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ZERO-SUM RISK-SENSITIVE STOCHASTIC DIFFERENTIAL GAMES WITH REFLECTING DIFFUSIONS IN THE ORTHANT*

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Abstract. We study zero-sum games with risk-sensitive cost criterion on the infinite horizon where the state is a controlled reflecting diffusion in the nonnegative orthant. We consider two cost evaluation criteria: discounted cost and ergodic cost. Under certain assumptions, we establish the existence of saddle point equilibria. We obtain our results by studying the corresponding Hamilton–Jacobi–Isaacs equations. For the ergodic cost criterion, exploiting the stochastic representation of the principal eigenfunction, we have completely characterized saddle point equilibrium in the space of stationary Markov strategies.

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1. Introduction

This paper is devoted to studying zero-sum risk-sensitive stochastic differential games where the state process is a controlled reflecting diffusion process in the nonnegative orthant $\overline{D} \subset \mathbb{R}^d$. Such problems arise in communication systems with heavy traffic [41]. In communication networks different users may have different objectives leading to conflicts. Thus the analysis of such problems is often carried out using game theoretic framework. To motivate the problem treated in this paper we first summarize the network model studied in [41]. Consider a sequence of open queuing networks consisting of M users and d servers with increasing traffic intensity, where each user controls arrival rates implicitly and service rates explicitly. Suppose $Q_i^n(t)$ denote the number of customers at the ith service station at time t of the nth network and let $X_i^n(t) = \frac{1}{\sqrt{n}} Q_i^n(t)$. Let $X^n(\cdot) = (X_1^n(\cdot), \dots, X_d^n(\cdot))$. Then under certain heavy traffic conditions, the process $X_n(\cdot)$ converges weakly to a process $X(\cdot) = (X_1(\cdot), \dots, X_d(\cdot))$ given by

$$\mathrm{d}X_i(t) = b_i(X(t), u_1(t), \dots, u_M(t))\mathrm{d}t + \int_0^t \sqrt{\lambda_i(X_i(s))}\, \mathrm{d}W_i(s) + \sum_{j=1}^d \int_0^t \sqrt{p_{ji}\mu_j(X_j(s))}\, \mathrm{d}W_{ji}(s)$$

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$$
\begin{aligned}
&- \sum_{j=0}^{d} \sqrt{p_{ij}\mu_i(X_i(s))}\, \mathrm{d}W_{ij}(s) + \xi_i(t) - \sum_{j=1}^{d} p_{ji}\xi_j(s),\\
\mathrm{d}\xi_i(t) &= I_{\{X_i(t)=0\}}\, d\xi_i(t),\ \ t \geq 0,\ \ i = 1, \ldots, d,
\end{aligned}
$$

where $W_i(\cdot), W_{ij}(\cdot)$ are independent Wiener processes, $u_i(t), i = 1, \ldots, M, t \geq 0$, are actions chosen by the users at time t, and

$$
b_i(x, u_i, \ldots, u_M) = \bar{\lambda}_i(x, u_i, \ldots, u_M) + \sum_{j=1}^{d} p_{ji}\bar{\mu}_j(x, u_i, \ldots, u_M) - \bar{\mu}_i(x, u_i, \ldots, u_M).
$$

Here the functions $\lambda_i, \bar{\lambda}_i$ are associated with the arrival rates whereas $\mu_i, \bar{\mu}_i$ are associated with the service rates; $[p_{ij}]$ is the routing matrix. The above stochastic differential equation (SDE) represents a controlled diffusion process in the nonnegative orthant. By a solution to the above SDE, we mean a pair of continuous time processes $(X(\cdot), \xi(\cdot))$, where $X(t)$ takes values in the nonnegative orthant. The process $\xi(t)$ is a nondecreasing process which increases only at the boundary of the orthant.

We now describe the differential game problem. Each user (referred to as a player) considers the rest of the players as a single superplayer and tries to find a minimax equilibrium. This gives him an "optimal" strategy against the worst case scenario, *i.e.*, the aim of each player is to guarantee the best performance under the worst case behaviour of the superplayer. We can view the situation as follows: each player takes the rest of the players as his adversary. Since the actions of the superplayer are not completely known to the particular player, to achieve his security strategy against the worst case scenario, he assumes that he controls the arrival process, and the superplayer tries to block him by controlling the service time. Thus the particular player, say Player 1, controls the arrival process of the network, and the superplayer controls the service time process through their actions. Thus the drift b takes the form:

$$
b_i(x, u_1, u_2) = \bar{\lambda}_i(u_1) + \sum_{j=1}^{d} p_{ji}\bar{\mu}_j(x_j, u_2) - \bar{\mu}_i(x_i, u_2),
$$

where u_1 denotes the action of Player 1 and u_2 denotes the action of the superplayer, $u_1, u_2 \in [0, 1]$. We assign a cost to the particular player, *i.e.* Player 1 against the superplayers, as

$$
r(x, u_1, u_2) = \gamma u_1 - \theta_1 u_2 - c(x),
$$

where c typically represents the holding cost and $\gamma \geq 0$, $\theta_1 \geq 0$ are constants. When the state is x and players chose actions u_1, u_2, Player 1 incurs a cost at the rate $r(x, u_1, u_2)$. Naturally Player 1 tries to minimize his total cost through his actions u_1 whereas the superplayer tries to maximize the same through his actions u_2. Thus we have reduced the M-player game to a two-player zero-sum game. An analogous model in discrete time was first studied by Altman [2] which has been extended to the continuous time case in [41]. We emphasize that we have presented only a bare sketch of the queuing network model. For more details we refer to [41].

The above problem motivates us to study a stochastic differential game where the state process is given by the following SDE:

$$
\begin{aligned}
\mathrm{d}X(t) &= \bar{b}(X(t), u_1(t), u_2(t))\mathrm{d}t + \sigma(X(t))\mathrm{d}W(t) - \gamma(X(t))d\xi(t),\\
d\xi(t) &= I_{\{X(t)\in\partial D\}}\mathrm{d}\xi(t),\\
\xi(0) &= 0, \qquad X(0) = x \in \overline{D},
\end{aligned}
$$

where u_i is a U_i-valued process which is an appropriate strategy taken by the Player i for some given compact metric space U_i, for $i = 1, 2$, $\bar{b} : \overline{D} \times U_1 \times U_2 \to \mathbb{R}^d$ is the drift vector, $\sigma : \overline{D} \to \mathbb{R}^{d\times d}$ is the diffusion term,

$\gamma : \mathbb{R}^d \to \mathbb{R}^d$ is a vector field which determines the direction of the reflection of the process, and $W(\cdot)$ is an $\mathbb{R}^d$-valued standard Wiener process. It is clear from the state equation that inside D the process $X(\cdot)$ behaves like a diffusion process and when it hits the ∂D it gets reflected inward in a direction determined by the vector field γ. The process $\xi(\cdot)$ changes only when $X(\cdot)$ hits ∂D. As described earlier, such reflecting diffusion processes arise in many applications including the heavy traffic analysis of queuing networks coming from problems in manufacturing systems and communications (see [18, 20, 41, 51]). It is known that except for a few special cases, the direct analysis of problems in queuing network are very difficult to analyze. Thus, one tries to find a continuous approximation of it which can provide a tractable solution. It has been proved in [47, 53, 66] for uncontrolled case and in [51, 52] for controlled case that as the traffic intensity goes to unity, a suitably scaled and normalized sequence of queue length processes in open queuing networks, converge weakly in the Skorohod topology to a certain reflected diffusion processes when the networks are close to heavy traffic.

Associated to the above state dynamics we now briefly formulate the game problems. In this paper, we consider a particular type of non-standard zero-sum risk sensitive stochastic differential games on the infinite planning horizon with two cost evaluation criteria: risk-sensitive discounted cost and risk-sensitive ergodic (average) cost.

The risk-sensitive α-discounted cost for the zero-sum game is given by

$$\mathcal{J}_\alpha^{u_1,u_2}(\theta,x) \;:=\; \frac{1}{\theta}\ln E_x^{u_1,u_2}\left[e^{\theta\int_0^\infty e^{-\alpha t}\bar{r}(X(t),u_1(t),u_2(t))\mathrm{d}t}\right], x \in \overline{D},$$

and the same for the ergodic cost criterion is given by

$$\rho^{u_1,u_2}(\theta,x) \;= \limsup_{T\to\infty}\frac{1}{\theta T}\log E_x^{u_1,u_2}\left[e^{\theta\int_0^T \bar{r}(X(t),u_1(t),u_2(t))\mathrm{d}t}\right], x \in \overline{D}$$

where $\bar{r}$ is the running cost function. Player 1 wishes to minimize the above cost over his admissible strategies whereas Player 2 tries to maximize the same. Such a problem is relevant in a worst-case scenario, *e.g.*, in financial applications where an investor is trying to minimize his/her long-term portfolio loss against an antagonistic market, which by default is the maximizer in this case [12]. Such a problem is also relevant for the communication networks where each user treats the other users as a single (antagonistic) super-player [12]. A strategy u_1^* is called optimal (for the discounted cost criterion) for Player 1 if for $(\theta,x) \in (0,\Theta)\times\overline{D}$

$$\mathcal{J}_\alpha^{u_1^*,u_2}(\theta,x) \le \sup_{u_2}\inf_{u_1}\mathcal{J}_\alpha^{u_1,u_2}(\theta,x) := \underline{\mathcal{J}}_\alpha(\theta,x)\,(\text{lower value})$$

for any u_2. Similarly a strategy u_2^* is called optimal (for the discounted cost criterion) for Player 2 if for $(\theta,x) \in (0,\Theta)\times\overline{D}$

$$\mathcal{J}_\alpha^{u_1,u_2^*}(\theta,x) \ge \inf_{u_1}\sup_{u_2}\mathcal{J}_\alpha^{u_1,u_2}(\theta,x) := \bar{\mathcal{J}}_\alpha(\theta,x)\,(\text{upper value})$$

for any u_1. The game has value if $\bar{\mathcal{J}}_\alpha(\theta,x) = \underline{\mathcal{J}}_\alpha(\theta,x)$ for all $(\theta,x) \in (0,\Theta)\times\overline{D}$. A pair of strategies (u_1^*,u_2^*) for which this value is attained is called a saddle point equilibrium. For each of the cost evaluation criterion we study the zero-sum game via the corresponding Hamilton Jacobi Isaacs (HJI) equations to be described later. In this paper, for α-discounted cost evaluation criterion, we first derive that the corresponding HJI equation which is a non-linear second order parabolic partial differential equation with oblique boundary condition (instead of the elliptic p.d.e as in the risk-neutral case) and then establish the existence of a unique solution in an appropriate function space. We then obtain a saddle point equilibrium via appropriate outer maximizing/minimizing selectors of the Hamiltonian associated with the HJI equation. Using principal eigenvalue approach, we establish the existence of a solution to the ergodic HJI equation in an appropriate function space. A saddle point equilibrium is then given by maximizing/minimizing selector of the Hamiltonian of the ergodic HJI equation.

In order to study risk-sensitive stochastic optimal control problem in $\mathbb{R}^d$, similar eigenvalue approach has been used in [8], [16]. To prove the uniqueness of principal eigenvector in a certain class of functions, authors in [8], have exploited its stochastic representation. Without using any kind of blanket stability assumptions in [5], authors have studied risk-sensitive stochastic optimal control problem in $\mathbb{R}^d$. Applying nonlinear version of Krein-Rutman theorem the authors in [9], [6] have proved the existence of principal eigenpair for the ergodic HJB equation in smooth bounded domain when the direction of reflection is co-normal. Similar risk-sensitive zero-sum game problems where the state dynamics given by nondegenerate diffusions without state constraints have been studied in [12], [17]. The authors in [17], have completely characterized saddle point equilibria in the space of stationary Markov strategies. They have obtained their result by using principal eigenvalue approach. In [38], [64] the authors have considered one controller case of this problem. In this paper we extend the results to the two controller case with strictly opposing interests. The authors in [40] have studied zero-sum risk-sensitive stochastic differential games with reflecting (nondegenerate) diffusions in a smooth bounded domain. The risk-neutral counterpart of this problem has been studied in [39] the smooth bounded domain case and in [41] the positive orthant case. To study risk-neutral zero-sum game problems authors in [19], have introduced an occupation measure based approach.

Before presenting a brief survey of literature on stochastic differential games, we observe that for a (standard) zero-sum game (see [12]) we have

$$\bar{r}_1(x, u_1, u_2) + \bar{r}_2(x, u_1, u_2) = 0$$

for all $x \in \overline{D}, u_1 \in U_1, u_2 \in U_2$, where $\bar{r}_i : \overline{D} \times U_1 \times U_2 \to \mathbb{R}_+$, $i = 1, 2$, are the running cost functions. However, owing to the multiplicative nature of the evaluation criterion, we cannot say that this implies that the sum of the risk-sensitive discounted (or ergodic) cost is zero. In this case, if we set $\bar{r}_1 = -\bar{r}_2 = \bar{r}$, then for $\theta > 0$, Player 1 is risk-averse whereas Player 2 is risk-seeking. Thus, for risk-sensitive criterion standard zero-sum games must be studied via Nash equilibria for nonzero-sum games. Hence the zero-sum case studied in this paper is not a special case of the non-zero sum case.

We now present a brief survey of literature on zero-sum deterministic and (risk neutral) stochastic differential games. The origin of (zero-sum) differential games can be traced back to the late 1940's in the pioneering work done by Isaacs in RAND Corporation in USA. These works first appeared in a series of RAND corporation memoranda [48]. Isaacs incorporated these works and his subsequent research work in a book [49], which stimulated further work and interest in this field. Pontryagin and his collaborators [62], [61] have carried out their work on differential games independently in the erstwhile Soviet Union. The main motivation at that time was the study of military problems and pursuit-evasion games were the most important examples. Isaacs treated differential games in open loop strategies. Using formal arguments he derived the Hamilton-Jacobi-Bellman-Isaacs (HJBI) equations. If the HJBI equations has a smooth solution, then the differential game has a value. There are simple counterexamples to show that (barring a few specific cases) the theory of differential games as formulated by Isaacs is, in general, intractable from a mathematical point of view. To circumvent this difficulty various approaches have been carried out. Notable contributions have been made by Fleming [32], [33], Friedman [36], Roxin [67], Varaiya [70], Varaiya-Lin [71], Krasovskii-Subbotin [50], Berkovitz [15], Elliott-Kalton [30] and many others. Of all these approaches, Elliot-Kalton approach has become by far the most popular, presumably because of its close connections with viscosity solutions [28], [34]. One of the first major contributions to zero-sum stochastic differential games is due to Elliott [27]. Using feedback strategies and assuming Isaacs minimax condition, the existence of a value is established for nondegenerate controlled diffusion processes using a martingale method [27]; see also [29]. The pde approach to stochastic differential games involving nondegenerate diffusions is due to Bensoussan and Friedman; see[14], [37] and the references therein. The same problem with degenerate diffusions is, however, significantly more difficult from a technical point of view; see [35], [59], [69]. Note that in the nondegenerate case, under feedback strategies as in [27], the corresponding SDE has a unique weak solution under standard assumptions, thanks to Girsanov theorem. This is not the case for degenerate controlled diffusions. In a seminal work [35] Elliott-Kalton strategy is suitably extended to the stochastic setup to establish the dynamic programming principle (DPP) which in turn leads to viscosity solutions of the HJBI equations. Note that DPP is based on concatenation of controls and strategies. In the stochastic setup, concatenation of strategies runs into a serious technical issue. In [35], this issue is circumvented by introducing suitable r-strategies and appropriate approximation procedure. The work has led to various refinements of Elliott-Kalton strategies, *e.g.* pathwise strategies with delay in [22], [23], relaxed strategies in [68]. A new approach based on backward stochastic differential equations (BSDE) is introduced by Hamedene and Lepeltier [43], [44], [45] which was further refined in [21] using relaxed strategies with delay; see [63] as well. Note that all the works mentioned thus far is on finite horizon. On the infinite horizon the stochastic differential games (both zero-sum and nozero sum cases) involving nondegenerate diffusions is treated in [19] with both discounted and ergodic payoff criteria. Combining pde and probabilistic arguments the existence of saddle point/Nash equilibria is established in relaxed feedback strategies for relevant cases.

Though there is considerable literature on risk-sensitive optimal control, the corresponding literature on risk-sensitive stochastic differential games is rather sparse. In [26], risk-sensitive stochastic differential games (both zero and nonzero sum) for nondegenerate diffusions on the finite horizon is studied via BSDE. In [11], a particular class of stochastic differential games (both zero-sum and nonzero-sum) is studied for the nondegenerate diffusions on the finite horizon. In [12], [17], the zero-sum case for controlled diffusions is studied on the infinite horizon using pde approach. In [40], both zero-sum and nonzero-sum problems are studied for controlled reflecting diffusions in a smooth bounded domain. To our knowledge, there is no work on risk-sensitive stochastic differential games involving degenerate diffusions.

The rest of the paper is structured as follows. Section 2 deals with a detailed description of the problem. In Section 3 we study the α-discounted cost evaluation criterion. The ergodic cost evaluation criterion is analyzed in Section 4. We conclude our paper in Section 5 with some concluding remarks.

2. Problem description

Let $D = \{x \in \mathbb{R}^d : x_i > 0, \forall\, i = 1, 2, \ldots, d\}$. Let U_i, $i = 1, 2$, be compact metric spaces. Let $V_i = \mathcal{P}(U_i)$ be the space of probability measures on U_i with topology of weak convergence. Let

$$\bar{b} = (\bar{b}_1, \cdots, \bar{b}_d) : \overline{D} \times U_1 \times U_2 \to \mathbb{R}^d,$$

$$\sigma : \overline{D} \to \mathbb{R}^{d\times d},\ \sigma = [\sigma_{ij}(\cdot)]_{1\le i,j\le d},$$

and let γ be an $\mathbb{R}^d$-valued function defined in a neighbourhood of ∂D. We consider a stochastic differential game whose state is evolving according to a controlled reflected diffusion in the orthant $\overline{D}$, given by the solution of the reflected stochastic differential equation (RSDE)

$$\left.\begin{aligned} \mathrm{d}X(t) &= b(X(t), v_1(t), v_2(t))\mathrm{d}t + \sigma(X(t))\mathrm{d}W(t) - \gamma(X(t))\mathrm{d}\xi(t),\\ \mathrm{d}\xi(t) &= I_{\{X(t)\in\partial D\}}\mathrm{d}\xi(t),\\ \xi(0) &= 0, \qquad X(0) = x \in \overline{D}, \end{aligned}\right\} \tag{2.1}$$

where the drift term $b(x, v_1, v_2)$ is given by

$$b(x, v_1, v_2) = \int_{U_2}\int_{U_1} \bar{b}(x, u_1, u_2)v_1(\mathrm{d}u_1)v_2(\mathrm{d}u_2),$$

$W = (W_1, \cdots, W_d)$ is an $\mathbb{R}^d$-valued standard Wiener process and $v_i(t) = f_i(t, X([0, t]))$, where $f_i : [0, \infty) \times C([0, \infty); \overline{D}) \to V_i$ is a measurable map and $X([0, t])(s) = X(t \wedge s)$, $\forall\ s \in [0, \infty)$. $v_i, i = 1, 2$, above is called an admissible (feedback) strategy for the ith player. For a physical interpretation of this class of strategies we refer to [19]. Let $\mathcal{A}_i$, $i = 1, 2$, denote the set of all admissible strategies of the ith player. An admissible strategy $v_i \in \mathcal{A}_i$ is said to be a Markov strategy if $v_i(t) = \overline{v}_i(t, X(t))$ for a measurable $\overline{v}_i : [0, \infty) \times \overline{D} \to V_i$. By an abuse of notation the map $\overline{v}_i$ itself is called a Markov strategy for the ith player. If $\overline{v}_i$ has no explicit time dependence then the Markov strategy $\overline{v}_i$ is called a stationary Markov strategy. Let $\mathcal{M}_i, \mathcal{S}_i$, $i = 1, 2$, denote the set of Markov, stationary Markov strategies, respectively, for the ith player.

We now proceed to prove the existence of a solution of (2.1). To this end we approximate $\overline{D}$ by appropriate smooth domains. Define

$$D_l^{'} = D \cap B(0, l), \text{ for } l = 1, 2, \ldots .$$

where $B(0, l) = \{x \in \mathbb{R}^d : ||x|| < l\}$. From Theorem (A2)(ii) and the remark in p.28 of [24] there exist domains $D_{l,m} \subset \mathbb{R}^d$ with C^∞ boundary satisfying the following conditions:

(i) The distance $d(\partial D_{l,m}, \partial D_l^{'}) < \frac{1}{m}$, $l \geq 1$,

(ii) $D_{l,m_1} \subset D_{l,m_2}$, $m_1 \leq m_2, l \geq 1$ and $D_{l_1,m} \subset D_{l_2,m}$, $l_1 \leq l_2, m \geq 1$.

Define

$$D_m = \cup_{l=1}^{\infty} D_{l,m}, m \geq 1.$$

From the construction it is clear that

(i) $D_m \uparrow \overline{D}$ where for each $m \geq 1$; D_m is a domain with C^∞ boundary.
(ii) For any smooth (*i.e.*, C^2) compact subset $C \subset \overline{D}$, we have $C \subset D_{l,m}$ for sufficiently large l, m.

Now we make the following assumptions to ensure the existence of a unique solution of (2.1).

(A0) (i) The function $\bar{b}$ is bounded, jointly continuous, Lipschitz continuous in its first argument uniformly with respect to the rest.

(ii) For $i, j = 1, 2, \ldots, d$, the functions σ_{ij} are bounded and Lipschitz continuous.

(iii) The function $a := \sigma\sigma^\perp$ (where $\sigma^\perp$ is the transpose of σ) is uniformly elliptic with ellipticity constant $\delta_0\,(> 0)$, *i.e.*,

$$za(x)z^\perp \geq \delta_0 |z|^2,\ x \in \overline{D},\ z \in \mathbb{R}^d.$$

(A1) The function $\gamma = (\gamma_1, \ldots, \gamma_d)$ is such that each of the component $\gamma_i \in C_b(\mathbb{R}^d)$, and there exist $\delta_1 > 0$ such that:

(i)

$$\gamma(x) \cdot \eta^i(x) \geq \delta_1 > 0 \text{ for all } x \in \Sigma_i,\ i = 1, \ldots, d,$$

where $\Sigma_i = \{x \in \mathbb{R}^d : x_i = 0\}$, and $\eta^i(\cdot)$ denotes the outward normal to Σ_i.

(ii) for all m sufficiently large

$$\gamma(x) \cdot \eta^m(x) \geq \delta_1 > 0 \text{ for all } x \in \partial D_m \cap G_j,\ \ j = 1, \ldots, d,$$

where G_j is a fixed neighbourhood of Σ_j and $\eta^m(\cdot)$ denotes the outward normal to ∂D_m.

Set

$$\Sigma = \{x \in \cup_{i=1}^d \Sigma_i : x \notin \cap_{j=1}^k \Sigma_{l_j},\ k \geq 2,\ l_j \in \{1, 2, \ldots, d\}\},$$

which is the smooth part of ∂D.

Under (A0) and (A1), for any pair of admissible strategies $(v_1, v_2) \in \mathcal{A}_1 \times \mathcal{A}_2$, it has been proved in [10] that (2.1) has a unique weak solution in $\overline{D}$. The existence of a solution of (2.1) is usually achieved in the following steps: First consider the zero drift case and prove the existence of a unique strong solution as follows.:

(i) Prove that (2.1) has a weak solution in the smooth domain $\overline{D}_m, m \geq 1$;
(ii) using convergence arguments prove that (2.1) has a weak solution in $\overline{D}$;
(iii) prove the pathwise uniqueness as in Lemma 3.3 of [10].

Now to prove the existence of a unique weak solution under an admissible strategy of (2.1) with nonzero drift use Girsanov transformation method; see pp. 42–44 in [7]. Adapting the approach by Zovokin and Veretenikov (see for example [73]) one can prove that under a Markov control (2.1) has a unique strong solution. For more details see Theorem 3.2 in [10].

Throughout this paper we make the following assumption.

(A2) Let $\Sigma^{'}$ denotes the non-smooth part of ∂D (clearly the surface measure of $\Sigma^{'}$ is zero). We assume that $P_x^{v_1,v_2}(X(t) \in \Sigma^{'}$ for some $t \geq 0) = 0$, for each $x \in D$, where $P_x^{v_1,v_2}$ is the probability measure on the space over which $X(\cdot)$ is a weak solution of (2.1) corresponding to a strategy pair (v_1, v_2) and initial state x.

We refer to [41], [57] for sufficient conditions ensuring **(A2)**.

Let $\bar{r} : \overline{D} \times U_1 \times U_2 \to \mathbb{R}_+$ be the running cost function. Define $r : \overline{D} \times V_1 \times V_2 \to \mathbb{R}_+$ by

$$r(x, v_1, v_2) = \int_{U_2} \int_{U_1} \bar{r}(x, u_1, u_2) v_1(\mathrm{d}u_1) v_2(\mathrm{d}u_2).$$

We assume that

(A3) $\bar{r}$ is bounded, jointly continuous and Lipschitz continuous in its first argument uniformly with respect to the rest.

We study two cost evaluation criteria: discounted cost and ergodic cost.

2.1. Discounted cost criterion

For $(v_1, v_2) \in \mathcal{A}_1 \times \mathcal{A}_2$, the risk-sensitive α- discounted cost for initial condition $x \in \overline{D}$ is given by

$$\mathcal{J}_\alpha^{v_1,v_2}(\theta, x) \ := \ \frac{1}{\theta} \ln E_x^{v_1,v_2} \left[e^{\theta \int_0^\infty e^{-\alpha t} r(X(t), v_1(t), v_2(t)) \mathrm{d}t} \right], x \in \overline{D}, \tag{2.2}$$

where $X(\cdot)$ is the solution of the (RSDE)(2.1) corresponding to $(v_1, v_2) \in \mathcal{A}_1 \times \mathcal{A}_2$, $E_x^{v_1,v_2}$ is the expectation with respect to the law of the process (2.1) with initial condition x and $\alpha > 0$ is the discount factor. The risk-sensitive parameter $\theta \in (0, \Theta), \Theta > 0$ is chosen by the minimizer. For a given choice of θ, the minimizer (Player 1) tries to minimize (2.2) over his/her admissible strategies $\mathcal{A}_1$, whereas maximizer (Player 2) tries to maximize the same over $\mathcal{A}_2$. A strategy $v_1^* \in \mathcal{A}_1$ is said to be optimal for Player 1 if for $(\theta, x) \in (0, \Theta) \times \overline{D}$

$$\mathcal{J}_\alpha^{v_1^*,v_2}(\theta, x) \leq \sup_{v_2 \in \mathcal{A}_2} \inf_{v_1 \in \mathcal{A}_1} \mathcal{J}_\alpha^{v_1,v_2}(\theta, x) := \underline{\mathcal{J}}_\alpha(\theta, x) \, (\text{lower value})$$

for any $v_2 \in \mathcal{A}_2$. Similarly, a strategy $v_2^* \in \mathcal{A}_2$ is said to be optimal for Player 2 if for $(\theta, x) \in (0, \Theta) \times \overline{D}$

$$\mathcal{J}_\alpha^{v_1,v_2^*}(\theta, x) \geq \inf_{v_1 \in \mathcal{A}_1} \sup_{v_2 \in \mathcal{A}_2} \mathcal{J}_\alpha^{v_1,v_2}(\theta, x) := \bar{\mathcal{J}}_\alpha(\theta, x) \, (\text{upper value})$$

for any $v_1 \in \mathcal{A}_1$. If $\bar{\mathcal{J}}_\alpha(\theta, x) = \underline{\mathcal{J}}_\alpha(\theta, x) := \mathcal{J}_\alpha(\theta, x)$ for all $(\theta, x) \in (0, \Theta) \times \overline{D}$ then the game is said to admit a value, and the function $\mathcal{J}_\alpha(\theta, x)$ is called an α-discounted value function. A pair of strategies (v_1^*, v_2^*) for which this value is attained is called a saddle point equilibrium, and then v_1^* is optimal for Player 1 and v_2^* is optimal for Player 2.

2.2. Ergodic cost criterion

For $(v_1, v_2) \in \mathcal{A}_1 \times \mathcal{A}_2$, the risk sensitive ergodic cost is defined by

$$\rho^{v_1,v_2}(\theta, x) \;= \limsup_{T\to\infty} \frac{1}{\theta T} \log E_x^{v_1,v_2}\Big[e^{\theta\int_0^T r(X(t),v_1(t),v_2(t))\mathrm{d}t}\Big], x \in \overline{D}. \tag{2.3}$$

Optimal strategies, saddle point equilibrium *etc.*, for this cost criterion are defined analogously.

For $(v_1, v_2) \in V_1 \times V_2$ and for a suitable function $f : \overline{D} \to \mathbb{R}$, write

$$L^{v_1,v_2} f(x) = \int_{U_1}\int_{U_2} Lf(x, u_1, u_2)v_1(\mathrm{d}u_1)v_2(\mathrm{d}u_2),$$

where

$$Lf(x, u_1, u_2) = L^{u_1,u_2} f(x) = \langle \bar{b}(x, u_1, u_2), \nabla f(x)\rangle + \frac{1}{2}\,\mathrm{trace}(a(x)\nabla^2 f(x)). \tag{2.4}$$

Define

$$\mathcal{L}f(x, u_1, u_2) = L^{u_1,u_2} f(x) + \bar{r}(x, u_1, u_2)f(x). \tag{2.5}$$

3. Analysis of discounted cost criterion

In this section, we consider the discounted cost criterion. To carry out our analysis for the α-discounted cost criterion we use the criterion

$$\beta_\alpha^{v_1,v_2}(\theta, x) \;:=\; E_x^{v_1,v_2}\Big[e^{\theta\int_0^\infty e^{-\alpha t} r(X(t),v_1(t),v_2(t))\mathrm{d}t}\Big]. \tag{3.1}$$

Since logarithm is an increasing function, optimal strategies for the criterion (2.2) are also optimal strategies for the above criterion. Corresponding to above cost criterion, the value functions are defined by

$$\inf_{v_1\in\mathcal{A}_1}\sup_{v_2\in\mathcal{A}_2} \beta_\alpha^{v_1,v_2}(\theta, x) := \bar{u}_\alpha(\theta, x),$$

and

$$\sup_{v_2\in\mathcal{A}_2}\inf_{v_1\in\mathcal{A}_1} \beta_\alpha^{v_1,v_2}(\theta, x) := \underline{u}_\alpha(\theta, x).$$

Usually the value function of a differential game is associated with the solution of a nonlinear partial differential equation which is referred to as HJI equation. Using dynamic programming heuristics as in [38], [58], the HJI equation for α-discounted cost criterion is given by

$$\begin{aligned}
\alpha\theta\frac{\partial u_\alpha(\theta,x)}{\partial\theta} &= \inf_{v_1\in V_1}\sup_{v_2\in V_2}\Big[\langle b(x,v_1,v_2), \nabla u_\alpha(\theta,x)\rangle + \theta r(x,v_1,v_2)u_\alpha\Big] + \frac{1}{2}\,\mathrm{trace}(a(x)\nabla^2 u_\alpha(\theta,x))\\
&= \sup_{v_2\in V_2}\inf_{v_1\in V_1}\Big[\langle b(x,v_1,v_2), \nabla u_\alpha(\theta,x)\rangle + \theta r(x,v_1,v_2)u_\alpha\Big] + \frac{1}{2}\,\mathrm{trace}(a(x)\nabla^2 u_\alpha(\theta,x)),\\
u_\alpha(0,x) &= 1 \text{ on } \overline{D},\;\; \nabla u_\alpha(\theta,x)\cdot\gamma(x) = 0 \text{ on } (0,\Theta)\times\partial D.
\end{aligned} \tag{3.2}$$

For a smooth solution u_α, the equality of “inf sup” and “sup inf” follows from Fan’s minimax theorem [31].

The singularity in θ at 0 and the non-smooth nature of the orthant pose technical difficulties in solving the p.d.e (3.2). To overcome these difficulties we use suitable approximation arguments which involves approximation of (3.2) by a family of p.d.e.s in the smooth bounded domains $D_{l,m}$. Consider the p.d.e.

$$\begin{aligned}\alpha\theta\frac{\partial u^{\kappa}_{\alpha,l,m}(\theta,x)}{\partial\theta} &= \inf_{v_1\in V_1}\sup_{v_2\in V_2}\Big[\langle b(x,v_1,v_2),\nabla u^{\kappa}_{\alpha,l,m}(\theta,x)\rangle + \theta r(x,v_1,v_2)u^{\kappa}_{\alpha,l,m}\Big] \\ &\quad + \frac{1}{2}\ \mathrm{trace}(a(x)\nabla^2 u^{\kappa}_{\alpha,l,m}(\theta,x)), \\ &= \sup_{v_2\in V_2}\inf_{v_1\in V_1}\Big[\langle b(x,v_1,v_2),\nabla u^{\kappa}_{\alpha,l,m}(\theta,x)\rangle + \theta r(x,v_1,v_2)u^{\kappa}_{\alpha,l,m}\Big] \\ &\quad + \frac{1}{2}\ \mathrm{trace}(a(x)\nabla^2 u^{\kappa}_{\alpha,l,m}(\theta,x)), \\ u^{\kappa}_{\alpha,l,m}(\kappa,x) &= e^{\dfrac{\kappa\|r\|_\infty}{\alpha}} \ \text{ on } \overline{D}_{l,m},\quad \nabla u^{\kappa}_{\alpha,l,m}(\theta,x)\cdot\gamma(x) = 0 \ \text{ on } (\kappa,\Theta)\times\partial D_{l,m},\end{aligned} \tag{3.3}$$

for each $l,m\geq 1$ and $0<\kappa<\Theta$. We now want to prove that these family of p.d.e.s admits a solution in a suitable class of functions.

Lemma 3.1. *Under (A0)–(A3), (3.3) has a unique solution $u^{\kappa}_{\alpha,l,m}\in W^{1,2,p}((\kappa,\Theta)\times D_{l,m})$, $2\leq p<\infty$, and*

$$\|u^{\kappa}_{\alpha,l,m}\|_{(\kappa,\Theta)\times\overline{D}_{l,m},\infty} \ \leq\ e^{\frac{\Theta\|r\|_\infty}{\alpha}},\ \text{for all } m,l\geq 1, 0<\kappa<\Theta. \tag{3.4}$$

$$\|\frac{\partial u^{\kappa}_{\alpha,l,m}}{\partial\theta}\|_{(\kappa,\Theta)\times\overline{D}_{l,m},\infty} \ \leq\ 3e^{\frac{(\Theta+3)\|r\|_\infty}{\alpha}}\frac{\|r\|_\infty}{\alpha},\ \text{for all } m,l\geq 1, 0<\kappa<\Theta. \tag{3.5}$$

Proof. Follows from Theorem 4.1 in [40]. □

Following theorem proves the existence of a solution to the limiting p.d.e. of the above family of p.d.e.s.

Theorem 3.2. *Under (A0)–(A3), (3.2) admits a solution u_α in $W^{1,2,p}_{loc}((0,\Theta)\times(D\cup\Sigma))\cap C^{0,1}((0,\Theta)\times(D\cup\Sigma))$, $2\leq p<\infty$.*

Proof. Let Q be an open bounded domain with C^2 boundary in $\overline{D}$. Then there exists a positive integer M_1, N_1 such that $\overline{Q}\subset D_{l,m}$ for all $m\geq M_1, l\geq N_1$. Lemma 3.1 implies that the p.d.e (3.3) has a unique solution $u^{\kappa}_{\alpha,l,m}$ in $W^{1,2,p}((\kappa,\Theta)\times D_{l,m})$, $2\leq p<\infty$ and the estimates (3.4), (3.5) hold. Furthermore, by an application of a standard measurable selection theorem [13], there exist measurable maps $\bar{v}_{i,l,m}:(0,\Theta)\times\overline{D}_{l,m}\to V_i$, $i=1,2$, such that

$$\begin{aligned}&\inf_{v_1\in V_1}\sup_{v_2\in V_2}\Big[\langle b(x,v_1,v_2),\nabla u^{\kappa}_{\alpha,l,m}(\theta,x)\rangle + \theta r(x,v_1,v_2)u^{\kappa}_{\alpha,l,m}\Big] = \\ &\sup_{v_2\in V_2}\Big[\langle b(x,\bar{v}_{1,l,m}(\theta,x),v_2),\nabla u^{\kappa}_{\alpha,l,m}(\theta,x)\rangle + \theta r(x,\bar{v}_{1,l,m}(\theta,x),v_2)u^{\kappa}_{\alpha,l,m}\Big],\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}&\sup_{v_2\in V_2}\inf_{v_1\in V_1}\Big[\langle b(x,v_1,v_2),\nabla u^{\kappa}_{\alpha,l,m}(\theta,x)\rangle + \theta r(x,v_1,v_2)u^{\kappa}_{\alpha,l,m}\Big] = \\ &\inf_{v_1\in V_1}\Big[\langle b(x,v_1,\bar{v}_{2,l,m})(\theta,x),\nabla u^{\kappa}_{\alpha,l,m}(\theta,x)\rangle + \theta r(x,v_1,\bar{v}_{2,l,m}(\theta,x))u^{\kappa}_{\alpha,l,m}\Big].\end{aligned} \tag{3.7}$$

We rewrite the p.d.e (3.3) as a parametric family of p.d.e.s as follows:

$$\frac{1}{2}\ \text{trace}(a(x)\nabla^2 u^{\kappa}_{\alpha,l,m}) + \langle b(x,\bar{v}_{1,l,m}(\theta,x),\bar{v}_{2,l,m}(\theta,x)),\nabla u^{\kappa}_{\alpha,l,m}\rangle = f_{l,m},$$
$$u^{\kappa}_{\alpha,l,m}(\kappa,x) = e^{\frac{\kappa\|r\|_\infty}{\alpha}}\ \text{on}\ \overline{D}_{l,m},\quad \nabla u^{\kappa}_{\alpha,l,m}(\theta,x)\cdot\gamma(x) = 0\ \text{on}\ (\kappa,\Theta)\times\partial D_{l,m},$$

where

$$f_{l,m}(\theta,x) = \alpha\theta\frac{\partial u^{\kappa}_{\alpha,l,m}(\theta,x)}{\partial\theta} - \theta r(x,\bar{v}_{1,l,m}(\theta,x),\bar{v}_{2,l,m}(\theta,x))u^{\kappa}_{\alpha,l,m}(\theta,x).$$

Let $\tilde{b}_{l,m}(\theta,x) = b(x,\bar{v}_{1,l,m}(\theta,x),\bar{v}_{2,l,m}(\theta,x))$. Then by our assumptions and Lemma 3.1, it is clear that for each $\theta\in(0,\Theta)$

$$\sup_{l,m}\|\tilde{b}_{l,m}(\theta,\cdot)\|_{\infty;D_{l,m}} < \infty,\ \sup_{l,m}\|f_{l,m}(\theta,\cdot)\|_{\infty;D_{l,m}} < \infty.$$

Now using Theorem 9.11 from [42] (see also,[38] Subsect. (1.6)), we obtain

$$\|u^{\kappa}_{\alpha,l,m}\|_{1,2,p;(\kappa,\Theta)\times Q} < \hat{\tilde{k}}_1,\ \text{for all}\ \ m\geq M_1, l\geq N_1, p\geq 2, \tag{3.8}$$

where the constant $\hat{\tilde{k}}_1$ is independent of m,l.

We now choose an increasing sequence $\{Q_n\}$ of bounded domains from D such that $D\cup\Sigma = \cup_{n\geq 1}\overline{Q}_n$ and $\partial Q_n\cap\partial D$ is a C^2 portion of ∂D. Therefore a standard diagonalization procedure implies that there exists $u^{\kappa}_{\alpha,m}\in W^{1,2,p}_{loc}((\kappa,\Theta)\times D_m)$, $2\leq p<\infty$, such that along a subsequence as $l\to\infty$

$$u^{\kappa}_{\alpha,l,m} \longrightarrow u^{\kappa}_{\alpha,m}\ \text{ weakly in } W^{1,2,p}((\kappa,\Theta)\times Q). \tag{3.9}$$

As in (3.8), we have

$$\|u^{\kappa}_{\alpha,m}\|_{1,2,p;(\kappa,\Theta)\times Q} < \hat{\tilde{k}}_1,\ \text{for all}\ \ m\geq M_1, p\geq 2.$$

Again repeating the diagonalization argument, there exists $u^{\kappa}_{\alpha}\in W^{1,2,p}_{loc}((\kappa,\Theta)\times(D\cup\Sigma))$, $2\leq p<\infty$, such that as $m\to\infty$

$$u^{\kappa}_{\alpha,m} \longrightarrow u^{\kappa}_{\alpha}\ \text{ weakly in } W^{1,2,p}((\kappa,\Theta)\times Q). \tag{3.10}$$

By the parabolic version of Morrey's lemma ([72], pp. 26-27), $W^{1,2,p}((\kappa,\Theta)\times Q)$ for $d+2<p<\infty$, is compactly contained in $C^{\frac{\hat{\alpha}}{2},\hat{\alpha}}((\kappa,\Theta)\times\overline{Q})$, $0<\hat{\alpha}<2-\frac{d+2}{p}$. Thus, along a suitable subsequence, we get

$$\lim_{m\to\infty}\lim_{l\to\infty} u^{\kappa}_{\alpha,l,m} = u^{\kappa}_{\alpha}\ \text{ in } C^{\frac{\hat{\alpha}}{2},\hat{\alpha}}((\kappa,\Theta)\times\overline{Q}). \tag{3.11}$$

Using (A1), (A3) and (3.11), we have

$$\begin{aligned}&\lim_{m\to\infty}\lim_{l\to\infty}\inf_{v_1\in V_1}\sup_{v_2\in V_2}\Big[\langle b(x,v_1,v_2),\nabla u^{\kappa}_{\alpha,l,m}(\theta,x)\rangle+\theta r(x,v_1,v_2)u^{\kappa}_{\alpha,l,m}\Big]=\\&\inf_{v_1\in V_1}\sup_{v_2\in V_2}\Big[\langle b(x,v_1,v_2),\nabla u^{\kappa}_{\alpha}(\theta,x)\rangle+\theta r(x,v_1,v_2)u^{\kappa}_{\alpha}\Big],\ \text{and}\\&\lim_{m\to\infty}\lim_{l\to\infty}\sup_{v_2\in V_2}\inf_{v_1\in V_1}\Big[\langle b(x,v_1,v_2),\nabla u^{\kappa}_{\alpha,l,m}(\theta,x)\rangle+\theta r(x,v_1,v_2)u^{\kappa}_{\alpha,l,m}\Big]=\\&\sup_{v_2\in V_2}\inf_{v_1\in V_1}\Big[\langle b(x,v_1,v_2),\nabla u^{\kappa}_{\alpha}(\theta,x)\rangle+\theta r(x,v_1,v_2)u^{\kappa}_{\alpha}\Big],\ \text{a.e.}\end{aligned}\tag{3.12}$$

Using (3.10), (3.12), letting $l\to\infty$ and then $m\to\infty$ in (3.3), we obtain

$$\alpha\theta\frac{\partial u^{\kappa}_{\alpha}(\theta,x)}{\partial\theta}=\inf_{v_1\in V_1}\sup_{v_2\in V_2}\Big[\langle b(x,v_1,v_2),\nabla u^{\kappa}_{\alpha}(\theta,x)\rangle+\theta r(x,v_1,v_2)u^{\kappa}_{\alpha}\Big]+\frac{1}{2}\operatorname{trace}(a(x)\nabla^2u^{\kappa}_{\alpha}(\theta,x)),\tag{3.13}$$

in the sense of distribution. Since $u^{\kappa}_{\alpha}\in W^{1,2,p}((\kappa,\Theta)\times Q)$ for any compact subset $\overline{Q}$ of $\overline{D}$ with C^2 smooth boundary, u^{κ}_{α} satisfies (3.13) strongly. Since $u^{\kappa}_{\alpha,l,m}(\kappa,x)=e^{\frac{\kappa\|r\|_{\infty}}{\alpha}}$ on $\overline{D}_{l,m}$, for all $l\geq N_1, m\geq M_1$, by (3.11), we have $u^{\kappa}_{\alpha}(\kappa,x)=e^{\frac{\kappa\|r\|_{\infty}}{\alpha}}$ on $\overline{D}$.

We now want to prove that $\nabla u^{\kappa}_{\alpha}(\theta,x)\cdot\gamma(x)=0$ a.e. on $(\kappa,\Theta)\times\partial D$. From the construction of $D_{l,m}$ it is clear that for any point $\tilde{x}_0\in\Sigma$, there exists a sequence $\{x_{l,m}\}_{l,m}$ such that $x_{l,m}\in\partial D_{l,m}$ and $x_{l,m}\longrightarrow\tilde{x}_0$ as $m,l\to\infty$. Since γ is continuous and $u^{\kappa}_{\alpha}(\theta,\cdot)\in C^1(D\cup\Sigma)$, thus from (3.11), we get

$$\nabla u^{\kappa}_{\alpha}(\theta,\tilde{x}_0)\cdot\gamma(\tilde{x}_0)=\lim_{m\to\infty}\lim_{l\to\infty}\nabla u^{\kappa}_{\alpha,l,m}(\theta,x_{l,m})\cdot\gamma(x_{l,m})=0.$$

Hence $\nabla u^{\kappa}_{\alpha}(\theta,x)\cdot\gamma(x)=0$ a.e. on $(\kappa,\Theta)\times\partial D$ since the surface measure of Σ' (non-smooth part of ∂D) is zero. Therefore by an application of Fan's minimax theorem [31], we have

$$\begin{aligned}\alpha\theta\frac{\partial u^{\kappa}_{\alpha}(\theta,x)}{\partial\theta}&=\inf_{v_1\in V_1}\sup_{v_2\in V_2}\Big[\langle b(x,v_1,v_2),\nabla u^{\kappa}_{\alpha}(\theta,x)\rangle+\theta r(x,v_1,v_2)u^{\kappa}_{\alpha}\Big]+\frac{1}{2}\operatorname{trace}(a(x)\nabla^2u^{\kappa}_{\alpha}(\theta,x)),\\&=\sup_{v_2\in V_2}\inf_{v_1\in V_1}\Big[\langle b(x,v_1,v_2),\nabla u^{\kappa}_{\alpha}(\theta,x)\rangle+\theta r(x,v_1,v_2)u^{\kappa}_{\alpha}\Big]+\frac{1}{2}\operatorname{trace}(a(x)\nabla^2u^{\kappa}_{\alpha}(\theta,x)),\\u^{\kappa}_{\alpha}(\kappa,x)&=e^{\frac{\kappa\|r\|_{\infty}}{\alpha}}\ \text{on}\ \overline{D},\ \ \nabla u^{\kappa}_{\alpha}(\theta,x)\cdot\gamma(x)=0\ \text{on}\ (\kappa,\Theta)\times\partial D,\end{aligned}\tag{3.14}$$

has a solution in $W^{1,2,p}_{loc}((\kappa,\Theta)\times(D\cup\Sigma))\cap C^{0,1}((\kappa,\Theta)\times(D\cup\Sigma))$, $p\geq 2$.

We now extend the function u^{κ}_{α} to whole of $(0,\Theta)$ as follows

$$\bar{u}^{\kappa}_{\alpha}(\theta,x)\ =\ \begin{cases}u^{\kappa}_{\alpha}(\theta,x) & \text{if}\quad \theta>\kappa\\ e^{\frac{\kappa\|r\|_{\infty}}{\alpha}} & \text{if}\quad \theta\leq\kappa.\end{cases}$$

Thus, $\bar{u}^{\kappa}_{\alpha}$ is nonnegative, bounded, continuous. As in (3.4), (3.5) and (3.8) it is clear that for each compact $Q\subset\overline{D}$ with C^2 smooth boundary,

$$\sup_{0<\kappa<\Theta}\|\bar{u}^{\kappa}_{\alpha}\|_{2,p;Q}\ <\ \infty,\ \text{ and }\ \sup_{0<\kappa<\Theta}\|\frac{\partial\bar{u}^{\kappa}_{\alpha}}{\partial\theta}\|_{(0,\Theta)\times D\cup\Sigma,\infty}\ <\ \infty.$$

One can see that for each $0<\theta<\Theta$, the function $\bar{u}^{\kappa}_{\alpha}$ satisfies the following p.d.e. in the almost everywhere sense

$$\begin{aligned}\alpha\theta\frac{\partial\bar{u}^{\kappa}_{\alpha}(\theta,x)}{\partial\theta} &= \inf_{v_1\in V_1}\sup_{v_2\in V_2}\Big[\langle b(x,v_1,v_2),\nabla\bar{u}^{\kappa}_{\alpha}(\theta,x)\rangle+\theta r(x,v_1,v_2)\bar{u}^{\kappa}_{\alpha}\Big]+\frac{1}{2}\operatorname{trace}(a(x)\nabla^2\bar{u}^{\kappa}_{\alpha}(\theta,x))\\ &\quad -\theta e^{\frac{\kappa\|r\|_{\infty}}{\alpha}}\inf_{v_1\in V_1}\sup_{v_2\in V_2} r(x,v_1,v_2)I_{\{\theta\leq\kappa\}},\\ \bar{u}^{\kappa}_{\alpha}(\kappa,x) &= e^{\frac{\kappa\|r\|_{\infty}}{\alpha}} \text{ on } \overline{D},\ \nabla\bar{u}^{\kappa}_{\alpha}(\theta,x)\cdot\gamma(x)=0 \text{ on } (0,\Theta)\times\partial D. \end{aligned} \tag{3.15}$$

Thus, $\bar{u}^{\kappa}_{\alpha}$ is a solution to (3.15) in $W^{1,2,p}_{loc}((0,\Theta)\times(D\cup\Sigma))\cap C^{\frac{\hat{\alpha}}{2},\hat{\alpha}}((0,\Theta)\times\overline{Q})$ for each bounded C^2 domain Q in D. Now multiplying (3.15) by a test function $\varphi\in C^{\infty}_c((0,\theta)\times(D\cup\Sigma))$ and integrating over $(0,\theta)\times(D\cup\Sigma)$, we get

$$\begin{aligned}&-\int_0^{\Theta}\int_{D\cup\Sigma}\alpha\theta\frac{\partial\bar{u}^{\kappa}_{\alpha}}{\partial\theta}\varphi \mathrm{d}\theta\mathrm{d}x+\int_0^{\Theta}\int_{D\cup\Sigma}\inf_{v_1\in V_1}\sup_{v_2\in V_2}\Big[\langle b(x,v_1,v_2),\nabla\bar{u}^{\kappa}_{\alpha}\rangle\\ &+\theta r(x,v_1,v_2)\bar{u}^{\kappa}_{\alpha}\Big]\varphi\mathrm{d}\theta\mathrm{d}x+\frac{1}{2}\int_0^{\Theta}\int_{D\cup\Sigma}\operatorname{trace}(a(x)\nabla^2\bar{u}^{\kappa}_{\alpha})\varphi\mathrm{d}\theta\mathrm{d}x\\ &=\int_0^{\kappa}\int_{D\cup\Sigma}\inf_{v_1\in V_1}\sup_{v_2\in V_2}\theta r(x,v_1,v_2)e^{\frac{\kappa\|r\|_{\infty}}{\alpha}}\varphi\mathrm{d}\theta\mathrm{d}x. \end{aligned}\tag{3.16}$$

In view of the above estimates, there exists $u_{\alpha}\in W^{1,2,p}_{loc}((0,\Theta)\times(D\cup\Sigma))\cap C^{\frac{\hat{\alpha}}{2},\hat{\alpha}}((0,\Theta)\times\overline{Q})$ for each bounded C^2 domain Q in D satisfying the following limiting equation as $\kappa\to\infty$ in the above equation

$$\begin{aligned}&-\int_0^{\Theta}\int_{D\cup\Sigma}\alpha\theta\frac{\partial u_{\alpha}}{\partial\theta}\varphi\mathrm{d}\theta\mathrm{d}x+\int_0^{\Theta}\int_{D\cup\Sigma}\inf_{v_1\in V_1}\sup_{v_2\in V_2}\Big[\langle b(x,v_1,v_2),\nabla u_{\alpha}\rangle\\ &+\theta r(x,v_1,v_2)u_{\alpha}\Big]\varphi\mathrm{d}\theta\mathrm{d}x+\frac{1}{2}\int_0^{\Theta}\int_{D\cup\Sigma}\operatorname{trace}(a(x)\nabla^2 u_{\alpha})\varphi\mathrm{d}\theta\mathrm{d}x=0. \end{aligned}\tag{3.17}$$

Now since $u_{\alpha}\in W^{1,2,p}_{loc}((0,\Theta)\times(D\cup\Sigma))\cap C^{\frac{\hat{\alpha}}{2},\hat{\alpha}}((0,\Theta)\times\overline{Q})$ for each bounded C^2 domain Q in D, is a solution of the following equation

$$\begin{aligned}\alpha\theta\frac{\partial u_{\alpha}(\theta,x)}{\partial\theta} &= \inf_{v_1\in V_1}\sup_{v_2\in V_2}\Big[\langle b(x,v_1,v_2),\nabla u_{\alpha}(\theta,x)\rangle+\theta r(x,v_1,v_2)u_{\alpha}\Big]+\frac{1}{2}\operatorname{trace}(a(x)\nabla^2 u_{\alpha}(\theta,x))\\ u_{\alpha}(0,x)&=1 \text{ on } \overline{D}. \end{aligned}\tag{3.18}$$

Let Q be a domain with Lipschitz boundary in D such that $\partial Q\cap\partial D$ is a smooth part of ∂D. For fixed $\theta\in(0,\Theta)$ we know $\bar{u}^{\kappa}_{\alpha}(\theta,\cdot),u^{\kappa}_{\alpha}(\theta,\cdot)\in W^{2,p}(Q),\ p\geq 2$. As an application of Morrey Lemma ([54], pp. 335–339), $W^{2,p}(Q)$ for $p>d$, compactly embedded in $C^{1,\hat{\alpha}}(\overline{Q})$. Thus, for fixed $\theta\in(0,\Theta)$

$$\bar{u}^{\kappa}_{\alpha}(\theta,\cdot)\longrightarrow u_{\alpha}(\theta,\cdot) \text{ in } C^{1,\hat{\alpha}}(\overline{Q}).$$

Therefore since $\nabla\bar{u}^{\kappa}_{\alpha}(\theta,x)\cdot\gamma(x)=0$ on $(0,\Theta)\times\partial D$ for all $\kappa>0$, we have $\nabla u_{\alpha}(\theta,x)\cdot\gamma(x)=0$ on $(0,\Theta)\times\partial D\cap\partial Q$. Since Q is arbitrary, it implies that $\nabla u_{\alpha}(\theta,x)\cdot\gamma(x)=0$ on $(0,\Theta)\times\partial D$. Thus, by making use of Fan's minimax theorem [31], we have proved the existence of a solution $u_{\alpha}\in W^{1,2,p}_{loc}((0,\Theta)\times(D\cup\Sigma))\cap C^{0,1}((0,\Theta)\times(D\cup\Sigma)),\ p\geq 2$, to (3.2). □

Next theorem proves the existence of saddle point equilibria in the space of Markov strategies.

Theorem 3.3. *Assume (A0)–(A3). Then (3.2) admits a unique solution* $u_\alpha \in W^{1,2,p}_{loc}((0,\Theta)\times D\cup\Sigma)\cap C^{0,0}((0,\Theta)\times\overline{D}),\ p\geq 2$, *which is given by*

$$u_\alpha(\theta,x) = \inf_{v_1\in\mathcal{A}_1}\sup_{v_2\in\mathcal{A}_2} E^{v_1,v_2}_x\Big[e^{\theta\int_0^\infty e^{-\alpha t} r(X(t),v_1(t),v_2(t))\mathrm{d}t}\Big]$$
$$= \sup_{v_2\in\mathcal{A}_2}\inf_{v_1\in\mathcal{A}_1} E^{v_1,v_2}_x\Big[e^{\theta\int_0^\infty e^{-\alpha t} r(X(t),v_1(t),v_2(t))\mathrm{d}t}\Big].$$

Furthermore, u_α *is the value function for the discounted cost criterion and a saddle point equilibrium exists in* $\mathcal{M}_1\times\mathcal{M}_2$.

Proof. It is clear that for fixed $\theta\in(0,\Theta)$ and sufficiently small κ, $\bar{u}^\kappa_\alpha = u^\kappa_\alpha$. Arguing as in Theorem 4.1 from [40], we have the following representation

$$u^\kappa_\alpha(\theta,x) = \inf_{v_1\in\mathcal{A}_1}\sup_{v_2\in\mathcal{A}_2} E^{v_1,v_2}_x\Big[e^{\frac{\kappa\|r\|_\infty}{\alpha}} e^{\theta\int_0^{T_\kappa} e^{-\alpha t} r(X(t),v_1(t),v_2(t))\mathrm{d}t}\Big]$$
$$= \sup_{v_2\in\mathcal{A}_2}\inf_{v_1\in\mathcal{A}_1} E^{v_1,v_2}_x\Big[e^{\frac{\kappa\|r\|_\infty}{\alpha}} e^{\theta\int_0^{T_\kappa} e^{-\alpha t} r(X(t),v_1(t),v_2(t))\mathrm{d}t}\Big],$$

where $T_\kappa = \frac{ln(\frac{\theta}{\kappa})}{\alpha}$. From the above representation

$$1\leq u^\kappa_\alpha \leq e^{\frac{\kappa\|r\|_\infty}{\alpha}} e^{\frac{\theta\|r\|_\infty(1-e^{-\alpha T_\kappa})}{\alpha}} = e^{\frac{\theta\|r\|_\infty}{\alpha}}\ (\text{since } e^{-\alpha T_\kappa} = \frac{\kappa}{\theta}), \tag{3.19}$$

for every $\kappa>0$ and $(\theta,x)\in(0,\Theta)\times\overline{D}$. Since (3.2) has a solution in $u_\alpha\in W^{1,2,p}_{loc}((0,\Theta)\times D\cup\Sigma)\cap C^{0,1}((0,\Theta)\times D\cup\Sigma),\ p\geq 2$, by a measurable selection theorem [13] there exist measurable maps $\bar{v}_i : (0,\Theta)\times\overline{D}\to V_i,\ i=1,2$, such that

$$\inf_{v_1\in V_1}\sup_{v_2\in V_2}\Big[\langle b(x,v_1,v_2),\nabla u_\alpha(\theta,x)\rangle + \theta r(x,v_1,v_2)u_\alpha\Big] =$$
$$\sup_{v_2\in V_2}\Big[\langle b(x,\bar{v}_1(\theta,x),v_2),\nabla u_\alpha(\theta,x)\rangle + \theta r(x,\bar{v}_1(\theta,x),v_2)u_\alpha\Big], \tag{3.20}$$

and

$$\sup_{v_2\in V_2}\inf_{v_1\in V_1}\Big[\langle b(x,v_1,v_2),\nabla u_\alpha(\theta,x)\rangle + \theta r(x,v_1,v_2)u_\alpha\Big] =$$
$$\inf_{v_1\in V_1}\Big[\langle b(x,v_1,\bar{v}_2(\theta,x)),\nabla u_\alpha(\theta,x)\rangle + \theta r(x,v_1,\bar{v}_2(\theta,x))u_\alpha\Big]. \tag{3.21}$$

Let $v^*_i : \mathbb{R}_+\times\overline{D}\to V_i$, $i=1,2$, be defined as follows

$$v^*_i(t,x) = \bar{v}_i(\theta e^{-\alpha t},x),\ i=1,2.$$

Let $(v^*_1,v_2)\in\mathcal{A}_1\times\mathcal{A}_2$ and $X(\cdot)$ be the process given by the solution of (2.1) corresponding to (v^*_1,v_2). Associated to $X(\cdot)$ we define a sequence of stopping times as follows:

$$\tau_k = \begin{cases} 0; & \text{if } x\in\bar{Q}^c_k,\\ \inf\{t\geq 0 : X(t)\in\bar{Q}^c_k\}; & \text{if } x\in\bar{Q}_k,\end{cases}$$

keeping in mind that infimum of an empty set is ∞. Since $D \cup \Sigma = \cup_{n\geq 1}\overline{Q}_n$ where $\{Q_n\}_n$ is an increasing sequence of bounded domains from D, therefore, we have $\lim_{k\to\infty} \tau_k = \infty$ almost surely. Applying Itô−Dynkin formula to the function

$$e^{\int_0^t \theta(s) r(X(s), v_1^*(\theta(s), X(s)), v_2(s))\mathrm{d}s}\, u_\alpha(\theta(t), X(t)),$$

where $\theta(t) = \theta e^{-\alpha t}$, we get

$$\begin{aligned} d\Big(e^{\int_0^t \theta(s) r(X(s), v_1^*(\theta(s), X(s)), v_2(s))\mathrm{d}s}\, u^\kappa_\alpha(\theta(t), X(t))\Big) &= e^{\int_0^t \theta(s) r(X(s), v_1^*(\theta(s), X(s)), v_2(s))\mathrm{d}s}\, d(u^\kappa_\alpha(\theta(t), X(t))) \\ + \theta(t) r(X(t), v_1^*(\theta(t), X(t)), v_2(t)) u^\kappa_\alpha(\theta(t), X(t)) &e^{\int_0^t \theta(s) r(X(s), v_1^*(\theta(s), X(s)), v_2(s))\mathrm{d}s}\mathrm{d}t \end{aligned}$$

where

$$\begin{aligned} d(u^\kappa_\alpha(\theta(t), X(t)) &= (\nabla u^\kappa_\alpha(\theta(t), X(t))^\perp \sigma(X(t))\mathrm{d}W(t) - [\gamma(X(t)) \cdot \nabla u^\kappa_\alpha(\theta(t), X(t))]\, I_{\{X(t)\in\partial D\}}\mathrm{d}\xi(t) \\ &+ \Big[L^{v_1^*(\theta(t), X(t)), v_2(t)} u^\kappa_\alpha(\theta(t), X(t)) - \alpha\theta(t)\frac{\partial}{\partial\theta} u^\kappa_\alpha(\theta(t), X(t))\Big]\mathrm{d}t, \end{aligned}$$

and L is defined as in (2.4). By Sobolev embedding theorem ∇u_α is continuous on $\overline{Q}_k$. Thus, ∇u_α is bounded on $\overline{Q}_k$. Therefore the stochastic integral

$$\int_0^{T\wedge\tau_k} e^{\int_0^t \theta(s) r(X(s), v_1^*(\theta(s), X(s)), v_2(s))\mathrm{d}s} (\nabla u^\kappa_\alpha(\theta(t), X(t))^\perp \sigma(X(t))\mathrm{d}W(t)$$

is a zero-mean martingale for each k. Now since u_α satisfies (3.2), using (3.20), we obtain

$$E_x^{v_1^*, v_2}\Big[e^{\theta\int_0^{T\wedge\tau_k} e^{-\alpha t} r(X(t), v_1^*(\theta(t), X(t)), v_2(t))\mathrm{d}t} u_\alpha(\theta(T\wedge\tau_k), X(T\wedge\tau_k)\Big] - u_\alpha(\theta, x) \leq 0.$$

Since, v_2 is an arbitrary admissible control, we get

$$u_\alpha(\theta, x) \geq \sup_{v_2\in\mathcal{A}_2} E_x^{v_1^*, v_2}\Big[e^{\theta\int_0^{T\wedge\tau_k} e^{-\alpha t} r(X(t), v_1^*(\theta(t), X(t)), v_2(t))\mathrm{d}t} u_\alpha(\theta(T\wedge\tau_k), X(T\wedge\tau_k)\Big].$$

Using the fact that $\theta(t) \to 0$ as $t \to \infty$ and $u_\alpha(0, x) = 1$ for all $x \in \overline{D}$, letting $k \to \infty$ and then $T \to \infty$, we have

$$u_\alpha(\theta, x) \geq \sup_{v_2\in\mathcal{A}_2} E_x^{v_1^*, v_2}\Big[e^{\theta\int_0^{\infty} e^{-\alpha t} r(X(t), v_1^*(\theta(t), X(t)), v_2(t))\mathrm{d}t}\Big]. \tag{3.22}$$

By the analogous arguments, using (3.21), one can prove that

$$u_\alpha(\theta, x) \leq \inf_{v_1\in\mathcal{A}_1} E_x^{v_1, v_2^*}\Big[e^{\theta\int_0^{\infty} e^{-\alpha t} r(X(t), v_1(t), v_2^*(\theta(t), X(t)))\mathrm{d}t}\Big]. \tag{3.23}$$

Combining (3.22) and (3.23), we obtain

$$\begin{aligned} u_\alpha(\theta,x) &= \inf_{v_1\in\mathcal{A}_1}\sup_{v_2\in\mathcal{A}_2} E_x^{v_1,v_2}\Big[e^{\theta\int_0^\infty e^{-\alpha t} r(X(t),v_1(t),v_2(t))\mathrm{d}t}\Big] \\ &= \sup_{v_2\in\mathcal{A}_2}\inf_{v_1\in\mathcal{A}_1} E_x^{v_1,v_2}\Big[e^{\theta\int_0^\infty e^{-\alpha t} r(X(t),v_1(t),v_2(t))\mathrm{d}t}\Big] \\ &= E_x^{v_1^*,v_2^*}\Big[e^{\theta\int_0^\infty e^{-\alpha t} r(X(t),v_1^*(\theta e^{-\alpha t},X(t)),v_2^*(\theta e^{-\alpha t},X(t))\mathrm{d}t}\Big]. \end{aligned} \tag{3.24}$$

Therefore $u_\alpha(\theta,x)$ is the value function of the α-discounted cost criterion and $(v_1^*,v_2^*)\in\mathcal{M}_1\times\mathcal{M}_2$ given above forms a saddle point equilibrium. □

We now illustrate our results with an example.

3.1. Example 1

Here we are considering a simplified version of the model considered in [41]. Consider a sequence of open queuing networks with one server. We parametrize the sequence by n; in the heavy traffic limit $n\to\infty$. Let $C^n(t)$ denotes the number of customers at the service station at time t of the nth network. Define $X^n(t)=\frac{C^n(t)}{\sqrt{n}}$ to be the corresponding normalized process. Let the intensity of the arrival process be given by

$$A^n(x,u_1,u_2)=\sqrt{n}\bar{A}xu_1+nA,\quad x\ge 0,$$

where u_1 denotes the action of the user (Player 1) from the action space $[0,1]$ and $\bar{A},A$ are positive constants. The intensity of the service time process is given by

$$S^n(x,u_1,u_2)=\sqrt{n}\bar{S}xu_2+nS,\quad x\ge 0,$$

where u_2 denotes the action of the superuser (Player 2) from the action space $[0,1]$ and $\bar{S},S$ are positive constants. In this heavy traffic queue the arrival and service processes are controlled counting processes. In the limiting situation state of the queue (*i.e.* the limit of $X^n(\cdot)$) is evolving according to a controlled reflected stochastic differential equation (for detail see [41]):

$$\begin{aligned} &\mathrm{d}X(t)=(\bar{A}v_1(t)-\bar{S}v_2(t))X(t)\mathrm{d}t+(\sqrt{A},\sqrt{S})\cdot\mathrm{d}W(t)+\mathrm{d}\xi(t),\\ &\mathrm{d}\xi(t)=I_{\{X(t)=0\}}\mathrm{d}\xi(t),\quad t\ge 0,\\ &\xi(0)=0,\qquad X(0)=x,\quad x\ge 0, \end{aligned} \tag{3.25}$$

where $W=(W_1,W_2)$ is an $\mathbb{R}^2$- valued standard Wiener process and v_i is a $\mathcal{P}([0,1])$- valued process which satisfies the usual nonanticipative condition for $i=1,2$. The running cost function is given by

$$\bar{r}=\gamma u_1-\theta_1 u_2-c(x),\quad \gamma,\theta>0,$$

where $c(x):\mathbb{R}_+\to\mathbb{R}$, is bounded, Lipschitz continuous function and γ,θ_1 are suitable constants. In the above expression for running cost the term $c(\cdot)$ denotes the holding cost. Now the HJI equation for α-discounted cost criterion is given by

$$\begin{aligned}
\alpha\theta\frac{\partial u_\alpha(\theta,x)}{\partial\theta} &= \inf_{v_1\in\mathcal{P}([0,1])}\sup_{v_2\in\mathcal{P}([0,1])}\Big[(\bar{A}v_1-\bar{S}v_2)x\frac{\partial u_\alpha(\theta,x)}{\partial x}\\
&\quad+\theta(\gamma v_1-\theta_1 v_2-c(x))u_\alpha\Big]+\frac{1}{2}(A+S)\frac{\partial^2 u_\alpha(\theta,x)}{\partial^2 x},\\
&= \sup_{v_2\in\mathcal{P}([0,1])}\inf_{v_1\in\mathcal{P}([0,1])}\Big[(\bar{A}v_1-\bar{S}v_2)x\frac{\partial u_\alpha(\theta,x)}{\partial x}\\
&\quad+\theta(\gamma v_1-\theta_1 v_2-c(x))u_\alpha\Big]+\frac{1}{2}(A+S)\frac{\partial^2 u_\alpha(\theta,x)}{\partial^2 x},\\
u_\alpha(0,x) &= 1 \text{ on } \overline{\mathbb{R}}_+,\quad \frac{\partial u_\alpha(\theta,0)}{\partial x}=0 \text{ on } (0,\Theta).
\end{aligned}\tag{3.26}$$

Clearly all the assumptions (A0)–(A3) are satisfied. Therefore (3.26) admits a unique solution $u_\alpha \in W^{1,2,p}_{loc}((0,\Theta)\times(0,\infty))\cap C^{0,0}((0,\Theta)\times[0,\infty))$, $p\geq 2$. Now since $\mathcal{P}([0,1])$ is convex and compact, the infimum and supremum are attained at the extreme points. Thus, rewriting (3.26), we get

$$\begin{aligned}
\alpha\theta\frac{\partial u_\alpha(\theta,x)}{\partial\theta} &= \inf_{v_1\in[0,1]}\Big[\bar{A}v_1 x\frac{\partial u_\alpha(\theta,x)}{\partial x}+\theta\gamma v_1 u_\alpha\Big]+\sup_{v_2\in[0,1]}\Big[-\bar{S}v_2 x\frac{\partial u_\alpha(\theta,x)}{\partial x}\\
&\quad-\theta\theta_1 v_2 u_\alpha\Big]-\theta c(x)u_\alpha+\frac{1}{2}(A+S)\frac{\partial^2 u_\alpha(\theta,x)}{\partial^2 x},\\
u_\alpha(0,x) &= 1 \text{ on } \overline{\mathbb{R}}_+,\quad \frac{\partial u_\alpha(\theta,0)}{\partial x}=0 \text{ on } (0,\Theta).
\end{aligned}\tag{3.27}$$

From Theorem 3.3, we have that the optimal strategy for Player 1 is given by

$$v_1^* = \begin{cases} 1; & \text{if } \ (\bar{A}x\frac{\partial u_\alpha(\theta,x)}{\partial x}+\theta u_\alpha\gamma)<0,\\ y; & \text{if } \ (\bar{A}x\frac{\partial u_\alpha(\theta,x)}{\partial x}+\theta u_\alpha\gamma)=0,\\ 0; & \text{if } \ (\bar{A}x\frac{\partial u_\alpha(\theta,x)}{\partial x}+\theta u_\alpha\gamma)>0,\end{cases}$$

where y is any value in $[0,1]$. The optimal strategy for Player 2 is given by

$$v_2^* = \begin{cases} 1; & \text{if } \ (\bar{S}x\frac{\partial u_\alpha(\theta,x)}{\partial x}+\theta\theta_1 u_\alpha)<0,\\ y; & \text{if } \ (\bar{S}x\frac{\partial u_\alpha(\theta,x)}{\partial x}+\theta\theta_1 u_\alpha)=0,\\ 0; & \text{if } \ (\bar{S}x\frac{\partial u_\alpha(\theta,x)}{\partial x}+\theta\theta_1 u_\alpha)>0.\end{cases}$$

4. Analysis of ergodic cost criterion

In this section, we prove that for the ergodic cost criterion the value of the game and saddle point equilibrium exist. To carryout our analysis we make the following additional assumptions.

(A4)(Stability assumption) There exists a stochastic Lyapunov type function $V:\overline{D}\to[1,\infty)$, with the following properties

(i) $V\in C^2(\overline{D})$, $\lim_{\|x\|\to\infty} V(x)=\infty$,
(ii) $\nabla V\cdot\gamma\geq 0$ on ∂D.
(iii) $L^{u_1,u_2}V(x)<\tilde{\alpha}I_K-2\delta V(x)$, $\forall\ (x,u_1,u_2)\in\overline{D}\times U_1\times U_2$, for some compact set $K\subset D$, and suitable constants $\tilde{\alpha}\geq 0$, $\delta>0$.

We also make the following technical assumption about the running cost function.
(A5)(Small Cost Condition) $\theta\|r\|_\infty < \delta$, $\theta \in (0, \Theta)$ and δ as in (A4).

Since for the ergodic cost criterion we always fix the risk-sensitive parameter θ, thus for notational simplicity we assume that $\theta = 1$. Let $x_0 \in D$ be fixed point. Choose $m_1, l_1 > 0$ large enough such that $x_0 \in D_{l,m}$ $\forall\, l \geq l_1, m \geq m_1$. Following the proof technique of Lemma 3.2 in [64], we now want to prove the existence of a solution to the ergodic HJI equation in smooth bounded domain.

Theorem 4.1. *Assume (A0)–(A3). Then for each $l \geq l_1$, $m \geq m_1$, there exists $(\rho^{l,m}, u^{l,m}) \in \mathbb{R} \times W^{2,q}(D_{l,m})$, $q \geq d+1$ such that*

$$\begin{aligned}\rho^{l,m}u^{l,m} &= \inf_{v_1\in V_1}\sup_{v_2\in V_2}\Big[\langle b(x,v_1,v_2),\nabla u^{l,m}\rangle + r(x,v_1,v_2)u^{l,m}\Big] + \frac{1}{2}\operatorname{trace}(a(x)\nabla^2 u^{l,m}),\\ &= \sup_{v_2\in V_2}\inf_{v_1\in V_1}\Big[\langle b(x,v_1,v_2),\nabla u^{l,m}\rangle + r(x,v_1,v_2)u^{l,m}\Big] + \frac{1}{2}\operatorname{trace}(a(x)\nabla^2 u^{l,m}),\\ &\quad \|u^{l,m}\|_{q;D_{l,m}} = 1,\ \nabla u^{l,m}\cdot\gamma = 0 \ \text{ on } \partial D_{l,m}.\end{aligned} \tag{4.1}$$

Proof. From Theorem 2.2. in [4] (see also, [3], Thm. 12.1; [46], Prop. 2.3), it follows that for each $(\tilde v_1, \tilde v_2) \in \mathcal{S}_1 \times \mathcal{S}_2$ and $l \geq l_1$, $m \geq m_1$, the following p.d.e.

$$\begin{aligned}&\rho^{l,m}_{\tilde v_1,\tilde v_2}u^{l,m}_{\tilde v_1,\tilde v_2} = \langle b(x,\tilde v_1(x),\tilde v_2(x)),\nabla u^{l,m}_{\tilde v_1,\tilde v_2}\rangle + r(x,\tilde v_1(x),\tilde v_2(x))u^{l,m}_{\tilde v_1,\tilde v_2} + \frac{1}{2}\operatorname{trace}(a(x)\nabla^2 u^{l,m}_{\tilde v_1,\tilde v_2}),\\ &\nabla u^{l,m}_{\tilde v_1,\tilde v_2}\cdot\gamma = 0 \text{ on } \partial D_{l,m},\end{aligned} \tag{4.2}$$

has a principal eigenpair $(\rho^{l,m}_{\tilde v_1,\tilde v_2}, u^{l,m}_{\tilde v_1,\tilde v_2}) \in \mathbb{R}\times W^{2,p}(D_{l,m})$, $2 \leq p < \infty$, $u^{l,m}_{\tilde v_1,\tilde v_2} > 0$ in $\overline{D}_{l,m}$. Since any positive constant multiple of $u^{l,m}_{\tilde v_1,\tilde v_2}$ is also a solution of (4.2). Thus, multiplying by a suitable positive constant one can obtain $\|u^{l,m}_{\tilde v_1,\tilde v_2}\|_{q;D_{l,m}} = 1$, for $q \geq d+1$. Moreover, by Itô-Krylov formula ([14], Corollary 4.1, p.89), we have

$$\rho^{l,m}_{\tilde v_1,\tilde v_2} = \limsup_{T\to\infty}\frac{1}{T}\log E^{\tilde v_1,\tilde v_2}_{x,l,m}\big[e^{\int_0^T r(X_{l,m}(s),\tilde v_1(X_{l,m}(s)),\tilde v_2(X_{l,m}(s)))\mathrm{d}s}\big], \tag{4.3}$$

where $X_{l,m}$ is the process given by

$$\begin{aligned}\mathrm{d}X_{l,m}(t) &= b(X_{l,m}(t),\tilde v_1(X_{l,m}(t)),\tilde v_2(X_{l,m}(t)))\mathrm{d}t + \sigma(X_{l,m}(t))\mathrm{d}W(t) - \gamma(X_{l,m}(t))\mathrm{d}\xi(t),\\ \mathrm{d}\xi(t) &= I_{\{X_{l,m}(t)\in\partial D_{l,m}\}}\mathrm{d}\xi(t),\\ \xi(0) &= 0,\quad X_{l,m}(0) = x \in \overline{D}_{l,m},\end{aligned} \tag{4.4}$$

and $E^{\tilde v_1,\tilde v_2}_{x,l,m}$ is the expectation with respect to the law of $X_{l,m}$ with initial condition x.

From (4.3), we have $0 \leq \rho^{l,m}_{\tilde v_1,\tilde v_2} \leq \|r\|_\infty$ for all $(\tilde v_1,\tilde v_2) \in \mathcal{S}_1\times\mathcal{S}_2$, $m \geq m_1$ and $l \geq l_1$. Define $\rho^{l,m}_{*,\tilde v_2} = \inf_{\tilde v_1\in\mathcal{S}_1}\rho^{l,m}_{\tilde v_1,\tilde v_2}$. Thus, there exists a sequence $\{\tilde v_{1,n}\}$ in $\mathcal{S}_1$ such that $\rho^{l,m}_{*,\tilde v_2} = \lim_{n\to\infty}\rho^{l,m}_{\tilde v_{1,n},\tilde v_2}$. We know that for each such $(\tilde v_{1,n},\tilde v_2)$ there exists a principal eigenpair $(\rho^{l,m}_{\tilde v_{1,n},\tilde v_2}, u^{l,m}_{\tilde v_{1,n},\tilde v_2}) \in \mathbb{R}\times W^{2,p}(D_{l,m})$, $\infty > p \geq 2$, $u^{l,m}_{\tilde v_{1,n},\tilde v_2} > 0$ satisfying the following p.d.e.:

$$\begin{aligned}&(\rho^{l,m}_{\tilde v_{1,n},\tilde v_2} - r(x,\tilde v_{1,n}(x),\tilde v_2(x)))u^{l,m}_{\tilde v_{1,n},\tilde v_2} = \langle b(x,\tilde v_{1,n}(x),\tilde v_2(x)),\nabla u^{l,m}_{\tilde v_{1,n},\tilde v_2}\rangle + \frac{1}{2}\operatorname{trace}(a(x)\nabla^2 u^{l,m}_{\tilde v_{1,n},\tilde v_2}),\\ &\|u^{l,m}_{\tilde v_{1,n},\tilde v_2}\|_{q;D_{l,m}} = 1,\ \nabla u^{l,m}_{\tilde v_{1,n},\tilde v_2}\cdot\gamma = 0 \,\text{on } \partial D_{l,m}.\end{aligned} \tag{4.5}$$

Since left hand side of (4.5) is bounded uniformly in n in $L^q(D_{l,m}), q \geq d+1$, by Theorem 1.1, Remark 1.1(3) in [25] (see also [1]), we get

$$\|u^{l,m}_{\tilde{v}_{1,n},\tilde{v}_2}\|_{2,q;D_{l,m}} \leq \bar{C}_2,$$

where $\bar{C}_2$ is a constant independent of n. Thus, we can extract a subsequence (denoting by the same notation without loss of generality) $\{u^{l,m}_{\tilde{v}_{1,n},\tilde{v}_2}\}$ which weakly converges to $u^{l,m}_{*,\tilde{v}_2} \in W^{2,q}(D_{l,m})$, $q \geq d+1$. Also, since $\mathcal{S}_1$ is metrizable with a compact metric, along a further subsequence $\tilde{v}_{1,n} \to v_{1,*}$ in $\mathcal{S}_1$ (see [7], Lem. 2.4.1, p. 57). Thus, multiplying both side of (4.5) by test functions and integrating and then letting $n \to \infty$, it follows that $u^{l,m}_{*,\tilde{v}_2} \in W^{2,q}(D_{l,m})$, $q \geq d+1$ and it satisfies the following p.d.e.:

$$\rho^{l,m}_{*} u^{l,m}_{*,\tilde{v}_2} = \Big[\langle b(x, v_{1,*}(x), \tilde{v}_2(x)), \nabla u^{l,m}_{*,\tilde{v}_2}\rangle + r(x, v_{1,*}(x), \tilde{v}_2(x)) u^{l,m}_{*}\Big] + \frac{1}{2}\ \mathrm{trace}(a(x)\nabla^2 u^{l,m}_{*,\tilde{v}_2}). \tag{4.6}$$

Since $\nabla u^{l,m}_{\tilde{v}_{1,n},\tilde{v}_2} \to \nabla u^{l,m}_{*,\tilde{v}_2}$ in $C(\bar{D}_{l,m})$ and $\nabla u^{l,m}_{\tilde{v}_{1,n},\tilde{v}_2} \cdot \gamma = 0$ on $\partial D_{l,m}$, therefore $\nabla \psi^{l,m}_{*,\tilde{v}_2} \cdot \gamma = 0$ on $\partial D_{l,m}$. We know from Theorem 2.2. in [4], that (4.6) has only one eigenvalue with strictly positive eigenfunction. Since $\|u^{l,m}_{\tilde{v}_{1,n},\tilde{v}_2}\|_{q;D_{l,m}} = 1$ and $u^{l,m}_{\tilde{v}_{1,n},\tilde{v}_2} \to u^{l,m}_{*,\tilde{v}_2}$ in $W^{1,q}(D_{l,m})$ strongly, it follows that $\|u^{l,m}_{*,\tilde{v}_2}\|_{q;D_{l,m}} = 1$. Also, since $u^{l,m}_{*,\tilde{v}_2} \geq 0$, there exists a compact set $\hat{K}$ such that $u^{l,m}_{*,\tilde{v}_2} > 0$ on $\hat{K}$. Therefore Harnack's inequality ([42], Cor. 9.25) implies that $u^{l,m}_{*,\tilde{v}_2}(x) > 0$ in $D_{l,m}$. From Theorem 6.1 in [3] (see also [4], Lem. 2.3.; [42], Lem. 3.4, p.34) since $\nabla u^{l,m}_{*,\tilde{v}_2} \cdot \gamma = 0$ on $\partial D_{l,m}$, it follows that $u^{l,m}_{*,\tilde{v}_2}(x) > 0$ in $\overline{D}_{l,m}$. Let $X^*_{l,m}$ be the solution of (4.4) corresponding to $(v_{1,*}, \tilde{v}_2) \in \mathcal{S}_1 \times \mathcal{S}_2$. By Itô-Krylov formula ([14], Cor. 4.1, p.89), we obtain for any $T > 0$

$$\begin{aligned} u^{l,m}_{*,\tilde{v}_2}(x) &= E^{v_{1,*},\tilde{v}_2}_{x,l,m}\Big[e^{\int_0^T (r(X^*_{l,m}(s), v_{1,*}(X^*_{l,m}(s)), \tilde{v}_2(X^*_{l,m}(s))) - \rho^{l,m}_{*,\tilde{v}_2})\mathrm{d}s} u^{l,m}_{*,\tilde{v}_2}(X^*_{l,m}(T))\Big] \\ &\geq \inf_{y \in \overline{D}_{l,m}} u^{l,m}_{*,\tilde{v}_2}(y) E^{v_{1,*},\tilde{v}_2}_{x,l,m}\Big[e^{\int_0^T (r(X^*_{l,m}(s), v_{1,*} X^*_{l,m}(s)), \tilde{v}_2(X^*_{l,m}(s))) - \rho^{l,m}_{*,\tilde{v}_2})\mathrm{d}s}\Big]. \end{aligned}$$

Taking logarithm on both side, dividing by T and letting $T \to \infty$, we get

$$\rho^{l,m}_{*,\tilde{v}_2} \geq \limsup_{T \to \infty} \frac{1}{T} \log E^{v_{1,*},\tilde{v}_2}_{x,l,m}\Big[e^{\int_0^T r(X^*_{l,m}(s), v_{1,*}(X^*_{l,m}(s)), \tilde{v}_2(X^*_{l,m}(s)))\mathrm{d}s}\Big] = \rho^{l,m}_{v_{1,*},\tilde{v}_2}. \tag{4.7}$$

This implies $\rho^{l,m}_{*,\tilde{v}_2} = \rho^{l,m}_{v_{1,*},\tilde{v}_2}$ (since $\rho^{l,m}_{*,\tilde{v}_2} = \inf_{\tilde{v}_1 \in \mathcal{S}_1} \rho^{l,m}_{\tilde{v}_1,\tilde{v}_2}$). Now we want to prove that

$$\begin{aligned} &\inf_{v_1 \in V_1} \Big[\langle b(x, v_1, \tilde{v}_2(x)), \nabla u^{l,m}_{*,\tilde{v}_2}\rangle + r(x, v_1, \tilde{v}_2(x)) u^{l,m}_{*,\tilde{v}_2}\Big] \\ &= \Big[\langle b(x, v_{1,*}(x), \tilde{v}_2(x)), \nabla u^{l,m}_{*,\tilde{v}_2}\rangle + r(x, v_{1,*}(x), \tilde{v}_2(x)) u^{l,m}_{*,\tilde{v}_2}\Big]. \end{aligned}$$

But, we have

$$\begin{aligned} &\inf_{v_1 \in V_1} \Big[\langle b(x, v_1, \tilde{v}_2(x)), \nabla u^{l,m}_{*,\tilde{v}_2}\rangle + r(x, v_1, \tilde{v}_2(x)) u^{l,m}_{*,\tilde{v}_2}\Big] \\ &\leq \Big[\langle b(x, v_{1,*}(x), \tilde{v}_2(x)), \nabla u^{l,m}_{*,\tilde{v}_2}\rangle + r(x, v_{1,*}(x), \tilde{v}_2(x)) u^{l,m}_{*,\tilde{v}_2}\Big]. \end{aligned}$$

A standard measurable selection theorem ensures that there exists $\hat{v}_1 \in \mathcal{S}_1$ such that

$$\begin{aligned}&\inf_{v_1\in V_1}\Big[\langle b(x,v_1,\tilde{v}_2(x)),\nabla u^{l,m}_{*,\tilde{v}_2}\rangle + r(x,v_1,\tilde{v}_2(x))u^{l,m}_{*,\tilde{v}_2}\Big]\\ &= \Big[\langle b(x,\hat{v}_1(x),\tilde{v}_2(x)),\nabla u^{l,m}_{*,\tilde{v}_2}\rangle + r(x,\hat{v}_1(x),\tilde{v}_2(x))u^{l,m}_{*,\tilde{v}_2}\Big].\end{aligned}$$

This implies

$$\rho^{l,m}_{*,\tilde{v}_2}u^{l,m}_{*,\tilde{v}_2} \geq \Big[\langle b(x,\hat{v}_1(x),\tilde{v}_2(x)),\nabla u^{l,m}_{*,\tilde{v}_2}\rangle + r(x,\hat{v}_1(x),\tilde{v}_2(x))u^{l,m}_{*,\tilde{v}_2}\Big] + \frac{1}{2}\,\mathrm{trace}(a(x)\nabla^2_x u^{l,m}_{*,\tilde{v}_2}). \tag{4.8}$$

Let $\hat{X}_{l,m}$ be the solution of (4.4) corresponding to $(\hat{v}_1,\tilde{v}_2)\in\mathcal{S}_1\times\mathcal{S}_2$. Using (4.8) we can show as in (4.7)) that

$$\rho^{l,m}_{*,\tilde{v}_2} \geq \limsup_{T\to\infty}\frac{1}{T}\log E^{\hat{v}_1,\tilde{v}_2}_{x,l,m}\Big[e^{\int_0^T r(\hat{X}_{l,m}(s),\hat{v}_1(\hat{X}_{l,m}(s)),\tilde{v}_2(\hat{X}_{l,m}(s)))\mathrm{d}s}\Big] = \rho^{l,m}_{\hat{v}_1,\tilde{v}_2}. \tag{4.9}$$

Thus, we obtain $\rho^{l,m}_{*,\tilde{v}_2} = \rho^{l,m}_{\hat{v}_1,\tilde{v}_2}$.

We know that for $(\hat{v}_1,\tilde{v}_2)\in\mathcal{S}_1\times\mathcal{S}_2$, there exists $(\rho^{l,m}_{\hat{v}_1,\tilde{v}_2},u^{l,m}_{\hat{v}_1,\tilde{v}_2})\in\mathbb{R}\times W^{2,p}(D_{l,m})$, $\infty>p\geq 2$, $u^{l,m}_{\hat{v}_1,\tilde{v}_2}>0$ satisfying

$$\begin{aligned}\rho^{l,m}_{\hat{v}_1,\tilde{v}_2}u^{l,m}_{\hat{v}_1,\tilde{v}_2} &= \Big[\langle b(x,\hat{v}_1(x),\tilde{v}_2(x)),\nabla u^{l,m}_{\hat{v}_1,\tilde{v}_2}\rangle + r(x,\hat{v}_1(x),\tilde{v}_2(x))u^{l,m}_{\hat{v}_1,\tilde{v}_2}\Big] + \frac{1}{2}\,\mathrm{trace}(a(x)\nabla^2 u^{l,m}_{\hat{v}_1,\tilde{v}_2}),\\ \|u^{l,m}_{\hat{v}_1,\tilde{v}_2}\|_{q;D_{l,m}} &= 1,\ \nabla u^{l,m}_{\hat{v}_1,\tilde{v}_2}\cdot\gamma = 0\,\text{on }\partial D_{l,m}.\end{aligned} \tag{4.10}$$

We already know that $\rho^{l,m}_{*,\tilde{v}_2} = \rho^{l,m}_{\hat{v}_1,\tilde{v}_2}$. Next we want to prove that $u^{l,m}_{\hat{v}_1,\tilde{v}_2} = u^{l,m}_{*,\tilde{v}_2}$. Let $k_1 = \inf_{\bar{D}_{l,m}}\big(\frac{u^{l,m}_{*,\tilde{v}_2}}{u^{l,m}_{\hat{v}_1,\tilde{v}_2}}\big)$ and $\bar{u}^{l,m}_{\hat{v}_1,\tilde{v}_2} = k_1 u^{l,m}_{\hat{v}_1,\tilde{v}_2}$. Then $u^{l,m}_{*,\tilde{v}_2}(x) - \bar{u}^{l,m}_{\hat{v}_1,\tilde{v}_2}(x)\geq 0$ in $\overline{D}_{l,m}$ and attains its minimum value 0 in $D_{l,m}$ (since $\nabla(u^{l,m}_{*,\tilde{v}_2}(x) - \bar{u}^{l,m}_{\hat{v}_1,\tilde{v}_2})\cdot\gamma = 0$ on $\partial D_{l,m}$ (see, [42], Lem. 3.4, p.34)). Now from (4.8) and (4.10), we have

$$\begin{aligned}&\frac{1}{2}\,\mathrm{trace}(a(x)\nabla^2(u^{l,m}_{*,\tilde{v}_2} - \bar{u}^{l,m}_{\hat{v}_1,\tilde{v}_2})) + \Big[\langle b(x,\hat{v}_1(x),\tilde{v}_2(x)),\nabla(u^{l,m}_{*,\tilde{v}_2} - \bar{u}^{l,m}_{\hat{v}_1,\tilde{v}_2})\rangle\\ &-(r(x,\hat{v}_1(x),\tilde{v}_2(x)) - \rho^{l,m}_{*,\tilde{v}_2})^-(u^{l,m}_{*,\tilde{v}_2} - \bar{u}^{l,m}_{\hat{v}_1,\tilde{v}_2})\Big] \leq -(r(x,\hat{v}_1(x),\tilde{v}_2(x)) - \rho^{l,m}_{*,\tilde{v}_2})^+(u^{l,m}_{*,\tilde{v}_2} - \bar{u}^{l,m}_{\hat{v}_1,\tilde{v}_2}) \leq 0.\end{aligned}$$

Thus, by strong maximum principle as in Theorem 9.6 from [42] (see also [65], Chap. 2), we obtain $u^{l,m}_{*\tilde{v}_2} = u^{l,m}_{\hat{v}_1,\tilde{v}_2}$ (since $\|u^{l,m}_{*,\tilde{v}_2}\|_{q;D_{l,m}} = \|u^{l,m}_{\hat{v}_1,\tilde{v}_2}\|_{q;D_{l,m}} = 1$). Therefore, we have the following

$$\begin{aligned}\rho^{l,m}_{*,\tilde{v}_2}u^{l,m}_{*,\tilde{v}_2} &= \Big[\langle b(x,v_{1,*}(x),\tilde{v}_2(x)),\nabla u^{l,m}_{*,\tilde{v}_2}\rangle + r(x,v_{1,*}(x),\tilde{v}_2(x))u^{l,m}_{1,*}\Big] + \frac{1}{2}\,\mathrm{trace}(a(x)\nabla^2 u^{l,m}_{*,\tilde{v}_2})\\ &\geq \inf_{v_1\in V_1}\Big[\langle b(x,v_1,\tilde{v}_2(x)),\nabla u^{l,m}_{*,\tilde{v}_2}\rangle + r(x,v_1,\tilde{v}_2(x))u^{l,m}_{*,\tilde{v}_2}\Big] + \frac{1}{2}\,\mathrm{trace}(a(x)\nabla^2 u^{l,m}_{*,\tilde{v}_2})\\ &= \Big[\langle b(x,\hat{v}_1(x),\tilde{v}_2(x)),\nabla u^{l,m}_{*,\tilde{v}_2}\rangle + r(x,\hat{v}_1(x),\tilde{v}_2(x))u^{l,m}_{*,\tilde{v}_2}\Big] + \frac{1}{2}\,\mathrm{trace}(a(x)\nabla^2 u^{l,m}_{*,\tilde{v}_2})\\ &= \rho^{l,m}_{*,\tilde{v}_2}u^{l,m}_{*,\tilde{v}_2}.\end{aligned}$$

Thus, we obtain

$$\rho^{l,m}_{*,\tilde{v}_2} u^{l,m}_{*,\tilde{v}_2} = \inf_{v_1 \in V_1} \Big[\langle b(x, v_1, \tilde{v}_2(x)), \nabla u^{l,m}_{*,\tilde{v}_2} \rangle + r(x, v_1, \tilde{v}_2(x)) u^{l,m}_{*,\tilde{v}_2} \Big] + \frac{1}{2} \operatorname{trace}(a(x) \nabla^2 u^{l,m}_{*,\tilde{v}_2}). \tag{4.11}$$

Let $(\bar{\rho}^{l,m}_{\tilde{v}_2}, \bar{u}^{l,m}_{\tilde{v}_2})$ be any other solution of (4.11), *i.e.*,

$$\begin{aligned} \bar{\rho}^{l,m}_{\tilde{v}_2} \bar{u}^{l,m}_{\tilde{v}_2} &= \inf_{v_1 \in V_1} \Big[\langle b(x, v_1, \tilde{v}_2(x)), \nabla \bar{u}^{l,m}_{\tilde{v}_2} \rangle + r(x, v_1, \tilde{v}_2(x)) \bar{u}^{l,m}_{\tilde{v}_2} \Big] + \frac{1}{2} \operatorname{trace}(a(x) \nabla^2 \bar{u}^{l,m}_{\tilde{v}_2}), \\ \|\bar{u}^{l,m}_{\tilde{v}_2}\|_{q,D_{l,m}} &= 1, \ \nabla \bar{u}^{l,m}_{\tilde{v}_2} \cdot \gamma = 0 \ \text{ on } \partial D_{l,m}. \end{aligned}$$

Let $\hat{v}_1 \in \mathcal{S}_1$ be a minimizing selector of (4.11). By Itô-Krylov formula (as in (4.7)), we get

$$\bar{\rho}^{l,m}_{\tilde{v}_2} \leq \limsup_{T \to \infty} \frac{1}{T} \log E^{\hat{v}_1, \tilde{v}_2}_{x,l,m} \big[e^{\int_0^T r(\hat{X}_{l,m}(s), \hat{v}_1(\hat{X}_{l,m}(s)), \tilde{v}_2(\hat{X}_{l,m}(s))) \mathrm{d}s} \big] = \rho^{l,m}_{*,\tilde{v}_2}.$$

Thus, $\rho^{l,m}_{*,\tilde{v}_2} = \bar{\rho}^{l,m}_{\tilde{v}_2}$. By an analogous application of the strong maximum principle one can prove that $\bar{u}^{l,m}_{\tilde{v}_2} = u^{l,m}_{*,\tilde{v}_2}$.
Define

$$\rho^{l,m}_{*} = \sup_{\tilde{v}_2 \in \mathcal{S}_2} \rho^{l,m}_{*,\tilde{v}_2}.$$

Then there exists a sequence $\{\tilde{v}_{2,n}\}$ in $\mathcal{S}_2$ such that $\rho^{l,m}_{*} = \lim_{n \to \infty} \rho^{l,m}_{*,\tilde{v}_{2,n}}$. Repeating the above arguments it follows that there exists a unique solution $(\rho^{l,m}_{*}, u^{l,m}_{*}) \in \mathbb{R} \times W^{2,q}(D_{l,m}), \ q \geq d+1$ to

$$\begin{aligned} \rho^{l,m}_{*} u^{l,m}_{*} &= \sup_{v_2 \in V_2} \inf_{v_1 \in V_1} \Big[\langle b(x, v_1, v_2), \nabla u^{l,m}_{*} \rangle + r(x, v_1, v_2) u^{l,m}_{*} \Big] + \frac{1}{2} \operatorname{trace}(a(x) \nabla^2 u^{l,m}_{*}), \\ \|u^{l,m}_{*}\|_{q;D_{l,m}} &= 1, \nabla u^{l,m}_{*} \cdot \gamma = 0 \ \text{ on } \partial D. \end{aligned}$$

Finally, by an application of Fan's mini-max theorem one can show that the pair $(\rho^{l,m}_{*}, u^{l,m}_{*}) \in \mathbb{R} \times W^{2,q}(D_{l,m}), \ q \geq d+1$, satisfies (4.1). □

From our proof it is clear that $\rho^{l,m} \leq \|r\|_\infty$ for all $m \geq m_1$ and $l \geq l_1$. From the above theorem, we have the following.

Theorem 4.2. *Assume (A0)–(A3). Then for each $l \geq l_1, m \geq m_1$, there exists $(\rho^{l,m}, u^{l,m}) \in \mathbb{R} \times W^{2,q}(D_{l,m}), \ q \geq d+1$, such that*

$$\begin{aligned} \rho^{l,m} u^{l,m} &= \inf_{v_1 \in V_1} \sup_{v_2 \in V_2} \Big[\langle b(x, v_1, v_2), \nabla u^{l,m} \rangle + r(x, v_1, v_2) u^{l,m} \Big] + \frac{1}{2} \operatorname{trace}(a(x) \nabla^2 u^{l,m}), \\ &= \sup_{v_2 \in V_2} \inf_{v_1 \in V_1} \Big[\langle b(x, v_1, v_2), \nabla u^{l,m} \rangle + r(x, v_1, v_2) u^{l,m} \Big] + \frac{1}{2} \operatorname{trace}(a(x) \nabla^2 u^{l,m}), \\ & u^{l,m}(x_0) = 1, \ \nabla u^{l,m} \cdot \gamma = 0 \ \textit{ on } \partial D_{l,m}. \end{aligned} \tag{4.12}$$

Proof. From Theorem 4.1, it is clear that any positive multiple of $u^{l,m}$ is also a solution to

$$\rho^{l,m}u^{l,m} = \inf_{v_1\in V_1}\sup_{v_2\in V_2}\Big[\langle b(x,v_1,v_2),\nabla u^{l,m}\rangle + r(x,v_1,v_2)u^{l,m}\Big] + \frac{1}{2}\ \text{trace}(a(x)\nabla^2 u^{l,m}),$$
$$= \sup_{v_2\in V_2}\inf_{v_1\in V_1}\Big[\langle b(x,v_1,v_2),\nabla u^{l,m}\rangle + r(x,v_1,v_2)u^{l,m}\Big] + \frac{1}{2}\ \text{trace}(a(x)\nabla^2 u^{l,m}),$$

with $\nabla u^{l,m}\cdot\gamma = 0$ on $\partial D_{l,m}$. Then multiplying by suitable positive constant one can easily show that there exists a solution to (4.12). In view of the uniqueness result of Theorem 4.1, it is easy to see that the solution of (4.12) is unique, because if $(\hat{\rho}^{l,m},\hat{u}^{l,m})\in\mathbb{R}\times W^{2,q}(D_{l,m})$, $q\geq d+1$, is any other solution of (4.12), then for some suitable positive constants $\bar{k}_1,\bar{k}_2$ the pairs $(\hat{\rho}^{l,m},\bar{k}_1\hat{u}^{l,m}),(\rho^{l,m},\bar{k}_2u^{l,m})$ are solution of (4.1). Therefore $\hat{\rho}^{l,m}=\rho^{l,m}$ and $\bar{k}_1\hat{u}^{l,m}=\bar{k}_2u^{l,m}$. Since $\hat{u}^{l,m}(x_0)=u^{l,m}(x_0)=1$, we have $\hat{u}^{l,m}=u^{l,m}$. This completes the proof. □

This extends the result of [60] from neumann boundary condition to oblique boundary condition. Now we want to prove the existence of a solution to the following limiting ergodic HJI equation.

Theorem 4.3. *Assume (A0)–(A3). Then there exists* $(\rho,u)\in\mathbb{R}\times W^{2,q}_{loc}(D\cup\Sigma)$, $q\geq d+1$, *satisfying*

$$\rho u = \inf_{v_1\in V_1}\sup_{v_2\in V_2}\Big[\langle b(x,v_1,v_2),\nabla u\rangle + r(x,v_1,v_2)u\Big] + \frac{1}{2}\ \text{trace}(a(x)\nabla^2 u),$$
$$= \sup_{v_2\in V_2}\inf_{v_1\in V_1}\Big[\langle b(x,v_1,v_2),\nabla u\rangle + r(x,v_1,v_2)u\Big] + \frac{1}{2}\ \text{trace}(a(x)\nabla^2 u),$$
$$u(x_0)=1, \nabla u\cdot\gamma = 0 \ \text{ on } \partial D. \tag{4.13}$$

Proof. Since $0\leq\rho^{l,m}\leq\|r\|_\infty$, there exists a constant ρ such that along a subsequance $\rho^{l,m}\to\rho$ as $l,m\to\infty$. Let Q be an open bounded domain with C^2 boundary in D. Then there exist positive integers M_1,N_1 such that $\overline{Q}\subset D_{l,m}$ for all $m\geq M_1, l\geq N_1$. Let $M=\max\{m_1,M_1\}$ and $N=\max\{l_1,N_1\}$. Theorem 4.1 ensures that the p.d.e (4.1) has a unique solution $u^{l,m}$ in $W^{2,q}(D_{l,m})$, $q\geq d+1$. Let $(\bar{v}_1^{l,m},\bar{v}_2^{l,m})\in\mathcal{S}_1\times\mathcal{S}_2$ be a minimax selector of (4.1). Therefore $\bar{v}_1^{l,m}\in\mathcal{S}_1$ is an outer minimizing selector of (4.1) and $\bar{v}_2^{l,m}\in\mathcal{S}_2$ is an outer maximizing selector of (4.1), *i.e.*,

$$\inf_{v_1\in V_1}\sup_{v_2\in V_2}\Big[\langle b(x,v_1,v_2),\nabla u^{l,m}\rangle + r(x,v_1,v_2)u^{l,m}\Big]$$
$$= \sup_{v_2\in V_2}\Big[\langle b(x,\bar{v}_1^{l,m}(x),v_2),\nabla u^{l,m}\rangle + r(x,\bar{v}_1^{l,m}(x),v_2)u^{l,m}\Big], \tag{4.14}$$

$$\sup_{v_2\in V_2}\inf_{v_1\in V_1}\Big[\langle b(x,v_1,v_2),\nabla u^{l,m}\rangle + r(x,v_1,v_2)u^{l,m}\Big]$$
$$= \inf_{v_1\in V_1}\Big[\langle b(x,v_1,\bar{v}_2^{l,m}(x)),\nabla u^{l,m}\rangle + r(x,v_1,\bar{v}_2^{l,m}(x))u^{l,m}\Big]. \tag{4.15}$$

Existence of such measurable selectors are ensured by a standard measurable selection theorem. Using (4.14), (4.15), we rewrite the p.d.e (4.1) as a parametric family of linear elliptic p.d.e.s as follows:

$$g_{l,m}(x) = \langle b(x,\bar{v}_1^{l,m}(x),\bar{v}_2^{l,m}(x)),\nabla u^{l,m}(x)\rangle + \frac{1}{2}\ \text{trace}(a(x)\nabla^2 u^{l,m}(x)),$$
$$u^{l,m}(x_0)=1, \nabla u^{l,m}(x)\cdot\gamma(x) = 0 \text{ on } \partial D_{l,m},$$

where $g_{l,m}(x) = (\rho^{l,m} - r(x, \bar{v}_1^{l,m}(x), \bar{v}_2^{l,m}(x))u^{l,m}(x)$. Let $b_{l,m}(x) = b(x, \bar{v}_1^{l,m}(x), \bar{v}_2^{l,m}(x))$, $\bar{g}_{l,m}(x) = (\rho^{l,m} - r(x, \bar{v}_1^{l,m}(x), \bar{v}_2^{l,m}(x))$. Then by our assumptions it is clear that, $\sup_{l,m} \|b_{l,m}\|_{\infty;D} < \infty$, $\sup_{l,m} \|\bar{g}_{l,m}\|_{\infty;D} < \infty$. Without loss of generality we may assume that $x_0 \in Q$. Now since $u^{l,m}(x_0) = 1$, by Harnack's inequality ([42], Cor. 9.25) (see also, [38], Thm. 1.8), we have

$$\sup_{Q} u^{l,m} \leq c_3, \;\forall\; l \geq N,\, m \geq M,$$

for some constant c_3 independent of l, m. Therefore, using Theorem 9.11 from [42] (see also,[38], Subsection (1.6)), we get

$$\|u^{l,m}\|_{2,q;Q} < \hat{C}_4, \text{ for all } \; m \geq M, l \geq N, q \geq d+1, \tag{4.16}$$

where the constant $\hat{C}_4$ is independent of m, l. Let $\{Q_n\}$ be a sequence of bounded domains from D such that $D \cup \Sigma = \cup_{n\geq 1} \overline{Q}_n$. Therefore by a standard diagonalization procedure there exists $u^m \in W^{2,q}_{loc}(D_m)$, $q \geq d+1$, such that along a subsequence as $l \to \infty$

$$u^{l,m} \longrightarrow u^m \;\text{ weakly in } W^{2,p}(Q). \tag{4.17}$$

Now (4.16) implies that $\|u^m\|_{2,q;Q} < \hat{C}_4$, for all $m \geq M, q \geq d+1$. Again repeating the diagonalization argument, there exists $u \in W^{2,q}_{loc}(D \cup \Sigma)$, $q \geq d+1$ such that as $m \to \infty$

$$u^m \longrightarrow u \;\text{ weakly in } W^{2,q}(Q). \tag{4.18}$$

By Sobolev embedding theorems, we have $W^{2,q}(Q)$ for $q \geq d+1$ is compactly contained $C^{1,\hat{\alpha}}(\overline{Q})$, $0 < \hat{\alpha} < 1$. Thus, along a suitable subsequence, we get

$$\lim_{m\to\infty} \lim_{l\to\infty} u^{l,m} = u \;\text{ in } C^{1,\hat{\alpha}}(\overline{Q}). \tag{4.19}$$

Therefore from (4.19), we obtain the following:

$$\begin{aligned}
&\lim_{m\to\infty} \lim_{l\to\infty} \inf_{v_1\in V_1} \sup_{v_2\in V_2} \Big[\langle b(x, v_1, v_2), \nabla u^{l,m}(x)\rangle + r(x, v_1, v_2)u^{l,m}\Big] \\
&= \inf_{v_1\in V_1} \sup_{v_2\in V_2} \Big[\langle b(x, v_1, v_2), \nabla u(x)\rangle + r(x, v_1, v_2)u\Big], \text{ and} \\
&\lim_{m\to\infty} \lim_{l\to\infty} \sup_{v_2\in V_2} \inf_{v_1\in V_1} \Big[\langle b(x, v_1, v_2), \nabla u^{l,m}(x)\rangle + r(x, v_1, v_2)u^{l,m}\Big] \\
&= \sup_{v_2\in V_2} \inf_{v_1\in V_1} \Big[\langle b(x, v_1, v_2), \nabla u(x)\rangle + r(x, v_1, v_2)u\Big].
\end{aligned} \tag{4.20}$$

Now multiplying (4.1), for $l \geq N, m \geq M$, by a test function $\varphi \in C_c^\infty(D \cup \Sigma)$ and integrating over $D \cup \Sigma$, we get

$$\begin{aligned}
&-\int_{D\cup\Sigma} \rho^{l,m} u^{l,m}(x)\varphi(x)\mathrm{d}x + \int_{D\cup\Sigma} \sup_{v_2\in V_2} \inf_{v_1\in V_1} \Big[\langle b(x, v_1, v_2), \nabla u^{l,m}(x)\rangle \\
&+ r(x, v_1, v_2)u^{l,m}(x)\Big]\varphi(x)dx + \frac{1}{2} \int_{D\cup\Sigma} \operatorname{trace}(a(x)\nabla^2 u^{l,m}(x))\varphi(x)\mathrm{d}x \\
&= 0.
\end{aligned} \tag{4.21}$$

Using (4.19) and (4.20), making $l \to \infty$ and then $m \to \infty$ in (4.21), we obtain

$$\begin{aligned} & - \int_{D\cup\Sigma} \rho u(x)\varphi(x)\mathrm{d}x + \int_{D\cup\Sigma} \sup_{v_2\in V_2}\inf_{v_1\in V_1} \Big[\langle b(x,v_1,v_2), \nabla u(x)\rangle \\ & + r(x,v_1,v_2)u(x)\Big]\varphi(x)dx + \frac{1}{2}\int_{D\cup\Sigma} \mathrm{trace}(a(x)\nabla^2 u(x))\varphi(x)\mathrm{d}x \\ & = 0. \end{aligned} \tag{4.22}$$

Since $\varphi \in C_c^\infty(D\cup\Sigma)$ is arbitrary and $u \in W^{2,q}_{loc}(D\cup\Sigma)$, $q \geq d+1$, we have

$$\rho u(x) = \sup_{v_2\in V_2}\inf_{v_1\in V_1} \Big[\langle b(x,v_1,v_2), \nabla u(x)\rangle + r(x,v_1,v_2)u(x)\Big] + \frac{1}{2}\,\mathrm{trace}(a(x)\nabla^2 u(x)),\ \text{a.e. in}\ \ D. \tag{4.23}$$

We know that $u^{l,m}(x_0) = 1$ for all $m \geq M, l \geq N$, thus $u(x_0) = 1$. Now it is clear from the construction of $D_{l,m}$ that for any point $\tilde{x} \in \Sigma$ there exist a sequence $\{x_{l,m}\}_{l,m}$ such that $x_{l,m} \in \partial D_{l,m}$ and $x_{l,m} \longrightarrow \tilde{x}$ as $m, l \to \infty$. Since γ is continuous and $u \in C^1(D\cup\Sigma)$, thus (4.19) implies that

$$\nabla u(\tilde{x})\cdot\gamma(\tilde{x}) = \lim_{m\to\infty}\lim_{l\to\infty} \nabla u^{l,m}(x_{l,m})\cdot\gamma(x_{l,m}) = 0.$$

Since the surface measure of Σ' (non-smooth part of ∂D) is zero, we obtain $\nabla u(x)\cdot\gamma(x) = 0$ a.e. on ∂D. We now obtain our desired result by an application of Fan's minimax theorem [31]. □

Now we prove the representation of the eigenfunction u of (4.13). The assumptions (A4) and (A5) are crucially used in what follows.

Lemma 4.4. *Assume (A0)–(A5). Then there exists a compact set $\mathcal{Q} \subset D$ such that for any mini-max selector $v^* = (v_1^*, v_2^*)$ of (4.13) and any compact ball $\mathcal{Q}_1 \supset \mathcal{Q}$, we have*

$$u(x) = E_x^{v^*}\Big[e^{\int_0^{\tau_1^c}(r(X(t),v^*(X(t)))-\rho)\mathrm{d}t}u(X(\tau_1^c))\Big],\ \ \forall\ x \in \mathcal{Q}_1^c, \tag{4.24}$$

where $\tau_1^c := \tau(\mathcal{Q}_1^c) = \inf\{t \geq 0 : X(t) \in \mathcal{Q}_1\}$.

Proof. We know that $0 \leq \rho^{l,m} \leq \|r\|_\infty$, therefore $0 \leq \rho \leq \|r\|_\infty$. By (A5) $0 \leq \rho \leq \|r\|_\infty < \delta$, therefore for some suitable choice of a compact set $\mathcal{Q}$ and a constant $\hat{\beta} \in (0,1)$, the following holds

$$\left(\max_{(u_1,u_2)\in U_1\times U_2} r(x,u_1,u_2) - \rho\right) < \hat{\beta}\delta,\ \ \forall\ x \in \mathcal{Q}^c. \tag{4.25}$$

Also, since $\rho^{l,m} \to \rho$ and $\rho^{l,m} \leq \|r\|_\infty$, we have (4.25) for ρ replaced by $\rho^{l,m}$ for large l, m. Without loss of generality we are assuming that $K \subset \mathcal{Q}$, where K is the compact set specified in $(A4)$. Choose l, m sufficiently large such that $\mathcal{Q} \subset D_{l,m}$. Let $\tilde{v}^{l,m} = (\tilde{v}_1^{l,m}, \tilde{v}_2^{l,m})$ be a mini-max selector of (4.1). Applying Itô-Krylov formula [([14] Cor. 4.1, p.89), we get for $x \in \mathcal{Q}^c$

$$\begin{aligned} u^{l,m}(x) &= E_{x,l,m}^{\tilde{v}^{l,m}}\big[e^{\int_0^{\tau_{l,m}^c}(r(X_{l,m}(s),\tilde{v}^{l,m}(X_{l,m}(s)))-\rho^{l,m})\mathrm{d}s}u^{l,m}(X_{l,m}(\tau_{l,m}^c))\big], \\ &\leq \sup_{y\in\mathcal{Q}} u^{l,m}(y) E_{x,l,m}^{\tilde{v}^{l,m}}\big[e^{\hat{\beta}\delta\tau_{l,m}^c}\big], \\ &\leq \sup_{y\in\mathcal{Q}} u^{l,m}(y)\left(E_{x,l,m}^{\tilde{v}^{l,m}}\big[e^{\delta\tau_{l,m}^c}\big]\right)^{\hat{\beta}},\ \text{(by Jensen's inequality)}, \end{aligned}$$

$$\leq \frac{\sup_{y\in\mathcal{Q}} u^{l,m}(y)}{\inf_{y\in\mathcal{Q}} V^{\hat{\beta}}(y)} \left(E^{\tilde{v}^{l,m}}_{x,l,m}\big[e^{\delta\tau^c} V(X_{l,m}(\tau^c_{l,m}))\big]\right)^{\hat{\beta}},$$
$$\leq \hat{C}_1 (V(x))^{\hat{\beta}}, \tag{4.26}$$

where $\tau^c_{l,m}$ is the hitting time of the process $X_{l,m}(\cdot)$ to the set $\mathcal{Q}$, $\hat{C}_1$ is a constant (using Harnack's inequality one can choose $\hat{C}_1$ to be independent of l, m) and the last inequality follows using (A4) and applying Itô-Krylov formula to $e^{\delta t} V(X_{l,m}(t))$.

Let $\mathcal{Q}_1$ be a compact set such that $\mathcal{Q}_1 \supset \mathcal{Q}$. We choose an increasing sequence $\{K_n\}$ of smooth bounded domains from D such that $\cup_n \overline{K}_n = D \cup \Sigma$. Let $\tau_n = \tau(K_n)$. It is clear that under (A0), (A4), $\tau_n \to \infty$ as $n \to \infty$. Now by Itô-Krylov formula ([38], Thm. 1.14) for the process (2.1) corresponding to a mini-max selector $v^* \in \mathcal{S}_1 \times \mathcal{S}_2$ of (4.13), we obtain

$$d(e^{\int_0^t \big(r(X(s),v^*(X(s)))-\rho\big)\mathrm{d}s} u(X(t)) = e^{\int_0^t \big(r(X(s),v^*(X(s)))-\rho\big)\mathrm{d}s} d(u(X(t)) + \\ \big(r(X(s), v^*(X(s))) - \rho\big) u(X(t)) e^{\int_0^t \big(r(X(s),v^*(X(s)))-\rho\big)\mathrm{d}s}$$

for $t \leq \tau_n$, $n \geq 1$, where

$$d(u(X(t)) = (\nabla u(X(t)))^{\perp} \sigma(X(t))\mathrm{d}W(t) - [\gamma(X(t)) \cdot \nabla u(X(t))]\, I_{\{X(t)\in\partial D\}} \mathrm{d}\xi(t) \\ + \Big[Lu(X(t), v^*(X(t)))\Big]\mathrm{d}t,$$

and L is as in (2.4). As we have $u \in W^{2,q}_{loc}(D \cup \Sigma)$, $p \geq d+1$, thus ∇u is bounded on each of the compact subset of $(D \cup \Sigma)$. Therefore

$$\int_0^{\tau_1^c \wedge \tau_n \wedge T} e^{\int_0^t \big(r(X(s),v^*(X(s)))-\rho\big)\mathrm{d}s} (\nabla u(X(t)))^{\perp} \sigma(X(t))\mathrm{d}W(t)$$

is a zero-mean martingale. It is clear that for n large enough, $\mathcal{Q}_1 \subset K_n$. Now using the fact that u is a solution to (4.13), it follows that for $x \in \mathcal{Q}_1^c \cap K_n$

$$E^{v^*}_x \Big[e^{\int_0^{\tau_1^c \wedge \tau_n \wedge T} (r(X(t),v^*(X(t)))-\rho)\mathrm{d}t} \psi(X(\tau_1^c \wedge \tau_n \wedge T))\Big] \\ = E^{v^*}_x \Big[\int_0^{\tau_1^c \wedge \tau_n \wedge T} e^{\int_0^t (r(X(s),v^*(X(s)))-\rho)\mathrm{d}s} [L\psi(X(t), v^*(X(t))) \\ + (r(X(t), v^*(X(t))) - \rho)\psi(X(t))\mathrm{d}t\Big] + \psi(x) \\ = \psi(x).$$

Hence it follows that

$$u(x) = E^{v^*}_x \Big[e^{\int_0^{\tau_1^c \wedge \tau_n \wedge T} (r(X(t),v^*(X(t)))-\rho)\mathrm{d}t} u(X(\tau_1^c \wedge \tau_n \wedge T))\Big]. \tag{4.27}$$

Now applying Itô-Krylov formula and using (A4), one can show that for $x \in \mathcal{Q}_1^c$

$$E^{v^*}_x \Big[e^{\delta \tau_1^c}\Big] < \infty. \tag{4.28}$$

Since for each n the function u is bounded in $\mathcal{Q}_1^c \cap K_n$, by dominated convergence theorem, letting $T \to \infty$ in (4.27), we get

$$u(x) = E_x^{v^*}\Big[e^{\int_0^{\tau_1^c \wedge \tau_n}(r(X(t),v^*(X(t)))-\rho)\mathrm{d}t} u(X(\tau_1^c \wedge \tau_n))\Big]. \tag{4.29}$$

We have

$$E_x^{v^*}\Big[e^{\delta(\tau_1^c \wedge \tau_n)}\Big] = E_x^{v^*}\Big[e^{\delta\tau_n} I_{\{\tau_n < \tau_1^c\}}\Big] + E_x^{v^*}\Big[e^{\delta\tau_1^c} I_{\{\tau_1^c < \tau_n\}}\Big]. \tag{4.30}$$

By monotone convergence theorem it follows that $E_x^{v^*}\Big[e^{\delta\tau_1^c \wedge \tau_n}\Big] \to E_x^{v^*}\Big[e^{\delta\tau_1^c}\Big]$ and $E_x^{v^*}\Big[e^{\delta\tau_1^c} I_{\{\tau_1^c<\tau_n\}}\Big] \to E_x^{v^*}\Big[e^{\delta\tau_1^c}\Big]$ as $n \to \infty$. Therefore from (4.30), it is clear that for $x \in \mathcal{Q}^c$

$$\lim_{n\to\infty} E_x^{v^*}\Big[e^{\delta\tau_n} I_{\{\tau_n < \tau_1^c\}}\Big] = 0. \tag{4.31}$$

Now for each $k \geq 1$ define $\partial K_n(k) := \{y \in \partial K_n : u(y) \geq k\}$. We now have the following

$$\begin{aligned}
& E_x^{v^*}\Big[e^{\int_0^{\tau_n}(r(X(t),v^*(X(t)))-\rho)\mathrm{d}t} u(X(\tau_n)) I_{\{\tau_n<\tau_1^c\}}\Big] \\
& \leq E_x^{v^*}\Big[e^{\delta\tau_n} u(X(\tau_n)) I_{\{x\in\partial K_n(k)^c\}} I_{\{\tau_n<\tau_1^c\}}\Big] + E_x^{v^*}\Big[e^{\delta\tau_n} u(X(\tau_n)) I_{\{x\in\partial K_n(k)\}} I_{\{\tau_n<\tau_1^c\}}\Big], \\
& \leq k E_x^{v^*}\Big[e^{\delta\tau_n} I_{\{\tau_n<\tau_1^c\}}\Big] + E_x^{v^*}\Big[e^{\delta\tau_n} u(X(\tau_n)) I_{\{x\in\partial K_n(k)\}} I_{\{\tau_n<\tau_1^c\}}\Big], \\
& \leq k E_x^{v^*}\Big[e^{\delta\tau_n} I_{\{\tau_n<\tau_1^c\}}\Big] + \hat{C}_1 E_x^{v^*}\Big[e^{\delta\tau_n} (V(X(\tau_n)))^{\hat{\beta}} I_{\{x\in\partial K_n(k)\}} I_{\{\tau_n<\tau_1^c\}}\Big], \text{ (by (4.26))}, \\
& \leq k E_x^{v^*}\Big[e^{\delta\tau_n} I_{\{\tau_n<\tau_1^c\}}\Big] + \hat{C}_1 \left(\frac{k}{\hat{C}_1}\right)^{\frac{\hat{\beta}-1}{\hat{\beta}}} E_x^{v^*}\Big[e^{\delta\tau_n} (V(X(\tau_n)))^{\hat{\beta}} I_{\{x\in\partial K_n(k)\}} I_{\{\tau_n<\tau_1^c\}}\Big], \\
& \leq k E_x^{v^*}\Big[e^{\delta\tau_n} I_{\{\tau_n<\tau_1^c\}}\Big] + \hat{C}_1 \left(\frac{k}{\hat{C}_1}\right)^{\frac{\hat{\beta}-1}{\hat{\beta}}} V(x).
\end{aligned} \tag{4.32}$$

In view of (4.31) letting $n \to \infty$ and then $k \to \infty$ in (4.32), we get

$$\lim_{n\to\infty} E_x^{v^*}\Big[e^{\int_0^{\tau_n}(r(X(t),v^*(X(t)))-\rho)\mathrm{d}t} u(X(\tau_n)) I_{\{\tau_n<\tau_1^c\}}\Big] = 0.$$

Therefore from (4.29), we obtain the required representation (4.24). □

Remark 4.5. From (4.26), we have that for all $x \in \mathcal{Q}^c$, $u^{l,m}(x) \leq \hat{C}_1 V^{\hat{\beta}}(x)$. Since $V(x) \geq 1$ and $\hat{\beta} \in (0,1)$, one can see that for all $x \in \mathcal{Q}^c$, $u^{l,m}(x) \leq \hat{C}_1 V(x)$. Also since $\hat{C}_1 = \frac{\sup_{y\in\mathcal{Q}} u^{l,m}(y)}{\inf_{y\in\mathcal{Q}} V^{\hat{\beta}}(y)}$, it is clear that $u^{l,m}(x) \leq \hat{C}_1 V^{\hat{\beta}}(x)$ for all $x \in \mathcal{Q}$. Thus, $u^{l,m}(x) \leq \hat{C}_1 V^{\hat{\beta}}(x)$ for all $x \in D_{l,m}$. This implies $u(x) \leq \hat{C}_1 V^{\hat{\beta}}(x)$ for all $x \in D$ (since u is a subsequential limit of $u^{l,m}$). This intern implies $u(x) \leq \hat{C}_1 V(x)$ for all $x \in D$.

Following [14], we now approximate the running cost function in the following way: Let $\{\phi_n\}$ be a sequence of test functions such that $\phi_n = 1$ in K_n and $\phi_n = 0$ in K_{n+1}^c. As we know $\|r\|_\infty < \delta$, thus one can choose a constant $\delta_2 > 0$ small enough such that $\|r\|_\infty + \delta_2 < \delta$. For all $(x, u_1, u_2) \in D \times U_1 \times U_2$ define

$$r_n(x, u_1, u_2) = \phi_n(x) r(x, u_1, u_2) + (1 - \phi_n(x))(\|r\|_\infty + \delta_2), \ \forall \ n \in \mathbb{N}.$$

It is easy see that all the results of Theorems 4.1–4.3 and Lemma 4.4 hold for r replaced by r_n. Collecting all the results, we have the following lemma

Lemma 4.6. *Assume (A0)–(A5). Then for each fixed $\tilde{v}_1 \in \mathcal{S}_1$ and $n \in \mathbb{N}$ there exists $(\rho_n^{\tilde{v}_1}, u_n^{\tilde{v}_1}) \in \mathbb{R} \times W_{loc}^{2,q}(D \cup \Sigma)$, $q \geq d+1$, $u_n^{\tilde{v}_1} > 0$, satisfying*

$$\begin{aligned}\rho_n^{\tilde{v}_1} u_n^{\tilde{v}_1} &= \sup_{v_2 \in V_2} \Big[\langle b(x, \tilde{v}_1(x), v_2), \nabla u_n^{\tilde{v}_1}\rangle + r_n(x, \tilde{v}_1(x), v_2) u_n^{\tilde{v}_1}\Big] + \frac{1}{2}\ \mathrm{trace}(a(x)\nabla^2 u_n^{\tilde{v}_1}),\\ u_n^{\tilde{v}_1}(x_0) &= 1,\ \nabla u_n^{\tilde{v}_1}(x) \cdot \gamma(x) = 0 \ \text{ on } \partial D.\end{aligned} \tag{4.33}$$

Moreover, $u_n^{\tilde{v}_1} \leq \hat{C}_1 V^{\hat{\beta}}$ outside some compact set, for some constant $\hat{C}_1$ which is independent of n.

Proof. The existence of a solution follows by the similar arguments as in Theorem 4.3. Following the similar steps as in Lemma 4.4, one can show that $u_n^{\tilde{v}_1} \leq \hat{C}_1 V^{\hat{\beta}}$ outside some compact set for some constant $\hat{C}_1$ which is independent of n. Here without loss of generality we are denoting the constant by the same notation as in Lemma 4.4. □

The next theorem proves that any mini-max selector of (4.13) is a saddle point equilibrium.

Theorem 4.7. *Assume (A0)–(A5). Then any mini-max selector $(v_1^*, v_2^*) \in \mathcal{S}_1 \times \mathcal{S}_2$ of (4.13) is a saddle point equilibrium and ρ as in (4.13) is the corresponding value of the game.*

Proof. Since we know that $u \in W_{loc}^{2,q}(D \cup \Sigma) \cap O(V)$, $q \geq d+1$, is a solution to (4.13), by standard measurable selection theorem and Fan's mini-max theorem there exist $(v_1^*, v_2^*) \in \mathcal{S}_1 \times \mathcal{S}_2$ such that

$$\begin{aligned}&\sup_{v_2 \in V_2} \Big[\langle b(x, v_1^*(x), v_2), \nabla u\rangle + r(x, v_1^*(x), v_2) u\Big]\\ &\qquad = \inf_{v_1 \in V_1} \sup_{v_2 \in V_2} \Big[\langle b(x, v_1, v_2), \nabla u\rangle + r(x, v_1, v_2) u\Big]\\ &\qquad = \sup_{v_2 \in V_2} \inf_{v_1 \in V_1} \Big[\langle b(x, v_1, v_2), \nabla u\rangle + r(x, v_1, v_2) u\Big]\\ &\qquad = \inf_{v_1 \in V_1} \Big[\langle b(x, v_1, v_2^*(x)), \nabla u\rangle + r(x, v_1, v_2^*(x)) u\Big].\end{aligned} \tag{4.34}$$

Since $u \in W_{loc}^{2,q}(D \cup \Sigma) \cap O(V)$, $q \geq d+1$, ∇u is bounded on each of the compact subset $(D \cup \Sigma) \cap \bar{B}_R$. Thus,

$$\int_0^{T \wedge \tau_R} e^{\int_0^t \big(r(X(s), v_1(s), v_2^*(X(s))) - \rho\big)\mathrm{d}s} (\nabla u(X(t)))^{\perp} \sigma(X(t)) \mathrm{d}W(t)$$

is a zero-mean martingale. Now applying Itô-Krylov formula, using (4.13) and (4.34), we have

$$\begin{aligned}&E_x^{v_1, v_2^*}\Big[e^{\int_0^{T\wedge\tau_R} (r(X(t), v_1(t), v_2^*(X(t))) - \rho)\mathrm{d}t} u(X(T \wedge \tau_R))\Big]\\ &= \ E_x^{v_1, v_2^*}\Big[\int_0^{T\wedge\tau_R} e^{\int_0^t (r(X(s), v_1(s), v_2^*(X(s))) - \rho)\mathrm{d}s} [Lu(X(t), v_1(t), v_2^*(X(t)))\\ &\qquad + (r(X(t), v_1(t), v_2^*(X(t))) - \rho) u(X(t))\mathrm{d}t\Big] + u(x)\\ &\geq u(x).\end{aligned}$$

Since $u(x) \leq \hat{C}_1 V^{\hat{\beta}}(x)$ in D (see Rem. 4.5)

$$\begin{aligned} u(x) &\leq \hat{C}_1 E_x^{v_1,v_2^*}\Big[e^{\int_0^{T\wedge\tau_R}(r(X(t),v_1(t),v_2^*(X(t)))-\rho)\mathrm{d}t}V^{\hat{\beta}}(X(T\wedge\tau_R))\Big] \\ &\leq \hat{C}_1 E_x^{v_1,v_2^*}\Big[e^{\int_0^{T\wedge\tau_R}(r(X(t),v_1(t),v_2^*(X(t)))-\rho)\mathrm{d}t}V^{\hat{\beta}}(X(T\wedge\tau_R))I_{\{T\leq\tau_R\}}\Big] \\ &+ \hat{C}_1 E_x^{v_1,v_2^*}\Big[e^{\int_0^{T\wedge\tau_R}(r(X(t),v_1(t),v_2^*(X(t)))-\rho)\mathrm{d}t}V^{\hat{\beta}}(X(\tau_R))I_{\{\tau_R\leq T\}}\Big]. \end{aligned} \tag{4.35}$$

Now arguing as in Lemma 4.4, we obtain

$$\lim_{R\to\infty} E_x^{v_1,v_2^*}\Big[e^{\int_0^{T\wedge\tau_R}(r(X(t),v_1(t),v_2^*(X(t)))-\rho)\mathrm{d}t}V^{\hat{\beta}}(X(\tau_R))I_{\{\tau_R\leq T\}}\Big] = 0. \tag{4.36}$$

Letting $R\to\infty$ in (4.35) using (4.36), we get (since $0\leq\hat{\beta}<1$ and $V\geq 1$)

$$u(x) \leq \hat{C}_1 E_x^{v_1,v_2^*}\Big[e^{\int_0^{T}(r(X(t),v_1(t),v_2^*(X(t)))-\rho)\mathrm{d}t}V(X(T))\Big]. \tag{4.37}$$

Using Itô-Krylov formula and (A4), it follows that

$$E_x^{v_1,v_2^*}\Big[e^{\int_0^{T} r(X(t),v_1(t),v_2^*(X(t)))\mathrm{d}t}V(X(T))\Big] \leq (V(x)+\tilde{\alpha}T)E_x^{v_1,v_2^*}\Big[e^{\int_0^{T} r(X(t),v_1(t),v_2^*(X(t)))\mathrm{d}t}\Big]. \tag{4.38}$$

Thus, from (4.37) and (4.38), we have

$$u(x) \leq \hat{C}_1(V(x)+\tilde{\alpha}T)e^{-\rho T}E_x^{v_1,v_2^*}\Big[e^{\int_0^{T} r(X(t),v_1(t),v_2^*(X(t)))\mathrm{d}t}\Big]. \tag{4.39}$$

Now by taking logarithm both sides of (4.39), dividing by T and then letting $T\to\infty$, we get

$$\rho \leq \limsup_{T\to\infty}\frac{1}{T}\log E_x^{v_1,v_2^*}\Big[e^{\int_0^{T} r(X(t),v_1(t),v_2^*(X(t)))\mathrm{d}t}\Big].$$

Thus,

$$\rho \leq \inf_{v_1\in\mathcal{A}_1}\limsup_{T\to\infty}\frac{1}{T}\log E_x^{v_1,v_2^*}\Big[e^{\int_0^{T} r(X(t),v_1(t),v_2^*(X(t)))\mathrm{d}t}\Big]. \tag{4.40}$$

Now (4.13) and (4.34) implies that

$$\begin{aligned} \rho u &= \sup_{v_2\in V_2}\Big[\langle b(x,v_1^*(x),v_2),\nabla u\rangle + r(x,v_1^*(x),v_2)u\Big] + \frac{1}{2}\,\mathrm{trace}(a(x)\nabla^2 u), \\ u(x_0) &= 1,\ \ \nabla u\cdot\gamma = 0 \text{ on } \partial D. \end{aligned} \tag{4.41}$$

From Lemma 4.6, we know that for each $n\geq 1$ and fixed $v_1^*\in\mathcal{S}_1$ there exist $(\rho_n^{v_1^*},u_n^{v_1^*})\in\mathbb{R}\times W_{loc}^{2,q}(D\cup\Sigma)$, $q\geq d+1$, $u_n^{v_1^*}>0$, satisfying

$$\begin{aligned} \rho_n^{v_1^*}u_n^{v_1^*} &= \sup_{v_2\in V_2}\Big[\langle b(x,v_1^*(x),v_2),\nabla u_n^{v_1^*}\rangle + r_n(x,v_1^*(x),v_2)u_n^{v_1^*}\Big] + \frac{1}{2}\,\mathrm{trace}(a(x)\nabla^2 u_n^{v_1^*}), \\ u_n^{v_1^*}(x_0) &= 1,\ \ \nabla u_n^{v_1^*}(x)\cdot\gamma(x) = 0 \ \text{ on } \ \partial D. \end{aligned} \tag{4.42}$$

Furthermore, $u_n^{v_1^*} \leq \hat{C}_1 V^{\hat{\beta}}$, outside a compact set for some constant $\hat{C}_1$ (denoting by same notation as in Lem. 4.4) independent of n. For any maximizing selector $\tilde{v}_2 \in \mathcal{S}_2$ of (4.42), by closely mimicking the steps as we have used to derive (4.40), one can show that

$$\rho_n^{v_1^*} \leq \limsup_{T\to\infty} \frac{1}{T} \log E_x^{v_1^*,\tilde{v}_2}\Big[e^{\int_0^T r_n(X(t),v^*(X(t)),\tilde{v}_2(X(t)))\mathrm{d}t}\Big]. \tag{4.43}$$

From the construction it is clear that $\|r_n\|_\infty \leq \|r\|_\infty + \delta_2$. Thus, from (4.43), we have $\rho_n \leq \|r\|_\infty + \delta_2$. Let $\hat{K} = \overline{K}_{n+1}$. Therefore from our construction it is easy to see that $\inf_{(u_1,u_2)\in U_1\times U_2} r_n(x,u_1,u_2) - \rho_n^{v_1^*} \geq 0$ for all $x \in \hat{K}^c$. Let $\tau_{\hat{K}}^c = \inf\{t \geq 0 : X(t) \in \hat{K}\}$. Without loss of generality we may assume that $\hat{K} \supset \mathcal{Q}$ (this is true for large n). Thus, following the arguments as in Lemma 4.4, we obtain

$$\begin{aligned} u_n^{v_1^*}(x) &= E_x^{v_1^*,\tilde{v}_2}\Big[e^{\int_0^{\tau_{\hat{K}}^c}(r_n(X(t),v_1^*(X(t)),\tilde{v}_2(X(t)))-\rho_n^{v_1^*})\mathrm{d}t} u_n^{v_1^*}(X(\tau_{\hat{K}}^c))\Big], \\ &\geq \inf_{\hat{K}} u_n^{v_1^*}, \ \ \forall \ x \in \hat{K}^c. \end{aligned}$$

Using Itô-Krylov's formula and Fatou's lemma, from (4.42) for any $v_2 \in \mathcal{A}_2$, we get

$$\begin{aligned} u_n^{v_1^*}(x) &\geq E_x^{v_1^*,v_2}\Big[e^{\int_0^T(r_n(X(t),v_1^*(X(t)),v_2(t))-\rho_n^{v_1^*})\mathrm{d}t} u_n^{v_1^*}(X(T))\Big], \\ &\geq \inf_{\hat{K}} u_n^{v_1^*} E_x^{v_1^*,v_2}\Big[e^{\int_0^T(r_n(X(t),v_1^*(X(t)),v_2(t))-\rho_n^{v_1^*})\mathrm{d}t}\Big]. \end{aligned}$$

Taking logarithm on both sides, dividing by T and then letting $T \to \infty$, it follows that

$$\begin{aligned} \rho_n^{v_1^*} &\geq \limsup_{T\to\infty} \frac{1}{T} \log E_x^{v_1^*,v_2}\Big[e^{\int_0^T r_n(X(t),v_1^*(X(t)),v_2(t))\mathrm{d}t}\Big], \\ &\geq \sup_{v_2\in\mathcal{A}_2} \limsup_{T\to\infty} \frac{1}{T} \log E_x^{v_1^*,v_2}\Big[e^{\int_0^T r(X(t),v_1^*(X(t)),v_2(t))\mathrm{d}t}\Big]. \end{aligned} \tag{4.44}$$

Applying Harnack's inequality and Sobolev estimate on (4.42) (after choosing some maximizing selector), it is easy to see as in Theorem 4.3 that $u_n^{v_1^*}$ is uniformly bounded in $W^{2,q}_{loc}(D\cup\Sigma)$, $q \geq d+1$. Therefore, one can extract a subsequence of $\{u_n^{v_1^*}\}$ that converges weakly in $W^{2,q}_{loc}(D\cup\Sigma)$, $q \geq d+1$ to some $u_*^{v_1^*} \in W^{2,q}_{loc}(D\cup\Sigma)$, $q \geq d+1$ and strongly in $C^{1,\hat{\alpha}}_{loc}(D\cup\Sigma)$, $\hat{\alpha} \in (0,1)$. It follows from (4.43) and (4.44) that $\{\rho_n^{v_1^*}\}$ is a bounded sequence. Therefore, along a further subsequence it converges to a constant $\rho_*^{v_1^*}$. Following the similar steps as in Theorem 4.3, letting $n \to \infty$ in (4.42), we obtain $(\rho_*^{v_1^*}, u_*^{v_1^*}) \in \mathbb{R} \times W^{2,q}_{loc}(D\cup\Sigma)$, $q \geq d+1$, $u_*^{v_1^*} > 0$, satisfies

$$\begin{aligned} \rho_*^{v_1^*} u_*^{v_1^*} &= \sup_{v_2\in V_2}\Big[\langle b(x,v_1^*(x),v_2), \nabla u_*^{v_1^*}\rangle + r(x,v_1^*(x),v_2) u_*^{v_1^*}\Big] + \frac{1}{2}\ \mathrm{trace}(a(x)\nabla^2 u_*^{v_1^*}), \\ u_*^{v_1^*}(x_0) &= 1,\ \nabla u_*^{v_1^*}\cdot\gamma = 0 \ \text{ on } \partial D. \end{aligned} \tag{4.45}$$

Since $u_n^{v_1^*} \leq \hat{C}_1 V^{\hat{\beta}}$, outside some suitable compact set for some constant $\hat{C}_1$ independent of n, thus the limit satisfies $u_*^{v_1^*} \leq \hat{C}_1 V^{\hat{\beta}}$, outside some suitable compact set. Therefore by the similar arguments as in Lemma 4.4, it follows that for any maximizing selector $\tilde{v}_2^* \in \mathcal{S}_2$ of (4.45), $u_*^{v_1^*}$ admits the following stochastic representation

$$u_*^{v_1^*}(x) = E_x^{v_1^*,\tilde{v}_2^*}\Big[e^{\int_0^{\tau_1^c}(r(X(t),v_1^*(X(t)),\tilde{v}_2^*(X(t)))-\rho_*^{v_1^*})\mathrm{d}t} u_*^{v_1^*}(X(\tau_1^c))\Big],\ \ \forall\ x \in \mathcal{Q}_1^c, \tag{4.46}$$

for some compact set $\mathcal{Q}_1$ (without loss of generality we are using the same notation as in Lem. 4.4). We now want to show that for any other solution $(\tilde{\rho}_*^{v_1^*}, \tilde{u}_*^{v_1^*}) \in \mathbb{R} \times W^{2,q}_{loc}(D \cup \Sigma)$, $q \geq d+1$, $\tilde{u}_*^{v_1^*} > 0$ of (4.45), we have $\rho_*^{v_1^*} \leq \tilde{\rho}_*^{v_1^*}$. In particular, we want to show that $\rho_*^{v_1^*} \leq \rho$ (in view of (4.41)). If not, then $\rho < \rho_*^{v_1^*}$. By similar arguments as in Lemma 4.4, from (4.41) for $\tilde{v}_2^* \in \mathcal{S}_2$, it follows that

$$\begin{aligned} u(x) &\geq E_x^{v_1^*, \tilde{v}_2^*}\Big[e^{\int_0^{\tau_1^c}(r(X(t), v_1^*(X(t)), \tilde{v}_2^*(X(t))) - \rho)\mathrm{d}t} u_*^{v_1^*}(X(\tau_1^c))\Big], \\ &\geq E_x^{v_1^*, \tilde{v}_2^*}\Big[e^{\int_0^{\tau_1^c}(r(X(t), v_1^*(X(t)), \tilde{v}_2^*(X(t))) - \rho_*^{v_1^*})\mathrm{d}t} u_*^{v_1^*}(X(\tau_1^c))\Big] \;\forall\; x \in \mathcal{Q}_1^c. \end{aligned} \tag{4.47}$$

Now from (4.46) and (4.47), we have

$$(u - u_*^{v_1^*})(x) \geq E_x^{v_1^*, \tilde{v}_2^*}\Big[e^{\int_0^{\tau_1^c}(r(X(t), v_1^*(X(t)), \tilde{v}_2^*(X(t))) - \rho_*^{v_1^*})\mathrm{d}t}(u - u_*^{v_1^*})(X(\tau_1^c))\Big].$$

Therefore, one can easily see that $u(x) - u_*^{v_1^*}(x) \geq 0$ for all $x \in D$ if it holds in $\mathcal{Q}_1$. Now multiplying $u_*^{v_1^*}$ by a suitable positive constant (say, $\hat{k}_1 = \inf_{\mathcal{Q}_1} \frac{u}{u_*^{v_1^*}}$), we have $u(x) - \tilde{u}_*^{v_1^*}(x) \geq 0$ in $\mathcal{Q}_1$ and attains its minimum value 0 in $\mathcal{Q}_1$, where $\tilde{u}_*^{v_1^*} = \hat{k}_1 u_*^{v_1^*}$. It is clear that $\tilde{u}_*^{v_1^*}$ also satisfies (4.45). Now from (4.45) (for $\tilde{u}_*^{v_1^*}$) and (4.41), it follows that

$$\begin{aligned} &\frac{1}{2}\mathrm{trace}(a(x)\nabla^2(u - \tilde{u}_*^{v_1^*})) + \Big[\langle b(x, v_1^*(x), \tilde{v}_2^*(x)), \nabla(u - \tilde{u}_*^{v_1^*})\rangle - \\ &(r(x, v_1^*(x), \tilde{v}_2^*(x)) - \hat{\rho})^-(u - \tilde{u}_*^{v_1^*})\Big] \leq -(r(x, v_1^*(x), \tilde{v}_2^*(x)) - \rho)^+(u - \tilde{u}_*^{v_1^*}) - (\rho_*^{v_1^*} - \rho)\tilde{u}_*^{v_1^*} \leq 0. \end{aligned}$$

By strong maximum principle as in Corollary 1.21 from [56] (see also [55], Cor. 2.4), we have $u = \tilde{u}_*^{v_1^*}$. Since $u(x_0) = u_*^{v_1^*}(x_0) = 1$, it follows that $u = u_*^{v_1^*}$. Thus, from (4.41) and (4.45), we get

$$\rho u = \rho_*^{v_1^*} u.$$

Since $u > 0$, one can see that $\rho = \rho_*^{v_1^*}$. This contradict the fact that $\rho < \rho_*^{v_1^*}$. Therefore we have $\rho \geq \rho_*^{v_1^*}$. Now combining (4.40) and (4.44), we obtain the following

$$\begin{aligned} \sup_{v_2 \in \mathcal{A}_2} \limsup_{T \to \infty} \frac{1}{T} \log E_x^{v_1^*, v_2}\Big[e^{\int_0^T r(X(t), v_1^*(X(t)), v_2(t))\mathrm{d}t}\Big] &\leq \rho_*^{v_1^*} \leq \rho \\ &\leq \inf_{v_1 \in \mathcal{A}_1} \limsup_{T \to \infty} \frac{1}{T} \log E_x^{v_1, v_2^*}\Big[e^{\int_0^T r(X(t), v_1(t), v_2^*(X(t)))\mathrm{d}t}\Big]. \end{aligned}$$

Therefore, we have

$$\rho = \limsup_{T \to \infty} \frac{1}{T} \log E_x^{v_1^*, v_2^*}\Big[e^{\int_0^T r(X(t), v_1^*(X(t)), v_2^*(X(t)))\mathrm{d}t}\Big]. \tag{4.48}$$

This completes the proof. □

Now we completely characterize the saddle point equilibrium in the space of stationary Markov controls.

Theorem 4.8. *Assume (A0)–(A5). Then for any saddle point equilibrium $(\bar{v}_1^*, \bar{v}_2^*) \in \mathcal{S}_1 \times \mathcal{S}_2$, i.e.,*

$$\rho^{\bar{v}_1^*, \bar{v}_2^*} \leq \rho^{v_1, \bar{v}_2^*} \ \ \forall\, v_1 \in \mathcal{A}_1,$$
$$\rho^{\bar{v}_1^*, \bar{v}_2^*} \geq \rho^{\bar{v}_1^*, v_2} \ \ \forall\, v_2 \in \mathcal{A}_2,$$

$\bar{v}_1^*$ is an outer minimizing selector of (4.13) and $\bar{v}_2^*$ is an outer maximizing selector of (4.13).

Proof. Since $(\bar{v}_1^*, \bar{v}_2^*) \in \mathcal{S}_1 \times \mathcal{S}_2$ is a saddle point equilibrium thus, we have

$$\inf_{\bar{v}_1 \in \mathcal{A}_1} \sup_{\bar{v}_2 \in \mathcal{A}_2} \rho^{\bar{v}_1, \bar{v}_2}(x) \leq \sup_{\bar{v}_2 \in \mathcal{A}_2} \rho^{\bar{v}_1^*, \bar{v}_2}(x) \leq \rho^{\bar{v}_1^*, \bar{v}_2^*}(x) \leq \inf_{\bar{v}_1 \in \mathcal{A}_1} \rho^{\bar{v}_1, \bar{v}_2^*}(x) \leq \sup_{\bar{v}_2 \in \mathcal{A}_2} \inf_{\bar{v}_1 \in \mathcal{A}_1} \rho^{\bar{v}_1, \bar{v}_2}(x), \tag{4.49}$$

for all $x \in \overline{D}$. Since $\inf_{\bar{v}_1 \in \mathcal{A}_1} \sup_{\bar{v}_2 \in \mathcal{A}_2} \rho^{\bar{v}_1, \bar{v}_2}(x) \geq \sup_{\bar{v}_2 \in \mathcal{A}_2} \inf_{\bar{v}_1 \in \mathcal{A}_1} \rho^{\bar{v}_1, \bar{v}_2}(x)$, all the above inequalities are actually equalities and equals to ρ (as in (4.13), follows from Thm. 4.7). Arguing as in Theorem 4.1 and Theorem 4.3 one can prove that there exist $(\rho, \tilde{u}) \in \mathbb{R} \times W^{2,q}_{loc}(D \cup \Sigma) \cap O(V)$, $q \geq d+1$, satisfying

$$\begin{aligned} \rho\tilde{u} &= \inf_{v_1 \in V_1} \Big[\langle b(x, v_1, \bar{v}_2^*(x)), \nabla u\rangle + r(x, v_1, \bar{v}_2^*(x))\tilde{u}\Big] + \frac{1}{2}\,\mathrm{trace}(a(x)\nabla^2 \tilde{u}), \\ \tilde{u}(x_0) &= 1, \nabla\tilde{u}\cdot\gamma = 0 \,\text{on}\, \partial D. \end{aligned} \tag{4.50}$$

Now for $v_1^* \in \mathcal{S}_1$ as in Theorem 4.7, we have

$$\rho\tilde{u} \leq \Big[\langle b(x, v_1^*(x), \bar{v}_2^*(x)), \nabla\tilde{u}\rangle + r(x, v_1^*(x), \bar{v}_2(x))\tilde{u}\Big] + \frac{1}{2}\,\mathrm{trace}(a(x)\nabla^2\tilde{u}), \ \text{ a.e. } \ x \in D. \tag{4.51}$$

From (4.13) and (4.34) it follows that

$$\begin{aligned} \rho u &= \sup_{v_2 \in V_2} \Big[\langle b(x, v_1^*(x), v_2), \nabla u\rangle + r(x, v_1^*(x), v_2)u\Big] + \frac{1}{2}\,\mathrm{trace}(a(x)\nabla^2 u), \\ &\geq \Big[\langle b(x, v_1^*(x), \bar{v}_2^*(x)), \nabla u\rangle + r(x, v_1^*(x), \bar{v}_2^*(x))u\Big] + \frac{1}{2}\,\mathrm{trace}(a(x)\nabla^2 u), \\ u(x_0) &= 1, \ \ \nabla u\cdot\gamma = 0 \,\text{on}\, \partial D. \end{aligned} \tag{4.52}$$

Arguing as in Lemma 4.4, it follows that

$$\tilde{u}(x) \leq E_x^{v_1^*, \bar{v}_2^*}\Big[e^{\int_0^{\tau_1^c}(r(X(t), v_1^*(X(t)), \bar{v}_2^*(X(t))) - \rho)\mathrm{d}t}\tilde{u}(X(\tau_1^c))\Big].$$

Again applying Itô-Krylov and using Fatou's lemma, from (4.52), we obtain

$$u(x) \geq E_x^{v_1^*, \bar{v}_2^*}\Big[e^{\int_0^{\tau_1^c}(r(X(t), v_1^*(X(t)), \bar{v}_2^*(X(t))) - \rho)\mathrm{d}t}u(X(\tau_1^c))\Big].$$

Thus,

$$(u - \tilde{u})(x) \geq E_x^{v_1^*, \bar{v}_2^*}\Big[e^{\int_0^{\tau_1^c}(r(X(t), v_1^*(X(t)), \bar{v}_2^*(X(t))) - \rho)\mathrm{d}t}(u - \tilde{u})(X(\tau_1^c))\Big]. \tag{4.53}$$

It is clear from (4.53) that $(u - \tilde{u})(x) \geq 0$ in D if it is true in $\mathcal{Q}_1$. Let $\bar{\tilde{u}} = k_2\tilde{u}$ where $k_2 = \inf_{\mathcal{Q}_1}(\frac{u}{\tilde{u}})$. By the choice of above constant, we have $u(x) - \bar{\tilde{u}}(x) \geq 0$ in $\mathcal{Q}_1$ and attains its minimum value 0 in $\mathcal{Q}_1$. From (4.50) and

(4.52), it follows that

$$\frac{1}{2}\,\mathrm{trace}(a(x)\nabla^2(u-\tilde{u})) + \Big[\langle b(x,v_1^*(x),\bar{v}_2^*(x)),\nabla(u-\tilde{u})\rangle - (r(x,v_1^*(x),\bar{v}_2^*(x))-\rho)^-(u-\tilde{u})\Big] \le -(r(x,v_1^*(x),\bar{v}_2^*(x))-\rho)^+(u-\tilde{u}) \le 0$$

Now by maximum principle as in Corollary 1.21 from [56] (see also, [55], Cor. 2.4), we have $u = \tilde{u}$ (since $u(x_0) = \tilde{u}(x_0) = 1$). It is now clear from (4.13) and (4.50) that $\bar{v}_2^*$ is an outer maximizing selector of (4.13). Similarly one can prove that $\bar{v}_1^*$ is an outer minimizing selector of (4.13). □

5. Conclusions

We have established the existence of values and saddle point equilibria for risk-sensitive zero-sum stochastic differential games where the state is a controlled reflecting diffusion processes in orthant. We have studied two cost evaluation criteria, *viz.*, discounted and ergodic cost. Under fairly general conditions we have established our results for discounted cost criterion. In this article, we have crucially used (A5) for the analysis of ergodic cost criterion. It will be interesting to establish analogous results without (A5).

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