+\varepsilon)^\alpha \hat{W}_x\Big)_x = s^d\theta^d(x+\varepsilon)^{-\gamma} w e^{2s\varphi} + \hat{h}\mathbf{1}_{\omega^{(1)}}, & (x,t)\in\Omega_T,\\ \hat{W}(0,t)=\hat{W}(1,t)=0, & t\in(0,T),\\ \hat{W}(x,0)=\hat{W}(x,T)=0, & x\in\Omega. \end{cases} \tag{3.12}$$

We define the space

$$E_0 = \{\zeta\in C^\infty(\overline{\Omega}_T)\mid \zeta(0,t)=\zeta(1,t)=0,\ t\in(0,T)\}.$$

We introduce a bilinear form $\kappa: E_0\times E_0\to\mathbb{R}$ defined by

$$\kappa(\zeta_1,\zeta_2)=\int_{\Omega_T} s^{d-3}\theta^{d-3}\mathcal{L}[\zeta_1]\mathcal{L}[\zeta_2]e^{2s\varphi}\mathrm{d}x\mathrm{d}t + \int_{\omega_T^{(1)}} s^d\theta^d\zeta_1\zeta_2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t$$

and a linear form $l: E_0\to\mathbb{R}$ defined by

$$l(\zeta)=\int_{\Omega_T} s^d\theta^d(x+\varepsilon)^{-\gamma} w\zeta e^{2s\varphi}\mathrm{d}x\mathrm{d}t.$$

By Carleman estimate (3.11), we obtain that $\kappa(\cdot,\cdot)^{\frac{1}{2}}$ defines a norm in the space E_0. Then we introduce E to denote the closure of E_0 with respect to the norm $\kappa(\cdot,\cdot)^{\frac{1}{2}}$, *i.e.* $\|\zeta\|_E=\kappa(\zeta,\zeta)^{\frac{1}{2}}$, which is a Hilbert space with the inner product $\kappa(\cdot,\cdot)$. Obviously, the bilinear form κ is bounded and coercive. Moreover, by applying Carleman estimate (3.11) to ζ, we have

$$\begin{aligned} |l(\zeta)| \le& \left(\int_{\Omega_T} s^d\theta^d(x+\varepsilon)^{-\gamma}|w|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_{\Omega_T} s^d\theta^d(x+\varepsilon)^{-\gamma}|\zeta|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}}\\ \le& C\left(\int_{\Omega_T} s^d\theta^d(x+\varepsilon)^{-\gamma}|w|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}}\kappa(\zeta,\zeta)^{\frac{1}{2}}, \end{aligned} \tag{3.13}$$

which implies

$$|l(\zeta)|\le C\|\zeta\|_E. \tag{3.14}$$

Here we have used

$$\left(\int_{\Omega_T}(x+\varepsilon)^{-\gamma}|w|^2\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}}\le\|w\|_{L^2(0,T;H^1_\alpha(\Omega))}$$

by Remark 2.1.

Therefore, by Lax-Milgram's lemma, we obtain that for any $0<\varepsilon<1$, there exists a unique $\hat{\zeta}\in E$ such that

$$\kappa(\hat{\zeta},\zeta)=l(\zeta),\quad \forall\zeta\in E. \tag{3.15}$$

That is

$$\mathcal{L}^*\left[s^{d-3}\theta^{d-3}\mathcal{L}[\hat{\zeta}]e^{2s\varphi}\right]=s^d\theta^d(x+\varepsilon)^{-\gamma}we^{2s\varphi}-s^d\theta^d\hat{\zeta}e^{2s\varphi}\mathbf{1}_{\omega^{(1)}} \tag{3.16}$$

in the distribution sense, where $\mathcal{L}^*$ is the dual operator of $\mathcal{L}$, *i.e.* $\mathcal{L}^*[y] = y_t - ((x+\varepsilon)^\alpha y_x)_x$. Therefore, we find that

$$(\hat{W}, \hat{h}) := (s^{d-3}\theta^{d-3}\mathcal{L}[\hat{\zeta}]e^{2s\varphi}, -s^d\theta^d\hat{\zeta}e^{2s\varphi})$$

is a solution of the null controllability problem (3.12). Further, letting $\zeta = \hat{\zeta}$ in (3.15) and using (3.11), (3.13) and Young's inequality, we find that

$$\begin{aligned}
&\kappa(\hat{\zeta},\hat{\zeta}) = l(\hat{\zeta}) \\
\le&\epsilon_1 \int_{\Omega_T} s^d\theta^d(x+\varepsilon)^{-\gamma}|\hat{\zeta}|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t + C(\epsilon_1)\int_{\Omega_T} s^d\theta^d(x+\varepsilon)^{-\gamma}|w|^2e^{2s\varphi}\mathrm{d}x\mathrm{d}t \\
\le&\epsilon_1 C\kappa(\hat{\zeta},\hat{\zeta}) + C(\epsilon_1)\int_{\Omega_T} s^d\theta^d(x+\varepsilon)^{-\gamma}|w|^2e^{2s\varphi}\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{3.17}$$

Choosing ϵ_1 sufficiently small such that $\epsilon_1 C < \frac{1}{2}$, we obtain that

$$\begin{aligned}
&\int_{\Omega_T} s^{-d+3}\theta^{-d+3}|\hat{W}|^2e^{-2s\varphi}\mathrm{d}x\mathrm{d}t + \int_{\omega_T^{(1)}} s^{-d}\theta^{-d}|\hat{h}|^2e^{-2s\varphi}\mathrm{d}x\mathrm{d}t \\
\le& C\int_{\Omega_T} s^d\theta^d(x+\varepsilon)^{-\gamma}|w|^2e^{2s\varphi}\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{3.18}$$

Now we multiply the equation of $\hat{W}$ in (3.12) by w. After integration by parts, we obtain the following duality between w and $\hat{W}$:

$$\int_{\Omega_T} w\left(s^d\theta^d(x+\varepsilon)^{-\gamma}we^{2s\varphi} + \hat{h}\mathbf{1}_{\omega^{(1)}}\right)\mathrm{d}x\mathrm{d}t = \int_{\Omega_T} B_0\hat{W}\mathrm{d}x\mathrm{d}t - \int_{\Omega_T} B_1\hat{W}_x\mathrm{d}x\mathrm{d}t. \tag{3.19}$$

By Young's inequality, we further find that

$$\begin{aligned}
&\int_{\Omega_T} s^d\theta^d(x+\varepsilon)^{-\gamma}|w|^2e^{2s\varphi}\mathrm{d}x\mathrm{d}t \\
\le&\epsilon_2\int_{\Omega_T} s^{-d+3}\theta^{-d+3}|\hat{W}|^2e^{-2s\varphi}\mathrm{d}x\mathrm{d}t + \epsilon_2\int_{\omega_T^{(1)}} s^{-d}\theta^{-d}|\hat{h}|^2e^{-2s\varphi}\mathrm{d}x\mathrm{d}t \\
&+ C(\epsilon_2)\int_{\omega_T^{(1)}} s^d\theta^d|w|^2e^{2s\varphi}\mathrm{d}x\mathrm{d}t + \epsilon_3\int_{\Omega_T} s^{-d+\frac{1}{2}}\theta^{-d+\frac{1}{2}}(x+\varepsilon)^{\gamma}|\hat{W}_x|^2e^{-2s\varphi}\mathrm{d}x\mathrm{d}t \\
&+ C(\epsilon_2)\int_{\Omega_T} s^{d-3}\theta^{d-3}|B_0|^2e^{2s\varphi}\mathrm{d}x\mathrm{d}t + C(\epsilon_3)\int_{\Omega_T} s^{d-\frac{1}{2}}\theta^{d-\frac{1}{2}}(x+\varepsilon)^{-\gamma}|B_1|^2e^{2s\varphi}\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{3.20}$$

Substituting (3.18) into (3.20) and choosing ϵ_2 sufficiently small, we have

$$\begin{aligned}
&\int_{\Omega_T} s^d\theta^d(x+\varepsilon)^{-\gamma}|w|^2e^{2s\varphi}\mathrm{d}x\mathrm{d}t \\
\le& C\int_{\omega_T^{(1)}} s^d\theta^d|w|^2e^{2s\varphi}\mathrm{d}x\mathrm{d}t + \epsilon_3 C\int_{\Omega_T} s^{-d+\frac{1}{2}}\theta^{-d+\frac{1}{2}}(x+\varepsilon)^{\gamma}|\hat{W}_x|^2e^{-2s\varphi}\mathrm{d}x\mathrm{d}t \\
&+ C\int_{\Omega_T} s^{d-3}\theta^{d-3}|B_0|^2e^{2s\varphi}\mathrm{d}x\mathrm{d}t + C(\epsilon_3)\int_{\Omega_T} s^{d-\frac{1}{2}}\theta^{d-\frac{1}{2}}(x+\varepsilon)^{-\gamma}|B_1|^2e^{2s\varphi}\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{3.21}$$

Next, we estimate $\int_{\Omega_T} s^{-d+\frac{1}{2}}\theta^{-d+\frac{1}{2}}(x+\varepsilon)^{\gamma}|\hat{W}_x|^2e^{-2s\varphi}\mathrm{d}x\mathrm{d}t$. To do this, we multiply the equation of $\hat{W}$ in (3.12) by $s^{-d+\frac{1}{2}}\theta^{-d+\frac{1}{2}}\hat{W}e^{-2s\varphi}$ and use (3.6) to obtain

$$\begin{aligned}
&\int_{\Omega_T} s^{-d+\frac{1}{2}}\theta^{-d+\frac{1}{2}}(x+\varepsilon)^{\alpha}|\hat{W}_x|^2e^{-2s\varphi}\mathrm{d}x\mathrm{d}t\\
=&\int_{\Omega_T} s^{-d+\frac{1}{2}}\theta^{-d+\frac{1}{2}}\hat{W}e^{-2s\varphi}\left(-\hat{W}_t+s^d\theta^d(x+\varepsilon)^{-\gamma}we^{2s\varphi}+\hat{h}\mathbf{1}_{\omega^{(1)}}\right)\mathrm{d}x\mathrm{d}t\\
&-\int_{\Omega_T} s^{-d+\frac{1}{2}}\theta^{-d+\frac{1}{2}}(x+\varepsilon)^{\alpha}\left(e^{-2s\varphi}\right)_x\hat{W}\hat{W}_x\mathrm{d}x\mathrm{d}t\\
\le&\int_{\Omega_T} s^{-d+3}\theta^{-d+3}|\hat{W}|^2e^{-2s\varphi}\mathrm{d}x\mathrm{d}t+\frac{1}{2}\int_{\Omega_T} s^{d-2}\theta^{d-2}(x+\varepsilon)^{-2\gamma}|w|^2e^{2s\varphi}\mathrm{d}x\mathrm{d}t\\
&+\int_{\omega_T^{(1)}} s^{-d}\theta^{-d}|\hat{h}|^2e^{-2s\varphi}\mathrm{d}x\mathrm{d}t+\frac{1}{2}\int_{\Omega_T} s^{-d+\frac{1}{2}}\theta^{-d+\frac{1}{2}}(x+\varepsilon)^{\alpha}|\hat{W}_x|^2e^{-2s\varphi}\mathrm{d}x\mathrm{d}t
\end{aligned}\tag{3.22}$$

for all sufficiently large s. By using (3.18) and (3.22), and noticing that $(x+\varepsilon)^{\gamma}\le C(x+\varepsilon)^{\alpha}$ due to $\gamma\ge\alpha$, we obtain

$$\begin{aligned}
&\int_{\Omega_T} s^{-d+\frac{1}{2}}\theta^{-d+\frac{1}{2}}(x+\varepsilon)^{\gamma}|\hat{W}_x|^2e^{-2s\varphi}\mathrm{d}x\mathrm{d}t\\
\le&\int_{\Omega_T} s^{d-2}\theta^{d-2}(x+\varepsilon)^{-2\gamma}|w|^2e^{2s\varphi}\mathrm{d}x\mathrm{d}t+C\int_{\Omega_T} s^{d}\theta^{d}(x+\varepsilon)^{-\gamma}|w|^2e^{2s\varphi}\mathrm{d}x\mathrm{d}t.
\end{aligned}\tag{3.23}$$

Substituting (3.23) into (3.21) and choosing ϵ_3 sufficiently small such that $\epsilon_3 C^2<\frac{1}{2}$ to absorb the second term on the right-hand side of (3.23) by the term on the left-hand side of (3.21), we obtain

$$\begin{aligned}
&\int_{\Omega_T} s^{d}\theta^{d}(x+\varepsilon)^{-\gamma}|w|^2e^{2s\varphi}\mathrm{d}x\mathrm{d}t\\
\le&\epsilon_3 C\int_{\Omega_T} s^{d-2}\theta^{d-2}(x+\varepsilon)^{-2\gamma}|w|^2e^{2s\varphi}\mathrm{d}x\mathrm{d}t+C(\epsilon_3)\int_{\Omega_T} s^{d-\frac{1}{2}}\theta^{d-\frac{1}{2}}(x+\varepsilon)^{-\gamma}|B_1|^2e^{2s\varphi}\mathrm{d}x\mathrm{d}t\\
&+C\int_{\Omega_T} s^{d-3}\theta^{d-3}|B_0|^2e^{2s\varphi}\mathrm{d}x\mathrm{d}t+C\int_{\omega_T^{(1)}} s^{d}\theta^{d}|w|^2e^{2s\varphi}\mathrm{d}x\mathrm{d}t.
\end{aligned}\tag{3.24}$$

Since the term $\int_{\Omega_T} s^{d-2}\theta^{d-2}(x+\varepsilon)^{-2\gamma}|w|^2e^{2s\varphi}\mathrm{d}x\mathrm{d}t$ on the right-hand side of (3.24) could not be absorbed directly by the term on the left-hand side of (3.24). We have to use $\int_{\Omega_T} s^{d-2}\theta^{d-2}(x+\varepsilon)^{\alpha}|w_x|^2e^{2s\varphi}\mathrm{d}x\mathrm{d}t$ to eliminate it. To do this, we consider another null controllability problem

$$\begin{cases}
\bar{W}_t-\left((x+\varepsilon)^{\alpha}\bar{W}_x\right)_x=-s^{d-2}\theta^{d-2}\left(e^{2s\varphi}(x+\varepsilon)^{\alpha}w_x\right)_x+\bar{h}\mathbf{1}_{\omega^{(1)}}, & (x,t)\in\Omega_T,\\
\bar{W}(0,t)=\bar{W}(L,t)=0, & t\in(0,T),\\
\bar{W}(x,0)=\bar{W}(x,T)=0, & x\in\Omega.
\end{cases}\tag{3.25}$$

Define linear form $\tilde{l}:E\to\mathbb{R}$ by

$$\tilde{l}(\zeta)=-\int_{\Omega_T} s^{d-2}\theta^{d-2}\left(e^{2s\varphi}(x+\varepsilon)^{\alpha}w_x\right)_x\zeta\mathrm{d}x\mathrm{d}t.$$

Applying Carleman estimate (3.11) to ζ, we have

$$
\begin{aligned}
&|\tilde{l}(\zeta)|\\
=&\int_{\Omega_T} s^{d-2}\theta^{d-2}(x+\varepsilon)^{\alpha} w_x\zeta_x e^{2s\varphi}\mathrm{d}x\mathrm{d}t\\
\leq&\left(\int_{\Omega_T} s^{d-2}\theta^{d-2}(x+\varepsilon)^{\alpha}|w_x|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_{\Omega_T} s^{d-2}\theta^{d-2}(x+\varepsilon)^{\alpha}|\zeta_x|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}}\\
\leq&C\|w\|_{L^2(0,T;H^1_\alpha(\Omega))}\|\zeta\|_E.
\end{aligned}
\tag{3.26}
$$

Then by an argument similar to (3.13)-(3.18), we can obtain that there exists a solution $(\bar{W},\bar{h})$ of (3.25) such that

$$
\begin{aligned}
&\int_{\Omega_T} s^{-d+3}\theta^{-d+3}|\bar{W}|^2 e^{-2s\varphi}\mathrm{d}x\mathrm{d}t+\int_{\omega_T^{(1)}} s^{-d}\theta^{-d}|\bar{h}|^2 e^{-2s\varphi}\mathrm{d}x\mathrm{d}t\\
\leq&C\int_{\Omega_T} s^{d-2}\theta^{d-2}(x+\varepsilon)^{\alpha}|w_x|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{3.27}
$$

We multiply the equation of $\bar{W}$ in (3.25) by w and integrate by parts to obtain

$$
\int_{\Omega_T} s^{d-2}\theta^{d-2}(x+\varepsilon)^{\alpha}|w_x|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t=\int_{\Omega_T} w\left(\bar{W}_t-\left((x+\varepsilon)^{\alpha}\bar{W}_x\right)_x-\bar{h}\mathbf{1}_{\omega^{(1)}}\right).
\tag{3.28}
$$

By applying the equation of w and Young's inequality again, we further have

$$
\begin{aligned}
&\int_{\Omega_T} s^{d-2}\theta^{d-2}(x+\varepsilon)^{\alpha}|w_x|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t\\
=&\int_{\Omega_T} B_0\bar{W}\mathrm{d}x\mathrm{d}t-\int_{\Omega_T} B_1\bar{W}_x\mathrm{d}x\mathrm{d}t-\int_{\omega_T^{(1)}}\bar{h}w\mathrm{d}x\mathrm{d}t\\
\leq&\epsilon_4\int_{\Omega_T} s^{-d+3}\theta^{-d+3}|\bar{W}|^2 e^{-2s\varphi}\mathrm{d}x\mathrm{d}t+\epsilon_4\int_{\omega_T^{(1)}} s^{-d}\theta^{-d}|\bar{h}|^2 e^{-2s\varphi}\mathrm{d}x\mathrm{d}t\\
&+\epsilon_5\int_{\Omega_T} s^{-d+1}\theta^{-d+1}(x+\varepsilon)^{\gamma}|\bar{W}_x|^2 e^{-2s\varphi}\mathrm{d}x\mathrm{d}t+C(\epsilon_4)\int_{\omega_T^{(1)}} s^{d}\theta^{d}|w|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t\\
&+C(\epsilon_4)\int_{\Omega_T} s^{d-3}\theta^{d-3}|B_0|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t+C(\epsilon_5)\int_{\Omega_T} s^{d-1}\theta^{d-1}(x+\varepsilon)^{-\gamma}|B_1|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{3.29}
$$

Using (3.27) and choosing ϵ_4 sufficiently small, we have

$$
\begin{aligned}
&\int_{\Omega_T} s^{d-2}\theta^{d-2}(x+\varepsilon)^{\alpha}|w_x|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t\\
\leq&\epsilon_5 C\int_{\Omega_T} s^{-d+1}\theta^{-d+1}(x+\varepsilon)^{\gamma}|\bar{W}_x|^2 e^{-2s\varphi}\mathrm{d}x\mathrm{d}t+C\int_{\omega_T^{(1)}} s^{d}\theta^{d}|w|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t\\
&+C\int_{\Omega_T} s^{d-3}\theta^{d-3}|B_0|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t+C(\epsilon_5)\int_{\Omega_T} s^{d-1}\theta^{d-1}(x+\varepsilon)^{-\gamma}|B_1|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{3.30}
$$

Multiplying the equation of $\bar{W}$ in (3.25) by $s^{-d+1}\theta^{-d+1}\bar{W}e^{-2s\varphi}$ and using Young's inequality, we obtain

$$\begin{aligned}
&\int_{\Omega_T} s^{-d+1}\theta^{-d+1}(x+\varepsilon)^{\alpha}|\bar{W}_x|^2 e^{-2s\varphi}\mathrm{d}x\mathrm{d}t\\
=&-\int_{\Omega_T} s^{-d+1}\theta^{-d+1}(x+\varepsilon)^{\alpha}\bar{W}\bar{W}_x\left(e^{-2s\varphi}\right)_x\mathrm{d}x\mathrm{d}t-\int_{\Omega_T} s^{-d+1}\theta^{-d+1}\bar{W}\bar{W}_t e^{-2s\varphi}\mathrm{d}x\mathrm{d}t\\
&+\int_{\Omega_T} s^{-1}\theta^{-1}(x+\varepsilon)^{\alpha}w_x\left(\bar{W}e^{-2s\varphi}\right)_x e^{2s\varphi}\mathrm{d}x\mathrm{d}t+\int_{\omega_T^{(1)}} s^{-d+1}\theta^{-d+1}\bar{W}he^{-2s\varphi}\mathrm{d}x\mathrm{d}t\\
\le&C\int_{\Omega_T} s^{-d+3}\theta^{-d+3}|\bar{W}|^2 e^{-2s\varphi}\mathrm{d}x\mathrm{d}t+C\int_{\omega_T^{(1)}} s^{-d}\theta^{-d}|\bar{h}|^2 e^{-2s\varphi}\mathrm{d}x\mathrm{d}t\\
&+\frac{1}{2}\int_{\Omega_T} s^{-d+1}\theta^{-d+1}(x+\varepsilon)^{\alpha}|\bar{W}_x|^2 e^{-2s\varphi}\mathrm{d}x\mathrm{d}t+C\int_{\Omega_T} s^{d-3}\theta^{d-3}(x+\varepsilon)^{\alpha}|w_x|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t.
\end{aligned}\tag{3.31}$$

Then substituting (3.31) into (3.30) and applying $(x+\varepsilon)^{\gamma}\le C(x+\varepsilon)^{\alpha}$ due to $\gamma\ge\alpha$ yields

$$\begin{aligned}
&\int_{\Omega_T} s^{d-2}\theta^{d-2}(x+\varepsilon)^{\alpha}|w_x|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t\\
\le&\epsilon_5 C\left(\int_{\Omega_T} s^{-d+3}\theta^{-d+3}|\bar{W}|^2 e^{-2s\varphi}\mathrm{d}x\mathrm{d}t+\int_{\omega_T^{(1)}} s^{-d}\theta^{-d}|\bar{h}|^2 e^{-2s\varphi}\mathrm{d}x\mathrm{d}t\right)\\
&+C\int_{\Omega_T} s^{d-3}\theta^{d-3}|B_0|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t+C(\epsilon_5)\int_{\Omega_T} s^{d-1}\theta^{d-1}(x+\varepsilon)^{-\gamma}|B_1|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t\\
&+C\int_{\omega_T^{(1)}} s^{d}\theta^{d}|w|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

which leads to

$$\begin{aligned}
&\int_{\Omega_T} s^{d-2}\theta^{d-2}(x+\varepsilon)^{\alpha}|w_x|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t\\
\le&C\int_{\Omega_T} s^{d-3}\theta^{d-3}|B_0|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t+C\int_{\Omega_T} s^{d-1}\theta^{d-1}(x+\varepsilon)^{-\gamma}|B_1|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t\\
&+C\int_{\omega_T^{(1)}} s^{d}\theta^{d}|w|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t,
\end{aligned}\tag{3.32}$$

if ϵ_5 is chosen to be sufficiently small.

From (3.24) and (3.32), it follows that

$$\begin{aligned}
&\int_{\Omega_T} s^{d}\theta^{d}(x+\varepsilon)^{-\gamma}|w|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t+\int_{\Omega_T} s^{d-2}\theta^{d-2}(x+\varepsilon)^{\alpha}|w_x|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t\\
\le&\epsilon_3 C\int_{\Omega_T} s^{d-2}\theta^{d-2}(x+\varepsilon)^{-2\gamma}|w|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t+C\int_{\Omega_T} s^{d-3}\theta^{d-3}|B_0|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t\\
&+C(\epsilon_3)\int_{\Omega_T} s^{d-\frac{1}{2}}\theta^{d-\frac{1}{2}}(x+\varepsilon)^{-\gamma}|B_1|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t+C\int_{\omega_T^{(1)}} s^{d}\theta^{d}|w|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t.
\end{aligned}\tag{3.33}$$

Using Lemma 5.1, (3.6) and $2-2\gamma>\alpha$, we have

$$\begin{aligned}
&\int_{\Omega_T} s^{d-2}\theta^{d-2}(x+\varepsilon)^{-2\gamma}|w|^2e^{2s\varphi}\mathrm{d}x\mathrm{d}t\\
\leq&\frac{4}{(2\gamma-1)^2}\int_{\Omega_T} s^{d-2}\theta^{d-2}(x+\varepsilon)^{2-2\gamma}\left|(we^{s\varphi})_x\right|^2\mathrm{d}x\mathrm{d}t\\
\leq&C\int_{\Omega_T} s^{d}\theta^{d}(x+\varepsilon)^{-\gamma}|w|^2e^{2s\varphi}\mathrm{d}x\mathrm{d}t+C\int_{\Omega_T} s^{d-2}\theta^{d-2}(x+\varepsilon)^{\alpha}|w_x|^2e^{2s\varphi}\mathrm{d}x\mathrm{d}t. \qquad (3.34)
\end{aligned}$$

Finally, substituting (3.34) into (3.33) and choosing ϵ_3 sufficiently small to absorb the terms on the right-hand side of (3.34) by the terms on the left-hand side of (3.33), we then obtain (3.7). This completes the proof of Theorem 3.1. □

3.2. Carleman estimate for the approximate adjoint system

In this subsection, we will prove a uniform Carleman estimate in ε for the approximate nondegenerate adjoint system (3.1), *i.e.* the following Theorem 3.3.

Theorem 3.3. *Let (A1)-(A3) be held and $\gamma=\max\{\alpha_1,\alpha_2,\cdots,\alpha_m\}$, $\beta=\frac{2}{9}-\frac{2}{3}\gamma$. Then for any $\varepsilon\in(0,1)$, there exist positive constants $\tilde{l}_1=\tilde{l}_1(m)$, $\lambda_3=\lambda_3(m,\omega,\Omega,T,\alpha_j,a_{ij},b_{ij})$, $s_3=s_3(m,\omega,\Omega,T,\alpha_j,a_{ij},b_{ij},\lambda)$ and $C=C(m,\omega,\Omega,T,\alpha_j,a_{ij},b_{ij},\lambda)$ such that*

$$\sum_{j=1}^{m}\mathcal{I}^{\varepsilon}_{\gamma,\alpha_j,3(m+1-j)}(v_j^{\varepsilon})\leq C\int_{\omega_T} s^{\tilde{l}_1}\theta^{\tilde{l}_1}|v_1^{\varepsilon}|^2e^{2s\varphi}\mathrm{d}x\mathrm{d}t \qquad (3.35)$$

for all $\lambda\geq\lambda_3$, $s\geq s_3$ and all $(v_1^{\varepsilon},\cdots,v_m^{\varepsilon})^T\in\left(L^2(0,T;H^2(\Omega))\cap H^1(0,T;L^2(\Omega))\right)^m$ satisfy (3.1).

Remark 3.3. Because system (3.1) is linear and nondegenerate, by the standard theory for parabolic equation we could obtain the existence and uniqueness of solution $(v_1^{\varepsilon},\cdots,v_m^{\varepsilon})^T\in\left(L^2(0,T;H^2(\Omega))\cap H^1(0,T;L^2(\Omega))\right)^m$. Therefore, the regularity of $(v_1^{\varepsilon},\cdots,v_m^{\varepsilon})^T$ we assumed in Theorem 3.3 is reasonable.

We are now in a position to prove Theorem 3.3. The proof follows the ideas already used to deal with nondegenerate parabolic system in [17] or [12]. However in our degenerate case we need a uniform Carleman estimate in ε. All along the following proof, C will be a generic constant depending on $m,\omega,\Omega,T,\alpha_j,a_{ij},b_{ij}$ and λ, but independent of ε.

Proof of Theorem 3.3. Notice that $\alpha_j<\gamma<\frac{1}{3}$ and $\beta\in\left(0,\frac{2}{9}-\frac{2}{3}\alpha_j\right)$ for all $1\leq j\leq m$. Then we can apply Carleman estimate (3.7) to v_j^{ε} with $d=3(m+1-j)$, and use the structure of A and B to obtain

$$\begin{aligned}
&\mathcal{I}^{\varepsilon}_{\gamma,\alpha_j,3(m+1-j)}(v_j^{\varepsilon})\\
\leq&C\sum_{i=1}^{m}\|a_{ij}\|^2_{L^\infty(\Omega_T)}\int_{\Omega_T} s^{3(m-j)}\theta^{3(m-j)}|v_i^{\varepsilon}|^2e^{2s\varphi}\mathrm{d}x\mathrm{d}t\\
&+C\sum_{i=1}^{m}\|b_{ij}\|^2_{L^\infty(\Omega_T)}\int_{\Omega_T} s^{3(m+1-j)-\frac{1}{2}}\theta^{3(m+1-j)-\frac{1}{2}}(x+\varepsilon)^{-\gamma}|v_i^{\varepsilon}|^2e^{2s\varphi}\mathrm{d}x\mathrm{d}t\\
&+C\int_{\omega_T^{(1)}} s^{3(m+1-j)}\theta^{3(m+1-j)}|v_j^{\varepsilon}|^2e^{2s\varphi}\mathrm{d}x\mathrm{d}t\\
\leq&C\sum_{i=1}^{j}\int_{\Omega_T} s^{3(m-j)}\theta^{3(m-j)}|v_i^{\varepsilon}|^2e^{2s\varphi}\mathrm{d}x\mathrm{d}t+C^{(j)}\int_{\Omega_T} s^{3(m-j)}\theta^{3(m-j)}|v_{j+1}^{\varepsilon}|^2e^{2s\varphi}\mathrm{d}x\mathrm{d}t
\end{aligned}$$

$$+ C\sum_{i=1}^{j}\int_{\Omega_T} s^{3(m+1-j)-\frac{1}{2}}\theta^{3(m+1-j)-\frac{1}{2}}(x+\varepsilon)^{-\gamma}|v_i^\varepsilon|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t$$
$$+ C\int_{\omega_T^{(1)}} s^{3(m+1-j)}\theta^{3(m+1-j)}|v_j^\varepsilon|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t, \qquad 1\le j\le m \tag{3.36}$$

for all large s. Here and in the following, we set $v_{m+1}^\varepsilon = 0$. Obviously, we have $(x+\varepsilon)^{-\gamma} > \frac{1}{2}$ for any $\varepsilon \in (0,1)$. Then multiplying (3.36) by $2^{j-1}C^{(1)}C^{(2)}\cdots C^{(j-1)}$ when $j\ge 2$, and then summing up the results over j from 1 to m, we find that

$$\sum_{j=1}^{m}\mathcal{I}^\varepsilon_{\gamma,\alpha_j,3(m+1-j)}(v_j^\varepsilon) \le C\sum_{j=1}^{m}\int_{\omega_T^{(1)}} s^{3(m+1-j)}\theta^{3(m+1-j)}|v_j^\varepsilon|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t. \tag{3.37}$$

In order to obtain the null controllability by a single control, we need to eliminate the local integral $\int_{\omega_T^{(1)}} s^{3(m+1-j)}\theta^{3(m+1-j)}|v_j^\varepsilon|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t$ for $2\le j\le m$ in (3.37). To do this, we first prove that for any $\epsilon>0$ and any $l\in\mathbb{N}$, there exists $l_j = l_j(l,m,j)$ such that

$$\begin{aligned}&\int_{\mathcal{O}_T^{(2)}} s^l\theta^l|v_j^\varepsilon|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t\\ \le&\epsilon\mathcal{I}^\varepsilon_{\gamma,\alpha_{j-1},3(m+1-(j-1))}(v_{j-1}^\varepsilon)+\epsilon\mathcal{I}^\varepsilon_{\gamma,\alpha_j,3(m+1-j)}(v_j^\varepsilon)+\epsilon\mathcal{I}^\varepsilon_{\gamma,\alpha_{j+1},3(m+1-(j+1))}(v_{j+1}^\varepsilon)\\ &+C(\epsilon)\sum_{i=1}^{j-1}\int_{\mathcal{O}_T^{(1)}} s^{l_j}\theta^{l_j}|v_i^\varepsilon|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t, \quad 2\le j\le m\end{aligned} \tag{3.38}$$

for any two open sets $\mathcal{O}^{(1)}$ and $\mathcal{O}^{(2)}$ satisfying $\mathcal{O}^{(2)}\Subset\mathcal{O}^{(1)}\Subset\omega$.

Let $\xi\in C^2(\overline{\Omega})$ such that $0\le\xi\le 1$ in Ω, $\xi\equiv 1$ in $\mathcal{O}^{(2)}$, $\xi\equiv 0$ in $\overline{\Omega\setminus\mathcal{O}^{(1)}}$ and

$$\frac{\xi_x}{\xi^{\frac{1}{2}}}\in L^\infty(\Omega_T). \tag{3.39}$$

By (A3), we obtain

$$\begin{aligned}&-\partial_t v_{j-1}^\varepsilon - \left((x+\varepsilon)^{\alpha_{j-1}}v_{j-1,x}^\varepsilon\right)_x\\ =&-\sum_{i=1}^{j-1}(b_{i,j-1}v_i^\varepsilon)_x + \sum_{i=1}^{j-1}a_{i,j-1}v_i^\varepsilon + a_{j,j-1}v_j^\varepsilon, \quad 1\le j\le m.\end{aligned} \tag{3.40}$$

Multiplying (3.40) by $s^l\theta^l\xi v_j^\varepsilon e^{2s\varphi}$, we obtain

$$\begin{aligned}&\int_{\Omega_T} s^l\theta^l\xi a_{j,j-1}|v_j^\varepsilon|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t\\ =&\int_{\Omega_T} s^l\theta^l\xi\left[-\partial_t v_{j-1}^\varepsilon - \left((x+\varepsilon)^{\alpha_{j-1}}v_{j-1,x}^\varepsilon\right)_x + \sum_{i=1}^{j-1}(b_{i,j-1}v_i^\varepsilon)_x - \sum_{i=1}^{j-1}a_{i,j-1}v_i^\varepsilon\right]v_j^\varepsilon e^{2s\varphi}\mathrm{d}x\mathrm{d}t\\ =&I_1+\cdots+I_4.\end{aligned} \tag{3.41}$$

For I_1, by integrating by parts with respect to t and using Young's inequality, we have

$$
\begin{aligned}
I_1 =& \int_{\Omega_T} s^l \xi v_{j-1}^{\varepsilon} \left(\theta^l v_j^{\varepsilon} e^{2s\varphi}\right)_t \mathrm{d}x\mathrm{d}t \\
\leq & C\int_{\Omega_T} s^{l+1}\theta^{l+2}\xi |v_{j-1}^{\varepsilon}||v_j^{\varepsilon}| e^{2s\varphi}\mathrm{d}x\mathrm{d}t + \int_{\Omega_T} s^l\theta^l \xi v_{j-1}^{\varepsilon}\partial_t v_j^{\varepsilon} e^{2s\varphi}\mathrm{d}x\mathrm{d}t \\
\leq & \frac{\epsilon}{8}\mathcal{I}^{\varepsilon}_{\gamma,\alpha_j,3(m+1-j)}(v_j^{\varepsilon}) + C(\epsilon)\int_{\Omega_T} s^{l_j^{(1)}}\theta^{l_j^{(1)}}\xi|v_{j-1}^{\varepsilon}|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t \\
& + \int_{\Omega_T} s^l\theta^l \xi v_{j-1}^{\varepsilon}\partial_t v_j^{\varepsilon} e^{2s\varphi}\mathrm{d}x\mathrm{d}t
\end{aligned} \tag{3.42}
$$

with $l_j^{(1)} = 2(l+2) - 3(m+1-j)$. To control the last term on the right-hand side of (3.42), we multiply the equation of v_j^{ε} by $s^l\theta^l\xi v_{j-1}^{\varepsilon}e^{2s\varphi}$ to obtain

$$
\begin{aligned}
& \int_{\Omega_T} s^l\theta^l \xi v_{j-1}^{\varepsilon}\partial_t v_j^{\varepsilon} e^{2s\varphi}\mathrm{d}x\mathrm{d}t \\
=& \int_{\Omega_T} s^l\theta^l\xi\left[-\left((x+\varepsilon)^{\alpha_j} v_{j,x}^{\varepsilon}\right)_x + \sum_{i=1}^{j-1}(b_{ij}v_i^{\varepsilon})_x + (b_{jj}v_j^{\varepsilon})_x\right] v_{j-1}^{\varepsilon}e^{2s\varphi}\mathrm{d}x\mathrm{d}t \\
& + \int_{\Omega_T} s^l\theta^l\xi\left[-\sum_{i=1}^{j-1} a_{ij}v_i^{\varepsilon} - a_{jj}v_j^{\varepsilon} - a_{j+1,j}v_{j+1}^{\varepsilon}\right] v_{j-1}^{\varepsilon}e^{2s\varphi}\mathrm{d}x\mathrm{d}t \\
=& \int_{\Omega_T} s^l\theta^l\left[(x+\varepsilon)^{\alpha_j} v_{j,x}^{\varepsilon} - \sum_{i=1}^{j-1} b_{ij}v_i^{\varepsilon} - b_{jj}v_j^{\varepsilon}\right]\left(\xi v_{j-1}^{\varepsilon}e^{2s\varphi}\right)_x\mathrm{d}x\mathrm{d}t \\
& + \int_{\Omega_T} s^l\theta^l\xi\left[-\sum_{i=1}^{j-1} a_{ij}v_i^{\varepsilon} - a_{jj}v_j^{\varepsilon} - a_{j+1,j}v_{j+1}^{\varepsilon}\right] v_{j-1}^{\varepsilon}e^{2s\varphi}\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{3.43}
$$

Using (3.6), (3.39) and Young's inequality, we further obtain

$$
\begin{aligned}
& \int_{\Omega_T} s^l\theta^l \xi v_{j-1}^{\varepsilon}\partial_t v_j^{\varepsilon} e^{2s\varphi}\mathrm{d}x\mathrm{d}t \\
\leq & \int_{\Omega_T} s^{l+1}\theta^{l+1}(x+\varepsilon)^{\alpha_j}\left(\xi|v_{j-1,x}^{\varepsilon}| + \xi|v_{j-1}^{\varepsilon}| + \xi^{\frac{1}{2}}\left|\frac{\xi_x}{\xi^{\frac{1}{2}}}v_{j-1}^{\varepsilon}\right|\right)|v_{j,x}^{\varepsilon}|e^{2s\varphi}\mathrm{d}x\mathrm{d}t \\
& + \sum_{i=1}^{j-1}\int_{\Omega_T} s^{l+1}\theta^{l+1}\left(\xi|v_{j-1,x}^{\varepsilon}| + \xi|v_{j-1}^{\varepsilon}| + \xi^{\frac{1}{2}}\left|\frac{\xi_x}{\xi^{\frac{1}{2}}}v_{j-1}^{\varepsilon}\right|\right)|v_i^{\varepsilon}|e^{2s\varphi}\mathrm{d}x\mathrm{d}t \\
& + C\int_{\Omega_T} s^l\theta^l\xi|v_{j-1,x}^{\varepsilon}||v_j^{\varepsilon}|e^{2s\varphi}\mathrm{d}x\mathrm{d}t + C\int_{\Omega_T} s^l\theta^l\xi\left(|v_j^{\varepsilon}| + |v_{j+1}^{\varepsilon}|\right)|v_{j-1}^{\varepsilon}|e^{2s\varphi}\mathrm{d}x\mathrm{d}t \\
\leq & \frac{\epsilon}{8}\mathcal{I}^{\varepsilon}_{\gamma,\alpha_j,3(m+1-j)}(v_j^{\varepsilon}) + \epsilon\mathcal{I}^{\varepsilon}_{\gamma,\alpha_{j+1},3(m+1-(j+1))}(v_{j+1}^{\varepsilon}) + C(\epsilon)\int_{\Omega_T} s^{l_j^{(2)}}\theta^{l_j^{(2)}}\xi|v_{j-1,x}^{\varepsilon}|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t \\
& + C(\epsilon)\sum_{i=1}^{j-1}\int_{\Omega_T} s^{l_j^{(2)}}\theta^{l_j^{(2)}}\xi^{\frac{1}{2}}|v_i^{\varepsilon}|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t
\end{aligned} \tag{3.44}
$$

with $l_j^{(2)} = \max\{l+1, 2l+1-3(m-j)\}$. Therefore, substituting (3.44) into (3.42) yields

$$
\begin{aligned}
I_1 \leq & \frac{\epsilon}{4}\mathcal{I}^{\varepsilon}_{\gamma,\alpha_j,3(m+1-j)}(v_j^{\varepsilon}) + \epsilon\mathcal{I}^{\varepsilon}_{\gamma,\alpha_{j+1},3(m+1-(j+1))}(v_{j+1}^{\varepsilon}) + C(\epsilon)\int_{\Omega_T} s^{l_j^{(3)}}\theta^{l_j^{(3)}}\xi|v_{j-1,x}^{\varepsilon}|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t \\
& + C(\epsilon)\sum_{i=1}^{j-1}\int_{\Omega} s^{l_j^{(3)}}\theta^{l_j^{(3)}}\xi^{\frac{1}{2}}|v_i^{\varepsilon}|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t
\end{aligned}
\tag{3.45}
$$

with $l_j^{(3)} = \max\{l_j^{(1)}, l_j^{(2)}\}$. Using integration by parts and Young's inequality, we have the following estimate for I_2

$$
\begin{aligned}
I_2 = & \int_{\Omega_T} s^l\theta^l(x+\varepsilon)^{\alpha_{j-1}}v_{j-1,x}^{\varepsilon}\left(\xi v_j^{\varepsilon}e^{2s\varphi}\right)_x\mathrm{d}x\mathrm{d}t \\
\leq & \int_{\Omega_T} s^{l+1}\theta^{l+1}(x+\varepsilon)^{\alpha_{j-1}}|v_{j-1,x}^{\varepsilon}|\left(\xi^{\frac{1}{2}}|\frac{\xi_x}{\xi^{\frac{1}{2}}}v_j^{\varepsilon}| + \xi|v_j^{\varepsilon}| + \xi|v_{j,x}^{\varepsilon}|\right)e^{2s\varphi}\mathrm{d}x\mathrm{d}t \\
\leq & \frac{\epsilon}{8}\mathcal{I}^{\varepsilon}_{\gamma,\alpha_j,3(m+1-j)}(v_j^{\varepsilon}) + C(\epsilon)\int_{\Omega_T} s^{l_j^{(4)}}\theta^{l_j^{(4)}}\xi\left(1+(x+\varepsilon)^{2\alpha_{j-1}-\alpha_j}\right)|v_{j-1,x}^{\varepsilon}|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t
\end{aligned}
\tag{3.46}
$$

with $l_j^{(4)} = 2(l+1) - 3(m+1-j) + 2$. Similarly,

$$
\begin{aligned}
& I_3 + I_4 \\
= & \sum_{i=1}^{j-1}\int_{\Omega_T} s^l\theta^l\left[-b_{i,j-1}v_i^{\varepsilon}\left(\xi v_j^{\varepsilon}e^{2s\varphi}\right)_x - a_{i,j-1}\xi v_i^{\varepsilon}v_j^{\varepsilon}e^{2s\varphi}\right]\mathrm{d}x\mathrm{d}t \\
\leq & \frac{\epsilon}{8}\mathcal{I}^{\varepsilon}_{\gamma,\alpha_j,3(m+1-j)}(v_j^{\varepsilon}) + C(\epsilon)\sum_{i=1}^{j-1}\int_{\Omega_T} s^{l_j^{(4)}}\theta^{l_j^{(4)}}\xi\left(1+(x+\varepsilon)^{-\alpha_j}\right)|v_i^{\varepsilon}|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{3.47}
$$

Therefore, using (3.41), (3.45)–(3.47) and noticing that $|a_{j,j-1}| > 0$ in $\mathcal{O}^{(2)}$ due to (1.8), we obtain

$$
\begin{aligned}
& \int_{\mathcal{O}_T^{(2)}} s^l\theta^l|v_j^{\varepsilon}|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t \\
\leq & \frac{\epsilon}{2}\mathcal{I}^{\varepsilon}_{\gamma,\alpha_j,3(m+1-j)}(v_j^{\varepsilon}) + \epsilon\mathcal{I}^{\varepsilon}_{\gamma,\alpha_{j+1},3(m+1-(j+1))}(v_{j+1}^{\varepsilon}) \\
& + C(\epsilon)\int_{\Omega_T} s^{l_j^{(5)}}\theta^{l_j^{(5)}}\xi\left(1+(x+\varepsilon)^{2\alpha_{j-1}-\alpha_j}\right)|v_{j-1,x}^{\varepsilon}|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t \\
& + C(\epsilon)\sum_{i=1}^{j-1}\int_{\Omega_T} s^{l_j^{(5)}}\theta^{l_j^{(5)}}\xi^{\frac{1}{2}}\left(1+(x+\varepsilon)^{-\alpha_j}\right)|v_i^{\varepsilon}|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t
\end{aligned}
\tag{3.48}
$$

with $l_j^{(5)} = \max\{l_j^{(3)}, l_j^{(4)}\}$.

On the other hand, by multiplying the equation of v_{j-1} by $s^{l_j^{(5)}}\theta^{l_j^{(5)}}\xi(x+\varepsilon)^{\alpha_{j-1}-\alpha_j}v_{j-1}^{\varepsilon}e^{2s\varphi}$ we find that

$$
\begin{aligned}
& \int_{\Omega_T} s^{l_j^{(5)}}\theta^{l_j^{(5)}}\xi(x+\varepsilon)^{2\alpha_{j-1}-\alpha_j}|v_{j-1,x}^{\varepsilon}|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t \\
= & -\int_{\Omega_T} s^{l_j^{(5)}}\theta^{l_j^{(5)}}\left(\xi(x+\varepsilon)^{\alpha_{j-1}-\alpha_j}e^{2s\varphi}\right)_x(x+\varepsilon)^{\alpha_{j-1}}v_{j-1}^{\varepsilon}v_{j-1,x}^{\varepsilon}\mathrm{d}x\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2}\int_{\Omega_T} s^{l_j^{(5)}}\xi(x+\varepsilon)^{\alpha_{j-1}-\alpha_j}\left(\theta^{l_j^{(5)}}e^{2s\varphi}\right)_t|v_{j-1}^\varepsilon|^2\mathrm{d}x\mathrm{d}t\\
&+\sum_{i=1}^{j-1}\int_{\Omega_T} s^{l_j^{(5)}}\theta^{l_j^{(5)}}b_{i,j-1}v_i^\varepsilon\left(\xi(x+\varepsilon)^{\alpha_{j-1}-\alpha_j}v_{j-1}^\varepsilon e^{2s\varphi}\right)_x\mathrm{d}x\mathrm{d}t\\
&+\int_{\Omega_T} s^{l_j^{(5)}}\theta^{l_j^{(5)}}\xi(x+\varepsilon)^{\alpha_{j-1}-\alpha_j}\left(\sum_{i=1}^{j-1}a_{i,j-1}v_i^\varepsilon+a_{j,j-1}v_j^\varepsilon\right)v_{j-1}^\varepsilon e^{2s\varphi}\mathrm{d}x\mathrm{d}t\\
\le&\epsilon^*\mathcal{I}^\varepsilon_{\gamma,\alpha_{j-1},3(m+1-(j-1))}(v_{j-1}^\varepsilon)+\epsilon^*\mathcal{I}^\varepsilon_{\gamma,\alpha_j,3(m+1-j)}(v_j^\varepsilon)\\
&+C(\epsilon^*)\sum_{i=1}^{j-1}\int_{\Omega_T} s^{l_j}\theta^{l_j}\xi(x+\varepsilon)^{3\alpha_{j-1}-2\alpha_j-2}|v_i^\varepsilon|^2\mathrm{d}x\mathrm{d}t
\end{aligned}
\tag{3.49}
$$

with $l_j=\max\{l_j^{(5)}+2,2l_j^{(5)}+1-3(m+1-j)\}$. Similarly,

$$
\begin{aligned}
&\int_{\Omega_T} s^{l_j^{(5)}}\theta^{l_j^{(5)}}\xi|v_{j-1,x}^\varepsilon|^2e^{2s\varphi}\mathrm{d}x\mathrm{d}t\\
\le&\epsilon^*\mathcal{I}^\varepsilon_{\gamma,\alpha_{j-1},3(m+1-(j-1))}(v_{j-1}^\varepsilon)+\epsilon^*\mathcal{I}^\varepsilon_{\gamma,\alpha_j,3(m+1-j)}(v_j^\varepsilon)\\
&+C(\epsilon^*)\sum_{i=1}^{j-1}\int_{\Omega_T} s^{l_j}\theta^{l_j}\xi(x+\varepsilon)^{-\alpha_{j-1}-2}|v_i^\varepsilon|^2\mathrm{d}x\mathrm{d}t
\end{aligned}
\tag{3.50}
$$

Therefore, by using (3.48)–(3.50), together with $\mathrm{Supp}(\xi)\subset\mathcal{O}^{(1)}$ and

$$
(x+\varepsilon)^{3\alpha_{j-1}-2\alpha_j-2}+(x+\varepsilon)^{-\alpha_j}+(x+\varepsilon)^{-\alpha_{j-1}-2}\le C,\quad x\in\mathcal{O}^{(1)}
$$

due to $0\notin\mathcal{O}^{(1)}$, we obtain

$$
\begin{aligned}
&\int_{\mathcal{O}_T^{(2)}} s^l\theta^l|v_j^\varepsilon|^2e^{2s\varphi}\mathrm{d}x\mathrm{d}t\\
\le&2\epsilon^*C(\epsilon)\mathcal{I}^\varepsilon_{\gamma,\alpha_{j-1},3(m+1-(j-1))}(v_{j-1}^\varepsilon)+(\frac{\epsilon}{2}+2\epsilon^*C(\epsilon))\mathcal{I}^\varepsilon_{\gamma,\alpha_j,3(m+1-j)}(v_j^\varepsilon)\\
&+\epsilon\mathcal{I}^\varepsilon_{\gamma,\alpha_{j+1},3(m+1-(j+1))}(v_{j+1}^\varepsilon)+C(\epsilon,\epsilon^*)\sum_{i=1}^{j-1}\int_{\mathcal{O}_T^{(1)}} s^{l_j}\theta^{l_j}|v_i^\varepsilon|^2e^{2s\varphi}\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{3.51}
$$

For any $\epsilon>0$, we can choose ϵ^* sufficiently small such that $\max\{2\epsilon^*C(\epsilon),\frac{\epsilon}{2}+2\epsilon^*C(\epsilon)\}<\epsilon$ to obtain (3.38).

Now we finish the proof of Theorem 3.3 by means of (3.38). Letting $j=m$, $l=3(m+1-m)$, $\mathcal{O}^{(2)}=\omega^{(1)}\subset\mathcal{O}^{(1)}=\varpi^{(m-1)}\subset\omega$, we deduce from (3.38) that

$$
\begin{aligned}
&\int_{\omega_T^{(1)}} s^{3(m+1-m)}\theta^{3(m+1-m)}|v_m^\varepsilon|^2e^{2s\varphi}\mathrm{d}x\mathrm{d}t\\
\le&\epsilon\mathcal{I}^\varepsilon_{\gamma,\alpha_{m-1},3(m+1-(m-1))}(v_{m-1}^\varepsilon)+\epsilon\mathcal{I}^\varepsilon_{\gamma,\alpha_j,3(m+1-m)}(v_m^\varepsilon)\\
&+C(\epsilon)\sum_{i=1}^{m-1}\int_{\varpi_T^{(m-1)}} s^{l_m}\theta^{l_m}|v_i^\varepsilon|^2e^{2s\varphi}\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{3.52}
$$

Substituting (3.52) into (3.37) and choosing ϵ sufficiently small, we have

$$\sum_{j=1}^{m}\mathcal{I}^{\varepsilon}_{\gamma,\alpha_j,3(m+1-j)}(v_j)\leq C\sum_{j=1}^{m-1}\int_{\omega_T^{(m-1)}} s^{\tilde{l}_{m-1}}\theta^{\tilde{l}_{m-1}}|v_j^{\varepsilon}|^2e^{2s\varphi}\mathrm{d}x\mathrm{d}t \tag{3.53}$$

with $\tilde{l}_{m-1}=\max\{l_m,3m\}$. Repeating the above procedure finite times, we can obtain (3.35) and complete the proof of Theorem 3.3. □

4. Proof of Theorem 1.1

In this section, we will show the null controllability for system (1.1), *i.e.* Theorem 1.1. We first give the same result for the nongenerate approximate version of (1.1)

$$\begin{cases}\partial_t u_1^{\varepsilon}=\left((x+\varepsilon)^{\alpha_1}u_{1,x}^{\varepsilon}\right)_x+\sum\limits_{i=1}^{m}b_{1i}u_{i,x}^{\varepsilon}+\sum\limits_{i=1}^{m}a_{1i}u_i^{\varepsilon}+f^{\varepsilon}\mathbf{1}_{\omega}, & (x,t)\in\Omega_T,\\ \partial_t u_2^{\varepsilon}=\left((x+\varepsilon)^{\alpha_2}u_{2,x}^{\varepsilon}\right)_x+\sum\limits_{i=1}^{m}b_{2i}u_{i,x}^{\varepsilon}+\sum\limits_{i=1}^{m}a_{2i}u_i^{\varepsilon}, & (x,t)\in\Omega_T,\\ \cdots & \\ \partial_t u_m^{\varepsilon}=\left((x+\varepsilon)^{\alpha_m}u_{m,x}^{\varepsilon}\right)_x+\sum\limits_{i=1}^{m}b_{mi}u_{i,x}^{\varepsilon}+\sum\limits_{i=1}^{m}a_{mi}u_i^{\varepsilon}, & (x,t)\in\Omega_T,\\ u_j^{\varepsilon}(0,t)=u_j^{\varepsilon}(1,t)=0,\quad t\in(0,T),\quad u_j^{\varepsilon}(x,0)=u_{j,0}^{\varepsilon}(x),\quad x\in\Omega,\quad 1\leq j\leq m, & \end{cases} \tag{4.1}$$

where $u_{0,j}^{\varepsilon}\in H^2(\Omega)\cap H_0^1(\Omega)$ such that

$$u_{j,0}^{\varepsilon}\to u_{j,0},\quad \text{strongly in } L^2(\Omega) \text{ for } 1\leq j\leq m.$$

Theorem 4.1. *Let (A1)–(A3) be held. Then there exists a control $f^{\varepsilon}\in L^2(\omega_T)$ such that the corresponding solution $(u_1^{\varepsilon},\cdots,u_m^{\varepsilon})^T$ of system (4.1) satisfies*

$$u_j^{\varepsilon}(x,T)=0,\quad x\in\Omega,\quad 1\leq j\leq m. \tag{4.2}$$

Moreover, f^{ε} satisfies the following estimate

$$\|f^{\varepsilon}\|_{L^2(\omega_T)}\leq C\sum_{j=1}^{m}\|u_{j,0}\|_{L^2(\Omega)}, \tag{4.3}$$

where C is depending on $m,\omega,\Omega,T,\alpha_j,a_{ij},b_{ij}$, but independent of ε.

It is well known that the key ingredient for proving Theorem 4.1 is to obtain the observability inequality for the corresponding adjoint system (3.1).

Lemma 4.2. *Let (A1)–(A3) be held. Then the solution of the adjoint system (3.1) satisfies*

$$\sum_{j=1}^{m}\int_{\Omega}|v_i^{\varepsilon}|^2(x,0)\mathrm{d}x\leq C\int_{\omega_T}|v_1^{\varepsilon}|^2\mathrm{d}x\mathrm{d}t, \tag{4.4}$$

where C is depending on $m,\omega,\Omega,T,\alpha_j,a_{ij},b_{ij}$, but independent of ε.

Proof. By a similar argument to (2.9), we obtain for $0 \le \tau < \tilde{\tau} \le T$ that

$$\sum_{j=1}^{m}\left(\int_{\Omega}|v_j^\varepsilon|^2(x,\tau)\mathrm{d}x+\int_{\tau}^{\tilde{\tau}}\int_{\Omega}(x+\varepsilon)^{\alpha_j}|v_{j,x}^\varepsilon|^2\mathrm{d}x\mathrm{d}t\right)$$
$$\le \sum_{j=1}^{m}\int_{\Omega}|v_j^\varepsilon|^2(x,\tilde{\tau})\mathrm{d}x + C\int_{\tau}^{\tilde{\tau}}\int_{\Omega}|v_j^\varepsilon|^2\mathrm{d}x\mathrm{d}t. \tag{4.5}$$

Then applying Gronwall inequality yields that

$$\sum_{j=1}^{m}\int_{\Omega}|v_j^\varepsilon|^2(x,\tau)\mathrm{d}x \le e^{C(\tilde{\tau}-\tau)}\sum_{j=1}^{m}\int_{\Omega}|v_j^\varepsilon|^2(x,\tilde{\tau})\mathrm{d}x, \quad 0\le\tau<\tilde{\tau}\le T. \tag{4.6}$$

Letting $\tau=0$ and integrating over $\left[\frac{T}{3},\frac{2T}{3}\right]$ with respect to $\tilde{\tau}$, we find that

$$\frac{T}{3}\sum_{j=1}^{m}\int_{\Omega}|v_j^\varepsilon|^2(x,0)\mathrm{d}x \le C\sum_{j=1}^{m}\int_{\frac{T}{3}}^{\frac{2T}{3}}\int_{\Omega}|v_j^\varepsilon|^2\mathrm{d}x\mathrm{d}t. \tag{4.7}$$

On the other hand, by Carleman estimate (3.35) we obtain

$$s^3\sum_{j=1}^{m}\int_{\Omega_T}\theta^3|v_j^\varepsilon|^2e^{2s\varphi}\mathrm{d}x\mathrm{d}t \le C\int_{\omega_T}s^{\tilde{l}_1}\theta^{\tilde{l}_1}|v_1^\varepsilon|^2e^{2s\varphi}\mathrm{d}x\mathrm{d}t \tag{4.8}$$

for all $\lambda \ge \lambda_3$, $s \ge s_3$. We fix $\lambda=\lambda_3$ and $s=s_3$. By

$$\theta^3e^{2s\varphi} \ge \left(\frac{4}{T^2}\right)^{12}\sum_{i=1}^{2}\exp\left(-4s_3\left(\frac{9}{2T^2}\right)^4e^{\lambda_3\|\eta_i\|_{C(\overline{\Omega})}}\right), \quad t\in\left[\frac{T}{3},\frac{2T}{3}\right],$$

we further have

$$\sum_{j=1}^{m}\int_{\Omega_T}|v_j^\varepsilon|^2e^{2s\varphi}\mathrm{d}x\mathrm{d}t \le C(T,\lambda_3,s_3)\int_{\omega_T}s_3^{\tilde{l}_1}\theta^{\tilde{l}_1}|v_1^\varepsilon|^2e^{2s\varphi}\mathrm{d}x\mathrm{d}t$$
$$\le C(T,\lambda_3,s_3,\tilde{l}_1)\int_{\omega_T}\left(\frac{1}{t^4(T-t)^4}\right)^{\tilde{l}_1}e^{-\frac{M}{t^4(T-t)^4}}|v_1^\varepsilon|^2\mathrm{d}x\mathrm{d}t, \tag{4.9}$$

where $M := 2s_3\min\{e^{\lambda_3\|\eta_1\|_{C(\overline{\Omega})}}, e^{\lambda_3\|\eta_2\|_{C(\overline{\Omega})}}\}$. Since $\max_{t\in[0,T]}\left(\frac{1}{t^4(T-t)^4}\right)^{\tilde{l}_1}e^{-\frac{M}{t^4(T-t)^4}} < \infty$, we deduce from (4.9) that

$$\sum_{j=1}^{m}\int_{\Omega_T}|v_j^\varepsilon|^2e^{2s\varphi}\mathrm{d}x\mathrm{d}t \le C\int_{\omega_T}|v_1^\varepsilon|^2\mathrm{d}x\mathrm{d}t. \tag{4.10}$$

Finally, we obtain the desired estimate (4.4) from (4.7) and (4.10) and then complete the proof of Lemma 4.2. □

By the observability inequality (4.4) and a classical argument, one can deduce the null controllability result of system (4.1) for any $0 < \varepsilon < 1$ and the uniform estimate (4.3), *i.e.* Theorem 4.1.

Proof of Theorem 1.1. Similar to (2.4), we have the following uniformly estimates in ε

$$\begin{aligned}&\mathrm{Sup}_{t\in(0,T)}\sum_{j=1}^{m}\int_{\Omega}|u_j^{\varepsilon}|^2(x,t)\mathrm{d}x+\sum_{j=1}^{m}\int_{\Omega_T}(x+\varepsilon)^{\alpha_j}|u_{j,x}^{\varepsilon}|^2\mathrm{d}x\mathrm{d}t\\&\leq C\sum_{j=1}^{m}\left(\|u_{j,0}\|^2_{L^2(\Omega)}+\|f^{\varepsilon}\|^2_{L^2(\omega_T)}\right).\end{aligned}\tag{4.11}$$

Together with Theorem 4.1 and the uniform estimate (4.3) for f^{ε}, we get a control $f\in L^2(\omega_T)$ (the weak limit of a subsequence of f^{ε} in $L^2(\omega_T)$ as $\varepsilon\to 0$) that drives the corresponding solution $(u_1,\cdots,u_m)$ to zero at time T. Finally, from (4.3) we immediately deduce (1.11) and then complete the proof of Theorem 1.1. □

Appendix A

Here, we prove Lemma 3.2. To do this, we need the following Hardy inequality.

Lemma A.1. *Let $\gamma\in[0,1)\cup(1,2]$ and $z\in H_0^1(\Omega)$. Then for any $\varepsilon\in(0,1)$, we have*

$$\int_{\Omega}(x+\varepsilon)^{-\gamma}|z|^2\mathrm{d}x\leq\frac{4}{(\gamma-1)^2}\int_{\Omega}(x+\varepsilon)^{2-\gamma}|z_x|^2\mathrm{d}x.\tag{A.1}$$

For $\gamma\in(1,2]$, (A.1) was proved by Cannarsa, Martinez and Vancostenoble, see Lemma 6.8 in [8]. Similar process is also applied to prove the same result for $\gamma\in[0,1)$. So we omit the proof of this lemma.

Proof of Lemma 3.2. Without loss of generality, we assume $d=3$. Indeed, if (3.11) holds for $d=3$, then we can easily prove (3.11) for general d by the change of variables $\tilde{y}=s^{\frac{d-3}{2}}\theta^{\frac{d-3}{2}}y$.

We split the proof into the following three steps.

Step 1. *Carleman estimate for degenerate part.*

Let

$$Y(x,t)=e^{s\varphi_1(x,t)}y(x,t),\quad (x,t)\in\Omega_T.\tag{A.2}$$

Then Y satisfies

$$e^{s\varphi_1}\mathcal{L}[y]=\mathcal{L}_1[Y]+\mathcal{L}_2[Y]\tag{A.3}$$

with

$$\begin{aligned}\mathcal{L}_1[Y]&=-Y_t+2s(x+\varepsilon)^{\alpha}\varphi_{1,x}Y_x+s((x+\varepsilon)^{\alpha}\varphi_{1,x})_xY,\\\mathcal{L}_2[Y]&=-((x+\varepsilon)^{\alpha}Y_x)_x+s\varphi_{1,t}Y-s^2(x+\varepsilon)^{\alpha}\varphi_{1,x}^2Y.\end{aligned}$$

Moreover, we have

$$Y(0,t)=Y(1,t)=0,\quad t\in(0,T),\tag{A.4}$$

$$Y(x,0)=Y(x,T)=0,\quad x\in\Omega.\tag{A.5}$$

Then, by (**??**) we find that

$$\|e^{s\varphi_1}\mathcal{L}[y]\|^2_{L^2(\Omega_T)}=\|\mathcal{L}_1[Y]\|^2_{L^2(\Omega_T)}+\|\mathcal{L}_2[Y]\|^2_{L^2(\Omega_T)}+2(\mathcal{L}_1[Y],\mathcal{L}_1[Y])_{L^2(\Omega_T)}.\tag{A.6}$$

By a calculation similar to [2], we obtain

$$(\mathcal{L}_1[Y], \mathcal{L}_1[Y])_{L^2(\Omega_T)} = \{D.T.\} + \{B.T.\}, \tag{A.7}$$

where

$$\begin{aligned}
\{D.T.\} &= \int_{\Omega_T} s^3 \left(2(x+\varepsilon)^{2\alpha}\varphi_{1,xx} + \alpha(x+\varepsilon)^{2\alpha-1}\varphi_{1,x}\right)\varphi_{1,x}^2 |Y|^2 \mathrm{d}x\mathrm{d}t \\
&\quad + \int_{\Omega_T} s\left(2(x+\varepsilon)^{2\alpha}\varphi_{1,xx} + \alpha(x+\varepsilon)^{2\alpha-1}\varphi_{1,x}\right)|Y_x|^2 \mathrm{d}x\mathrm{d}t \\
&\quad - \int_{\Omega_T} 2s^2(x+\varepsilon)^{\alpha}\varphi_{1,x}\varphi_{1,tx}Y^2 \mathrm{d}x\mathrm{d}t + \int_{\Omega_T} \frac{s}{2}\varphi_{1,tt}Y^2 \mathrm{d}x\mathrm{d}t \\
&\quad + \int_{\Omega_T} s(x+\varepsilon)^{\alpha}\left((x+\varepsilon)^{\alpha}\varphi_{1,x}\right)_{xx} YY_x \mathrm{d}x\mathrm{d}t, \\
\{B.T.\} &= \int_0^T \left[(x+\varepsilon)^{\alpha}Y_xY_t + s^2(x+\varepsilon)^{\alpha}\varphi_{1,t}\varphi_{1,x}Y^2 - s^3(x+\varepsilon)^{2\alpha}\varphi_{1,x}^3Y^2\right]_{x=0}^{x=1} \mathrm{d}t \\
&\quad - \int_0^T \left[s(x+\varepsilon)^{2\alpha}\varphi_{1,x}|Y_x|^2 + s(x+\varepsilon)^{\alpha}\left((x+\varepsilon)^{\alpha}\varphi_{1,x}\right)_x YY_x\right]_{x=0}^{x=1} \mathrm{d}t.
\end{aligned}$$

For $Y \in L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))$ such that (A.4) we have

$$\{B.T.\} = -\int_0^T \left[s\lambda(1+\beta)(x+\varepsilon)^{2\alpha+\beta}\theta|Y_x|^2\right]_{x=0}^{x=1} \mathrm{d}t \geq -C\int_0^T s\theta|Y_x(1,t)|^2 \mathrm{d}t. \tag{A.8}$$

By using

$$\begin{cases}
\varphi_{1,x} = \lambda(1+\beta)(x+\varepsilon)^{\beta}\phi_1\theta, \\
\varphi_{1,xx} = \left(\lambda^2(1+\beta)^2(x+\varepsilon)^{2\beta} + \lambda\beta(1+\beta)(x+\varepsilon)^{\beta-1}\right)\phi_1\theta, \\
\theta > 0, \quad |\theta_t| \leq C\theta^2, \quad |\theta_{tt}| \leq C\theta^3,
\end{cases} \tag{A.9}$$

we obtain

$$\begin{aligned}
\{D.T.\} \geq C_{\alpha,\beta}^{(1)} &\int_{\Omega_T} s^3\lambda^3(x+\varepsilon)^{2\alpha+3\beta-1}\phi_1^3\theta^3|Y|^2 \mathrm{d}x\mathrm{d}t \\
&+ C_{\alpha,\beta}^{(2)}\int_{\Omega_T} s\lambda(x+\varepsilon)^{2\alpha+\beta-1}\phi_1\theta|Y_x|^2 \mathrm{d}x\mathrm{d}t + X_1 + X_2,
\end{aligned} \tag{A.10}$$

where

$$C_{\alpha,\beta}^{(1)} = (\alpha+2\beta)\,(1+\beta)^3, \quad C_{\alpha,\beta}^{(2)} = (\alpha+2\beta)(1+\beta),$$

and

$$\begin{aligned}
X_1 &= \int_{\Omega_T} \left(-2s^2(x+\varepsilon)^{\alpha}\varphi_{1,x}\varphi_{1,tx} + \frac{s}{2}\varphi_{1,tt}\right)|Y|^2 \mathrm{d}x\mathrm{d}t, \\
X_2 &= \int_{\Omega_T} s(x+\varepsilon)^{\alpha}\left((x+\varepsilon)^{\alpha}\varphi_{1,x}\right)_{xx} YY_x \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Now we estimate X_1 and X_2. By the definition of φ_1 and (A.9), we have

$$|\varphi_{1,tx}| \leq C\lambda(x+\varepsilon)^{\beta}\phi_1\theta^2, \quad |\varphi_{1,tt}| \leq C(\lambda)\theta^2. \tag{A.11}$$

Since $0<\alpha<\frac{1}{3}$ and $0<\beta<\frac{2}{9}-\frac{2\alpha}{3}$, we have $0<2\alpha+3\beta<1$ and further $(x+\varepsilon)^{\alpha+2\beta}+1 \leq C(x+\varepsilon)^{2\alpha+3\beta-1}$. Therefore, together with (A.9), we find that

$$\begin{aligned}|X_1| &\leq C\int_{\Omega_T} s^2\lambda^2(x+\varepsilon)^{\alpha+2\beta}\phi_1^2\theta^3|Y|^2\mathrm{d}x\mathrm{d}t + C(\lambda)\int_{\Omega_T} s\theta^2|Y|^2\mathrm{d}x\mathrm{d}t\\ &\leq C^*(\lambda)\int_{\Omega_T} s^2(x+\varepsilon)^{2\alpha+3\beta-1}\phi_1^2\theta^3|Y|^2\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{A.12}$$

A direct calculation gives

$$\begin{aligned}&|(x+\varepsilon)^{\alpha}\left((x+\varepsilon)^{\alpha}\varphi_{1,x}\right)_{xx}|\\ \leq& \left(C^{(3)}_{\alpha,\beta}\lambda(x+\varepsilon)^{2\alpha+\beta-2}+C\lambda^2(x+\varepsilon)^{2\alpha+2\beta-1}+C\lambda^3(x+\varepsilon)^{2\alpha+3\beta}\right)\phi_1\theta,\end{aligned}$$

where

$$C^{(3)}_{\alpha,\beta} = (1+\beta)(\alpha+\beta)(1-\alpha-\beta).$$

Then, for X_2, we have

$$\begin{aligned}&|X_2|\\ \leq& \int_{\Omega_T}\left(C^{(3)}_{\alpha,\beta}s\lambda(x+\varepsilon)^{2\alpha+\beta-2}+Cs\lambda^3(x+\varepsilon)^{2\alpha+2\beta-1}\right)\phi_1\theta|Y||Y_x|\mathrm{d}x\mathrm{d}t := J_1+J_2.\end{aligned} \tag{A.13}$$

By Young's inequality and Lemma 5.1, we have

$$\begin{aligned}&J_1\\ \leq& \int_{\Omega_T}\epsilon C^{(3)}_{\alpha,\beta}s\lambda(x+\varepsilon)^{2\alpha+\beta-1}\phi_1\theta|Y_x|^2\mathrm{d}x\mathrm{d}t+\int_{\Omega_T}\frac{1}{4\epsilon}C^{(3)}_{\alpha,\beta}s\lambda(x+\varepsilon)^{2\alpha+\beta-3}\phi_1\theta|Y|^2\mathrm{d}x\mathrm{d}t\\ \leq& \int_{\Omega_T}\epsilon C^{(3)}_{\alpha,\beta}s\lambda(x+\varepsilon)^{2\alpha+\beta-1}\phi_1\theta|Y_x|^2\mathrm{d}x\mathrm{d}t\\ &+\int_{\Omega_T}\frac{1}{4\epsilon}C^{(3)}_{\alpha,\beta}\frac{4}{(2-2\alpha-\beta)^2}s\lambda(x+\varepsilon)^{2\alpha+\beta-1}\theta\left|\left(\phi_1^{\frac{1}{2}}Y\right)_x\right|^2\mathrm{d}x\mathrm{d}t\\ \leq& \int_{\Omega_T}\left[\epsilon C^{(3)}_{\alpha,\beta}+\frac{1}{\epsilon}C^{(3)}_{\alpha,\beta}\frac{1}{(2-2\alpha-\beta)^2}\right]s\lambda(x+\varepsilon)^{2\alpha+\beta-1}\phi_1\theta|Y_x|^2\mathrm{d}x\mathrm{d}t\\ &+C(\epsilon)\int_{\Omega_T}s\lambda^3(x+\varepsilon)^{2\alpha+3\beta-1}\phi_1\theta|Y|^2\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{A.14}$$

We fix $\epsilon=\frac{1}{2-2\alpha-\beta}$. Therefore, for $\alpha\in\left(0,\frac{1}{3}\right)$ and $\beta\in\left(0,\frac{2}{9}-\frac{2\alpha}{3}\right)$,

$$\epsilon C^{(3)}_{\alpha,\beta}+\frac{1}{\epsilon}C^{(3)}_{\alpha,\beta}\frac{1}{(2-2\alpha-\beta)^2} = (1+\beta)(\alpha+\beta)\left(\frac{2-2\alpha-2\beta}{2-2\alpha-\beta}\right) < C^{(2)}_{\alpha,\beta}-\beta,$$

then we have

$$J_1 \leq C \int_{\Omega_T} s\lambda^3 (x+\varepsilon)^{2\alpha+3\beta-1} \phi_1 \theta |Y|^2 \mathrm{d}x\mathrm{d}t + \int_{\Omega_T} (C^{(2)}_{\alpha,\beta} - \beta) s\lambda (x+\varepsilon)^{2\alpha+\beta-1} \phi_1 \theta |Y_x|^2 \mathrm{d}x\mathrm{d}t. \tag{A.15}$$

Obviously,

$$J_2 \leq C \int_{\Omega_T} s\lambda^6 (x+\varepsilon)^{2\alpha+3\beta-1} \phi_1 \theta |Y|^2 \mathrm{d}x\mathrm{d}t + C \int_{\Omega_T} s (x+\varepsilon)^{2\alpha+\beta-1} \phi_1 \theta |Y_x|^2 \mathrm{d}x\mathrm{d}t. \tag{A.16}$$

Thus, from (A.15) and (A.16), it follows that

$$|X_2| \leq C \int_{\Omega_T} s\lambda^6 (x+\varepsilon)^{2\alpha+3\beta-1} \phi_1 \theta |Y|^2 \mathrm{d}x\mathrm{d}t + \int_{\Omega_T} \left[(C^{(2)}_{\alpha,\beta} - \beta) s\lambda + Cs \right] (x+\varepsilon)^{2\alpha+\beta-1} \phi_1 \theta |Y_x|^2 \mathrm{d}x\mathrm{d}t. \tag{A.17}$$

Choosing $\lambda \geq \lambda_2 := \frac{2C}{\beta}$ and then

$$C^{(2)}_{\alpha,\beta} s\lambda - \left[(C^{(2)}_{\alpha,\beta} - \beta) s\lambda + Cs \right] > Cs$$

and $s \geq s_2 := \max\{C^*(\lambda), \lambda^6\}$, and substituting (A.12) and (A.17) into (A.10), we have

$$\{D.T.\} \geq C \int_{\Omega_T} s^3 (x+\varepsilon)^{2\alpha+3\beta-1} \phi_1^3 \theta^3 |Y|^2 \mathrm{d}x\mathrm{d}t + C \int_{\Omega_T} s (x+\varepsilon)^{2\alpha+\beta-1} \phi_1 \theta |Y_x|^2 \mathrm{d}x\mathrm{d}t \tag{A.18}$$

for all $\lambda \geq \lambda_2$ and $s \geq s_2$.

Substituting (A.8) and (A.18) into (A.6) and replacing Y by $ye^{s\varphi_1}$ yields that

$$\int_{\Omega_T} \left(s^3\theta^3 (x+\varepsilon)^{2\alpha+3\beta-1} |y|^2 + s\theta (x+\varepsilon)^{2\alpha+\beta-1} |y_x|^2 \right) e^{2s\varphi_1} \mathrm{d}x\mathrm{d}t \leq C \int_{\Omega_T} |\mathcal{L}[y]|^2 e^{2s\varphi_1} \mathrm{d}x\mathrm{d}t + C \int_0^T s\theta |y_x(1,t)|^2 e^{2s\varphi_1(1,t)} \mathrm{d}t. \tag{A.19}$$

Then by using an argument similar to Proposition 3 in [2], we obtain the Carleman estimate for degenerate part $(0, x_1^{(3)}) \times (0,T)$

$$\int_0^T \int_0^{x_1^{(3)}} \left(s^3\theta^3 (x+\varepsilon)^{2\alpha+3\beta-1} |y|^2 + s\theta (x+\varepsilon)^{2\alpha+\beta-1} |y_x|^2 \right) e^{2s\varphi} \mathrm{d}x\mathrm{d}t \leq C \int_{\Omega_T} \chi^2 |\mathcal{L}[y]|^2 e^{2s\varphi_1} \mathrm{d}x\mathrm{d}t + C \int_{\omega_T^{(3)}} \left(|y_x|^2 + |y|^2 \right) e^{2s\varphi_1} \mathrm{d}x\mathrm{d}t, \tag{A.20}$$

where we have used $\varphi = \varphi_1$ for $x \in (0, x_1^{(3)})$.

Step 2. *Carleman estimate for nondegenerate part.*

Now we derive the Carleman estimate for nondegenerate part $(x_2^{(3)}, 1) \times (0, T)$. To do this, letting $z = (1-\chi)y$, then we have

$$-z_t - ((x+\varepsilon)^\alpha z_x)_x = (1-\chi)\mathcal{L}[y] + ((x+\varepsilon)^\alpha \chi_x y)_x + \chi_x (x+\varepsilon)^\alpha y_x, \quad (x,t) \in \Omega_T, \tag{A.21}$$

and

$$z(0,t) = z(1,t) = 0, \quad t \in (0,T). \tag{A.22}$$

By the classic Carleman estimate, *e.g.* [34], we have

$$\begin{aligned}\int_{\Omega_T} \left(s^3\theta^3|z|^2 + s\theta|z_x|^2\right) e^{2s\varphi_2} \mathrm{d}x\mathrm{d}t \le C \int_{\Omega_T} (1-\chi)^2 |\mathcal{L}[y]|^2 e^{2s\varphi_2} \mathrm{d}x\mathrm{d}t \\ + C\int_0^T \int_{x_1^{(3)}}^{x_2^{(3)}} \left(|y_x|^2 + |y|^2\right) e^{2s\varphi_2} \mathrm{d}x\mathrm{d}t + C \int_{\omega_T^{(2)}} s^3\theta^3 |y|^2 e^{2s\varphi_2} \mathrm{d}x\mathrm{d}t.\end{aligned} \tag{A.23}$$

Since $(x+\varepsilon)^{2\alpha+3\beta-1} > C$, $(x+\varepsilon)^{2\alpha+\beta-1} > C$ and $\varphi_2 = \varphi$ for $x \in (x_2^{(3)}, 1)$, together with (A.23) we find that

$$\begin{aligned}&\int_0^T \int_{x_2^{(3)}}^1 \left(s^3\theta^3 (x+\varepsilon)^{2\alpha+3\beta-1}|y|^2 + s\theta (x+\varepsilon)^{2\alpha+\beta-1}|y_x|^2\right) e^{2s\varphi} \mathrm{d}x\mathrm{d}t \\ \le& C\int_{\Omega_T} (1-\chi)^2 |\mathcal{L}[y]|^2 e^{2s\varphi_2} \mathrm{d}x\mathrm{d}t + C \int_{\omega_T^{(2)}} \left(|y_x|^2 + s^3\theta^3|y|^2\right) e^{2s\varphi_2} \mathrm{d}x\mathrm{d}t.\end{aligned} \tag{A.24}$$

Step 3. *End of the proof.*

Combining (A.20) and (A.24) and adding to both sides of the inequality the term

$$\int_0^T \int_{x_1^{(3)}}^{x_2^{(3)}} \left(s^3\theta^3 (x+\varepsilon)^{2\alpha+3\beta-1}|y|^2 + s\theta (x+\varepsilon)^{2\alpha+\beta-1}|y_x|^2\right) e^{2s\varphi} \mathrm{d}x\mathrm{d}t,$$

we obtain

$$\begin{aligned}&\int_{\Omega_T} \left(s^3\theta^3 (x+\varepsilon)^{2\alpha+3\beta-1}|y|^2 + s\theta (x+\varepsilon)^{2\alpha+\beta-1}|y_x|^2\right) e^{2s\varphi} \mathrm{d}x\mathrm{d}t \\ \le& C\int_{\Omega_T} |\mathcal{L}[y]|^2 e^{2s\varphi} \mathrm{d}x\mathrm{d}t + C\int_{\omega_T^{(2)}} s\theta |y_x|^2 e^{2s\varphi} \mathrm{d}x\mathrm{d}t + C\int_{\omega_T^{(2)}} s^3\theta^3 |y|^2 e^{2s\varphi} \mathrm{d}x\mathrm{d}t.\end{aligned} \tag{A.25}$$

Here we have used that $\varphi_1 = \varphi_2 = \varphi$ in $\omega^{(3)}$ due to $\eta_1 = \eta_2$ in $\omega^{(3)}$. Then by using Caccioppoli equality [2]

$$\int_{\omega_T^{(2)}} s\theta |y_x|^2 e^{2s\varphi} \mathrm{d}x\mathrm{d}t \le C \int_{\omega_T^{(1)}} \left(s^3\theta^3|y|^2 + |\mathcal{L}[y]|^2\right) e^{2s\varphi} \mathrm{d}x\mathrm{d}t, \tag{A.26}$$

we have

$$\begin{aligned}&\int_{\Omega_T}\left(s^3\theta^3(x+\varepsilon)^{2\alpha+3\beta-1}|y|^2+s\theta(x+\varepsilon)^{2\alpha+\beta-1}|y_x|^2\right)e^{2s\varphi}\mathrm{d}x\mathrm{d}t\\ \leq& C\int_{\Omega_T}|\mathcal{L}[y]|^2e^{2s\varphi}\mathrm{d}x\mathrm{d}t+C\int_{\omega_T^{(1)}}s^3\theta^3|y|^2e^{2s\varphi}\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{A.27}$$

Since for any $\alpha\in(0,\frac{1}{3})$ and any $\beta\in\left(0,\frac{2}{9}-\frac{2\alpha}{3}\right)$,

$$\alpha>2\alpha+\beta-1\quad\text{and}\quad-\gamma>2\alpha+3\beta-1\ \text{ for any }\gamma\in\left[\alpha,\frac{1}{3}\right),\tag{A.28}$$

we then obtain the desired estimate (3.11) and complete the proof of Lemma 3.2. □

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