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EXPLICIT DECAY RATE FOR A DEGENERATE HYPERBOLIC-PARABOLIC COUPLED SYSTEM*

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Abstract. This paper studies the stability of a 1-dim system which comprises a wave equation and a degenerate heat equation in two connected bounded intervals. The coupling between these two different components occurs at the interface with certain transmission conditions. We find an explicit polynomial decay rate for solutions of this system. This rate depends on the degree of the degeneration for the diffusion coefficient near the interface. Besides, the well-posedness of this degenerate coupled system is proved by the semigroup theory.

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1. Introduction

1.1. System

Consider the system:

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = 0 & \text{in } (-1,0)\times\mathbb{R}^+, \\ \theta_t(x,t) - (a(x)\theta_x(x,t))_x = 0 & \text{in } (0,1)\times\mathbb{R}^+, \\ u(-1,t)=0, \quad \theta(1,t)=0 & \text{in } \mathbb{R}^+, \\ u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x) & \text{in } (-1,0), \\ \theta(x,0)=\theta_0(x) & \text{in } (0,1), \end{cases} \tag{1.1}$$

with the transmission conditions at the interface:

$$u_t(0,t)=\theta(0,t), \quad u_x(0,t)=(a\theta_x)(0,t) \ \text{ in } \ \mathbb{R}^+. \tag{1.2}$$

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Here, the initial data u_0 and u_1 and θ_0 are in some spaces which will be given later; the variable diffusion coefficient a is given by

$$a(x) = mx^{\alpha},\ x \in (0,1] \ \text{ with constants } \ 0 \le \alpha < 1, m > 0. \tag{1.3}$$

From (1.3), we see that the parabolic equation in (1.1) degenerates at $x = 0$. The degenerate parabolic equations connect with several applied domains such as image processing, population genetics and some kinds of physical situations [7, 9, 12]. The system (1.1)–(1.2) can be viewed as a simplification of a model of the interactions between the fluid and the elastic structure along their interface (see [2, 24, 25]). Thus, we can view the states u, u_t and θ as the displacement of the elastic structure, the velocity of the elastic structure and the velocity of the fluid. *It deserves mentioning that when $\alpha > 0$, the main operator in the second equation of (1.1) has some degenerate property.*

1.2. Aim and motivation

The aim of this paper is to give an explicit decay rate for the system (1.1)–(1.2), in term of $\alpha \in [0, 1)$. The motivation of our study arises from what follows: First, the stability and the null controllability for the system (1.1)–(1.2) with $a \equiv m$ (which corresponds to the case that $\alpha = 0$) were studied by Zhang and Zuazua in [26] (see also [24]). Second, the method used in [26] seems not to work for the case when the diffusion coefficient a depends on x.

1.3. Main result

First of all, we successfully put the system (1.1)–(1.2) into the abstract framework:

$$\begin{cases} \frac{\mathrm{d}U(t)}{\mathrm{d}t} = \mathcal{A}U(t), & t > 0, \\ U(0) = U_0 \in \mathcal{H}, \end{cases} \tag{1.4}$$

where the operator $\mathcal{A}$ (with its domain $\mathcal{D}(\mathcal{A})$) generates a C_0-semigroup $\{S(t)\}_{t\ge 0}$ in a real Hilbert space $\mathcal{H}$. This will be seen in Section 2. The main result of this paper is as:

Theorem 1.1. *There is a constant $C > 0$ so that*

$$\|S(t)W_0\|_{\mathcal{H}} \le Ct^{-\frac{3-\alpha}{2(1-\alpha)}}\|W_0\|_{\mathcal{D}(\mathcal{A})} \ \textit{ for each } \ W_0 \in \mathcal{D}(\mathcal{A}).$$

Remark 1.2. On one hand, we see from (1.3) that the bigger α is, the more degenerated the second equation in (1.1) is. On the other hand, we find from Theorem 1.1 that the bigger α is, the faster the system (1.1)–(1.2) decays. Hence, the more degenerated the second equation in (1.1) is, the faster the system (1.1)–(1.2) decays. This phenomenon coincides with that in [13] for the degenerate heat equation in non-cylindrical domain (linear moving boundary) where the decay rate is also increasing as α varies from 0 to 2. Besides, when $\alpha \to 1^-$, we have $-\frac{3-\alpha}{2(1-\alpha)} \to -\infty$. Thus, it is reasonable to predict that the exponential stability could be achieved when $\alpha = 1$. However, it is still open that whether the solution to system (1.1)–(1.2) decays exponentially or not when $\alpha = 1$.

1.4. Novelty

We present an explicit relationship between the decay rate of the solutions to the hyperbolic-parabolic coupled system and the degree of the degeneration of the diffusion coefficient at the interface. This seems to be new. Indeed, the studies on large time behavior of the solutions to the degenerate hyperbolic-parabolic coupled systems have not been touched upon. (At least, we do not find such studies in any published literature.)

1.5. Related works

- For 1-dim hyperbolic-parabolic systems (with constant diffusion coefficient), the studies on the decay rates (as well as the null controllability) were done in [26] where the optimal polynomial decay rate t^{-2} was derived. For n-dim hyperbolic-parabolic systems, a nearly optimal polynomial decay rate $t^{-(1-\epsilon)}$ was obtained in [11]. We also would like to mention [2, 3, 22, 24, 25, 27] in this direction.

  It deserves mentioning that the decay rate in our Theorem 1.1 is weaker than that in [26]. This may be caused by our method. However, the diffusion coefficient at the interface in the current paper has degeneration, while that in [26] is a positive constant.
- For more realistic fluid-structure PDE models where the heat equations are replaced by the Navier-Stokes equations or the Stokes equations, the studies on the well-posedness and the decay rate were done in [4, 10, 14, 15] (see also [25]).
- For the studies on the large time behavior of some other kinds of transmission problems, we would like to mention [16–19, 23] and the references therein.
- For the studies on the well-posedness and the controllability of degenerate heat equations, we would like to mention [1, 7, 8].

1.6. Plan of this paper

The rest of this paper is organized as follows: Section 2 proves the well-posedness of the system (1.1)–(1.2). Section 3 presents an explicit polynomial decay rate for this system. The last section is appendix which proves a lemma.

2. Well-posedness

We put the system (1.1)–(1.2) into an abstract framework in the following manner: Let

$$\mathcal{H} = V^1(-1,0) \times L^2(-1,0) \times L^2(0,1), \tag{2.1}$$

where

$$V^k(-1,0) := \{f \in H^k(-1,0) \mid f(-1) = 0\},\ k = 1,2. \tag{2.2}$$

One can easily check that $\mathcal{H}$ is a real Hilbert space $\mathcal{H}$ with the inner product:

$$((u,v,\theta),(\widetilde{u},\widetilde{v},\widetilde{\theta}))_{\mathcal{H}} = \int_{-1}^{0} (u_x\widetilde{u}_x + v\widetilde{v})\mathrm{d}x + \int_0^1 \theta\widetilde{\theta}\mathrm{d}x, \quad (u,v,\theta),(\widetilde{u},\widetilde{v},\widetilde{\theta}) \in \mathcal{H}.$$

Let $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ be defined by

$$\mathcal{A}(u,v,\theta) := (v, u_{xx}, (a\theta_x)_x), \quad (u,v,\theta) \in \mathcal{D}(\mathcal{A}), \tag{2.3}$$

where

$$\mathcal{D}(\mathcal{A}) := \{(u,v,\theta) \in \mathcal{H} \ : \ (u,v,\theta) \text{ is in } \hat{\mathcal{H}} \text{ and satisfies } BC \text{ and } TC\}. \tag{2.4}$$

In (2.4),

$$\hat{\mathcal{H}} := \left\{ (u,v,\theta) \in V^2(-1,0) \times V^1(-1,0) \times L^2(0,1) \ : \ \begin{array}{l} a^{\frac{1}{2}}\theta_x \in L^2(0,1), \\ (a\theta_x)_x \in L^2(0,1),\ \theta \in W^{1,1}(0,1) \end{array} \right\}; \tag{2.5}$$

BC means the boundary condition:

$$\theta(1) = 0; \tag{2.6}$$

TC means the transmission conditions:

$$v(0) = \theta(0) \ \text{ and } \ u_x(0) = (a\theta_x)(0). \tag{2.7}$$

Then one can directly check that the system (1.1)–(1.2) can be rewritten as (1.4). Further, we have

Proposition 2.1. *The operator $\mathcal{A}$ generates a C_0 semigroup $\{S(t)\}_{t\geq 0}$ of contractions on $\mathcal{H}$.*

Proof. First, we have that the operator $\mathcal{A}$ is closed and $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}$ (see Lem. A.1 in Appendix). Moreover, from (2.3), one can directly check

$$(\mathcal{A}Z, Z)_{\mathcal{H}} = -\int_0^1 a|\theta_x|^2 \mathrm{d}x \leq 0 \ \text{ for each } \ Z = (u, v, \theta) \in \mathcal{D}(\mathcal{A}), \tag{2.8}$$

from which, it follows that $\mathcal{A}$ is dissipative in $\mathcal{H}$.

Next, we will prove

$$0 \in \rho(\mathcal{A}). \tag{2.9}$$

When (2.9) is proved, we can use the Lumer-Phillips theorem [21] to see that $\mathcal{A}$ generates a C_0 semigroup on $\mathcal{H}$. The proof of (2.9) is organized by two steps.

Step 1. *We prove that $\mathcal{A}$ is injective.*

Let $W = (\hat{u}, \hat{v}, \hat{\theta}) \in \mathcal{D}(\mathcal{A})$ satisfy $\mathcal{A}W = 0$. Then by (2.3), we have

$$\begin{cases} \hat{v}(x) = 0 & x \in [-1, 0], \\ \hat{u}_{xx}(x) = 0 & x \in [-1, 0], \\ (a\hat{\theta}_x)_x(x) = 0 & x \in [0, 1]. \end{cases} \tag{2.10}$$

According to $(2.10)_3$ and (1.3), there is a constant c_1 so that

$$\hat{\theta}_x(x) = \frac{c_1}{a(x)} \ \text{ for each } \ x \in (0, 1].$$

Thus, by (2.7) and $(2.10)_1$, we have

$$\hat{\theta}(x) = c_1 \int_0^x \frac{1}{a(s)} \mathrm{d}s \ \text{ for each } \ x \in (0, 1]. \tag{2.11}$$

Meanwhile, by (2.6), we see $\hat{\theta}(1) = 0$. This, along with (2.11) and (1.3), yields $c_1 = 0$. Then by making use of (2.11) again, we get

$$\hat{\theta}(x) = 0 \ \text{ for each } \ x \in (0, 1],$$

which, together with the fact that $\hat{\theta} \in W^{1,1}(0, 1)$ (see (2.4)), yields

$$\hat{\theta}(x) = 0 \ \text{ for each } \ x \in [0, 1]. \tag{2.12}$$

Next, by (2.4) and $(2.10)_3$, we have

$$\hat{u}_x(0) = (a\hat{\theta}_x)(0) = 0.$$

This, along with $(2.10)_2$, indicates that

$$\hat{u}_x(x) = 0 \ \text{ for each } \ x \in [-1,0]. \tag{2.13}$$

Meanwhile, from (2.4) and (2.2), we find $\hat{u}(-1) = 0$. This, combined with (2.13), shows that

$$\hat{u}(x) = 0 \ \text{ for each } \ x \in [-1,0]. \tag{2.14}$$

Finally, by (2.14), $(2.10)_1$ and (2.12), it follows that $W = 0$. Hence, $\mathcal{A}$ is injective in $\mathcal{H}$.
Step 1. *We prove that $\mathcal{A}$ is surjective.*
Arbitrarily fix $(f_1, f_2, f_3) \in \mathcal{H} \setminus \{0\}$. Let

$$G := -\frac{1-\alpha}{m}[\int_0^1 \frac{1}{a(s)} \int_0^s f_3(r)\mathrm{d}r\mathrm{d}s + f_1(0)]. \tag{2.15}$$

One can easily check that $|G| < \infty$. Let

$$\begin{cases} \widetilde{u}(x) = (x+1)G - \int_{-1}^x \int_s^0 f_2(r)\mathrm{d}r\mathrm{d}s & x \in [-1,0], \\ \widetilde{v}(x) = f_1(x) & x \in [-1,0], \\ \widetilde{\theta}(x) = -G\int_x^1 \frac{1}{a(s)}\mathrm{d}s - \int_x^1 \frac{1}{a(s)} \int_0^s f_3(r)\mathrm{d}r\mathrm{d}s & x \in [0,1]. \end{cases} \tag{2.16}$$

From (2.16) and (2.15), after direct computations, we can see that

$$(\widetilde{u}, \widetilde{v}, \widetilde{\theta}) \in \mathcal{D}(\mathcal{A}) \ \text{ and } \ \mathcal{A}(\widetilde{u}, \widetilde{v}, \widetilde{\theta}) = (f_1, f_2, f_3).$$

(We omit the details here.) Hence, $\mathcal{A}$ is surjective.
Thus, we have finished the proof of Proposition 2.1. $\square$

We now give two lemmas. The first one, *i.e.*, Lemma 2.2, is a direct consequence of Theorem 1.1 in [5].

Lemma 2.2. *Let $\zeta(\cdot)$ and $\eta(\cdot)$ be two measurable and positive functions over $(0,1)$. Let*

$$L_{2,\zeta}(0,1) := \{f \ : \ \|f\|_{2,\zeta} < \infty\},$$

where

$$\|f\|_{2,\zeta} = \left(\int_0^1 |f(x)|^2 \zeta^2(x)\mathrm{d}x\right)^{\frac{1}{2}}, \ \ f \in L_{2,\zeta}(0,1).$$

Let Q be defined by $Qf(x) := \int_x^1 f(s)\mathrm{d}s$, $x \in (0,1)$. Then the following statements are equivalent:
(a) $\|Qf\|_{2,\eta} \le C\|f\|_{2,\zeta}$ for any $f \in L_{2,\zeta}(0,1)$;
(b) $K = \sup\limits_{0\le x\le 1} \Big(\int_0^x [\eta(s)]^2\mathrm{d}s\Big)^{\frac{1}{2}} \Big(\int_x^1 [\zeta(s)]^{-2}\mathrm{d}s\Big)^{\frac{1}{2}} < \infty$.

The second one is as:

Lemma 2.3. *Given two constants $\beta > -1$ and $\alpha < 1$, there is $C = C(\alpha, \beta) > 0$ so that*

$$\left(\int_0^1 x^{\beta}|\xi(x)|^2 \mathrm{d}x\right)^{\frac{1}{2}} \leq C\left(\int_0^1 x^{\alpha}|\xi_x(x)|^2 \mathrm{d}x\right)^{\frac{1}{2}}, \tag{2.17}$$

for any $\xi(x) \in W^{1,1}(0,1)$ satisfying that $x^{\frac{\alpha}{2}}\xi_x(x) \in L^2(0,1)$ and $\xi(1) = 0$.

Proof. Arbitrarily fix $\beta > -1$, $\alpha < 1$ and $\xi(x) \in W^{1,1}(0,1)$ satisfying that $x^{\frac{\alpha}{2}}\xi_x(x) \in L^2(0,1)$ and $\xi(1) = 0$. Let

$$\eta(x) := x^{\frac{\beta}{2}},\ x \in (0,1);\ \ \zeta(x) := x^{\frac{\alpha}{2}},\ \ x \in (0,1);\ \ f(x) := \xi_x(x),\ x \in (0,1).$$

Two observations are given in order. First, one can easily see

$$\sup_{r\in[0,1]}\left(\int_0^r x^{\beta}\mathrm{d}x\right)\left(\int_r^1 x^{-\alpha}\mathrm{d}x\right) = \sup_{r\in[0,1]} \frac{r^{\beta+1}(1-r^{-\alpha+1})}{(\beta+1)(1-\alpha)} < \infty,$$

which leads to (b) in Lemma 2.2. Second, we have

$$f \in L_{2,\zeta}(0,1) \ \text{ and } \ Qf(x) = \int_x^1 f(s)\mathrm{d}s = -\xi(x),\ x \in (0,1).$$

From these observations, we can apply Lemma 2.2 to obtain

$$\left(\int_0^1 x^{\beta}|\xi(x)|^2 \mathrm{d}x\right)^{\frac{1}{2}} = \|Qf\|_{2,\eta} \leq C\|f\|_{2,\zeta} = C\left(\int_0^1 x^{\alpha}|\xi_x(x)|^2 \mathrm{d}x\right)^{\frac{1}{2}},$$

which yields (2.17) and ends the proof of Lemma 2.3. □

We next give a proposition.

Proposition 2.4. *The resolvent set of $\mathcal{A}$ contains the whole imaginary axis, i.e., $i\mathbb{R} \in \rho(\mathcal{A})$.*

Proof. By contradiction, we suppose that $i\mathbb{R} \not\subseteq \rho(\mathcal{A})$. Then since $0 \in \rho(\mathcal{A})$ and $\rho(\mathcal{A})$ is open in $\mathbb{C}$, we must have

$$0 < \hat{\omega} < \infty,$$

where

$$\hat{\omega} := \sup\{R > 0\ :\ [-Ri, Ri] \subset \rho(\mathcal{A})\}.$$

Thus, by a standard way involving the use of the Banach-Steinhaus theorem (see Page 84, [20]), we can find $W_n = (u_n, v_n, \theta_n) \in D(\mathcal{A})$ (with $\|W_n\|_{\mathcal{H}} = 1$) and $\omega_n \in \mathbb{R}$ (with $\omega_n \to \widehat{\omega}$) so that

$$\lim_{n\to\infty} \|(i\omega_n I - \mathcal{A})W_n\|_{\mathcal{H}} = 0. \tag{2.18}$$

Notice that (2.18) is equivalent to what follows:

$$i\omega_n u_n - v_n = h_{1n} \to 0 \quad \text{in } H^1(-1,0), \tag{2.19}$$

$$i\omega_n v_n - u_{n,xx} = h_{2n} \to 0 \quad \text{in } L^2(-1,0), \tag{2.20}$$

$$i\omega_n\theta_n - (a\theta_{n,x})_x = h_{3n} \to 0 \quad \text{in } L^2(0,1). \tag{2.21}$$

We aim to show

$$\|\theta_n\|_{L^2(0,1)} = o(1) \ \text{ and } \ \int_{-1}^{0} (|v_n|^2 + |u_{n,x}|^2)\mathrm{d}x = o(1). \tag{2.22}$$

Here and in what follows, $o(1)$ denotes a quantity which goes to zero as n tends to ∞. When (2.22) is proved, we can use it to get $\|W_n\|_{\mathcal{H}} = o(1)$, which contradicts the fact that $\|W_n\|_{\mathcal{H}} = 1$. Thus we have finished the proof of Proposition 2.4.

We now show the first equality in (2.22). By (1.3), (2.8) and (2.18), we obtain

$$\mathcal{R}e((i\omega_n - \mathcal{A})W_n, W_n)_{\mathcal{H}} = ||a^{\frac{1}{2}}\theta_{n,x}||^2_{L^2(0,1)} = ||\sqrt{m}x^{\frac{\alpha}{2}}\theta_{n,x}||^2_{L^2(0,1)} = o(1). \tag{2.23}$$

This, along with Lemma 2.3 (where $\beta = -\alpha$), yields

$$\|x^{-\frac{\alpha}{2}}\theta_n\|_{L^2(0,1)} \le C\|x^{\frac{\alpha}{2}}\theta_{n,x}\|_{L^2(0,1)} = o(1). \tag{2.24}$$

Thus, by (2.24), we have

$$\|\theta_n\|^2_{L^2(0,1)} = (x^{\frac{\alpha}{2}}\theta_n, x^{-\frac{\alpha}{2}}\theta_n) \le \|x^{\frac{\alpha}{2}}\theta_n\|_{L^2(0,1)}\|x^{-\frac{\alpha}{2}}\theta_n\|_{L^2(0,1)} \le C\|\theta_n\|_{L^2(0,1)}\|x^{\frac{\alpha}{2}}\theta_{n,x}\|_{L^2(0,1)} = o(1).$$

Hence, the first equality in (2.22) holds.

We next show the second equality in (2.22). The proof is organized by several steps.

Step 1. *We show*

$$(a\theta_{n,x})(0) = o(1). \tag{2.25}$$

First, by (2.23) and (1.3), we obtain

$$\|a\theta_{n,x}\|_{L^2(0,1)} = \|mx^{\alpha}\theta_{n,x}\|_{L^2(0,1)} \le c\|\sqrt{m}x^{\frac{\alpha}{2}}\theta_{n,x}\|_{L^2(0,1)} = o(1). \tag{2.26}$$

Second, by (2.21) and the first equality in (2.22), we see

$$\|(a\theta_{n,x})_x\|_{L^2(0,1)} = o(1). \tag{2.27}$$

Third, by the Gagliardo-Nirenberg inequality (see [20], Page 11), we can find $C_1 > 0$ and $C_2 > 0$ (independent of n) so that

$$\|a\theta_{n,x}\|_{L^\infty(0,1)} \le C_1\|(a\theta_{n,x})_x\|^{\frac{1}{2}}_{L^2(0,1)}\|a\theta_{n,x}\|^{\frac{1}{2}}_{L^2(0,1)} + C_2\|a\theta_{n,x}\|_{L^2(0,1)}. \tag{2.28}$$

Finally, by (2.28), (2.26) and (2.27), we get

$$\|a\theta_{n,x}\|_{L^\infty(0,1)} = o(1),$$

which, along with the fact that $a\theta_{n,x} \in H^1[0,1] \subset C[0,1]$, implies (2.25).

Step 2. *We show*

$$|\theta_n(0)| = o(1). \tag{2.29}$$

By Lemma 2.3 (where $\beta = -\alpha$), we obtain

$$|(\theta_n, \theta_{n,x})_{L^2(0,1)}| \le \|x^{-\frac{\alpha}{2}}\theta_n\|_{L^2(0,1)}\|x^{\frac{\alpha}{2}}\theta_{n,x}\|_{L^2(0,1)} \le C\|x^{\frac{\alpha}{2}}\theta_{n,x}\|_{L^2(0,1)}\|x^{\frac{\alpha}{2}}\theta_{n,x}\|_{L^2(0,1)},$$

which together with (2.23), yields

$$(\theta_n, \theta_{n,x})_{L^2(0,1)} = o(1).$$

This leads to

$$\mathcal{R}e(\theta_n, \theta_{n,x})_{L^2(0,1)} = \frac{1}{2}(|\theta_n(1)|^2 - |\theta_n(0)|^2) = o(1),$$

which, along with the fact that $\theta_n(1) = 0$ (see (2.6)), gives (2.29).
Step 3. *We end the proof of the second equality in* (2.22).
Multiplying (2.20) by $(1+x)u_{n,x}$, integrating it over $(-1,0)$, then using (2.19), we get

$$\int_{-1}^{0}(|v_n|^2 + |u_{n,x}|^2)\mathrm{d}x - (|v_n(0)|^2 + |u_{n,x}(0)|^2) = o(1). \tag{2.30}$$

Since $v_n(0) = \theta_n(0)$ and $u_{n,x}(0) = (a\theta_{n,x})(0)$ (see (2.7)), we can substitute (2.25) and (2.29) into (2.30) to get the second equality in (2.22).
We have completed the proof of Proposition 2.4. □

3. Explicit decay rate

In this section, we will prove Theorem 1.1. For this purpose, we introduce the next key lemma which is quoted from [6] (see also [17] for the weaker result).

Lemma 3.1. *Let A generate a bounded C_0 semigroup $\{S(t)\}_{t\ge 0}$ on a Hilbert space H. Suppose that $\{i\beta | \beta \in \mathbb{R}\} \subset \rho(A)$. Let $\gamma > 0$. Then the following statements are equivalent:*
(i) There is a constant $c > 0$ so that $\|S(t)U_0\|_H \le ct^{-\frac{1}{\gamma}}\|U_0\|_{\mathcal{D}(A)}$ for each $U_0 \in \mathcal{D}(A)$;
(ii) It holds that $\limsup\limits_{\beta\to\infty} \frac{1}{\beta^\gamma}\|(i\beta - A)^{-1}\|_H < \infty$.

Proof of Theorem 1.1. From Proposition 2.4 and Lemma 3.1, we see that it suffices to show

$$\limsup_{\omega\to\infty} \frac{1}{\omega^{\frac{2(1-\alpha)}{3-\alpha}}}\|(i\omega I - \mathcal{A})^{-1}\|_{\mathcal{H}} < \infty. \tag{3.1}$$

By contradiction, we suppose that (3.1) is not true. Then according to the Banach-Steinhaus theorem, there is $\{\omega_n\}_{n=1}^{\infty} \subset \mathbb{R}$ (with $\omega_n \to \infty$) and $\widetilde{X} \in \mathcal{H}$ so that

$$\lim_{n\to\infty} \frac{1}{\omega_n^{\frac{2(1-\alpha)}{3-\alpha}}}\|(i\omega_n I - \mathcal{A})^{-1}\widetilde{X}\|_{\mathcal{H}} = \infty. \tag{3.2}$$

Set

$$\widetilde{F}_n = \frac{1}{\omega_n^{\frac{2(1-\alpha)}{3-\alpha}}}(i\omega_n I - \mathcal{A})^{-1}\widetilde{X}.$$

Since $i\omega_n \in \rho(\mathcal{A})$, we have

$$\widetilde{F}_n \in \mathcal{D}(\mathcal{A}) \ \text{ and } \ \omega_n^{\frac{2(1-\alpha)}{3-\alpha}}(i\omega_n I - \mathcal{A})\widetilde{F}_n = \widetilde{X}.$$

Besides, by (3.2), we find

$$\|F_n\|_{\mathcal{H}} \to \infty, \ \text{ as } \ n \to \infty.$$

Thus, we have

$$\omega_n^{\frac{2(1-\alpha)}{3-\alpha}}(i\omega_n I - \mathcal{A})\frac{\widetilde{F}_n}{\|\widetilde{F}_n\|_{\mathcal{H}}} = \frac{\widetilde{X}}{\|\widetilde{F}_n\|_{\mathcal{H}}} \to 0 \quad \text{in } \mathcal{H}, \quad as\ n \to \infty.$$

Set

$$W_n = (u_n, v_n, \theta_n) = \frac{\widetilde{F}_n}{\|\widetilde{F}_n\|_{\mathcal{H}}}.$$

Then we have that $W_n \in \mathcal{D}(\mathcal{A})$; $\|W_n\|_{\mathcal{H}} = 1$ and

$$\lim_{n\to\infty} \omega_n^{\frac{2(1-\alpha)}{3-\alpha}} \|(i\omega_n I - \mathcal{A})W_n\|_{\mathcal{H}} = 0, \tag{3.3}$$

$i.e.$, when $n \to \infty$,

$$\omega_n^{\frac{2(1-\alpha)}{3-\alpha}}[i\omega_n u_n - v_n] = f_{1n} \to 0 \quad \text{in } H^1(-1,0), \tag{3.4}$$

$$\omega_n^{\frac{2(1-\alpha)}{3-\alpha}}[i\omega_n v_n - u_{n,xx}] = f_{2n} \to 0 \quad \text{in } L^2(-1,0), \tag{3.5}$$

$$\omega_n^{\frac{2(1-\alpha)}{3-\alpha}}[i\omega_n \theta_n - (a\theta_{n,x})_x] = f_{3n} \to 0 \quad \text{in } L^2(0,1). \tag{3.6}$$

By the dissipativity of the system (1.1)–(1.2) (see (2.8)) and by (3.3), we see

$$\mathcal{R}e(\omega_n^{\frac{2(1-\alpha)}{3-\alpha}}(i\omega_n - \mathcal{A})W_n, W_n)_{\mathcal{H}} = \omega_n^{\frac{2(1-\alpha)}{3-\alpha}} ||a^{\frac{1}{2}}\theta_{n,x}||^2_{L^2(0,1)} = o(1). \tag{3.7}$$

We claim

$$\|W_n\|_{\mathcal{H}} = o(1). \tag{3.8}$$

When this is done, we get a contradiction (since $\|W_n\|_{\mathcal{H}} = 1$ for all n) which leads to (3.1).

The rest is to show (3.8). This will be carried out by several steps.

Step 1. *We show the boundedness of* $\{|(a\theta_{n,x})(0)|\}_{n=1}^{\infty}$ *and* $\{|\theta_n(0)|\}_{n=1}^{\infty}$.

Multiplying (3.5) by $\frac{(1+x)u_{n,x}}{\omega_n^{\frac{2(1-\alpha)}{3-\alpha}}}$, integrating over $(-1,0)$, and then combining (3.4), we find

$$\int_{-1}^{0}(|v_n|^2 + |u_{n,x}|^2)\mathrm{d}x - (|v_n(0)|^2 + |u_{n,x}(0)|^2) = o(1). \tag{3.9}$$

Since $\|(u_n, v_n, \theta_n)\|_{\mathcal{H}} = 1$, we can see from (3.9) that

$$|v_n(0)| < C \text{ and } |u_{n,x}(0)| < C \text{ for all } n.$$

Here and in what follows, C stands for a positive constant independent of n. Then using the transmission conditions: $\theta_n(0) = v_n(0)$, $(a\theta_{n,x})(0) = u_{n,x}(0)$, we obtain that

$$|(a\theta_{n,x})(0)| < C \text{ and } |\theta_n(0)| < C \text{ for all } n. \tag{3.10}$$

Step 2. *We show*

$$|\omega_n{}^{\frac{1-\alpha}{3-\alpha}}\theta_n(0)| = o(1) \;\; \textit{and} \;\; \omega_n^{\frac{1}{2}+\frac{1-\alpha}{2(3-\alpha)}}\|\theta_n\|_{L^2(0,1)} = o(1). \tag{3.11}$$

To show the first equality in (3.11), we note that $\theta_n(1) = 0$ and

$$\theta_n(1) - \theta_n(0) = \int_0^1 \theta_{n,x}(x)\mathrm{d}x. \tag{3.12}$$

From (3.12), (1.3), (3.7) and the Hölder inequality, we find

$$\begin{aligned} |\theta_n(0)| &= |\int_0^1 a^{-\frac{1}{2}}a^{\frac{1}{2}}\theta_{n,x}(x)\mathrm{d}x| \\ &\le \sqrt{\int_0^1 a^{-1}\mathrm{d}s} \cdot \sqrt{\int_0^1 |a^{\frac{1}{2}}\theta_{n,x}(x)|^2\mathrm{d}x} \\ &= \sqrt{\frac{m}{-\alpha+1}} \cdot \omega_n^{-\frac{1-\alpha}{3-\alpha}}o(1) \\ &= \omega_n^{-\frac{1-\alpha}{3-\alpha}}o(1), \end{aligned} \tag{3.13}$$

which leads to the first estimate in (3.11).

To show the second estimate in (3.11), we use (3.6) to obtain

$$\omega_n^{\frac{1-\alpha}{3-\alpha}}(i\omega_n\theta_n, \theta_n) - \omega_n^{\frac{1-\alpha}{3-\alpha}}((a\theta_{n,x})_x, \theta_n) = \omega_n^{-\frac{1-\alpha}{3-\alpha}}(f_{3n}, \theta_n) = o(1). \tag{3.14}$$

(Here and in what follows, $(\cdot,\cdot)$ denotes the inner product in $L^2(0,1)$.) Using integration by parts in (3.14), we have

$$\omega_n^{\frac{1-\alpha}{3-\alpha}}(i\omega_n\theta_n, \theta_n) + \omega_n^{\frac{1-\alpha}{3-\alpha}}(a\theta_{n,x})(0)\theta_n(0) + \omega_n^{\frac{1-\alpha}{3-\alpha}}(a\theta_{n,x}, \theta_{n,x}) = o(1).$$

This, along with (3.10), (3.13) and (3.7), yields

$$\omega_n^{\frac{1-\alpha}{3-\alpha}}(i\omega_n\theta_n, \theta_n) = o(1),$$

which leads to the second estimate in (3.11).

Step 3. *We show*

$$(a\theta_{n,x})(0) = o(1). \tag{3.15}$$

Notice that for any $r > 0$,

$$\begin{aligned}
\min_{x\in[\frac{1}{2}\omega_n^{-r},\omega_n^{-r}]} |(a\theta_{n,x})(x)| &\leq \frac{1}{\frac{1}{2}\omega_n^{-r}} \int_{\frac{1}{2}\omega_n^{-r}}^{\omega_n^{-r}} |a\theta_{n,x}|\mathrm{d}x \\
&\leq \frac{1}{\frac{1}{2}\omega_n^{-r}} \sqrt{\frac{1}{2}\omega_n^{-r}} \sqrt{\int_{\frac{1}{2}\omega_n^{-r}}^{\omega_n^{-r}} |a\theta_{n,x}|^2\mathrm{d}x} \\
&\leq \sqrt{2}\omega_n^{\frac{r}{2}} \|a\theta_{n,x}\|_{L^2(\frac{1}{2}\omega_n^{-r},\omega_n^{-r})} \\
&\leq \sqrt{2}\omega_n^{\frac{r}{2}} (\max_{x\in[\frac{1}{2}\omega_n^{-r},\omega_n^{-r}]} |\sqrt{m}x^{\frac{\alpha}{2}}|)\|a^{\frac{1}{2}}\theta_{n,x}\|_{L^2(0,1)} \\
&\leq \sqrt{2m}\omega_n^{\frac{r}{2}}\omega_n^{-\frac{\alpha r}{2}} \|a^{\frac{1}{2}}\theta_{n,x}\|_{L^2(0,1)} \\
&= \omega_n^{\frac{(1-\alpha)r}{2}-\frac{1-\alpha}{3-\alpha}} o(1).
\end{aligned} \tag{3.16}$$

(We used (3.7) in the last equality of (3.16).) Let $r = \frac{2}{3-\alpha}$. One can directly check that

$$\frac{(1-\alpha)r}{2} - \frac{1-\alpha}{3-\alpha} = \frac{1-r}{2} - \frac{1-\alpha}{2(3-\alpha)} = 0. \tag{3.17}$$

Then it follows from (3.16) and (3.17) that

$$\min_{x\in[\frac{1}{2}\omega_n^{-r},\omega_n^{-r}]} |(a\theta_{n,x})(x)| = o(1).$$

Thus, there is a sequence $\{\zeta_n\}_{n=1}^{\infty} \subset [\frac{1}{2}\omega_n^{-r}, \omega_n^{-r}]$ so that

$$|(a\theta_{n,x})(\zeta_n)| = o(1). \tag{3.18}$$

Meanwhile, dividing (3.6) by $\omega_n^{\frac{2(1-\alpha)}{3-\alpha}}$ and then integrating from 0 to ζ_n, we find

$$i\omega_n \int_0^{\zeta_n} \theta_n(s)\mathrm{d}s - (a\theta_{n,x})(\zeta_n) + (a\theta_{n,x})(0) = o(1). \tag{3.19}$$

By (3.19) and (3.18), we see

$$(a\theta_{n,x})(0) = -i\omega_n \int_0^{\zeta_n} \theta_n(s)\mathrm{d}s + o(1). \tag{3.20}$$

From (3.20) and the second estimate in (3.11), we obtain

$$
\begin{aligned}
|(a\theta_{n,x})(0)| &\le \left|\omega_n^{1-(\frac{1}{2}+\frac{1-\alpha}{2(3-\alpha)})}\int_0^{\zeta_n}\omega_n^{\frac{1}{2}+\frac{1-\alpha}{2(3-\alpha)}}\theta_n(s)\mathrm{d}s\right| + o(1)\\
&\le \omega_n^{1-(\frac{1}{2}+\frac{1-\alpha}{2(3-\alpha)})}\sqrt{\zeta_n}\cdot\sqrt{\int_0^{\zeta_n}|\omega_n^{\frac{1}{2}+\frac{1-\alpha}{2(3-\alpha)}}\theta_n(s)|^2\mathrm{d}s} + o(1)\\
&\le \omega_n^{1-(\frac{1}{2}+\frac{1-\alpha}{2(3-\alpha)})}\omega_n^{-\frac{r}{2}}o(1) + o(1)\\
&= \omega_n^{\frac{1-r}{2}-\frac{1-\alpha}{2(3-\alpha)}}o(1) + o(1).
\end{aligned}
$$

This, along with (3.17), leads to (3.15).
Step 4. *We show*

$$\|u_{n,x}\|_{L^2(-1,0)} \text{ and } \|v_n\|_{L^2(-1,0)} = o(1). \tag{3.21}$$

Using the transmission conditions: $v_n(0) = \theta_n(0)$, $u_{n,x}(0) = (a\theta_{n,x})(0)$ (see (2.7)), and substituting the first estimate in (3.11) and (3.15) into (3.9), we have

$$\int_{-1}^0 (|v_n|^2 + |u_{n,x}|^2)\mathrm{d}x = o(1),$$

which leads to (3.21).
Step 5. *We show* (3.8).
By the second estimate in (3.11) and (3.21), we obtain (3.8) at once.
Hence, we have completed the proof of Theorem 1.1. □

Appendix A

Lemma A.1. *The operator $\mathcal{A}$ is closed and $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}$.*

Proof. We organize the proof by two parts.
Part I. We show that $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}$, that is, $\overline{\mathcal{D}(\mathcal{A})} = \mathcal{H}$.

Arbitrarily fix $W := (u, v, \theta) \in \mathcal{H}$. We aim to find a sequence $Y_n = (\hat{u}^n, \hat{v}^n, \hat{\theta}^n) \in \mathcal{D}(\mathcal{A})$ so that $Y_n \to W$ in $\mathcal{H}$, *i.e.*, given $\varepsilon > 0$, there is a positive integer $N(\varepsilon)$ so that

$$\|W - Y_n\|_{\mathcal{H}} = [\int_{-1}^0 |u_x - \hat{u}_x^n|^2\mathrm{d}x + \int_{-1}^0 |v - \hat{v}^n|^2\mathrm{d}x + \int_0^1 |\theta - \hat{\theta}^n|^2\mathrm{d}x]^{\frac{1}{2}} < \varepsilon \quad \forall\, n > N. \tag{A.1}$$

This will be carried out by several steps.
Step I.1. *We choose $(\hat{u}^n, \hat{v}^n, \hat{\theta}^n)$.*

By the density of $C_0^\infty(-1,0)$ in $L^2(-1,0)$, we can find two sequences of smooth functions: $\{\hat{u}^n\}_{n=1}^\infty$ and $\{\hat{v}^n\}_{n=1}^\infty$, with $\hat{u}^n(-1) = \hat{v}^n(-1) = 0$, and a positive integer N_1 so that

$$\|u_x - \hat{u}_x^n\|_{L^2(-1,0)} < \varepsilon, \quad \|v - \hat{v}^n\|_{L^2(-1,0)} < \varepsilon \quad \text{as } n > N_1. \tag{A.2}$$

Similarly, we can find a sequence $\{\widetilde{\theta}_1^n\}_{n=1}^\infty \in C_0^\infty(0,1)$, with $\widetilde{\theta}_1^n(1) = 0$, and a positive integer N_2 so that

$$\|\theta - \widetilde{\theta}_1^n\|_{L^2(0,1)} < \varepsilon \quad \text{as } n > N_2. \tag{A.3}$$

Then we choose

$$\hat{\theta}^n(x) := \begin{cases} \widetilde{\theta}_1^n(x) & x \in [\frac{1}{n}, 1], \\ f_1^n(x) & x \in [\frac{1}{2n}, \frac{1}{n}], \\ f_2^n(x) & x \in [0, \frac{1}{2n}], \end{cases} \tag{A.4}$$

where

$$f_2^n(x) = \hat{v}^n(0) + \frac{1}{m(1-\alpha)}\hat{u}_x^n(0)x^{1-\alpha} \quad x \in [0, \frac{1}{2n}]; \tag{A.5}$$

f_1^n is a smooth function which connects smoothly $\widetilde{\theta}_1^n$ and f_2^n. Indeed, f_1^n can be chosen in the following manner:

$$f_1^n(x) := b_1^n x^3 + b_2^n x^2 + b_3^n x + b_4^n, \tag{A.6}$$

where

$$\begin{cases} b_1^n = 4n^3(-4\widetilde{\theta}_1^n(\frac{1}{n}) + \frac{1}{n}\widetilde{\theta}_{1,x}^n(\frac{1}{n}) + 4f_2^n(\frac{1}{2n}) + \frac{1}{n}f_{2,x}^n(\frac{1}{2n})), \\ b_2^n = -2n^2(-18\widetilde{\theta}_1^n(\frac{1}{n}) + \frac{4}{n}\widetilde{\theta}_{1,x}^n(\frac{1}{n}) + 18f_2^n(\frac{1}{2n}) + \frac{5}{n}f_{2,x}^n(\frac{1}{2n})), \\ b_3^n = -n(24\widetilde{\theta}_1^n(\frac{1}{n}) - \frac{5}{n}\widetilde{\theta}_{1,x}^n(\frac{1}{n}) - 24f_2^n(\frac{1}{2n}) - \frac{8}{n}f_{2,x}^n(\frac{1}{2n})), \\ b_4^n = 5\widetilde{\theta}_1^n(\frac{1}{n}) - \frac{1}{n}\widetilde{\theta}_{1,x}^n(\frac{1}{n}) - 4f_2^n(\frac{1}{2n}) - \frac{2}{n}f_{2,x}^n(\frac{1}{2n}). \end{cases} \tag{A.7}$$

Step I.2. *We show that* $(\hat{u}^n, \hat{v}^n, \hat{\theta}^n) \in \hat{\mathcal{H}}$.

By (2.5), we only need to show

$$(\hat{u}^n, \hat{v}^n, \hat{\theta}^n) \in V^2(-1,0) \times V^1(-1,0) \times L^2(0,1) \tag{A.8}$$

and

$$a^{\frac{1}{2}}\hat{\theta}_x^n \in L^2(0,1),\ (a\hat{\theta}_x^n)_x \in L^2(0,1),\ \hat{\theta}^n \in W^{1,1}(0,1). \tag{A.9}$$

To this end, several observations are given in order.

Observation One: It is clear that

$$\hat{u}^n \in V^2(-1,0);\ \hat{v}^n \in V^1(-1,0);\ \ \hat{\theta}^n \in L^2(0,1). \tag{A.10}$$

Observation Two: Since

$$\|\frac{1}{a(x)}\|_{L^1(0,1)} = \|\frac{1}{m}x^{-\alpha}\|_{L^1(0,1)} = \frac{m}{-\alpha+1}, \text{ when } 0 \le \alpha < 1, \tag{A.11}$$

we have

$$\frac{1}{a(x)} := \frac{1}{m}x^{-\alpha} \in L^1(0,1), \text{ when } 0 \le \alpha < 1.$$

Observation Three: A direct calculation yields that

$$\int_0^{\frac{1}{2n}} a(x)|f_{2,x}^n(x)|^2 \mathrm{d}x = \int_0^{\frac{1}{2n}} mx^\alpha |\frac{1}{m}\hat{u}_x^n(0)x^{-\alpha}|^2 \mathrm{d}x \le |\hat{u}_x^n(0)|^2 \int_0^{\frac{1}{2n}} \frac{1}{m}x^{-\alpha}\mathrm{d}x < \infty.$$

This, along with (A.4) and the facts that $\widetilde{\theta}_1^n \in C_0^\infty(0,1)$ and f_1^n is smooth, yields

$$\int_0^1 a(x)|\hat{\theta}_x^n(x)|^2 \mathrm{d}x < \infty,$$

which leads to

$$a^{\frac{1}{2}}\hat{\theta}_x^n \in L^2(0,1).$$

Observation Four: By (A.5), we have

$$\int_0^{\frac{1}{2n}} |(af_{2,x}^n)_x|^2 \mathrm{d}x = 0,$$

which, together with (A.4) and our choices of $\widetilde{\theta}_1^n$ and f_1^n, implies that

$$(a\hat{\theta}_x^n)_x \in L^2(0,1).$$

Observation Five: By the Hölder inequality, Observation Three, (A.11) and (1.3), we have

$$\int_0^1 |\hat{\theta}_x^n(x)|\mathrm{d}x \le \sqrt{\frac{m}{1-\alpha}} \cdot \sqrt{\int_0^1 a(x)|\hat{\theta}_x^n(x)|^2 \mathrm{d}x} < \infty,$$

which, along with (A.10), leads to

$$\hat{\theta}^n(x) \in W^{1,1}(0,1).$$

Now from the above observations, (A.8) and (A.9), we see that $(\hat{u}^n, \hat{v}^n, \hat{\theta}^n) \in \hat{\mathcal{H}}$.

Step I.3. *We prove* $(\hat{u}^n, \hat{v}^n, \hat{\theta}^n) \in \mathcal{D}(\mathcal{A})$.

By *Step I.2* and (2.4), we only need to show that $(\hat{u}^n, \hat{v}^n, \hat{\theta}^n)$ satisfies conditions BC and TC (see (2.6) and (2.7)). First, by (A.4), we have $\hat{\theta}^n(1) = \widetilde{\theta}_1^n(1) = 0$, which along with (2.6), leads to BC. Next, we show that $(\hat{u}^n, \hat{v}^n, \hat{\theta}^n)$ holds TC. Indeed, by (A.5), we have

$$f_{2,x}^n(x) = \frac{1}{m}\hat{u}_x^n(0)x^{-\alpha} \quad \text{and} \quad (af_{2,x}^n)(x) = mx^\alpha f_{2,x}^n(x) = mx^\alpha \frac{1}{m}\hat{u}_x^n(0)x^{-\alpha} = \hat{u}_x^n(0). \tag{A.12}$$

From (A.12), we see

$$(a\hat{\theta}_x^n)(0) = (af_{2,x}^n)(0) = \hat{u}_x^n(0). \tag{A.13}$$

Meanwhile, by (A.5), we can easily check

$$\hat{\theta}^n(0) = f_2^n(0) = \hat{v}^n(0). \tag{A.14}$$

From (A.13), (A.14) and (2.7), we get TC.

Hence, we conclude that $(\hat{u}^n, \hat{v}^n, \hat{\theta}^n) \in \mathcal{D}(\mathcal{A})$.

Step I.4. *We show (A.1).*

First of all, according to (A.4) and (A.5), there is $C > 0$ so that

$$|\widetilde{\theta}_1^n(x)| < C,\ |\widetilde{\theta}_{1,x}^n(x)| < C \quad \forall x \in [0,1], \quad \text{and} \quad |f_2^n(x)| < C \quad \forall x \in [0, \frac{1}{2n}]. \tag{A.15}$$

Set

$$M_1 := \max_{x\in[\frac{1}{2n},\frac{1}{n}],\ n\in\mathbb{Z}} |\widetilde{\theta}_1^n - f_1^n| \ \text{ and } \quad M_2 := \max_{x\in[0,\frac{1}{2n}],\ n\in\mathbb{Z}} |\widetilde{\theta}_1^n - f_2^n|. \tag{A.16}$$

We claim that M_1 and M_2 are finite, that is, for some $M > 0$,

$$M_1 < M \ \text{ and } \ M_2 < M. \tag{A.17}$$

In fact, the second inequality in (A.17) is clear. To see the first one in (A.17), we use (A.12) to find

$$|\frac{1}{n} f_{2,x}^n(\frac{1}{2n})| = \frac{2^\alpha}{m} |\hat{u}_x^n(0)| (\frac{1}{n})^{1-\alpha} < C.$$

Thus, by (A.7) and (A.15), we can find some positive constant $\widetilde{C}$ so that

$$|b_1^n| \le \widetilde{C} n^3, \quad |b_2^n| \le \widetilde{C} n^2, \quad |b_3^n| \le \widetilde{C} n, \quad |b_4^n| \le \widetilde{C}.$$

The above, along with (A.6), yields that when $x \in [\frac{1}{2n}, \frac{1}{n}]$,

$$|f_1^n(x)| \le |b_1^n| \frac{1}{n^3} + |b_2^n| \frac{1}{n^2} + |b_3^n| \frac{1}{n} + |b_4^n| \le 4\widetilde{C},$$

which, together with (A.16), leads to the first inequality in (A.17).

Next, we choose N_3 so that $\frac{1}{n} < \varepsilon$ *as* $n > N_3$. Then by (A.3), we have as $n > \max\{N_2, N_3\}$,

$$\begin{aligned}
&\int_0^1 |\theta - \hat{\theta}^n|^2 \mathrm{d}x \\
&= \int_{\frac{1}{n}}^1 |\theta - \widetilde{\theta}_1^n|^2 \mathrm{d}x + \int_{\frac{1}{2n}}^{\frac{1}{n}} |\theta - \widetilde{\theta}_1^n + \widetilde{\theta}_1^n - f_1^n|^2 \mathrm{d}x + \int_0^{\frac{1}{2n}} |\theta - \widetilde{\theta}_1^n + \widetilde{\theta}_1^n - f_2^n|^2 \mathrm{d}x \\
&\le \int_{\frac{1}{n}}^1 |\theta - \widetilde{\theta}_1^n|^2 \mathrm{d}x + 2\int_{\frac{1}{2n}}^{\frac{1}{n}} (|\theta - \widetilde{\theta}_1^n|^2 + |\widetilde{\theta}_1^n - f_1^n|^2) \mathrm{d}x + 2\int_0^{\frac{1}{2n}} (|\theta - \widetilde{\theta}_1^n|^2 + |\widetilde{\theta}_1^n - f_2^n|^2) \mathrm{d}x \\
&\le 2\varepsilon^2 + M^2\varepsilon + M^2\varepsilon.
\end{aligned} \tag{A.18}$$

Finally, by (A.2) and (A.18), we see that when $n > N := \max\{N_1, N_2, N_3\}$,

$$\|W - Y_n\|_{\mathcal{H}}^2 < 4\varepsilon^2 + 2M^2\varepsilon,$$

which leads to (A.1).

Step I.5. *From conclusions in above steps, we see that* $\overline{\mathcal{D}(\mathcal{A})} = \mathcal{H}$.

Part II. We show that $\mathcal{A}$ is a closed operator on $\mathcal{H}$.

Let $\{W_n\}_{n=1}^{\infty} \subset \mathcal{D}(\mathcal{A})$ satisfy that

$$W_n \to W \ \text{ and } \ \mathcal{A}W_n \to \hat{F} \text{ in } \mathcal{H}, \text{ as } n \to \infty.$$

Write

$$W_n := (u_n, v_n, \theta_n);\ W := (u, v, \theta) \ \text{ and } \ \hat{F} := (\hat{f}_1, \hat{f}_2, \hat{f}_3).$$

Then we have that

$$\mathcal{A}W_n = (v_n, u_{n,xx}, (a\theta_{n,x})_x) \in \mathcal{H}; \tag{A.19}$$

$$\begin{cases} u_n & \to & u & \text{in } V^1(-1,0), \\ v_n & \to & v & \text{in } L^2(-1,0), & \quad \textit{as } n \to \infty; \\ \theta_n & \to & \theta & \text{in } L^2(0,1), \end{cases} \tag{A.20}$$

and

$$\begin{cases} v_n & \to & \hat{f}_1 & \text{in } V^1(-1,0), \\ u_{n,xx} & \to & \hat{f}_2 & \text{in } L^2(-1,0), & \quad \textit{as } n \to \infty. \\ (a\theta_{n,x})_x & \to & \hat{f}_3 & \text{in } L^2(0,1), \end{cases} \tag{A.21}$$

It suffices to show

$$W \in \mathcal{D}(\mathcal{A}), \ \text{ and } \ \mathcal{A}W = \hat{F}, \tag{A.22}$$

i.e.,

$$v = \hat{f}_1, \quad u_{xx} = \hat{f}_2, \quad (a\theta_x)_x = \hat{f}_3. \tag{A.23}$$

The proof of (A.22) is organized by the following two steps.
Step II.1. *We show $\mathcal{A}W = \hat{F}$, i.e., (A.23).*
First, by $(A.20)_2$ and $(A.21)_1$, we can easily check

$$v = \hat{f}_1 \ \text{ in } \ V^1(-1,0), \tag{A.24}$$

which is the first equality in (A.23).
Second, we will show the second equality in (A.23), *i.e.*,

$$u_{xx} = \hat{f}_2 \ \text{ in } \ L^2(-1,0). \tag{A.25}$$

To this end, we let

$$G_n := -\frac{1-\alpha}{m}\Big[\int_0^1 \frac{1}{a(s)}\int_0^s (a\theta_{n,r})_r \mathrm{d}r\mathrm{d}s + v_n(0)\Big]. \tag{A.26}$$

Since $(u_n, v_n, \theta_n) \in \mathcal{D}(\mathcal{A})$, we can easily check from (A.19), (2.1), (2.4), (2.6) and (2.7) that

$$u_{n,x}(x) = G_n - \int_x^0 u_{n,ss}\mathrm{d}s,\ x \in [-1, 0], \tag{A.27}$$

$$\theta_n(x) = -G_n \int_x^1 \frac{1}{a(s)}\mathrm{d}s - \int_x^1 \frac{1}{a(s)} \int_0^s (a\theta_{n,r})_r \mathrm{d}r\mathrm{d}s,\ x \in [0, 1]. \tag{A.28}$$

We now claim that

$$G_n \to G := -\frac{1-\alpha}{m}[\int_0^1 \frac{1}{a(s)} \int_0^s \hat{f}_3(r)\mathrm{d}r\mathrm{d}s + \hat{f}_1(0)] < \infty, \quad as\ n \to \infty \tag{A.29}$$

and that when $x \in [-1, 0]$,

$$\int_x^0 u_{n,ss}\mathrm{d}s \to \int_x^0 \hat{f}_2(s)\mathrm{d}s, \quad as\ n \to \infty. \tag{A.30}$$

To show (A.29), we use by the Hölder inequality to find

$$\begin{aligned} &|\int_x^1 \Big[\frac{1}{a(s)} \int_0^s (a\theta_{n,r})_r \mathrm{d}r - \frac{1}{a(s)} \int_0^s \hat{f}_3 \mathrm{d}r\Big]\mathrm{d}s| \\ \leq &\int_x^1 \frac{1}{a(s)}\Big[\int_0^s |(a\theta_{n,r})_r - \hat{f}_3|\mathrm{d}r\Big]\mathrm{d}s \\ \leq &\frac{m}{1-\alpha} \cdot (\int_0^1 |(a\theta_{n,x})_x - \hat{f}_3|^2\mathrm{d}x)^{\frac{1}{2}}. \end{aligned} \tag{A.31}$$

Since $\frac{1}{a(x)} \in L^1(0, 1)$ (see (A.11)), it follows from $(A.21)_3$ and (A.31) that when $x \in [0, 1]$,

$$|\int_x^1 \Big[\frac{1}{a(s)} \int_0^s (a\theta_{n,r})_r \mathrm{d}r - \frac{1}{a(s)} \int_0^s \hat{f}_3 \mathrm{d}r\Big]\mathrm{d}s| \to 0, \quad as\ n \to \infty, \tag{A.32}$$

which, together with (A.24) and (A.26), implies (A.29). To show (A.30), we use the Hölder inequality and $(A.21)_2$ to see that for any $x \in [0, 1]$,

$$|\int_x^0 u_{n,ss}(s)\mathrm{d}s - \int_x^0 \hat{f}_2(s)\mathrm{d}s| \leq \int_x^0 |u_{n,ss} - \hat{f}_2|\mathrm{d}s \leq (\int_{-1}^0 |u_{n,ss} - \hat{f}_2|^2\mathrm{d}s)^{\frac{1}{2}} \to 0 \quad as\ n \to \infty,$$

which leads to (A.30). Now, by (A.29), (A.30) and $(A.20)_1$, we can pass to the limit in (A.27) to get

$$u_x = G - \int_x^0 \hat{f}_2(s)\mathrm{d}s \ \text{ for a.e. } x \in (-1, 0). \tag{A.33}$$

Since $\hat{f}_2 \in L^2(-1, 0)$, we get (A.25) from (A.33) at once.

Third, we will show the third equality in (A.23), *i.e.*,

$$(a\theta_x)_x = \hat{f}_3 \ \text{ in } \ L^2(0, 1). \tag{A.34}$$

Indeed, by (A.32), (A.29) and (A.28), we get that when $x \in [0,1]$,

$$\theta_n(x) \to -G \int_x^1 \frac{1}{a(s)} \mathrm{d}s - \int_x^1 \frac{1}{a(s)} \int_0^s \hat{f}_3(r) \mathrm{d}r \mathrm{d}s, \quad as\ n \to \infty.$$

This, along with $(\text{A.20})_3$, yields

$$\theta(x) = -G \int_x^1 \frac{1}{a(s)} \mathrm{d}s - \int_x^1 \frac{1}{a(s)} \int_0^s \hat{f}_3(r) \mathrm{d}r \mathrm{d}s \ \text{ for a.e. } \ x \in (0,1). \tag{A.35}$$

From (A.35), we find

$$a(x)\theta_x(x) = G + \int_0^x \hat{f}_3(s) \mathrm{d}s \ \text{ for a.e. } \ x \in (0,1). \tag{A.36}$$

Since $\hat{f}_3 \in L^2(0,1)$, (A.34) follows from (A.36) at once.

Now, (A.23) follows from (A.24), (A.25) and (A.34).

Step II.2. *We verify that $W := (u, v, \theta) \in \mathcal{D}(\mathcal{A})$.*

This will be done, if we can show that $(u, v, \theta) \in \hat{\mathcal{H}}$ and that (u, v, θ) satisfies BC and TC which are given by (2.6) and (2.7) respectively. (See (2.4).)

First, we show $(u, v, \theta) \in \hat{\mathcal{H}}$. (Recall that $\hat{\mathcal{H}}$ is given by (2.5).) To this end, several observations are given in order.

Observation One: We have

$$u \in V^2(-1,0),\ v \in V^1(-1,0),\ \theta \in L^2(0,1),\ (a\theta_x)_x \in L^2(0,1). \tag{A.37}$$

Indeed, since $u_n(-1) = 0$ for all n, it follows from $(\text{A.20})_1$ that $u(-1) = 0$, which together with (A.25), leads to

$$u \in V^2(-1,0). \tag{A.38}$$

By (A.38), (A.24), $(\text{A.20})_3$ and (A.34), we get (A.37).

Observation Two: We have

$$\theta \in W^{1,1}(0,1). \tag{A.39}$$

Indeed, since $\frac{1}{a} \in L^1(0,1)$ and $\hat{f}_3 \in L^2(0,1)$, it follows from (A.36) that

$$\begin{aligned}
\int_0^1 |\theta_x| \mathrm{d}x &= \int_0^1 |\frac{1}{a} G + \frac{1}{a} \int_0^x \hat{f}_3(s) \mathrm{d}s| \mathrm{d}x \\
&\le |G| \frac{m}{1-\alpha} + \int_0^1 \left| \frac{1}{a} \int_0^x \hat{f}_3(s) \mathrm{d}s \right| \mathrm{d}x \\
&\le |G| \frac{m}{1-\alpha} + \frac{m}{1-\alpha} \cdot \sup_{x \in [0,1]} \int_0^x |\hat{f}_3(s)| \mathrm{d}s \\
&\le |G| \frac{m}{1-\alpha} + \frac{m}{1-\alpha} (\int_0^1 |\hat{f}_3(s)|^2 \mathrm{d}s)^{\frac{1}{2}} \\
&< \infty,
\end{aligned}$$

which, along with the third conclusion in (A.37), leads to (A.39).

Observation Three: We have

$$a^{\frac{1}{2}}\theta_x \in L^2(0,1). \tag{A.40}$$

Indeed, since $\frac{1}{a} \in L^1(0,1)$ and $\hat{f}_3 \in L^2(0,1)$, it follows from (A.36) that

$$\begin{aligned}\int_0^1 a|\theta_x|^2 \mathrm{d}x &= \int_0^1 \frac{1}{a}|G + \int_0^x \hat{f}_3(s)\mathrm{d}s|^2 \mathrm{d}x \\ &\le 2|G|^2 \frac{m}{1-\alpha} + 2\int_0^1 \frac{1}{a}\left|\int_0^x \hat{f}_3(s)\mathrm{d}s\right|^2 \mathrm{d}x \\ &\le 2|G|^2 \frac{m}{1-\alpha} + 2\int_0^1 \frac{1}{a}\int_0^x |\hat{f}_3(s)|^2 \mathrm{d}s\mathrm{d}x \\ &\le \frac{2m}{1-\alpha}(|G|^2 + \int_0^1 |\hat{f}_3(s)|^2\mathrm{d}s) \\ &< \infty,\end{aligned}$$

which leads to (A.40). Now, by (2.5), (A.37), (A.39) and (A.40), we are led to $(u, v, \theta) \in \hat{\mathcal{H}}$.

Second, we have BC for (u, v, θ). Indeed, this follows from (A.35) and (2.6) at once.

Third, we show that (u, v, θ) satisfies TC (see (2.7)). Indeed, by (A.24), (A.35) and (A.29), we can check directly that $\theta(0) = v(0)$. Meanwhile, by (A.33) and (A.36), we have $(a\theta_x)(0) = u_x(0)$. These, along with (2.7), show that (u, v, θ) satisfies TC.

In summary, we conclude that $(u, v, \theta) \in \mathcal{D}(\mathcal{A})$, and hence, $\mathcal{A}$ is a closed operator in $\mathcal{H}$.

Thus, we have completed the proof of Lemma A.1. □

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