ESAIM: COCV 26 (2020) 121
https://doi.org/10.1051/cocv/2020039

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

REMARKS ON NONLINEAR ELASTIC WAVES WITH NULL CONDITIONS

Dongbing Zha[1] and Weimin Peng[2,*]

Abstract. For the Cauchy problem of nonlinear elastic wave equations for 3D isotropic, homogeneous and hyperelastic materials with null conditions, global existence of classical solutions with small initial data was proved in R. Agemi (Invent. Math. **142** (2000) 225–250) and T. C. Sideris (Ann. Math. **151** (2000) 849–874) independently. In this paper, we will give some remarks and an alternative proof for it. First, we give the explicit variational structure of nonlinear elastic waves. Thus we can identify whether materials satisfy the null condition by checking the stored energy function directly. Furthermore, by some careful analyses on the nonlinear structure, we show that the Helmholtz projection, which is usually considered to be ill-suited for nonlinear analysis, can be in fact used to show the global existence result. We also improve the amount of Sobolev regularity of initial data, which seems optimal in the framework of classical solutions.

Mathematics Subject Classification. 35L52, 35Q74.

Received February 4, 2020. Accepted June 29, 2020.

1. Introduction and main result

For isotropic, homogeneous and hyperelastic materials, the motion for the displacement is governed by the nonlinear elastic wave equation which is a second-order quasilinear hyperbolic system. Some physical backgrounds of the nonlinear elastic waves can be found in [4, 5, 16].

Researches on long time existence of classical solutions for nonlinear elastic waves can trace back to Fritz John's pioneering works on elastodynamics (see [13]). For the Cauchy problem of nonlinear elastic waves, John [8] proved that in the radially symmetric case, a genuine nonlinearity condition will lead to the formation of singularities for small initial data. John [9] also showed that the equations have almost global classical solutions for small initial data. Then Klainerman and Sideris [14] simplified John's proof (see also other simplified proofs in [6, 15, 22, 24]). Agemi [1] and Sideris [20] proved independently that for certain classes of materials satisfying some kind of null condition, which is the complement of the genuine nonlinearity condition, there exist global classical solutions with small initial data (see also a previous result in [19]). Some low regularity global existence results in the radially symmetric case can be found in [26]. Some results in two dimensional case can be found in [23, 25].

Keywords and phrases: Nonlinear elastic waves, Helmholtz projection, null conditions, global existence.

[1] Department of Mathematics and Institute for Nonlinear Sciences, Donghua University, Shanghai 201620, P.R. China.
[2] College of Science, University of Shanghai for Science and Technology, Shanghai 200093, P.R. China.

* Corresponding author: weiminpeng7@usst.edu.cn

Article published by EDP Sciences

The purpose of this paper is to give some remarks and an alternative proof on the global existence result for nonlinear elastic waves under the null condition. First, it is well-known that hyperelastic materials can be defined by the stored energy function. Thus in order to identify whether materials satisfy the null condition, it is convenient to check the stored energy function directly. For this purpose, it is necessary to give the variational structure of nonlinear elastic waves. Based on [1, 19], we write down the quadratic and cubic nonlinearity in the stored energy function explicitly. Furthermore, since nonlinear elastic waves are coupled even at linear level, in [20] in order to develop some weighted L^2 estimates (Klainerman–Sideris type estimates) and realize the null condition, a local decomposition is employed. While the nonlocal Helmholtz projection, which decomposes elastic waves into their longitudinal and transverse components, is usually considered to be ill-suited for nonlinear analysis. By some careful analyses on the nonlinear structure of nonlinear elastic waves, together with some new observations on the commutation relations between the Helmholtz projection and the vector fields corresponding to the symmetries of nonlinear elastic waves, and Klainerman–Sideris type estimates for wave operator [14], we can show that the Helmholtz decomposition can be also used to prove the global existence for nonlinear elastic waves with null conditions. We also improve the amount of Sobolev regularity of initial data, which seems optimal in the framework of classical solutions.

In fact, the main motivation of providing a new proof for the global existence result for nonlinear elastic waves via the Helmholtz projection is to attack the problem of asymptotic behavior of global solutions. We conjecture that the global solution will converge to a solution of the homogeneous linear elastic wave equation as time tends to infinity in the energy sense. The analogue of multiple-speeds wave systems has been verified in [11]. We hope that the Helmholtz decomposition can be used to study the asymptotic behavior of global solutions for nonlinear elastic waves in the future.

The outline of this paper is as follows. The remainder of this introduction will be devoted to the description of the basic notation which will be used in the sequel, derivation of the equations of motion and a statement of the global existence theorem. Section 2 introduces some necessary tools used to prove the global existence, including some commutation relations, decay estimates for null form nonlinearity, weighted Sobolev inequalities and Klainerman–Sideris type estimates. Then, the global existence theorem will be proved in Section 3.

1.1. Notation

Denote the space gradient and space-time gradient by $\nabla = (\partial_1, \partial_2, \partial_3)$ and $\partial = (\partial_t, \nabla)$, respectively. The angular momentum operators (generator of the spatial rotation) are the vector fields $\Omega = (\Omega_{ij} : 1 \le i < j \le 3)$, where

$$\Omega_{ij} = x_i\partial_j - x_j\partial_i. \tag{1.1}$$

Denote the generators of simultaneous rotations by $\widetilde{\Omega} = (\widetilde{\Omega}_{ij} : 1 \le i < j \le 3)$, where

$$\widetilde{\Omega}_{ij} = \Omega_{ij} I + U_{ij}, \tag{1.2}$$

and $U_{ij} = e_i \otimes e_j - e_j \otimes e_i$, $\{e_i\}_{i=1}^3$ is the standard basis on $\mathbb{R}^3$. The scaling operator is

$$S = t\partial_t + r\partial_r, \tag{1.3}$$

where $r = |x|,\ \partial_r = \omega \cdot \nabla,\ \omega = (\omega_1, \omega_2, \omega_3),\ \omega_i = x_i/r, i = 1, 2, 3$, and we will use

$$\widetilde{S} = S - 1. \tag{1.4}$$

Denote the collection of vector fields by $\Gamma = (\partial_t, \nabla, \widetilde{\Omega}, \widetilde{S}) = (\Gamma_0, \dots, \Gamma_7)$. It is easy to verify that the commutator of Γ and ∇ is in the span of ∇. Schematically, we write

$$\big[\Gamma, \nabla\big] = \nabla. \tag{1.5}$$

By Γ^a, $a = (a_1, \dots, a_k)$, we denote an ordered product of $k = |a|$ vector fields $\Gamma_{a_1} \cdots \Gamma_{a_k}$.

Denote the basic energy corresponding to the linear elastic wave operator (see Sect. 1.2) by

$$E_1(u(t)) = \frac{1}{2}\int_{\mathbb{R}^3} \big(|\partial_t u(t,x)|^2 + c_2^2|\nabla u(t,x)|^2 + (c_1^2 - c_2^2)(\nabla\cdot u(t,x))^2\big)\mathrm{d}x \tag{1.6}$$

and the higher order version by

$$E_\kappa(u(t)) = \sum_{|a|\le \kappa-1} E_1(\Gamma^a u(t)). \tag{1.7}$$

The solution will be constructed in the space $\dot{H}^\kappa_\Gamma(T)$, which is the closure of the set $C^\infty\big([0,T); C_c^\infty(\mathbb{R}^3;\mathbb{R}^3)\big)$ in the norm $\sup\limits_{0\le t<T} E_\kappa^{1/2}(u(t))$.

Set

$$\Lambda = (\nabla, \widetilde{\Omega}, r\partial_r - 1) = (\Lambda_1, \dots, \Lambda_7). \tag{1.8}$$

Define the time-independent spaces

$$H^\kappa_\Lambda = \{f \in L^2(\mathbb{R}^3;\mathbb{R}^3) : \Lambda^a f \in L^2(\mathbb{R}^3;\mathbb{R}^3), |a| \le \kappa\} \tag{1.9}$$

with the norm

$$\|f\|_{H^\kappa_\Lambda} = \Big(\sum_{|a|\le\kappa} \|\Lambda^a f\|^2_{L^2(\mathbb{R}^3)}\Big)^{1/2}. \tag{1.10}$$

As in [17], we will introduce the Banach space $L^{p,q}(\mathbb{R}^3)$ equipped with the norm

$$\|f\|_{L^{p,q}(\mathbb{R}^3)} = \|f(r\omega) r^{\frac{2}{p}}\|_{L^p_r(0,\infty; L^q_\omega(S^2))}, \tag{1.11}$$

where $1 \le p, q \le +\infty$.

Denote the Riesz transformation by

$$R_k = \frac{\partial_k}{\sqrt{-\Delta}},\ k = 1,2,3. \tag{1.12}$$

We also employ the notation $R = (R_1, R_2, R_3)$. We will use the Helmholtz decomposition, which projects any vector field onto curl-free and divergence-free components. Specifically speaking, for any function $u \in C^\infty(\mathbb{R}^3;\mathbb{R}^3)$ with sufficient decay at infinity, we have

$$u = u_{cf} + u_{df}, \tag{1.13}$$

with

$$u_{cf}=-R(R\cdot u),\quad u_{df}=R\wedge(R\wedge u), \tag{1.14}$$

$\wedge$ being the usual vector cross product. And it holds that

$$\nabla\wedge u_{cf}=0,\quad \nabla\cdot u_{df}=0, \tag{1.15}$$

$$\|u\|^2_{L^2(\mathbb{R}^3)}=\|u_{cf}\|^2_{L^2(\mathbb{R}^3)}+\|u_{df}\|^2_{L^2(\mathbb{R}^3)}. \tag{1.16}$$

The proof of above facts can be found in [2].

Using the Helmholtz decomposition, we will also define the following weighted L^2 norm

$$\mathcal{X}_\kappa(u(t))=\sum_{\beta=0}^{3}\sum_{l=1}^{3}\sum_{|a|\le\kappa-2}\big(\|\langle c_1t-r\rangle\partial_\beta\partial_l\Gamma^a u_{cf}(t)\|_{L^2}+\|\langle c_2t-r\rangle\partial_\beta\partial_l\Gamma^a u_{df}(t)\|_{L^2}\big), \tag{1.17}$$

where $\langle\cdot\rangle=(1+|\cdot|^2)^{1/2}$. Note that the above definition of weighted L^2 norm $\mathcal{X}_\kappa(u(t))$ is different from the corresponding one in [20] (see (1.6) in [20]), where a local decomposition is employed.

1.2. The equations of motion

Consider a homogeneous elastic material filling in the whole space $\mathbb{R}^3$. Assume that its density in its undeformed state is unity. Let $y:\mathbb{R}\times\mathbb{R}^3\longrightarrow\mathbb{R}^3$ be the smooth deformation of the material that evolves with time, which is an orientation preserving diffeomorphism taking a material point $x\in\mathbb{R}^3$ in the reference configuration to its position $y(t,x)\in\mathbb{R}^3$ at time t. The deformation gradient is then the matrix $F=\nabla y$ with components $F_{il}=\partial_l y^i$.

For the materials under consideration, the potential energy density is characterized by a stored energy function $W(F)$. Then we have the Lagrangian

$$\mathscr{L}(y)=\iint\frac{1}{2}|y_t|^2-W(\nabla y)\ \mathrm{d}x\mathrm{d}t. \tag{1.18}$$

A material is frame indifferent, if the conditions

$$W(F)=W(QF) \tag{1.19}$$

hold for every orthogonal matrix Q, and it material is isotropic, if the conditions

$$W(F)=W(FQ) \tag{1.20}$$

hold for every orthogonal matrix Q. It is well-known that (1.19) and (1.20) imply that the stored energy function $W(F)=\overline{\sigma}(\iota_1,\iota_2,\iota_3)$, where ι_1,ι_2,ι_3 are the principal invariants of the (left) Cauchy–Green strain tensor FF^T. By applying Hamilton's principle to (1.18), we can get the corresponding Euler–Lagrange equation[1]

$$\frac{\partial^2 y^i}{\partial t^2}-\frac{\partial}{\partial x^l}\left(\frac{\partial W}{\partial F_{il}}(\nabla y)\right)=0. \tag{1.21}$$

[1] Repeated indices are always summed.

We will consider displacements $u(t,x) = y(t,x) - x$ from the reference configuration. The displacement gradient matrix $G = \nabla u$ satisfies $G = F - I$, and $C = FF^T - I = G + G^T + GG^T$. Consequently we have

$$W(F) = \sigma(k_1, k_2, k_3), \tag{1.22}$$

where k_1, k_2, k_3 are the principal invariants of C. For the displacement, we have the Lagrangian

$$\widetilde{\mathscr{L}}(u) = \mathscr{L}(y) = \iint \frac{1}{2}|u_t|^2 - \sigma(k_1, k_2, k_3)\, \mathrm{d}x\mathrm{d}t. \tag{1.23}$$

Then the PDE's can be formulated as the nonlinear system

$$\frac{\partial^2 u^i}{\partial t^2} - \frac{\partial}{\partial x^l}\frac{\partial \sigma}{\partial G_{il}} = 0. \tag{1.24}$$

Now in order to give the variational structure of nonlinear elastic waves, we need to represent $\sigma(k_1, k_2, k_3)$ by $G = \nabla u$ explicitly. We will consider only small displacements from the reference configuration. In three space dimensions, the global existence of small amplitude solutions to nonlinear hyperbolic systems hinges on the specific form of the quadratic portion of the nonlinearity. Such compatibility conditions are often referred to as null conditions. See [3, 7, 10, 12, 18, 21] etc. From the analytical point of view, therefore, it is enough to truncate (1.24) to third order in u, the higher order corrections having no influence on the existence of small solutions. And we will truncate $\sigma(k_1, k_2, k_3)$ in (1.23) to fourth order in u.

Let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of C. We use the following formulas for the principal invariants:

$$k_1 = \lambda_1 + \lambda_2 + \lambda_3 = \mathrm{tr}\ C, \tag{1.25}$$

$$k_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 = \frac{(\mathrm{tr}\ C)^2 - \mathrm{tr}\ C^2}{2}, \tag{1.26}$$

$$k_3 = \lambda_1\lambda_2\lambda_3 = \det C = \frac{(\mathrm{tr}\ C)^3 - 3(\mathrm{tr}\ C)(\mathrm{tr}\ C^2) + 2\mathrm{tr}\ C^3}{6}. \tag{1.27}$$

Noting that

$$\mathrm{tr}\ C = 2\mathrm{tr}\ G + \mathrm{tr}\ GG^T, \tag{1.28}$$

$$(\mathrm{tr}\ C)^2 = 4(\mathrm{tr}\ G)^2 + 4\mathrm{tr}\ G\ \mathrm{tr}\ GG^T + (\mathrm{tr}\ GG^T)^2, \tag{1.29}$$

$$(\mathrm{tr}\ C)^3 = 8(\mathrm{tr}\ G)^3 + 12(\mathrm{tr}\ G)^2\ \mathrm{tr}\ GG^T + 6\mathrm{tr}\ G(\mathrm{tr}\ GG^T)^2 + (\mathrm{tr}\ GG^T)^3, \tag{1.30}$$

$$\mathrm{tr}\ C^2 = 2(\mathrm{tr}\ G^2 + \mathrm{tr}\ GG^T) + 4\mathrm{tr}\ G^2G^T + \mathrm{tr}\ (GG^T)^2, \tag{1.31}$$

$$\mathrm{tr}\ C^3 = 2\mathrm{tr}\ G^3 + 6\mathrm{tr}\ G^2G^T + h.o.t., \tag{1.32}$$

we get

$$k_1 = 2\mathrm{tr}\ G + \mathrm{tr}\ GG^T, \tag{1.33}$$

$$\begin{aligned} k_2 = {} & 2(\mathrm{tr}\ G)^2 - (\mathrm{tr}\ G^2 + \mathrm{tr}\ GG^T) + 2(\mathrm{tr}\ G\ \mathrm{tr}\ GG^T - \mathrm{tr}\ G^2G^T) \\ & + \frac{1}{2}\big((\mathrm{tr}\ GG^T)^2 - \mathrm{tr}\ (GG^T)^2\big), \end{aligned} \tag{1.34}$$

$$k_3 = \frac{4}{3}(\mathrm{tr}\ G)^3 - 2(\mathrm{tr}\ G)(\mathrm{tr}\ G^2 + \mathrm{tr}\ GG^T) + \frac{2}{3}\mathrm{tr}\ G^3 + 2\mathrm{tr}\ G^2G^T + h.o.t. \tag{1.35}$$

From (1.33), (1.34) and (1.35) it is apparent that $k_1 = \mathscr{O}(|G|), k_2 = \mathscr{O}(|G|^2), k_3 = \mathscr{O}(|G|^3)$. Therefore, the relevant terms in the Taylor expansion at $k_i = 0$ are

$$\sigma(k_1, k_2, k_3) = (\sigma_0 + \sigma_1 k_1) + \left(\frac{1}{2}\sigma_{11} k_1^2 + \sigma_2 k_2\right) + \left(\frac{1}{6}\sigma_{111} k_1^3 + \sigma_{12} k_1 k_2 + \sigma_3 k_3\right) + h.o.t., \tag{1.36}$$

with the constants σ_0, σ_1 etc., standing for the partial derivatives of σ at $k_i = 0$. Without loss of generality, we assume that $\sigma_0 = 0$. And we impose the condition $\sigma_1 = 0$, which implies that the reference configuration is in a stress-free state. Denote

$$\sigma(k_1, k_2, k_3) = l_2(G) + l_3(G) + \mathscr{O}(|G|^4), \tag{1.37}$$

where $l_i(G)(i = 2, 3)$ represents the homogeneous ith order part of $\sigma(k_1, k_2, k_3)$ with respect to $G = \nabla u$. By (1.33), (1.34) and (1.35), after a bit of calculation, we see that

$$l_2(G) = 2(\sigma_{11} + \sigma_2)(\text{tr } G)^2 - \sigma_2(\text{tr } G^2 + \text{tr } GG^T), \tag{1.38}$$

$$\begin{aligned} l_3(G) = &\left(\frac{4}{3}\sigma_{111} + 4\sigma_{12} + \frac{4}{3}\sigma_3\right)(\text{tr } G)^3 + (-2\sigma_{12} - 2\sigma_3)\text{tr } G \text{ tr } G^2 + \frac{2}{3}\sigma_3 \text{tr } G^3 \\ &+ (2\sigma_{11} - 2\sigma_{12} + 2\sigma_2 - 2\sigma_3)\text{tr } G \text{ tr } GG^T + (-2\sigma_2 + 2\sigma_3)\text{tr } G^2 G^T \\ = &\left(\frac{4}{3}\sigma_{111} + 4\sigma_{12} + \frac{4}{3}\sigma_3\right)(\text{tr } G)^3 + \left(-2\sigma_{12} - \frac{4}{3}\sigma_3\right)\text{tr } G \text{ tr } G^2 - \frac{2}{3}\sigma_3(\text{tr } G \text{ tr } G^2 - \text{tr } G^3) \\ &+ (2\sigma_{11} - 2\sigma_{12})\text{tr } G \text{ tr } GG^T + (2\sigma_2 - 2\sigma_3)(\text{tr } G \text{ tr } GG^T - \text{tr } G^2 G^T). \end{aligned} \tag{1.39}$$

Our task now is to represent $l_i(G)(i = 2, 3)$ via $G = \nabla u$ explicitly. Denote the null forms

$$Q_{ij}(v, w) = \partial_i v \partial_j w - \partial_i w \partial_j v, \ 1 \le i, j \le 3. \tag{1.40}$$

First it is easy to see that

$$\text{tr } G = \nabla \cdot u, \tag{1.41}$$

$$\text{tr } G^2 = (\nabla \cdot u)^2 - 2 \sum_{1 \le i < j \le 3} Q_{ij}(u^i, u^j), \tag{1.42}$$

$$\text{tr } GG^T = |\nabla u|^2 = (\nabla \cdot u)^2 + |\nabla \wedge u|^2 - 2 \sum_{1 \le i < j \le 3} Q_{ij}(u^i, u^j). \tag{1.43}$$

It follows from (1.41), (1.42) and (1.43) that

$$l_2(\nabla u) = (2\sigma_{11} + \sigma_2)(\nabla \cdot u)^2 - \sigma_2 |\nabla u|^2 + 2\sigma_2 \sum_{1 \le i < j \le 3} Q_{ij}(u^i, u^j). \tag{1.44}$$

Next we compute $l_3(\nabla u)$. According to (1.41) and (1.42), we can get

$$(\text{tr } G)^3 = (\nabla \cdot u)^3, \tag{1.45}$$

$$\text{tr } G \text{ tr } G^2 = (\nabla \cdot u)^3 - 2(\nabla \cdot u) \sum_{1 \le i < j \le 3} Q_{ij}(u^i, u^j). \tag{1.46}$$

We can also show that

$$\begin{aligned}\text{tr } G\text{tr } G^2 - \text{tr } G^3 &= \partial_i u^i \partial_k u^j \partial_j u^k - \partial_j u^i \partial_k u^j \partial_i u^k = \partial_k u^j (\partial_i u^i \partial_j u^k - \partial_j u^i \partial_i u^k) \\ &= \partial_k u^j Q_{ij}(u^i, u^k). \end{aligned} \tag{1.47}$$

Since (1.41) and (1.43), it follows that

$$\text{tr } G \text{ tr } GG^T = (\nabla \cdot u)^3 + (\nabla \cdot u)|\nabla u|^2 - 2(\nabla \cdot u) \sum_{1\le i<j\le 3} Q_{ij}(u^i, u^j). \tag{1.48}$$

We can also show that

$$\begin{aligned}\text{tr } G \text{ tr } GG^T - \text{tr } G^2G^T &= \partial_i u^i \partial_k u^j \partial_k u^j - \partial_i u^j \partial_k u^i \partial_k u^j = \partial_k u^j (\partial_i u^i \partial_k u^j - \partial_i u^j \partial_k u^i) \\ &= \partial_k u^j Q_{ik}(u^i, u^j). \end{aligned} \tag{1.49}$$

So it is a consequence of (1.45)–(1.49) that

$$\begin{aligned} l_3(\nabla u) &= d_1(\nabla \cdot u)^3 + d_2(\nabla \cdot u)|\nabla \wedge u|^2 + d_3(\nabla \cdot u)Q_{ij}(u^i, u^j) \\ &\quad + d_4 \partial_k u^j Q_{ij}(u^i, u^k) + d_5 \partial_k u^j Q_{ik}(u^i, u^j), \end{aligned} \tag{1.50}$$

where

$$\begin{cases} d_1 = \frac{4}{3}\sigma_{111} + 2\sigma_{11}, \\ d_2 = 2(\sigma_{11} - \sigma_{12}), \\ d_3 = -2\sigma_{11} + 4\sigma_{12} + \frac{4}{3}\sigma_3, \\ d_4 = -\frac{2}{3}\sigma_3, \\ d_5 = 2(\sigma_2 - \sigma_3). \end{cases} \tag{1.51}$$

By the above discussion, in view of (1.23), (1.37), (1.44) and (1.50), we can get the following nonlinear elastic wave equation for homogeneous, isotropic and hyperelastic materials

$$Lu = N(u, u). \tag{1.52}$$

Here the linear elastic wave operator

$$L = (\partial_t^2 - c_2^2 \Delta)I - (c_1^2 - c_2^2)\nabla \otimes \nabla, \tag{1.53}$$

where $c_1^2 = 4\sigma_{11}$ and $c_2^2 = -2\sigma_2$ satisfy $0 < c_2 < c_1$, the nonlinearity

$$N(u, v) = N_1(u, v) + N_2(u, v), \tag{1.54}$$

with

$$\begin{aligned} N_1(u, v) &= 3d_1 \nabla\big((\nabla \cdot u)\ (\nabla \cdot v)\big) + d_2 \nabla\big((\nabla \wedge u) \cdot (\nabla \wedge v)\big) \\ &\quad - d_2 \nabla \wedge \big((\nabla \cdot u)(\nabla \wedge v)\big) - d_2 \nabla \wedge \big((\nabla \cdot v)(\nabla \wedge u)\big) \end{aligned} \tag{1.55}$$

and

$$
\begin{aligned}
N_2(u,v)^i = & \left(d_3+\frac{d_4}{2}\right)\big(Q_{ij}(\partial_k u^k,v^j)+Q_{ij}(\partial_k v^k,u^j)-Q_{jk}(\partial_i u^k,v^j)-Q_{jk}(\partial_i v^k,u^j)\big)\\
&+\frac{d_5}{2}\big(Q_{ij}(\partial_j u^k,v^k)+Q_{ij}(\partial_j v^k,u^k)+2Q_{jk}(\partial_j u^i,v^k)+2Q_{jk}(\partial_j v^i,u^k)\big)\\
&-\frac{d_5}{2}\big(Q_{jk}(\partial_j u^k,v^i)+Q_{jk}(\partial_j v^k,u^i)\big). \qquad (1.56)
\end{aligned}
$$

We also write the nonlinear term N as the following form

$$N(u,v)^i=\partial_l(g^{ijk}_{lmn}\partial_m u^j\partial_n v^k), \qquad (1.57)$$

where the coefficients are constants and are symmetric with respect to pairs of indices

$$g^{ijk}_{lmn}=g^{jik}_{mln}=g^{kji}_{nml}, \qquad (1.58)$$

and $g=(g^{ijk}_{lmn})$ is a six order isotropic tensor.

We say that the nonlinear elastic wave equation (1.52) satisfies the null condition if

$$d_1=\frac{4}{3}\sigma_{111}+2\sigma_{11}=0, \qquad (1.59)$$

which is equivalent to

$$g^{ijk}_{lmn}\omega_i\omega_j\omega_k\omega_l\omega_m\omega_n=0,\quad \text{for all } \omega\in S^2. \qquad (1.60)$$

See [1, 19, 20]. In view of (1.50), we can see that the null condition (1.59) is just used to rule out the term $(\nabla\cdot u)^3$ in the stored energy function. Thus we can identify whether materials satisfy the null condition by checking the cubic term in the stored energy function directly.

1.3. The global existence theorem

Theorem 1.1. *Assume that the null condition* (1.59) *is satisfied. Then the Cauchy problem for* (1.52)–(1.56) *with initial data*

$$\partial u(0)\in H^{\kappa-1}_{\Lambda},\ \ \kappa\geq 5 \qquad (1.61)$$

admits a unique global solution $u\in \dot H^{\kappa}_{\Gamma}(T)$ *for every* $T>0$, *if*

$$E_{\kappa-1}(u(0))\exp\left[C_0E^{1/2}_{\kappa}(u(0))\right]\leq\varepsilon^2 \qquad (1.62)$$

and ε *is sufficiently small, depending on* C_0. *The solution satisfies the bounds*

$$E_{\kappa-1}(u(t))\leq\varepsilon^2\quad \text{and}\quad E_{\kappa}(u(t))\leq 2E_{\kappa}(u(0))\langle t\rangle^{C_0\varepsilon}, \qquad (1.63)$$

for every $t\geq 0$.

Remark 1.2. In [20], the global existence for nonlinear elastic waves with null conditions was established when the amount of Sobolev regularity of initial data $\kappa\geq 9$. Here it is only required that $\kappa\geq 5$, which seems optimal in the framework of classical solutions. In [6], the almost global existence of classical solutions for nonlinear elastic waves without null conditions was proved under the assumption that the amount of Sobolev regularity of initial data $\kappa \geq 4$. The key tool in [6] is some new space-time L^2 estimates for perturbed linear elastic waves. The Helmholtz decomposition also plays a key role in the proof of these estimates.

2. Preliminaries

2.1. Commutation

We first give the following commutation relations between the Riesz transformation and vector fields Γ.

Lemma 2.1. *We have*

$$[\partial_t, R_k] = [\partial_l, R_k] = 0, \tag{2.1}$$
$$[S, R_k] = 0, \tag{2.2}$$
$$[\Omega_{ij}, R_k] = -\delta_{ik} R_j + \delta_{jk} R_i. \tag{2.3}$$

Proof. (2.1) is obvious. In order to show (2.2) and (2.3), we first prove the following commutation relation

$$[x_i\partial_j, R_k] = -\delta_{ik} R_j - R_i R_j R_k. \tag{2.4}$$

Denote the Fourier transformation of f by $\widehat{f}(\xi)$. By the properties of Fourier transformation, we have

$$\begin{aligned}
&\widehat{[x_i\partial_j, R_k]} f = \widehat{x_i\partial_j R_k} f - R_k\widehat{(x_i\partial_j f)} \\
&= -\sqrt{-1}\partial_{\xi_i}(\xi_j \frac{\xi_k}{|\xi|}\widehat{f}) + \sqrt{-1}\frac{\xi_k}{|\xi|}\partial_{\xi_i}(\xi_j\widehat{f}) \\
&= -\sqrt{-1}\partial_{\xi_i}(\frac{\xi_k}{|\xi|})(\xi_j\widehat{f}) = -\sqrt{-1}(\delta_{ik}\frac{\xi_j}{|\xi|} - \frac{\xi_i}{|\xi|}\frac{\xi_j}{|\xi|}\frac{\xi_k}{|\xi|})\widehat{f}.
\end{aligned} \tag{2.5}$$

Then (2.4) follows from (2.5). Noting that

$$[S, R_k] = \sum_{i=1}^{3} [x_i\partial_i, R_k], \tag{2.6}$$

we can get (2.3) by (2.4) and the fact that $\sum_{i=1}^{3} R_i^2 = -I$. While (2.3) is a consequence of the following relation

$$[\Omega_{ij}, R_k] = [x_i\partial_j, R_k] - [x_j\partial_i, R_k] \tag{2.7}$$

and (2.4). □

The following lemma, which asserts that the Helmholtz projection is commutative with vector fields Γ, is crucial for our argument.

Lemma 2.2. *We have*

$$(\Gamma^a u)_{cf} = \Gamma^a u_{cf}, \tag{2.8}$$
$$(\Gamma^a u)_{df} = \Gamma^a u_{df}. \tag{2.9}$$

Proof. We only need to prove the case $|a| = 1$.It follows from (1.14), (2.1) and (2.2) that

$$(\partial_t u)_{cf} = \partial_t u_{cf},\ (\partial_t u)_{df} = \partial_t u_{df}, \tag{2.10}$$
$$(\partial_l u)_{cf} = \partial_l u_{cf},\ (\partial_l u)_{df} = \partial_l u_{df}, \tag{2.11}$$
$$(\widetilde{S}u)_{cf} = \widetilde{S}u_{cf},\ (\widetilde{S}u)_{df} = \widetilde{S}u_{df}. \tag{2.12}$$

The remaining task is to show

$$(\widetilde{\Omega}_{ij}u)_{cf} = \widetilde{\Omega}_{ij}u_{cf} \tag{2.13}$$

and

$$(\widetilde{\Omega}_{ij}u)_{df} = \widetilde{\Omega}_{ij}u_{df}. \tag{2.14}$$

We only prove (2.13). The proof of (2.14) is similar. We have

$$\big(\widetilde{\Omega}_{ij}u_{cf}\big)_k = \Omega_{ij}(u_{cf})_k + \big(U_{ij}u_{cf}\big)_k = -\Omega_{ij}(R_kR_lu_l) - \big(U_{ij}(RR\cdot u)\big)_k \tag{2.15}$$

and

$$\big((\widetilde{\Omega}_{ij}u)_{cf}\big)_k = -R_kR_l(\widetilde{\Omega}_{ij}u)_l = -R_kR_l(\Omega_{ij}u_l) - R_kR_l(U_{ij}u)_l. \tag{2.16}$$

It follows from (2.3) that

$$\begin{aligned}\Omega_{ij}(R_kR_lu_l) &= R_kR_l(\Omega_{ij}u_l) + R_k[\Omega_{ij}, R_l]u_l + [\Omega_{ij}, R_k]R_lu_l\\ &= R_kR_l(\Omega_{ij}u_l) + R_k(\delta_{jl}R_i - \delta_{il}R_j)u_l + (\delta_{jk}R_i - \delta_{ik}R_j)R_lu_l\\ &= R_kR_l(\Omega_{ij}u_l) + (R_iR_ku_j - R_jR_ku_i) + (\delta_{jk}R_i - \delta_{ik}R_j)R_lu_l.\end{aligned} \tag{2.17}$$

We can also check that

$$\big(U_{ij}(RR\cdot u)\big)_k = -\delta_{jk}(RR\cdot u)_i + \delta_{ik}(RR\cdot u)_j = (-\delta_{jk}R_i + \delta_{ik}R_j)R_lu_l \tag{2.18}$$

and

$$R_kR_l(U_{ij}u)_l = R_kR_l(\delta_{il}u_j - \delta_{jl}u_i) = R_iR_ku_j - R_jR_ku_i. \tag{2.19}$$

The combination of (2.15)–(2.19) gives (2.13). □

Lemma 2.3. *For any solution u of (1.52) in $\dot{H}^\kappa_\Gamma(T)$, we have*

$$L\Gamma^a u = \sum_{b+c=a} N(\Gamma^b u, \Gamma^c u), \tag{2.20}$$

in which the sum extends over ordered partitions of the sequences a, with $|a| \le k-1$.

Proof. See Proposition 3.1 in [20]. □

2.2. Null form estimates

Lemma 2.4. *We have*

$$|N_2(u,v)| \le \frac{C}{r}\sum_{|a|\le 1}\big(|\widetilde{\Omega}^a u||\nabla^2 v| + |\widetilde{\Omega}^a v||\nabla^2 u| + |\nabla\widetilde{\Omega}^a u||\nabla v| + |\nabla\widetilde{\Omega}^a v||\nabla u|\big). \tag{2.21}$$

Proof. See Proposition 3.2 in [20]. □

2.3. Sobolev inequalities

Lemma 2.5. *For $u\in C^\infty(\mathbb{R}^3;\mathbb{R}^3)$ with sufficiently decay ant infinity, $r=|x|, \rho=|y|, \alpha=1,2,$ we have*

$$r^{1/2}\|u(r\omega)\|_{L^2_\omega} \le C\|\nabla u\|_{L^2}, \tag{2.22}$$

$$r^{1/2}|u(x)| \le C\sum_{|a|\le 1}\|\nabla\widetilde{\Omega}^a u\|_{L^2}, \tag{2.23}$$

$$r|u(x)| \le C\sum_{|a|\le 1}\|\partial_r\widetilde{\Omega}^a u\|^{1/2}_{L^2(|y|\ge r)}\cdot\sum_{|a|\le 2}\|\widetilde{\Omega}^a u\|^{1/2}_{L^2(|y|\ge r)}, \tag{2.24}$$

$$r\langle c_\alpha t - r\rangle^{1/2}|u(x)| \le C\sum_{|a|\le 1}\|\langle c_\alpha t-\rho\rangle\partial_r\widetilde{\Omega}^a u\|_{L^2(|y|\ge r)} + C\sum_{|a|\le 2}\|\widetilde{\Omega}^a u\|_{L^2(|y|\ge r)}, \tag{2.25}$$

$$\begin{aligned} r\langle c_\alpha t - r\rangle|u(x)| \le{}& C\sum_{|a|\le 1}\|\langle c_\alpha t-\rho\rangle\partial_r\widetilde{\Omega}^a u\|_{L^2(|y|\ge r)} \\ &+ C\sum_{|a|\le 2}\|\langle c_\alpha t-\rho\rangle\widetilde{\Omega}^a u\|_{L^2(|y|\ge r)}. \end{aligned} \tag{2.26}$$

Proof. See Lemma 3.3 in [20]. □

Lemma 2.6. *For $u\in \dot{H}^\kappa_\Gamma(T)$ with $\mathcal{X}_\kappa(u(t))<+\infty$, we have*

$$r^{1/2}\|\Gamma^a u(t,r\omega)\|_{L^2_\omega} \le CE_\kappa^{1/2}(u(t)),\ |a|+1\le\kappa, \tag{2.27}$$

$$\langle r\rangle^{1/2}|\Gamma^a u(t,x)| \le CE_\kappa^{1/2}(u(t)),\ |a|+2\le\kappa, \tag{2.28}$$

$$\langle r\rangle|\partial\Gamma^a u(t,x)| \le CE_\kappa^{1/2}(u(t)),\ |a|+3\le\kappa, \tag{2.29}$$

$$\begin{aligned} &\langle r\rangle\langle c_1 t-r\rangle^{1/2}|\partial\Gamma^a u_{cf}(t,x)| + \langle r\rangle\langle c_2 t-r\rangle^{1/2}|\partial\Gamma^a u_{df}(t,x)| \\ &\quad\le CE_\kappa^{1/2}(u(t)) + C\mathcal{X}_\kappa(u(t)), |a|+3\le\kappa, \end{aligned} \tag{2.30}$$

$$\begin{aligned} &\langle r\rangle\langle c_1 t-r\rangle|\partial\nabla\Gamma^a u_{cf}(t,x)| + \langle r\rangle\langle c_2 t-r\rangle|\partial\nabla\Gamma^a u_{df}(t,x)| \\ &\quad\le C\mathcal{X}_\kappa(u(t)), |a|+4\le\kappa. \end{aligned} \tag{2.31}$$

Proof. It is obvious that (2.22) implies (2.27) directly. For other inequalities, we first consider the case $r\ge 1$. In this case, (2.28) is just a consequence of (2.23). While (2.29) follows from (2.24) and the commutation property (1.5). Similarly, (2.31) results from (2.26) and (1.5). On the other hand, (2.30) follows from the combination of (2.25), (1.5), Lemma 2.2 and (1.16).

In the case $r\le 1$, (2.28) and (2.29) are immediate consequences of the Sobolev inequality

$$\|u\|_{L^\infty(\mathbb{R}^3)} \le C\sum_{|b|\le 1}\|\nabla\nabla^b u\|_{L^2(\mathbb{R}^3)}. \tag{2.32}$$

To obtain the other inequalities, following [20], we define a smooth cut-off function

$$\zeta(r) = \begin{cases} 1, & r \leq 1, \\ 0, & r \geq 2. \end{cases} \tag{2.33}$$

By (2.32) and (2.25), we obtain

$$\begin{aligned}
&\langle c_1 t - r\rangle^{1/2} |\partial \Gamma^a u_{cf}(t,x)| \\
&\leq C\langle t\rangle^{1/2} \zeta(r) |\partial \Gamma^a u_{cf}(t,x)| \\
&\leq C\langle t\rangle^{1/2} \sum_{|b|\leq 1} \|\nabla \nabla^b (\zeta \partial \Gamma^a u_{cf})\|_{L^2} \\
&\leq C\langle t\rangle^{1/2} \sum_{|b|\leq 1} \|\partial \nabla \nabla^b \Gamma^a u_{cf}\|_{L^2(|y|\leq 2)} + C\langle t\rangle^{1/2} \|\partial \Gamma^a u_{cf}\|_{L^2(1\leq |y|\leq 2)} \\
&\leq C \sum_{|b|\leq 1} \|\langle c_1 t - \rho\rangle^{1/2} \partial \nabla \nabla^b \Gamma^a u_{cf}\|_{L^2(|y|\leq 2)} + C\|\rho \langle c_1 t - \rho\rangle^{1/2} \partial \Gamma^a u_{cf}\|_{L^\infty(1\leq |y|\leq 2)} \\
&\leq C E_\kappa^{1/2}(u(t)) + C\mathcal{X}_\kappa(u(t)).
\end{aligned} \tag{2.34}$$

Similarly, we can also get

$$\langle c_2 t - r\rangle^{1/2} |\partial \Gamma^a u_{df}(t,x)| \leq C E_\kappa^{1/2}(u(t)) + C\mathcal{X}_\kappa(u(t)). \tag{2.35}$$

The combination of (2.34) and (2.35) gives (2.30). We can also prove (2.31) by similar argument. ☐

2.4. Klainerman-Sideris type estimates

We first need the Klainerman-Sideris type estimate for wave operator, which was first established in [14].

Lemma 2.7. *For $u \in \dot{H}^2_\Gamma(T)$, we have*

$$\|\langle ct - r\rangle \partial \nabla u(t)\|_{L^2} \leq C \sum_{|a|\leq 1} \|\partial \Gamma^a u(t)\|_{L^2} + Ct\|\Box_c u(t)\|_{L^2}, \tag{2.36}$$

where $\Box_c = \partial_t^2 - c^2\Delta$ is the wave operator with wave speed $c > 0$.

Lemma 2.8. *For $u \in \dot{H}^2_\Gamma(T)$, we have*

$$\mathcal{X}_2(u(t)) \leq C E_2^{1/2}(u(t)) + Ct\|Lu(t)\|_{L^2}. \tag{2.37}$$

Proof. It follows from Lemma 2.7 that

$$\|\langle c_1 t - r\rangle \partial \nabla u_{cf}(t)\|_{L^2} \leq C \sum_{|a|\leq 1} \|\partial \Gamma^a u_{cf}(t)\|_{L^2} + Ct\|\Box_{c_1} u_{cf}(t)\|_{L^2}. \tag{2.38}$$

By (2.8) and (1.16), we see that

$$\|\partial \Gamma^a u_{cf}(t)\|_{L^2} = \|(\partial \Gamma^a u)_{cf}(t)\|_{L^2} \leq \|\partial \Gamma^a u(t)\|_{L^2} \leq E_2^{1/2}(u(t)). \tag{2.39}$$

Noting that u_{cf} is curl-free, by (1.16) we obtain

$$\|\Box_{c_1} u_{cf}(t)\|_{L^2} = \|Lu_{cf}(t)\|_{L^2} = \|(Lu)_{cf}(t)\|_{L^2} \le \|Lu(t)\|_{L^2}. \tag{2.40}$$

The combination of (2.38), (2.39) and (2.40) yields

$$\|\langle c_1 t - r\rangle \partial\nabla u_{cf}(t)\|_{L^2} \le CE_2^{1/2}(u(t)) + Ct\|Lu(t)\|_{L^2}. \tag{2.41}$$

Similarly, we can also get

$$\|\langle c_2 t - r\rangle \partial\nabla u_{df}(t)\|_{L^2} \le CE_2^{1/2}(u(t)) + Ct\|Lu(t)\|_{L^2}. \tag{2.42}$$

We deduce (2.37) from (2.41) and (2.42). □

Lemma 2.9. *If $u \in \dot{H}^5_\Gamma(T)$ is a solution of (1.52), and*

$$\varepsilon_1 \equiv \sup_{0\le t<T} E_4^{1/2}(u(t)) \tag{2.43}$$

is sufficiently small, then for $0 \le t < T$, we have

$$\mathcal{X}_4(u(t)) \le CE_4^{1/2}(u(t)), \tag{2.44}$$

$$\mathcal{X}_5(u(t)) \le CE_5^{1/2}(u(t)). \tag{2.45}$$

Proof. By applying Lemma 2.8 to $\Gamma^a u$ and summering over $|a| \le \kappa - 2$, and noting Lemma 2.3, we have

$$\begin{aligned}\mathcal{X}_\kappa(u(t)) &\le CE_\kappa^{1/2}(u(t)) + Ct\sum_{|a|\le\kappa-2}\|L\Gamma^a u(t)\|_{L^2}\\ &\le CE_\kappa^{1/2}(u(t)) + Ct\sum_{|a|\le\kappa-2}\sum_{b+c=a}\|N(\Gamma^b u, \Gamma^c u)\|_{L^2}\\ &\le CE_\kappa^{1/2}(u(t)) + Ct\sum_{|a|\le\kappa-2}\sum_{b+c=a}\|\nabla\Gamma^b u\nabla^2\Gamma^c u\|_{L^2}.\end{aligned} \tag{2.46}$$

We first consider the case $\kappa = 4$. Note that $\langle t\rangle \le C\langle r\rangle\langle c_1 t - r\rangle$, $\langle t\rangle \le C\langle r\rangle\langle c_2 t - r\rangle$. For $|a| \le 2, b + c = a$, if $|b| \le 1$, it follows from (2.29) that

$$\begin{aligned}&t\|\nabla\Gamma^b u\nabla^2\Gamma^c u\|_{L^2}\\ &\le C\|\langle r\rangle\nabla\Gamma^b u\|_{L^\infty}\big(\|\langle c_1 t - r\rangle\nabla^2\Gamma^c u_{cf}\|_{L^2} + \|\langle c_2 t - r\rangle\nabla^2\Gamma^c u_{df}\|_{L^2}\big)\\ &\le CE_4^{1/2}(u(t))\mathcal{X}_4(u(t)).\end{aligned} \tag{2.47}$$

If $|b| = 2, c = 0$, (2.31) yields

$$\begin{aligned}&t\|\nabla\Gamma^b u\nabla^2\Gamma^c u\|_{L^2}\\ &\le C\|\nabla\Gamma^b u\|_{L^2}\big(\|\langle r\rangle\langle c_1 t - r\rangle\nabla^2\Gamma^c u_{cf}\|_{L^\infty} + \|\langle r\rangle\langle c_2 t - r\rangle\nabla^2\Gamma^c u_{df}\|_{L^\infty}\big)\\ &\le CE_4^{1/2}(u(t))\mathcal{X}_4(u(t)).\end{aligned} \tag{2.48}$$

Thus we have

$$\mathcal{X}_4(u(t)) \le CE_4^{1/2}(u(t)) + CE_4^{1/2}(u(t))\mathcal{X}_4(u(t)). \tag{2.49}$$

Noting the smallness condition (2.43), we get (2.44).

Now we consider the case $\kappa = 5$. For $|a| \le 3, b + c = a$, if $|b| \le 1$, similarly to (2.47), by (2.29) we have

$$\begin{aligned} &t\|\nabla\Gamma^b u\nabla^2\Gamma^c u\|_{L^2} \\ &\le C\|\langle r\rangle\nabla\Gamma^b u\|_{L^\infty}\big(\|\langle c_1 t - r\rangle\nabla^2\Gamma^c u_{cf}\|_{L^2} + \|\langle c_2 t - r\rangle\nabla^2\Gamma^c u_{df}\|_{L^2}\big) \\ &\le CE_4^{1/2}(u(t))\mathcal{X}_5(u(t)). \end{aligned} \tag{2.50}$$

If $|b| = 2, |c| \le 1$ or $|b| = 3, c = 0$, (2.31) leads to

$$\begin{aligned} &t\|\nabla\Gamma^b u\nabla^2\Gamma^c u\|_{L^2} \\ &\le C\|\nabla\Gamma^b u\|_{L^2}\big(\|\langle r\rangle\langle c_1 t - r\rangle\nabla^2\Gamma^c u_{cf}\|_{L^\infty} + \|\langle r\rangle\langle c_2 t - r\rangle\nabla^2\Gamma^c u_{df}\|_{L^\infty}\big) \\ &\le CE_4^{1/2}(u(t))\mathcal{X}_5(u(t)). \end{aligned} \tag{2.51}$$

Thus we obtain

$$\mathcal{X}_5(u(t)) \le CE_5^{1/2}(u(t)) + CE_4^{1/2}(u(t))\mathcal{X}_5(u(t)). \tag{2.52}$$

Then the smallness condition (2.43) implies (2.45). □

3. Proof of Theorem 1.1

In this section we will complete the proof of Theorem 1.1 by a bootstrap argument. Without loss of generality, we only consider the case $\kappa = 5$. Assume that $u \in \dot{H}^5_\Gamma(T)$ is a local solution of (1.52). There are two key steps in the proof. First, under the assumption on the lower order energy

$$E_4(u(t)) \le 4\varepsilon^2, \tag{3.1}$$

we will prove the higher order energy estimate

$$E_5(u(t)) \le 2E_5(u(0))\langle t\rangle^{C_0\varepsilon}. \tag{3.2}$$

Second, we will show that (3.2) implies

$$E_4(u(t)) \le \varepsilon^2. \tag{3.3}$$

In order to achieve the above goal, we will derive a pair of coupled differential inequalities for the lower order energy $E_4(u(t))$ and the modified higher order energy $\widetilde{E}_5(u(t))$.

3.1. Higher order energy estimate

Noting the symmetry condition (1.58), by Lemma 2.3 and the energy method we can get

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{E}_5(u(t)) \le C\sum_{|a|\le 4}\sum_{\substack{b+c=a\\ c\ne a}}\|\partial\Gamma^a u\nabla\Gamma^b u\partial\nabla\Gamma^c u\|_{L^1}, \tag{3.4}$$

where the modified higher order energy

$$\widetilde{E}_5(u(t)) = E_5(u(t)) + \sum_{|a|\le 4} g_{lmn}^{ijk} \int_{\mathbb{R}^3} \partial_l \Gamma^a u^i \partial_m \Gamma^a u^j \partial_n u \mathrm{d}x. \tag{3.5}$$

The smallness of lower order energy and Sobolev embedding $H^2(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$ imply

$$\frac{1}{\sqrt{2}} E_5(u(t)) \le \widetilde{E}_5(u(t)) \le \sqrt{2} E_5(u(t)). \tag{3.6}$$

Without loss of generality, we only consider the case $|a| = 4$. By (2.29), (2.31) and Lemma 2.9, we have

$$\begin{aligned}
&\|\partial\Gamma^a u \nabla\Gamma^b u \partial\nabla\Gamma^c u\|_{L^1} \\
&\le C\langle t\rangle^{-1} \|\partial\Gamma^a u\|_{L^2} \\
&\begin{cases}
\|\langle r\rangle \nabla\Gamma^b u\|_{L^\infty} \big(\|\langle c_1 t - r\rangle \partial\nabla\Gamma^c u_{cf}\|_{L^2} + \|\langle c_2 t - r\rangle \partial\nabla\Gamma^c u_{df}\|_{L^2}\big), |b| = 1, |c| = 3 \\
\|\langle r\rangle \nabla\Gamma^b u\|_{L^\infty} \big(\|\langle c_1 t - r\rangle \partial\nabla\Gamma^c u_{cf}\|_{L^2} + \|\langle c_2 t - r\rangle \partial\nabla\Gamma^c u_{df}\|_{L^2}\big), |b| = 2, |c| = 2 \\
\|\nabla\Gamma^b u\|_{L^2} \big(\|\langle r\rangle\langle c_1 t - r\rangle \partial\nabla\Gamma^c u_{cf}\|_{L^\infty} + \|\langle r\rangle\langle c_2 t - r\rangle \partial\nabla\Gamma^c u_{df}\|_{L^\infty}\big), |b| = 3, |c| = 1 \\
\|\nabla\Gamma^b u\|_{L^2} \big(\|\langle r\rangle\langle c_1 t - r\rangle \partial\nabla\Gamma^c u_{cf}\|_{L^\infty} + \|\langle r\rangle\langle c_2 t - r\rangle \partial\nabla\Gamma^c u_{df}\|_{L^\infty}\big), |b| = 4, |c| = 0
\end{cases} \\
&\le C\langle t\rangle^{-1} E_5^{1/2}(u(t)) \begin{cases}
E_4^{1/2}(u(t)) \mathcal{X}_5(u(t)), \ |b| = 1, |c| = 3 \\
E_5^{1/2}(u(t)) \mathcal{X}_4(u(t)), \ |b| = 2, |c| = 2 \\
E_4^{1/2}(u(t)) \mathcal{X}_5(u(t)), \ |b| = 3, |c| = 1 \\
E_5^{1/2}(u(t)) \mathcal{X}_4(u(t)), \ |b| = 4, |c| = 0
\end{cases} \\
&\le C\langle t\rangle^{-1} E_4^{1/2}(u(t)) E_5(u(t)).
\end{aligned} \tag{3.7}$$

Hence by (3.4), (3.7) and (3.6) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \widetilde{E}_5(u(t)) \le C\langle t\rangle^{-1} E_4^{1/2}(u(t)) \widetilde{E}_5(u(t)). \tag{3.8}$$

3.2. Lower order energy estimate

Lemma 2.3 and the energy method yield

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} E_4(u(t)) = & \sum_{|a|\le 3} \sum_{b+c=a} \int_{\mathbb{R}^3} \langle \partial_t \Gamma^a u, N(\Gamma^b u, \Gamma^c u)\rangle \mathrm{d}x \\
= & \sum_{|a|\le 3} \sum_{b+c=a} \int_{\mathbb{R}^3} \langle \partial_t \Gamma^a u, N_1(\Gamma^b u, \Gamma^c u)\rangle \mathrm{d}x \\
& + \sum_{|a|\le 3} \sum_{b+c=a} \int_{\mathbb{R}^3} \langle \partial_t \Gamma^a u, N_2(\Gamma^b u, \Gamma^c u)\rangle \mathrm{d}x.
\end{aligned} \tag{3.9}$$

Estimates of null form nonlinearity. We first estimate the second part on the right hand side of (3.9). We have

$$\int_{r\le \frac{\langle c_2 t\rangle}{2}} \langle \partial_t \Gamma^a u, N_2(\Gamma^b u, \Gamma^c u)\rangle \mathrm{d}x \le \|\partial_t \Gamma^a u\|_{L^2} \|\nabla\Gamma^b u \nabla^2 \Gamma^c u\|_{L^2(r\le \frac{\langle c_2 t\rangle}{2})}. \tag{3.10}$$

Note that $r \le \frac{\langle c_2 t\rangle}{2}$ implies $\langle t\rangle \le C\langle c_1t-r\rangle, \langle t\rangle \le C\langle c_2t-r\rangle$. For $|a|\le 3, b+c=a$, if $|b|\le 1$, due to (2.30) and Lemma 2.9, it follows that

$$\begin{aligned}
&\|\nabla\Gamma^b u\nabla^2\Gamma^c u\|_{L^2(r\le\frac{\langle c_2t\rangle}{2})}\\
&\le C\langle t\rangle^{-3/2}\big(\|\langle c_1t-r\rangle^{1/2}\nabla\Gamma^b u_{cf}\|_{L^\infty}+\|\langle c_2t-r\rangle^{1/2}\nabla\Gamma^b u_{df}\|_{L^\infty}\big)\\
&\ \big(\|\langle c_1t-r\rangle\nabla^2\Gamma^c u_{cf}\|_{L^2}+\|\langle c_2t-r\rangle\nabla^2\Gamma^c u_{df}\|_{L^2}\big)\\
&\le C\langle t\rangle^{-3/2}\big(E_4^{1/2}(u(t))+\mathcal{X}_4(u(t))\big)\mathcal{X}_5(u(t))\\
&\le C\langle t\rangle^{-3/2}E_4^{1/2}(u(t))E_5^{1/2}(u(t)).
\end{aligned}\tag{3.11}$$

If $|b|=2, |c|\le 1$ or $|b|=3, c=0$, we conclude from Sobolev embeddings $\dot H^1(\mathbb{R}^3)\hookrightarrow L^6(\mathbb{R}^3), H^1(\mathbb{R}^3)\hookrightarrow L^3(\mathbb{R}^3)$, Lemma 2.2, (1.16) and Lemma 2.9 that

$$\begin{aligned}
&\|\nabla\Gamma^b u\nabla^2\Gamma^c u\|_{L^2(r\le\frac{\langle c_2t\rangle}{2})}\\
&\le C\langle t\rangle^{-2}\big(\|\langle c_1t-r\rangle\nabla\Gamma^b u_{cf}\|_{L^6}+\|\langle c_2t-r\rangle\nabla\Gamma^b u_{df}\|_{L^6}\big)\\
&\ \big(\|\langle c_1t-r\rangle\nabla^2\Gamma^c u_{cf}\|_{L^3}+\|\langle c_2t-r\rangle\nabla^2\Gamma^c u_{df}\|_{L^3}\big)\\
&\le C\langle t\rangle^{-2}\big(\|\langle c_1t-r\rangle\nabla\Gamma^b u_{cf}\|_{\dot H^1}+\|\langle c_2t-r\rangle\nabla\Gamma^b u_{df}\|_{\dot H^1}\big)\\
&\ \big(\|\langle c_1t-r\rangle\nabla^2\Gamma^c u_{cf}\|_{H^1}+\|\langle c_2t-r\rangle\nabla^2\Gamma^c u_{df}\|_{H^1}\big)\\
&\le C\langle t\rangle^{-2}\big(E_4^{1/2}(u(t))+\mathcal{X}_5(u(t))\big)\big(E_3^{1/2}(u(t))+\mathcal{X}_4(u(t))\big)\\
&\le C\langle t\rangle^{-2}E_4^{1/2}(u(t))E_5^{1/2}(u(t)).
\end{aligned}\tag{3.12}$$

The combination of (3.10)–(3.12) gives

$$\int_{r\le\frac{\langle c_2t\rangle}{2}}\langle\partial_t\Gamma^a u, N_2(\Gamma^b u,\Gamma^c u)\rangle \mathrm{d}x\le C\langle t\rangle^{-3/2}E_4(u(t))E_5^{1/2}(u(t)).\tag{3.13}$$

Noting Lemma 2.4, we have

$$\begin{aligned}
&\int_{r\ge\frac{\langle c_2t\rangle}{2}}\langle\partial_t\Gamma^a u, N_2(\Gamma^b u,\Gamma^c u)\rangle \mathrm{d}x\\
&\le C\langle t\rangle^{-1}\|\partial_t\Gamma^a u\|_{L^2}\big(\|\Gamma^{b+1}u\nabla^2\Gamma^c u\|_{L^2(r\ge\frac{\langle c_2t\rangle}{2})}+\|\nabla\Gamma^b u\nabla\Gamma^{c+1}u\|_{L^2(r\ge\frac{\langle c_2t\rangle}{2})}\big)
\end{aligned}\tag{3.14}$$

It follows from (2.28), (2.27) and Sobolev embedding on the sphere $H^2(S^2)\hookrightarrow L^\infty(S^2)$ that

$$\begin{aligned}
&\|\Gamma^{b+1}u\nabla^2\Gamma^c u\|_{L^2(r\ge\frac{\langle c_2t\rangle}{2})}\\
&\le C\langle t\rangle^{-1/2}\begin{cases}\|r^{1/2}\Gamma^{b+1}u\|_{L^\infty}\|\nabla^2\Gamma^c u\|_{L^2}, |b|\le 2\\ \|r^{1/2}\Gamma^{b+1}u\|_{L^{\infty,2}}\|\nabla^2\Gamma^c u\|_{L^{2,\infty}}, |b|=3\end{cases}\\
&\le C\langle t\rangle^{-1/2}E_4^{1/2}(u(t))E_5^{1/2}(u(t)).
\end{aligned}\tag{3.15}$$

Similarly, we can also get

$$\|\nabla\Gamma^b u\nabla\Gamma^{c+1}u\|_{L^2(r\ge\frac{\langle c_2t\rangle}{2})}\le C\langle t\rangle^{-1/2}E_4^{1/2}(u(t))E_5^{1/2}(u(t)).\tag{3.16}$$

By (3.14)–(3.16), we have

$$\int_{r\geq\frac{\langle c_2t\rangle}{2}}\langle\partial_t\Gamma^a u, N_2(\Gamma^b u,\Gamma^c u)\rangle \mathrm{d}x\leq C\langle t\rangle^{-3/2}E_4(u(t))E_5^{1/2}(u(t)). \tag{3.17}$$

In view of (3.13) and (3.17), we obtain

$$\sum_{|a|\leq 3}\sum_{b+c=a}\int_{\mathbb{R}^3}\langle\partial_t\Gamma^a u, N_2(\Gamma^b u,\Gamma^c u)\rangle \mathrm{d}x\leq C\langle t\rangle^{-3/2}E_4(u(t))E_5^{1/2}(u(t)). \tag{3.18}$$

Estimates of other nonlinearity. The last task is to treat the first part on the right hand side of (3.9), which is the key point in our argument. In view of (1.55), by integration by parts we have

$$\begin{aligned}
&\sum_{|a|\leq 3}\sum_{b+c=a}\int_{\mathbb{R}^3}\langle\partial_t\Gamma^a u, N_1(\Gamma^b u,\Gamma^c u)\rangle \mathrm{d}x\\
&= d_2\sum_{|a|\leq 3}\sum_{b+c=a}\int_{\mathbb{R}^3}\langle\partial_t\Gamma^a u, \nabla\big((\nabla\wedge\Gamma^b u)\cdot(\nabla\wedge\Gamma^c u)\big)\rangle \mathrm{d}x\\
&\quad -2d_2\sum_{|a|\leq 3}\sum_{b+c=a}\int_{\mathbb{R}^3}\langle\partial_t\Gamma^a u, \nabla\wedge\big((\nabla\cdot\Gamma^b u)(\nabla\wedge\Gamma^c u)\big)\rangle \mathrm{d}x\\
&= -d_2\sum_{|a|\leq 3}\sum_{b+c=a}\int_{\mathbb{R}^3}\langle(\partial_t\nabla\cdot\Gamma^a u)(\nabla\wedge\Gamma^b u), \nabla\wedge\Gamma^c u\rangle \mathrm{d}x\\
&\quad -2d_2\sum_{|a|\leq 3}\sum_{b+c=a}\int_{\mathbb{R}^3}\langle\partial_t\Gamma^a u, (\nabla\nabla\cdot\Gamma^b u)\wedge(\nabla\wedge\Gamma^c u)\rangle \mathrm{d}x\\
&\quad -2d_2\sum_{|a|\leq 3}\sum_{b+c=a}\int_{\mathbb{R}^3}\langle\partial_t\Gamma^a u, (\nabla\cdot\Gamma^b u)\nabla\wedge(\nabla\wedge\Gamma^c u)\rangle \mathrm{d}x.
\end{aligned} \tag{3.19}$$

We should point out that in all three terms on the right hand side of (3.19), divergence terms and curl terms appear simultaneously. We will see that this is compatible with the using of Helmholtz decomposition, in order to get enough decay in time.

We first focus on the first term on the right hand side of (3.19). For $|a|\leq 3, b+c=a$, by Lemma 2.2, (2.30) and Lemma 2.9, it follows that

$$\begin{aligned}
&\int_{r\leq\frac{\langle c_2t\rangle}{2}}\langle(\partial_t\nabla\cdot\Gamma^a u)(\nabla\wedge\Gamma^b u), \nabla\wedge\Gamma^c u\rangle \mathrm{d}x\\
&\leq C\langle t\rangle^{-3/2}\|\langle c_1t-r\rangle\partial_t\nabla\cdot\Gamma^a u_{cf}\|_{L^2}\begin{cases}\|\langle c_2t-r\rangle^{1/2}\nabla\wedge\Gamma^b u_{df}\|_{L^\infty}\|\nabla\wedge\Gamma^c u\|_{L^2}, & |b|\leq 1\\ \|\nabla\wedge\Gamma^b u\|_{L^2}\|\langle c_2t-r\rangle^{1/2}\nabla\wedge\Gamma^c u_{df}\|_{L^\infty}, & |c|\leq 1\end{cases}\\
&\leq C\langle t\rangle^{-3/2}\|\langle c_1t-r\rangle\partial\nabla\Gamma^a u_{cf}\|_{L^2}\begin{cases}\|\langle c_2t-r\rangle\rangle^{1/2}\nabla\Gamma^b u_{df}\|_{L^\infty}\|\nabla\Gamma^c u\|_{L^2}, & |b|\leq 1\\ \|\nabla\Gamma^b u\|_{L^2}\|\langle c_2t-r\rangle\rangle^{1/2}\nabla\Gamma^c u_{df}\|_{L^\infty}, & |c|\leq 1\end{cases}\\
&\leq C\langle t\rangle^{-3/2}E_4^{1/2}(u(t))\big(E_4^{1/2}(u(t))+\mathcal{X}_4(u(t))\big)\mathcal{X}_5(u(t))\\
&\leq C\langle t\rangle^{-3/2}E_4(u(t))E_5^{1/2}(u(t)).
\end{aligned} \tag{3.20}$$

Noting that when $\frac{\langle c_2 t\rangle}{2} \leq r \leq \frac{\langle (c_1+c_2)t\rangle}{2}$, we have $\langle t\rangle \leq C\langle r\rangle, \langle t\rangle \leq C\langle c_1 t - r\rangle$. The combination of Lemma 2.2, (2.29) and Lemma 2.9 yields

$$\begin{aligned}
&\int_{\frac{\langle c_2 t\rangle}{2}\leq r\leq \frac{\langle (c_1+c_2)t\rangle}{2}} \langle (\partial_t \nabla\cdot\Gamma^a u)(\nabla\wedge\Gamma^b u), \nabla\wedge\Gamma^c u\rangle \mathrm{d}x\\
&\leq C\langle t\rangle^{-2}\|\langle c_1 t - r\rangle \partial_t\nabla\cdot\Gamma^a u_{cf}\|_{L^2}\begin{cases}\|\langle r\rangle\nabla\wedge\Gamma^b u\|_{L^\infty}\|\nabla\wedge\Gamma^c u\|_{L^2},\ |b|\leq 1\\ \|\nabla\wedge\Gamma^b u\|_{L^2}\|\langle r\rangle\nabla\wedge\Gamma^c u\|_{L^\infty},\ |c|\leq 1\end{cases}\\
&\leq C\langle t\rangle^{-2}\|\langle c_1 t - r\rangle \partial\nabla\Gamma^a u_{cf}\|_{L^2}\begin{cases}\|\langle r\rangle\nabla\Gamma^b u\|_{L^\infty}\|\nabla\Gamma^c u\|_{L^2},\ |b|\leq 1\\ \|\nabla\Gamma^b u\|_{L^2}\|\langle r\rangle\nabla\Gamma^c u\|_{L^\infty},\ |c|\leq 1\end{cases}\\
&\leq C\langle t\rangle^{-2}E_4(u(t))\mathcal{X}_5(u(t))\\
&\leq C\langle t\rangle^{-2}E_4(u(t))E_5^{1/2}(u(t)). \qquad (3.21)
\end{aligned}$$

Note that $r \geq \frac{\langle (c_1+c_2)t\rangle}{2}$ implies $\langle t\rangle \leq C\langle r\rangle, \langle t\rangle \leq C\langle c_2 t - r\rangle$. It follows from Lemma 2.2, (2.30) and Lemma 2.9 that

$$\begin{aligned}
&\int_{r\geq \frac{\langle (c_1+c_2)t\rangle}{2}} \langle (\partial_t \nabla\cdot\Gamma^a u)(\nabla\wedge\Gamma^b u), \nabla\wedge\Gamma^c u\rangle \mathrm{d}x\\
&\leq C\langle t\rangle^{-3/2}\|\partial_t\nabla\cdot\Gamma^a u\|_{L^2}\begin{cases}\|\langle r\rangle\langle c_2 t - r\rangle^{1/2}\nabla\wedge\Gamma^b u_{df}\|_{L^\infty}\|\nabla\wedge\Gamma^c u\|_{L^2},\ |b|\leq 1\\ \|\nabla\wedge\Gamma^b u\|_{L^2}\|\langle r\rangle\langle c_2 t - r\rangle^{1/2}\nabla\wedge\Gamma^c u_{df}\|_{L^\infty},\ |c|\leq 1\end{cases}\\
&\leq C\langle t\rangle^{-3/2}\|\partial\nabla\Gamma^a u\|_{L^2}\begin{cases}\|\langle r\rangle\langle c_2 t - r\rangle^{1/2}\nabla\Gamma^b u_{df}\|_{L^\infty}\|\nabla\Gamma^c u\|_{L^2},\ |b|\leq 1\\ \|\nabla\Gamma^b u\|_{L^2}\|\langle r\rangle\langle c_2 t - r\rangle^{1/2}\nabla\Gamma^c u_{df}\|_{L^\infty},\ |c|\leq 1\end{cases}\\
&\leq C\langle t\rangle^{-3/2}\big(E_4^{1/2}(u(t)) + \mathcal{X}_4(u(t))\big)E_4^{1/2}(u(t))E_5^{1/2}(u(t))\\
&\leq C\langle t\rangle^{-3/2}E_4(u(t))E_5^{1/2}(u(t)). \qquad (3.22)
\end{aligned}$$

The combination of (3.20)–(3.22) leads to

$$\sum_{|a|\leq 3}\sum_{b+c=a}\int_{\mathbb{R}^3}\langle (\partial_t \nabla\cdot\Gamma^a u)(\nabla\wedge\Gamma^b u), \nabla\wedge\Gamma^c u\rangle \mathrm{d}x \leq C\langle t\rangle^{-3/2}E_4(u(t))E_5^{1/2}(u(t)). \qquad (3.23)$$

Now we will treat the second term on the right hand side of (3.19). Similarly to (3.13), we have

$$\begin{aligned}
&\int_{r\leq\frac{\langle c_2 t\rangle}{2}}\langle\partial_t\Gamma^a u,(\nabla\nabla\cdot\Gamma^b u)\wedge(\nabla\wedge\Gamma^c u)\rangle\mathrm{d}x\\
&\leq\|\partial_t\Gamma^a u\|_{L^2}\|\nabla^2\Gamma^b u\nabla\Gamma^c u\|_{L^2(r\leq\frac{\langle c_2 t\rangle}{2})}\\
&\leq C\langle t\rangle^{-3/2}E_4(u(t))E_5^{1/2}(u(t)).
\end{aligned}\tag{3.24}$$

By Lemma 2.2, (2.31) and Lemma 2.9, we obtain

$$\begin{aligned}
&\int_{\frac{\langle c_2 t\rangle}{2}\leq r\leq\frac{\langle(c_1+c_2)t\rangle}{2}}\langle\partial_t\Gamma^a u,(\nabla\nabla\cdot\Gamma^b u)\wedge(\nabla\wedge\Gamma^c u)\rangle\mathrm{d}x\\
&\leq C\langle t\rangle^{-2}\|\partial_t\Gamma^a u\|_{L^2}\begin{cases}\|\langle r\rangle\langle c_1t-r\rangle\nabla\nabla\cdot\Gamma^b u_{cf}\|_{L^\infty}\|\nabla\wedge\Gamma^c u\|_{L^2}, & |b|\leq 1\\ \|\langle c_1t-r\rangle\nabla\nabla\cdot\Gamma^b u_{cf}\|_{L^2}\|\langle r\rangle\nabla\wedge\Gamma^c u\|_{L^\infty}, & |c|\leq 1\end{cases}\\
&\leq C\langle t\rangle^{-2}\|\partial_t\Gamma^a u\|_{L^2}\begin{cases}\|\langle r\rangle\langle c_1t-r\rangle\nabla^2\Gamma^b u_{cf}\|_{L^\infty}\|\nabla\Gamma^c u\|_{L^2}, & |b|\leq 1\\ \|\langle c_1t-r\rangle\nabla^2\Gamma^b u_{cf}\|_{L^2}\|\langle r\rangle\nabla\Gamma^c u\|_{L^\infty}, & |c|\leq 1\end{cases}\\
&\leq C\langle t\rangle^{-2}E_4(u(t))\mathcal{X}_5(u(t))\\
&\leq C\langle t\rangle^{-2}E_4(u(t))E_5^{1/2}(u(t)).
\end{aligned}\tag{3.25}$$

We conclude from Lemma 2.2, (2.30), (2.22), (1.16) and Lemma 2.9 that

$$\begin{aligned}
&\int_{r\geq\frac{\langle(c_1+c_2)t\rangle}{2}}\langle\partial_t\Gamma^a u,(\nabla\nabla\cdot\Gamma^b u)\wedge(\nabla\wedge\Gamma^c u)\rangle\mathrm{d}x\\
&\leq C\langle t\rangle^{-3/2}\|\partial_t\Gamma^a u\|_{L^2}\begin{cases}\|\nabla\nabla\cdot\Gamma^b u\|_{L^2}\|\langle r\rangle\langle c_2t-r\rangle^{1/2}\nabla\wedge\Gamma^c u_{df}\|_{L^\infty}, & |c|\leq 2\\ \|\nabla\nabla\cdot\Gamma^b u\|_{L^{2,\infty}}\|r^{1/2}\langle c_2t-r\rangle\nabla\wedge\Gamma^c u_{df}\|_{L^{\infty,2}}, & |c|=3\end{cases}\\
&\leq C\langle t\rangle^{-3/2}\|\partial_t\Gamma^a u\|_{L^2}\begin{cases}\|\nabla^2\Gamma^b u\|_{L^2}\|\langle r\rangle\langle c_2t-r\rangle^{1/2}\nabla\Gamma^c u_{df}\|_{L^\infty}, & |c|\leq 2\\ \|\nabla^2\Gamma^b u\|_{L^{2,\infty}}\|\langle c_2t-r\rangle\nabla\Gamma^c u_{df}\|_{\dot H^1}, & |c|=3\end{cases}\\
&\leq C\langle t\rangle^{-3/2}E_4^{1/2}(u(t))\big(E_4^{1/2}(u(t))+\mathcal{X}_4(u(t))\big)\big(E_5^{1/2}(u(t))+\mathcal{X}_5(u(t))\big)\\
&\leq C\langle t\rangle^{-3/2}E_4(u(t))E_5^{1/2}(u(t)).
\end{aligned}\tag{3.26}$$

Here we also use the Sobolev embedding on the sphere $H^2(S^2)\hookrightarrow L^\infty(S^2)$ in the case $|c|=3$.

Due to (3.24)–(3.26), we obtain

$$\sum_{|a|\leq 3}\sum_{b+c=a}\int_{\mathbb{R}^3}\langle\partial_t\Gamma^a u,(\nabla\nabla\cdot\Gamma^b u)\wedge(\nabla\wedge\Gamma^c u)\rangle\mathrm{d}x\leq C\langle t\rangle^{-3/2}E_4(u(t))E_5^{1/2}(u(t)).\tag{3.27}$$

Similarly to (3.27), we can also get the following upper bound for the third term on the right hand side of (3.19)

$$\sum_{|a|\leq 3}\sum_{b+c=a}\int_{\mathbb{R}^3}\langle\partial_t\Gamma^a u,(\nabla\cdot\Gamma^b u)\nabla\wedge(\nabla\wedge\Gamma^c u)\rangle\mathrm{d}x\leq C\langle t\rangle^{-3/2}E_4(u(t))E_5^{1/2}(u(t)).\tag{3.28}$$

In view of (3.19), (3.23), (3.27) and (3.28), we have

$$\sum_{|a|\le 3}\sum_{b+c=a}\int_{\mathbb{R}^3}\langle \partial_t\Gamma^a u, N_1(\Gamma^b u,\Gamma^c u)\rangle \mathrm{d}x \le C\langle t\rangle^{-3/2}E_4(u(t))E_5^{1/2}(u(t)). \tag{3.29}$$

The combination of (3.9), (3.18), (3.29) and (3.6) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}E_4(u(t)) \le C\langle t\rangle^{-3/2}\widetilde{E}_5^{1/2}(u(t))E_4(u(t)). \tag{3.30}$$

3.3. Conclusion of the proof

Now we have arrived at the coupled pair of differential inequalities

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{E}_5(u(t)) \le C\langle t\rangle^{-1}E_4^{1/2}(u(t))\widetilde{E}_5(u(t)), \tag{3.31}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}E_4(u(t)) \le C\langle t\rangle^{-3/2}\widetilde{E}_5^{1/2}(u(t))E_4(u(t)). \tag{3.32}$$

By (3.31), Gronwall inequality and the assumption (3.1), we can get

$$\widetilde{E}_5(u(t)) \le \widetilde{E}_5(u(0))\langle t\rangle^{C_0\varepsilon}, \tag{3.33}$$

where we have taken $C_0 = 2C$. Noting (3.6) and (3.33), we also see that

$$E_5(u(t)) \le 2E_5(u(0))\langle t\rangle^{C_0\varepsilon}. \tag{3.34}$$

Inserting (3.33) into (3.32), we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_4(u(t)) &\le C\langle t\rangle^{-3/2+\sqrt{C_0\varepsilon}}\widetilde{E}_5^{1/2}(u(0))E_4(u(t))\\ &\le C_0\langle t\rangle^{-3/2+\sqrt{C_0\varepsilon}}E_5^{1/2}(u(0))E_4(u(t)).\end{aligned} \tag{3.35}$$

Take ε sufficiently small such that $C_0\varepsilon \le \frac{1}{16}$. Then Gronwall inequality gives

$$E_4(u(t)) \le E_4(u(t))\exp\left[C_0E_5^{1/2}(u(0))\right] \le \varepsilon^2. \tag{3.36}$$

Thus we have completed the proof of Theorem 1.1.

Acknowledgements. The authors would like to express their sincere gratitude to the referees for their helpful comments and suggestions on this paper. The first author was supported by National Natural Science Foundation of China No.11801068 and the Fundamental Research Funds for the Central Universities. The second author was supported by National Natural Science Foundation of China No.11901391.

References

[1] R. Agemi, Global existence of nonlinear elastic waves *Invent. Math.* **142** (2000) 225–250.
[2] A.J. Chorin, J.E. Marsden, A mathematical introduction to fluid mechanics, *Texts in Applied Mathematics*, 3rd edn. Vol. 4, Springer-Verlag, New York (1993).
[3] D. Christodoulou, Global solutions of nonlinear hyperbolic equations for small initial data. *Comm. Pure Appl. Math.* **39** (1986) 267–282.

[4] P.G. Ciarlet, Mathematical elasticity, Three-dimensional elasticity, *Studies in Mathematics and its Applications*, Vol. 20, North-Holland Publishing Co., Amsterdam (1988).

[5] M.E. Gurtin, Topics in finite elasticity, *CBMS-NSF Regional Conference Series in Applied Mathematics*, Vol. 35, Society for Industrial and Applied Mathematics (SIAM), Philadelphia (1981).

[6] K. Hidano and D. Zha, Space-time L^2 estimates, regularity and almost global existence for elastic waves. *Forum Math.* **30** (2018) 1291–1307.

[7] F. John, Blow-up for quasilinear wave equations in three space dimensions. *Comm. Pure Appl. Math.* **34** (1981) 29–51.

[8] F. John, Formation of singularities in elastic waves, in Trends and applications of pure mathematics to mechanics (Palaiseau, 1983), *Lecture Notes in Physics*, Vol. 195, Springer, Berlin (1984) 194–210.

[9] F. John, Almost global existence of elastic waves of finite amplitude arising from small initial disturbances. *Comm. Pure Appl. Math.* **41** (1988) 615–666.

[10] F. John and S. Klainerman, Almost global existence to nonlinear wave equations in three space dimensions. *Comm. Pure Appl. Math.* **37** (1984) 443–455.

[11] S. Katayama, Asymptotic behavior for systems of nonlinear wave equations with multiple propagation speeds in three space dimensions. *J. Differ. Equ.* **255** (2013) 120–150.

[12] S. Klainerman, The null condition and global existence to nonlinear wave equations, in Nonlinear systems of partial differential equations in applied mathematics, Part 1 (Santa Fe, N.M., 1984), *Lectures in Appl. Math.*, Vol. 23. American Mathematics Society, Providence, RI (1986) 293–326.

[13] S. Klainerman, On the work and legacy of Fritz John, 1934–1991. *Comm. Pure Appl. Math.* **51** (1998) 991–1017.

[14] S. Klainerman and T.C. Sideris, On almost global existence for nonrelativistic wave equations in 3D. *Comm. Pure Appl. Math.* **49** (1996) 307–321.

[15] H. Kubo, Lower bounds for the lifespan of solutions to nonlinear wave equations in elasticity, in Evolution equations of hyperbolic and Schrödinger type, Progress in Mathematics, Vol. 301. Birkhäuser/Springer Basel AG, Basel (2012) 187–212.

[16] T. Li and T. Qin, Physics and Partial Differential Equations. Volume 1. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, Higher Education Press, Beijing (2012).

[17] T. Li, X. Yu, Life-span of classical solutions to fully nonlinear wave equations. *Comm. Part. Differ. Equ.* **16** (1991) 909–940.

[18] T.C. Sideris, Global behavior of solutions to nonlinear wave equations in three dimensions. *Comm. Part. Differ. Equ.* **8** (1983) 1291–1323.

[19] T.C. Sideris, The null condition and global existence of nonlinear elastic waves. *Invent. Math.* **123** (1996) 323–342.

[20] T.C. Sideris, Nonresonance and global existence of prestressed nonlinear elastic waves. *Ann. Math.* **151** (2000) 849–874.

[21] T.C. Sideris and S.-Y. Tu, Global existence for systems of nonlinear wave equations in 3D with multiple speeds. S*IAM J. Math. Anal.* **33** (2001) 477–488.

[22] J. Xin and T. Qin, Almost global existence for the initial value problem of nonlinear elastodynamic system. *J. Math. Anal. Appl.* **339** (2008) 517–529.

[23] D. Zha, Remarks on nonlinear elastic waves in the radial symmetry in 2-D. *Discrete Contin. Dyn. Syst.* **36** (2016) 4051–4062.

[24] D. Zha, Space–time L^2 estimates for elastic waves and applications. *J. Differ. Equ.* **263** (2017) 1947–1965.

[25] D. Zha, On nonlinear elastic waves in 2-D. *J. Differ. Equ.* **268** (2020) 1250–1269.

[26] D. Zha and K. Hidano, Global solutions to systems of quasilinear wave equations with low regularity data and applications. *J. Math. Pures. Appl.* **142** (2020) 146–183.