ESAIM: COCV 26 (2020) 123
https://doi.org/10.1051/cocv/2020031

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

QUASISTATIC EVOLUTION FOR DISLOCATION-FREE FINITE PLASTICITY

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Abstract. We investigate quasistatic evolution in finite plasticity under the assumption that the plastic strain is compatible. This assumption is well-suited to describe the special case of dislocation-free plasticity and entails that the plastic strain is the gradient of a plastic deformation map. The total deformation can be then seen as the composition of a plastic and an elastic deformation. This opens the way to an existence theory for the quasistatic evolution problem featuring both Lagrangian and Eulerian variables. A remarkable trait of the result is that it does not require second-order gradients.

Mathematics Subject Classification. 35Q74, 49J40, 74C15.

Received October 8, 2019. Accepted May 20, 2020.

1. Introduction

The elastoplastic behavior of a crystalline solid under the action of external loads results from a combination of reversible elastic and irreversible plastic effects [44]. The state of the body is specified in terms of its deformation $y : \Omega \to \mathbb{R}^3$ from a reference configuration $\Omega \subset \mathbb{R}^3$. Elastic and plastic effects are classically assumed to combine *via* the *Kröner-Lee-Liu multiplicative decomposition* of the total strain $\nabla y = F_{\rm e} F_{\rm p}$ [37, 39, 40]. Here, the elastic strain $F_{\rm e}$ describes the elastic response of the medium, whereas the plastic strain $F_{\rm p}$ records the accumulation of plastic distortion [31]. In metals, it is usually assumed that plastic effects induce no volume change, namely $\det F_{\rm p} = 1$ [61].

Elastoplastic evolution results from the competition of elastic-energy storage and plastic-dissipation mechanisms. As such, a common and successful approach to the description of elastoplasticity of crystalline materials is *via* variational methods [55]. The energy of the specimen is often assumed to be of the form

$$\int_\Omega W_{\rm e}(\nabla y F_{\rm p}^{-1})\,{\rm d}x + \int_\Omega W_{\rm p}(F_{\rm p})\,{\rm d}x, \tag{1.1}$$

where $W_{\rm e}$ is the elastic-energy density, a function of the elastic strain $F_{\rm e} = \nabla y F_{\rm p}^{-1}$, and $W_{\rm p}$ is a hardening-energy density. In the *incremental* setting of the elastoplastic evolution problem, given external loads and boundary conditions, one minimizes the energy, augmented by a dissipation term $\mathcal{D}(F_{\rm p0}, F_{\rm p})$ [47]. The latter measures the

Keywords and phrases: Elasticity, plasticity, quasistatic evolution.

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distance of the actual plastic strain F_p from the previous $F_{\mathrm{p}0}$. This inspires different solution notions on the time-continuous, *quasistatic* evolution level [51].

In view of the mathematical treatment of finite plasticity, one is hence confronted with the necessity of controlling the product $\nabla y F_\mathrm{p}^{-1}$. This is indeed a critical point, for weak topologies are not sufficient in order to identify this product within a corresponding limit passage. Such observation has sparked the interest for so-called *second-order* theories, where a term featuring the gradient ∇F_p is included in the energy. This gradient term models nonlocal effects caused by short-range interactions among dislocations [16, 29, 30]. From a mathematical standpoint, the presence of the gradient ∇F_p in the energy contributes strong compactness for F_p, which then allows to pass to the limit in the product $\nabla y F_\mathrm{p}^{-1}$.

To date, multidimensional existence results for incremental and quasistatic evolutions are just a few and all hinge on second-order theories [15, 25, 26, 41, 50, 52]. However, in finite plasticity these second-order gradient theories are still debated from the modeling standpoint. In particular, it is not clear which function of the gradient should be used. We refer to [1, 36, 67] for attempts to derive it from statistical physics, revealing the complexity of this issue. A related approach to nonlocal models in damage and plasticity was undertaken in [7], see also [18, 19, 22, 44].

Our aim is to investigate existence for quasistatic evolutions not relying on second-order theories, namely in absence of a regularizing gradient term ∇F_p. This follows the analysis of [64], where the same issue was considered at the incremental level. The price to pay for allowing such an existence result is that of restricting the analysis to the case of *compatible* plastic strains F_p, namely to impose $\operatorname{curl} F_\mathrm{p} = 0$. This case corresponds to *dislocation-free* elastoplastic evolution. Albeit not generic, such situation still includes plastic slips [56] and may actually occur in ductile metals [34, 43]. This is particularly relevant in case of small bodies. Indeed, dislocation dynamics is strongly size-dependent [28, 66] so that very small dislocation-free bodies may plasticize without nucleating dislocations.

In case of compatibility, one can identify the plastic strain F_p with a gradient of a *plastic deformation* $y_\mathrm{p} : \Omega \to y_\mathrm{p}(\Omega) \subset \mathbb{R}^3$, mapping indeed the reference configuration to the so-called *intermediate* one. At the same time, this defines an *elastic deformation* $y_\mathrm{e} : y_\mathrm{p}(\Omega) \to \mathbb{R}^3$ from the intermediate to the actual configuration such that the decomposition

$$y = y_\mathrm{e} \circ y_\mathrm{p} \tag{1.2}$$

holds. The latter of course entails the multiplicative decomposition $\nabla y = \nabla y_\mathrm{e} \nabla y_\mathrm{p}$ *via* the classical chain rule. On the other hand, by assuming y_p to be injective, it allows for rewriting the energy in (1.1), by a change of variables, as

$$\int_{y_\mathrm{p}(\Omega)} W_\mathrm{e}(\nabla y_\mathrm{e})\,\mathrm{d}\xi + \int_\Omega W_\mathrm{p}(\nabla y_\mathrm{p})\,\mathrm{d}x. \tag{1.3}$$

This reformulation of the energy is particularly advantageous from the mathematical viewpoint, for it does not feature the product term $\nabla y F_\mathrm{p}^{-1}$ anymore. This in turn allows for an existence theory *via* classical variational methods, even in absence of strong compactness for $F_\mathrm{p} = \nabla y_\mathrm{p}$. Indeed, in two space dimensions, by assuming $F_\mathrm{p} = \nabla y_\mathrm{p}$ one would even be able to directly identify the limit in $\nabla y (\nabla y_\mathrm{p})^{-1}$ *via* the classical div-curl lemma as ∇y is curl-free and $\operatorname{div} (\nabla y_\mathrm{p})^{-T} = 0$ if $\det \nabla y_\mathrm{p} = 1$, see also [11].

Arguing *via* reformulation (1.3) calls for the treatment of both *Lagrangian* and *Eulerian* terms, respectively defined on the reference and on the intermediate configuration, which itself depends on part of the solution. This kind of mixed Lagrangian-Eulerian problems has to be traced back at least to [13, 21] for the case of defective crystals and has recently attracted attention in connection with nematic elastomers [5, 6], magnetoelasticity [6, 38, 59, 60, 63], solid-solid phase change [27, 62] and, as already mentioned, incremental finite plasticity [64]. A decomposition of type (1.2) has recently also been used as a starting point to model dissolution-precipitation creep [35].

The main result of this paper is the existence of *incrementally approximable quasistatic evolutions*, see Theorem 2.4. These time-continuous solutions are assumed to be limits of time-discrete trajectories, hence the term *incrementally approximable*. The reader is referred to [14] for another use of incrementally approximable solution, although in a different quasistatic setting. Solutions feature stability and energy balance on the time-discrete level as well as *semistability* relation with respect to elastic deformations and an *energy inequality* in the time-continuous limit. This solution notion is weaker than the concept of *energetic solutions* [51], featuring full stability and energy equality instead. Still, it implies the validity of the quasistatic equilibrium system as well as the dissipative character of the evolution, see the discussion in Section 2.7. In a different mechanical context, let us remark that semistability and energy inequality are also at the basis of the notion of weak solution for the strain-gradient viscoplasticity model in Definition 2.1 from [57].

The existence proof follows the classical time-discretization strategy. Discrete-in-time solutions are found by solving incremental problems on a given time partition and a quasistatic evolution is then recovered as the fineness of the partitions tends to zero. In order to check for the energy inequality, the lower semicontinuity of the energy and dissipation functionals plays a crucial role. This results from the *weak compactness* of the minors of $\nabla y_{\rm e}$ and $\nabla y_{\rm p}$ (see (1.3)) under the assumption of *polyconvex* densities [2]. The passage to the limit in the discrete semistability requires an ad hoc recovery-sequence construction, which in turn hinges upon the possibility of extending elastic deformations to a neighborhood of the intermediate configuration. In order to be able to achieve this, intermediate configurations are asked to have regular boundaries. More precisely, they are restricted to belong to a certain uniform subclass of Jones domains [33], see Section 2.3.

The mechanical model and its variational formulation are introduced in Section 2 and the main result is stated in Section 2.7. The existence proof is then detailed in Section 3.

2. Main result

This section brings us to the formulation of our main result, Theorem 2.4. We start by introducing our assumptions and basic framework in Sections 2.1–2.6 and end with our main statement in Section 2.7.

2.1. Notation

In what follows, we denote by $\mathbb{R}^{d\times d}$ the Euclidean space of $d\times d$ real matrices and by $\mathrm{SL}(d), \mathrm{GL}(d)$, and $\mathrm{SO}(d)$ its subspaces of matrices with unit determinant, invertible matrices, and proper rotations, respectively. Using $\mathcal{L}^d$ and $\mathcal{H}^{d-1}$ we refer to the d-dimensional Lebesgue measure and the $(d-1)$-dimensional Hausdorff measure. The norm on a generic Banach space E is denoted by $\|\cdot\|_E$ and we use the standard notation for Sobolev and Lebesgue spaces. By default, we denote by $f_n \to f$ strong convergence, whereas $f_n \rightharpoonup f$ means weak convergence.

2.2. Deformations

Let $d \geq 2$ and $\Omega \subset \mathbb{R}^d$ be a non-empty, open, simply connected, bounded domain with Lipschitz boundary. The boundary is essentially split into a Dirichlet part Γ_D and a Neumann part Γ_N, namely $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$ with Γ_D and Γ_N open in $\partial\Omega$ and $\Gamma_D \cap \Gamma_N = \varnothing$ where $\mathcal{H}^{d-1}(\Gamma_D) > 0$. We indicate by $y : \Omega \to \mathbb{R}^d$ the deformation of the body Ω.

The crucial assumption of our theory is that the deformation y can be decomposed into elastic and plastic deformations $y_{\rm e}$ and $y_{\rm p}$ as in (1.2). As mentioned, this follows from the standard multiplicative decomposition $\nabla y = F_{\rm e} F_{\rm p}$ in case $F_{\rm p}$ is curl-free. Indeed, if $F_{\rm p} = \nabla y_{\rm p}$ for some plastic deformation $y_{\rm p}$ one can easily check [64] that $F_{\rm e} = \nabla y_{\rm e}$ for some elastic deformation $y_{\rm e}$, so that the multiplicative decomposition $\nabla y = \nabla y_{\rm e} \nabla y_{\rm p}$ follows by (1.2) and the classical chain rule. We now detail our assumptions on $y_{\rm p}$ and $y_{\rm e}$.

Plastic deformations. We assume that the plastic deformation fulfills

$$y_{\rm p} \in W^{1,q_{\rm p}}(\Omega;\mathbb{R}^d) \quad \text{for some } q_{\rm p} > d(d-1)$$

and that it is locally volume preserving, namely, $\det \nabla y_{\rm p} = 1$ almost everywhere (a.e.) in Ω [42, 61]. This implies that $y_{\rm p}$ is Hölder continuous with exponent $1 - d/q_{\rm p}$ and almost everywhere differentiable ([20], Lem. 2.7). From now on, when writing $y_{\rm p}$ we always mean its continuous representative.

The map $y_{\rm p}$ possesses the so-called Lusin's N*-property*, namely $\mathcal{L}^d(E) = 0 \Rightarrow \mathcal{L}^d(y_{\rm p}(E)) = 0$ for all measurable $E \subset \mathbb{R}^d$, as well as the corresponding N^{-1}*-property*, *i.e.* $\mathcal{L}^d(E') = 0 \Rightarrow \mathcal{L}^d(y_{\rm p}^{-1}(E')) = 0$ for all measurable $E' \subset \mathbb{R}^d$, see page 296 in [24]. Moreover, $y_{\rm p}$ is *locally invertible almost everywhere* ([20], Thm. 3.1, Cor. 3.3). This means that for a.e. $x \in \Omega$ there exists a ball $B \subset \mathbb{R}^d$ centered at $y_{\rm p}(x)$, an open neighborhood $U \subset \Omega$ of x, and a local inverse $y_{\rm p}^{-1} : B \to U$ with $y_{\rm p}^{-1} \in W^{1,q_{\rm p}/(d-1)}(B;\mathbb{R}^d)$ such that $y_{\rm p}|_U$ and $y_{\rm p}^{-1}$ are onto, $y_{\rm p}^{-1} \circ y_{\rm p} = \mathrm{id}$ a.e. in U, $y_{\rm p} \circ y_{\rm p}^{-1} = \mathrm{id}$ a.e. in B, and $\nabla y_{\rm p}^{-1} = (\nabla y_{\rm p})^{-1} \circ y_{\rm p}^{-1}$ a.e. in B.

In view of changing from Lagrangian to Eulerian variables, we require $y_{\rm p}$ to be *injective almost everywhere*, namely, that there exists a negligible set N such that $y_{\rm p}$ is injective on $\Omega \setminus N$. This property is implemented by imposing the classical *Ciarlet-Nečas condition* [10]

$$\mathcal{L}^d(\Omega) = \int_\Omega \det(\nabla y_{\rm p}(x))\,\mathrm{d}x \le \mathcal{L}^d(y_{\rm p}(\Omega)). \tag{2.1}$$

In this setting, (2.1) and injectivity almost everywhere are actually equivalent ([23], Prop. 15). Using injectivity almost everywhere, we get the change of variables formula

$$\int_E \phi(y_{\rm p}(x))\,\mathrm{d}x = \int_{y_{\rm p}(E)} \phi(\xi)\,\mathrm{d}\xi \tag{2.2}$$

for every measurable function $\phi : \Omega \to \mathbb{R}^d$ and all measurable $E \subset \mathbb{R}^d$, see Lemma 2.4 in [20]. Note that, here and in the following, we use the shorthand $\mathrm{d}x$ for $\mathrm{d}\mathcal{L}^d(x)$ when integrating with respect to Lagrangian coordinates $x \in \Omega$, and $\mathrm{d}\xi$ for $\mathrm{d}\mathcal{L}^d(\xi)$ in case of Eulerian coordinates, namely for integration on the intermediate configuration $y_{\rm p}(\Omega)$.

If $y_{\rm p} \in W^{1,d}(\Omega;\mathbb{R}^d)$ with distortion $K := |\nabla y_{\rm p}|^d / \det \nabla y_{\rm p} \in L^p(\Omega;\mathbb{R})$ for $p > d-1$, then $y_{\rm p}$ is either constant or open ([32], Thm. 3.4). Since in our setting $q_{\rm p} > d(d-1)$ and $\det \nabla y_{\rm p} = 1$, this integrability requirement is exactly fulfilled. Moreover, by the Ciarlet-Nečas condition (2.1), $y_{\rm p}$ cannot be constant, which shows that $y_{\rm p}$ is open and injective almost everywhere. This implies that $y_{\rm p}$ is (globally) injective ([27], Lem. 3.3), and that $y_{\rm p}$ is actually a *homeomorphism* having inverse $y_{\rm p}^{-1}$ of regularity

$$y_{\rm p}^{-1} \in W^{1,q_{\rm p}/(d-1)}(y_{\rm p}(\Omega);\mathbb{R}^d).$$

Note that, if the plastic deformation $y_{\rm p}$ at the boundary $\partial\Omega$ was coinciding with that of a homeomorphism on $\overline{\Omega}$, given the integrability of the distortion one could resort to the invertibility theory by Ball [3] to deduce that $y_{\rm p}$ is actually a homeomorphism, even without asking for the Ciarlet-Nečas condition (2.1). In our case however, we cannot assume to be able to prescribe $y_{\rm p}(\partial\Omega)$, for $y_{\rm p}$ is an internal variable. In fact, since the problem is formulated in terms of $\nabla y_{\rm p}$ only, we later ask for the normalization condition $\int_\Omega y_{\rm p}(x)\mathrm{d}x = 0$.

Elastic deformations. Given a plastic deformation $y_{\rm p}$, we assume the elastic deformation $y_{\rm e}$, defined on the intermediate configuration $y_{\rm p}(\Omega)$, to satisfy

$$y_{\rm e} \in W^{1,q_{\rm e}}(y_{\rm p}(\Omega);\mathbb{R}^d) \quad \text{for some } q_{\rm e} > d.$$

By using the local invertibility of $y_{\rm p}$, one checks that the chain rule

$$\nabla y(x) = \nabla y_{\rm e}(y_{\rm p}(x))\,\nabla y_{\rm p}(x) \tag{2.3}$$

holds for almost every $x \in \Omega$, see [64] for details. We can use the change of variables formula (2.2) together with the chain rule (2.3) and Hölder's inequality to estimate

$$\int_\Omega |\nabla y(x)|^q \,\mathrm{d}x \le \left(\int_{y_\mathrm{p}(\Omega)} |\nabla y_\mathrm{e}(\xi)|^{q_\mathrm{e}} \,\mathrm{d}\xi\right)^{q/q_\mathrm{e}} \left(\int_\Omega |\nabla y_\mathrm{p}(x)|^{q_\mathrm{p}} \,\mathrm{d}x\right)^{q/q_\mathrm{p}}, \tag{2.4}$$

where q is defined by

$$\frac{1}{q} = \frac{1}{q_\mathrm{e}} + \frac{1}{q_\mathrm{p}}.$$

2.3. Domains

In order to carry on our existence proof, some regularity for the intermediate configurations $y_\mathrm{p}(\Omega)$ is needed. Our goal is to find conditions under which $y_\mathrm{p}(\Omega)$ is a *Sobolev extension domain*. These are open subsets of $\mathbb{R}^d$ allowing the extension of Sobolev functions to the whole space. More precisely, $\omega \subset \mathbb{R}^d$ is a $W^{1,p}$*-extension domain*, if and only if one can define a bounded linear operator

$$E : W^{1,p}(\omega; \mathbb{R}^d) \to W^{1,p}(\mathbb{R}^d; \mathbb{R}^d)$$

such that

$$Eu = u \quad \text{in } \omega$$

for every $u \in W^{1,p}(\omega; \mathbb{R}^d)$. Additionally, we need to ensure that the class of intermediate domains is closed under Hausdorff convergence of sets, in order to guarantee that the state space is closed. The Hausdorff distance of two non-empty, compact subsets X, Y of $\mathbb{R}^d$ is defined as

$$d_\mathrm{H}(X, Y) := \inf\{\nu \ge 0 : X \subset B_\nu(Y), Y \subset B_\nu(X)\},$$

where $B_\nu(X) := \{z \in \mathbb{R}^d : \text{ there exists } x \in X \text{ such that } |x - z| < \nu\} = X + B_\nu(0)$ is an ν-fattening of the set X. It is easy to see that

$$d_\mathrm{H}(X, Y) = \max\left\{\sup_{x\in X} \operatorname{dist}(x, Y), \ \sup_{y\in Y} \operatorname{dist}(y, X)\right\},$$

where $\operatorname{dist}(x, Y) := \inf_{y\in Y} |x - y|$. We remark that, if $y_\mathrm{p}^n \rightharpoonup y_\mathrm{p}$ in $W^{1,q_\mathrm{p}}(\Omega; \mathbb{R}^d)$, then y_p^n converges uniformly to y_p on $\overline{\Omega}$ by the compact Sobolev embedding $W^{1,q_\mathrm{p}}(\Omega; \mathbb{R}^d) \subset\subset C^0(\overline{\Omega}; \mathbb{R}^d)$. This implies Hausdorff convergence of the intermediate configurations, namely

$$d_\mathrm{H}(\overline{y_\mathrm{p}^n(\Omega)}, \overline{y_\mathrm{p}(\Omega)}) \to 0, \quad d_\mathrm{H}(\partial y_\mathrm{p}^n(\Omega), \partial y_\mathrm{p}(\Omega)) \to 0 \tag{2.5}$$

as n tends to ∞, see Lemma 3.1 below.

It is well-known that Lipschitz domains are $W^{1,p}$-extension domains for every $1 \le p \le \infty$ [9, 65]. However, the class of Lipschitz domains is not closed under Hausdorff convergence. We hence focus here on a larger class of domains, the (ε, δ)*-domains* introduced by Jones in [33] and defined below. These possess the extension property ([33], Thm. 1) and include Lipschitz domains. By restricting to the subclass of uniform (ε, δ)-domains, we obtain uniformly bounded extension operators as well as closedness with respect to Hausdorff convergence.

Definition 2.1 ($\mathcal{J}_{\varepsilon,\delta}$ domains)**.** We say that a bounded, open set $\omega \subset \mathbb{R}^d$ is an (ε,δ)-domain, denoted $\omega \in \mathcal{J}_{\varepsilon,\delta}$, if for every $x, y \in \omega$ with $|x-y| < \delta$ there exists a Lipschitz curve $\gamma \in W^{1,\infty}([0,1];\omega)$ with $\gamma(0) = x$, and $\gamma(1) = y$ satisfying the following two conditions:

$$\ell(\gamma) := \int_0^1 |\dot\gamma(s)| \, \mathrm{d}s \le \frac{1}{\varepsilon}|x-y| \tag{2.6}$$

and

$$\operatorname{dist}(\gamma(t), \partial\omega) \ge \varepsilon \frac{|x-\gamma(t)||\gamma(t)-y|}{|x-y|} \quad \forall t \in [0,1]. \tag{2.7}$$

One can immediately see that these classes of domains are nicely ordered in the sense that if ω is an (ε', δ')-domain for some $\varepsilon' \ge \varepsilon$ and $\delta' \ge \delta$, then ω is also an (ε,δ)-domain. More precisely,

$$\mathcal{J}_{\varepsilon,\delta} = \bigcup_{\varepsilon' \ge \varepsilon, \delta' \ge \delta} \mathcal{J}_{\varepsilon',\delta'}.$$

2.4. States

Let $\varepsilon, \delta > 0$. We define the set of admissible states as

$$\mathcal{Q} := \Big\{ (y_\mathrm{e}, y_\mathrm{p}) \in W^{1,q_\mathrm{e}}(y_\mathrm{p}(\Omega);\mathbb{R}^d) \times W^{1,q_\mathrm{p}}(\Omega;\mathbb{R}^d) : \\ y_\mathrm{p}(\Omega) \in \mathcal{J}_{\varepsilon,\delta}, \quad \int_\Omega y_\mathrm{p} \, \mathrm{d}x = 0, \quad \det \nabla y_\mathrm{p} = 1 \text{ a.e. in } \Omega, \quad \mathcal{L}^d(\Omega) \le \mathcal{L}^d(y_\mathrm{p}(\Omega)) \Big\}.$$

The state space $\mathcal{Q}$ is equipped with the weak topology of $W^{1,q_\mathrm{e}}_{\mathrm{loc}}(y_\mathrm{p}(\Omega);\mathbb{R}^d) \times W^{1,q_\mathrm{p}}(\Omega;\mathbb{R}^d)$. More precisely, we write that $(y_\mathrm{e}^n, y_\mathrm{p}^n)_{n\in\mathbb{N}} \subset \mathcal{Q}$ *converges to* $(y_\mathrm{e}, y_\mathrm{p})$ *in* $\mathcal{Q}$, if

$$\begin{aligned} &y_\mathrm{p}^n \rightharpoonup y_\mathrm{p} \text{ in } W^{1,q_\mathrm{p}}(\Omega;\mathbb{R}^d), \\ &y_\mathrm{e}^n \rightharpoonup y_\mathrm{e} \text{ in } W^{1,q_\mathrm{e}}(K;\mathbb{R}^d) \text{ for every } K \subset\subset y_\mathrm{p}(\Omega). \end{aligned}$$

Note that, since $W^{1,q_\mathrm{p}}(\Omega;\mathbb{R}^d) \subset\subset C^0(\overline{\Omega};\mathbb{R}^d)$, for every $K \subset\subset y_\mathrm{p}(\Omega)$ there exists $n_K \in \mathbb{N}$ such that $K \subset y_\mathrm{p}^n(\Omega)$ for all $n \ge n_K$. In Section 3.1 below, we prove (sequential) closedness of $\mathcal{Q}$ under this convergence. The constraint $y_\mathrm{p}(\Omega) \in \mathcal{J}_{\varepsilon,\delta}$ is global in nature and is expected to be not restrictive in most practical cases.

2.5. Energy

The stored energy corresponding to the state $(y_\mathrm{e}, y_\mathrm{p}) \in \mathcal{Q}$ consists of three parts: an elastic energy, which is defined on the intermediate configuration $y_\mathrm{p}(\Omega)$ and depends on the elastic strain ∇y_e, a kinematic hardening energy, depending solely on the plastic strain ∇y_p, and a soft elastic boundary condition defined on the Dirichlet boundary Γ_D. More precisely, the stored energy of the system reads

$$\mathcal{W}(y_\mathrm{e}, y_\mathrm{p}) = \int_{y_\mathrm{p}(\Omega)} W_\mathrm{e}(\nabla y_\mathrm{e}(\xi)) \, \mathrm{d}\xi + \int_\Omega W_\mathrm{p}(\nabla y_\mathrm{p}(x)) \, \mathrm{d}x + \int_{\Gamma_D} |y_\mathrm{e}(y_\mathrm{p}(x)) - x| \, \mathrm{d}\mathcal{H}^{d-1}(x).$$

Our motivation for choosing the soft elastic boundary condition above is twofold. On the one hand, we believe soft elastic boundary conditions to be often more realistic from the modeling point of view than the classical *hard device* one $y_\mathrm{e}(y_\mathrm{p}(x)) = x$. On the other hand, our analysis relies on a construction of a suitable recovery sequence for deformations. In case of hard-device boundary conditions, the recovery sequence would hence have to satisfy these as an additional constraint, a possibility that we cannot cover at the moment.

The system is driven by a time-dependent body force $f : [0,T] \times \Omega \to \mathbb{R}^d$ and a boundary traction $g : [0,T] \times \Gamma_N \to \mathbb{R}^d$ (provided $\Gamma_N \neq \varnothing$) which result in an external loading ℓ, and loading energy defined as

$$\langle \ell(t), y \rangle = \int_\Omega f(t,x) \cdot y(x) \,\mathrm{d}x + \int_{\Gamma_N} g(t,x) \cdot y(x) \,\mathrm{d}\mathcal{H}^{d-1}(x).$$

The total energy of the system is then given by

$$\mathcal{E}(t, y_\mathrm{e}, y_\mathrm{p}) = \mathcal{W}(y_\mathrm{e}, y_\mathrm{p}) - \langle \ell(t), y_\mathrm{e} \circ y_\mathrm{p} \rangle.$$

We assume the elastic energy to have q_e-*growth* and the plastic energy density to be *coercive*, *i.e.*

$$c|F_\mathrm{e}|^{q_\mathrm{e}} - \frac{1}{c} \le W_\mathrm{e}(F_\mathrm{e}) \le \frac{1}{c}(1 + |F_\mathrm{e}|^{q_\mathrm{e}}), \tag{2.8a}$$

$$c|F_\mathrm{p}|^{q_\mathrm{p}} - \frac{1}{c} \le W_\mathrm{p}(F_\mathrm{p}) \tag{2.8b}$$

for some constant $c > 0$ and every $F_\mathrm{e} \in \mathbb{R}^{d\times d}$, $F_\mathrm{p} \in \mathrm{SL}(d)$. This is combined with the structural assumption of *polyconvexity*, namely

$$W_\mathrm{e}(F_\mathrm{e}) = \widehat{W}_\mathrm{e}(F_\mathrm{e}, \operatorname{cof} F_\mathrm{e}, \det F_\mathrm{e}), \tag{2.9a}$$

$$W_\mathrm{p}(F_\mathrm{p}) = \widehat{W}_\mathrm{p}(F_\mathrm{p}, \operatorname{cof} F_\mathrm{p}) \tag{2.9b}$$

where $\widehat{W}_\mathrm{e} : \mathbb{R}^{d\times d} \times \mathbb{R}^{d\times d} \times \mathbb{R} \to \mathbb{R}$ and $\widehat{W}_\mathrm{p} : \mathbb{R}^{d\times d} \times \mathbb{R}^{d\times d} \to \mathbb{R}$ are convex. We remark that the notation corresponds to space dimension $d = 3$, as the minors of a matrix are then given by determinant, cofactor, and the matrix itself. For $d = 2$ the dependence on the cofactor matrix could be dropped and in dimensions $d > 3$ the definition of polyconvexity could be generalized by including further minors. Although not directly needed for the analysis, we may assume the energy to be frame-indifferent. This corresponds to asking the elastic energy density to satisfy the assumption $W_\mathrm{e}(RF_\mathrm{e}) = W_\mathrm{e}(F_\mathrm{e})$ for all $R \in \mathrm{SO}(d)$, $F_\mathrm{e} \in \mathbb{R}^{d\times d}$. We further assume

$$f \in W^{1,1}(0,T; L^{(q^*)'}(\Omega;\mathbb{R}^d)), \quad g \in W^{1,1}(0,T; L^{(q^\#)'}(\Gamma_N;\mathbb{R}^d)),$$

where q^* and $q^\#$ denote the *Sobolev exponent* and the *trace exponent*, respectively, and prime stands for conjugation [58]. Let us remark that the assumptions on q_e and q_p contribute the following lower bounds on the mentioned exponents:

$$q > d-1, \qquad q^* > d(d-1), \qquad q^\# > (d-1)^2.$$

These assumptions ensure that the loading is absolutely continuous in time, namely,

$$\ell \in W^{1,1}(0,T; (W^{1,q}(\Omega;\mathbb{R}^d))^*) \tag{2.10}$$

where $*$ denotes the dual space.

Remark 2.2. *Locking materials* may also be considered. These materials are characterized by a tolerance $M > 0$ and internal energy defined as above if $\|\nabla y_\mathrm{p}\|_{L^\infty(\Omega;\mathbb{R}^d)} \le M$, and $\mathcal{W}(y_\mathrm{e}, y_\mathrm{p}) = \infty$ otherwise. This would force the plastic deformations to be (uniformly) Lipschitz continuous.

2.6. Dissipation

We follow the by-now classical approach by Mielke [46–48] and endow $\mathrm{SL}(d)$ with a Finsler-type metric, in order to quantify distances between different plastic states. Indeed, we define the (local) *dissipation distance* $\Delta : (\mathrm{SL}(d))^2 \to [0,\infty]$ as

$$\Delta(F_{\mathrm{p}0}, F_{\mathrm{p}1}) = \inf\left\{ \int_0^1 R(P(t),\dot P(t))\,\mathrm{d}t \;:\; P \in C^1([0,1];\mathrm{SL}(d)),\;\; P(i) = F_{\mathrm{p}i},\;\text{ for } i=0,1\right\},$$

where the *dissipation potential*

$$R : \mathrm{SL}(d) \times \mathbb{R}^{d\times d} \to [0,\infty],$$

is convex and positively 1-homogeneous in the rate, namely,

$$R(P,\lambda \dot P) = \lambda R(P,\dot P) \quad \text{ for all } \lambda \ge 0,$$

and satisfies the *plastic indifference* assumption

$$R(PQ,\dot P Q) = R(P,\dot P) \quad \text{ for all } Q \in \mathrm{SL}(d).$$

The latter corresponds to asking that dissipation mechanisms are independent from *prior* plastic deformations. The latter are here modeled by a given $Q \in \mathrm{SL}(d)$ representing a plastic deformation occurred before P, hence to be multiplied to P and $\dot P$ from the *right*, see also [41, 51, 52, 54].

These properties imply that there exists a convex, positively 1-homogeneous function $\widehat R : \mathbb{R}^{d\times d} \to [0,\infty]$ such that

$$R(P,\dot P) = \widehat R(\dot P P^{-1}),$$

see [47] or Section 4.2.1.1 in [51] and that Δ satisfies the *triangle inequality*

$$\Delta(F_{\mathrm{p}0}, F_{\mathrm{p}2}) \le \Delta(F_{\mathrm{p}0}, F_{\mathrm{p}1}) + \Delta(F_{\mathrm{p}1}, F_{\mathrm{p}2}),$$

as well as

$$\Delta(F_{\mathrm{p}0}, F_{\mathrm{p}1}) = \Delta(I, F_{\mathrm{p}1}F_{\mathrm{p}0}^{-1})$$

for all $F_{\mathrm{p}i} \in \mathrm{SL}(d)$, $i = 0,1,2$, where I is the identity matrix.

We assume the function $D : \mathrm{SL}(d) \to [0,\infty]$ defined as

$$D(F_{\mathrm{p}}) := \Delta(I, F_{\mathrm{p}})$$

to be *polyconvex*. Namely, we suppose that there exists a convex function $\widehat D : \mathbb{R}^{d\times d} \times \mathbb{R}^{d\times d} \to [0,\infty]$ such that

$$D(F_{\mathrm{p}}) = \widehat D(F_{\mathrm{p}}, \operatorname{cof} F_{\mathrm{p}}). \tag{2.11}$$

Eventually, we define the (global) *dissipation distance* between plastic strain states $F_{\mathrm{p}0}, F_{\mathrm{p}1} : \Omega \to \mathrm{SL}(d)$ as

$$\mathcal{D}(F_{\mathrm{p}0}, F_{\mathrm{p}1}) = \int_\Omega D(F_{\mathrm{p}1}(x)(F_{\mathrm{p}0}(x))^{-1})\,\mathrm{d}x$$

and the *total dissipation* of a plastic evolution $y_\mathrm{p} : [0,T] \to \mathcal{Q}$ from s to t as

$$\mathrm{Diss}_\mathcal{D}(\nabla y_\mathrm{p}; s,t) = \sup\left\{ \sum_{j=1}^N \mathcal{D}(\nabla y_\mathrm{p}(t_{i-1}), \nabla y_\mathrm{p}(t_i)) : s = t_0 < \cdots < t_N = t,\ N \in \mathbb{N} \right\}.$$

Before closing this subsection, let us collect some remarks on the above setting.

To actually compute D or Δ from R is a delicate task and explicit forms are available in von-Mises-type settings only [46, 47]. A complete characterization of polyconvex functions D is presently available in the case of 2d isotropic hardening only, see [49]. In 3d, a family of functions is conjectured to be polyconvex in Remark 4.2 in [50].

Note that, in case $\widehat{R}$ is bounded by the norm, namely if there exists $c > 0$ such that $\widehat{R}(T) \le c|T|$ for all $T \in \mathbb{R}^{d\times d}$, the polyconvex function D turns out to be linearly bounded from above ([45], Prop. 2.2). Hence, Corrollary 5.9, p. 169 in [12] implies that D is actually convex.

An example with D convex is the case of *single-slip crystal plasticity* [29]. Here, the plastic-strain rate $\dot{P}P^{-1}$ is imposed to have the form

$$\dot{P}P^{-1} = \dot{\gamma} e_1 \otimes e_2 \tag{2.12}$$

with given orthogonal unit vectors e_1, e_2 describing the glide direction and normal of the slip plane, and slip rate $\dot{\gamma}$. Taking $\widehat{R}(\cdot) = \kappa|\cdot|$, we compute

$$D(F_\mathrm{p}) = \begin{cases} \kappa|\gamma|, & \text{if } F_\mathrm{p} = I + \gamma e_1 \otimes e_2, \\ \infty, & \text{else,} \end{cases}$$

where $\kappa > 0$ is the plastic yield.

Eventually, we would like to mention that some of the requirements on D could be weakened at the expense of weaker results. In particular, the Finsler structure in Δ can be dispensed of, and, still retaining plastic indifference, one could directly define a pseudodistance between plastic states as

$$\Delta(F_{\mathrm{p}0}, F_{\mathrm{p}1}) = \widehat{R}(F_{\mathrm{p}1}F_{\mathrm{p}0}^{-1} - I).$$

This gives a convex function $F_\mathrm{p} \mapsto \Delta(I, F_\mathrm{p})$ which still allows to solve the discrete incremental problem, obtain estimates, and pass to limits. Discrete stability would however not follow from minimality and one would have to reduce to semistability, also at the discrete level. The limiting time-continuous trajectory would still be semistable and fulfill the energy inequality, see Definition 2.3 below.

2.7. Main results

Let a partition $\Pi = \{0 = t_0 < t_1 < \cdots < t_N = T\},\ N \in \mathbb{N}$ and an initial condition $(y_{\mathrm{e}0}, y_{\mathrm{p}0}) \in \mathcal{Q}$ be given and let $(y_{\mathrm{e}i}, y_{\mathrm{p}i}) \in \mathcal{Q}$, $i = 1, \dots, N$ solve the incremental minimization problem

$$(y_{\mathrm{e}i}, y_{\mathrm{p}i}) \in \underset{(y_\mathrm{e}, y_\mathrm{p})\in\mathcal{Q}}{\operatorname{argmin}} \Big(\mathcal{E}(t_i, y_\mathrm{e}, y_\mathrm{p}) + \mathcal{D}(\nabla y_{\mathrm{p}(i-1)}, \nabla y_\mathrm{p}) \Big). \tag{2.13}$$

Define the right-continuous, piecewise-constant interpolant

$$\begin{aligned} &(y_\mathrm{e}, y_\mathrm{p})(t) = (y_{\mathrm{e}(i-1)}, y_{\mathrm{p}(i-1)}) \quad \text{for } t \in [t_{i-1}, t_i), \qquad i = 1, \dots, N, \\ &(y_\mathrm{e}, y_\mathrm{p})(T) = (y_{\mathrm{e}N}, y_{\mathrm{p}N}) \end{aligned} \tag{2.14}$$

and set

$$y(t) = y_e(t) \circ y_p(t).$$

We refer to any such interpolation $(y_e, y_p) : [0, T] \to \mathcal{Q}$ as to an *incremental solution*. This solution depends on the choice of minimizers in (2.13) and on the partition Π.

By *formally* deriving first-order optimality conditions for (2.13), one obtains the elastic equilibrium and a discrete flow rule for y_p, similarly as in (2.16) in [64]. Relating the minimization to such first-order conditions is however far from being immediate, see the remarks in [4] in the mere elasticity setting. In Theorem 2.4 below, we prove that incremental solutions converge as the time-step size tends to zero to semistable dissipative trajectories satisfying the energy inequality. In order to specify this, we give the following definition inspired by Definition 2.12 fom [14]. The definition features the map $t \mapsto \delta(t)$, playing the role of the dissipated energy in the time interval $[0, t]$.

Definition 2.3 (Incrementally approximable solutions)**.** We call $(y_e, y_p, \delta) : [0, T] \to \mathcal{Q} \times [0, \infty)$ an *incrementally approximable quasistatic evolution* if there exists a sequence of partitions $(\Pi_n)_{n\in\mathbb{N}}$ with fineness $\max_{i=1,\dots,N(n)}(t_i^n - t_{i-1}^n)$ tending to 0 as n goes to ∞ and a corresponding sequence of incremental solutions $(y_e^n, y_p^n)_{n\in\mathbb{N}} \subset \mathcal{Q}$, such that, up to not relabeled subsequences

– (Convergences of the plastic strain) for every $s, t \in [0, T]$

$$y_p^n(t) \rightharpoonup y_p(t) \text{ in } W^{1,q_p}(\Omega), \tag{2.15a}$$

$$\mathrm{Diss}_{\mathcal{D}}(\nabla y_p^n; 0, t) \to \delta(t), \tag{2.15b}$$

$$\mathrm{Diss}_{\mathcal{D}}(\nabla y_p; s, t) \le \delta(t) - \delta(s); \tag{2.15c}$$

– (Convergences of the states) for every $t \in [0, T]$ there exists a t-dependent subsequence n_k^t such that

$$(y_e^{n_k^t}(t), y_p^{n_k^t}(t)) \to (y_e(t), y_p(t)) \ \text{ in } \mathcal{Q};$$

– (Discrete stability) for all $t \in [0, T]$,

$$\mathcal{E}(t, y_e^n(t), y_p^n(t)) \le \mathcal{E}(t, \widehat{y}_e, \widehat{y}_p) + \mathcal{D}(\nabla y_p^n(t), \nabla \widehat{y}_p) \quad \text{for all } (\widehat{y}_e, \widehat{y}_p) \in \mathcal{Q}; \tag{S$_{\mathrm{discr}}$}$$

– (Discrete energy inequality) for every $s, t \in \Pi_n$, $s \le t$,

$$\begin{aligned}\mathcal{E}(t, y_e^n(t), y_p^n(t) - \mathcal{E}(s, y_e^n(s), y_p^n(s)) + \mathrm{Diss}_{\mathcal{D}}(\nabla y_p^n; s, t)\\ \le -\int_s^t \langle \dot{\ell}(r), (y_e \circ y_p)(r) \rangle \, \mathrm{d}r;\end{aligned} \tag{E$_{\mathrm{discr}}$}$$

– (Semistability) for all $t \in [0, T]$,

$$\mathcal{E}(t, y_e(t), y_p(t)) \le \mathcal{E}(t, \widehat{y}_e, y_p(t)) \quad \text{for all } \widehat{y}_e \text{ with } (\widehat{y}_e, y_p(t)) \in \mathcal{Q}, \tag{S$_{\mathrm{semi}}$}$$

– (Energy inequality) for all $t \in [0, T]$,

$$\mathcal{E}(t, y_e(t), y_p(t)) + \delta(t) \le \mathcal{E}(0, y_e(0), y_p(0)) - \int_0^t \langle \dot{\ell}(s), (y_e \circ y_p)(s) \rangle \, \mathrm{d}s. \tag{E}$$

Theorem 2.4 (Existence of incrementally approximable solutions). *Let $\Omega \subset \mathbb{R}^d$ be as in Section 2.2. Let $q_{\rm e} > d$, $q_{\rm p} > d(d-1)$ and define $\mathcal{Q}$ as in Section 2.4, $\mathcal{E}$ as in Section 2.5, and $\mathcal{D}$ as in Section 2.6. Assume $W_{\rm e} : \mathbb{R}^{d\times d} \to \mathbb{R}, W_{\rm p} : {\rm SL}(d) \to \mathbb{R}$ and $D : {\rm SL}(d) \to [0,\infty]$ to be polyconvex, see (2.9) and (2.11). Moreover, let $W_{\rm e}$ satisfy the growth condition (2.8a) and $W_{\rm p}$ satisfy the coercivity bound (2.8b). Further, assume that ℓ fulfills regularity assumption (2.10). Let $(y_{\rm e0}, y_{\rm p0}) \in \mathcal{Q}$ be initial data satisfying the semistability condition at time 0, namely*

$$\mathcal{E}(0, y_{\rm e0}, y_{\rm p0}) \leq \mathcal{E}(0, \widehat{y}_{\rm e}, y_{\rm p0}) \quad \textit{for all } (\widehat{y}_{\rm e}, y_{\rm p0}) \in \mathcal{Q}.$$

Then, there exists an incrementally approximable quasistatic evolution $(y_{\rm e}, y_{\rm p}, \delta) : [0,T] \to \mathcal{Q} \times [0,\infty)$ with $(y_{\rm e}(0), y_{\rm p}(0)) = (y_{\rm e0}, y_{\rm p0})$.

Incrementally approximable quasistatic evolutions fulfill the *semistability* condition ($\mathrm{S}_{\rm semi}$) with respect to elastic deformations, as well as an *energy inequality* (E). These properties are close to the solution concept discussed in [57] in the context of viscoplasticity but considerably weaker than the classical notion of *energetic solutions* [51]. There, the trajectory is required to be stable with respect to *both* plastic and elastic deformation and energy equality holds. We refer to Chapter 3 in [51] for a detailed discussion about different solution concepts for rate-independent systems. As mentioned, the function δ in Theorem 2.4 plays the role of a limit dissipation. The property (2.15c) expresses the regularity of solutions in time and is a direct consequence of Helly's Selection Principle; see Theorem 2.1.24 in [51] for a similar result and (3.20c) below. In particular, the map δ is nondecreasing and bounded, hence the map $t \mapsto {\rm Diss}_{\mathcal{D}}(\nabla y_{\rm p}; 0, t)$ is of bounded variation by (2.15c).

Despite the limitations of the solution concept, the fact that incrementally approximable solutions are indeed limits of incremental solutions guarantees that plasticity actually occurs, whenever necessary. In particular, the purely elastic evolution $\nabla y_{\rm p}(t) = I$, which fulfills ($\mathrm{S}_{\rm semi}$) for compatible initial data, may fail to be incrementally approximable for loadings exceeding the plastic-activation threshold. Indeed, in this case one would obtain a strictly positive $\delta(t)$.

In order to give an elementary example demonstrating this fact, we present a simplified argument, by reducing to one space dimension and to a single material point. In this frame, by choosing energy densities to be quadratic, setting most constants to 1, and indicating by $\kappa \geq 1$ the activation stress, the incremental problem (2.13) can be recast in terms of the deformation strain $f \in \mathbb{R}$ and the plastic strain $p > 0$ (we neglect the isochoric constraint, since we are in one space dimension) as

$$(f_i, p_i) \in \underset{f\in\mathbb{R},\, p>0}{\rm argmin} \left(\frac{1}{2}|fp^{-1}|^2 + \frac{1}{2}p^2 - \ell(t_i)f + \kappa|\log p - \log p_{i-1}| \right) \quad \text{for } i = 1, \ldots, N$$

where the initial values (f_0, p_0) with $p_0 = 1$ and the loading $\ell(t) = \lambda t$, $\lambda > 0$, are given. In this pure-traction case, the incrementally approximable solution is unique and reads

$$f(t) = \begin{cases} \lambda t & \text{for } \ 0 \leq t < (1+\kappa)/\lambda, \\ \lambda^2 t^2 - \lambda t & \text{for } \ t \geq (1+\kappa)/\lambda, \end{cases}$$
$$p(t) = \begin{cases} 1 & \text{for } \ 0 \leq t < (1+\kappa)/\lambda, \\ \sqrt{\lambda t - 1} & \text{for } \ t \geq (1+\kappa)/\lambda. \end{cases}$$

For the specific reference choice $\kappa = \lambda = 1$, this solution is depicted in Figure 1 in the classical deformation stress-strain diagram. The response is elastic for $t < (1+\kappa)/\lambda = 2$. The onset of plasticization at stress level $\kappa = 1$, namely for $t = 2$, as well as the nonlinear character of the plastic model are clearly visible.

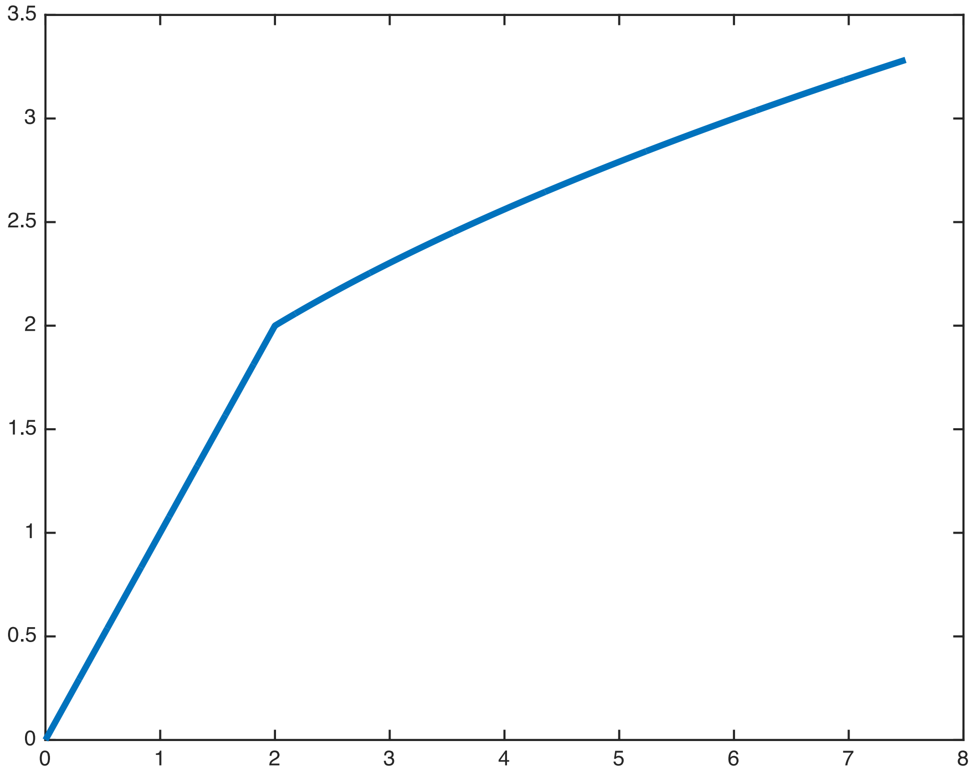

FIGURE 1. Stress-strain diagram for the one-dimensional traction test: loading $\ell(t)$ in terms of the deformation $f(t)$.

3. PROOFS

The proof of Theorem 2.4 is detailed along the whole section and consists of several parts. In Section 3.1 we discuss the closure of the state space $\mathcal{Q}$. In Section 3.2 we show the existence of incrementally approximable solutions. In Sections 3.3–3.4 the validity of energy inequality (E) and semistability ($\mathrm{S}_{\mathrm{semi}}$) is checked by passing the corresponding approximate properties of the incremental scheme to the limit.

3.1. Closedness of the state space

Consider a sequence $(y_{\mathrm{e}}^n, y_{\mathrm{p}}^n) \in \mathcal{Q}$ converging to $(y_{\mathrm{e}}, y_{\mathrm{p}})$ in $\mathcal{Q}$ in the sense of Section 2.4. Then $y_{\mathrm{p}}^n \rightharpoonup y_{\mathrm{p}}$ in $W^{1,q_{\mathrm{p}}}(\Omega;\mathbb{R}^d)$ implies the weak convergence of $\det \nabla y_{\mathrm{p}}^n$ to $\det \nabla y_{\mathrm{p}}$ in $L^{q_{\mathrm{p}}/d}(\Omega;\mathbb{R})$. Hence $\det \nabla y_{\mathrm{p}} = 1$ almost everywhere and $\mathcal{L}^d(\Omega) \le \mathcal{L}^d(y_{\mathrm{p}}(\Omega))$ by [10]. Moreover, strong convergence in $L^{q_{\mathrm{p}}}(\Omega)$ implies $\int_\Omega y_{\mathrm{p}}(x)\,\mathrm{d}x = 0$. The fact that $y_{\mathrm{p}}(\Omega) \in \mathcal{J}_{\varepsilon,\delta}$ follows from Lemmas 3.1-3.2 below. In Lemma 3.1 we show the set convergence (2.5) by exploiting the fact that the plastic deformations are homeomorphisms, see Section 2.2. In Lemma 3.2 we show that $\mathcal{J}_{\varepsilon,\delta}$ is closed under this convergence.

Lemma 3.1 (Hausdorff convergence of intermediate configurations)**.** *Let $\Omega \subset \mathbb{R}^d$ be open and bounded. Let $y, y_n \in C^0(\overline{\Omega};\mathbb{R}^d)$, $n \in \mathbb{N}$, be such that y_n converges to y uniformly on $\overline{\Omega}$ and $y|_\Omega : \Omega \to y(\Omega)$, $y_n|_\Omega : \Omega \to y_n(\Omega)$ are homeomorphisms for every $n \in \mathbb{N}$. Then*

$$d_{\mathrm{H}}(\overline{y_n(\Omega)}, \overline{y(\Omega)}) \to 0, \quad d_{\mathrm{H}}(\partial y_n(\Omega), \partial y(\Omega)) \to 0$$

as n tends to ∞.

Proof of Lemma 3.1. By uniform convergence, we get that $d_{\mathrm{H}}(y_n(\overline{\Omega}), y(\overline{\Omega}))$ and $d_{\mathrm{H}}(y_n(\partial\Omega), y(\partial\Omega))$ tend to 0. One is left to show that $y(\overline{\Omega}) = \overline{y(\Omega)}$ and $y(\partial\Omega) = \partial y(\Omega)$.

Ad $y(\overline{\Omega}) = \overline{y(\Omega)}$: we observe that, by continuity of y and compactness of $\overline{\Omega}$, the set $y(\overline{\Omega})$ is closed. Therefore $\overline{y(\Omega)} \subset y(\overline{\Omega})$. In order to check the opposite inclusion, let $z \in y(\overline{\Omega})$ and choose $x \in y^{-1}(z) \subset \overline{\Omega}$ and $x_n \in \Omega$ converging to x. Then, $y(x_n) \in y(\Omega)$ and by the continuity of y up to the boundary, $y(x_n)$ converges to $y(x)$, showing that $z = y(x) \in \overline{y(\Omega)}$.

Ad $y(\partial\Omega) = \partial y(\Omega)$: we use the fact that, as y is a homeomorphism, the set $y(\Omega)$ is open. Thus, $\partial y(\Omega) = y(\overline{\Omega}) \setminus y(\Omega)$. We claim that $y(\overline{\Omega}) \setminus y(\Omega) = y(\partial\Omega)$. Indeed, if $z \in y(\Omega)$, then there exists an open neighborhood U of z such that $U \subset y(\Omega)$. Since y is a homeomorphism, $V = y^{-1}(U)$ is an open neighborhood of $y^{-1}(z)$ such that $V \subset \Omega$, implying that $y^{-1}(z) \notin \partial\Omega$, *i.e.* $z \notin y(\partial\Omega)$. This shows $y(\overline{\Omega}) \setminus y(\Omega) \supset y(\partial\Omega)$. On the other hand, $y(\overline{\Omega}) \setminus y(\Omega) \subset y(\overline{\Omega} \setminus \Omega) = y(\partial\Omega)$. This concludes the proof. □

Lemma 3.2 (Closedness of $\mathcal{J}_{\varepsilon,\delta}$ under Hausdorff convergence)**.** *Let $\omega_n \in \mathcal{J}_{\varepsilon,\delta}$ converge to ω in the sense that*

$$d_{\mathrm{H}}(\overline{\omega}, \overline{\omega}_n) \to 0 \tag{3.1}$$

and

$$d_{\mathrm{H}}(\partial\omega, \partial\omega_n) \to 0 \tag{3.2}$$

as n tends to ∞. Then $\omega \in \mathcal{J}_{\varepsilon,\delta}$.

Proof of Lemma 3.2. Let $x, y \in \omega$ with $|x - y| < \delta$. By convergence (3.1), for every $\nu > 0$ there exists $N_\nu \in \mathbb{N}$ such that for all $n \geq N_\nu$ we have $\omega \subset \overline{\omega} \subset B_\nu(\overline{\omega}_n) = B_\nu(\omega_n)$. Therefore, we can choose a (not relabeled) subsequence and $x_n, y_n \in \omega_n$ such that x_n and y_n converge to x and y, respectively, and $|x_n - y_n| < \delta$ for every $n \in \mathbb{N}$. We now use the assumption that $\omega_n \in \mathcal{J}_{\varepsilon,\delta}$ and find $\gamma_n \in W^{1,\infty}([0,1]; \omega_n)$ such that $\gamma_n(0) = x_n, \gamma_n(1) = y_n$,

$$\ell(\gamma_n) \leq \frac{1}{\varepsilon}|x_n - y_n| < \frac{\delta}{\varepsilon}, \tag{3.3}$$

and

$$\operatorname{dist}(\gamma_n(t), \partial\omega_n) \geq \varepsilon \frac{|x_n - \gamma_n(t)||\gamma_n(t) - y_n|}{|x_n - y_n|} \quad \forall t \in [0,1]. \tag{3.4}$$

Set $L := \delta/\varepsilon$. From condition (3.3) we see that $\sup_{n\in\mathbb{N}} \ell(\gamma_n) \leq L$. Now consider the parametrizations by arclength with constant extension at the endpoint denoted by $\tilde{\gamma}_n : [0, L] \to \omega_n$. By definition, these satisfy

$$|\dot{\tilde{\gamma}}_n(s)| = \begin{cases} 1, & \text{if } s \in [0, \ell(\gamma_n)], \\ 0, & \text{if } s \in (\ell(\gamma_n), L]. \end{cases}$$

We use the Arzelà-Ascoli Theorem to extract a (not relabeled) subsequence and find $\tilde{\gamma} \in W^{1,\infty}([0, L]; \overline{\omega})$ such that

$$\dot{\tilde{\gamma}}_n \overset{*}{\rightharpoonup} \dot{\tilde{\gamma}} \quad \text{in } L^\infty(0, L), \tag{3.5}$$

$$\tilde{\gamma}_n \to \tilde{\gamma} \quad \text{in } C^0([0, L]). \tag{3.6}$$

Define now $\gamma(t) := \tilde\gamma(t/L)$. Then $\gamma \in W^{1,\infty}([0,1];\overline{\omega})$ and by weak lower-semicontinuity we get

$$\ell(\gamma) = \int_0^L |\dot{\tilde\gamma}(s)|\,\mathrm{d}s \overset{(3.5)}{\leq} \liminf_{n\to\infty} \int_0^L |\dot{\tilde\gamma}_n(s)|\,\mathrm{d}s = \liminf_{n\to\infty} \ell(\gamma_n) \overset{(3.3)}{\leq} \frac{1}{\varepsilon} \lim_{n\to\infty} |x_n - y_n| = \frac{1}{\varepsilon}|x-y|.$$

Notice that, as soon as we prove condition (2.7), $\gamma([0,1]) \subset \omega$ follows. In order to show (2.7), we fix $s \in [0,L]$. By compactness of the boundary $\partial\omega$ we can choose $z \in \partial\omega$ such that

$$\operatorname{dist}(\tilde\gamma(s), \partial\omega) = |\tilde\gamma(s) - z|. \tag{3.7}$$

We further choose $z_n \in \partial\omega_n$ such that

$$|z_n - z| \leq d_{\mathrm{H}}(\partial\omega, \partial\omega_n)$$

for every $n \in \mathbb{N}$. Then, by the triangle inequality

$$\begin{aligned} |\tilde\gamma(s) - z| &\geq |\tilde\gamma_n(s) - z_n| - |\tilde\gamma(s) - \tilde\gamma_n(s)| - |z_n - z| \\ &\geq \operatorname{dist}(\tilde\gamma_n(s), \partial\omega_n) - \|\tilde\gamma - \tilde\gamma_n\|_{C^0([0,1])} - d_{\mathrm{H}}(\partial\omega, \partial\omega_n). \end{aligned}$$

Using assumption (3.2), condition (3.4), and convergence (3.6), we deduce that

$$|\tilde\gamma(s) - z| \geq \varepsilon \frac{|x_n - \tilde\gamma_n(s)||\tilde\gamma_n(s) - y_n|}{|x_n - y_n|}.$$

Passage to the limit on the right-hand side concludes the proof of (2.7). □

3.2. Existence of incremental solutions

In the following $C > 0$ denotes a positive real constant which may change from line to line, whereas $c > 0$ denotes the constant used in assumptions (2.8a)-(2.8b). For the purpose of readability, we abbreviate $\|f\|_{L^p(\Omega;\mathbb{R}^d)}$ by $\|f\|_{L^p(\Omega)}$ and $\|g\|_{L^p(\Gamma_D;\mathbb{R}^d)}$ by $\|g\|_{L^p(\Gamma_D)}$.

Let $\Pi = \{0 = t_0 < t_1 < \cdots < t_N = T\}$ be a partition of $[0,T]$. Given $i \in \{1,\ldots,N\}$ and $(y_{\mathrm{e}(i-1)}, y_{\mathrm{p}(i-1)}) \in \mathcal{Q}$ we aim at proving that minimizers of the *incremental problem*

$$(y_{\mathrm{e}i}, y_{\mathrm{p}i}) \in \operatorname*{argmin}_{(y_{\mathrm{e}},y_{\mathrm{p}})\in\mathcal{Q}} \Big(\mathcal{E}(t_i, y_{\mathrm{e}}, y_{\mathrm{p}}) + \mathcal{D}(\nabla y_{\mathrm{p}(i-1)}, \nabla y_{\mathrm{p}})\Big) \tag{3.8}$$

exist.

We follow the Direct Method of the Calculus of Variations: Let $(y_{\mathrm{e}}^n, y_{\mathrm{p}}^n)_{n\in\mathbb{N}} \subset \mathcal{Q}$ be an infimizing sequence for (3.8). Here, we use that $\mathcal{Q}$ is non-empty, since $(T^{-1}, T) \in \mathcal{Q}$, where $T(x) = x - \bar{x}$ is a translation and $\bar{x}$ is the barycenter of Ω. As $\mathcal{D}$ is nonnegative, we can assume without loss of generality that $\mathcal{E}(t_i, y_{\mathrm{e}}^n, y_{\mathrm{p}}^n) \leq C$. We aim at showing the following compactness result:

$$\mathcal{E}(t_i, y_{\mathrm{e}}^n, y_{\mathrm{p}}^n) \leq C \quad \Longrightarrow \quad (y_{\mathrm{e}}^n, y_{\mathrm{p}}^n) \to (y_{\mathrm{e}}, y_{\mathrm{p}}) \text{ in } \mathcal{Q} \tag{3.9}$$

along a not relabeled subsequence. Indeed, the energy bound $\mathcal{E}(t_i, y_{\mathrm{e}}^n, y_{\mathrm{p}}^n) \leq C$ together with the growth assumption (2.8a) and the coercivity (2.8b) entails

$$c\|\nabla y_{\mathrm{e}}^n\|^{q_{\mathrm{e}}}_{L^{q_{\mathrm{e}}}(y_{\mathrm{p}}^n(\Omega))} + \|\nabla y_{\mathrm{p}}^n\|^{q_{\mathrm{p}}}_{L^{q_{\mathrm{p}}}(\Omega)} + c\|y_{\mathrm{e}}^n \circ y_{\mathrm{p}}^n - \mathrm{id}\|_{L^1(\Gamma_D)} \leq C + \langle \ell(t), y_{\mathrm{e}}^n \circ y_{\mathrm{p}}^n \rangle.$$

By the regularity assumption (2.10) and the chain rule estimate (2.4), we can bound

$$\begin{aligned}|\langle \ell(t), y_{\mathrm{e}}^n\circ y_{\mathrm{p}}^n\rangle| &\le \|\ell(t)\|_{(W^{1,q}(\Omega))^*}\|y_{\mathrm{e}}^n\circ y_{\mathrm{p}}^n\|_{W^{1,q}(\Omega)}\\ &\le C\|\nabla y_{\mathrm{e}}^n\|_{L^{q_{\mathrm{e}}}(y_{\mathrm{p}}^n(\Omega))}\|\nabla y_{\mathrm{p}}^n\|_{L^{q_{\mathrm{p}}}(\Omega)}\\ &\le \frac{c}{2}\|\nabla y_{\mathrm{e}}^n\|^{q_{\mathrm{e}}}_{L^{q_{\mathrm{e}}}(y_{\mathrm{p}}^n(\Omega))}+\frac{c}{2}\|\nabla y_{\mathrm{p}}^n\|^{q_{\mathrm{p}}}_{L^{q_{\mathrm{p}}}(\Omega)}+C\end{aligned}$$

and conclude that

$$\|\nabla y_{\mathrm{e}}^n\|^{q_{\mathrm{e}}}_{L^{q_{\mathrm{e}}}(y_{\mathrm{p}}^n(\Omega))}+\|\nabla y_{\mathrm{p}}^n\|^{q_{\mathrm{p}}}_{L^{q_{\mathrm{p}}}(\Omega)}+\|y_{\mathrm{e}}^n\circ y_{\mathrm{p}}^n-\mathrm{id}\|_{L^1(\Gamma_D)}\le C. \tag{3.10}$$

Since y_{p}^n has zero mean, the Poincaré-Wirtinger inequality implies that y_{p}^n is bounded in $W^{1,q_{\mathrm{p}}}(\Omega;\mathbb{R}^d)$. On the other hand $y^n := y_{\mathrm{e}}^n\circ y_{\mathrm{p}}^n$ is subject to the elastic Dirichlet boundary condition on Γ_D and we have the following result.

Lemma 3.3 (Generalized Poincaré inequality)**.** *Let $\Omega\subset\mathbb{R}^d$ be as in Section 2.2 and $q\ge 1$. Then, there exists a constant $C_{Poincaré}>0$ such that*

$$\|y\|_{W^{1,q}(\Omega)}\le C_{Poincaré}\Big(\|\nabla y\|_{L^q(\Omega)}+\|y-\mathrm{id}\|_{L^1(\Gamma_D)}\Big)$$

for every $y\in W^{1,q}(\Omega;\mathbb{R}^d)$.

Proof of Lemma 3.3. We argue by contradiction. Let the sequence $(y_k)_{k\in\mathbb{N}}\subset W^{1,q}(\Omega;\mathbb{R}^d)$ be such that

$$\|\nabla y_k\|_{L^q(\Omega)}+\|y_k-\mathrm{id}\|_{L^1(\Gamma_D)}<\frac{1}{k}\|y_k\|_{L^q(\Omega)}. \tag{3.11}$$

We claim that $\|y_k\|_{L^q(\Omega)}\to\infty$. If this would not be the case, we would have $\|y_k\|_{W^{1,q}(\Omega)}\le C$ and can pick a (not relabeled) subsequence y_k converging to y weakly in $W^{1,q}(\Omega)$. By the trace theorem, the traces would also converge strongly in $L^1(\partial\Omega)$. Moreover, by (3.11),

$$\|\nabla y\|_{L^q(\Omega)}+\|y-\mathrm{id}\|_{L^1(\Gamma_D)}\le\liminf_{k\to\infty}\Big(\|\nabla y_k\|_{L^q(\Omega)}+\|y_k-\mathrm{id}\|_{L^1(\Gamma_D)}\Big)=0.$$

This would imply $\nabla y=0$ in Ω and $y=\mathrm{id}$ on Γ_D. A contradiction. Hence $\|y_k\|_{L^q(\Omega)}\to\infty$.
We now rescale the sequence by setting

$$w_k:=\frac{y_k}{\|y_k\|_{L^q(\Omega)}}$$

and note that

$$\|w_k\|_{L^q(\Omega)}=1\qquad\text{and}\qquad\|\nabla w_k\|_{L^q(\Omega)}+\|w_k-\lambda_k\mathrm{id}\|_{L^1(\Gamma_D)}<\frac{1}{k}$$

where $\lambda_k=\|y_k\|^{-1}_{L^q(\Omega)}$ tends to 0. Then, we choose a (not relabeled) subsequence w_k converging to some w weakly in $W^{1,q}(\Omega)$, strongly in $L^q(\Omega)$, and such that the traces converge strongly in $L^1(\partial\Omega)$. This leads to $\nabla w=0$ in Ω and $w=0$ on Γ_D. Since Ω is connected, this forces $w=0$ in Ω and contradicts the fact that $\|w\|_{L^q(\Omega)}=\lim_{k\to\infty}\|w_k\|_{L^q(\Omega)}=1$. □

We use Lemma 3.3 in combination with the chain rule (2.3) and Hölder's inequality to estimate

$$\|y^n\|^q_{W^{1,q}(\Omega)} \leq C \left(\|\nabla y^n_{\rm e}\|^{q_{\rm e}}_{L^{q_{\rm e}}(y^n_{\rm p}(\Omega))} + \|\nabla y^n_{\rm p}\|^{q_{\rm p}}_{L^{q_{\rm p}}(\Omega)} + \|y^n_{\rm e} \circ y^n_{\rm p} - {\rm id}\|^q_{L^1(\Gamma_D)} \right) \overset{(3.10)}{\leq} C. \tag{3.12}$$

We further remark that $W^{1,q}(\Omega;\mathbb{R}^d)$ embeds into $L^{q^*}(\Omega;\mathbb{R}^d)$ with $q^* > q_{\rm e}$. This leads to

$$\|y^n_{\rm e}\|_{L^{q_{\rm e}}(y^n_{\rm p}(\Omega))} = \|y^n\|_{L^{q_{\rm e}}(\Omega)} \leq C\|y^n\|_{W^{1,q}(\Omega)} \leq C.$$

Altogether, we conclude that

$$\|y^n\|_{W^{1,q}(\Omega)} + \|y^n_{\rm e}\|_{W^{1,q_{\rm e}}(y^n_{\rm p}(\Omega))} + \|y^n_{\rm p}\|_{W^{1,q_{\rm p}}(\Omega)} \leq C.$$

This bound implies that there exists a (not relabeled) subsequence such that $(y^n_{\rm e}, y^n_{\rm p})$ converges to $(y_{\rm e}, y_{\rm p})$ in $\mathcal{Q}$ which concludes the proof of (3.9).

In Section 3.1 we have seen that $\mathcal{Q}$ is closed under this convergence, consequently $(y_{\rm e}, y_{\rm p}) \in \mathcal{Q}$. Furthermore, by the continuity of the trace operator, we have $y^n \to y$ strongly in $L^{q^\#}(\partial\Omega)$, where $q^\# > (d-1)^2 \geq 1$. This yields

$$\int_{\Gamma_D} |y(x) - x| \,{\rm d}\mathcal{H}^{d-1}(x) = \lim_{n\to\infty} \int_{\Gamma_D} |y^n(x) - x| \,{\rm d}\mathcal{H}^{d-1}(x). \tag{3.13}$$

By the weak continuity of the loading term, we have that $\langle \ell(t_i), y^n \rangle$ converges to $\langle \ell(t_i), y \rangle$. The weak continuity of the minors entails that ${\rm cof}\, \nabla y^n_{\rm p} \rightharpoonup {\rm cof}\, \nabla y_{\rm p}$ in $L^{q_{\rm p}/(d-1)}(\Omega;\mathbb{R}^d)$. In combination with polyconvexity (2.9b), we deduce

$$\int_\Omega W_{\rm p}(\nabla y_{\rm p}(x)) \,{\rm d}x \leq \liminf_{n\to\infty} \int_\Omega W_{\rm p}(\nabla y^n_{\rm p}(x)) \,{\rm d}x. \tag{3.14}$$

For every fixed $K \subset\subset y_{\rm p}(\Omega)$, again by weak continuity of the minors (recall that $q_{\rm e} > d$) and polyconvexity (2.9a), we have

$$\int_K W_{\rm el}(\nabla y_{\rm e}(\xi)) \,{\rm d}\xi \leq \liminf_{n\to\infty} \int_{y^n_{\rm p}(\Omega)} W_{\rm el}(\nabla y^n_{\rm e}(\xi)) \,{\rm d}\xi. \tag{3.15}$$

Letting K tend to $y_{\rm p}(\Omega)$ in (3.15), together with (3.13) and (3.14) we have shown lower semi-continuity of the energy, namely

$$\mathcal{E}(t_i, y_{\rm e}, y_{\rm p}) \leq \liminf_{n\to\infty} \mathcal{E}(t_i, y^n_{\rm e}, y^n_{\rm p}). \tag{3.16}$$

In a similar way, using polyconvexity of D (2.11), we get

$$\begin{aligned}
\mathcal{D}(\nabla y_{{\rm p}(i-1)}, \nabla y_{\rm p}) &= \int_\Omega D(\nabla y_{\rm p} (\nabla y_{{\rm p}(i-1)})^{-1}) \,{\rm d}x \\
&= \int_\Omega \widehat{D}(\nabla y_{\rm p} (\nabla y_{{\rm p}(i-1)})^{-1}, {\rm cof}\,(\nabla y_{\rm p} (\nabla y_{{\rm p}(i-1)})^{-1})) \,{\rm d}x \\
&= \int_\Omega \widehat{D}(\nabla y_{\rm p} (\nabla y_{{\rm p}(i-1)})^{-1}, {\rm cof}\,(\nabla y_{\rm p})\, {\rm cof}\,(\nabla y_{{\rm p}(i-1)})^{-1}) \,{\rm d}x
\end{aligned}$$

$$
\begin{aligned}
&\leq \liminf_{n\to\infty} \int_\Omega \widehat{D}(\nabla y_{\rm p}^n (\nabla y_{{\rm p}(i-1)})^{-1}, \mathrm{cof}\,(\nabla y_{\rm p}^n)\mathrm{cof}\,(\nabla y_{{\rm p}(i-1)})^{-1})\,\mathrm{d}x \\
&= \liminf_{n\to\infty} \mathcal{D}(\nabla y_{{\rm p}(i-1)}, \nabla y_{\rm p}^n),
\end{aligned}
$$

where we also used the fact that

$$
\begin{aligned}
&\mathrm{cof}\,(\nabla y_{\rm p}^n)\mathrm{cof}\,(\nabla y_{{\rm p}(i-1)})^{-1} = \mathrm{cof}\,(\nabla y_{\rm p}^n)\nabla y_{{\rm p}(i-1)}^T \\
&\rightharpoonup \mathrm{cof}\,(\nabla y_{\rm p})\nabla y_{{\rm p}(i-1)}^T = \mathrm{cof}\,(\nabla y_{\rm p})\mathrm{cof}\,(\nabla y_{{\rm p}(i-1)})^{-1}
\end{aligned}
$$

in $L^{q_{\rm p}/d}(\Omega;\mathbb{R}^{d\times d})$. This shows that $(y_{\rm e}, y_{\rm p})$ is a minimizer of (3.8).

The discrete stability condition ($\mathrm{S}_{\mathrm{discr}}$) can be deduced easily by testing (3.8) with a competitor $(\widehat{y}_{\rm e}, \widehat{y}_{\rm p}) \in \mathcal{Q}$ and using the triangle inequality for $\mathcal{D}$. The discrete energy inequality ($\mathrm{E}_{\mathrm{discr}}$) is shown below in (3.17).

3.3. Energy inequality

Take a sequence of partitions $\Pi_n = \{0 = t_0^n < t_1^n < \cdots < t_{N(n)}^n = T\}$, $n \in \mathbb{N}$, with fineness $\max_{i=1,\ldots,N(n)}(t_i^n - t_{i-1}^n)$ tending to 0 as n goes to ∞. For fixed n we iteratively choose $(y_{{\rm e}i}^n, y_{{\rm p}i}^n) \in \mathcal{Q}$, $i = 1,\ldots,N(n)$, solving the incremental minimization problem (3.8), consider the right-continuous, piecewise constant approximation as in (2.14), and set

$$
y^n(t) = y_{\rm e}^n(t) \circ y_{\rm p}^n(t).
$$

Testing (3.8) against $(y_{{\rm e}(i-1)}^n, y_{{\rm p}(i-1)}^n)$, we get

$$
\mathcal{E}(t_i^n, y_{{\rm e}i}^n, y_{{\rm p}i}^n) - \mathcal{E}(t_{i-1}^n, y_{{\rm e}(i-1)}^n, y_{{\rm p}(i-1)}^n) + \mathcal{D}(\nabla y_{{\rm p}(i-1)}^n, \nabla y_{{\rm p}i}^n) \leq \int_{t_{i-1}^n}^{t_i^n} \partial_t \mathcal{E}(s, y_{{\rm e}(i-1)}^n, y_{{\rm p}(i-1)}^n)\,\mathrm{d}s,
$$

where $\partial_t \mathcal{E}(s, y_{{\rm e}(i-1)}^n, y_{{\rm p}(i-1)}^n) = -\langle \dot{\ell}(s), y_{{\rm e}(i-1)}^n \circ y_{{\rm p}(i-1)}^n \rangle = -\langle \dot{\ell}(s), y^n(s)\rangle$.

Summing over i, we arrive at

$$
\mathcal{E}(t_k^n, y_{{\rm e}k}, y_{{\rm p}k}) - \mathcal{E}(t_j^n, y_{{\rm e}j}, y_{{\rm p}j}) + \mathrm{Diss}_{\mathcal{D}}(\nabla y_{\rm p}^n; t_j^n, t_k^n) \leq -\int_{t_j^n}^{t_k^n} \langle \dot{\ell}(s), y^n(s)\rangle\,\mathrm{d}s \tag{3.17}
$$

for every $0 \leq j \leq k \leq N(n)$. We estimate the right-hand side by

$$
|\langle \dot{\ell}(s), y^n(s)\rangle| \leq \zeta(s)\|y^n(s)\|_{W^{1,q}(\Omega)},
$$

where $\zeta(s) := \|\dot{\ell}(s)\|_{W^{1,q}(\Omega)^*}$, by assumption (2.10), is integrable. By estimate (3.12) and assumptions (2.8a)-(2.8b), we have

$$
\|y^n(s)\|_{W^{1,q}(\Omega)} \leq C(1 + \mathcal{E}(s, y_{\rm e}^n(s), y_{\rm p}^n(s))).
$$

Therefore, altogether

$$
\mathcal{E}(t, y_{\rm e}^n(t), y_{\rm p}^n(t)) - \mathcal{E}(s, y_{\rm e}^n(s), y_{\rm p}^n(s)) + \mathrm{Diss}_{\mathcal{D}}(\nabla y_{\rm p}^n; s, t) \leq C\int_s^t \zeta(r)(1 + \mathcal{E}(r, y^n(r)))\,\mathrm{d}r
$$

for every $s, t \in \Pi_n$, $s \le t$. By virtue of Gronwall's inequality, using the integrability of ζ, we find

$$\sup_{t\in\Pi_n} \mathcal{E}(t, y_{\mathrm{e}}^n(t), y_{\mathrm{p}}^n(t)) \le C.$$

Since $\mathcal{E}$ is absolutely continuous in time and the approximate solution $(y_{\mathrm{e}}^n, y_{\mathrm{p}}^n)$ is piecewise constant, we deduce

$$\sup_{t\in[0,T]} \mathcal{E}(t, y_{\mathrm{e}}^n(t), y_{\mathrm{p}}^n(t)) + \mathrm{Diss}_{\mathcal{D}}(\nabla y_{\mathrm{p}}^n; 0, T) \le C. \tag{3.18}$$

We now prepare an intermediate result.

Lemma 3.4 (Lower-semicontinuity of $\mathcal{D}$ in both arguments)**.** *Let $y_{\mathrm{p}}^n \rightharpoonup y_{\mathrm{p}}$ and $y_{\mathrm{p0}}^n \rightharpoonup y_{\mathrm{p0}}$ in $W^{1,q_{\mathrm{p}}}(\Omega)$ with $q_{\mathrm{p}} > d(d-1)$ such that* $\det \nabla y_{\mathrm{p0}}^n = 1$ *a.e. and $|\Omega| \le |y_{\mathrm{p0}}^n(\Omega)|$ for every $n \in \mathbb{N}$. Then,*

$$\mathcal{D}(\nabla y_{\mathrm{p0}}, \nabla y_{\mathrm{p}}) \le \liminf_{n\to\infty} \mathcal{D}(\nabla y_{\mathrm{p0}}^n, \nabla y_{\mathrm{p}}^n).$$

Proof of Lemma 3.4. We rely on the assumption $q_{\mathrm{p}} \ge d(d-1)$ and define

$$v^n = y_{\mathrm{p}}^n \circ (y_{\mathrm{p0}}^n)^{-1},$$

where the global inverse $(y_{\mathrm{p0}}^n)^{-1}$ is bounded in $W^{1,q_{\mathrm{p}}/(d-1)}(y_{\mathrm{p0}}^n(\Omega))$, since we have that $(\nabla y_{\mathrm{p0}}^n)^{-1} = (\mathrm{cof}\, \nabla y_{\mathrm{p0}}^n)^T$. We rewrite

$$\mathcal{D}(\nabla y_{\mathrm{p0}}^n, \nabla y_{\mathrm{p}}^n) = \int_{y_{\mathrm{p0}}^n(\Omega)} D(\nabla v^n(\xi))\, \mathrm{d}\xi$$

and estimate

$$\|v^n\|_{L^{q_{\mathrm{p}}/d}(y_{\mathrm{p0}}^n(\Omega))} = \|y_{\mathrm{p}}^n\|_{L^{q_{\mathrm{p}}/d}(\Omega)} \le |\Omega|^{(d-1)/q_{\mathrm{p}}} \|y_{\mathrm{p}}^n\|_{L^{q_{\mathrm{p}}}(\Omega)} \le C, \tag{3.19a}$$

$$\begin{aligned} \|\nabla v^n\|_{L^{q_{\mathrm{p}}/d}(y_{\mathrm{p0}}^n(\Omega))} &= \|\nabla y_{\mathrm{p}}^n (\nabla y_{\mathrm{p0}}^n)^{-1}\|_{L^{q_{\mathrm{p}}/d}(\Omega)} \\ &\le \|\nabla y_{\mathrm{p}}^n\|_{L^{q_{\mathrm{p}}}(\Omega)} \|\mathrm{cof}\, \nabla y_{\mathrm{p0}}^n\|_{L^{q_{\mathrm{p}}/(d-1)}(\Omega)} \le C. \end{aligned} \tag{3.19b}$$

Let K be a compact subset of $y_{\mathrm{p0}}(\Omega)$. Since $y_{\mathrm{p0}}^n \to y_{\mathrm{p0}}$ uniformly in $\overline{\Omega}$, there exists $n_K \in \mathbb{N}$ such that for all $n \ge n_K$, we have $K \subset y_{\mathrm{p0}}^n(\Omega)$. Using estimates (3.19), we choose a (not relabeled) subsequence such that

$$v^n \rightharpoonup v \quad \text{in } W^{1,q_{\mathrm{p}}/d}(K),$$

where $v = y_{\mathrm{p}} \circ y_{\mathrm{p0}}^{-1}$ on K. As $q_{\mathrm{p}}/d > d-1$, we conclude, by using the polyconvexity (2.11) and the weak continuity of the minors of ∇v, that

$$\int_K D(\nabla v(\xi))\, \mathrm{d}\xi \le \liminf_{n\to\infty} \int_K D(\nabla v^n(\xi))\, \mathrm{d}\xi \le \liminf_{n\to\infty} \int_{y_{\mathrm{p0}}^n(\Omega)} D(\nabla v^n(\xi))\, \mathrm{d}\xi.$$

Now it suffices to consider an increasing sequence of compact subsets exhausting $y_{\mathrm{p0}}(\Omega)$. By further extracting a diagonal sequence, we get that $y_{\mathrm{p}} = v \circ y_{\mathrm{p0}}$ on Ω and the statement follows. □

We proceed with the proof of the energy inequality by noting that (3.18) together with Lemma 3.4 allows us to use Helly's Selection Principle ([51], Thm. B.5.13). Namely, there exists a subsequence $(n_k)_{k\in\mathbb{N}}$, a function

$y_{\mathrm{p}} : [0,T] \to W^{1,q_{\mathrm{p}}}(\Omega)$, and a nondecreasing function $\delta : [0,T] \to [0,\infty)$ such that

$$y_{\mathrm{p}}^{n_k}(t) \rightharpoonup y_{\mathrm{p}}(t) \text{ in } W^{1,q_{\mathrm{p}}}(\Omega), \tag{3.20a}$$

$$\mathrm{Diss}_{\mathcal{D}}(\nabla y_{\mathrm{p}}^{n_k}; 0, t) \to \delta(t), \tag{3.20b}$$

$$\mathrm{Diss}_{\mathcal{D}}(\nabla y_{\mathrm{p}}; s, t) \leq \delta(t) - \delta(s) \tag{3.20c}$$

for every $s, t \in [0,T]$. By defining $\theta_n(s) := -\langle \dot{\ell}(s), y^n(s)\rangle$ and observing that θ_n is equiintegrable, we can use the Dunford-Pettis Theorem (see [17] or [51, Thm. B.3.8]) to extract a further (not relabeled) subsequence satisfying

$$\theta_{n_k} \rightharpoonup \theta \quad \text{in } L^1(0,T). \tag{3.21}$$

Fix now some $t \in [0,T]$ and define

$$\tau^n := \min\{\tau \in \Pi_n : \tau \geq t\}$$

such that $\tau^n \geq t, \tau^n \to t$. We can directly pass to the lim inf in the dissipation

$$\delta(t) \overset{(3.20b)}{=} \lim_{k\to\infty} \mathrm{Diss}_{\mathcal{D}}(\nabla y_{\mathrm{p}}^{n_k}; 0, t) \leq \liminf_{k\to\infty} \mathrm{Diss}_{\mathcal{D}}(\nabla y_{\mathrm{p}}^{n_k}; 0, \tau^{n_k}).$$

Moreover, by the energy bound (3.18), we can follow the argument leading to (3.9) and choose a t-dependent subsequence $(N_k^t)_{k\in\mathbb{N}}$ of $(n_k)_{k\in\mathbb{N}}$ such that $(y_{\mathrm{e}}^{N_k^t}(t), y_{\mathrm{p}}^{N_k^t}(t))$ converges to $(y_{\mathrm{e}}(t), y_{\mathrm{p}}(t))$ in $\mathcal{Q}$ and $y^n(t) \rightharpoonup y_{\mathrm{e}}(t) \circ y_{\mathrm{p}}(t) =: y(t)$ in $W^{1,q}(\Omega;\mathbb{R}^d)$. Additionally, by extracting a further subsequence, we guarantee that

$$\theta^{N_k^t}(t) \to \limsup_{k\to\infty} \theta^{n_k}(t) := \theta_{\sup}(t).$$

Since $y^{N_k^t}(t) \rightharpoonup y(t)$ in $W^{1,q}(\Omega)$, it easily follows that

$$\theta_{\sup}(t) = \lim_{k\to\infty} \theta^{N_k^t}(t) = \lim_{k\to\infty} \langle \dot{\ell}(t), y^{N_k^t}(t)\rangle = \langle \dot{\ell}(t), y(t)\rangle.$$

Furthermore,

$$\mathcal{E}(t, y_{\mathrm{e}}(t), y_{\mathrm{p}}(t)) \leq \liminf_{n\to\infty} \mathcal{E}(\tau^{N_k^t}, y_{\mathrm{e}}^{N_k^t}(t), y_{\mathrm{p}}^{N_k^t}(t)),$$

see the discussion in Section 3.2 leading to (3.16) and notice that $(y_{\mathrm{e}}^n, y_{\mathrm{p}}^n)(t) = (y_{\mathrm{e}}^n, y_{\mathrm{p}}^n)(\tau^n)$. At this point, we can pass to the lim sup on the right-hand side of inequality (3.17), using convergence (3.21) and $\theta \leq \theta_{\sup}$. As the energy is continuous in t, we conclude that

$$\mathcal{E}(t, y_{\mathrm{e}}(t), y_{\mathrm{p}}(t)) - \mathcal{E}(0, y_{\mathrm{e}0}, y_{\mathrm{p}0}) + \delta(t) \leq -\int_0^t \langle \dot{\ell}(s), y(s)\rangle \,\mathrm{d}s$$

as desired.

3.4. Semistability

In this section, we prove the semistability condition ($\mathrm{S}_{\mathrm{semi}}$). Fix $t \in [0,T]$ and define τ^n as above. Let $(y_\mathrm{e}^n(t), y_\mathrm{p}^n(t))$ be the right-continuous approximation defined in (2.14) and note that $(y_\mathrm{e}^n, y_\mathrm{p}^n)(t) = (y_\mathrm{e}^n, y_\mathrm{p}^n)(\tau^n)$. By testing the minimum in (3.8) at time τ^n against competitors $(\widehat{y}_\mathrm{e}^n, y_\mathrm{p}^n(t))$ having the same plastic component, we get the following *discrete* semistability:

$$\mathcal{E}(\tau_n, y_\mathrm{e}^n(t), y_\mathrm{p}^n(t)) \leq \mathcal{E}(\tau_n, \widehat{y}_\mathrm{e}^n, y_\mathrm{p}^n(t)) \tag{3.22}$$

for every $\widehat{y}_\mathrm{e}^n$ satisfying $(\widehat{y}_\mathrm{e}^n, y_\mathrm{p}^n(t)) \in \mathcal{Q}$. By following the discussion of Sections 3.2-3.3, we can choose a (not relabeled) subsequence such that $(y_\mathrm{e}^n(t), y_\mathrm{p}^n(t)) \to (y_\mathrm{e}(t), y_\mathrm{p}(t))$ in $\mathcal{Q}$. Note that this subsequence may be t-dependent as in Section 3.3. We aim at showing the corresponding *limit* semistability:

$$\mathcal{E}(t, y_\mathrm{e}(t), y_\mathrm{p}(t)) \leq \mathcal{E}(t, \widehat{y}_\mathrm{e}, y_\mathrm{p}(t)) \tag{3.23}$$

for every $\widehat{y}_\mathrm{e}$ satisfying $(\widehat{y}_\mathrm{e}, y_\mathrm{p}(t)) \in \mathcal{Q}$. This is done by passing to the limit in (3.22) with a suitable recovery sequence in the spirit of [53].

Lemma 3.5 (Existence of recovery sequences)**.** *Let $(y_\mathrm{e}^n, y_\mathrm{p}^n)_{n\in\mathbb{N}} \subset \mathcal{Q}$ converge to $(y_\mathrm{e}, y_\mathrm{p})$ in $\mathcal{Q}$. Then, for every $\widehat{y}_\mathrm{e}$ with $(\widehat{y}_\mathrm{e}, y_\mathrm{p}) \in \mathcal{Q}$ there exists a sequence $\widehat{y}_\mathrm{e}^n$ with $(\widehat{y}_\mathrm{e}^n, y_\mathrm{p}^n) \in \mathcal{Q}$ satisfying*

$$\limsup_{n\to\infty} \Big(\mathcal{E}(\tau_n, \widehat{y}_\mathrm{e}^n, y_\mathrm{p}^n) - \mathcal{E}(\tau_n, y_\mathrm{e}^n, y_\mathrm{p}^n) \Big) \leq \mathcal{E}(t, \widehat{y}_\mathrm{e}, y_\mathrm{p}) - \mathcal{E}(t, y_\mathrm{e}, y_\mathrm{p}). \tag{3.24}$$

Proof of Lemma 3.5. Let $\widehat{y}_\mathrm{e}$ be such that $(\widehat{y}_\mathrm{e}, y_\mathrm{p}) \in \mathcal{Q}$. As the energy $\mathcal{E}$ is absolutely continuous with respect to time (2.10), relation (3.24) follows as soon as we find $\widehat{y}_\mathrm{e}^n$ with $(\widehat{y}_\mathrm{e}^n, y_\mathrm{p}^n) \in \mathcal{Q}$ satisfying

$$\limsup_{n\to\infty} \Big(\mathcal{E}(t, \widehat{y}_\mathrm{e}^n, y_\mathrm{p}^n) - \mathcal{E}(t, y_\mathrm{e}^n, y_\mathrm{p}^n) \Big) \leq \mathcal{E}(t, \widehat{y}_\mathrm{e}, y_\mathrm{p}) - \mathcal{E}(t, y_\mathrm{e}, y_\mathrm{p}) \tag{3.25}$$

where now time is fixed.

In order to check for (3.25), due to the cancellation of the kinematic hardening energy W_p and the weak lower-semicontinuity (3.16), it suffices to show that there exists a sequence $(\widehat{y}_\mathrm{e}^n, y_\mathrm{p}^n) \in \mathcal{Q}$ such that

$$\limsup_{n\to\infty} \int_{y_\mathrm{p}^n(\Omega)} W_\mathrm{e}(\nabla \widehat{y}_\mathrm{e}^n(\xi))\, \mathrm{d}\xi \leq \int_{y_\mathrm{p}(\Omega)} W_\mathrm{e}(\nabla \widehat{y}_\mathrm{e}(\xi))\, \mathrm{d}\xi \tag{3.26}$$

and

$$\lim_{n\to\infty} \int_{\Gamma_D} |\widehat{y}_\mathrm{e}^n(y_\mathrm{p}^n(x)) - x|\, \mathrm{d}\mathcal{H}^{d-1}(x) = \int_{\Gamma_D} |\widehat{y}_\mathrm{e}(y_\mathrm{p}(x)) - x|\, \mathrm{d}\mathcal{H}^{d-1}(x). \tag{3.27}$$

We may assume without loss of generality that

$$\int_{y_\mathrm{p}(\Omega)} W_\mathrm{e}(\nabla \widehat{y}_\mathrm{e}(\xi))\, \mathrm{d}\xi < \infty.$$

In particular, $\widehat{y}_\mathrm{e} \in W^{1,q_\mathrm{e}}(y_\mathrm{p}(\Omega); \mathbb{R}^d)$ by the growth assumption (2.8a). To define the recovery sequence $\widehat{y}_\mathrm{e}^n$ we use the fact that $y_\mathrm{p}(\Omega) \in \mathcal{J}_{\varepsilon,\delta}$. By the extension property of (ε,δ)-domains, there exists a bounded linear operator

$$E : W^{1,q_\mathrm{e}}(y_\mathrm{p}(\Omega); \mathbb{R}^d) \to W^{1,q_\mathrm{e}}(\mathbb{R}^d; \mathbb{R}^d) \tag{3.28}$$

such that $Eu = u$ in $y_{\rm p}(\Omega)$, and a constant C solely depending on $\varepsilon, \delta, q_{\rm e}$, and d such that

$$\|Eu\|_{W^{1,q_{\rm e}}(\mathbb{R}^d)} \le C\|u\|_{W^{1,q_{\rm e}}(y_{\rm p}(\Omega))} \tag{3.29}$$

for every $u \in W^{1,q_{\rm e}}(y_{\rm p}(\Omega);\mathbb{R}^d)$ ([33], Thm. 1).
Set now

$$\widehat{y}_{\rm e}^n := E\widehat{y}_{\rm e}\big|_{y_{\rm p}^n(\Omega)}$$

and note that this test is admissible, namely $(\widehat{y}_{\rm e}^n, y_{\rm p}^n) \in \mathcal{Q}$. Then, we split

$$\int_{y_{\rm p}^n(\Omega)} W_{\rm e}(\nabla\widehat{y}_{\rm e}^n(\xi))\,{\rm d}\xi = \int_{y_{\rm p}^n(\Omega)\cap y_{\rm p}(\Omega)} W_{\rm e}(\nabla\widehat{y}_{\rm e}(\xi))\,{\rm d}\xi + \int_{y_{\rm p}^n(\Omega)\setminus y_{\rm p}(\Omega)} W_{\rm e}(\nabla(E\widehat{y}_{\rm e})(\xi))\,{\rm d}\xi.$$

We use the growth condition (2.8a) to control

$$\left|\int_{y_{\rm p}^n(\Omega)\setminus y_{\rm p}(\Omega)} W_{\rm e}(\nabla(E\widehat{y}_{\rm e})(\xi))\,{\rm d}\xi\right| \le \frac{1}{c}\int_{y_{\rm p}^n(\Omega)\setminus y_{\rm p}(\Omega)} \left(1+|\nabla(E\widehat{y}_{\rm e})(\xi)|^{q_{\rm e}}\right)\,{\rm d}\xi \tag{3.30}$$

and use the convergence $\mathcal{L}^d(y_{\rm p}^n(\Omega)\setminus y_{\rm p}(\Omega)) \to 0$ as well as $E\widehat{y}_{\rm e} \in W^{1,q_{\rm e}}(\mathbb{R}^d;\mathbb{R}^d)$, by bound (3.29), to deduce the convergence to 0 of the right-hand side of (3.30). Since $y_{\rm p}^n \to y_{\rm p}$ uniformly in $\overline{\Omega}$, we have $\mathcal{L}^d(y_{\rm p}^n(\Omega)\triangle y_{\rm p}(\Omega)) \to 0$ as $n \to \infty$, where $\triangle$ denotes the symmetric difference. Thus,

$$\lim_{n\to\infty}\int_{y_{\rm p}^n(\Omega)\cap y_{\rm p}(\Omega)} W_{\rm e}(\nabla\widehat{y}_{\rm e}(\xi))\,{\rm d}\xi = \int_{y_{\rm p}(\Omega)} W_{\rm e}(\nabla\widehat{y}_{\rm e}(\xi))\,{\rm d}\xi.$$

This proves inequality (3.26) and we are left with checking the convergence (3.27). Observe that by the chain rule (2.3)-(2.4) and the boundedness of the extension (3.29), we have

$$\begin{aligned}
&\|\widehat{y}_{\rm e}^n\circ y_{\rm p}^n\|_{L^q(\Omega)} = \|E\widehat{y}_{\rm e}\|_{L^q(y_{\rm p}^n(\Omega))} \le |y_{\rm p}^n(\Omega)|^{1/q_{\rm p}}\|E\widehat{y}_{\rm e}\|_{L^{q_{\rm e}}(\mathbb{R}^d)} \le C,\\
&\|\nabla(\widehat{y}_{\rm e}^n\circ y_{\rm p}^n)\|_{L^q(\Omega)} \le \|\nabla E\widehat{y}_{\rm e}\|_{L^{q_{\rm e}}(y_{\rm p}^n(\Omega))}\|\nabla y_{\rm p}^n\|_{L^{q_{\rm p}}(\Omega)} \le C.
\end{aligned}$$

The latter shows that $\widehat{y}_{\rm e}^n\circ y_{\rm p}^n$ is bounded in $W^{1,q}(\Omega;\mathbb{R}^d)$. Moreover, we know that $y_{\rm p}^n$ converges uniformly to $y_{\rm p}$ on $\overline{\Omega}$. Hence we can choose a (not relabeled) subsequence such that $E\widehat{y}_{\rm e}\circ y_{\rm p}^n \rightharpoonup \widehat{y}_{\rm e}\circ y_{\rm p}$ in $W^{1,q}(\Omega;\mathbb{R}^d)$. This implies that $E\widehat{y}_{\rm e}\circ y_{\rm p}^n \to \widehat{y}_{\rm e}\circ y_{\rm p}$ in $L^{q^\#}(\partial\Omega)$ with $q^\# > (d-1)^2 \ge 1$, so that the convergence

$$\int_{\Gamma_D} |E\widehat{y}_{\rm e}(y_{\rm p}^n(x)) - x|\,{\rm d}\mathcal{H}^{d-1}(x) \to \int_{\Gamma_D} |\widehat{y}_{\rm e}(y_{\rm p}(x)) - x|\,{\rm d}\mathcal{H}^{d-1}(x)$$

holds. □

By using Lemma 3.5, starting from the discrete semistability (3.22), we readily check its time-continuous counterpart (3.23) for all times. This concludes the proof of Theorem 2.4.

Remark 3.6. We wish to point out that, to our best knowledge, necessary and (nontrivial) sufficient conditions allowing for an extension of a deformation mapping (as in (3.28)) but respecting additionally the orientation-preservation of the map are not known in general. We refer to [8] for some discussion on this issue.

Acknowledgements. This work has been funded by the Austrian Science Fund (FWF) and Czech Science Foundation (GAČR) through the joint international project FWF-GAČR I 4052, 19-29646L, and the scientific and technological cooperation project OeAD-MŠMT CZ 17/2016, 8J19AT013. Support by the Austrian Science Fund (FWF) projects F 65 and W 1245 is also acknowledged.

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