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LOCAL FEEDBACK STABILIZATION OF TIME-PERIODIC EVOLUTION EQUATIONS BY FINITE DIMENSIONAL CONTROLS

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Abstract. We study the feedback stabilization around periodic solutions of parabolic control systems with unbounded control operators, by controls of finite dimension. We prove that the stabilization of the infinite dimensional system relies on the stabilization of a finite dimensional control system obtained by projection and next transformed *via* its Floquet-Lyapunov representation. We emphasize that this approach allows us to define feedback control laws by solving Riccati equations of finite dimension. This approach, which has been developed in the recent years for the boundary stabilization of autonomous parabolic systems, seems to be totally new for the stabilization of periodic systems of infinite dimension. We apply results obtained for the linearized system to prove a local stabilization result, around periodic solutions, of the Navier-Stokes equations, by finite dimensional Dirichlet boundary controls.

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1. Introduction

In this paper, we are interested in the local feedback stabilization of some time-periodic semilinear parabolic control systems. We consider systems for which the underlying linearized system is a time periodic control system in a real Hilbert space H of the form

$$y'(t) = A(t)y(t) + B(t)u(t), \quad \forall\, t \geq 0, \quad y(0) = y_0 \in H, \tag{1.1}$$

where $A(\cdot)$ is T-periodic and, for each t, $(A(t), \mathcal{D}(A(t)))$ generates an analytic semigroup on H. The control operator $B(\cdot)$ is also T-periodic. For each t, $B(t)$ is an unbounded operator from a real Hilbert space $\mathcal{U}$ into H, and $u(t) \in \mathcal{U}$ is the control.

In a series of recent papers dealing with the feedback stabilization of autonomous systems, feedback control laws are determined by solving finite dimensional Riccati equations [10, 32, 36]. In those papers, the idea is to look for controls of finite dimension. This approach has several advantages. First of all, these feedback laws

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are well adapted to locally stabilize semilinear parabolic systems. And from a numerical viewpoint, even if the semidiscrete approximation of the infinite dimensional system is of very large dimension, the Riccati equation used to define the feedback control law is of very small dimension, and therefore very easy to solve (see [4]).

In the present paper, we would like to adapt to time periodic systems the approach developed in [10, 36] for autonomous systems.

By controls of finite dimension, we mean controls of the form

$$u(t) = \sum_{j=1}^{K} f_j(t) u_j(t), \tag{1.2}$$

where the functions $u_j(t) \in \mathcal{U}$ are considered as actuators, and are chosen to satisfy some stabilizability property. The family $\{u_j(t) \in \mathcal{U} \mid 1 \le j \le K\}$ can also be chosen independent of the time variable t, see ([7], Cor. 3.12). For $1 \le j \le K$, the functions $f_j \in L^2(0,\infty)$ are the control variables that we are going to look for in feedback form. When u is of the form (1.2), we can rewrite the system (1.1) in the form

$$y'(t) = A(t)y(t) + \mathbb{B}(t)f(t), \quad \forall\, t \ge 0, \quad y(0) = y_0 \in H, \tag{1.3}$$

where $f(t) = (f_1(t), \cdots, f_K(t)) \in \mathbb{R}^K$ and

$$\mathbb{B}(t)f(t) = \sum_{j=1}^{K} f_j(t) B(t) u_j(t). \tag{1.4}$$

In order to determine feedback control laws stabilizing system (1.3), we need to prove that the pair $(A(t), \mathbb{B}(t))_{t\ge 0}$ is stabilizable. In [7], for a time-periodic linear system corresponding to a periodic operator $A(\cdot)$ and a periodic control operator $B(\cdot)$, satisfying Assumptions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$, various equivalent necessary and sufficient conditions for open-loop stabilizability of $(A(t), B(t))_{t\ge 0}$ have been proved. Further, seeking finite-dimensional stabilizing controls in the form (1.2), as a linear combination of actuators, the existence of such families of actuators has been established in [7] along with a lower bound for the dimension of the family of actuators. However, unlike in the autonomous case, for time-varying systems the feedback stabilizability of $(A(t), \mathbb{B}(t))_{t\ge 0}$ may not be equivalent to open-loop stabilizability (see [7], Rem. 3.2). The main goals of the present paper are to obtain feedback stabilizability results for system (1.3) and to determine explicit feedback laws.

More precisely, the main results of the paper are the following ones:

– When the pair $(A(t), \mathbb{B}(t))$ is open-loop stabilizable with a prescribed exponential decay rate $\sigma \ge 0$, we determine, *via* Riccati equations of finite dimension, feedback control laws able to stabilize the system (1.3) (Thms. 3.8 and 3.10(i)).

– We study the regularity properties of the evolution operator of the associated closed-loop linear system in Theorems 3.10(ii) and 3.11.

– We show that these results may be used to prove a local stabilization result for the two dimensional Navier-Stokes equations by a Dirichlet boundary control (Thm. 4.13).

To the best of our knowledge, Theorem 4.13 is the first local feedback stabilization result of a semilinear parabolic system in a neighborhood of a periodic solution, with an unbounded control operator.

Let us give some detailed comments on the above results and on the outline of the paper. In Sections 2.1 and 2.2, we state assumptions on $(A(t))_{t\in\mathbb{R}}$ and $(B(t))_{t\in\mathbb{R}}$, and we state results on the evolution operator generated by $(A(t))_{t\in\mathbb{R}}$. The results in Section 2.3 are borrowed from ([7], Section 2.4), and we repeat them for the convenience of the reader. In particular, we introduce the decomposition of $H = H_u(t) \oplus H_s(t)$ into its *time dependent stable and unstable subspaces* for $A(t) + \sigma I$, where $\sigma \ge 0$ is the prescribed exponential decay rate, together with the associated projections $P_u(t)$ and $P_s(t)$, and their properties.

In Section 3, using the Floquet-Lyapunov representation of the unstable component $y_u(t) = P_u(t)y(t)$ of system (1.3), we obtain a control system in $H_{\mathbb{C},u}(0) = P_u(0)H_{\mathbb{C}}$, where $H_{\mathbb{C}} = H + iH$ is the complexified space associated with H. We determine a feedback control law stabilizing this system in $H_{\mathbb{C},u}(0)$, from which we determine a feedback law stabilizing system (1.3) in H, with the prescribed exponential decay rate σ. The regularity properties that we prove for the associated closed-loop linear system are useful to study the local feedback stabilization of nonlinear systems of the form

$$y'(t) = A(t)y(t) + N(y(t)) + B(t)u(t) \quad \text{in } (0,\infty), \quad y(0) = y_0 \in H, \tag{1.5}$$

or

$$y'(t) = A(t)y(t) + N(y(t), u(t)) + B(t)u(t) \quad \text{in } (0,\infty), \quad y(0) = y_0 \in H. \tag{1.6}$$

In Section 4, we apply the results of Section 3 to prove the local stabilization of the Navier-Stokes equations, around a time periodic solution, by a Dirichlet boundary control. As in the autonomous case [8, 10, 35, 36], we have to deal with a system of the form (1.6). To prove that the pair $(A(t), B(t))$ of the Oseen system, coming from the linearization of the Navier-Stokes equations around that periodic solution, is stabilizable with a prescribed exponential decay rate $\sigma \geq 0$, we have to verify the following stabilizability criterion borrowed from ([7], Thm. 3.3):

$$\begin{aligned}
&\text{If } (\lambda, q) \in \mathbb{C} \times C([0,T]; H_{\mathbb{C}}) \text{ is a solution to the eigenvalue problem} \\
&\lambda \in \mathbb{C},\ |\lambda| > e^{-\sigma T} \text{ with } \sigma \geq 0, \quad -q' = A^*(t)q \ \text{ in } (0,T), \quad q(0) = \lambda q(T), \\
&\text{and if } \quad B^*(\cdot)q \equiv 0 \text{ in } (0,T), \quad \text{then } q = 0.
\end{aligned} \tag{1.7}$$

In Proposition 4.4, we prove that this stabilizability criterion is satisfied. Next, in Lemma 4.5, using a result from ([7], Cor. 3.19), we deduce that there exists a family $(u_j(t))_{j=1}^K$ of regular controls for which the pair $(A(t), \mathbb{B}(t))$ is stabilizable with the prescribed exponential decay rate $\sigma \geq 0$. We underline that, if K is large enough, the family can even be chosen independent of the time variable, see Remark 4.6. In contrast, we show in Proposition 4.7 that the Oseen operator satisfies a unique continuation property stronger than (1.7), allowing to choose a family reduced to a single element ($K = 1$) so that the corresponding pair $(A(t), \mathbb{B}(t))$ is stabilizable with the prescribed exponential decay rate $\sigma \geq 0$.

Next, we prove in Theorem 4.13 that the feedback control law determined in Section 3 is able to locally stabilize the Navier-Stokes equations (Thm. 4.13). To shorten the paper, we do not study the local feedback stabilization of general systems of the form (1.5) or (1.6). We have only studied the particular case of the two dimensional Navier-Stokes equations.

For dynamical systems, periodic solutions play a major role in many applications [14]. Stabilizing nonlinear infinite dimensional dynamical systems in a neighborhood of periodic solutions is a challenging problem. For example, understanding what is called the auto-regulation for blood flow in the brain may be viewed as a feedback stabilization of a control dynamical system (see e.g. [5]). The feedback stabilization of linear parabolic systems is studied in [3], and of semilinear parabolic systems in [11] for bounded control operators. In those papers, feedback control laws are determined through the solutions to infinite dimensional Riccati equations. The idea of splitting the solution to system (1.1) into its unstable and stable components is widely used in the literature (see [11, 26]). But as far as we know, there is no paper using this decomposition to determine a feedback control law based on the solution of a finite dimensional Riccati equation for systems of the form (1.1), even in the case of a bounded control operator.

We are not aware of works dealing with the local feedback stabilization of time-periodic semilinear parabolic systems in the case of unbounded control operators. However we would like to mention a recent work in which the local feedback stabilization to a given non-stationary trajectory of the Navier-Stokes system, using a finite dimensional boundary control, in a two or three dimensional bounded domain, has been studied [37]. The novelty is that the non-stationary trajectory is not time periodic. However, even if the control is of finite dimension, the feedback is obtained by solving a Riccati equation set in the whole infinite dimensional space. Moreover, contrarily to what we do in the periodic case, the total number of controls is not explicitly known. As mentioned in ([7], Sect. 1), in many applications it is meaningful to stabilize a system around a time-periodic target trajectory. In that case our approach, based on the spectral analysis of the Poincaré map, allows us to determine feedback laws stabilizing an infinite dimensional system by looking at feedback laws stabilizing a projected system of finite dimension. This leads to solve Riccati equations of finite dimension. Thus, in the case of periodic system, our approach seems to be more appropriate.

2. Time-periodic linear system

2.1. Notations

For a Hilbert space X, $\|\cdot\|_X$ and $(\cdot,\cdot)_X$ denote the norm and the scalar product of X respectively, $C_{\rm per}([0,T];X)$ (respectively $C^1_{\rm per}([0,T];X)$, $C^\alpha_{\rm per}([0,T];X)$ with $\alpha\in(0,1)$) denotes the space of T-periodic functions belonging to $C(\mathbb{R};X)$ (respectively $C^1(\mathbb{R};X)$, $C^\alpha(\mathbb{R};X)$). Similarly, for $s>0$, $H^s_{\rm per}(0,T;X)$ denotes the space of T-periodic functions belonging to $H^s_{\rm loc}(\mathbb{R};X)$.

Recall that $\mathcal{L}(X,Y)$ denotes the space of bounded linear operators from X into Y equipped with the uniform operator topology. In $C([0,T];\mathcal{L}_s(X,Y))$, $\mathcal{L}_s(X,Y)$ denotes the space of bounded linear operators from X into Y equipped with the strong operator topology, *i.e.*, $U\in C([0,T];\mathcal{L}_s(X,Y))$ means that for each $x\in X$, $t\mapsto U(t)x$ is continuous from $[0,T]$ into Y. We use the shorter expressions $\mathcal{L}(X)=\mathcal{L}(X,X)$ and $\mathcal{L}_s(X)=\mathcal{L}_s(X,X)$.

We sometimes use C as a generic positive constant which may vary in different estimates.

2.2. Time-periodic linear operator and related evolution operator

Let $\{(A(t),D(A(t))\}_{t\in\mathbb{R}}$ be a family of unbounded operators in H. We denote by $\{(A^*(t),D(A^*(t))\}_{t\in\mathbb{R}}$ the family of adjoints with respect to H. We make the following assumptions.

Assumption ($\mathcal{A}_1$). (i) The mapping $t\mapsto A(t)$ is T-periodic, *i.e.*,

$$A(t+T)=A(t),\quad \forall\, t\in\mathbb{R}.$$

(ii) For all $t\in\mathbb{R}$, the domain of $A(t)$ is $\mathcal{D}(A(t))=D_A$, the domain of $A^*(t)$ is $\mathcal{D}(A^*(t))=D_{A^*}$, where D_A and D_{A^*} are two dense subspaces in H, compactly embedded in H, equipped with the norms

$$\|x\|_{D_A}\stackrel{\rm def}{=}\|x\|_H+\|A(0)x\|_H,\quad \|x\|_{D_{A^*}}\stackrel{\rm def}{=}\|x\|_H+\|A^*(0)x\|_H.$$

Moreover, there exists a constant $c>1$ such that

$$\forall\, t\in[0,T],\ \forall x\in D_A,\quad c^{-1}\|x\|_{D_A}\le\|x\|_H+\|A(t)x\|_H\le c\|x\|_{D_A},$$

$$\forall\, t\in[0,T],\ \forall x\in D_{A^*},\quad c^{-1}\|x\|_{D_{A^*}}\le\|x\|_H+\|A^*(t)x\|_H\le c\|x\|_{D_{A^*}}.$$

(iii) For some $\alpha\in(0,1)$, the mapping $t\mapsto A(t)$ belongs to $C^\alpha_{\rm per}([0,T];\mathcal{L}(D_A,H))$ and the mapping $t\mapsto A^*(t)$ belongs to $C^\alpha_{\rm per}([0,T];\mathcal{L}(D_{A^*},H))$.

(iv) For all $t\in[0,T]$, $(A(t),D_A)$ generates a uniformly (in t) analytic semigroup $\{e^{sA(t)}\}_{s\ge0}$ on H: There exist $M>0$, $\omega\in\mathbb{R}$ and $\vartheta\in(\frac{\pi}{2},\pi]$ such that, for all $t\in[0,T]$, the resolvent set of $A(t)$ contains a sector $S_{\omega,\vartheta}=\{\lambda\in\mathbb{C}\mid\lambda\neq\omega,\ |\arg(\lambda-\omega)|<\vartheta\}$ and

$$\|(\lambda I-A(t))^{-1}\|_{\mathcal{L}(H)}\le\frac{M}{|\lambda-\omega|},\quad\forall\,\lambda\in S_{\omega,\vartheta},\quad\forall\, t\in[0,T],\tag{2.1}$$

where I denotes the identity operator in H. In the following we fix $\lambda_0 > \omega$ and we use the notation

$$\widehat{A}(t) \overset{\text{def}}{=} \lambda_0 I - A(t). \tag{2.2}$$

(v) For all $t \in [0,T]$, $\widehat{A}(t)$ (defined in (2.2)) has uniformly (in t) bounded imaginary powers: There exist $\varepsilon > 0$ and $N \geq 1$ such that, for all $t \in [0,T]$,

$$\forall \tau \in [-\varepsilon, \varepsilon],\ \widehat{A}^{\imath\tau}(t) \in \mathcal{L}(H) \quad \text{and} \quad \|\widehat{A}^{\imath\tau}(t)\|_{\mathcal{L}(H)} \leq N. \tag{2.3}$$

Note that (i), (iv) and (v) in Assumption $(\mathcal{A}_1)$ imply that $A^*(\cdot)$ is T-periodic, and that $A^*(t)$ satisfies the same uniform (in t) estimate as $A(\cdot)$ in (2.1), with the same constant $M > 0$, and therefore it generates a uniformly (in t) analytic semigroup on H. Moreover $\widehat{A}^*(t)$, the adjoint of $\widehat{A}(t)$ defined in (2.2), has uniformly (in t) bounded imaginary powers with the same constant N as in (2.3).

In addition, Assumption $(\mathcal{A}_1)_{(\text{v})}$, implies that, for $\theta \in (0,1)$, the space $\mathcal{D}(\widehat{A}^\theta(t))$ coincides with the (complex) interpolation space $[H, \mathcal{D}(A(t))]_\theta$ and the constants $c_2 > c_1 > 0$ of the norm equivalence $c_1\|\cdot\|_{[H,\mathcal{D}(A(t))]_\theta} \leq \|\widehat{A}^\theta(t)\cdot\|_H \leq c_2\|\cdot\|_{[H,\mathcal{D}(A(t))]_\theta}$ are independent of t (see ([39], Thm. 1.15.3, p. 103) or [41]). The same remark also holds true for $\widehat{A}^*$. In the following, we use the notation

$$D_A^\theta \overset{\text{def}}{=} [H, D_A]_\theta \quad \text{and} \quad D_{A^*}^\theta \overset{\text{def}}{=} [H, D_{A^*}]_\theta, \quad \theta \in [0,1].$$

Due to $(\mathcal{A}_1)_{(\text{ii})}$, for $\theta \in [0,1]$, the spaces $\mathcal{D}(\widehat{A}^\theta(t))$, $\mathcal{D}(\widehat{A}^{*\theta}(t))$ are independent of t and satisfy $\mathcal{D}(\widehat{A}^\theta(t)) = D_A^\theta$, $\mathcal{D}(\widehat{A}^{*\theta}(t)) = D_{A^*}^\theta$. Moreover, there exists $c_\theta > 1$ such that

$$\begin{aligned} &\forall\, t \in [0,T],\ \forall x \in D_A^\theta, \quad c_\theta^{-1}\|x\|_{D_A^\theta} \leq \|\widehat{A}^\theta(t)x\|_H \leq c_\theta\|x\|_{D_A^\theta}, \\ &\forall\, t \in [0,T],\ \forall x \in D_{A^*}^\theta, \quad c_\theta^{-1}\|x\|_{D_{A^*}^\theta} \leq \|\widehat{A}^{*\theta}(t)x\|_H \leq c_\theta\|x\|_{D_{A^*}^\theta}. \end{aligned} \tag{2.4}$$

We set

$$\Delta \overset{\text{def}}{=} \{(s,t) \in \mathbb{R}\times\mathbb{R} \mid s \leq t\} \quad \text{and} \quad \Delta^* \overset{\text{def}}{=} \{(s,t) \in \mathbb{R}\times\mathbb{R} \mid s < t\}.$$

Since $(A(t), D_A)_{t\in\mathbb{R}}$ satisfies the Assumption $(\mathcal{A}_1)$, there exists a unique parabolic evolution operator $\{S(t,s) \mid (s,t) \in \Delta\}$ for $A(\cdot)$ with regularity space D_A (in the sense of definition [6], II.2.1, p. 45). We now state some results mainly already introduced in [7], and which are needed in Section 3.

Proposition 2.1. *We have the following results:*

1. *The parabolic evolution operator $\{S(t,s) \mid (s,t) \in \Delta\}$ satisfies*

$$S \in C^1(\Delta^*; \mathcal{L}_s(D_A, H)) \cap C(\Delta; \mathcal{L}_s(H)) \cap C(\Delta^*; \mathcal{L}(H, D_A)). \tag{2.5}$$

2. *There exists $\tilde{\omega} > \omega$ such that, for all $\theta \in [0, 1+\alpha)$, we have*

$$\forall x \in H,\ \forall (s,t) \in \Delta^*, \quad \|\widehat{A}^\theta(t)S(t,s)x\|_H \leq \frac{C}{(t-s)^\theta} e^{\tilde{\omega}(t-s)}\|x\|_H, \tag{2.6}$$

where ω and α are introduced in Assumption $(\mathcal{A}_1)$.

3. For all θ, $\beta \in [0,1]$ and $x \in H$, we have

$$\|\widehat{A}^{\theta}(t)S(t,s)\widehat{A}^{\beta}(s)x\|_H \leq \frac{C}{(t-s)^{\theta+\beta}} e^{\tilde{\omega}(t-s)}\|x\|_H, \tag{2.7}$$

where $\tilde{\omega}$ is the same exponent as in (2.6).

Proof. The statements (2.5) and (2.6) for $\theta \in [0,1]$ are already stated in ([7], Prop. 2.1). Estimate (2.7) is stated in ([7], Cor. 2.3). Thus, we have only to prove (2.6) for $\theta \in (1, 1+\alpha)$. For that, we are going to use arguments from [6].

Let us set

$$\begin{aligned}
&\widehat{S}(t,s) = e^{-\lambda_0(t-s)}S(t,s), \quad E(t,s) = \widehat{A}(t)e^{-(t-s)\widehat{A}(t)} - \widehat{A}(s)e^{-(t-s)\widehat{A}(s)},\\
&a(t,s) = e^{-(t-s)\widehat{A}(s)}, \quad k(t,s) = -\left[\widehat{A}(t) - \widehat{A}(s)\right] a(t,s),\\
&w(t,s) = \sum_{n=1}^{\infty} \underbrace{k \star \cdots \star k}_{n}(t,s),
\end{aligned}$$

where $\lambda_0 > \omega$ is introduced in (2.2) and $h \star k(t,s) = \int_s^t h(t,\tau)k(\tau,s)d\tau$ for $(t,s) \in \Delta$.

First notice that we have

$$\|k(t,s)\|_{\mathcal{L}(H)} \leq C(t-s)^{\alpha-1} \quad \text{for } s < t.$$

This estimate can be obtained as in ([33], Chap. 5, Cor.6.3), with $(\mathcal{A}_1)$ and using the condition $\lambda_0 > \omega$.

From ([6], Chap. II, (4.3.12)), it follows that $\widehat{S}(t,s) = a + a \star w$. By calculating the derivative of this identity with respect to t, we obtain

$$\begin{aligned}
\widehat{A}(t)\widehat{S}(t,s) = &-\widehat{A}(s)e^{-(t-s)\widehat{A}(s)} + e^{-(t-s)\widehat{A}(t)}w(t,s)\\
&+ \int_s^t E(t,\tau)w(\tau,s)\mathrm{d}\tau + \int_s^t \widehat{A}(t)e^{-(t-\tau)\widehat{A}(t)}[w(t,s) - w(\tau,s)]\mathrm{d}\tau.
\end{aligned} \tag{2.8}$$

(The above expression can be deduced from ([6], Chap. II, (4.3.30)), and it is precisely given in ([33], p. 158) with different notations.) To prove (2.6) for $\theta \in (1, 1+\alpha)$, setting $\theta - 1 = \beta$, we have to prove that

$$\forall (s,t) \in \Delta^*, \quad \|\widehat{A}(t)\widehat{S}(t,s)\|_{\mathcal{L}(H;D_A^{\beta})} \leq \frac{C}{(t-s)^{\beta+1}} e^{\omega_1(t-s)} \quad \text{for some } \omega_1 > 0. \tag{2.9}$$

For that, we are going to use the identity stated in (2.8). We first notice that, using classical estimates for analytic semigroups and the uniform analyticity condition stated in $(\mathcal{A}_1)_{\text{iv}}$, we have

$$\|\widehat{A}(s)e^{-(t-s)\widehat{A}(s)}\|_{\mathcal{L}(H,D_A^{\beta})} \leq \frac{C}{(t-s)^{\beta+1}} \quad \text{for all } s < t,$$

$$\|\widehat{A}(t)e^{-(t-\tau)\widehat{A}(t)}\|_{\mathcal{L}(H,D_A^{\beta})} \leq \frac{C}{(t-\tau)^{\beta+1}} \quad \text{for all } \tau < t,$$

and

$$\|e^{-(t-s)\widehat{A}(t)}w(t,s)\|_{\mathcal{L}(H,D_A^{\beta})} \leq \frac{C}{(t-s)^{\beta}}\|w(t,s)\|_{\mathcal{L}(H)} \quad \text{for all } s < t.$$

A slight modification of the proof of ([6], II.4.3, Lem. 4.3.2) yields

$$\|E(t,\tau)\|_{\mathcal{L}(H,D_A^\beta)} \le C(t-\tau)^{\alpha-\beta-1}, \quad \text{for } \tau<t \text{ and } \beta\in(0,\alpha).$$

According to ([6], II.4.3, Lems. 4.3.1 and 4.3.3), we have

$$\|w(t,s)\|_{\mathcal{L}(H)} \le C(t-s)^{\alpha-1}e^{\omega_2(t-s)} \quad \text{for all } s<t,$$

and

$$\|w(t,s)-w(\tau,s)\|_{\mathcal{L}(H)} \le C(t-\tau)^{\beta'}(\tau-s)^{\alpha-\beta'-1}e^{\omega_2(t-s)} \quad \text{for all } s<\tau<t \text{ and } \beta<\beta'<\alpha,$$

for some $\omega_2>0$. Then, using these estimates in (2.8), and the well known identity

$$\int_s^t (t-\tau)^{\alpha'-1}(\tau-s)^{\beta'-1}\mathrm{d}\tau = (t-s)^{\alpha'+\beta'-1}\,\frac{\Gamma(\alpha')\Gamma(\beta')}{\Gamma(\alpha'+\beta')} \quad \text{for all } \alpha'>0,\ \beta'>0,$$

together with the obvious inequality $(t-s)^\alpha \le Ce^{\omega_3(t-s)}$ for some $\omega_3>0$, we obtain $\|\widehat{A}(t)\widehat{S}(t,s)\|_{\mathcal{L}(H,D_A^\beta)} \le C(t-s)^{-1-\beta}e^{\omega_1(t-s)}$ for some $\omega_1>0$, and the proof is complete. □

We denote by $S^*(t,s)$ the adjoint of $S(t,s)$.

Proposition 2.2. *The family $\{S^*(t,s)\mid -\infty<s\le t<\infty\}$ satisfies the following regularity property*

$$S^*\in C^1(\Delta^*;\mathcal{L}_s(D_{A^*},H))\cap C(\Delta;\mathcal{L}_s(H))\cap C(\Delta^*;\mathcal{L}(H,D_{A^*})). \tag{2.10}$$

For details, see ([7], Prop. 2.2).

2.3. Floquet theory

For all $t\in\mathbb{R}$, the Poincaré map is defined by

$$U(t)=S(T+t,t)\in\mathcal{L}(H). \tag{2.11}$$

The spectrum of the Poincaré map $U(\cdot)$ plays an important role in studying periodic evolution equations. Let us now recall some useful properties of $U(\cdot)$ (see *e.g.* [29]):

Proposition 2.3. *Let $U(\cdot)$ be the Poincaré map defined on $\mathbb{R}$ by (2.11). Then, we have the following properties:*

(i) *The map $U(\cdot)$ is T-periodic and it satisfies*

$$\begin{aligned} &U\in C^1_{\mathrm{per}}([0,T];\mathcal{L}_s(D_A,H))\cap C_{\mathrm{per}}([0,T];\mathcal{L}(H,D_A))\\ &\text{and}\quad AU\in L^\infty(0,T;\mathcal{L}(H,D_A^{\alpha-\epsilon}))\quad \text{for all } \epsilon\in(0,\alpha).\end{aligned} \tag{2.12}$$

(ii) *For all $t\in\mathbb{R}$, the spectrum of $U(t)$, except 0, is constituted of isolated eigenvalues independent of t, and with finite algebraic multiplicity. This set will be denoted by $\Sigma(U(t))=\{\lambda_j\mid j\in\mathbb{N}\}$, and the eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$ can be ordered in the following way*

$$\cdots\le|\lambda_{j+1}|\le|\lambda_j|\le\cdots\le|\lambda_1|, \quad \text{and } |\lambda_j|\to 0 \text{ as } j\to\infty.$$

Proof. All these results are stated in ([7], Prop. 2.6), except $(2.12)_2$ which is a consequence of (2.6) with $\theta=1+\alpha-\epsilon$. □

For any $\sigma \geq 0$, there exists $N_\sigma \in \mathbb{N}$ depending on σ such that

$$\cdots \leq |\lambda_{N_\sigma+1}| < e^{-\sigma T} \leq |\lambda_{N_\sigma}| \leq \cdots \leq |\lambda_1|.$$

In the following, without loss of generality, by choosing a larger value of σ if necessary, we assume that

$$|\lambda_{N_\sigma+1}| < e^{-\sigma T} < |\lambda_{N_\sigma}|. \tag{2.13}$$

Thus, for all $t \in \mathbb{R}$, Σ_u, the unstable part of the spectrum of $U(t)$, contains a finite number of eigenvalues $\{\lambda_j\}_{j=1}^{N_\sigma}$ of $U(t)$ lying outside the disk $\{\lambda \in \mathbb{C} \mid |\lambda| \leq e^{-\sigma T}\}$. For all $t \in \mathbb{R}$, let us define the projections

$$P_s(t) = \frac{1}{2\pi i}\int_\Gamma (\lambda I - U(t))^{-1} \mathrm{d}\lambda, \quad P_u(t) = I - P_s(t). \tag{2.14}$$

Here Γ is the circle centered at zero with radius $e^{-\sigma T}$, oriented counterclockwise.

For our later analysis we introduce the complexified spaces

$$H_\mathbb{C} = H + iH \quad \text{and} \quad \mathcal{U}_\mathbb{C} = \mathcal{U} + i\mathcal{U}.$$

The operators $U(\cdot)$, $A(\cdot)$, $S(\cdot,\cdot)$ and $B(\cdot)$ are defined in the complexified spaces by setting $U(\cdot)f = U(\cdot)\mathrm{Re}f + iU(\cdot)\mathrm{Im}f$, with similar definitions for $A(\cdot)$, $S(\cdot,\cdot)$ and $B(\cdot)$.

The decomposition of $H_\mathbb{C}$ associated to $P_u(t)$ and $P_s(t)$ is

$$H_\mathbb{C} = H_{\mathbb{C},u}(t) \oplus H_{\mathbb{C},s}(t), \quad \text{with} \quad H_{\mathbb{C},u}(t) = P_u(t)H_\mathbb{C} \text{ and } H_{\mathbb{C},s}(t) = P_s(t)H_\mathbb{C}, \tag{2.15}$$

for all $t \in \mathbb{R}$, and the associated decomposition of the real space H is

$$H = H_u(t) \oplus H_s(t), \quad \text{with} \quad H_u(t) = P_u(t)H \text{ and } H_s(t) = P_s(t)H, \text{ for all } t \in \mathbb{R}. \tag{2.16}$$

The following lemma is dedicated to classical properties satisfied by P_u and P_s. Similar statements hold for the complex case too.

Lemma 2.4. *Let $P_u(\cdot)$ and $P_s(\cdot)$ be the operators defined in (2.14), and $H_u(t)$ and $H_s(t)$ the spaces defined in (2.16). We have $H = H_u(t) \oplus H_s(t)$ for all $t \in \mathbb{R}$, and the following properties are satisfied.*
(i) *For all $s \in \mathbb{R}$, we have $H_u(s) \subset D_A$ and*

$$\begin{aligned}
&S(t,s) : H_u(s) \to H_u(t), \quad S(t,s) : H_s(s) \to H_s(t), \quad \forall\, t \geq s,\\
&S(t,s)P_u(s) = P_u(t)S(t,s), \quad S(t,s)P_s(s) = P_s(t)S(t,s), \quad \forall\, t \geq s.
\end{aligned}$$

(ii) *The projection operators $P_s(\cdot)$ and $P_u(\cdot)$ both belong to $C^1_{\mathrm{per}}([0,T];\mathcal{L}_s(D_A,H))$ and satisfy the following equalities in $\mathcal{L}_s(D_A,H)$:*

$$P_s' = AP_s - P_sA \quad \text{and} \quad P_u' = AP_u - P_uA. \tag{2.17}$$

Moreover, $P_u(\cdot)$ satisfies the following additional regularity properties:

$$\begin{aligned}
&P_u \in C^1_{\mathrm{per}}([0,T];\mathcal{L}_s(D_A,H)) \cap C_{\mathrm{per}}([0,T];\mathcal{L}(H,D_A)),\\
&\text{and} \quad AP_u \in L^\infty(0,T;\mathcal{L}(H,D_A^{\alpha-\epsilon})) \quad \text{for all } \epsilon \in (0,\alpha).
\end{aligned} \tag{2.18}$$

(iii) *For all* $t \in \mathbb{R}$ *and* $0 \le s \le t$, $S(t,s)|_{H_u(s)}$ *is a bounded operator. Moreover, for all* $\tau > 0$ *satisfying* $|\lambda_{N_\sigma+1}| < e^{-(\sigma+\tau)T}$, *we have*

$$\|S(t,s)x\|_H \le Ce^{-(\sigma+\tau)(t-s)}\|x\|_H, \quad \forall\, x \in H_s(s), \forall\, t \ge s, \tag{2.19}$$

where σ *is the exponent appearing in* (2.13). *Further, for all* $\theta \in [0,1]$ *and for all* $\tau > 0$ *satisfying* $|\lambda_{N_\sigma+1}| < e^{-(\sigma+\tau)T}$, *we also have*

$$\forall x \in H,\ \forall (s,t) \in \Delta^*, \quad \|S(t,s)P_s(s)\widehat{A}^\theta(s)\|_{\mathcal{L}(H)} \le \frac{C}{(t-s)^\theta} e^{-(\sigma+\tau)(t-s)}. \tag{2.20}$$

Proof. See ([7], Prop. 2.6), for all the above results, except the second statement of (2.18) which is an easy consequence of the second statement of (2.12) and of the expression of $P_u(t)$ stated in ([7], (2.26)). ☐

2.4. Control operator and extensions to $(D_{A^*})'$

We study (1.1) with a control operator $B(\cdot)$ satisfying the following assumption.

Assumption $(\mathcal{A}_2)$.

(i) The control operator $t \mapsto B(t)$ belongs to $C^\alpha_{\mathrm{per}}([0,T];\mathcal{L}(\mathcal{U},(D_{A^*})'))$, where $\alpha \in (0,1)$ is the exponent appearing in Assumption $(\mathcal{A}_1)_{(\mathrm{iii})}$.

(ii) There exists $0 < \delta < \alpha$ such that, for all $t \in [0,T]$, the mapping $\widehat{A}^{-\delta}(t)B(t) : \mathcal{U} \mapsto H$ is bounded uniformly in $t \in [0,T]$, where $\widehat{A}$ is defined in (2.2).

This kind of 'unbounded' control operator B arises in the case of boundary controls, see *e.g.* [28, 34]. We stress on the fact that the condition $\delta < \alpha$ in $(\mathcal{A}_2)_{(\mathrm{ii})}$ is required to prove (3.22) in Theorem 3.6. This last result is used in Proposition 3.9(iv), which is finally used in Theorem 3.10 to apply Amann's theory for evolution operators having a generator with variable domain (see [6], Chap. IV). Thus the condition $\delta < \alpha$ is needed in a crucial way to study, in Section 3, the well-posedness of the closed-loop control system.

As we consider equation (1.1) with forcing term $B(t)u(t) \in (D_{A^*})'$, we work with extensions of $A(t)$ and $S(t,s)$ to $(D_{A^*})'$. However, for $y_0 \in H$ and $u \in L^2(0,+\infty;\mathcal{U})$, we recall that – see ([7], Prop. 2.5) – the weak solution of (1.1) is the function $y \in C([0,+\infty);(D_{A^*})') \cap L^2_{\mathrm{loc}}([0,+\infty);H)$ defined by

$$y(t) = S(t,0)y_0 + \int_0^t S(t,s)B(s)u(s)\mathrm{d}s. \tag{2.21}$$

Next, to have a well-defined decomposition of (1.1) with forcing term in $L^1(\mathbb{R}^+;(D_{A^*})')$ we need to extend the projection operators $P_u(t)$ and $P_s(t)$ to $(D_{A^*})'$. For that, we first need to introduce their adjoints which are defined by

$$P_s^*(t) = \frac{1}{2\pi i}\int_\Gamma (\lambda I - U^*(t))^{-1}\mathrm{d}\lambda, \quad P_u^*(t) = I - P_s^*(t), \quad \text{for all } t \in \mathbb{R}, \tag{2.22}$$

where Γ is the circle centered at zero with radius $e^{-\sigma T}$, oriented counterclockwise. We have

$$U^* \in C^1_{\mathrm{per}}([0,T];\mathcal{L}_s(D_{A^*},H)) \cap C_{\mathrm{per}}([0,T];\mathcal{L}(H,D_{A^*})), \tag{2.23}$$

$$A^*U^* \in L^\infty(0,T;\mathcal{L}(H,D^{\alpha-\epsilon}_{A^*})),\ \epsilon \in (0,\alpha). \tag{2.24}$$

As for P_u, we deduce that

$$P_u^* \in C^1_{\text{per}}([0,T];\mathcal{L}_s(D_{A^*},H)) \cap C_{\text{per}}([0,T];\mathcal{L}(H,D_{A^*})), \tag{2.25}$$

$$A^* P_u^* \in L^\infty(0,T;\mathcal{L}(H,D_{A^*}^{\alpha-\epsilon})),\ \epsilon \in (0,\alpha). \tag{2.26}$$

This yields the following lemma.

Lemma 2.5. *The projection operators P_u and P_u^* belong to $W^{1,\infty}(0,T;\mathcal{L}(H))$. Moreover, for all $\theta \in (0,1)$, P_u belongs to $C^\theta_{\text{per}}([0,T];\mathcal{L}(H,D_A^{1-\theta}))$ and $P_u^* \in C^\theta_{\text{per}}([0,T];\mathcal{L}(H,D_{A^*}^{1-\theta}))$.*

Proof. Let us prove the lemma for P_u. The result for P_u^* can be proved analogously. From (2.26), with a duality argument, we first get $P_u A \in L^\infty(0,T;\mathcal{L}(H))$. Thus, from $(2.18)_2$ and (2.17), we deduce that $P_u' \in L^\infty(0,T;\mathcal{L}(H))$, and the first part of the lemma is proved.

Next, from $(2.18)_2$ and $(\mathcal{A}_1)_{(ii)}$, we deduce that $P_u \in L^\infty(0,T;\mathcal{L}(H,D_A))$. Moreover, with the Lipschitz continuity of P_u stated in the first part of the lemma, and the interpolation inequality

$$\|P_u(t)f - P_u(s)f\|_{D_A^{1-\theta}} \leq C\|P_u(t)f - P_u(s)f\|_H^\theta \|P_u(t)f - P_u(s)f\|_{D_A}^{(1-\theta)} \quad \text{for all } f \in D_A,$$

we deduce that $P_u \in C^\theta_{\text{per}}([0,T];\mathcal{L}(H,D_A^{1-\theta}))$. □

Since H is identified with H^*, in addition to (2.16), we have

$$H = H_u^*(t) \oplus H_s^*(t), \quad \text{with} \quad H_u^*(t) = P_u^*(t)H \text{ and } H_s^*(t) = P_s^*(t)H, \text{ for all } t \in \mathbb{R}. \tag{2.27}$$

Since $P_u^*(t) \in \mathcal{L}(H,D_{A^*})$, we can extend $P_u(t)$ from $(D_{A^*})'$ into H, and thus $P_s(t) = I - P_u(t)$ is extended as a bounded linear operator from $(D_{A^*})'$ into $(D_{A^*} \cap H_s^*(t))'$ (see [7], Sect. 2.4) for more details). Then, we have the following decomposition

$$(D_{A^*})' = H_u(t) \oplus (D_{A^*} \cap H_s^*(t))', \quad \forall\, t \in \mathbb{R},$$

and we set

$$B_u(t) = P_u(t)B(t) \in \mathcal{L}(\mathcal{U},H_u(t)), \quad \text{and} \quad B_s(t) = P_s(t)B(t) \in \mathcal{L}(\mathcal{U},(D_{A^*} \cap H_s(t))'). \tag{2.28}$$

As discussed in ([7], Sect. 2.4), the weak solution $y \in C([0,+\infty);(D_{A^*})')$ of (1.1) with initial condition $y_0 \in H$ can be decomposed as follows:

$$y(t) = y_u(t) + y_s(t), \quad \text{where } \ y_u(t) \stackrel{\text{def}}{=} P_u(t)y(t),\ y_s(t) \stackrel{\text{def}}{=} P_s(t)y(t), \tag{2.29}$$

$$y_0 = y_{0,u} + y_{0,s}, \quad \text{where } \ y_{0,u} \stackrel{\text{def}}{=} P_u(0)y_0, \quad y_{0,s} \stackrel{\text{def}}{=} P_s(0)y_0,$$

where $y_u(\cdot) \in C([0,+\infty);H)$ and $y_s(\cdot) \in C([0,+\infty);(D_{A^*})')$ satisfy

$$y_u(t) = S(t,0)y_{0,u} + \int_0^t S(t,s)B_u(s)u(s)\mathrm{d}s, \quad \forall\, t \geq 0, \quad y_u(0) = y_{0,u}, \tag{2.30}$$

$$y_s(t) = S(t,0)y_{0,s} + \int_0^t S(t,s)B_s(s)u(s)\mathrm{d}s, \quad \forall\, t \geq 0, \quad y_s(0) = y_{0,s}. \tag{2.31}$$

2.5. Perturbation of the generator of an analytic semigroup

In several models, $(A(t), D_A)$ is of the form

$$A(t) = -A_0 + A_1(t), \quad \mathcal{D}(A(t)) = D_A, \tag{2.32}$$

where $(-A_0, \mathcal{D}(A_0))$ is the infinitesimal generator of an analytic semigroup on H, with compact resolvent, with 0 in its resolvent set, with bounded imaginary powers for A_0, and where $\{(A_1(t), \mathcal{D}(A_1(t))\}_{t\geq 0}$ is a family of closed linear operators satisfying

$$\forall t \in \mathbb{R}, \quad \mathcal{D}(A_0^\beta) \subset \mathcal{D}(A_1(t)), \quad \mathcal{D}(A_0^{*\beta}) \subset \mathcal{D}(A_1^*(t)) \quad \text{for some } \beta \in (0,1). \tag{2.33}$$

In the following proposition we give sufficient conditions on A_0 and $B(\cdot)$ under which $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ are satisfied. Since its proof is straightforward, we skip it.

Proposition 2.6. *Let $(-A_0, \mathcal{D}(A_0))$ and $\{(A_1(t), \mathcal{D}(A_1(t))\}_{t\in\mathbb{R}}$ be as mentioned above. Let $\{(A(t), \mathcal{D}(A(t))\}_{t\in\mathbb{R}}$ be the family of linear operators defined by (2.32). Let $\alpha \in (0,1)$ and $\delta \in (0,1)$ be such that $\delta < \alpha$. Suppose that*

(a1) $A_1(\cdot)$ is T-periodic,
(a2) There exists $\beta \in (0,1)$ such that the mapping $t \mapsto A_1(t)$ belongs to $C^\alpha_{\mathrm{per}}([0,T]; \mathcal{L}(\mathcal{D}(A_0^\beta), H))$, and $t \mapsto A_1^*(t)$ belongs to $C^\alpha_{\mathrm{per}}([0,T]; \mathcal{L}(\mathcal{D}(A_0^{*\beta}), H))$.
(a3) There exists $c > 0$ such that, for all $t \in \mathbb{R}$, we have

$$\forall x \in \mathcal{D}(A_0^\beta), \quad \|A_1(t)x\|_H \leq c\|A_0^\beta x\|_H, \quad \forall x \in \mathcal{D}(A_0^{*\beta}), \quad \|A_1^*(t)x\|_H \leq c\|A_0^{*\beta} x\|_H.$$

Then, $D_A = \mathcal{D}(A_0)$, $D_{A^*} = \mathcal{D}(A_0^*)$, and $(A(\cdot), D_A)$ satisfies Assumption $(\mathcal{A}_1)$ with α as above.
Let us assume in addition that the family $\{B(t)\}_{t\in\mathbb{R}} \subset \mathcal{L}(\mathcal{U}, (\mathcal{D}(A_0^*))')$ satisfies the following assumptions:

(b1) $B(\cdot)$ is T-periodic,
(b2) The mapping $t \mapsto B(t)$ belongs to $C^\alpha_{\mathrm{per}}([0,T]; \mathcal{L}(\mathcal{U}, (\mathcal{D}(A_0^*))')$,
(b3) For all $t \in \mathbb{R}$, the mapping $A_0^{-\delta}B(t) : \mathcal{U} \mapsto H$ is bounded uniformly in $t \in \mathbb{R}$.

Then $B(\cdot)$ satisfies Assumption $(\mathcal{A}_2)$ with δ as above.

Warning. Throughout what follows, the exponents α and δ are given fixed and are those appearing in $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$.

3. Stabilization of the linearized system

Throughout this section, we assume that $A(\cdot)$ and $B(\cdot)$ satisfy the Assumptions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$, and that $\sigma \geq 0$ is given fixed so that the condition (2.13) is satisfied. We emphasize that $P_u(\cdot)$ and $P_s(\cdot)$, defined in (2.14), depend on σ.

Definition 3.1. The pair $(A(\cdot), B(\cdot))$ is said to be open-loop stabilizable in H, with the prescribed exponential decay rate σ, if, for all $y_0 \in H$, there exists $u \in L^2(0+\infty; \mathcal{U})$ such that $e^{\sigma(\cdot)}u$ belongs to $L^2(0+\infty; \mathcal{U})$ and the solution y of (1.1) obeys

$$\int_0^{+\infty} e^{2\sigma t}\|y(t)\|_H^2 \mathrm{d}t < +\infty. \tag{3.1}$$

Let us recall that the operator $\mathbb{B}(t)$ associated to a family $(u_j)_{j=1}^K$ in $C^\alpha_{\rm per}([0,T];\mathcal{U}^K)$ is defined by

$$\mathbb{B}(t)f = \sum_{j=1}^K f_j\, B(t)u_j(t) \quad \text{for all } f=(f_1,\cdots,f_K)\in\mathbb{R}^K.$$

Thus $\mathbb{B}\in C^\alpha_{\rm per}([0,T];\mathcal{L}(\mathbb{R}^K,(D_{A^*})'))$ satisfies Assumption $(\mathcal{A}_2)$. In addition to $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$, we assume that the following assumption is satisfied.

Assumption $(\mathcal{A}_3)$**.** The family $(u_j)_{j=1}^K$ in $C^\alpha_{\rm per}([0,T];\mathcal{U}^K)$ is such that the pair $(A(\cdot),\mathbb{B}(\cdot))$ is open-loop stabilizable in H, with the prescribed exponential decay rate $\sigma\ge 0$.

If, for $\sigma\ge 0$ given fixed, the pair $(A(\cdot),B(\cdot))$ satisfies the Assumptions $(\mathcal{A}_1)$–$(\mathcal{A}_2)$, and the Hautus criterion (1.7), due to ([7], Cor. 3.16), we can explicitly determine a family $(u_j)_{j=1}^K$ in $C^\alpha_{\rm per}([0,T];\mathcal{U}^K)$ so that $(\mathcal{A}_3)$ is satisfied.

We introduce the operators $\mathbb{B}_u(t)$ and $\mathbb{B}_s(t)$ defined by

$$\mathbb{B}_u(t)f = \sum_{j=1}^K f_j\, B_u(t)u_j(t) \quad\text{and}\quad \mathbb{B}_s(t)f = \sum_{j=1}^K f_j\, B_s(t)u_j(t) \quad \text{for all } f=(f_1,\cdots,f_K)\in\mathbb{R}^K.$$

We recall (see [7]) that y is the solution to system (1.3) if and only if $y(t)=y_u(t)+y_s(t)$, where, for all $t\ge 0$,

$$y_u(t) = S(t,0)P_u(0)y_0 + \int_0^t S(t,s)\mathbb{B}_u(s)f(s)\mathrm{d}s, \tag{3.2}$$

$$y_s(t) = S(t,0)P_s(0)y_0 + \int_0^t S(t,s)\mathbb{B}_s(s)f(s)\mathrm{d}s. \tag{3.3}$$

To determine a feedback control operator $\mathbb{K}\in C^\alpha_{\rm per}([0,T];\mathcal{L}(H,\mathbb{R}^K))$ such that $A(\cdot)+\sigma I+\mathbb{B}(\cdot)\mathbb{K}(\cdot)$ generates an evolution operator exponentially stable on H, we first look for a feedback stabilizing the system (3.2). We shall consider both the cases where $y_{0,u}\in H_{\mathbb{C},u}(0)$ and $y_{0,u}\in H_u(0)$. Next, we use the feedback control depending on $y_u(t)$ as control in system (3.3), and we prove that the corresponding solution y_s remains stable.

3.1. Feedback stabilization of the finite dimensional system

In order to stabilize equation (3.2), it is convenient to use the Floquet Theory which allows us to transform equation (3.2) into an equation in the time independent space $H_{\mathbb{C},u}(0)$.

Proposition 3.2. *There exist bounded linear operators* $\Lambda\in\mathcal{L}(H_{\mathbb{C},u}(0))$ *and* $Q(t)\in\mathcal{L}(H_{\mathbb{C},u}(0),H_{\mathbb{C},u}(t))$ *for all* $t\in\mathbb{R}$*, satisfying* $Q(T+t)=Q(t)$*,* $Q(0)=Q^{-1}(0)=I$ *and*

$$S(t,s)|_{H_{\mathbb{C},u}(s)} = Q(t)e^{(t-s)\Lambda}Q^{-1}(s), \quad \forall\, s\le t. \tag{3.4}$$

Moreover, we have

$$Q(t) = S(t,0)|_{H_{\mathbb{C},u}(0)}e^{-\Lambda t}, \qquad t\ge 0. \tag{3.5}$$

Proof. See ([26], Prop. 6.2, p. 46). □

Proposition 3.3. *Let* Q *be the family of operators introduced in* (3.5)*. The following results hold:*

(i) *The mappings Q, $Q^{-1}P_u$, $P_u^*Q^{*-1}$, where Q^* is the conjugate transpose of Q, satisfy*

$$Q \in C^1_{\rm per}([0,T];\mathcal{L}(H_{\mathbb{C},u}(0),H_{\mathbb{C}}))\cap C_{\rm per}([0,T];\mathcal{L}(H_{\mathbb{C},u}(0),D_A)), \tag{3.6}$$

$$\begin{aligned} Q^{-1}P_u \in\ & C^1_{\rm per}([0,T];\mathcal{L}(H_{\mathbb{C}},H_{\mathbb{C},u}(0)))\cap C_{\rm per}([0,T];\mathcal{L}(D'_{A^*},H_{\mathbb{C},u}(0)))\\ &\cap W^{1,\infty}(0,T;\mathcal{L}((D^{\alpha-\epsilon}_{A^*})',H_{\mathbb{C},u}(0))),\ \text{ for all } \epsilon\in(0,\alpha), \end{aligned}\tag{3.7}$$

$$\begin{aligned} P_u^*Q^{*-1} \in\ & C^1_{\rm per}([0,T];\mathcal{L}(H^*_{\mathbb{C},u}(0),H_{\mathbb{C}}))\cap C_{\rm per}([0,T];\mathcal{L}(H^*_{\mathbb{C},u}(0),D_{A^*}))\\ &\cap W^{1,\infty}(0,T;\mathcal{L}(H^*_{\mathbb{C},u}(0),D^{\alpha-\epsilon}_{A^*}),\ \text{ for all } \epsilon\in(0,\alpha). \end{aligned}\tag{3.8}$$

(ii) *Let y_u be the solution of (3.2), and let us set $z(t)=Q^{-1}(t)y_u(t)$. Then, the function z is the solution to the following system in $H_{\mathbb{C},u}(0)$:*

$$z'(t)=\Lambda z(t)+Q^{-1}(t)\mathbb{B}_u(t)f(t),\quad \forall\, t\ge 0,\quad z(0)=P_u(0)y_0. \tag{3.9}$$

Proof.

Step 1. Let us prove (3.6). Since $S(t,0)P_u(0)=P_u(t)S(t,0)$, we have

$$\forall x\in H_{\mathbb{C},u}(0)\quad Q(t)x=S(t,0)e^{-\Lambda t}x=P_u(t)S(t,0)e^{-\Lambda t}x.$$

Due to (2.5), $S(\cdot,0)$ belongs to $C_{\rm per}([0,T];\mathcal{L}_s(H_{\mathbb{C}}))$. Moreover $e^{-\Lambda\cdot}$ belongs to $C_{\rm per}([0,T];\mathcal{L}(H_{\mathbb{C},u}(0)))$ and P_u belongs to $C_{\rm per}([0,T];\mathcal{L}(H_{\mathbb{C}}(0),D_A))$ (see (2.18)). Therefore Q belongs to $C_{\rm per}([0,T];\mathcal{L}_s(H_{\mathbb{C},u}(0),D_A))$. Since $H_{\mathbb{C},u}(0)$ is of finite dimension, the strong continuity implies the uniform continuity, which implies that $Q\in C_{\rm per}([0,T];\mathcal{L}(H_{\mathbb{C},u}(0),D_A))$. The proof of the continuous differentiability of Q follows in the same way, by using (2.18), (2.5) and $(\mathcal{A}_1)_{(iii)}$, from the identity

$$\forall x\in H_{\mathbb{C},u}(0),\quad Q'(t)x=A(t)P_u(t)S(t,0)e^{-\Lambda t}x-S(t,0)\Lambda e^{-\Lambda t}x.$$

Step 2. Let us prove (3.8). From (3.4) and $P_u^*(t)S^*(T,t)=S^*(T,t)P_u^*(T)$ we deduce that

$$\forall x\in H^*_{\mathbb{C},u}(0),\ P_u^*(t)Q^{*-1}(t)x=P_u^*(t)S^*(T,t)e^{-(T-t)\Lambda^*}x=S^*(T,t)P_u^*(T)e^{-(T-t)\Lambda^*}x. \tag{3.10}$$

Then, for all $x\in H^*_{\mathbb{C},u}(0)$, we have

$$\begin{aligned} (P_u^*(t)Q^{*-1}(t)x)' &= -A^*(t)S^*(T,t)P_u^*(T)e^{-(T-t)\Lambda^*}x+S^*(T,t)\Lambda^*e^{-(T-t)\Lambda^*}x && (3.11)\\ &= -A^*(t)P_u^*(t)S^*(T,t)e^{-(T-t)\Lambda^*}x+S^*(T,t)\Lambda^*e^{-(T-t)\Lambda^*}x. && (3.12) \end{aligned}$$

From the first equality in (3.10) and (3.11), using (2.10) and (2.25), we deduce that $P_u^*Q^{*-1}\in C_{\rm per}([0,T];\mathcal{L}_s(H^*_{\mathbb{C},u}(0),D_{A^*}))$ and $P_u^*Q^{*-1}\in C^1_{\rm per}([0,T];\mathcal{L}_s(H^*_{\mathbb{C},u}(0),H_{\mathbb{C}}))$. Since $H^*_{\mathbb{C},u}(0)$ is of finite dimension, we have $\mathcal{L}_s(H^*_{\mathbb{C},u}(0),D_{A^*})=\mathcal{L}(H^*_{\mathbb{C},u}(0),D_{A^*})$ and $\mathcal{L}_s(H^*_{\mathbb{C},u}(0),H_{\mathbb{C}})=\mathcal{L}(H^*_{\mathbb{C},u}(0),H_{\mathbb{C}})$.

Finally, $(P_u^*Q^{*-1})'\in L^\infty(0,T;\mathcal{L}(H^*_{\mathbb{C},u}(0),D^{\alpha-\epsilon}_{A^*}))$ is a consequence of (3.11) and (2.26). Then (3.8) is proved.

Step 3. The regularities stated in (3.7) follow, by duality, from (3.8).

Step 4. Proof of (3.9). Using (3.2) and $y_u(t) = Q(t)z(t)$, we have $z(0) = y_{0,u}$, and, for all $t \geq 0$, we get

$$z(t) = Q^{-1}(t)S(t,0)P_u(0)y_0 + \int_0^t Q^{-1}(t)S(t,s)Q(s)Q^{-1}(s)\mathbb{B}_u(s)f(s)\,\mathrm{d}s.$$

Now from (3.4), we derive $Q^{-1}(t)S(t,s)Q(s)|_{H_{\mathbb{C},u}(0)} = e^{(t-s)\Lambda}$ for all $0 \leq s \leq t$. It yields

$$z(t) = e^{t\Lambda}P_u(0)y_0 + \int_0^t e^{(t-s)\Lambda}Q^{-1}(s)\mathbb{B}_u(s)f(s)\,\mathrm{d}s, \quad \forall\, t \geq 0.$$

Hence, z satisfies (3.9). □

Now we study the feedback stabilizability of the pair $(\Lambda, Q^{-1}(\cdot)\mathbb{B}_u(\cdot))$.

Proposition 3.4. *The pair* $(\Lambda, Q^{-1}(\cdot)\mathbb{B}_u(\cdot))$ *is open-loop stabilizable in* $H_{\mathbb{C},u}(0)$, *with the prescribed exponential decay rate* σ.

Proof. Since $(\mathcal{A}_3)$ is satisfied, from ([7], Thm. 3.3_2), it follows that $(A(\cdot), \mathbb{B}_u(\cdot))$ is stabilizable in $H_{\mathbb{C}}$, with the prescribed exponential decay rate σ. Then, for all $z_0 \in H_{\mathbb{C},u}(0)$, there exists $f \in L^2(0,\infty;\mathbb{C}^K)$ such that $e^{\sigma\,\cdot}f$ belongs to $L^2(0,\infty;\mathbb{C}^K)$ and such that the solution y_u to

$$y_u'(t) = A(t)y_u(t) + \mathbb{B}_u(t)f(t), \quad \forall\, t \geq 0, \quad y_u(0) = z_0, \tag{3.13}$$

satisfies

$$\int_0^\infty e^{2\sigma t}\|y_u(t)\|^2_{H_{\mathbb{C}}}\,\mathrm{d}t < \infty.$$

Since y_u is given by (3.2), with z_0 instead of $P_u(0)y_0$, we get $y_u(t) \in H_{\mathbb{C},u}(t)$ for all $t \geq 0$. By applying $Q^{-1}(t)$, we deduce that $z = Q^{-1}y_u$ is the solution of

$$z'(t) = \Lambda z(t) + Q^{-1}(t)\mathbb{B}_u(t)f(t), \quad \forall\, t \geq 0, \quad z(0) = z_0.$$

Moreover, from the uniform boundedness of $Q^{-1}P_u$ in $\mathcal{L}(H_{\mathbb{C}}, H_{\mathbb{C},u}(0))$, we obtain that

$$\int_0^\infty e^{2\sigma t}\|z(t)\|^2_{H_{\mathbb{C},u}(0)}\,\mathrm{d}t < \infty.$$

The proof is complete. □

We construct a feedback control using the solution of the following finite dimensional Riccati equation:

$$\begin{aligned}
&\Pi_u \in C^1_{\mathrm{per}}([0,T];\mathcal{L}(H_{\mathbb{C},u}(0), H^*_{\mathbb{C},u}(0))), \qquad \text{for all } t \in [0,T],\\
&\langle \Pi_u(t)\eta, \xi\rangle_H = \langle \eta, \Pi_u(t)\xi\rangle_H, \quad (\eta,\xi) \in H_{\mathbb{C},u}(0) \times H_{\mathbb{C},u}(0),\\
&\langle \Pi_u(t)\xi, \xi\rangle_H \geq 0, \quad \forall\, \xi \in H_u(0),\\
&\Pi_u'(t) + (\Lambda^* + \sigma I)\Pi_u(t) + \Pi_u(t)(\Lambda + \sigma I)\\
&\qquad -\Pi_u(t)(Q^{-1}(t)\mathbb{B}_u(t))(Q^{-1}(t)\mathbb{B}_u(t))^*\Pi_u(t) + Q^*(t)Q(t) = 0.
\end{aligned} \tag{3.14}$$

We look for solutions to equation (3.14) satisfying the following additional property:

$$\begin{aligned}&\text{The evolution operator generated by } \Lambda+\sigma I-Q^{-1}\mathbb{B}_u(Q^{-1}\mathbb{B}_u)^*\Pi_u\\ &\text{is uniformly exponentially stable on } H_{\mathbb{C},u}(0).\end{aligned} \tag{3.15}$$

The evolution operator involved in (3.15) is uniformly exponentially stable on $H_{\mathbb{C},u}(0)$, with the exponential decay rate $\tau>0$, if and only if the solution to the closed-loop system

$$z'(t)=[\Lambda-Q^{-1}(t)\mathbb{B}_u(t)(Q^{-1}(t)\mathbb{B}_u(t))^*\Pi_u(t)]z(t),\quad \forall\, t\geq 0,\quad z(0)=z_0\in H_{\mathbb{C},u}(0), \tag{3.16}$$

satisfies

$$\|z(t)\|_{H_\mathbb{C}}\leq Me^{-(\sigma+\tau)t}\|z_0\|_{H_\mathbb{C}},\quad \text{for all } t\geq 0. \tag{3.17}$$

Proposition 3.5. *We assume that Assumptions $(\mathcal{A}_1)$–$(\mathcal{A}_3)$ together with (2.13) are satisfied. The Riccati equation (3.14), with the additional condition (3.15), admits a unique solution.*

Proof. From Proposition 3.4 and ([7], Thm. 3.3) applied to the pair $(\Lambda, Q^{-1}(\cdot)\mathbb{B}_u(\cdot))$, it follows that this pair satisfies the following Hautus condition:

$$\begin{aligned}&\text{If } (\lambda,z)\in\mathbb{C}\times C([0,T];H_\mathbb{C}) \text{ is a solution to the eigenvalue problem}\\ &\lambda\in\mathbb{C},\ |\lambda|>e^{-\sigma T},\quad -z'=\Lambda^*z \ \text{ in } (0,T),\quad z(0)=\lambda z(T),\\ &\text{and if }\quad \mathbb{B}_u^*(\cdot)Q^{-*}(\cdot)z\equiv 0 \text{ in } (0,T),\quad \text{then } z=0.\end{aligned} \tag{3.18}$$

Thus, from ([1], Thm. 5.4.17), we deduce that equation (3.14) admits a unique semi-stabilizing solution $\Pi_u\in C^1_{\rm per}([0,T];\mathcal{L}(H_{\mathbb{C},u}(0),H^*_{\mathbb{C},u}(0)))$. Since there is no eigenvalue λ of the eigenvalue problem stated in (3.18) satisfying $|\lambda|=e^{-\sigma T}$, from ([1], Thm.5.4.17(i)), we deduce that the solution Π_u obeys (3.15). □

Let us set

$$\Pi(t)=P_u^*(t)Q^{-1*}(t)\Pi_u(t)Q^{-1}(t)P_u(t)\quad \text{for all } t\in[0,T]. \tag{3.19}$$

Theorem 3.6. *Let Π_u be a solution of (3.14). Then, Π, defined in (3.19), satisfies the following regularity properties:*

$$\Pi\in C^1_{\rm per}([0,T];\mathcal{L}(H_\mathbb{C}))\cap C_{\rm per}([0,T];\mathcal{L}(D'_{A^*},D_{A^*})). \tag{3.20}$$

$$\Pi\in W^{1,\infty}(0,T;\mathcal{L}((D^{\alpha-\epsilon}_{A^*})',D^{\alpha-\epsilon}_{A^*}))\quad \text{for all } \epsilon\in(0,\alpha), \tag{3.21}$$

$$\mathbb{B}^*\Pi\in C^\alpha_{\rm per}([0,T];\mathcal{L}(H,\mathbb{R}^K)), \tag{3.22}$$

$$\mathbb{B}_u\mathbb{B}_u^*\Pi\in C^\rho_{\rm per}([0,T];\mathcal{L}(H))\qquad \textit{with}\quad \rho=\frac{\alpha(1-\delta)}{1-\delta+\alpha}. \tag{3.23}$$

Proof.

Step 1. Since Π_u belongs to $C^1_{\rm per}([0,T];\mathcal{L}(H_{u,\mathbb{C}}(0),H^*_{u,\mathbb{C}}(0)))$, we deduce (3.20) and (3.21) from (3.7), (3.8) and (3.19).

Step 2. To prove (3.22), for $(s,t)\in[0,T]^2$, we write

$$\mathbb{B}^*(t)\Pi(t)-\mathbb{B}^*(s)\Pi(s)=(\mathbb{B}^*(t)-\mathbb{B}^*(s))\widehat{A}^{*-1}(t)\widehat{A}^*(t)\Pi(t)+\mathbb{B}^*(s)\widehat{A}^{*-\delta}(t)\widehat{A}^{*\delta}(t)(\Pi(t)-\Pi(s)).$$

Then (3.22) follows from the uniform boundedness of $\widehat{A}^*(t)\Pi(t)$ and that of $\mathbb{B}^*(s)\widehat{A}^{*-\delta}(t)$, which are respective consequences of (3.20) and of $(\mathcal{A}_2)_{\rm (ii)}$, combined with $(\mathcal{A}_2)_{\rm (i)}$ and (3.21). Indeed, as $\delta<\alpha$, (3.21) implies that Π is Lipschitz continuous with value in $\mathcal{L}(H,D^\delta_{A*})$.

Step 3. Let us prove (3.23). First, from Assumption $(\mathcal{A}_2)$, it follows that $\mathbb{B}$ belongs to $C^\alpha_{\rm per}([0,T];\mathcal{L}(\mathbb{R}^K,(D_{A^*})'))$ and to $L^\infty(0,T;\mathcal{L}(\mathbb{R}^K,(D^\delta_{A^*})'))$. With the interpolation inequality

$$\|\mathbb{B}(t)f-\mathbb{B}(s)f\|_{(D^{\theta+\delta(1-\theta)}_{A^*})'}\le C\|\mathbb{B}(t)f-\mathbb{B}(s)f\|^\theta_{(D_{A^*})'}\|\mathbb{B}(t)f-\mathbb{B}(s)f\|^{(1-\theta)}_{(D^\delta_{A^*})'},$$

for all $f\in\mathbb{R}^K$, we can prove that $\mathbb{B}$ belongs to $C^{\alpha\theta}_{\rm per}([0,T];\mathcal{L}(\mathbb{R}^K,(D^{\theta+\delta(1-\theta)}_{A^*})'))$ for any $\theta\in(0,1)$. By choosing $\theta=\frac{1-\delta}{1-\delta+\alpha}$, we obtain $\mathbb{B}\in C^\rho_{\rm per}([0,T];\mathcal{L}(\mathbb{R}^K,(D^{1-\rho}_{A^*})'))$ with $\rho=\frac{\alpha(1-\delta)}{1-\delta+\alpha}$. Next, from Lemma 2.5, it yields that $P^*_u\in C^\rho([0,T];\mathcal{L}(H,D^{1-\rho}_{A^*}))$. Then, by duality, $P_u\in C^\rho([0,T];\mathcal{L}((D^{1-\rho}_{A^*})',H))$. Hence, we deduce $P_u\mathbb{B}\in C^\rho_{\rm per}([0,T];\mathcal{L}(\mathbb{R}^K,H))$, and since $\mathbb{B}_u(t)\mathbb{B}^*_u(t)\Pi(t)=P_u(t)\mathbb{B}(t)\mathbb{B}^*(t)\Pi(t)$, the conclusion follows from (3.22). □

Due to (3.23), the family of operators $((A(t)-\mathbb{B}_u(t)\mathbb{B}^*_u(t)\Pi(t),D_A))_{t\in\mathbb{R}}$ satisfies the Assumption $(\mathcal{A}_1)$, and therefore it is the generator of an evolution operator on $H_\mathbb{C}$, with regularity space D_A.

We consider the following infinite dimensional Riccati equation

$$\begin{aligned}&\Pi\in C^1_{\rm per}([0,T];\mathcal{L}(H_\mathbb{C})),\quad \text{for all } t\in[0,T],\quad \Pi(t)=\Pi^*(t)\ge0,\\&\Pi'(t)+(A^*(t)+\sigma I)\Pi(t)+\Pi(t)(A(t)+\sigma I)-\Pi(t)\mathbb{B}_u(t)\mathbb{B}^*_u(t)\Pi(t)+P^*_u(t)P_u(t)=0,\\&\text{and the evolution operator generated by}\\&A(t)+\sigma I-\mathbb{B}_u(t)\mathbb{B}^*_u(t)\Pi(t)\quad\text{is uniformly exponentially stable on } H_\mathbb{C}.\end{aligned}\tag{3.24}$$

Equation $(3.24)_{2-3}$ must be interpreted as an equality in $\mathcal{L}(D_A,H_\mathbb{C})$.

In order to establish a relationship between equations (3.14) and (3.24), for any solution Π to equation (3.24), we introduce the operator $\widetilde{\Pi}_u$ defined by

$$\widetilde{\Pi}_u(t)=Q^*(t)\Pi(t)Q(t),\quad \text{for all}\, t\in[0,T].\tag{3.25}$$

Theorem 3.7. *Let Π_u be the solution of (3.14)–(3.15). Then, Π, defined in (3.19) is a solution of (3.24).*

Conversely, let Π be a solution of (3.24). Then $\widetilde{\Pi}_u$, defined by (3.25), is solution of (3.14)–(3.15). Consequently, (3.24) admits a unique solution.

Proof.

Step 1. Let us prove the first statement. Let Π_u be the solution of (3.14)–(3.15). From the equality $\frac{\rm d}{{\rm d}t}(Q^{-1}(t)P_u(t))=\Lambda Q^{-1}(t)P_u(t)-Q^{-1}(t)P_u(t)A(t)$ and (3.14), we deduce that $\Pi(t)$ satisfies $(3.24)_{1-3}$.

Next, to prove $(3.24)_4$, we first remark that the solution to

$$y'(t) = [A(t) - \mathbb{B}_u(t)(\mathbb{B}_u(t))^*\Pi(t)]y(t), \quad \forall\, t \geq 0, \quad y_u(0) = y_0 \in H_{\mathbb{C}}, \tag{3.26}$$

can be decomposed as $y = y_u + y_s$, where $y_s(t) = S(t,0)P_s(0)y_0$ and

$$y_u(t) = S(t,0)P_u(0)y_0 - \int_0^t S(t,s)\mathbb{B}_u(s)(\mathbb{B}_u(t))^*\Pi(t)y_u(s)\mathrm{d}s.$$

Due to (2.19), to prove $(3.24)_4$ it is sufficient to show that y_u satisfies

$$\|y_u(t)\|_{H_{\mathbb{C}}} \leq Me^{-(\sigma+\tau)t}\|y_0\|_{H_{\mathbb{C}}}, \quad \text{for all } t \geq 0, \tag{3.27}$$

for some positive constants M and τ independent of t. Let us set $z(t) = Q^{-1}(t)y_u(t)$. Then z is the solution of (3.16) and satisfies (3.17) with $z_0 = P_u(0)y_0$. We deduce (3.27) from the regularity of $Q(t)$ stated in (3.6).

Step 2. Let us prove the converse statement. Let Π be a solution to equation (3.24) and let $\widetilde{\Pi}_u(t)$ be defined by (3.25). By using the definition of $Q(t)$ in (3.5), and calculating its derivative, we can deduce that $\widetilde{\Pi}_u$ satisfies (3.14).

Let us prove that $\widetilde{\Pi}_u$ satisfies (3.15) or, equivalently, that the solution of (3.16) satisfies (3.17). By setting $z(t) = Q^{-1}(t)P_u(t)y(t)$ and using the uniform boundedness of $Q^{-1}(t)P_u(t)$ stated in (3.7), we deduce that z is the solution of

$$z'(t) = \Big(\Lambda - Q^{-1}(t)\mathbb{B}_u(t)(Q^{-1}(t)\mathbb{B}_u(t))^*\widetilde{\Pi}_u(t)\Big)z(t), \quad z(0) = P_u(0)y_0,$$

and that $z(t)$ obeys (3.17) with $z_0 = P_u(0)y_0$.

The uniqueness of solution to (3.24) follows from the uniqueness of solution to (3.14)–(3.15). The proof is complete. □

Theorem 3.8. *The mapping* Π *in* (3.19) *is a real valued operator, and it is the unique solution to the Riccati equation* (3.24)*.*

Proof. Let Π be a solution of (3.24). We can see that $\overline{\Pi}$, the conjugate of Π, also satisfies (3.24) and that $A(\cdot) + \sigma I - \mathbb{B}_u(\cdot)\mathbb{B}_u^*(\cdot)\overline{\Pi}$ is the generator of an exponentially stable evolution operator on $H_{\mathbb{C}}$. Setting $\widetilde{\Pi}_u = Q^*(t)\overline{\Pi}(t)Q(t)$ for all $t \in (0,T)$, from Theorem 3.7, it follows that $\widetilde{\Pi}_u$ is a solution to (3.14)–(3.15). From the uniqueness of solution to (3.14)–(3.15) it follows that $\Pi_u = \widetilde{\Pi}_u$, *i.e.*

$$Q^*(t)\Pi(t)Q(t) = Q^*(t)\overline{\Pi}(t)Q(t), \quad t \in (0,T).$$

Since, for all $t \in [0,T]$, $Q(t)$ and $Q^*(t)$ are invertible, we deduce that $\Pi = \overline{\Pi}$, and that the unique solution to the Riccati equation (3.24) is a real valued operator. □

3.2. Stabilization of the full system

For $t \in \mathbb{R}$, we introduce the unbounded linear operator $(A_\Pi(t), \mathcal{D}(A_\Pi(t)))$ in H defined by

$$\mathcal{D}(A_\Pi(t)) = \{x \in H \mid A(t)x - \mathbb{B}(t)\mathbb{B}^*(t)\Pi(t)x \in H\} \quad \text{and} \quad A_\Pi(t) = A(t) - \mathbb{B}(t)\mathbb{B}^*(t)\Pi(t).$$

The main properties of $A_\Pi(t)$ are stated in the following proposition:

Proposition 3.9. *Let $0 < \delta < \alpha < 1$ be the exponents appearing in Assumptions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$. The following results hold.*

(i) *For all $t \in \mathbb{R}$, the adjoint of the unbounded operator $(A_\Pi(t), \mathcal{D}(A_\Pi(t)))$ is defined by*

$$\mathcal{D}(A^*_\Pi(t)) = \mathcal{D}(A^*(t)) = D_{A^*} \quad \text{and} \quad A^*_\Pi(t) = A^*(t) - [\mathbb{B}^*(t)\Pi(t)]^*\mathbb{B}^*(t). \tag{3.28}$$

Moreover, there exists a constant $c > 0$ such that

$$\forall\, t \in \mathbb{R},\ \forall x \in D_{A^*}, \quad c^{-1}\|x\|_{D_{A^*}} \leq \|x\|_H + \|A^*_\Pi(t)x\|_H \leq c\|x\|_{D_{A^*}}. \tag{3.29}$$

(ii) *For all $t \in \mathbb{R}$, the operator $A_\Pi(t)$ generates a uniform (in t) analytic semigroup $\{e^{sA_\Pi(t)}\}_{s\geq 0}$ on H. In the following, we set*

$$\widehat{A}_\Pi(t) \overset{def}{=} \lambda_0 I - A_\Pi(t)$$

where $\lambda_0 \geq 0$ is chosen large enough such that, for all $t \in \mathbb{R}$, the semigroup $(e^{-s\widehat{A}_\Pi(t)})_{s\geq 0}$ is exponentially stable. Without loss of generality, we can suppose that the value λ_0 is the one appearing in (2.2).

(iii) *For all $t \in \mathbb{R}$, we have the following uniform (in t) topological equalities*

$$\forall \theta \in [0, 1-\delta], \qquad \mathcal{D}(\widehat{A}^\theta_\Pi(t)) = D^\theta_A, \tag{3.30}$$

$$\forall \theta \in [0, 1], \qquad \mathcal{D}(\widehat{A}^{*\theta}_\Pi(t)) = D^\theta_{A^*}. \tag{3.31}$$

(iv) *The mapping $t \mapsto \widehat{A}^{-1}_\Pi(t)$ belongs to $C^\alpha_{\mathrm{per}}([0,T]; \mathcal{L}(H, D^{1-\delta}_A))$. For $\theta \in (0,1)$, the mapping $t \mapsto \widehat{A}^{-\theta}_\Pi(t)$ belongs to $C^{\theta\alpha}_{\mathrm{per}}([0,T]; \mathcal{L}(H))$.*

Proof.

Step 1. Let us prove (i) and (ii). We notice that $[\mathbb{B}^*(t)\Pi(t)]^*\mathbb{B}^*(t)$ is a uniformly bounded operator from $D^\delta_{A^*}$ to H. Hence using the perturbation result from ([33], Chap. 3, Cor. 2.4, p. 81), we conclude that $(A^*_\Pi(t), \mathcal{D}(A^*_\Pi(t))) = (A^*_\Pi(t), \mathcal{D}(A^*(t)))$, and that $(A^*_\Pi(t), D_{A^*})$ is the infinitesimal generator of a uniformly (in t) analytic semigroup on H. The analyticity of $(A_\Pi(t), \mathcal{D}(A_\Pi(t)))$ follows by a duality argument.

Step 2. Let us prove (iii). Since $[\mathbb{B}^*(t)\Pi(t)]^*\mathbb{B}^*(t) \in \mathcal{L}(D^\delta_{A^*}, H)$ uniformly in t, a perturbation argument together with Assumption $(\mathcal{A}_1)_{(\mathrm{v})}$ allows us to deduce that $\widehat{A}^*_\Pi(t)$ has uniformly (in t) bounded imaginary powers (see [6], Thm. 4.8.7, p. 172 or [20], Prop. 2). By a duality argument it follows that $\widehat{A}_\Pi(t)$ has also uniformly (in t) bounded imaginary powers. Then using ([39], Thm. 1.15.3, p. 103) or [41], the following uniform (in t) topological equalities

$$[H, \mathcal{D}(\widehat{A}^*_\Pi(t))]_\theta = \mathcal{D}(\widehat{A}^{*\theta}_\Pi(t)), \qquad [H, \mathcal{D}(\widehat{A}_\Pi(t))]_\theta = \mathcal{D}(\widehat{A}^\theta_\Pi(t)),$$

are true for $\theta \in (0,1)$. Then, (3.31) follows from (i).

To prove (3.30), we first remark that

$$\widehat{A}_\Pi(t) = \widehat{A}(t)G(t) \quad \text{where} \quad G(t) \overset{\text{def}}{=} I + \widehat{A}^{-1}(t)\mathbb{B}(t)\mathbb{B}^*(t)\Pi(t). \tag{3.32}$$

We first need to prove that, for $\theta \in [0,1]$, the mapping $G(t)$ is an isomorphism from $\mathcal{D}(\widehat{A}^\theta_\Pi(t))$ into D^θ_A, uniformly in t, namely

$$\forall t \in \mathbb{R}, \quad \|G(t)x\|_{D^\theta_A} \leq C\|\widehat{A}^\theta_\Pi(t)x\|_H \quad \text{and} \quad \|\widehat{A}^\theta_\Pi(t)G^{-1}(t)x\|_H \leq C\|x\|_{D^\theta_A}. \tag{3.33}$$

For that it suffices to prove the inequalities for $\theta = 0$ and $\theta = 1$. Then, the result follows from an interpolation argument.

The first inequality in (3.33) for $\theta = 1$ is a direct consequence of the first equality in (3.32) and of $(\mathcal{A}_1)_{\rm (ii)}$. To prove the first inequality in (3.33) for $\theta = 0$, we first write

$$\widehat{A}^{-1}(t)\mathbb{B}(t)\mathbb{B}^*(t)\Pi(t) = \widehat{A}^{\delta-1}(t)(\widehat{A}^{-\delta}(t)\mathbb{B}(t))\mathbb{B}^*(t)\Pi(t). \tag{3.34}$$

By using the change of variable $x = A^{-(1-\delta)}(t)y$ in $(2.4)_1$, with $\theta = 1-\delta$, we deduce that $\widehat{A}^{-(1-\delta)}(t)$ is uniformly bounded in $\mathcal{L}(H, D_A^{1-\delta})$. Hence, with the uniform boundedness of $\mathbb{B}^*(t)\Pi(t) \in \mathcal{L}(H, \mathbb{R}^K)$ and of $\widehat{A}^{-\delta}(t)\mathbb{B} \in \mathcal{L}(\mathbb{R}^K, H)$, we deduce that the right hand side of (3.34) is also uniformly bounded in $\mathcal{L}(H, D_A^{1-\delta})$. Thus

$$\widehat{A}^{-1}(t)\mathbb{B}(t)\mathbb{B}^*(t)\Pi(t) \ \text{ is uniformly bounded in } \ \mathcal{L}(H, D_A^{1-\delta}). \tag{3.35}$$

Hence $G(t)$ is also uniformly bounded in $\mathcal{L}(H)$, which gives $(3.33)_1$ for $\theta = 0$.

Let us now prove the second inequality in (3.33). From (3.32) we deduce that $G(t) \in \mathcal{L}(\mathcal{D}(\widehat{A}_\Pi(t)), D_A)$ is invertible and that $G(t)^{-1} = \widehat{A}_\Pi(t)^{-1}\widehat{A}(t) \in \mathcal{L}(D_A, \mathcal{D}(\widehat{A}_\Pi(t)))$. Hence, the second uniform inequality in (3.33) follows from $(\mathcal{A}_1)_{\rm (ii)}$. Next, from the uniform boundedness of $\widehat{A}^*(t)$ in $\mathcal{L}(D_{A^*}, H)$ and of $\widehat{A}_\Pi^{-*}(t)$ in $\mathcal{L}(H, D_{A^*})$, we deduce that $G^{-*}(t) = \widehat{A}^*(t)\widehat{A}_\Pi^{-*}(t)$ is uniformly bounded in $\mathcal{L}(H)$. Then by duality, $G^{-1}(t)$ is also uniformly bounded in $\mathcal{L}(H)$, and the second inequality in (3.33) for $\theta = 0$ follows.

Next, let us prove (3.30). Suppose that $x \in \mathcal{D}(A_\Pi^\theta(t))$ for $\theta \in [0, 1-\delta]$. Then, from $G(t) \in \mathcal{L}(\mathcal{D}(A_\Pi^\theta(t)), D_A^\theta)$, we deduce that $G(t)x = x + \widehat{A}^{-1}(t)\mathbb{B}(t)\mathbb{B}^*(t)\Pi(t)x \in D_A^\theta$. Due to (3.35), $\widehat{A}^{-1}(t)\mathbb{B}(t)\mathbb{B}^*(t)\Pi(t)x$ belongs to $D_A^{1-\delta}$. Since $D_A^{1-\delta} \hookrightarrow D_A^\theta$, it follows that $x \in D_A^\theta$. We have proved the uniform continuous embedding $\mathcal{D}(A_\Pi^\theta(t)) \hookrightarrow D_A^\theta$.

Conversely, suppose that $x \in D_A^\theta$ with $\theta \in [0, 1-\delta]$. We have $\widehat{A}^{-1}(t)\mathbb{B}(t)\mathbb{B}^*(t)\Pi(t)x \in D_A^{1-\delta}$. Since $D_A^{1-\delta} \hookrightarrow D_A^\theta$, we deduce $G(t)x = x + \widehat{A}^{-1}(t)\mathbb{B}(t)\mathbb{B}^*(t)\Pi(t)x \in D_A^\theta$. Then from $G^{-1}(t) \in \mathcal{L}(D_A^\theta, \mathcal{D}(A_\Pi^\theta(t)))$, we deduce $x \in \mathcal{D}(A_\Pi^\theta(t))$. Then, we have proved the uniform continuous embedding $D_A^\theta \hookrightarrow \mathcal{D}(A_\Pi^\theta(t))$. The proof of (3.30) is compete.

Step 4. Let us prove the first part of (iv). We notice that

$$\begin{aligned}\widehat{A}_\Pi^{-1}(t) - \widehat{A}_\Pi^{-1}(s) &= G^{-1}(t)\widehat{A}^{-1}(t) - G^{-1}(s)\widehat{A}^{-1}(s)\\ &= (G^{-1}(t) - G^{-1}(s))\widehat{A}^{-1}(t) + G^{-1}(s)(\widehat{A}^{-1}(t) - \widehat{A}^{-1}(s))\\ &= G^{-1}(s)(G(s) - G(t))G^{-1}(t)\widehat{A}^{-1}(t) + G^{-1}(s)\widehat{A}^{-1}(s)(\widehat{A}(s) - \widehat{A}(t))\widehat{A}^{-1}(t).\end{aligned} \tag{3.36}$$

From (3.30) and (3.33), it follows that $G^{-1}(t)$ is uniformly bounded in $\mathcal{L}(D_A^{1-\delta})$. Moreover, we recall that $\widehat{A}^{-1}(t)$ is uniformly bounded in $\mathcal{L}(H, D_A)$ and that $\widehat{A} \in C^\alpha_{\rm per}([0,T]; \mathcal{L}(D_A, H))$. Hence, we will get the conclusion if we prove that $G \in C^\alpha_{\rm per}([0,T]; \mathcal{L}(D_A^{1-\delta}))$. For that, we write

$$\begin{aligned}G(s) - G(t) &= \widehat{A}^{-1}(s)\mathbb{B}(s)\mathbb{B}^*(s)\Pi(s) - \widehat{A}^{-1}(t)\mathbb{B}(t)\mathbb{B}^*(t)\Pi(t)\\ &= \widehat{A}^{-1}(t)(\widehat{A}(t) - \widehat{A}(s))\widehat{A}^{\delta-1}(s)\widehat{A}^{-\delta}(s)\mathbb{B}(s)\mathbb{B}^*(s)\Pi(s) + \widehat{A}^{-1}(t)(\mathbb{B}(s) - \mathbb{B}(t))\mathbb{B}^*(s)\Pi(s)\\ &\quad + \widehat{A}^{-1}(t)\mathbb{B}(t)(\mathbb{B}^*(s)\Pi(s) - \mathbb{B}^*(t)\Pi(t)).\end{aligned} \tag{3.37}$$

From $\widehat{A}^* \in C^\alpha_{\rm per}([0,T]; \mathcal{L}(D_{A^*}, H))$, with a duality argument, we get $\widehat{A} \in C^\alpha_{\rm per}([0,T]; \mathcal{L}(H, (D_{A*})'))$. Since $\widehat{A} \in C^\alpha_{\rm per}([0,T]; \mathcal{L}(D_A, H))$, with an interpolation argument, we deduce that $\widehat{A} \in C^\alpha_{\rm per}([0,T]; \mathcal{L}(D_A^{1-\delta}, (D_{A^*}^\delta)'))$. Then with the uniform boundedness of $\widehat{A}^{-\delta}(t)\mathbb{B}(t) \in \mathcal{L}([0,T]^K, H)$ and of $\mathbb{B}^*(t)\Pi(t) \in \mathcal{L}(H, \mathbb{R}^K)$ we deduce that the first term in the right hand side of (3.37) belongs to

$C^{\alpha}_{\rm per}([0,T];\mathcal{L}(D_A^{1-\delta}))$. To estimate the other terms we use $\mathbb{B}\in C^{\alpha}_{\rm per}([0,T];\mathcal{L}(\mathbb{R}^K,(D_{A^*})'))$, and $\mathbb{B}^*\Pi\in C^{\alpha}_{\rm per}([0,T];\mathcal{L}(H,\mathbb{R}^K))$ (see (3.22)), and we finally obtain $G\in C^{\alpha}_{\rm per}([0,T];\mathcal{L}(D_A^{1-\delta}))$. Thus, we have analyzed the different terms in (3.36), and the first part of (iv) is proved.

Step 5. To prove the second part of (iv), we start from the integral representation:

$$\widehat{A}_\Pi^{-\theta}(r)-\widehat{A}_\Pi^{-\theta}(s)=\frac{\sin(\pi\theta)}{\pi}\int_0^{+\infty}\lambda^{-\theta}((\lambda I+\widehat{A}_\Pi(r))^{-1}-(\lambda I+\widehat{A}_\Pi(s))^{-1})\mathrm{d}\lambda, \tag{3.38}$$

see ([6], (4.6.9), p.150). First, the uniform exponential stability of $(e^{-s\widehat{A}_\Pi(t)})_{s\geq 0}$ ensures that:

$$\|(\lambda I+\widehat{A}_\Pi(r))^{-1}\|_{\mathcal{L}(H)}+\|(\lambda I+\widehat{A}_\Pi(s))^{-1}\|_{\mathcal{L}(H)}\leq\frac{C}{\lambda}\quad\text{for all }\lambda>0. \tag{3.39}$$

Since $\widehat{A}_\Pi(\cdot)(\lambda I+\widehat{A}_\Pi(\cdot))^{-1}=I-\lambda(\lambda I+\widehat{A}_\Pi(\cdot))^{-1}$, by combining estimate (3.39) with

$$\begin{aligned}&(\lambda I+\widehat{A}_\Pi(r))^{-1}-(\lambda I+\widehat{A}_\Pi(s))^{-1}\\&=\widehat{A}_\Pi(r)(\lambda I+\widehat{A}_\Pi(r))^{-1}\left(\widehat{A}_\Pi^{-1}(r)-\widehat{A}_\Pi^{-1}(s)\right)\widehat{A}_\Pi(s)(\lambda I+\widehat{A}_\Pi^{-1}(s))^{-1},\end{aligned}$$

and the fact that $A_\Pi^{-1}\in C^{\alpha}(\mathbb{R};\mathcal{L}(H))$, we obtain

$$\|(\lambda I+\widehat{A}_\Pi(r))^{-1}-(\lambda I+\widehat{A}_\Pi(s))^{-1}\|_{\mathcal{L}(H)}\leq C|r-s|^{\alpha}. \tag{3.40}$$

As a consequence, by using estimate (3.39) for $\lambda\geq|r-s|^{-\alpha}$, and estimate (3.40) for $\lambda<|r-s|^{-\alpha}$, in the integral formula (3.38), we obtain

$$\begin{aligned}\|\widehat{A}_\Pi^{-\theta}(r)-\widehat{A}_\Pi^{-\theta}(s)\|_{\mathcal{L}(H)}&\leq C\int_0^{+\infty}\lambda^{-\theta}\|(\lambda I+\widehat{A}_\Pi(r))^{-1}-(\lambda I+\widehat{A}_\Pi(s))^{-1}\|_{\mathcal{L}(H)}\mathrm{d}\lambda\\&\leq C\left(|r-s|^{\alpha}\int_0^{1/|r-s|^{\alpha}}\lambda^{-\theta}\mathrm{d}\lambda+\int_{1/|r-s|^{\alpha}}^{+\infty}\lambda^{-\theta-1}\mathrm{d}\lambda\right)\leq C|r-s|^{\alpha\theta}.\end{aligned}$$

The proof is complete. □

Now, we state a stabilizability result for the evolution operator generated by $A_\Pi(\cdot)$.

Theorem 3.10. (i) *There exists a unique parabolic evolution operator $(s,t)\mapsto S_\Pi(t,s)$ for $(A_\Pi(t),\mathcal{D}(A_\Pi(t)))$, with regularity subspace $D_A^{1-\delta}$ (in the sense of [6], II.2.1, p. 45), and with an exponential decay rate $\sigma\geq 0$. In particular, there exists $\tau>0$ such that*

$$\forall x\in H,\ \forall (s,t)\in\Delta^*,\quad \|S_\Pi(t,s)x\|_H\leq Ce^{-(\sigma+\tau)(t-s)}\|x\|_H. \tag{3.41}$$

(ii) *Moreover, for $\theta\in[0,1]$ and $\beta\in[0,1]$, the mapping $(s,t)\mapsto\widehat{A}_\Pi^{\theta}(t)S_\Pi(t,s)\widehat{A}_\Pi^{\beta}(s)$ is continuous from Δ^* into $\mathcal{L}(H)$ and there exists $\tau>0$ such that*

$$\forall x\in H,\ \forall (s,t)\in\Delta^*,\quad \|\widehat{A}_\Pi^{\theta}(t)S_\Pi(t,s)\widehat{A}_\Pi^{\beta}(s)x\|_H\leq\frac{C}{(t-s)^{\theta+\beta}}e^{-(\sigma+\tau)(t-s)}\|x\|_H, \tag{3.42}$$

and

$$\forall x \in H,\ \forall (s,t) \in \Delta^*, \quad \|\widehat{A}_\Pi(t) S_\Pi(t,s) \widehat{A}_\Pi^{-1}(s) x\|_H \le Ce^{-(\sigma+\tau)(t-s)} \|x\|_H. \tag{3.43}$$

Proof.

Step 1. The existence of a unique parabolic evolution operator $(s,t) \mapsto S_\Pi(t,s)$ for A_Π, with regularity subspace $D_A^{1-\delta}$, follows from ([6], Thm. 2.3.2, p. 222). In fact, due to (3.30) in Proposition 3.9, $[H, \mathcal{D}(A_\Pi(t))]_{1-\delta} = D_A^{1-\delta}$ is independent of t. Further, point (iv) in Proposition 3.9, gives the Hölder exponent of the map $\widehat{A}_\Pi^{-1}(t)$ as α. By Assumption $(\mathcal{A}_2)_{(ii)}$, α is greater than δ. Thus ([6], Asm. (ii), p. 214) is also satisfied.

Step 2. Let us prove (3.41) for $s = 0$. The proof is similar for any $s \in \mathbb{R}$. We have $y(t) = y_u(t) + y_s(t)$, where (see (3.2) and (3.3)) y_u and y_s are respectively given by

$$y_u(t) = S(t,0)P_u(0)y_0 - \int_0^t S(t,s)\mathbb{B}_u(s)\mathbb{B}_u^*(s)\Pi(s)y_u(s)\mathrm{d}s,$$

$$y_s(t) = S(t,0)P_s(0)y_0 - \int_0^t S(t,s)\mathbb{B}_s(s)\mathbb{B}_u^*(s)\Pi(s)y_u(s)\mathrm{d}s.$$

From the uniform boundedness of $\mathbb{B}_u^*(t)\Pi(t)$ and from (3.27), the feedback control $f(t) = -\mathbb{B}_u^*(t)\Pi(t)y_u(t)$ satisfies

$$\|f(t)\|_{\mathbb{R}^K} \le Ce^{-(\sigma+\tau_0)t}\|y_0\|_H, \quad \text{for all } t \ge 0, \text{ and some } \tau_0 > 0. \tag{3.44}$$

Moreover, with such a notation we have the following integral representation for $y_s(t)$:

$$y_s(t) = S(t,0)P_s(0)y_0 + \int_0^t S(t,s)\mathbb{B}_s(s)f(s)\mathrm{d}s, \quad \forall\, t \ge 0. \tag{3.45}$$

Note that (2.19) guarantees that the first term in the right hand side of (3.45) satisfies the required exponential decay. Hence, we only have to focus on the asymptotic behavior of the integral term. For that we recall that $\mathbb{B}_s(s) = P_s(s)\mathbb{B}(s)$, and we write the integrand as

$$S(t,s)\mathbb{B}_s(s)f(s) = \Big(S(t,s)P_s(s)\widehat{A}^\delta(s)\Big)\Big(\widehat{A}^{-\delta}(s)\mathbb{B}(s)f(s)\Big), \quad \forall\, 0 \le s < t.$$

First, as an immediate consequence of Assumption $(\mathcal{A}_2)_{(\mathrm{ii})}$ we have the following bound

$$\forall\, s \ge 0, \quad \|(\widehat{A}(s))^{-\delta}\mathbb{B}(s)f(s)\|_H \le C\|f(s)\|_{\mathbb{R}^K}.$$

Using (2.20) with $0 < 2\tau < \tau_0$, it follows that

$$\|S(t,s)P_s(s)\widehat{A}^\delta(s)\|_{\mathcal{L}(H)} \le Ce^{-(\sigma+2\tau)(t-s)}|t-s|^{-\delta}.$$

Since $\delta < 1$, with (3.44), Young's inequality for convolution product and the condition $2\tau < \tau_0$, we obtain

$$\left\| \int_0^t S(t,s)\mathbb{B}_s(s)f(s)\mathrm{d}s \right\|_H \le C\int_0^t \frac{e^{-(\sigma+2\tau)(t-s)}}{|t-s|^\delta}\|f(s)\|_{\mathbb{R}^K}\mathrm{d}s$$
$$\le Ce^{-(\sigma+\tau)t}\left(\int_0^t e^{-(\tau_0-2\tau)s}\frac{e^{-\tau(t-s)}}{|t-s|^\delta}\,\mathrm{d}s\right)\|y_0\|_H \le Ce^{-(\sigma+\tau)t}\|y_0\|_H, \quad \forall\, t\ge 0.$$

Step 3. *Proof of* (3.42) *in the case* $|t-s|<1$. The continuity of $(s,t)\mapsto \widehat{A}_\Pi(t)S_\Pi(t,s)$ and estimate (3.42) for $\theta=0$ and $\theta=1$, and for $\beta=0$ and $|t-s|<1$, follow from ([6], Thm. 2.3.2, p. 222). Moreover, estimate (3.42), for $\theta\in(0,1)$, $\beta=0$ and $|t-s|<1$, follows from an interpolation argument.

By remarking that $\widehat{A}_\Pi^\theta(t)S_\Pi(t,s)=\widehat{A}_\Pi^{\theta-1}(t)\widehat{A}_\Pi(t)S_\Pi(t,s)$, the continuity of $(s,t)\mapsto \widehat{A}_\Pi^\theta(t)S_\Pi(t,s)$ follows from the continuity of $t\mapsto \widehat{A}_\Pi^{\theta-1}(t)$ stated in Proposition 3.9(iv).

Estimate (3.42), for $\theta\in(0,1)$, $\beta\in(0,1)$ and $|t-s|<1$, follows with an argument similar to the one used to obtain (2.7) (see [7], Cor. 2.3).

Step 4. *Proof of* (3.42) *in the case* $|t-s|\ge 1$. We have

$$\widehat{A}_\Pi^\theta(t)S_\Pi(t,s)\widehat{A}_\Pi^\beta(s)=\widehat{A}_\Pi^\theta(t)S_\Pi(t,t-1/2)S_\Pi(t-1/2,s+1/2)S_\Pi(s+1/2,s)\widehat{A}_\Pi^\beta(s).$$

To obtain (3.42) when $|t-s|\ge 1$, it is sufficient to use (3.41) to estimate $S_\Pi(t-1/2,s+1/2)$, and (3.42), already proved when $|t-s|<1$, to estimate the other terms.

Step 5. Let us prove (3.43). The estimate (3.43) for $|t-s|<1$ follows from ([6], Thm. 2.4.1, p. 228). Let us prove (3.43) in the case where $|t-s|\ge 1$. For all $x\in H$, with (3.42), we have

$$\|\widehat{A}_\Pi(t)S_\Pi(t,s)\widehat{A}_\Pi^{-1}(s)x\|_H \le C\frac{e^{-(\sigma+\tau)(t-s)}}{|t-s|}\|\widehat{A}_\Pi^{-1}(s)x\|_H \le Ce^{-(\sigma+\tau)(t-s)}\|x\|_H.$$

The proof is complete. □

To deal with nonlinear systems, it is convenient to consider the following nonhomogeneous closed-loop system

$$y'=A_\Pi(t)y+f(t), \quad y(0)=y_0. \tag{3.46}$$

Since $(s,t)\mapsto S_\Pi(t,s)$ is the evolution operator for $(A_\Pi(t),\mathcal{D}(A_\Pi(t)))$, with regularity subspace $D_A^{1-\delta}$, and since $(D_{A^*})'=(\mathcal{D}(A_\Pi^*(t)))'$, for $y_0\in H$ and $f\in L^2_{\mathrm{loc}}([0,+\infty);(D_{A^*})')$, the solution of (3.46) is the function $y\in C([0,+\infty);(D_{A^*})')$ defined by

$$y(t)=S_\Pi(t,0)y_0+\int_0^t S_\Pi(t,s)f(s)\mathrm{d}s.$$

Theorem 3.11. *Let* δ *be the exponent appearing in* $(\mathcal{A}_2)_{\mathrm{(ii)}}$. *Let* y_0 *belong to* H *and* $e^{\sigma t}f$ *belong to* $L^2\big(0,\infty;\big(D_{A^*}^{1/2}\big)'\big)$.

If $\delta>1/2$, *then the solution to* (3.46) *satisfies*

$$\|e^{\sigma\,\cdot}y\|_{C_b([0,\infty);H)}+\|e^{\sigma\,\cdot}y\|_{L^2(0,\infty;D_A^{1-\delta})}\le C\left(\|y_0\|_H+\|e^{\sigma\,\cdot}f\|_{L^2\left(0,\infty;\left(D_{A^*}^{1/2}\right)'\right)}\right). \tag{3.47}$$

If $\delta \le 1/2$, then, for all $\epsilon \in (0,1/2)$, the solution to (3.46) satisfies

$$\|e^{\sigma\,\cdot} y\|_{C_b([0,\infty);H)} + \|e^{\sigma\,\cdot} y\|_{L^2(0,\infty;D_A^{1/2-\epsilon})} \le C_\epsilon \left(\|y_0\|_H + \|e^{\sigma\,\cdot} f\|_{L^2\left(0,\infty;\left(D_{A^*}^{1/2}\right)'\right)} \right).$$

Proof. We give the proof in the case $\delta > 1/2$. The case $\delta \le 1/2$ can be obtained analogously by replacing $1-\delta$ by $1/2-\epsilon$ in the following.

Step 1. We consider the case when $f = 0$. From (3.41), (3.42) with $\theta = 1-\delta$ and $\beta = 0$, and from (3.30), it follows that

$$\|e^{\sigma\,\cdot} y\|_{C_b([0,\infty);H)} + \|e^{\sigma\,\cdot} y\|_{L^2(0,\infty;D_A^{1-\delta})} \le C\,\|y_0\|_H. \tag{3.48}$$

Step 2. We now study the case when $y_0 = 0$ and $f \neq 0$. Using (3.31), by duality, it can be shown that the norms $f \mapsto \|f\|_{\left(D_{A^*}^{1/2}\right)'}$ and $f \mapsto \|\widehat{A}_\Pi^{-1/2}(t) f\|_H$ are uniformly equivalent. Using (3.42) for $\theta = 0$ and $\beta = 1/2$, and Young's convolution in weak Lebesgue spaces, we have

$$\begin{aligned}
\|y(t)\|_H &= \left\| \int_0^t S_\Pi(t,s)\, f(s)\,\mathrm{d}s \right\|_H \\
&\le \int_0^t \left\| S_\Pi(t,s) \widehat{A}_\Pi^{1/2}(s) \widehat{A}_\Pi^{-1/2}(s) f(s) \right\|_H \mathrm{d}s \le C \int_0^t \frac{e^{-(\sigma+\tau)(t-s)}}{|t-s|^{1/2}}\, e^{-\sigma s}\, \|e^{\sigma s} f(s)\|_{\left(D_{A^*}^{1/2}\right)'}\,\mathrm{d}s \\
&\le C e^{-\sigma t} \int_0^t \frac{e^{-\tau(t-s)}}{|t-s|^{1/2}}\, \|e^{\sigma s} f(s)\|_{\left(D_{A^*}^{1/2}\right)'}\,\mathrm{d}s \le C e^{-\sigma t}\, \|e^{\sigma\,\cdot} f\|_{L^2\left(0,\infty;\left(D_{A^*}^{1/2}\right)'\right)}.
\end{aligned}$$

Thus the estimate of $\|e^{\sigma\cdot} y\|_{C_b([0,\infty);H)}$ is proved. For the second one, we first notice that, due to (3.30), the norms $y \mapsto \|y\|_{D_A^{1-\delta}}$ and $y \mapsto \|\widehat{A}_\Pi^{1-\delta}(t) y\|_H$ are equivalent. Thus, using (3.42) for $\theta = 1-\delta$ and $\beta = 1/2$, we can write

$$\begin{aligned}
\|y(t)\|_{D_A^{1-\delta}} &= \left\| \int_0^t S_\Pi(t,s)\, f(s)\,\mathrm{d}s \right\|_{D_A^{1-\delta}} \le \int_0^t \left\| \widehat{A}_\Pi^{1-\delta}(t)\, S_\Pi(t,s)\, \widehat{A}_\Pi^{1/2}(s) \widehat{A}_\Pi^{-1/2}(s)\, f(s) \right\|_H \mathrm{d}s \\
&\le C \int_0^t \frac{e^{-(\sigma+\tau)(t-s)}}{|t-s|^{3/2-\delta}}\, e^{-\sigma s}\, \|e^{\sigma s} f(s)\|^2_{\left(D_{A^*}^{1/2}\right)'}\,\mathrm{d}s \le C e^{-\sigma t} \int_0^t \frac{e^{-\tau(t-s)}}{|t-s|^{3/2-\delta}}\, \|e^{\sigma s} f(s)\|_{\left(D_{A^*}^{1/2}\right)'}\,\mathrm{d}s.
\end{aligned}$$

Since $\delta > 1/2$, the mapping $t \mapsto \frac{e^{-\tau t}}{|t|^{3/2-\delta}}$ belongs to $L^1(0,\infty)$, and the Young inequality yields

$$\|e^{\sigma\,\cdot} y\|_{L^2(0,\infty;D_A^{1-\delta})} \le C \|e^{\sigma\,\cdot} f\|_{L^2\left(0,\infty;\left(D_{A^*}^{1/2}\right)'\right)}.$$

The proof is complete. ◻

4. Incompressible Navier-Stokes equation

Let Ω be a bounded domain in $\mathbb{R}^2$, with a boundary $\partial\Omega$ of class $C^{2,1}$. Let us denote by n the outward unit normal to the boundary. We introduce the function spaces

$$\begin{aligned}
&V^r(\Omega) = \Big\{ y \in H^r(\Omega;\mathbb{R}^2) \mid \nabla\cdot y = 0 \text{ in } \Omega,\ \langle y\cdot n, 1\rangle_{H^{-\frac{1}{2}}(\partial\Omega), H^{\frac{1}{2}}(\partial\Omega)} = 0 \Big\}, \quad r \geq 0,\\
&V_n^r(\Omega) = \Big\{ y \in H^r(\Omega;\mathbb{R}^2) \mid \nabla\cdot y = 0 \quad \text{in } \Omega,\ y\cdot n = 0 \text{ on } \quad \partial\Omega \Big\}, \quad r \geq 0,\\
&V_0^r(\Omega) = \Big\{ y \in H^r(\Omega;\mathbb{R}^2) \mid \nabla\cdot y = 0 \text{ in } \Omega,\ y = 0 \text{ on } \partial\Omega \Big\}, \quad \tfrac{1}{2} < r < \tfrac{3}{2},\\
&V^r(\partial\Omega) = \Big\{ y \in H^r(\partial\Omega;\mathbb{R}^2) \mid \int_{\partial\Omega} y\cdot n = 0 \Big\}, \quad r \geq 0.
\end{aligned}$$

For $r < 0$, $V^r(\partial\Omega)$ denotes the dual space of $V^{-r}(\partial\Omega)$ with $V^0(\partial\Omega)$ as pivot space. In this section, we define the state space H and the control space $\mathcal{U}$ by

$$H = V_n^0(\Omega) \quad \text{and} \quad \mathcal{U} = V^0(\partial\Omega).$$

We recall that $L^2(\Omega;\mathbb{R}^2) = H \oplus \nabla H^1(\Omega)$. The orthogonal projector from $L^2(\Omega;\mathbb{R}^2)$ onto H, the so-called Leray projector, is denoted by $\mathbb{P}$. It is well known that

$$\mathbb{P} \in \mathcal{L}(H^{1/2-\varepsilon}(\Omega;\mathbb{R}^2), V_n^{1/2-\varepsilon}(\Omega)), \quad \text{for all } 0 < \varepsilon \leq \frac{1}{2}. \tag{4.1}$$

We consider a T-periodic smooth trajectory $(z_{\text{per}}, p_{\text{per}})$ satisfying

$$\begin{aligned}
&\frac{\partial z_{\text{per}}}{\partial t} - \nu\Delta z_{\text{per}} + (z_{\text{per}}\cdot\nabla) z_{\text{per}} + \nabla p_{\text{per}} = f_{\text{per}} \quad \text{in} \quad \Omega\times(0,\infty),\\
&\nabla\cdot z_{\text{per}} = 0 \quad \text{in} \quad \Omega\times(0,\infty),\\
&z_{\text{per}} = u_{\text{per}} \quad \text{on} \quad \partial\Omega\times(0,\infty).
\end{aligned} \tag{4.2}$$

More precisely, we assume that

$$z_{\text{per}} \in C^{\alpha}_{\text{per}}([0,T]; V^1(\Omega)) \cap L^\infty(\Omega\times(0,T);\mathbb{R}^2), \quad \text{for some } \frac{3}{4} < \alpha < 1. \tag{4.3}$$

The assumption $z_{\text{per}} \in L^\infty(\Omega\times(0,T);\mathbb{R}^2)$ is needed only in the proof of Proposition 4.4. In Section 4.4, we prove the existence of solutions to equation (4.2) for $u_{\text{per}} \in C^{\alpha-1/2}_{\text{per}}([0,T]; V^{3/2}(\partial\Omega)) \cap C^{\alpha+1/2}_{\text{per}}([0,T]; V^{-1/2}(\partial\Omega))$ and $f_{\text{per}} \in C^{\alpha-1/2}_{\text{per}}([0,T]; L^2(\Omega;\mathbb{R}^2))$, small enough in those norms.

In order to localize the boundary control in an open subset Γ_c of the boundary $\partial\Omega$, we introduce a localization operator $M \in \mathcal{L}(V^0(\partial\Omega))$ defined by

$$Mu(x) \ = \ m(x)u(x) \ - \ \frac{m(x)}{\int_{\partial\Omega} m} \left(\int_{\partial\Omega} mu\cdot n \right) \ n(x), \quad \forall\, x \in \partial\Omega, \tag{4.4}$$

where m is a smooth function with values in $[0,1]$ and with support in $\Gamma_c \subset \partial\Omega$, equal to 1 in Γ_1, a non empty open subset in Γ_c. We note that M is a self-adjoint operator.

Let (z, p) satisfy the incompressible Navier-Stokes system with an initial condition corresponding to a perturbation $w_0 \in V_n^0(\Omega)$ of $z_{\text{per}}(0)$:

$$\begin{aligned}
&\frac{\partial z}{\partial t} - \nu\Delta z + (z\cdot\nabla)z + \nabla p = f_{\text{per}} \quad \text{in} \quad \Omega\times(0,\infty),\\
&\nabla\cdot z = 0 \quad \text{in} \quad \Omega\times(0,\infty),\\
&z = Mu + u_{\text{per}} \quad \text{on} \quad \partial\Omega\times(0,\infty),\\
&\mathbb{P}z(\cdot,0) = w_0 + \mathbb{P}z_{\text{per}}(\cdot,0) \in V_n^0(\Omega).
\end{aligned} \tag{4.5}$$

Let us emphasize that, since the normal component of the boundary condition $z\cdot n = (Mu + u_{\text{per}})\cdot n$ is not necessarily zero, the solution $z(t)$ to system (4.5) does not belong to $V_n^0(\Omega)$ and the initial condition has to be stated for $\mathbb{P}z(\cdot,0)$ and not for $z(\cdot,0)$ (see [34], Rem. 2.4).

Our goal is to find u, as a feedback depending on $z - z_{\text{per}}$, able to stabilize $z - z_{\text{per}}$ to zero for some norm, with any prescribed exponential decay rate $\sigma \geq 0$. For that, we set

$$w = z - z_{\text{per}}, \quad q = p - p_{\text{per}}.$$

The pair (w, q) satisfies

$$\begin{aligned}
&\frac{\partial w}{\partial t} - \nu\Delta w + (w\cdot\nabla)w + (w\cdot\nabla)z_{\text{per}} + (z_{\text{per}}\cdot\nabla)w + \nabla q = 0 \quad \text{in} \quad \Omega\times(0,\infty),\\
&\nabla\cdot w = 0 \quad \text{in} \quad \Omega\times(0,\infty),\\
&w = Mu \quad \text{on} \quad \partial\Omega\times(0,\infty),\\
&\mathbb{P}w(\cdot,0) = w_0 \quad \text{in} \quad \Omega.
\end{aligned} \tag{4.6}$$

In the next subsections, we are going to see that the linearized system associated to (4.6) enters into the functional framework studied in Sections 2–3.

4.1. Stabilization of the linearized system

Let us first consider the linearized system associated to (4.6):

$$\begin{aligned}
&\frac{\partial w}{\partial t} - \nu\Delta w + (w\cdot\nabla)z_{\text{per}} + (z_{\text{per}}\cdot\nabla)w + \nabla q = 0 \quad \text{in} \quad \Omega\times(0,\infty),\\
&\nabla\cdot w = 0 \quad \text{in} \quad \Omega\times(0,\infty),\\
&w = Mu \quad \text{on} \quad \partial\Omega\times(0,\infty),\\
&\mathbb{P}w(\cdot,0) = w_0 \quad \text{in} \quad \Omega.
\end{aligned} \tag{4.7}$$

To write system (4.7) in the form (1.1), we first introduce the Stokes operator $(A_0, \mathcal{D}(A_0))$ in H defined by $\mathcal{D}(A_0) = V^2(\Omega)\cap V_0^1(\Omega)$ and

$$A_0 y = -\nu\mathbb{P}\Delta y, \quad \text{for all } y \in \mathcal{D}(A_0). \tag{4.8}$$

We also consider the family of closed linear operators in H, $(A_1(t), \mathcal{D}(A_1(t)))_{t\in\mathbb{R}}$, defined by

$$\begin{aligned}
&\mathcal{D}(A_1(t)) = V_0^{1+\varepsilon}(\Omega), \quad \text{with } 0 < \varepsilon < 1/2,\\
&A_1(t)y = -\mathbb{P}\Big((y\cdot\nabla)z_{\text{per}}(t) + (z_{\text{per}}(t)\cdot\nabla)y\Big), \quad \text{for all } y \in V_0^{1+\varepsilon}(\Omega).
\end{aligned} \tag{4.9}$$

The family of adjoint operators $(A_1^*(t), \mathcal{D}(A_1^*(t)))_{t\in\mathbb{R}}$ satisfies

$$V_0^{1+\varepsilon}(\Omega) \subset \mathcal{D}(A_1^*(t)) \quad \text{and}$$
$$A_1^*(t)y = -\mathbb{P}\Big((\nabla z_{\rm per}(t))^T y - (z_{\rm per}(t)\cdot\nabla)y\Big), \quad \text{for all } y \in V_0^{1+\varepsilon}(\Omega).$$

Next, we define the family of unbounded operators in H, $(A(t), \mathcal{D}(A(t)))_{t\geq 0}$, by setting $\mathcal{D}(A(t)) = V^2(\Omega)\cap V_0^1(\Omega) = D_A$ and

$$A(t) = -A_0 + A_1(t) \quad \text{for all } t \in \mathbb{R}. \tag{4.10}$$

Lemma 4.1. *We assume that (4.3) is satisfied. Then $(A(t), D_A)_{t\in\mathbb{R}}$ satisfies the assumptions $(a1)$, $(a2)$ and $(a3)$ of Proposition 2.6. Therefore $(A(t), D_A)_{t\in\mathbb{R}}$ generates a family of evolution operators $\{S(t,s) \mid -\infty < s \leq t < \infty\}$ on H.*

The family of operators $(A^*(t), \mathcal{D}(A^*(t)))_{t\in\mathbb{R}}$ is defined by $\mathcal{D}(A^*(t)) = D_{A^*} = D_A$ and

$$A^*(t)\xi = \nu\,\mathbb{P}\,\Delta\xi + \mathbb{P}[(\nabla z_{\rm per}(t))^T\xi - (z_{\rm per}(t)\cdot\nabla)\xi] \quad \text{for all } \xi \in D_{A^*}.$$

Proof. Since $z_{\rm per}$ is T-periodic with respect to time t, it is clear that $(A_1(t), V_0^{1+\varepsilon}(\Omega))_{t\in\mathbb{R}}$ is T-periodic, and $(a1)$ in Proposition 2.6 is satisfied. We recall that

$$\mathcal{D}(A_0^{1/2+\varepsilon/2}) = [V_n^0(\Omega), V^2(\Omega)\cap V_0^1(\Omega)]_{1/2+\varepsilon/2} = V_0^{1+\varepsilon}(\Omega) \quad \text{for all } 0<\varepsilon<1/2.$$

Thus, (2.33) is satisfied for $\beta = \frac{1}{2}+\frac{\varepsilon}{2}$. If $y \in V_0^{1+\varepsilon}(\Omega)$ with $0<\varepsilon<1/2$, we have

$$\|A_1(t)y\|_{V_n^0(\Omega)} + \|A_1^*(t)y\|_{V_n^0(\Omega)} \leq C\|z_{\rm per}(t)\|_{H^1(\Omega)}\,\|y\|_{V_0^{1+\varepsilon}(\Omega)},$$

see ([8], Sect. 2.2). Thus $(a3)$ in Proposition 2.6 is satisfied. Let us check that $t\mapsto A_1(t) \in C^\alpha_{\rm per}([0,T]; \mathcal{L}(\mathcal{D}(A_0^{1/2+\varepsilon/2}), H))$. This regularity result follows from the estimate

$$\|(A_1(t)-A_1(s))y\|_{V_n^0(\Omega)} \leq C\|z_{\rm per}(t)-z_{\rm per}(s)\|_{H^1(\Omega)}\,\|y\|_{V_0^{1+\varepsilon}(\Omega)},$$

and from (4.3). Proving that $t\mapsto A_1^*(t)$ belongs to $C^\alpha_{\rm per}([0,T]; \mathcal{L}(\mathcal{D}(A_0^{1/2+\varepsilon/2}), H))$ can be done in the same way. Therefore the assumptions $(a1)$, $(a2)$ and $(a3)$ in Proposition 2.6 are satisfied and the proof is complete. □

We choose $\lambda_0>0$ so that

$$((\lambda_0 I + A_0 - A_1(t))y, y)_{V_n^0(\Omega)} \geq \frac{\nu}{2}\|y\|^2_{V_0^1(\Omega)}, \tag{4.11}$$

for all $t\in[0,T]$ and all $y\in D_A$. As in (2.2), we set

$$\widehat{A}(t) \stackrel{\rm def}{=} \lambda_0 I - A(t) = \lambda_0 I + A_0 - A_1(t). \tag{4.12}$$

We introduce the Dirichlet map $G(t) \in \mathcal{L}(V^0(\partial\Omega), V^0(\Omega))$, defined by $G(t)u = w_u(t)$ for all $u\in V^0(\partial\Omega)$ and $t\in[0,T]$, where $(w_u(t), q_u(t))$ is the unique solution (uniqueness up to an additive constant for the pressure q_u) of the following equation

$$\begin{aligned} &\lambda_0 w_u - \nu\Delta w_u + (w_u\cdot\nabla)z_{\rm per}(t) + (z_{\rm per}(t)\cdot\nabla)w_u + \nabla q_u = 0 \quad \text{in} \quad \Omega,\\ &\nabla\cdot w_u = 0 \quad \text{in} \quad \Omega, \quad w = u \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{4.13}$$

The solution to equation (4.13) is defined by the transposition method (see *e.g.* [34], B.7).

Lemma 4.2. *The mapping* $t \mapsto G(t)$ *belongs to* $C^\alpha([0,T];\mathcal{L}(V^s(\partial\Omega), V^{s+\frac{1}{2}}(\Omega)))$*, for all* $s \in [0, \frac{3}{2}]$.

Proof.

Step 1. *Decomposition and preliminary estimates.* We look for the solution of equation (4.13) in the form $(w_u, q_u) = (v_u + w_a, p_u + q_a)$, where (v_u, p_u) is the solution to

$$\begin{aligned} &\lambda_0 v_u - \nu\Delta v_u + \nabla p_u = 0 \quad \text{in} \quad \Omega, \\ &\nabla\cdot v_u = 0 \quad \text{in} \quad \Omega, \quad v = u \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{4.14}$$

and (w_a, q_a) is the solution to

$$\begin{aligned} &\lambda_0 w_a - \nu\Delta w_a + (w_a\cdot\nabla)z_{\rm per} + (z_{\rm per}\cdot\nabla)w_a + \nabla q_a = F_u \quad \text{in} \quad \Omega, \\ &\nabla\cdot w_a = 0 \quad \text{in} \quad \Omega, \quad w_a = 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{4.15}$$

where $F_u = -(v_u\cdot\nabla)z_{\rm per} - (z_{\rm per}\cdot\nabla)v_u$. From ([34], Cor. A.1), it follows that

$$\|v_u\|_{C^\alpha([0,T];V^{s+\frac{1}{2}}(\Omega))} \le C\|u\|_{C^\alpha([0,T];V^s(\partial\Omega))} \quad \text{for all } s\in[0,3/2]. \tag{4.16}$$

In ([34], Cor. A.1), Ω is of class C^3, but Ω of class $C^{2,1}$ is sufficient. We have

$$\begin{aligned} &\|F_u(t)\|_{H^{-1}(\Omega;\mathbb{R}^2)} \le C\|v_u(t)\|_{V^{\frac{1}{2}}(\Omega)}\|z_{\rm per}(t)\|_{V^1(\Omega)} \quad \text{for all } t\in[0,T], \quad \text{and} \\ &\|F_u(t)\|_{L^2(\Omega;\mathbb{R}^2)} \le C_s\|v_u(t)\|_{V^{\frac{1}{2}+s}(\Omega)}\|z_{\rm per}(t)\|_{V^1(\Omega)} \text{ for all } s\in]1/2,3/2] \text{ and all } t\in[0,T]. \end{aligned}$$

Step 2. Let us prove the lemma for $s \in [0, 1/2]$. With (4.11) we can obtain the following estimate

$$\begin{aligned} \|w_a\|_{L^\infty(0,T;V^1(\Omega))} &\le C\|F_u\|_{L^\infty(0,T;H^{-1}(\Omega;\mathbb{R}^2))} \\ &\le C\|v_u\|_{L^\infty(0,T;V^{\frac{1}{2}}(\Omega))}\|z_{\rm per}\|_{L^\infty(0,T;V^1(\Omega))}. \end{aligned} \tag{4.17}$$

For τ, $t\in[0,T]$, the equation satisfied by $w_a(t)-w_a(\tau)$ can be written in the form

$$\begin{aligned} &\lambda_0(w_a(t)-w_a(\tau)) - \nu\Delta(w_a(t)-w_a(\tau)) + ((w_a(t)-w_a(\tau))\cdot\nabla)z_{\rm per}(t) \\ &\quad +(z_{\rm per}(t)\cdot\nabla)(w_a(t)-w_a(\tau)) + \nabla(q_a(t)-q_a(\tau)) \\ &= -(w_a(\tau)\cdot\nabla)(z_{\rm per}(t)-z_{\rm per}(\tau)) - ((z_{\rm per}(t)-z_{\rm per}(\tau))\cdot\nabla)w_a(\tau) \quad \text{in} \quad \Omega, \\ &\nabla\cdot(w_a(t)-w_a(\tau)) = 0 \quad \text{in} \quad \Omega, \quad w_a(t)-w_a(\tau) = 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{4.18}$$

Still using (4.11) we can obtain

$$\|w_a(t)-w_a(\tau)\|_{V^1(\Omega)} \le C\|w_a\|_{L^\infty(0,T;V^1(\Omega))}\|z_{\rm per}(t)-z_{\rm per}(\tau)\|_{V^1(\Omega)}.$$

From which we deduce

$$\|w_a\|_{C^\alpha([0,T];V^1(\Omega))} \le C\|w_a\|_{L^\infty(0,T;V^1(\Omega))}\|z_{\rm per}\|_{C^\alpha([0,T];V^1(\Omega))}. \tag{4.19}$$

The proof of the lemma for $s \in [0,1/2]$ follows from (4.16), (4.17), and (4.19).

Step 3. We prove the lemma for $s \in]1/2, 3/2]$. For all $0 < \varepsilon < 1/2$, we have

$$\|(w_a \cdot \nabla)z_{\text{per}} + (z_{\text{per}} \cdot \nabla)w_a\|_{L^\infty(0,T;H^{-\varepsilon}(\Omega;\mathbb{R}^2))} \leq C_\varepsilon \|w_a\|_{L^\infty(0,T;V^1(\Omega))} \|z_{\text{per}}\|_{L^\infty(0,T;V^1(\Omega))}.$$

Thus, for $s > 1/2$ and $0 < \varepsilon < 1/2$, it yields that

$$\begin{aligned}
&\|w_a\|_{L^\infty(0,T;V^{2-\varepsilon}(\Omega))} \leq C\|F_u + (w_a \cdot \nabla)z_{\text{per}} + (z_{\text{per}} \cdot \nabla)w_a\|_{L^\infty(0,T;H^{-\varepsilon}(\Omega;\mathbb{R}^2))} \\
&\leq C_\varepsilon \|v_u\|_{L^\infty(0,T;V^{\frac{1}{2}+s}(\Omega))} \|z_{\text{per}}\|_{L^\infty(0,T;V^1(\Omega))} + \|w_a\|_{L^\infty(0,T;V^1(\Omega))} \|z_{\text{per}}\|_{L^\infty(0,T;V^1(\Omega))}.
\end{aligned}$$

By reasoning as in the case when $s \in [0,1/2]$, we obtain the bound of w_a in $C^\alpha([0,T]; V^{2-\varepsilon}(\Omega))$. Next using that bound we can finally obtain the bound for w_a in $C^\alpha([0,T]; V^2(\Omega))$. □

We introduce the control operator $B(t)$ defined by

$$B(t)u = -\widehat{A}(t)\,\mathbb{P}\,G(t)\,Mu, \quad \text{for all } u \in \mathcal{U} = V^0(\partial\Omega), \tag{4.20}$$

where $\widehat{A}(t)$ is defined by (4.12). It is clear that the conditions $(b1)$ and $(b2)$ of Proposition 2.6 are satisfied by $B(t)$. From (4.20), Lemma 4.2, and (4.1), it follows that

$$\widehat{A}^{-\delta}(t)B(t) = \widehat{A}^{1-\delta}(t)\mathbb{P}G(t)M \ \in \mathcal{L}(V^0(\partial\Omega), H), \quad \text{for all} \quad \frac{3}{4} < \delta < 1 \quad \text{and all } t \in [0,T]. \tag{4.21}$$

Thus, we can choose δ such that $\frac{3}{4} < \delta < \alpha$, and the condition $(b3)$ follows from (4.21).

We are now in position to write system (4.7) in the form (1.1). Following the approach introduced in [34] for the autonomous Oseen system, it can be shown in the same way that a function $w \in L^2_{\text{loc}}([0,\infty); V^0(\Omega))$ is a solution of (4.7) in the transposition sense (see [34], Def. 2.2) if and only if $(\mathbb{P}w, (I-\mathbb{P})w)$ is a weak solution of the following system

$$\begin{aligned}
&(\mathbb{P}w)'(t) = A(t)\ (\mathbb{P}w(t)) + B(t)\,u(t) \ \text{ in } \ (D_{A^*})', \quad \forall\, t \geq 0, \quad \mathbb{P}w(0) = \mathbb{P}w_0, \\
&(I-\mathbb{P})w(t) = (I-\mathbb{P})G(t)Mu(t).
\end{aligned} \tag{4.22}$$

This equivalence is proved for the Stokes system in ([34], Thm. 2.4). The autonomous and nonautonomous Oseen systems are studied in ([34], Sect. 4), and the equivalence between the two notions of solutions can be adapted from ([34], Sect. 2).

Since $B \in C^\alpha_{\text{per}}([0,T]; \mathcal{L}(V, (D_{A^*})'))$, we note that $(4.22)_1$ is not valid in H, but in $(D_{A^*})'$.

Note that $(A(\cdot), D_A)$ and B satisfy the assumptions of Proposition 2.6, and therefore they also satisfy Assumptions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$. As in Section 2.3, the Poincaré map is defined by

$$U(t) = S(T+t, t),$$

where $\{S(t,s) \mid -\infty < s \leq t < \infty\}$ is the family of evolution operators on H generated by $(A(\cdot), D_A)$. Then $U(\cdot)$ satisfies Proposition 2.3.

From now on, we choose the prescribed exponential decay rate $\sigma \geq 0$. We denote by $N_\sigma \in \mathbb{N}$ the number for which $\{\lambda_n\}_{n\in\mathbb{N}}$, the spectrum of $U(0)$ except 0, satisfies (2.13), namely

$$\cdots \leq |\lambda_{N_\sigma+1}| < e^{-\sigma T} < |\lambda_{N_\sigma}| \leq \cdots \leq |\lambda_1|. \tag{4.23}$$

Now we need to verify the Hautus condition (1.7) to prove that the pair $(A(t), B(t))_{t\in\mathbb{R}}$ is open-loop stabilizable in $V_n^0(\Omega)$, with the prescribed decay rate σ. We first need to characterize $B^*(t)$ in the following proposition.

Proposition 4.3. *The adjoint of $G(t) \in \mathcal{L}(V^0(\partial\Omega), V^0(\Omega))$ is defined by*

$$G^*(t)f = \psi(t)n - \nu\frac{\partial\xi(t)}{\partial n}, \quad \forall\, f \in V^0(\Omega),$$

where $(\xi, \psi) \in C^\alpha([0,T]; (V_0^1(\Omega) \cap V^2(\Omega)) \times H^1(\Omega))$ is the solution of

$$\begin{aligned}&\lambda_0\xi - \nu\Delta\xi + (\nabla z_{\rm per}(t))^T\xi - (z_{\rm per}(t)\cdot\nabla)\xi + \nabla\psi = f \quad \text{in} \quad \Omega,\\ &\nabla\cdot\xi = 0 \quad \text{in} \quad \Omega, \quad \xi = 0 \quad \text{on} \quad \partial\Omega.\end{aligned} \tag{4.24}$$

The adjoint of $B(t) \in \mathcal{L}(V^0(\partial\Omega), (D_{A^*})')$ is defined by

$$B^*(t)\xi = M\left(\psi n - \nu\frac{\partial\xi}{\partial n}\right), \quad \forall\xi \in D_{A^*}, \quad \forall\, t \in [0,T],$$

where the pressure $\psi \in C^\alpha([0,T]; H^1(\Omega))$ is uniquely defined, up to an additive constant, by the equation

$$\begin{aligned}&\Delta\psi(t) = -\text{div}\,\Big((I - \mathbb{P})\Big[(\nabla z_{\rm per}(t))^T\xi - (z_{\rm per}(t).\nabla)\xi\Big]\Big) \quad \text{in} \quad \Omega,\\ &\frac{\partial\psi}{\partial n} = \Big[\nu\Delta\xi - (\nabla z_{\rm per}(t))^T\xi + (z_{\rm per}(t)\cdot\nabla)\xi\Big]\cdot n \quad \text{on} \quad \partial\Omega.\end{aligned} \tag{4.25}$$

Moreover, $t \mapsto B^*(t)$ belongs to $C^\alpha([0,T]; \mathcal{L}(V^{s+\frac{3}{2}}(\Omega) \cap V_0^1(\Omega), V^s(\partial\Omega)))$, for all $s \in [0, \frac{1}{2}]$.

Proof. The expressions of $G^*(t) \in \mathcal{L}(V^0(\Omega), V^0(\partial\Omega))$ and $B^*(t) \in \mathcal{L}(D_{A^*}, V^0(\partial\Omega))$ can be found in ([34], Lem. B.4). The regularity of $t \mapsto B^*(t)$ can be proved using the same type of arguments as those in the proof of Lemma 4.2. □

Proposition 4.4. *Let λ belong to $\{\lambda_j \mid 1 \le j \le N_\sigma\}$, where N_σ is characterized by (4.23). Let $\xi \in H^1(0,T;H) \cap L^2(0,T;D_{A^*})$ satisfy*

$$-\,\xi'(\cdot,t) = A^*(t)\xi(\cdot,t), \quad \text{for all } t \in [0,T], \quad \xi(\cdot,0) = \lambda\xi(\cdot,T). \tag{4.26}$$

If $B^*(t)\xi(x,t) = 0$ on $\Gamma_1 \times [0,T]$, then $\xi \equiv 0$ in $\Omega \times [0,T]$.

Proof. From the definition of $B^*(t)$, it follows that

$$\begin{aligned}&-\frac{\partial\xi}{\partial t} - \nu\Delta\xi + (\nabla z_{\rm per})^T\xi - (z_{\rm per}\cdot\nabla)\xi + \nabla\psi = 0 \text{ and } \nabla\cdot\xi = 0 \quad \text{in } \Omega\times(0,T),\\ &\xi = 0 \quad \text{and} \quad M\Big(\psi n - \nu\frac{\partial\xi}{\partial n}\Big) = 0 \quad \text{on } \partial\Omega\times(0,T),\end{aligned} \tag{4.27}$$

and ψ is the solution to equation (4.25). We set

$$\chi(x,t) = \psi(x,t)n - \nu\frac{\partial\xi(x,t)}{\partial n} \quad \text{for all } (x,t) \in \partial\Omega\times(0,T).$$

For all $x \in \partial\Omega$, we denote by $\tau(x)$ the unit tangent vector to $\partial\Omega$ for a given orientation. Using $(4.27)_2$ and the definition of M in (4.4), we get

$$m\chi \cdot \tau = 0 \quad \text{and} \quad m\chi \cdot n \int_{\partial\Omega} m = m \int_{\partial\Omega} m\chi \cdot n \quad \text{on } \partial\Omega \times (0,T). \tag{4.28}$$

Since $\xi(\cdot,t)$ is regular and is a divergence free vector field, using the decomposition $\xi = \xi_\tau \tau + \xi_n n$, we have

$$\operatorname{div} \xi = \frac{\partial \xi_\tau}{\partial \tau} + \frac{\partial \xi_n}{\partial n} + \xi \cdot n \operatorname{div}(n) = 0 \quad \text{on } \partial\Omega \times (0,T).$$

As $\xi = 0$ on $\partial\Omega \times (0,\infty)$, we have $\frac{\partial \xi_\tau}{\partial \tau} = 0$, and therefore $\frac{\partial \xi_n}{\partial n} = 0$ on $\partial\Omega \times (0,T)$. Since $\frac{\partial \xi}{\partial n} \cdot n = \frac{\partial \xi_n}{\partial n} = 0$ on $\partial\Omega \times (0,T)$, the second identity in (4.28) reduces to $m\psi \int_{\partial\Omega} m = m \int_{\partial\Omega} m\psi$, from which we deduce $\psi = C$ on $\Gamma_1 \times (0,T)$ as $m = 1$ on Γ_1.

We denote by $\Omega_e \supset \Omega$ an open set such that $\Omega_e \setminus \Omega$ is simply connected, $\partial(\Omega_e \setminus \Omega) = \Gamma_1 \cup \overline{\Gamma}_e$, where $\Gamma_e \cap \overline{\Omega} = \emptyset$. Since

$$\xi = 0 \quad \text{and} \quad \psi n - \nu \frac{\partial \xi}{\partial n} = C\, n \quad \text{on } \Gamma_1 \times (0,T), \tag{4.29}$$

we extend ξ and ψ to Ω_e by setting

$$(\xi_e, \psi_e) = (\xi, \psi) \quad \text{in } \Omega, \quad (\xi_e, \psi_e) = (0, C) \quad \text{in } \Omega_e \setminus \Omega,$$

and we extend z_{per} to $\Omega_e \setminus \Omega$ by a bounded function, still denoted by z_{per}. Due to (4.29), the pair (ξ_e, ψ_e) satisfies $(4.27)_1$ in $\Omega_e \times (0,T)$, and $\xi_e = 0$ in $\Omega_e \setminus \Omega$. Since the extension of z_{per} belongs to $L^\infty(\Omega_e \times (0,T))$, from ([21], Thm. 1.4), it follows that $\xi = 0$ in $\Omega \times (0,T)$. □

For any family $\mathbf{u} = (u_j)_{j=1}^K$ in $C^\alpha_{\text{per}}([0,T]; (V^{1/2}(\partial\Omega))^K)$, we recall that the operator $\mathbb{B}(\cdot)$ associated to $\mathbf{u}$ is defined in (1.4) by

$$\mathbb{B}(t) f = \sum_{j=1}^K f_j\, B(t) u_j(t) \quad \text{for all } f = (f_1, \cdots, f_K) \in \mathbb{R}^K. \tag{4.30}$$

The operator $\mathbb{B}(t)$ belongs to $\mathcal{L}(\mathbb{R}^K, (D_{A^*})')$ and its adjoint $\mathbb{B}^*(t) \in \mathcal{L}(D_{A^*}, \mathbb{R}^K)$ is defined by

$$\mathbb{B}^*(t)\xi = \left(\int_{\partial\Omega} B^*(t)\xi\, u_j(t) \right)_{1 \le j \le K}, \quad \text{for all } \xi \in D_{A^*} \text{ and all } t \in [0,T].$$

Lemma 4.5. *There exists a family* $\mathbf{u} = (u_j)_{j=1}^K$ *in* $C^\alpha_{\text{per}}([0,T]; (V^{1/2}(\partial\Omega))^K)$ *such that the pair* $(A(\cdot), \mathbb{B}(\cdot))$ *is open-loop stabilizable with the exponential decay rate* $\sigma \ge 0$*, and therefore Assumption* $(\mathcal{A}_3)$ *is satisfied.*

Proof. It follows from Proposition 4.4 and ([7], Cor. 3.19). □

Remark 4.6. The size K of the family given by the Lemma 4.5 can be chosen equal to the maximum of the geometric multiplicities of the eigenvalues $(\lambda_j)_{1 \le j \le N_\sigma}$. We also underline that the family $\mathbf{u} = (u_j)_{j=1}^K$ can be chosen independent of the time variable t if we choose K greater or equal than the sum of the geometric multiplicities of the eigenvalues $(\lambda_j)_{1 \le j \le N_\sigma}$. We refer to [7] for details.

In fact, we have the following unique continuation result stronger than the one stated in Proposition 4.4.

Proposition 4.7. *Let λ belong to $\{\lambda_j \mid 1 \le j \le N_\sigma\}$, where N_σ is characterized by (4.23). Let $\xi \in H^1(0,T;H) \cap L^2(0,T;D_{A^*})$ satisfy (4.26). For a nonempty open interval $(a,b) \subset (0,T)$, if $B^*(t)\xi(x,t) = 0$ on $\Gamma_1 \times [a,b]$, then $\xi \equiv 0$ in $\Omega \times [0,T]$.*

Proof. First, by following the lines of Proposition 4.4 but with (a,b) instead of $(0,T)$, from ([21], Thm. 1.4) we deduce that $\xi \equiv 0$ in $\Omega \times [a,b]$. Thus, since $\xi(\cdot,a) = 0$ in Ω we deduce that ξ (which is solution of a backward equation) also vanishes on $[0,a]$, namely $\xi \equiv 0$ in $\Omega \times [0,b]$. In particular, $\xi(\cdot,0) = 0$ in Ω. Finally, from $|\lambda| > 0$ and $0 = \xi(\cdot,0) = \lambda\xi(\cdot,T)$ we deduce that $\xi(\cdot,T) = 0$, and then that $\xi \equiv 0$ in $\Omega \times [0,T]$. □

The above proposition guarantees the existence of a family reduced to a single element ($K = 1$) for which $(A(\cdot), \mathbb{B}(\cdot))$ is open-loop stabilizable.

Lemma 4.8. *There exists a family $\mathbf{u} = (u)$ in $C^\alpha_{\mathrm{per}}([0,T]; V^{1/2}(\partial\Omega))$, reduced to a single element ($K = 1$), such that the pair $(A(\cdot), \mathbb{B}(\cdot))$ is open-loop stabilizable with the exponential decay rate $\sigma \ge 0$, and therefore Assumption $(\mathcal{A}_3)$ is satisfied.*

Proof. It follows from Proposition 4.7 and ([7], Cor. 3.19). □

From now on, the family $\mathbf{u} = (u_j)_{j=1}^K$ is that introduced in Lemma 4.5 or in Lemma 4.8, and $\mathbb{B}(t)$ is the associated control operator.

Let $\Pi(t) \in \mathcal{L}(H)$ be the solution to equation (3.24). Let us consider the linear homogeneous closed-loop system

$$\begin{aligned}
&\frac{\partial w}{\partial t} - \nu\Delta w + (w\cdot\nabla)z_{\mathrm{per}} + (z_{\mathrm{per}}\cdot\nabla)w + \nabla q = 0 \quad \text{in} \quad \Omega\times(0,\infty),\\
&\nabla\cdot w = 0 \quad \text{in} \quad \Omega\times(0,\infty),\\
&w(\cdot,t) = -\mathbb{B}^*(t)\Pi(t)\mathbb{P}w(t)\cdot\mathbf{u}(t) = -\sum_{j=1}^K \left(\mathbb{B}^*(t)\Pi(t)\mathbb{P}w(t)\right)_j u_j \quad \text{on} \quad \partial\Omega\times(0,\infty),\\
&\mathbb{P}w(0) = \mathbb{P}w_0 \quad \text{in} \quad \Omega.
\end{aligned} \tag{4.31}$$

As in Section 3.2, we introduce the family of operators $(A_\Pi(t), \mathcal{D}(A_\Pi(t)))$ defined by

$$A_\Pi(t) = A(t) - \mathbb{B}(t)\mathbb{B}^*(t)\Pi(t), \quad \mathcal{D}(A_\Pi(t)) = \{x \in H \mid A(t)x - \mathbb{B}(t)\mathbb{B}^*(t)\Pi(t)x \in H\}.$$

The system (4.31) is equivalent to

$$\begin{aligned}
(\mathbb{P}w)'(t) &= A_\Pi(t)\mathbb{P}w(t) \quad \text{in } (D_{A^*})', t > 0, \quad \mathbb{P}w(0) = w_0,\\
(I-\mathbb{P})w(t) &= -(I-\mathbb{P})G(t)\left(\mathbb{B}^*(t)\Pi(t)\mathbb{P}w(t)\cdot\mathbf{u}(t)\right), t > 0.
\end{aligned} \tag{4.32}$$

From Theorem 3.10, it follows that, for all $w_0 \in V^0_n(\Omega)$, the solution $\mathbb{P}w$ of equation $(4.32)_1$ satisfies

$$\|e^{\sigma\,\cdot}\mathbb{P}w(\cdot)\|_{C_b([0,\infty);H)} \le C\|w_0\|_H. \tag{4.33}$$

Before ending this subsection, let us recall some useful results. With the notation introduced in Sections 2–3, we have

$$\begin{aligned}
&V^1_0(\Omega) = D_A^{1/2} = D_{A^*}^{1/2}, \quad V^{-1}(\Omega) = \left(D_{A^*}^{1/2}\right)',\\
&\text{and} \quad V^{1/2-\varepsilon}(\Omega) = D_A^{1/4-\varepsilon/2} \quad \text{for all } 0 < \varepsilon \le 1/2.
\end{aligned} \tag{4.34}$$

4.2. Non-homogeneous closed-loop linearized system

Let us consider the following closed-loop non-homogeneous linear system

$$\begin{aligned} &\frac{\partial w}{\partial t} - \nu\Delta w + (w\cdot\nabla)z_{\text{per}} + (z_{\text{per}}\cdot\nabla)w + \nabla q = h \quad \text{in} \quad \Omega\times(0,\infty),\\ &\nabla\cdot w = 0 \quad \text{in} \quad \Omega\times(0,\infty),\\ &w(\cdot,t) = -\mathbb{B}^*(t)\Pi(t)\mathbb{P}w(t)\cdot\mathbf{u}(t) \quad \text{on} \quad \partial\Omega\times(0,\infty),\\ &\mathbb{P}w(\cdot,0) = w_0 \quad \text{in} \quad \Omega, \end{aligned} \tag{4.35}$$

where h is a given forcing term in $L^2(0,\infty;H^{-1}(\Omega;\mathbb{R}^2))$. The operator $\mathbb{P}$ can be extended as a continuous linear operator $\mathbb{P}\in\mathcal{L}(H^{-1}(\Omega;\mathbb{R}^2),V^{-1}(\Omega))$, where $V^{-1}(\Omega)$ is the dual of $V_0^1(\Omega)$, with $V_n^0(\Omega)$ as pivot space (see [32]). Thus $\mathbb{P}h$ belongs to $L^2(0,\infty;V^{-1}(\Omega))$. The above system is equivalent to

$$\begin{aligned} (\mathbb{P}w)'(t) &= A_\Pi(t)\mathbb{P}w(t) + \mathbb{P}h(t) \quad \text{in } (D_{A^*})', t>0, \quad \mathbb{P}w(0)=w_0,\\ (I-\mathbb{P})w(t) &= -(I-\mathbb{P})G(t)\left(\mathbb{B}^*(t)\Pi(t)\mathbb{P}w(t)\cdot\mathbf{u}(t)\right), t>0. \end{aligned} \tag{4.36}$$

We can write the equation satisfied by $w_u(\cdot) = P_u(\cdot)\mathbb{P}w(\cdot)$ as follows:

$$w_u'(t) = \big(A_u(t) - \mathbb{B}_u(t)\mathbb{B}_u^*(t)\Pi(t)\big)w_u(t) + P_u(t)\mathbb{P}h(t), t>0, \quad w_u(0) = P_u(0)w_0 = w_{0,u}. \tag{4.37}$$

We first prove the following.

Proposition 4.9. *If $e^{\sigma\,\cdot}h(\cdot)$ belongs to $L^2(0,\infty;H^{-1}(\Omega;\mathbb{R}^2))$, and if $w_0\in V_n^0(\Omega)$, then the solution w to system* (4.35) *belongs to $L^2(0,\infty;V^{1/2-\varepsilon}(\Omega))\cap C_b([0,\infty);L^2(\Omega;\mathbb{R}^2))$ and satisfies the estimates*

$$\begin{aligned} &\|e^{\sigma\,\cdot}\mathbb{P}w(\cdot)\|_{C_b([0,\infty);L^2(\Omega;\mathbb{R}^2))} + \|e^{\sigma\,\cdot}(I-\mathbb{P})w(\cdot)\|_{C_b([0,\infty);L^2(\Omega;\mathbb{R}^2))}\\ &\le C(\|w_0\|_H + \|e^{\sigma\,\cdot}h(\cdot)\|_{L^2(0,\infty;H^{-1}(\Omega;\mathbb{R}^2))}),\\ &\text{and}\\ &\|e^{\sigma\,\cdot}\mathbb{P}w(\cdot)\|_{L^2(0,\infty;V^{1/2-\varepsilon}(\Omega))} + \|e^{\sigma\,\cdot}(I-\mathbb{P})w(\cdot)\|_{L^2(0,\infty;V^{1/2-\varepsilon}(\Omega))}\\ &\le C_\varepsilon(\|w_0\|_H + \|e^{\sigma\,\cdot}h(\cdot)\|_{L^2(0,\infty;H^{-1}(\Omega;\mathbb{R}^2))}), \quad \text{for all } 0<\varepsilon\le 1/2. \end{aligned} \tag{4.38}$$

Proof. The estimates for $\mathbb{P}w(\cdot)$ follow from (4.34) and from Theorem 3.11, by choosing δ in (4.21) close enough to $3/4$ to satisfy $\frac14 - \frac{\varepsilon}{2} \le 1-\delta < \frac14$.

To estimate $(I-\mathbb{P})w(\cdot)$, we notice that $\mathbb{B}^*(\cdot)\Pi(\cdot)$ belongs to $C^\alpha_{\text{per}}([0,T];\mathcal{L}(H,\mathbb{R}^K))$ and $G(\cdot)$ belongs to $C^\alpha_{\text{per}}([0,T];\mathcal{L}(V^{1/2}(\partial\Omega),V^1(\Omega))$. Thus the estimates for $(I-\mathbb{P})w(\cdot)$ follow from $(4.36)_2$. □

Proposition 4.10. *If $e^{\sigma\,\cdot}h(\cdot)$ belongs to $L^2(0,\infty;H^{-1}(\Omega;\mathbb{R}^2))$, and if $w_0\in V_n^0(\Omega)$, then the solution w_u to equation* (4.37) *belongs to $L^2(0,\infty;H)$ and satisfies*

$$\|e^{\sigma\,\cdot}w_u(\cdot)\|_{C_b([0,\infty);H)} + \|e^{\sigma\,\cdot}w_u(\cdot)\|_{L^2(0,\infty;H)} \le C(\|w_0\|_H + \|e^{\sigma\,\cdot}h(\cdot)\|_{L^2(0,\infty;H^{-1}(\Omega;\mathbb{R}^2))}). \tag{4.39}$$

Moreover, the mapping $t\mapsto\mathbb{B}^*(t)\Pi(t)w_u(t)$ belongs to $H^{1/4}(0,\infty;\mathbb{R}^K)$, and

$$\|e^{\sigma\,\cdot}\mathbb{B}^*(\cdot)\Pi(\cdot)w_u(\cdot)\|_{H^{1/4}(0,\infty;\mathbb{R}^K)} \le C(\|w_0\|_H + \|e^{\sigma\,\cdot}h(\cdot)\|_{L^2(0,\infty;H^{-1}(\Omega;\mathbb{R}^2))}). \tag{4.40}$$

Proof.

Step 1. The estimate (4.39) follows from Proposition 4.9 and the fact that $P_u \in C_{\rm per}([0,T];\mathcal{L}(H))$ (actually $P_u \in C_{\rm per}([0,T];\mathcal{L}(H,D_A))$ see (2.18)).

Step 2. Let $(Q(t))_{t\in\mathbb{R}}$ and Λ be the operators introduced in Proposition 3.2. To prove (4.40), we set $z(t) = Q^{-1}(t)w_u(t)$. The equation satisfied by z is

$$z'(t) = (\Lambda - Q^{-1}(t)\mathbb{B}_u(t)(Q^{-1}(t)\mathbb{B}_u(t))^*\Pi_u(t)z(t) + Q^{-1}(t)P_u(t)h(t), t>0, \quad z(0) = w_{0,u},$$

and the equation satisfied by $e^{\sigma\,\cdot}z(\cdot)$ is

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(e^{\sigma\, t}z(t))\\ &= (\Lambda + \sigma I - Q^{-1}(t)\mathbb{B}_u(t)(Q^{-1}(t)\mathbb{B}_u(t))^*\Pi_u(t)(e^{\sigma\, t}z(t)) + Q^{-1}(t)P_u(t)e^{\sigma t}h(t).\end{aligned} \tag{4.41}$$

With (4.38) and the fact that $Q^{-1}P_u \in C^1_{\rm per}([0,T],\mathcal{L}(H_{\mathbb{C}},H_{\mathbb{C},u}(0)))$ (see (3.7)), we first get

$$\begin{aligned}&\|e^{\sigma\,\cdot}z(\cdot)\|_{C_b([0,\infty);H_{\mathbb{C},u}(0))} + \|e^{\sigma\,\cdot}z(\cdot)\|_{L^2(0,\infty;H_{\mathbb{C},u}(0))}\\ &\le C(\|w_0\|_H + \|e^{\sigma\,\cdot}h(\cdot)\|_{L^2(0,\infty;H^{-1}(\Omega;\mathbb{R}^2))}).\end{aligned} \tag{4.42}$$

From Proposition 3.3, it follows that $(Q^{-1}(\cdot)\mathbb{B}_u(\cdot))^*\Pi_u(\cdot)$ belongs to $C_b([0,\infty);\mathcal{L}(H_{\mathbb{C},u}(0)))$. Thus, we have

$$\|e^{\sigma\,\cdot}\mathbb{B}(\cdot)(Q^{-1}(\cdot)\mathbb{B}_u(\cdot))^*\Pi_u(\cdot)z(\cdot)\|_{L^2(0,\infty;D'_{A^*})} \le C(\|w_0\|_H + \|e^{\sigma\,\cdot}h(\cdot)\|_{L^2(0,\infty;H^{-1}(\Omega;\mathbb{R}^2))}).$$

Since $Q^{-1}P_u \in C_{\rm per}([0,T],\mathcal{L}(D'_{A^*},H_{\mathbb{C},u}(0)))$, see (3.7), we obtain

$$\begin{aligned}&\|e^{\sigma\,\cdot}Q^{-1}(\cdot)\mathbb{B}_u(\cdot)(Q^{-1}(\cdot)\mathbb{B}_u(\cdot))^*\Pi_u(t)z(\cdot)\|_{L^2(0,\infty;H_{\mathbb{C},u}(0))}\\ &\qquad \le C(\|w_0\|_H + \|e^{\sigma\,\cdot}h(\cdot)\|_{L^2(0,\infty;H^{-1}(\Omega;\mathbb{R}^2))}).\end{aligned}$$

Combining the above estimates for the RHS of equation (4.41), we obtain

$$\|e^{\sigma\,\cdot}z(\cdot)\|_{H^1(0,\infty;H_{\mathbb{C},u}(0))} \le C(\|w_0\|_H + \|e^{\sigma\,\cdot}h(\cdot)\|_{L^2(0,\infty;H^{-1}(\Omega;\mathbb{R}^2))}). \tag{4.43}$$

To estimate $\|e^{\sigma\,\cdot}\mathbb{B}^*(\cdot)\Pi(\cdot)w_u(\cdot)\|_{H^{1/4}(0,\infty;\mathbb{R}^K)}$, note that

$$\begin{aligned}&\|e^{\sigma\,\cdot}\mathbb{B}^*(\cdot)\Pi(\cdot)w_u(\cdot)\|^2_{H^{1/4}(0,\infty;\mathbb{R}^K)} = \|e^{\sigma\,\cdot}\mathbb{B}^*(\cdot)\Pi(\cdot)Q(\cdot)z(\cdot)\|^2_{H^{1/4}(0,\infty;\mathbb{R}^K)}\\ &= \int_0^\infty\int_0^\infty \frac{|e^{\sigma t}\mathbb{B}^*(t)\Pi(t)Q(t)z(t) - e^{\sigma s}\mathbb{B}^*(s)\Pi(s)Q(s)z(s)|^2}{|t-s|^{3/2}}\,\mathrm{d}s\,\mathrm{d}t\\ &\le C\Big(\int_0^\infty\int_0^\infty \frac{\|\mathbb{B}^*(t)\Pi(t)Q(t) - \mathbb{B}^*(s)\Pi(s)Q(s)\|^2_{\mathcal{L}(H_{\mathbb{C},u}(0),\mathbb{C}^K)}}{|t-s|^{3/2}}\|e^{\sigma t}z(t)\|^2_{H_{\mathbb{C},u}(0)}\,\mathrm{d}s\,\mathrm{d}t\\ &\quad + \int_0^\infty\int_0^\infty \|\mathbb{B}^*(s)\Pi(s)Q(s)\|^2_{\mathcal{L}(H_{\mathbb{C},u}(0),\mathbb{R}^K)}\frac{\|e^{\sigma t}z(t) - e^{\sigma s}z(s)\|^2_{H_{\mathbb{C},u}(0)}}{|t-s|^{3/2}}\Big)\,\mathrm{d}s\,\mathrm{d}t\\ &\le C\|\mathbb{B}^*(\cdot)\Pi(\cdot)Q(\cdot)\|_{C^\alpha_{\rm per}([0,T];\mathcal{L}(H_{\mathbb{C},u}(0),\mathbb{C}^K))}\|e^{\sigma\,\cdot}z(\cdot)\|_{H^1(0,\infty;H_{\mathbb{C},u}(0))},\end{aligned}$$

for $\alpha > \frac{3}{4}$. Thus, with (3.6), (3.22), and (4.43), we obtain

$$\|e^{\sigma\,\cdot}\mathbb{B}^*(\cdot)\Pi(\cdot)w_u(\cdot)\|_{H^{1/4}(0,\infty;\mathbb{R}^K)} \le C(\|w_0\|_H + \|e^{\sigma\,\cdot}h(\cdot)\|_{L^2(0,\infty;H^{-1}(\Omega;\mathbb{R}^2))}),$$

and the proof of (4.40) is complete. □

Proposition 4.11. *If $e^{\sigma\,\cdot}h(\cdot)$ belongs to $L^2(0,\infty;H^{-1}(\Omega;\mathbb{R}^2))$, and if $w_0 \in V_n^0(\Omega)$, then the solution w to system (4.35) satisfies the estimate*

$$\|e^{\sigma\,\cdot}w(\cdot)\|_{L^2(0,\infty;V^1(\Omega))} \le C(\|w_0\|_H + \|e^{\sigma\,\cdot}h(\cdot)\|_{L^2(0,\infty;H^{-1}(\Omega;\mathbb{R}^2))}). \tag{4.44}$$

Proof. We rewrite the system (4.35) in the form

$$\begin{aligned}
&\frac{\partial w}{\partial t} - \nu\Delta w + \sigma w + \nabla q = -(w\cdot\nabla)z_{\rm per} - (z_{\rm per}\cdot\nabla)w + \sigma w + h \quad \text{in} \quad \Omega\times(0,\infty),\\
&\nabla\cdot w = 0 \quad \text{in} \quad \Omega\times(0,\infty),\\
&w(\cdot,t) = -\mathbb{B}^*(t)\Pi(t)\mathbb{P}w(t)\cdot\mathbf{u}(t) \quad \text{on} \quad \partial\Omega\times(0,\infty),\\
&\mathbb{P}w(\cdot,0) = w_0 \quad \text{in} \quad \Omega,
\end{aligned}\tag{4.45}$$

and we have

$$\|e^{t(-A_0-\sigma I)}\|_{\mathcal{L}(H)} \le Ce^{-t(\sigma+\varepsilon)}, \quad \text{for some } \varepsilon > 0.$$

We set $w = \zeta + \xi$ and $q = \rho + p$, where

$$\begin{aligned}
&\frac{\partial \zeta}{\partial t} - \nu\Delta\zeta + \sigma\zeta + \nabla\rho = 0 \quad \text{in} \quad \Omega\times(0,\infty),\\
&\nabla\cdot w = 0 \quad \text{in} \quad \Omega\times(0,\infty),\\
&\zeta(\cdot,t) = -\mathbb{B}^*(t)\Pi(t)\mathbb{P}w(t)\cdot\mathbf{u}(t) \quad \text{on} \quad \partial\Omega\times(0,\infty),\\
&\zeta(0) = 0,
\end{aligned}\tag{4.46}$$

and

$$\begin{aligned}
&\frac{\partial \xi}{\partial t} - \nu\Delta\xi + \sigma\xi + \nabla p = -(w\cdot\nabla)z_{\rm per} - (z_{\rm per}\cdot\nabla)w + \sigma w + h \quad \text{in} \quad \Omega\times(0,\infty),\\
&\nabla\cdot\xi = 0 \quad \text{in} \quad \Omega\times(0,\infty),\\
&\xi(\cdot,t) = 0 \quad \text{on} \quad \partial\Omega\times(0,\infty),\\
&\mathbb{P}\xi(\cdot,0) = w_0 \quad \text{in} \quad \Omega.
\end{aligned}\tag{4.47}$$

We have $\mathbb{B}^*(\cdot)\Pi(\cdot)\mathbb{P}w(\cdot) = \mathbb{B}^*(\cdot)\Pi(\cdot)P_u(t)\mathbb{P}w(\cdot) = \mathbb{B}^*(\cdot)\Pi(\cdot)w_u(\cdot)$. Thus, from (4.40), using that $\mathbf{u} \in C^\alpha_{\rm per}([0,T];V^{1/2}(\partial\Omega;\mathbb{R}^K)$, it follows that

$$\|e^{\sigma\,\cdot}\mathbb{B}^*(\cdot)\Pi(\cdot)\mathbb{P}w(\cdot)\cdot\mathbf{u}(\cdot)\|_{H^{1/2,1/4}(\partial\Omega\times(0,\infty))} \le C(\|w_0\|_H + \|e^{\sigma\,\cdot}h(\cdot)\|_{L^2(0,\infty;H^{-1}(\Omega;\mathbb{R}^2))}).$$

Using ([34], Thms. 2.3 and 2.6), with the above estimate, it follows that

$$\|e^{\sigma\,\cdot}\zeta(\cdot)\|_{L^2(0,\infty;V^1(\Omega))} \le C(\|w_0\|_H + \|e^{\sigma\,\cdot}h(\cdot)\|_{L^2(0,\infty;H^{-1}(\Omega;\mathbb{R}^2))}).$$

To estimate ξ, using that $\operatorname{div} z_{\text{per}} = 0$, we notice that

$$\begin{aligned}&\|e^{\sigma\,\cdot}(-(w\cdot\nabla)z_{\text{per}} - (z_{\text{per}}\cdot\nabla)w)\|_{L^2(0,\infty;H^{-1}(\Omega;\mathbb{R}^2))}\\ &\leq C_\varepsilon \|e^{\sigma\,\cdot} w\|_{L^2(0,\infty;V^{1/2-\varepsilon}(\Omega))}\|z_{\text{per}}\|_{C^\alpha_{\text{per}}([0,T];V^1(\Omega))},\end{aligned}$$

for all $0<\varepsilon<1/2$, and

$$\|e^{\sigma\,\cdot}\sigma w\|_{L^2(0,\infty;H^{-1}(\Omega;\mathbb{R}^2))} \leq C\|e^{\sigma\,\cdot} w\|_{L^2(0,\infty;L^2(\Omega))}.$$

From these estimates, with (4.38), it follows that

$$\|e^{\sigma\,\cdot}\xi(\cdot)\|_{L^2(0,\infty;V^1_0(\Omega))} \leq C(\|w_0\|_H + \|e^{\sigma\,\cdot} h(\cdot)\|_{L^2(0,\infty;H^{-1}(\Omega;\mathbb{R}^2))}).$$

□

4.3. Local stabilization of the nonlinear system

To handle the nonlinear term in (4.6), we set

$$N(\zeta) = (\zeta\cdot\nabla)\zeta. \tag{4.48}$$

The nonlinear closed-loop system reads as follows

$$\begin{aligned}&\frac{\partial w}{\partial t} - \nu\Delta w + (w\cdot\nabla)z_{\text{per}} + (z_{\text{per}}\cdot\nabla)w + \nabla q = -N(w) \quad \text{in} \quad \Omega\times(0,\infty),\\ &\nabla\cdot w = 0 \quad \text{in} \quad \Omega\times(0,\infty),\\ &w(\cdot,t) = -\mathbb{B}^*(t)\Pi(t)w(t)\cdot\mathbf{u}(t) \quad \text{on} \quad \partial\Omega\times(0,\infty),\\ &\mathbb{P}w(0) = w_0 \quad \text{in} \quad \Omega.\end{aligned} \tag{4.49}$$

The above system is equivalent to

$$\begin{aligned}&(\mathbb{P}w)'(t) \;=\; A_\Pi(t)\mathbb{P}w(t) - \mathbb{P}N(w(t)) \quad \text{in } (D_{A^*})', t>0, \quad \mathbb{P}w(0) = w_0,\\ &(I-\mathbb{P})w(t) = -(I-\mathbb{P})G(t)\mathbb{B}^*(t)\Pi(t)\mathbb{P}w(t)\cdot\mathbf{u}(t), t>0.\end{aligned} \tag{4.50}$$

Lemma 4.12. *We have*

$$\begin{aligned}&\|e^{\sigma\,\cdot}N(\zeta)\|_{L^2(0,\infty;H^{-1}(\Omega;\mathbb{R}^2))} \leq C_1\,\|e^{\sigma\,\cdot}\zeta\|^2_{L^\infty(0,\infty;V^0(\Omega))\cap L^2(0,\infty;V^1(\Omega))},\\ &\text{and}\\ &\|e^{\sigma\,\cdot}(N(\zeta_1)-N(\zeta_2))\|_{L^2(0,\infty;H^{-1}(\Omega;\mathbb{R}^2))}\\ &\leq C_1\Big(\|e^{\sigma\,\cdot}\zeta_1\|_{L^\infty(0,\infty;V^0(\Omega))\cap L^2(0,\infty;V^1(\Omega))} + \|e^{\sigma\,\cdot}\zeta_2\|_{L^\infty(0,\infty;V^0(\Omega))\cap L^2(0,\infty;V^1(\Omega))}\Big)\\ &\quad\times\Big(\|e^{\sigma\,\cdot}(\zeta_1-\zeta_2)\|_{L^\infty(0,\infty;V^0_n(\Omega))\cap L^2(0,\infty;V^1(\Omega))}\Big),\end{aligned} \tag{4.51}$$

for all $e^{\sigma\,\cdot}\zeta$, $e^{\sigma\,\cdot}\zeta_1$ and $e^{\sigma\,\cdot}\zeta_2$ belonging to $L^2(0,\infty;V^1(\Omega))\cap L^\infty(0,\infty;V^0(\Omega))$, and some positive constant C_1.

Proof. We notice that $L^2(0,\infty;V^1(\Omega))\cap L^\infty(0,\infty;V^0(\Omega))\hookrightarrow L^4(0,\infty;L^4(\Omega))$, and, using that $\operatorname{div}\zeta=0$, we have

$$\begin{aligned}
&\|e^{\sigma\,\cdot}(\zeta\cdot\nabla)\xi\|_{L^2(0,\infty;H^{-1}(\Omega;\mathbb{R}^2))}=\left\|\sup_{\|\phi\|_{H^1_0(\Omega;\mathbb{R}^2)=1}} e^{\sigma\,\cdot}\int_\Omega(\zeta(\cdot)\otimes\xi(\cdot)):\nabla\phi\,\mathrm{d}x\right\|_{L^2(0,\infty)}\\
&\le\left\|e^{\sigma\,\cdot}\|\zeta(\cdot)\otimes\xi(\cdot)\|_{L^2(\Omega;\mathbb{R}^4)}\right\|_{L^2(0,\infty)}\\
&\le C\left\|e^{\sigma\,\cdot}\|\zeta(\cdot)\|_{L^4(\Omega;\mathbb{R}^2)}\|\xi(\cdot)\|_{L^4(\Omega;\mathbb{R}^2)}\right\|_{L^2(0,\infty)}\\
&\le C\|e^{\sigma\,\cdot}\zeta(\cdot)\|_{L^4(0,\infty;L^4(\Omega;\mathbb{R}^2))}\|\xi(\cdot)\|_{L^4(0,\infty;L^4(\Omega;\mathbb{R}^2))}\\
&\le C\|e^{\sigma\,\cdot}\zeta(\cdot)\|_{L^2(0,\infty;V^1(\Omega))\cap L^\infty(0,\infty;V^0(\Omega))}\|\xi(\cdot)\|_{L^2(0,\infty;V^1(\Omega))\cap L^\infty(0,\infty;V^0(\Omega))},
\end{aligned}$$

for all $\zeta\in L^2(0,\infty;V^1(\Omega))\cap L^\infty(0,\infty;V^0(\Omega))$ and $\xi\in L^2(0,\infty;V^1(\Omega))\cap L^\infty(0,\infty;V^0(\Omega))$. The estimates in (4.51) can be derived from the above one. □

Theorem 4.13. *There exist positive constants ρ_0 and μ_0 depending on σ, Ω such that, for all $\mu\in(0,\mu_0)$ and for all initial condition $w_0\in V^0_n(\Omega)$ satisfying*

$$\|w_0\|_{V^0_n(\Omega)}\le\rho_0\mu, \tag{4.52}$$

there exists a unique solution w of the closed-loop system (4.49) in the ball

$$\begin{aligned}
D_\mu=\{e^{\sigma\,\cdot}w\in L^2(0,\infty;V^1(\Omega))\cap C_b([0,\infty);V^0(\Omega))\mid&\\
\|e^{\sigma\,\cdot}w\|_{L^2(0,\infty;V^1(\Omega))}+\|e^{\sigma\,\cdot}w\|_{L^\infty(0,\infty;V^0(\Omega))}\le\mu\}.&
\end{aligned}\tag{4.53}$$

Proof. For a given $\mu>0$, we consider the mapping

$$F\ :\ \zeta\mapsto w_\zeta$$

where $\zeta\in D_\mu$ and w_ζ is the weak solution to the system

$$\begin{aligned}
(\mathbb{P}w_\zeta)'(t)\ &=\ A_\Pi(t)\mathbb{P}w_\zeta(t)-\mathbb{P}N(\zeta)\quad\text{in }(D_{A^*})',t>0,\quad\mathbb{P}w_\zeta(0)=w_0,\\
(I-\mathbb{P})w_\zeta(t)&=-(I-\mathbb{P})G(t)\mathbb{B}^*(t)\Pi(t)\mathbb{P}w_\zeta(t)\cdot\mathbf{u}(t),t>0.
\end{aligned}\tag{4.54}$$

As in ([32], Thm. 4.2), using estimates established in Lemma 4.12 for nonlinear term $N(w(\cdot))$, we can show that there exist positive constants ρ_0 and μ depending on σ, Ω, such that F is a strict contraction in D_μ provided that (4.52) is satisfied. The theorem follows from the Banach fixed point Theorem. □

4.4. Existence of time periodic solutions

Now, we state a theorem claiming the existence of a solution to system (4.2) under some smallness condition.

Theorem 4.14. *Let θ belong to $(1/2,1)$ and T be positive. There exists a positive constant $C_{\theta,T}$ such that, for all $u_{\rm per}\in C^\theta_{\rm per}([0,T];V^{3/2}(\partial\Omega))\cap C^{1+\theta}_{\rm per}([0,T];V^{-1/2}(\partial\Omega))$ and all $f_{\rm per}\in C^\theta_{\rm per}([0,T];L^2(\Omega;\mathbb{R}^2))$ such that*

$$\|u_{\rm per}\|_{C^\theta_{\rm per}([0,T];V^{3/2}(\partial\Omega))\cap C^{1+\theta}_{\rm per}([0,T];V^{-1/2}(\partial\Omega))}+\|f_{\rm per}\|_{C^\theta_{\rm per}([0,T];L^2(\Omega;\mathbb{R}^2))}\le C_{\theta,T},$$

the system (4.2) admits a solution $z_{\rm per}\in C^\theta_{\rm per}([0,T];V^2(\Omega))\cap C^{1+\theta}_{\rm per}([0,T];V^0(\Omega))$. In particular $z_{\rm per}\in C^\alpha_{\rm per}([0,T];V^1(\Omega))\cap L^\infty(\Omega\times(0,T);\mathbb{R}^2)$, with $\alpha=\theta+\frac{1}{2}>\frac{3}{4}$.

Proof. We refer to ([15], Thm. 1), where a similar result is proved for a more complex fluid-structure system. But the proof can be directly adapted to our system. The fact that $z_{\rm per}$ belongs to $C^\alpha_{\rm per}([0,T];V^1(\Omega))$ follows by interpolation. The fact that $z_{\rm per}$ belongs to $L^\infty(\Omega\times(0,T);\mathbb{R}^2)$ follows from the imbedding $C^\theta_{\rm per}([0,T];V^2(\Omega)) \hookrightarrow L^\infty(\Omega\times(0,T);\mathbb{R}^2)$. □

For other existence results for the Navier-Stokes system, we also refer to [23, 27]. Other results may be obtained without smallness condition but under geometrical conditions on Ω, see, *e.g.*, [25].

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References

[1] H. Abou-Kandil, G. Freiling, V. Ionescu and G. Jank, Matrix Equations in Control and Systems Theory, Systems & Control: Foundations & Applications. Birkhauser Basel (2003).
[2] P. Acquistapace, F. Flandoli and B. Terreni, Initial-boundary value problems and optimal control for nonautonomous parabolic systems. *SIAM J. Control Optim.* **29** (1991) 89–118.
[3] P. Acquistapace and B. Terreni, Infinite-horizon linear-quadratic regulator problems for nonautonomous parabolic systems with boundary control. *SIAM J. Control Optim.* **34** (1996) 1–30.
[4] C. Airiau, J.-M. Buchot, R.K. Dubey, M. Fournié, J.-P. Raymond and J. Weller-Calvo, Stabilization and best actuator location for the Navier-Stokes equations *SIAM J. Sci. Comput.* **39** (2017) B993–B1020.
[5] J. Alastruey, S.M. Moore, K.H. Parker, T. David, J. Peiró and S.J. Sherwin, Reduced modelling of blood flow in the cerebral circulation: coupling 1-D, 0-D and cerebral auto-regulation models. Internat. *J. Numer. Methods Fluids* **56** (2008) 1061–1067.
[6] H. Amann, Linear and quasilinear parabolic problems, Vol. I, volume 89 of *Monographs in Mathematics*. Birkhäuser Boston, Inc., Boston, MA (1995).
[7] M. Badra, D. Mitra, M. Ramaswamy and J.-P. Raymond, Stabilizabilty of time-periodic parabolic systems by finite dimensional controls. *SIAM J. Control Optim.* **58** (2020) 1735–1768.
[8] M. Badra, Local stabilization of the Navier-Stokes equations with a feedback controller localized in an open subset of the domain. *Numer. Funct. Anal. Optim.* **28** (2007) 559–589.
[9] M. Badra and T. Takahashi. On the Fattorini criterion for approximate controllability and stabilizability of parabolic systems. *ESAIM: COCV* **20** (2014) 924–956.
[10] M. Badra and T. Takahashi, Stabilization of parabolic nonlinear systems with finite dimensional feedback or dynamical controllers: application to the Navier-Stokes system. *SIAM J. Control Optim.* **49** (2011) 420–463.
[11] V. Barbu and G. Wang, Feedback stabilization of periodic solutions to nonlinear parabolic-like evolution systems. *Indiana Univ. Math. J.* **54** (2005) 1521–1546.
[12] A. Bensoussan, G. Da Prato, M. Delfour and S.K. Mitter, Representation and Control of Infinite Dimensional Systems, Second edition, Systems and Control: Foundations & Applications. Birkhäuser Boston, Inc., Boston, MA (2007).
[13] S. Bittanti, P. Colaneri and G. Guardabassi, Analysis of the periodic Lyapunov and Riccati equations via canonical decomposition. *SIAM J. Control Optim.* **24** (1998) 1138–1149.
[14] S. Bittanti and P. Colaneri, Periodic Systems, Filtering and Control. Springer, Berlin (2009).
[15] J.-J. Casanova, Existence of time-periodic strong solutions to a fluid-structure system. *Discrete Contin. Dyn. Syst.* **39** (2019) 3291–3313.
[16] P. Constantin and C. Foias, Navier-Stokes equations. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL (1988).
[17] G. Da Prato and A. Ichikawa. Quadratic control for linear time-varying systems. *SIAM J. Control Optim.* **28** (1990) 359–381.
[18] G. Da Prato, Synthesis of optimal control for an infinite-dimensional periodic problem. *SIAM J. Control Optim.* **25** (1987) 706–714.
[19] R. Datko, Uniform asymptotic stability of evolutionary processes in a Banach space. *SIAM J. Math. Anal.* **3** (1972) 428–445.
[20] R. Denk, M. Hieber and J. Prüss. $\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type. *Mem. Amer. Math. Soc.* **166** (2003).
[21] C. Fabre, Uniqueness results for Stokes equations and their consequences in linear and nonlinear control problems. *ESAIM: COCV* **1** (1995/96) 267–302.
[22] D. Henry, Geometric Theory of Semilinear Parabolic Equations, *Lecture Notes in Mathematics,* Vol. 840. Springer-Verlag, Berlin (1981).
[23] H. Kato, Existence of periodic solutions of the Navier-Stokes equations. *J. Math. Anal. Appl.* **208** (1997) 141–157.
[24] T. Kato, Perturbation Theory for Linear Operators, *Grundlehren Math. Wiss.*, Vol. 132. Springer-Verlag, New York (1966).
[25] T. Kobayashi, Time periodic solutions of the Navier-Stokes equations under general outflow condition. *Tokyo J. Math.* **32** (2009) 409–424.

[26] P. Koch Medina, Feedback stabilizability of time-periodic parabolic equations. Dynamics reported, 26–98, Dynam. Report. Expositions in Dynamical Systems (N.S.), 5. Springer, Berlin (1996).

[27] P. Kučera, The time-periodic solutions of the Navier-Stokes equations with mixed boundary conditions. *Discrete Contin. Dyn. Syst. Ser. S* **3** (2010) 325–337.

[28] I. Lasiecka and R. Triggiani, Control theory for partial differential equations: continuous and approximation theories. I, Vol. 74 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge (2000).

[29] A. Lunardi, Stabilizability of time-periodic parabolic equations. *SIAM J. Control Optim.* **29** (1991) 810–828.

[30] A. Lunardi, Bounded solutions of linear periodic abstract parabolic equations. *Proc. Roy. Soc. Edinburgh Sect. A* **110** (1988) 135–159.

[31] A. Lunardi, On the evolution operator for abstract parabolic equations. *Israel J. Math.* **60** (1987) 281–314.

[32] P.A. Nguyen, J.-P. Raymond, Boundary stabilization of the Navier-Stokes equations in the case of mixed boundary conditions. *SIAM J. Control Optim.* **53** (2015) 3006–3039.

[33] A. Pazy, Semigroups of linear operators and applications to partial differential equations. *Applied Mathematical Sciences*. Springer-Verlag, New York (1983).

[34] J.-P. Raymond, Stokes and Navier-Stokes equations with nonhomogeneous boundary conditions. *Ann. Inst. Henri Poincaré Anal. Non Linéaire* **24** (2007) 921–951.

[35] J.-P. Raymond, Feedback boundary stabilization of the two-dimensional Navier-Stokes equations. *SIAM J. Control Optim.* **45** (2006) 790–828.

[36] J.-P. Raymond and L. Thevenet, Boundary feedback stabilization of the two dimensional Navier-Stokes equations with finite dimensional controllers. *Discrete Contin. Dyn. Syst.* **27** (2010) 1159–1187.

[37] Sérgio S. Rodrigues, Feedback Boundary Stabilization to Trajectories for 3D Navier–Stokes Equations. To published in: *Appl. Math. Optim.* https://doi.org/10.1007/s00245-017-9474-5. (2018).

[38] R. Temam, Navier-Stokes equations. Theory and numerical analysis, Reprint of the 1984 edition. AMS Chelsea Publishing, Providence, RI (2001).

[39] H. Triebel. Interpolation theory, function spaces, differential operators, volume 18 of North-Holland Mathematical Library. North-Holland Publishing Co., Amsterdam-New York (1978).

[40] G. Wang and Y. Xu, Equivalent conditions on periodic feedback stabilization for linear periodic evolution equations. *J. Function. Anal.* **266** (2014) 5126–5173

[41] A. Yagi, Coïncidence entre des espaces d'interpolation et des domaines de puissances fractionnaires d'opérateurs. *C. R. Acad. Sci. Paris Sér. I Math.* **299** (1984) 173–176.