

STOCHASTIC OPTIMAL CONTROL OF A EVOLUTIONARY p -LAPLACE EQUATION WITH MULTIPLICATIVE LÉVY NOISE

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Abstract. In this article, we are interested in an initial value optimal control problem for a evolutionary p -Laplace equation driven by multiplicative Lévy noise. We first present wellposedness of a weak solution by using an implicit time discretization of the problem, along with the Jakubowski version of the Skorokhod theorem for a non-metric space. We then formulate associated control problem, and establish existence of an optimal solution by using variational method and exploiting the convexity property of the cost functional.

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1. INTRODUCTION

The last couple of decades have witnessed remarkable advances on the larger area of stochastic partial differential equations that are driven by Lévy noise. A worthy reference on this subject is [25]. In this article, we are interested in the specific problem of evolution equation with Lévy noise, and aim to prove existence of a weak optimal solution of an initial value control evolutionary p -Laplace equation driven by Lévy noise. A formal description of our problem as follows. Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$ be a filtered probability space satisfying the usual hypotheses *i.e.* $\{\mathcal{F}_t\}_{t \geq 0}$ is a right-continuous filtration such that \mathcal{F}_0 contains all the \mathbb{P} -null subsets of (Ω, \mathcal{F}) . In addition, let $N(dz, dt)$ be a time homogeneous Poisson random measure¹ on \mathbb{R} with intensity measure $m(dz)$ with respect to the same stochastic basis. We are interested in the initial value control evolution equation of the type

$$\begin{aligned} du - \operatorname{div}_x(|\nabla u|^{p-2} \nabla u + \vec{f}(u)) dt &= \int_{|z|>0} \eta(u; z) \tilde{N}(dz, dt) \quad \text{in } \Omega \times D_T, \\ u &= 0 \quad \text{on } \Omega \times \partial D_T, \\ u(0, \cdot) &= u_0(\cdot) + U(\cdot) \quad \text{in } \Omega \times D, \end{aligned} \tag{1.1}$$

where $p > 2$, $D_T = (0, T) \times D$ with $T > 0$ fixed, $D \subset \mathbb{R}^d$ is a bounded domain with Lipschitz boundary ∂D , $\partial D_T = (0, T) \times \partial D$, u is the unknown random scalar valued function, $\vec{f}: \mathbb{R} \rightarrow \mathbb{R}^d$ is a given flux function, and

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¹ For the definition of a time homogeneous Poisson random measure, we refer to see page 631 in [4].

$\tilde{N}(dz, dt) = N(dz, dt) - m(dz) dt$, the compensated time homogeneous Poisson random measure. Furthermore, $(u, z) \mapsto \eta(u; z)$ is a real valued function defined on the domain $\mathbb{R} \times \mathbb{R}$. The stochastic integral in the right hand side of (1.1) is defined in the Lévy-Itô sense.

We point out that adding a Brownian component to the Lévy noise term on the right hand side of (1.1) would make it more general, and the results of this paper are still valid under appropriate conditions.

The equation (1.1) could be viewed as a stochastic perturbation of a evolution p -Laplacian equation with nonlinear sources. Equations of this type arise in the field of mechanics, physics and biology [8, 31]. In the case $\eta = 0$, the equation (1.1) becomes a deterministic evolution p -Laplacian equation with nonlinear sources, and there is a plethora literature (see [20, 32] and references therein) for its wellposedness.

Due to more technical novelties, the study of wellposedness result in case of nonlinear p -evolutionary equation with nonlinear stochastic forcing is more subtle. The presence of nonlinearity in the drift and diffusion terms in equation prevents us to define a semi-group solution. Moreover, because of nonlinear perturbation $\operatorname{div}_x f(u)$ of p -Laplace operator ($p > 2$), one can not use the results of monotone or locally monotone SPDEs, see *e.g.*, [15, 23]. In a recent article [28], the authors have considered (1.1) with cylindrical Wiener process $W = \{W_t : t \in [0, T]\}$ in $L^2(D)$, and proved wellposedness of strong solution. In [28], existence of a martingale solution is shown by constructing an approximate solution (*via* implicit time discretization) and deriving its *a priori* estimates which are used to apply Jakubowski-Skorokhod theorem in a non-metric space. Then, using an argument of path-wise uniqueness and Gyöngy-Krylov characterization [12] of convergence in probability, the authors established wellposedness of strong solution.

In this paper, our goal is to find a weak admissible solution $\pi^* := (\Omega^*, \mathcal{F}^*, \mathbb{P}^*, \{\mathcal{F}_t^*\}, N^*, u^*, U^*)$ which minimizes

$$\mathcal{J}(\pi) = \mathbb{E} \left[\int_0^T \|u(t) - u_{\text{tar}}(t)\|_{L^2(D)}^2 dt + \|U\|_{W^{1,p}}^p + \Psi(u(T)) \right] \quad (1.2)$$

with $\pi = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}, N, u, U)$ subject to (1.1),

for a given deterministic target profile u_{tar} , and terminal payoff Ψ . The existing literature (see *e.g.* [21]) on stochastic optimal control with SPDEs mainly considers those which has a mild solution, which is not available for problem (1.1). For this reason, we use variational method to construct a minimizer π^* of (1.2), see also [6, 9]. Being motivated from [6, 9, 28], our aim is twofold:

- i) Firstly, we prove existence of a weak solution of the problem (1.1). We construct an approximate solutions $\tilde{u}_\tau := \{\tilde{u}_\tau(t); t \in [0, T]\}$ (*cf.* (3.5)) *via* implicit time discretization, and derive its *a priori* bounds which is used to show the tightness of the laws of sequence (\tilde{u}_τ) , denoted by $\mathcal{L}(\tilde{u}_\tau)$, in some appropriate space *via* Aldous condition (see Def. 3.3). We then use the Jakubowski version of the Skorokhod theorem in a non-metric space to show existence of a weak solution of (1.1). We use smooth approximation of absolute value function and then apply Itô-Lévy formula, and pass to the limit as approximation parameter goes to zero to show the path-wise uniqueness of weak solutions.
- ii) Secondly, we construct a minimizer π^* of the control problem (1.2) by considering a minimizing weak admissible solutions $\pi_n = (\Omega_n, \mathcal{F}_n, \mathbb{P}_n, \{\mathcal{F}_t^n\}, N_n, u_n, U_n)$ along with Skorokhod's theorem and exploiting the convexity property of the cost functional \mathcal{J} with respect to the control variable.

The remaining of the paper is organized as follows. We state the assumptions, detail the technical framework and state the main results in Section 2. In Section 3, we construct an approximate solutions, derive its *a priori* estimates, show the tightness of the laws of the approximate solutions in some space, and then apply the Jakubowski version of the Skorokhod theorem to have a existence of a weak solution of the problem. Moreover, path-wise uniqueness of weak solutions is shown in Section 3.5.1. The final section is devoted to establish existence of an optimal solution of the initial value control problem (1.2).

2. TECHNICAL FRAMEWORK AND STATEMENT OF THE MAIN RESULTS

Throughout this paper, we use the letter C to denote various generic constants. In the sequel, we denote by $\langle \cdot, \cdot \rangle$, the pairing between $W_0^{1,p}(D)$ and $W^{-1,p'}(D)$ where p' denotes the convex conjugate of p .

In the theory of stochastic evolution equations, two types of solution concept are considered namely strong solution and weak solution. A strong solution is typically an analytically weak solution (in space) on a given stochastic basis. In general, for a nonlinear non-Lipschitz drift operator, one may not able to prove existence of a strong solution, and therefore needs to consider concept of weak solution.

Definition 2.1. (Weak solution) A weak solution of (1.1) is a 7-tuple $\bar{\pi} = (\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}, \{\bar{\mathcal{F}}_t\}, \bar{N}, \bar{u}, \bar{U})$ such that

- i) $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ is a complete probability space endowed with the filtration $\{\bar{\mathcal{F}}_t\}$ satisfying the usual hypotheses.
- ii) \bar{N} is a time-homogeneous Poisson random measure on \mathbb{R} with intensity measure $m(dz)$ with respect to the filtration $\{\bar{\mathcal{F}}_t\}$.
- iii) \bar{U} is measurable with $\bar{\mathbb{P}}$ -a.s. $\omega \in \bar{\Omega}$, $\bar{U}(\omega, \cdot) \in W^{1,p}(D)$.
- iv) $\bar{u} : \bar{\Omega} \times [0, T] \rightarrow L^2(D)$ is an $W_0^{1,p}(D)$ -valued $\{\bar{\mathcal{F}}_t\}$ -predictable stochastic process such that $\bar{\mathbb{P}}$ -a.s.,
 - a) $\bar{u} \in L^p(0, T; W_0^{1,p}(D)) \cap L^\infty(0, T; L^2(D))$ and $\bar{u}(0, \cdot) = u_0 + \bar{U}$ in $L^2(D)$.
 - b) for all $t \in [0, T]$, there holds

$$\bar{u}(t) = u_0 + \bar{U} + \int_0^t \operatorname{div}_x (|\nabla \bar{u}|^{p-2} \nabla \bar{u} + \vec{f}(\bar{u})) ds + \int_0^t \int_{|z|>0} \eta(\bar{u}; z) \tilde{N}(dz, ds) \quad \text{in } L^2(D).$$

We show the wellposedness of weak solution of (1.1), in the sense of Definition 2.1, under the following assumptions:

- A.1 $u_0 \in L^2(D)$.
- A.2 $\vec{f} : \mathbb{R} \mapsto \mathbb{R}^d$ is C^2 and Lipschitz continuous with $\vec{f}(0) = 0$.
- A.3 $\eta(0; z) = 0$ for all $z \in \mathbb{R}$. Moreover, there exists positive constant $0 < \lambda^* < 1$ such that for all $u, v \in \mathbb{R}$ and $z \in \mathbb{R}$

$$|\eta(u; z) - \eta(v; z)| \leq \lambda^* |u - v| (1 \wedge |z|)^2.$$

- A.4 The Lévy measure $m(dz)$ is a Radon measure on $\mathbb{R} \setminus \{0\}$ with a possible singularity at $z = 0$, which satisfies

$$c_\eta := \int_{|z|>0} (1 \wedge |z|^2) m(dz) < +\infty.$$

Theorem 2.2. Let $p > 2$, and the assumptions A.1-A.4 be true. Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\})$ be a given filtered probability space satisfying the usual hypotheses and N be a time-homogeneous Poisson random measure on \mathbb{R} with intensity measure $m(dz)$ defined on $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\})$, and $U \in L^p(\Omega; W^{1,p}(D))$. Then there exists a unique weak solution $\bar{\pi} = (\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}, \{\bar{\mathcal{F}}_t\}, \bar{N}, \bar{u}, \bar{U})$ of the problem (1.1) in the sense of Definition 2.1. Moreover

- i) $\mathcal{L}(U) = \mathcal{L}(\bar{U})$ on $W^{1,p}(D)$ with $\bar{U} \in L^p(\bar{\Omega}; W^{1,p}(D))$, and
- ii) there exists a positive constant $C > 0$ such that

$$\bar{\mathbb{E}} \left[\sup_{0 \leq t \leq T} \|\bar{u}(t)\|_{L^2(D)}^2 + \int_0^T \|\bar{u}(t)\|_{W_0^{1,p}(D)}^p dt \right] \leq C \bar{\mathbb{E}} \left[\|u_0\|_{L^2(D)}^2 + \|U\|_{W^{1,p}(D)}^p \right]. \quad (2.1)$$

²Here we denote $x \wedge y := \min\{x, y\}$.

Remark 2.3. We remark that the assumption **A.3** is natural in the context of Lévy noise with the exception of $\lambda^* \in (0, 1)$, which is necessary to handle the nonlocal nature of the Itô-Lévy formula for the path-wise uniqueness in Section 3.5.1; see also Remark 1 in [2]. To simplify lot of technicalities, we have considered the assumption that $\eta(0, z) = 0$ for all $z \in \mathbb{R}$. We point out that the results of this paper are still valid under the weaker condition namely

$$|\eta(u; z)| \leq C(1 + |u|)(1 \wedge |z|) \quad \forall (u, z) \in \mathbb{R} \times \mathbb{R},$$

along with the second condition as stated in the assumption **A.3**, where $C > 0$ is a constant. Finally, the assumptions **A.1-A.4** collectively ensures existence of a weak solution of the problem (1.1).

Remark 2.4. In view of Remark 1.1 from [28], \mathbb{P} -a.s., $\bar{u} \in C_w([0, T]; L^2(D))$ ³ and hence for all $t \in [0, T]$, $u(t, \cdot)$ is a stochastic process with values in $L^2(D)$.

We denote by $\mathcal{U}_{\text{ad}}^w(u_0; T)$ the set of weak admissible solutions to the problem (1.1) in the sense Theorem 2.2. Then, the initial value control problem (1.2) can be re-written as follows.

Definition 2.5. Let the assumptions of Theorem 2.2 hold, and let Ψ be a given Lipschitz continuous function on $L^2(D)$ and $u_{\text{tar}} \in L^p(0, T; W_0^{1,p}(D))$ be a given deterministic target profile. A weak optimal solution of (1.2) is a 7-tuple $\pi^* := (\Omega^*, \mathcal{F}^*, \mathbb{P}^*, \{\mathcal{F}_t^*\}, N^*, u^*, U^*) \in \mathcal{U}_{\text{ad}}^w(u_0; T)$ such that

$$\mathcal{J}(\pi^*) = \inf_{\pi \in \mathcal{U}_{\text{ad}}^w(u_0; T)} \mathcal{J}(\pi) := \Lambda. \quad (2.2)$$

The associated control U^* in π^* , as in (2.2) is called weak optimal control of the control problem (1.2).

Theorem 2.6. *There exists a weak optimal solution π^* of (1.2) in the sense of Definition 2.5.*

3. WELLPOSEDNESS OF WEAK SOLUTION

In this section, we establish wellposedness of a weak solution for (1.1). To do this, we first construct an approximate solution *via* implicit time discretization scheme, and then derive necessary uniform bounds.

3.1. Implicit Euler scheme

For $N \in \mathbb{N}^*$, let $0 = t_0 < t_1 < \dots < t_N = T$ be a uniform partition of $[0, T]$ with mesh size $\tau := \frac{T}{N} = t_{k+1} - t_k$ for all $k = 0, 1, \dots, N - 1$.

For any $u_0 \in L^2(D)$, there exists a sequence $\{u_0^\tau\}_{\tau > 0} \in W_0^{1,p}(D)$, the unique solution to the problem $u_0^\tau - \tau \Delta_p u_0^\tau = u_0$, such that $u_0^\tau \rightarrow u_0$ in $L^2(D)$ as $\tau \rightarrow 0$. Moreover, the following estimate holds:

$$\frac{1}{2} \|u_0^\tau\|_{L^2(D)}^2 + \tau \|\nabla u_0^\tau\|_{L^p(D)}^p \leq \frac{1}{2} \|u_0\|_{L^2(D)}^2. \quad (3.1)$$

For its proof, we refer to see Lemma 30 in [28]. Set $\hat{u}_0 = u_0^\tau + U$. With this \hat{u}_0 , we introduce the following time discretization:

$$\hat{u}_{k+1} - \hat{u}_k - \tau \operatorname{div}_x (|\nabla \hat{u}_{k+1}|^{p-2} \nabla \hat{u}_{k+1} + \vec{f}(\hat{u}_{k+1})) = \int_{|z|>0} \int_{t_k}^{t_{k+1}} \eta(\hat{u}_k; z) \tilde{N}(dz, dt). \quad (3.2)$$

Proposition 3.1. *Let $\tau > 0$ be small and \hat{u}_0 is defined as above. Then, for any $k = 0, 1, 2, \dots, N - 1$, there exists a unique $\mathcal{F}_{t_{k+1}}$ -measurable $W_0^{1,p}(D)$ -valued random variable \hat{u}_{k+1} such that for any $v \in W_0^{1,p}(D)$, the*

³For any Banach space \mathbb{X} , $C_w([0, T]; \mathbb{X})$ denotes the Bochner space of weakly continuous functions with values in \mathbb{X} .

following variational formula holds

$$\begin{aligned} & \int_D \left((\hat{u}_{k+1} - \hat{u}_k)v + \tau \{ |\nabla \hat{u}_{k+1}|^{p-2} \nabla \hat{u}_{k+1} + \vec{f}(\hat{u}_{k+1}) \} \cdot \nabla v \right) dx \\ &= \int_D \int_{t_k}^{t_{k+1}} \int_{|z|>0} \eta(\hat{u}_k; z) v \tilde{N}(dz, ds) dx. \end{aligned} \quad (3.3)$$

i.e., \mathbb{P} -a.s., \hat{u}_{k+1} is a unique weak solution to the problem (3.2).

Proof. Let $\tau > 0$ be a fixed small number. Define an operator $\mathcal{A} : W_0^{1,p}(D) \rightarrow W^{-1,p'}(D)$ via

$$\langle \mathcal{A}u, v \rangle := \int_D \left(uv + \tau \{ |\nabla u|^{p-2} \nabla u + \vec{f}(u) \} \cdot \nabla v \right) dx, \quad \forall u, v \in W_0^{1,p}(D).$$

Then, \mathcal{A} is a coercive pseudo-monotone operator and hence by Brezis' theorem \mathcal{A} is onto $W^{-1,p'}(D)$, see Theorem 2.6 in [26]. Arguing similarly as in the proof of Lemma 1 from [28], we infer that \mathcal{A} is injective and $\mathcal{A}^{-1} : W^{-1,p'}(D) \rightarrow W_0^{1,p}(D)$ is continuous.

Let $X_k := \hat{u}_k + \int_{|z|>0} \int_{t_k}^{t_{k+1}} \eta(\hat{u}_k; z) \tilde{N}(dz, dt)$. Then, thanks to the assumption A.3 and Itô-Lévy isometry, we have

$$\begin{aligned} \mathbb{E}[\|X_k\|_{L^2(D)}^2] &\leq 2\mathbb{E}[\|\hat{u}_k\|_{L^2(D)}^2] + 2(\lambda^*)^2 \tau \mathbb{E} \left[\int_{|z|>0} \|\hat{u}_k\|_{L^2(D)}^2 (1 \wedge |z|^2) m(dz) \right] \\ &\leq C(\tau, \lambda, c_\eta) \mathbb{E}[\|\hat{u}_k\|_{L^2(D)}^2]. \end{aligned}$$

Therefore for a.s. $\omega \in \Omega$, $X_k \in L^2(D)$, and hence $\hat{u}_{k+1} = \mathcal{A}^{-1}X_k$. Note that \hat{u}_{k+1} is $\mathcal{F}_{t_{k+1}}$ -measurable if we assume that \hat{u}_k is \mathcal{F}_{t_k} -measurable. Thus the assertion follows by induction. This completes the proof. \square

3.2. A priori estimate

We choose a test function $v = \hat{u}_{k+1}$ in (3.3), and use Young's inequality, A.3-A.4, Itô-Lévy isometry, and the identity $(a-b)a = \frac{1}{2}[a^2 + (a-b)^2 - b^2]$ for all $a, b \in \mathbb{R}$ to have, after taking the expectation and recalling $\int_D \vec{f}(v) \cdot \nabla v dx = 0$ for any $v \in W_0^{1,p}(D)$,

$$\begin{aligned} & \frac{1}{2} \left\{ \mathbb{E}[\|\hat{u}_{k+1}\|_{L^2(D)}^2] + \mathbb{E}[\|\hat{u}_{k+1} - \hat{u}_k\|_{L^2(D)}^2] - \mathbb{E}[\|\hat{u}_k\|_{L^2(D)}^2] \right\} + \tau \mathbb{E}[\|\nabla \hat{u}_{k+1}\|_{L^p(D)}^p] \\ & \leq \mathbb{E} \left[\int_D \int_{t_k}^{t_{k+1}} \int_{|z|>0} \eta(\hat{u}_k; z) \hat{u}_k \tilde{N}(dz, ds) dx \right] + \frac{\varepsilon}{2} \mathbb{E}[\|\hat{u}_{k+1} - \hat{u}_k\|_{L^2(D)}^2] \\ & \quad + \frac{1}{2\varepsilon} \mathbb{E} \left[\int_D \left(\int_{t_k}^{t_{k+1}} \int_{|z|>0} \eta(\hat{u}_k; z) \tilde{N}(dz, ds) \right)^2 dx \right], \quad \text{for some } \varepsilon > 0 \\ & \leq \frac{1}{4} \mathbb{E}[\|\hat{u}_{k+1} - \hat{u}_k\|_{L^2(D)}^2] + C \tau \mathbb{E}[\|\hat{u}_k\|_{L^2(D)}^2] \quad \left(\text{for } \varepsilon = \frac{1}{2} \right). \end{aligned}$$

An application of discrete Gronwall's lemma then implies

$$\sup_{0 \leq n \leq N} \mathbb{E}[\|\hat{u}_n\|_{L^2(D)}^2] + \sum_{k=0}^{N-1} \mathbb{E}[\|\hat{u}_{k+1} - \hat{u}_k\|_{L^2(D)}^2] + \tau \sum_{k=0}^{N-1} \mathbb{E}[\|\nabla \hat{u}_{k+1}\|_{L^p(D)}^p] \leq C. \quad (3.4)$$

Moreover, we can easily show that $\mathbb{E} \left[\sup_{0 \leq n \leq N} \|\hat{u}_n\|_{L^2(D)}^2 \right] \leq C$.

We would like to define certain processes defined on the whole time interval $[0, T]$ in terms of the discrete solutions $\{\hat{u}_k\}$, and derive *a priori* estimate. Like in [28], we introduce the right-continuous step function $u_\tau(t)$, left-continuous $\{\mathcal{F}_t\}$ -adapted step function $\bar{u}_\tau(t)$, square-integrable $\{\mathcal{F}_t\}$ -martingale $B_\tau(t)$ and the piecewise affine functions $\tilde{u}_\tau(t)$ and $\tilde{B}_\tau(t)$ as

$$\begin{aligned} u_\tau(t) &:= \sum_{k=0}^{N-1} \hat{u}_{k+1} \mathbf{1}_{[t_k, t_{k+1})}(t); & \bar{u}_\tau(t) &:= \sum_{k=0}^{N-1} \hat{u}_k \mathbf{1}_{(t_k, t_{k+1}]}(t) \quad \text{with } \bar{u}_\tau(0) = \hat{u}_0, \\ B_\tau(t) &:= \int_0^t \int_{|z|>0} \eta(u_\tau(s); z) \tilde{N}(dz, ds), \end{aligned}$$

and

$$\begin{aligned} \tilde{u}_\tau(t) &:= \sum_{k=0}^{N-1} \left(\frac{\hat{u}_{k+1} - \hat{u}_k}{\tau} (t - t_k) + \hat{u}_k \right) \mathbf{1}_{[t_k, t_{k+1})}(t) \quad \text{with } \tilde{u}_\tau(T) = \hat{u}_N, \\ \tilde{B}_\tau(t) &:= \sum_{k=0}^{N-1} \left(\frac{B_\tau(t_{k+1}) - B_\tau(t_k)}{\tau} (t - t_k) + B_\tau(t_k) \right) \mathbf{1}_{[t_k, t_{k+1})}(t). \end{aligned} \tag{3.5}$$

A straightforward calculation shows that

$$\begin{cases} \sup_{t \in [0, T]} \mathbb{E} [\|u_\tau(t)\|_{L^2(D)}^2] = \max_{0 \leq k \leq N-1} \mathbb{E} [\|\hat{u}_{k+1}\|_{L^2(D)}^2], \\ \mathbb{E} \left[\sup_{t \in [0, T]} \|u_\tau(t)\|_{L^2(D)}^2 \right] \leq \mathbb{E} \left[\max_{0 \leq k \leq N-1} \|\hat{u}_{k+1}\|_{L^2(D)}^2 \right], \\ \mathbb{E} [\|u_\tau - \tilde{u}_\tau\|_{L^2(D_T)}^2] \leq \tau \sum_{k=0}^{N-1} \mathbb{E} [\|\hat{u}_{k+1} - \hat{u}_k\|_{L^2(D)}^2]. \end{cases}$$

In view of the above definitions and *a priori* estimate (3.4), we arrive at the following lemma.

Lemma 3.2. *There exists a constant $C > 0$, independent of τ , such that*

$$\begin{aligned} \sup_{t \in [0, T]} \mathbb{E} [\|u_\tau(t)\|_{L^2(D)}^2] &= \sup_{t \in [0, T]} \mathbb{E} [\|\tilde{u}_\tau(t)\|_{L^2(D)}^2] \leq C, \\ \mathbb{E} \left[\sup_{t \in [0, T]} \|u_\tau(t)\|_{L^2(D)}^2 \right] &= \mathbb{E} \left[\sup_{t \in [0, T]} \|\tilde{u}_\tau(t)\|_{L^2(D)}^2 \right] \leq C, \\ \mathbb{E} \left[\int_0^T \int_D |\nabla u_\tau(t)|^p dx dt \right] &\leq C; \quad \mathbb{E} \left[\int_0^T \int_D |\nabla \tilde{u}_\tau(t)|^p dx dt \right] \leq C, \\ \mathbb{E} [\|u_\tau - \tilde{u}_\tau\|_{L^2(D_T)}^2] &\leq C \tau. \end{aligned} \tag{3.6}$$

Thanks to (3.1) and (3.6), one can easily show the following estimate:

$$\begin{aligned} \mathbb{E} \left[\|\tilde{u}_\tau\|_{L^p(0, T; W_0^{1,p}(D))}^p \right] &\leq C \mathbb{E} \left[\int_0^T \|\nabla u_\tau(t)\|_{L^p(D)}^p dt + \tau \|\nabla \hat{u}_0\|_{L^p(D)}^p \right] \\ &\leq C \left(\|u_0\|_{L^2(D)}^2 + \mathbb{E} [\|U\|_{W^{1,p}(D)}^p] \right), \end{aligned} \tag{3.7}$$

for some constant $C > 0$, independent of τ .

3.3. Tightness of the sequence $\mathcal{L}(\tilde{u}_\tau)$

In this subsection, we will show that the laws of the sequence \tilde{u}_τ , denoted by $\mathcal{L}(\tilde{u}_\tau)$, is tight on some appropriate functional space. Since we are dealing the PDE with Lévy noise, appropriate spaces of càdlàg functions need to be consider. We would like to use Theorem 2 in [19], where tightness of the laws of the sequence of càdlàg functions are discussed in a general functional spaces. From Lemma 3.2 and (3.7), we define the following functional space, analogous to those considered in [5, 16, 17],

$$\mathcal{Z} := \mathbb{D}([0, T]; W^{-1,p'}(D)) \cap \mathbb{D}([0, T]; L_w^2(D)) \cap L_w^2(0, T; L^2(D)) \cap L^2(0, T; L^2(D))$$

equipped with the topology \mathcal{T} , the supremum of the corresponding topologies. The functional spaces $\mathbb{D}([0, T]; W^{-1,p'}(D))$, $L_w^2(0, T; L^2(D))$, and $\mathbb{D}([0, T]; L_w^2(D))$ endowed with the respective topologies are defined as

- 1). $\mathbb{D}([0, T]; W^{-1,p'}(D)) :=$ the space of càdlàg functions $u : [0, T] \rightarrow W^{-1,p'}(D)$ with the extended Skorokhod topology.⁴
- 2). $L_w^2(0, T; L^2(D)) :=$ the space $L^2(0, T; L^2(D))$ with the weak topology.
- 3). $\mathbb{D}([0, T]; L_w^2(D)) :=$ the space of all weakly càdlàg functions $u : [0, T] \rightarrow L^2(D)$ with the weakest topology such that for all $h \in L^2(D)$, the mapping $\mathbb{D}([0, T]; L_w^2(D)) \ni u \mapsto \int_D u(\cdot)h \, dx \in \mathbb{D}([0, T]; \mathbb{R})$ are continuous.

Definition 3.3. (Aldous condition) Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of càdlàg, $\{\mathcal{F}_t\}$ -adapted stochastic processes in a Banach space \mathbb{U} . We say that $(X_n)_{n \in \mathbb{N}}$ satisfies the Aldous condition if for every $\varepsilon > 0$ and $\gamma > 0$, there is $\delta > 0$ such that for every sequence $(\kappa_n)_{n \in \mathbb{N}}$ of $\{\mathcal{F}_t\}$ -stopping times with $\kappa_n \leq T$, one has

$$\sup_{n \in \mathbb{N}} \sup_{0 < \theta \leq \delta} \mathbb{P}\{\|X_n(\kappa_n + \theta) - X_n(\kappa_n)\|_{\mathbb{U}} \geq \gamma\} \leq \varepsilon.$$

The following lemma ensures the Aldous condition in a separable Banach space \mathbb{U} for the sequence $(X_n)_{n \in \mathbb{N}}$; cf. Lemma 9 in [19].

Lemma 3.4. Let $(\mathbb{U}, \|\cdot\|_{\mathbb{U}})$ be a separable Banach space and let $(X_n)_{n \in \mathbb{N}}$ be a sequence of \mathbb{U} -valued random variables. Assume that for every sequence (κ_n) of $\{\mathcal{F}_t\}$ -stopping times with $\kappa_n \leq T$ and $\theta \geq 0$, the following condition holds

$$\mathbb{E}\left[\|X_n(\kappa_n + \theta) - X_n(\kappa_n)\|_{\mathbb{U}}^\alpha\right] \leq C\theta^\zeta, \quad (3.8)$$

for some $\alpha, \zeta > 0$ and some constant $C > 0$. Then the sequence $(X_n)_{n \in \mathbb{N}}$ satisfies the Aldous condition.

In view of Lemma 2.5 from [18], Theorem 2 from [19], see also Lemma 3.3 from [5] and Lemma 7 from [19], we arrive at the following useful theorem regarding the criterion for the tightness in \mathcal{Z} . For its proof, consult Corollary 1 from [19].

Theorem 3.5. Let $(u_\tau)_{\tau > 0}$ be a sequence of càdlàg, $W^{-1,p'}(D)$ -valued stochastic processes such that

- i) there exists a constant $C_1 > 0$ such that

$$\sup_{\tau > 0} \mathbb{E}\left[\sup_{t \in [0, T]} \|u_\tau(t)\|_{L^2(D)}\right] \leq C_1,$$

⁴ For the Skorokhod topology, we refer to see [1, 27] and references therein.

ii) there exists a constant $C_2 > 0$ such that

$$\sup_{\tau > 0} \mathbb{E} \left[\int_0^T \|u_\tau(t)\|_{W_0^{1,p}(D)}^2 dt \right] \leq C_2,$$

iii) $(u_\tau)_{\tau > 0}$ satisfies the Aldous condition in $W^{-1,p'}(D)$.

Then the sequence $(\mathcal{L}(u_\tau))_{\tau > 0}$ is tight on $(\mathcal{Z}, \mathcal{T})$.

With the help of Theorem 3.5, we prove the tightness of the laws of the sequence $\{\tilde{u}_\tau\}$ in $(\mathcal{Z}, \mathcal{T})$.

Lemma 3.6. *The sequence $(\mathcal{L}(\tilde{u}_\tau))_{\tau > 0}$ is tight on $(\mathcal{Z}, \mathcal{T})$.*

Proof. Thanks to the *a priori* estimates (3.6) and (3.7), we see that assumptions i) and ii) of Theorem 3.5 hold for the sequence $(\tilde{u}_\tau)_{\tau > 0}$. Hence it suffices to prove that the sequence $(\tilde{u}_\tau)_{\tau > 0}$ satisfies the Aldous condition in $W^{-1,p'}(D)$. Note that, we can rewrite (3.2) in terms of u_τ , \tilde{u}_τ , and \tilde{B}_τ as

$$\begin{aligned} \tilde{u}_\tau(t) &= u_{0,\tau} + U + \int_0^t \operatorname{div}_x (|\nabla u_\tau(s)|^{p-2} \nabla u_\tau(s) + \vec{f}(u_\tau(s))) ds + \tilde{B}_\tau(t) \\ &:= \hat{u}_0 + T_1^\tau(t) + T_2^\tau(t). \end{aligned} \quad (3.9)$$

First note that, since the term \hat{u}_0 is independent of time, clearly (3.8) is satisfied for any α, ζ . In view of Lemma 3.4, we need to show that $T_1^\tau(t)$ and $T_2^\tau(t)$ satisfy the inequality (3.8) for a suitable choices of α, ζ . Let (κ_m) be a sequence of stopping times with $\kappa_m \leq T$, and $\theta > 0$. Then, by using (3.6) we have

$$\begin{aligned} &\mathbb{E} \left[\|T_1^\tau(\kappa_m + \theta) - T_1^\tau(\kappa_m)\|_{W^{-1,p'}(D)} \right] \\ &= \mathbb{E} \left[\left\| \int_{\kappa_m}^{\kappa_m + \theta} \operatorname{div}_x (|\nabla u_\tau(s)|^{p-2} \nabla u_\tau(s) + \vec{f}(u_\tau(s))) ds \right\|_{W^{-1,p'}(D)} \right] \\ &\leq \mathbb{E} \left[\int_{\kappa_m}^{\kappa_m + \theta} \left\| |\nabla u_\tau(s)|^{p-2} \nabla u_\tau(s) + \vec{f}(u_\tau(s)) \right\|_{L^{p'}(D)} ds \right] \\ &\leq C \mathbb{E} \left[\int_{\kappa_m}^{\kappa_m + \theta} \left(\|\nabla u_\tau(s)\|_{L^p(D)}^{p-1} + \|u_\tau(s)\|_{L^{p'}(D)} \right) ds \right] \\ &\leq C \theta^{\frac{1}{p}} \mathbb{E} \left[\int_0^T \left(\|\nabla u_\tau(s)\|_{L^p(D)}^p + \|u_\tau(s)\|_{L^{p'}(D)}^{p'} \right) ds \right] \leq C \theta^{\frac{1}{2}}. \end{aligned}$$

Thus $T_1^\tau(t)$ satisfies (3.8) with $\alpha = 1$ and $\zeta = \frac{1}{2}$. Again, thanks to Itô-Lévy isometry, the assumptions A.3-A.4, and (3.6), and since $W_0^{1,p}(D) \hookrightarrow L^2(D) \hookrightarrow W^{-1,p'}(D)$, we see that

$$\begin{aligned} &\mathbb{E} \left[\|T_2^\tau(\kappa_m + \theta) - T_2^\tau(\kappa_m)\|_{W^{-1,p'}(D)}^2 \right] \\ &\leq \mathbb{E} \left[\left\| \int_{\kappa_m}^{\kappa_m + \theta} \int_{|z|>0} \eta(\tilde{u}_\tau(s); z) \tilde{N}(dz, ds) \right\|_{W^{-1,p'}(D)}^2 \right] \\ &\leq C \mathbb{E} \left[\left\| \int_{\kappa_m}^{\kappa_m + \theta} \int_{|z|>0} \eta(\tilde{u}_\tau(s); z) \tilde{N}(dz, ds) \right\|_{L^2(D)}^2 \right] \\ &\leq C \mathbb{E} \left[\int_{\kappa_m}^{\kappa_m + \theta} \int_{|z|>0} \|\eta(\tilde{u}_\tau(s); z)\|_{L^2(D)}^2 m(dz) ds \right] \end{aligned}$$

$$\begin{aligned}
&\leq C\mathbb{E}\left[\int_{\kappa_m}^{\kappa_m+\theta}\int_{|z|>0}\|\bar{u}_\tau(s)\|_{L^2(D)}^2(1\wedge|z|^2)m(dz)ds\right] \\
&\leq Cc_\eta\mathbb{E}\left[\int_{\kappa_m}^{\kappa_m+\theta}\|\bar{u}_\tau(s)\|_{L^2(D)}^2ds\right]\leq C\theta\mathbb{E}\left[\sup_{s\in[0,T]}\|u_\tau(s)\|_{L^2(D)}^2\right]\leq C\theta.
\end{aligned}$$

Hence $T_2^\tau(t)$ satisfies (3.8) with $\alpha = 2$ and $\zeta = 1$. This completes the proof. \square

3.4. Construction of a martingale solution

Construction of a martingale solution is based on Skorokhod Theorem [14] for a non metric space. Note that \mathcal{Z} is a locally convex topological space and there exist a sequence of continuous functions $f_m : \mathcal{Z} \rightarrow \mathbb{R}$ that separates the points of \mathcal{Z} which generates the Borel σ -algebra; cf. Remark 2 from [19]. Let $\bar{\mathbb{N}}$ denotes the set of all extended natural numbers *i.e.*, $\bar{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$. For any measurable space $(S, \mathcal{B}(S))$, we denote by $M_{\bar{\mathbb{N}}}(S)$ the set of all $\bar{\mathbb{N}}$ -valued measures on $(S, \mathcal{B}(S))$ endowed with the σ -field $\mathcal{M}_{\bar{\mathbb{N}}}(S)$ generated by the projection maps $i_B : M_{\bar{\mathbb{N}}}(S) \ni \mu \mapsto \mu(B) \in \bar{\mathbb{N}}$ for all $B \in \mathcal{B}(S)$. Define $N_\tau(dz, dt) = N(dz, dt)$ for all $\tau > 0$. For the basic properties of the stochastic integral with respect to compensated Poisson random measure \tilde{N} , we refer to see [4, 13] and [25]. Since $M_{\bar{\mathbb{N}}}(\mathbb{R} \times [0, T])$ is a separable metric space, by Theorem 3.2 from [24], the laws of the family $\{N_\tau(dz, dt)\}$ is tight on $M_{\bar{\mathbb{N}}}(\mathbb{R} \times [0, T])$.

Define $U_\tau = U$ for all $\tau > 0$ and $\mathbb{X}_U = (W^{1,p}(D), w)$.⁵

Lemma 3.7. *The set $\{\mathcal{L}(U_\tau) : \tau > 0\}$ is tight in \mathbb{X}_U .*

Proof. Note that $\sup_{\tau>0}\mathbb{E}[\|U_\tau\|_{W^{1,p}(D)}^p] = \mathbb{E}[\|U\|_{W^{1,p}(D)}^p] < +\infty$. Now for any $R > 0$, the set

$$B_R := \{U \in W^{1,p}(D) : \|U\|_{W^{1,p}(D)} \leq R\}$$

is relatively compact in \mathbb{X}_U and

$$\mathbb{P}(\|U_\tau\|_{W^{1,p}(D)} \geq R) \leq \frac{1}{R^p}\mathbb{E}[\|U_\tau\|_{W^{1,p}(D)}^p] \leq \frac{C}{R^p},$$

which yields the proof. \square

By Lemma 3.6, the set of measures $(\mathcal{L}(\tilde{u}_\tau))_{\tau>0}$ is tight on $(\mathcal{Z}, \mathcal{T})$. Hence, in view of Lemma 3.7, the set $\{\mathcal{L}(\tilde{u}_\tau, U_\tau, N_\tau) : \tau > 0\}$ is tight on $\mathcal{X} := \mathcal{Z} \times \mathbb{X}_U \times M_{\bar{\mathbb{N}}}(\mathbb{R} \times [0, T])$. Note that the space \mathcal{X} is non-metric space, and hence our compactness argument is based on the Jakubowski-Skorokhod representation theorem. Moreover, by using ([19], Cor. 2), see also ([7], Thm. D1), we arrive at the following result.

Proposition 3.8. *There exist a subsequence of $\{\tau\}$, still we denote it by same $\{\tau\}$, a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ and, on this space \mathcal{X} -valued random variables (u_*, U_*, N_*) and $(u_\tau^*, U_\tau^*, N_\tau^*)$ such that*

- $\mathcal{L}(u_\tau^*, U_\tau^*, N_\tau^*) = \mathcal{L}(\tilde{u}_\tau, U_\tau, N_\tau)$ for all $\tau > 0$,
- $(u_\tau^*, U_\tau^*, N_\tau^*) \rightarrow (u_*, U_*, N_*)$ in \mathcal{X} $\bar{\mathbb{P}}$ -a.s. as $\tau \rightarrow 0$,
- $N_\tau^*(\bar{\omega}) = N_*(\bar{\omega})$ for all $\bar{\omega} \in \bar{\Omega}$.

Moreover, by Theorem 1.10.4 and Addendum 1.10.5 from [29], there exist a sequence of perfect functions $\phi_\tau : \bar{\Omega} \rightarrow \Omega$ such that

$$u_\tau^* = \tilde{u}_\tau \circ \phi_\tau, \quad U_\tau^* = U_\tau \circ \phi_\tau, \quad \mathbb{P} = \bar{\mathbb{P}} \circ \phi_\tau^{-1}. \quad (3.10)$$

⁵We denote by (\mathbb{Y}, w) the topological space \mathbb{Y} equipped with the weak topology.

Let $\bar{\mathbb{F}} := (\bar{\mathcal{F}}_t)_{t \in [0, T]}$ be the filtration defined by

$$\bar{\mathcal{F}}_t := \sigma\{(u_\tau^*(s), N_\tau^*(s), u_*(s)) : 0 \leq s \leq t\}, \quad t \in [0, T]. \quad (3.11)$$

Note that since $N_\tau^*(\bar{\omega}) = N_*(\bar{\omega})$ for all $\bar{\omega} \in \bar{\Omega}$, the filtration obtained by replacing N_τ^* by N_* in (3.11) is equal to $\bar{\mathbb{F}}$. Moreover, N_τ^* , N_* are time homogeneous Poisson random measures on \mathbb{R} over the stochastic basis $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}, \bar{\mathbb{F}})$ with intensity measure $m(dz)$; cf. Section 9 from [7].

Let us define

$$\begin{aligned} v_k &= \hat{u}_k \circ \phi_\tau, \quad k = 0, 1, \dots, N, \\ v_\tau(t) &= \sum_{k=0}^{N-1} v_{k+1} \mathbf{1}_{[t_k, t_{k+1})}(t) \quad t \in [0, T], \\ \bar{v}_\tau(t) &= \sum_{k=0}^{N-1} v_k \mathbf{1}_{(t_k, t_{k+1}]}(t) \quad t \in (0, T] \text{ with } \bar{v}_\tau(0) = u_{0, \tau} + U_\tau^*, \\ B_\tau^*(t) &= \int_0^t \int_{|z| > 0} \eta(\bar{v}_\tau(s); z) \tilde{N}_\tau^*(dz, ds). \end{aligned} \quad (3.12)$$

Note that, thanks to (3.10), (3.12) and (3.2), we have, for any $k = 0, 1, \dots, N-1$ and $\bar{\mathbb{P}}$ a.s.,

$$v_{k+1} - v_k - \tau \operatorname{div}_x(|\nabla v_{k+1}|^{p-2} \nabla v_{k+1} + \vec{f}(v_{k+1})) = \int_{|z| > 0} \int_{t_k}^{t_{k+1}} \eta(v_k; z) \tilde{N}_\tau^*(dz, dt), \quad (3.13)$$

$$u_\tau^*(t) = \sum_{k=0}^{N-1} \left(\frac{v_{k+1} - v_k}{\tau} (t - t_k) + v_k \right) \mathbf{1}_{[t_k, t_{k+1})}(t), \quad t \in [0, T] \text{ with } u_\tau^*(T) = v_N. \quad (3.14)$$

Moreover the estimate (3.4) remains valid for $v_k : k = 0, 1, \dots, N$. Furthermore, thanks to (3.6)-(3.7), and Proposition 3.8, there hold

$$\bar{\mathbb{E}} \left[\sup_{t \in [0, T]} \|u_\tau^*(t)\|_{L^2(D)}^2 \right] = \bar{\mathbb{E}} \left[\sup_{t \in [0, T]} \|v_\tau(t)\|_{L^2(D)}^2 \right] \leq C(\|u_0\|_{L^2(D)}^2 + \bar{\mathbb{E}}[\|U_*\|_{L^2(D)}^2]), \quad (3.15)$$

$$\bar{\mathbb{E}} \left[\|u_\tau^*\|_{L^p(0, T; W_0^{1, p}(D))} \right] \leq C \bar{\mathbb{E}} \left[\|u_0\|_{L^2(D)}^2 + \|U_*\|_{W^{1, p}(D)}^p \right]. \quad (3.16)$$

Lemma 3.9. *We have the following:*

- i) $u_\tau^* \rightarrow u_*$ in $L^q(\bar{\Omega}; L^2(0, T; L^2(D)))$ for all $1 \leq q < p$.
- ii) $v_\tau \rightarrow u_*$ in $L^2(\bar{\Omega}; L^2(0, T; L^2(D)))$.
- iii) $u_\tau^* \xrightarrow{*} u_*$ in $L_w^2(\bar{\Omega}; L^\infty(0, T; L^2(D)))$.

Proof. We use the estimates (3.15)-(3.16) to prove the lemma.

Proof of i.) In view of (3.14), (3.16) and the definition of v_τ in (3.12), we see that the sequence $\{u_\tau^*\}$ is uniformly bounded in $L^p(\bar{\Omega}; L^2(0, T; L^2(D)))$ and therefore equi-integrable in $L^q(\bar{\Omega}; L^2(0, T; L^2(D)))$ for all $1 \leq q < p$. Since $\bar{\mathbb{P}}$ -a.s., $u_\tau^* \rightarrow u_*$ in \mathcal{Z} (in particular, $u_\tau^* \rightarrow u_*$ in $L^2(0, T; L^2(D))$), by Vitali convergence theorem we conclude that i) holds as well.

Proof of ii). A straightforward calculation reveals that

$$\bar{\mathbb{E}} \left[\int_0^T \|u_\tau^*(t) - v_\tau\|_{L^2(D)}^2 dt \right] \leq C\tau. \quad (3.17)$$

Thanks to i), we see that $u_\tau^* \rightarrow u_*$ in $L^2(\bar{\Omega}; L^2(0, T; L^2(D)))$ and hence ii) follows from (3.17).

Proof of iii). Note that, by (3.15), the sequence $\{u_\tau^*\}$ is uniformly bounded in $L^2(\bar{\Omega}; L^\infty(0, T; L^2(D)))$. Since $L^2(\bar{\Omega}; L^\infty(0, T; L^2(D)))$ is isomorphic to the space $(L^2(\bar{\Omega}; L^1(0, T; L^2(D))))^*$, by Banach Alaoglu theorem there exist a subsequence, still denoted by $\{u_\tau^*\}$, and $Y \in L^2(\bar{\Omega}; L^\infty(0, T; L^2(D)))$ such that for all $\psi \in L^2(\bar{\Omega}; L^1(0, T; L^2(D)))$

$$\bar{\mathbb{E}} \left[\int_0^T \int_D u_\tau^*(t, x) \psi(t, x) dx dt \right] \rightarrow \bar{\mathbb{E}} \left[\int_0^T \int_D Y(t, x) \psi(t, x) dx dt \right].$$

Observe that, thanks to i), $u_\tau^* \rightharpoonup u_*$ in $L^2(\bar{\Omega}; L^2(0, T; L^2(D)))$. Since $L^2(\bar{\Omega}; L^2(0, T; L^2(D)))$ is a dense subspace of $L^2(\bar{\Omega}; L^1(0, T; L^2(D)))$, we conclude that

$$Y = u_* \quad \text{and} \quad u_* \in L^2(\bar{\Omega}; L^\infty(0, T; L^2(D))).$$

This completes the proof. \square

Lemma 3.10. *For all $\phi \in W_0^{1,p}(D)$, the following holds*

$$\lim_{\tau \rightarrow 0} \bar{\mathbb{E}} \left[\int_0^T \left| \int_0^t \int_{|z|>0} \langle \eta(\bar{v}_\tau(s); z) - \eta(u_*(s-); z), \phi \rangle \tilde{N}_*(dz, ds) \right|^2 dt \right] = 0. \quad (3.18)$$

Proof. By using A.3-A.4, we observe that for any $\phi \in L^2(D)$,

$$\begin{aligned} & \bar{\mathbb{E}} \left[\int_0^t \int_{|z|>0} \left| \left(\eta(\bar{v}_\tau(s); z) - \eta(u_*(s-); z), \phi \right)_{L^2(D)} \right|^2 m(dz) ds \right] \\ & \leq \|\phi\|_{L^2(D)}^2 \bar{\mathbb{E}} \left[\int_0^t \int_{|z|>0} \|\eta(\bar{v}_\tau(s); z) - \eta(u_*(s-); z)\|_{L^2(D)}^2 m(dz) ds \right] \\ & \leq C \|\phi\|_{L^2(D)}^2 \bar{\mathbb{E}} \left[\int_0^T \|\bar{v}_\tau(s) - u_*(s-)\|_{L^2(D)}^2 ds \right]. \end{aligned}$$

Note that by i) of Lemma 3.9, $u_\tau^* \rightarrow u_*$ in $L^2(\bar{\Omega}, L^2(0, T; L^2(D)))$, and the same holds for \bar{v}_τ . Hence

$$\lim_{\tau \rightarrow 0} \bar{\mathbb{E}} \left[\int_0^t \int_{|z|>0} \left| \left(\eta(\bar{v}_\tau(s); z) - \eta(u_*(s-); z), \phi \right)_{L^2(D)} \right|^2 m(dz) ds \right] = 0. \quad (3.19)$$

Moreover, by the assumptions A.3-A.4 and (3.15) along with the fact that $u_* \in L^2(\bar{\Omega}; L^\infty(0, T; L^2(D)))$, we have

$$\bar{\mathbb{E}} \left[\int_0^t \int_{|z|>0} \left| \left(\eta(\bar{v}_\tau(s); z) - \eta(u_*(s-); z), \phi \right)_{L^2(D)} \right|^2 m(dz) ds \right] \leq C \quad (3.20)$$

for some constant $C > 0$. Furthermore, by using the properties of the stochastic integral with respect to the compensated Poisson random measure and the fact that $N_\tau^* = N_*$, we have

$$\begin{aligned} & \bar{\mathbb{E}} \left[\left| \int_0^t \int_{|z|>0} \left(\eta(\bar{v}_\tau(s); z) - \eta(u_*(s-); z), \phi \right)_{L^2(D)} \tilde{N}_*(dz, ds) \right|^2 \right] \\ &= \bar{\mathbb{E}} \left[\int_0^t \int_{|z|>0} \left| \left(\eta(\bar{v}_\tau(s); z) - \eta(u_*(s-); z), \phi \right)_{L^2(D)} \right|^2 m(dz) ds \right]. \end{aligned}$$

Therefore by (3.19) and (3.20), we have for all $\phi \in L^2(D)$

$$\begin{aligned} & \lim_{\tau \rightarrow 0} \bar{\mathbb{E}} \left[\left| \int_0^t \int_{|z|>0} \left(\eta(\bar{v}_\tau(s); z) - \eta(u_*(s-); z), \phi \right)_{L^2(D)} \tilde{N}_*(dz, ds) \right|^2 \right] = 0 \\ \text{and, } & \bar{\mathbb{E}} \left[\left| \int_0^t \int_{|z|>0} \left(\eta(\bar{v}_\tau(s); z) - \eta(u_*(s-); z), \phi \right)_{L^2(D)} \tilde{N}_*(dz, ds) \right|^2 \right] \leq C. \end{aligned}$$

Thus, one can use dominated convergence theorem to conclude

$$\lim_{\tau \rightarrow 0} \bar{\mathbb{E}} \left[\int_0^T \left| \int_0^t \int_{|z|>0} \left(\eta(\bar{v}_\tau(s); z) - \eta(u_*(s-); z), \phi \right)_{L^2(D)} \tilde{N}_*(dz, ds) \right|^2 dt \right] = 0$$

for any $\phi \in L^2(D)$. Since $W_0^{1,p}(D) \subset L^2(D)$, (3.18) holds true for all $\phi \in W_0^{1,p}(D)$. This finishes the proof. \square

Define the piecewise affine function

$$b_\tau^*(t) := \sum_{k=0}^{N-1} \left(\frac{B_\tau^*(t_{k+1}) - B_\tau^*(t_k)}{\tau} (t - t_k) + B_\tau^*(t_k) \right) \mathbf{1}_{[t_k, t_{k+1})}(t), \quad t \in [0, T].$$

Lemma 3.11. *We have*

$$\bar{\mathbb{E}} \left[\int_0^T \|B_\tau^*(t) - b_\tau^*(t)\|_{L^2(D)}^2 dt \right] \leq C\tau.$$

Proof. Note that for any $t \in [t_k, t_{k+1})$, we have $\bar{\mathbb{P}}$ -a.s.,

$$\begin{aligned} & \|B_\tau^*(t) - b_\tau^*(t)\|_{L^2(D)}^2 \\ &= \left\| \int_{t_k}^t \int_{|z|>0} \eta(\bar{v}_\tau(s); z) \tilde{N}_*(dz, ds) - \frac{t - t_k}{\tau} \int_{t_k}^{t_{k+1}} \int_{|z|>0} \eta(\bar{v}_\tau(s); z) \tilde{N}_*(dz, ds) \right\|_{L^2(D)}^2 \\ &\leq 2 \left\| \int_{t_k}^{t_{k+1}} \int_{|z|>0} \eta(\bar{v}_\tau(s); z) \tilde{N}_*(dz, ds) \right\|_{L^2(D)}^2. \end{aligned} \tag{3.21}$$

Thanks to Itô-Lévy isometry, the assumption [A.3](#), and the estimate [\(3.15\)](#) along with [\(3.21\)](#), we obtain

$$\begin{aligned}
& \mathbb{E} \left[\int_0^T \|B_\tau^*(t) - b_\tau^*(t)\|_{L^2(D)}^2 dt \right] \\
&= \sum_{k=0}^{N-1} \mathbb{E} \left[\int_{t_k}^{t_{k+1}} \|B_\tau^*(t) - b_\tau^*(t)\|_{L^2(D)}^2 dt \right] \\
&\leq 2 \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \mathbb{E} \left[\left\| \int_{t_k}^{t_{k+1}} \int_{|z|>0} \eta(\bar{v}_\tau(s); z) \tilde{N}_*(dz, ds) \right\|_{L^2(D)}^2 \right] dt \\
&\leq 2 \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \mathbb{E} \left[\int_{t_k}^{t_{k+1}} \int_{|z|>0} \|\bar{v}_\tau(s)\|_{L^2(D)}^2 (1 \wedge |z|^2) m(dz) ds \right] dt \\
&\leq C\tau \mathbb{E} \left[\int_0^T \|\bar{v}_\tau(s)\|_{L^2(D)}^2 ds \right] \leq C\tau.
\end{aligned}$$

This completes the proof. \square

Lemma 3.12. *The following holds: for all $\phi \in W_0^{1,p}(D)$*

$$\lim_{\tau \rightarrow 0} \mathbb{E} \left[\left| (u_\tau^*(0) - u_*(0), \phi)_{L^2(D)} \right| \right] = 0, \quad (3.22)$$

$$\lim_{\tau \rightarrow 0} \mathbb{E} \left[\left| \int_0^T \left\langle b_\tau^*(t), \phi \right\rangle - \left\langle \int_0^t \int_{|z|>0} \eta(u_*(s-, \cdot); z) \tilde{N}_*(dz, ds), \phi \right\rangle dt \right| \right] = 0, \quad (3.23)$$

$$\lim_{\tau \rightarrow 0} \mathbb{E} \left[\left| \int_0^T \left\langle \operatorname{div}_x(\vec{f}(\bar{v}_\tau(s)) - \vec{f}(u_*(s))), \phi \right\rangle ds \right| dt \right] = 0, \quad (3.24)$$

and there exists $G \in L^{p'}(\bar{\Omega} \times D_T)^d$ such that

$$\lim_{\tau \rightarrow 0} \mathbb{E} \left[\left| \int_0^T \left\langle \operatorname{div}_x(|\nabla v_\tau(s)|^{p-2} \nabla v_\tau(s) - G(s)), \phi \right\rangle ds \right| dt \right] = 0. \quad (3.25)$$

Proof. We prove [\(3.22\)](#)-[\(3.25\)](#) step by step.

Proof of [\(3.22\)](#): Note that \mathbb{P} -a.s., $u_\tau^* \rightarrow u_*$ in $\mathbb{D}([0, T]; L_w^2(D))$ and u_* is right continuous at $t = 0$. Thus, for any $\phi \in W_0^{1,p}(D)$ there holds \mathbb{P} -a.s., $(u_\tau^*(0), \phi)_{L^2(D)} \rightarrow (u_*(0), \phi)_{L^2(D)}$. Therefore, one can use [\(3.15\)](#) and Vitali theorem to conclude [\(3.22\)](#).

Proof of [\(3.23\)](#): Notice that, for any $\phi \in W_0^{1,p}(D)$

$$\begin{aligned}
& \left\langle b_\tau^*(t), \phi \right\rangle - \left\langle \int_0^t \int_{|z|>0} \eta(u_*(s-, \cdot); z) \tilde{N}_*(dz, ds), \phi \right\rangle \\
&= \left\langle b_\tau^*(t) - B_\tau^*(t) + B_\tau^*(t) - \int_0^t \int_{|z|>0} \eta(u_*(s-, \cdot); z) \tilde{N}_*(dz, ds), \phi \right\rangle \\
&\leq \|\phi\|_{W_0^{1,p}(D)} \|b_\tau^*(t) - B_\tau^*(t)\|_{W^{-1,p}(D)} + \left| \left\langle B_\tau^*(t) - \int_0^t \int_{|z|>0} \eta(u_*(s-, \cdot); z) \tilde{N}_*(dz, ds), \phi \right\rangle \right| \\
&\leq C \|\phi\|_{W_0^{1,p}(D)} \|b_\tau^*(t) - B_\tau^*(t)\|_{L^2(D)} \\
&\quad + \left| \int_0^t \int_{|z|>0} \langle \eta(\bar{v}_\tau(s, \cdot); z) - \eta(u_*(s-, \cdot); z), \phi \rangle \tilde{N}_*(dz, ds) \right|.
\end{aligned}$$

One can use Lemmas 3.10 and 3.11 to arrive at (3.23).

Proof of (3.24): Since \vec{f} is Lipschitz continuous, we have

$$\mathbb{E} \left[\int_0^T \left| \int_0^t \langle \operatorname{div}_x(\vec{f}(\bar{v}_\tau(s)) - \vec{f}(u_*(s))), \phi \rangle ds \right| dt \right] \leq C \|\phi\|_{W_0^{1,p}(D)} \|\bar{v}_\tau - u_*\|_{L^2(\bar{\Omega} \times D_T)},$$

and hence (3.24) holds by recalling that $\bar{v}_\tau \rightarrow u_*$ in $L^2(\bar{\Omega} \times D_T)$.

Proof of (3.25): Thanks to ii) of Lemma 3.9 and the estimate (3.16), there exists a not relabeled subsequence of $\{v_\tau\}$ such that $\nabla v_\tau \rightharpoonup \nabla u_*$ in $L^p(\bar{\Omega} \times D_T)^d$ for $\tau \rightarrow 0$. Moreover, since $\left| |\nabla v_\tau|^{p-2} \nabla v_\tau \right|^{p'} = |\nabla v_\tau|^p$, there exists $G \in L^{p'}(\bar{\Omega} \times D_T)^d$ such that $|\nabla v_\tau|^{p-2} \nabla v_\tau \rightharpoonup G$ in $L^{p'}(\bar{\Omega} \times D_T)^d$ for the same subsequence and $\tau \rightarrow 0$. Thus, it is easy to conclude that for any $\phi \in W_0^{1,p}(D)$

$$\lim_{\tau \rightarrow 0} \mathbb{E} \left[\int_0^T \left| \int_0^t \langle \operatorname{div}_x(|\nabla v_\tau|^{p-2} \nabla v_\tau(s) - G(s)), \phi \rangle ds \right| dt \right] = 0,$$

i.e., (3.25) holds true. This completes the proof. \square

3.5. Proof of Theorem 2.2

In this subsection, we use Lemmas 3.9 and 3.12 to prove existence of a weak solution of (1.1) in the sense of Definition 2.1 in three steps. Moreover, we show path-wise uniqueness of weak solutions of the problem (1.1) with respect to the same stochastic basis and a given control.

Step i): We define the functionals for all $\phi \in W_0^{1,p}(D)$,

$$\begin{aligned} \mathcal{K}_\tau(\tilde{u}_\tau, U, \tilde{N}; \phi) &= (\hat{u}_0, \phi)_{L^2(D)} + \int_0^t \langle \operatorname{div}_x(|\nabla u_\tau|^{p-2} \nabla u_\tau(s)), \phi \rangle ds \\ &\quad + \int_0^t \langle \operatorname{div}_x \vec{f}(u_\tau(s)), \phi \rangle ds + \langle \tilde{B}_\tau(t), \phi \rangle, \\ \mathcal{K}_\tau^*(u_\tau^*, U_\tau^*, \tilde{N}_\tau^*; \phi) &= (u_\tau^*(0), \phi)_{L^2(D)} + \int_0^t \langle \operatorname{div}_x(|\nabla v_\tau|^{p-2} \nabla v_\tau(s)), \phi \rangle ds \\ &\quad + \int_0^t \langle \operatorname{div}_x \vec{f}(v_\tau(s)), \phi \rangle ds + \langle b_\tau^*(t), \phi \rangle, \\ \mathcal{K}_*(u_*, U_*, \tilde{N}_*; \phi) &= (u_*(0), \phi)_{L^2(D)} + \int_0^t \langle \operatorname{div}_x(G(s) + \vec{f}(u_*(s))), \phi \rangle ds \\ &\quad + \left\langle \int_0^t \int_{|z|>0} \eta(u_*(s-, \cdot); z) \tilde{N}_*(dz, ds), \phi \right\rangle. \end{aligned}$$

In view of Lemma 3.12, we conclude that

$$\lim_{\tau \rightarrow 0} \left\| \mathcal{K}_\tau^*(u_\tau^*, U_\tau^*, \tilde{N}_\tau^*; \phi) - \mathcal{K}_*(u_*, U_*, \tilde{N}_*; \phi) \right\|_{L^1(\bar{\Omega} \times [0, T])} = 0. \quad (3.26)$$

Thanks to the definition of $\mathcal{K}_\tau(\tilde{u}_\tau, U, \tilde{N}; \phi)$ and the equality (3.9), we have: $\bar{\mathbb{P}}$ -a.s., $(\tilde{u}_\tau(t), \phi)_{L^2(D)} = \mathcal{K}_\tau(\tilde{u}_\tau, U, \tilde{N}; \phi)$ for all $t \in [0, T]$. More precisely,

$$\mathbb{E} \left[\int_0^T \left| (\tilde{u}_\tau(t), \phi)_{L^2(D)} - \mathcal{K}_\tau(\tilde{u}_\tau, U, \tilde{N}; \phi) \right| dt \right] = 0.$$

Since $\mathcal{L}(u_\tau^*, U_\tau^*, N_\tau^*) = \mathcal{L}(\tilde{u}_\tau, U_\tau, N_\tau)$ with $N_\tau(dz, dt) = N(dz, dt)$ and $U_\tau = U$ for all $\tau > 0$, we directly have

$$\int_0^T \bar{\mathbb{E}} \left[\left\| (u_\tau^*(t), \phi)_{L^2(D)} - \mathcal{K}_\tau^*(u_\tau^*, U_\tau^*, \tilde{N}_\tau^*; \phi) \right\| \right] dt = 0. \quad (3.27)$$

Again, thanks to i) of Lemma 3.9, we see that

$$\lim_{\tau \rightarrow 0} \left\| (u_\tau^*(\cdot), \phi)_{L^2(D)} - (u_*(\cdot), \phi)_{L^2(D)} \right\|_{L^1(\bar{\Omega} \times [0, T])} = 0. \quad (3.28)$$

We combine (3.26)-(3.28) to conclude that $\bar{\mathbb{P}}$ -a.s., for a.e. $t \in [0, T]$, and $\phi \in W_0^{1,p}(D)$

$$\begin{aligned} (u_*(t), \phi)_{L^2(D)} &= (u_*(0), \phi)_{L^2(D)} + \int_0^t \langle \operatorname{div}_x(G(s) + \vec{f}(u_*(s))), \phi \rangle ds \\ &\quad + \left\langle \int_0^t \int_{|z|>0} \eta(u_*(s-, \cdot); z) \tilde{N}_*(dz, ds), \phi \right\rangle. \end{aligned}$$

Note that, $u_\tau^*(0) = u_{0,\tau} + U_\tau^*$ and $\bar{\mathbb{P}}$ -a.s., $U_\tau^* \rightarrow U_*$ in \mathbb{X}_U . Since $u_{0,\tau} \rightarrow u_0$ in $L^2(D)$, by using i) of Lemma 3.9, we infer that $u_*(0) = u_0 + U_*$. Hence, we obtain

$$\begin{aligned} (u_*(t), \phi)_{L^2(D)} &= (u_0 + U_*, \phi)_{L^2(D)} + \int_0^t \langle \operatorname{div}_x(G(s) + \vec{f}(u_*(s))), \phi \rangle ds \\ &\quad + \left\langle \int_0^t \int_{|z|>0} \eta(u_*(s-, \cdot); z) \tilde{N}_*(dz, ds), \phi \right\rangle. \end{aligned} \quad (3.29)$$

Since $u_* \in \mathbb{D}([0, T]; L_w^2(D))$, (3.29) holds true for all $t \in [0, T]$ and all $\phi \in W_0^{1,p}(D)$.

Step ii): We wish to identify the function $G \in L^{p'}(\bar{\Omega} \times D_T)^d$. We take the L^2 -scalar product with v_{k+1} in (3.13) and use the identity $(a - b)a = \frac{1}{2}(|a|^2 - |b|^2 + |a - b|^2) \quad \forall a, b \in \mathbb{R}$ to have

$$\begin{aligned} &\frac{1}{2} \bar{\mathbb{E}} \left[\|v_{k+1}\|_{L^2(D)}^2 - \|v_k\|_{L^2(D)}^2 + \|v_{k+1} - v_k\|_{L^2(D)}^2 \right] + \tau \bar{\mathbb{E}} \left[\int_D |\nabla v_{k+1}|^{p-2} \nabla v_{k+1} \cdot \nabla v_{k+1} dx \right] \\ &- \bar{\mathbb{E}} \left[\left(\int_{t_k}^{t_{k+1}} \int_{|z|>0} \eta(v_k; z) \tilde{N}_\tau^*(dz, dt), v_{k+1} - v_k \right)_{L^2(D)} \right] = 0. \end{aligned} \quad (3.30)$$

Since

$$\begin{aligned} &- \bar{\mathbb{E}} \left[\left(\int_{t_k}^{t_{k+1}} \int_{|z|>0} \eta(v_k; z) \tilde{N}_\tau^*(dz, dt), v_{k+1} - v_k \right)_{L^2(D)} \right] \\ &= -\frac{1}{2} \bar{\mathbb{E}} \left[\|v_{k+1} - v_k\|_{L^2(D)}^2 \right] - \frac{1}{2} \bar{\mathbb{E}} \left[\left\| \int_{t_k}^{t_{k+1}} \int_{|z|>0} \eta(v_k; z) \tilde{N}_\tau^*(dz, dt) \right\|_{L^2(D)}^2 \right] \\ &\quad + \frac{1}{2} \bar{\mathbb{E}} \left[\left\| \int_{t_k}^{t_{k+1}} \int_{|z|>0} \eta(v_k; z) \tilde{N}_\tau^*(dz, dt) - (v_{k+1} - v_k) \right\|_{L^2(D)}^2 \right], \end{aligned}$$

by summing over $k = 0, 1, \dots, N-1$ in (3.30) and using the fact that $v_N = u_\tau^*(T)$, we get

$$\begin{aligned} & \frac{1}{2} \bar{\mathbb{E}} \left[\|u_\tau^*(T)\|_{L^2(D)}^2 \right] + \bar{\mathbb{E}} \left[\int_{D_T} |\nabla v_\tau(t)|^{p-2} \nabla v_\tau(t) \cdot \nabla v_\tau(t) \, dx \, dt \right] \\ & - \frac{1}{2} \sum_{k=0}^{N-1} \bar{\mathbb{E}} \left[\left\| \int_{t_k}^{t_{k+1}} \int_{|z|>0} \eta(v_k; z) \tilde{N}_\tau^*(dz, dt) \right\|_{L^2(D)}^2 \right] \leq \frac{1}{2} \bar{\mathbb{E}} \left[\|u_{0,\tau} + U_\tau^*\|_{L^2(D)}^2 \right]. \end{aligned} \quad (3.31)$$

Thanks to Itô-Lévy isometry, we see that

$$\begin{aligned} & \sum_{k=0}^{N-1} \bar{\mathbb{E}} \left[\left\| \int_{t_k}^{t_{k+1}} \int_{|z|>0} \eta(v_k; z) \tilde{N}_\tau^*(dz, dt) \right\|_{L^2(D)}^2 \right] \\ & = \bar{\mathbb{E}} \left[\int_0^T \int_{|z|>0} \|\eta(\bar{v}_\tau(t); z)\|_{L^2(D)}^2 m(dz) \, dt \right]. \end{aligned} \quad (3.32)$$

Again, an application of Itô-Lévy formula ([11], similar to Thm. 3.4) to the functional $\|u_*(t)\|_2^2$ in (3.29) yields

$$\begin{aligned} & \frac{1}{2} \bar{\mathbb{E}} \left[\|u_*(T)\|_{L^2(D)}^2 \right] + \bar{\mathbb{E}} \left[\int_{D_T} G \cdot \nabla u_*(s-) \, dx \, ds \right] - \frac{1}{2} \bar{\mathbb{E}} \left[\int_0^T \int_{|z|>0} \|\eta(u_*(s-); z)\|_{L^2(D)}^2 m(dz) \, ds \right] \\ & = \frac{1}{2} \bar{\mathbb{E}} \left[\|u_0 + U_*\|_{L^2(D)}^2 \right]. \end{aligned} \quad (3.33)$$

Combining (3.31), (3.32), and (3.33) we obtain

$$\begin{aligned} & \frac{1}{2} \bar{\mathbb{E}} \left[\|u_\tau^*(T)\|_{L^2(D)}^2 - \|u_*(T)\|_{L^2(D)}^2 \right] + \bar{\mathbb{E}} \left[\int_{D_T} |\nabla v_\tau(t)|^{p-2} \nabla v_\tau(t) \cdot \nabla v_\tau(t) \, dx \, dt \right] \\ & - \frac{1}{2} \bar{\mathbb{E}} \left[\int_0^T \int_{|z|>0} \left(\|\eta(\bar{v}_\tau(s); z)\|_{L^2(D)}^2 - \|\eta(u_*(s-); z)\|_{L^2(D)}^2 \right) m(dz) \, ds \right] \\ & \leq \bar{\mathbb{E}} \left[\int_{D_T} G \cdot \nabla u_* \, dx \, dt \right] + \frac{1}{2} \left\{ \bar{\mathbb{E}} \left[\|u_{0,\tau} + U_\tau^*\|_{L^2(D)}^2 \right] - \bar{\mathbb{E}} \left[\|u_0 + U_*\|_{L^2(D)}^2 \right] \right\}. \end{aligned}$$

Note that

$$\begin{cases} \liminf_{\tau>0} \bar{\mathbb{E}} \left[\|u_\tau^*(T)\|_{L^2(D)}^2 - \|u_*(T)\|_{L^2(D)}^2 \right] \geq 0, \\ \bar{\mathbb{E}} \left[\|u_{0,\tau} + U_\tau^*\|_{L^2(D)}^2 \right] \rightarrow \bar{\mathbb{E}} \left[\|u_0 + U_*\|_{L^2(D)}^2 \right], \end{cases}$$

and thanks to ii) of Lemma 3.9 along with the assumptions A.3 and A.4, it follows that

$$\bar{\mathbb{E}} \left[\int_0^T \int_{|z|>0} \|\eta(\bar{v}_\tau(t); z)\|_{L^2(D)}^2 m(dz) \, dt \right] \rightarrow \bar{\mathbb{E}} \left[\int_0^T \int_{|z|>0} \|\eta(u_*(t-); z)\|_{L^2(D)}^2 m(dz) \, dt \right].$$

Thus, one arrives at the following inequality

$$\limsup_{\tau>0} \bar{\mathbb{E}} \left[\int_{D_T} |\nabla v_\tau(t)|^{p-2} \nabla v_\tau(t) \cdot \nabla v_\tau(t) \, dx \, dt \right] \leq \bar{\mathbb{E}} \left[\int_{D_T} G \cdot \nabla u_* \, dx \, dt \right]. \quad (3.34)$$

We recall that $\nabla v_\tau \rightharpoonup \nabla u_*$ in $L^p(\bar{\Omega} \times D_T)^d$ and $|\nabla v_\tau|^{p-2} \nabla v_\tau \rightharpoonup G$ in $L^{p'}(\bar{\Omega} \times D_T)^d$. Note that for any $p > 2$,

$$2^{2-p} |\xi - \zeta|^p \leq (|\xi|^{p-2} \xi - |\zeta|^{p-2} \zeta) \cdot (\xi - \zeta) \quad \text{for any } \xi, \zeta \in \mathbb{R}^d. \quad (3.35)$$

Since $p > 2$, by using the inequality (3.35), there exists a constant $C > 0$, independent of τ , such that

$$\begin{aligned} & C \limsup_{\tau \rightarrow 0} \bar{\mathbb{E}} \left[\int_{D_T} |\nabla v_\tau - \nabla u_*|^p dx dt \right] \\ & \leq \limsup_{\tau \rightarrow 0} \bar{\mathbb{E}} \left[\int_{D_T} \left(|\nabla v_\tau|^{p-2} \nabla v_\tau - |\nabla u_*|^{p-2} \nabla u_* \right) \cdot \nabla (v_\tau - u_*) dx dt \right] \\ & \leq \limsup_{\tau > 0} \bar{\mathbb{E}} \left[\int_{D_T} |\nabla v_\tau(t)|^{p-2} \nabla v_\tau(t) \cdot \nabla v_\tau(t) dx dt \right] - \bar{\mathbb{E}} \left[\int_{D_T} G \cdot \nabla u_* dx dt \right] \leq 0, \end{aligned}$$

where the last inequality follows from (3.34). Therefore, since $\nabla v_\tau \rightharpoonup \nabla u_*$ in $L^p(\bar{\Omega} \times D_T)^d$ we conclude that $\nabla v_\tau \rightarrow \nabla u_*$ in $L^p(\bar{\Omega} \times D_T)^d$. Note that, $\left| |\nabla v_\tau|^{p-2} \nabla v_\tau \right|^{p'} = |\nabla v_\tau|^p$, and therefore, $\| |\nabla v_\tau|^{p-2} \nabla v_\tau \|_{L^{p'}(\bar{\Omega} \times D_T)^d} \rightarrow \| |\nabla u_*|^{p-2} \nabla u_* \|_{L^{p'}(\bar{\Omega} \times D_T)^d}$. Hence, by uniform convexity of the space $L^{p'}(\bar{\Omega} \times D_T)^d$, where p' is the conjugate index of $p > 2$, we conclude that $|\nabla v_\tau|^{p-2} \nabla v_\tau \rightarrow |\nabla u_*|^{p-2} \nabla u_*$ in $L^{p'}(\bar{\Omega} \times D_T)^d$. In other words, $G = |\nabla u_*|^{p-2} \nabla u_*$.

Step iii): With the identification of G , it follows from (3.29) that the system $\bar{\pi} := (\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}, \bar{\mathbb{F}}, N_*, u_*, U_*)$ is a weak solution of the problem (1.1). Moreover, since

$$\mathcal{L}(U_\tau^*) = \mathcal{L}(U_\tau) \text{ on } W^{1,p}(D) \text{ with } U_\tau = U, \text{ and } \mathbb{P}\text{-a.s., } U_\tau^* \rightarrow U_* \text{ in } \mathbb{X}_U,$$

we see that i) in Theorem 2.2 holds. Furthermore, one can use Proposition 3.8 and the estimates (3.15)-(3.16) to arrive at ii), Theorem 2.2. This completes the existence proof.

3.5.1. On path-wise uniqueness of weak solutions

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}, N, u_1, U)$ and $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}, N, u_2, U)$ be two weak solutions of (1.1) with a given control U . Let us introduce the convex approximation of the absolute value function. Let $\beta : \mathbb{R} \rightarrow \mathbb{R}$ be a C^∞ function satisfying

$$\beta(0) = 0, \quad \beta(-r) = \beta(r), \quad \beta'(-r) = -\beta'(r), \quad \beta'' \geq 0,$$

and

$$\beta'(r) = \begin{cases} -1 & \text{when } r \leq -1, \\ \in [-1, 1] & \text{when } |r| < 1, \\ +1 & \text{when } r \geq 1. \end{cases}$$

For any $\vartheta > 0$, define $\beta_\vartheta : \mathbb{R} \rightarrow \mathbb{R}$ by $\beta_\vartheta(r) = \vartheta \beta(\frac{r}{\vartheta})$. Then

$$|r| - M_1 \vartheta \leq \beta_\vartheta(r) \leq |r| \quad \text{and} \quad |\beta_\vartheta''(r)| \leq \frac{M_2}{\vartheta} \mathbf{1}_{|r| \leq \vartheta}, \quad (3.36)$$

where $M_1 = \sup_{|r| \leq 1} |r| - \beta(r)$ and $M_2 = \sup_{|r| \leq 1} |\beta''(r)|$.

We apply Itô-Lévy formula to the functional $\int_D \beta_\vartheta(u_1(t) - u_2(t)) dx$ along with the integration by parts formula to have

$$\begin{aligned}
& \int_D \beta_\vartheta(u_1(t) - u_2(t)) dx \\
&= - \int_0^t \int_D (|\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_2|^{p-2} \nabla u_2) \cdot \nabla(u_1 - u_2)(s) \beta_\vartheta''(u_1 - u_2) dx ds \\
&\quad - \int_0^t \int_D (\vec{f}(u_1(s, x)) - \vec{f}(u_2(s, x))) \cdot \nabla(u_1(s, x) - u_2(s, x)) \beta_\vartheta''(u_1 - u_2) dx ds \\
&\quad + \int_0^t \int_{|z|>0} \int_D \int_0^1 \beta_\vartheta' \left((u_1 - u_2)(s-, x) + \lambda(\eta(u_1(s-, x); z) - \eta(u_2(s-, x); z)) \right) \\
&\quad \times (\eta(u_1(s-, x); z) - \eta(u_2(s-, x); z)) d\lambda dx \tilde{N}(dz, ds) \\
&\quad + \int_0^t \int_{|z|>0} \int_D \int_0^1 (1 - \lambda) \beta_\vartheta'' \left((u_1 - u_2)(s-, x) + \lambda(\eta(u_1(s-, x); z) - \eta(u_2(s-, x); z)) \right) \\
&\quad \times (\eta(u_1(s-, x); z) - \eta(u_2(s-, x); z))^2 d\lambda dx m(dz) ds.
\end{aligned}$$

Again we use the inequality (3.35), and the fact that $\beta_\vartheta'' \geq 0$ to have

$$\begin{aligned}
& - (|\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_2|^{p-2} \nabla u_2) \cdot \nabla(u_1 - u_2) \beta_\vartheta''(u_1 - u_2) \\
& \leq -C |\nabla(u_1 - u_2)|^p \beta_\vartheta''(u_1 - u_2) \leq 0,
\end{aligned}$$

and therefore, we obtain

$$\begin{aligned}
& \mathbb{E} \left[\int_D \beta_\vartheta(u_1(t) - u_2(t)) dx \right] \\
& \leq \mathbb{E} \left[- \int_0^t \int_D (\vec{f}(u_1(s, x)) - \vec{f}(u_2(s, x))) \cdot \nabla(u_1(s, x) - u_2(s, x)) \beta_\vartheta''(u_1 - u_2) dx ds \right] \\
& \quad + \mathbb{E} \left[\int_0^t \int_{|z|>0} \int_D \int_0^1 (1 - \lambda) \beta_\vartheta'' \left(u_1(s, x) - u_2(s, x) + \lambda(\eta(u_1(s, x); z) - \eta(u_2(s, x); z)) \right) \right. \\
& \quad \quad \quad \left. \times (\eta(u_1(s, x); z) - \eta(u_2(s, x); z))^2 d\lambda dx m(dz) ds \right] \\
& \equiv \mathcal{A} + \mathcal{B}.
\end{aligned} \tag{3.37}$$

Since $\beta_\vartheta''(r) \leq \frac{M_2}{\vartheta} \mathbf{1}_{\{|r| \leq \vartheta\}}$ and \vec{f} is a Lipschitz continuous function, we have \mathbb{P} -a.s.,

$$\begin{aligned}
& (\vec{f}(u_1) - \vec{f}(u_2)) \cdot \nabla(u_1(s, x) - u_2(s, x)) \beta_\vartheta''(u_1 - u_2) \\
& \leq c_f |u_1 - u_2| |\nabla(u_1 - u_2)| \frac{M_2}{\vartheta} \mathbf{1}_{\{|u_1 - u_2| \leq \vartheta\}} \rightarrow 0 \quad (\vartheta \rightarrow 0)
\end{aligned}$$

for almost every $(t, x) \in D_T$. Moreover

$$|\vec{f}(u_1) - \vec{f}(u_2)| |\nabla(u_1(s, x) - u_2(s, x))| \beta_\vartheta''(u_1 - u_2) \leq M_2 |\nabla(u_1 - u_2)| \in L^1(\Omega \times D_T).$$

Thus, by dominated convergence theorem we conclude that $\mathcal{A} \rightarrow 0$ as $\vartheta \rightarrow 0$.

Next we move on to estimate \mathcal{B} . Let

$$a = u_1(s, x) - u_2(s, x) \quad \text{and} \quad b = \eta(u_1(s, x); z) - \eta(u_2(s, x); z).$$

Then, we have, in view of the assumption [A.3](#),

$$\begin{aligned} \mathcal{B} &= \mathbb{E} \left[\int_0^t \int_{|z|>0} \int_D \int_0^1 (1-\lambda) b^2 \beta_\vartheta''(a + \lambda b) \, d\lambda \, dx \, m(dz) \, ds \right] \\ &\leq \mathbb{E} \left[\int_0^t \int_{|z|>0} \int_D \int_0^1 (1-\lambda) a^2 \beta_\vartheta''(a + \lambda b) (1 \wedge |z|^2) \, d\lambda \, dx \, m(dz) \, ds \right]. \end{aligned} \quad (3.38)$$

We need to find a suitable upper bound on $a^2 \beta_\vartheta''(a + \lambda b)$. Note that β_ϑ'' is non-negative and symmetric around zero. Thus we may assume, without loss of generality, that $a \geq 0$. Then by the assumption [A.3](#)

$$u_1(s, x) - u_2(s, x) + \lambda b \geq (1 - \lambda^*)(u_1(s, x) - u_2(s, x))$$

for $\lambda \in [0, 1]$. In other words

$$0 \leq a \leq (1 - \lambda^*)^{-1}(a + \lambda b). \quad (3.39)$$

We combine [\(3.38\)](#) and [\(3.39\)](#) to obtain

$$\mathcal{B} \leq C(\lambda^*) \mathbb{E} \left[\int_0^t \int_{|z|>0} \int_D \int_0^1 (1-\lambda) (a + \lambda b)^2 \beta_\vartheta''(a + \lambda b) (1 \wedge |z|^2) \, d\lambda \, dx \, m(dz) \, ds \right].$$

In view of [\(3.36\)](#), and the assumption on η that $\eta(0, z) = 0$ for all $z \in \mathbb{R}$, we see that for each $\lambda \in [0, 1]$

$$(a + \lambda b)^2 \beta_\vartheta''(a + \lambda b) \leq |a + \lambda b| \mathbf{1}_{\{0 < |a + \lambda b| < \vartheta\}} \leq |a + \lambda b| \in L^1(\Omega \times D_T)$$

for $m(dz)$ -almost every $z \in \mathbb{R}$. Again $|a + \lambda b| \mathbf{1}_{\{0 < |a + \lambda b| < \vartheta\}} \rightarrow 0$ as $\vartheta \rightarrow 0$ for almost every (s, x) and almost surely. We apply dominated convergence theorem, along with the assumption [A.4](#) to conclude that $\mathcal{B} \rightarrow 0$ as $\vartheta \rightarrow 0$. Let us explain how one can assume, without loss of generality, that $a \geq 0$. Observe that if $a \leq 0$, then by setting $\tilde{a} := -a \geq 0$, one arrives at the inequality

$$0 \leq \tilde{a} \leq (1 - \lambda^*)^{-1}(\tilde{a} - \lambda b) \quad \text{for } \lambda \in [0, 1].$$

Since β_ϑ'' is non-negative and symmetric around zero, we then obtain

$$a^2 \beta_\vartheta''(a + \lambda b) = \tilde{a}^2 \beta_\vartheta''(\tilde{a} - \lambda b) \leq C(\lambda^*) (\tilde{a} - \lambda b)^2 \beta_\vartheta''(\tilde{a} - \lambda b).$$

One can give a similar argument, used for the case $a \geq 0$, to conclude that $\mathcal{B} \rightarrow 0$ as $\vartheta \rightarrow 0$. Putting things together and passing to the limit in [\(3.37\)](#), we have

$$\mathbb{E} \left[\int_D |u_1(t, x) - u_2(t, x)| \, dx \right] = 0.$$

In other words, \mathbb{P} -a.s., $u_1(t, x) = u_2(t, x)$ for almost every (t, x) . This yields the uniqueness of path-wise weak solution of the underlying problem [\(1.1\)](#) with respect to the same stochastic basis. This completes the proof of [Theorem 2.2](#).

Remark 3.13. Existence of weak solution and path-wise uniqueness guarantee uniqueness in law due to classical Yamada-Watanabe technique [30] for finite dimensional case; for the infinite dimensional case, see *e.g.*, Theorems 2 and 11 from [22].

Remark 3.14. Thanks to Skorokhod parameterization, see *e.g.*, [3, 10], one can prove the following theorem as a generalization of Theorem 2.2:

Let the assumptions **A.1-A.4** be true and $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\})$ be a given filtered probability space satisfying the usual hypotheses. Let N be a time-homogeneous Poisson random measure on \mathbb{R} with intensity measure $m(dz)$ defined on $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\})$, and μ be a probability measure on $L^2(D)$ such that $\int_{L^2(D)} \Phi(v) \mu(dv) < \infty$ where $\Phi(v) = \|v\|_{W^{1,p}(D)}^p$. Then, for the problem (1.1), there exists a weak solution $\hat{\pi} = (\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}, \{\hat{\mathcal{F}}_t\}, \hat{N}, \hat{u}, \hat{U})$ in the sense of Definition 2.1 such that (2.1) holds and $\mu = \mathcal{L}(\hat{U})$ on $L^2(D)$.

Remark 3.15. In the current paper, we are essentially using Itô-Lévy isometry for the proof. In particular, to show that $T_2^r(t)$ satisfies the inequality (3.8), and to identify the weak limit G of the sequences $|\nabla v_\tau|^{p-2} \nabla v_\tau$, we have used the Itô-Lévy isometry. Therefore, it is not clear whether one can work with Banach setting instead of Hilbert assumptions.

4. EXISTENCE OF OPTIMAL CONTROL: PROOF OF THEOREM 2.6

The objective of this section is to prove existence of a weak optimal solution of (1.2) in the sense of Definition 2.5 *i.e.*, Theorem 2.6.

Proof. We prove Theorem 2.6 in several steps.

Step I): In view of Theorem 2.2, there exists a weak solution of (1.1) with $U = 0$, and satisfies the estimate ii) of Theorem 2.2. Since Ψ is Lipschitz continuous and $u_{\text{tar}} \in L^p(0, T; W_0^{1,p}(D))$, it follows easily that Λ defined in (2.2) is finite. Thus, there exists a minimizing sequence of weak admissible solutions $\pi_n = (\Omega_n, \mathcal{F}_n, \mathbb{P}_n, \mathbb{F}_n = \{\mathcal{F}_t^n\}, N_n, u_n, U_n)$ such that $\Lambda = \lim_{n \rightarrow \infty} \mathcal{J}(\pi_n)$. Since for each $n \in \mathbb{N}$, $\pi_n \in \mathcal{U}_{\text{ad}}^w(u_0; T)$, we have, \mathbb{P}_n -a.s. for all $t \in [0, T]$

$$\begin{aligned} u_n(t) &= u_0 + U_n + \int_0^t \operatorname{div}_x(|\nabla u_n|^{p-2} \nabla u_n + \vec{f}(u_n)) \, ds + \int_0^t \int_{|z|>0} \eta(u_n; z) \tilde{N}_n(dz, ds) \\ &= u_0 + U_n + T_{1,n}(t) + T_{2,n}(t). \end{aligned} \quad (4.1)$$

Moreover, since Λ is finite, one has the following estimates (uniform in n):

$$\begin{cases} \sup_n \mathbb{E}_n [\|U_n\|_{W^{1,p}(D)}^p] \leq C, \\ \sup_n \mathbb{E}_n \left[\sup_{0 \leq t \leq T} \|u_n(t)\|_{L^2(D)}^2 + \int_0^T \|u_n(t)\|_{W_0^{1,p}(D)}^p \, dt \right] \leq C, \end{cases} \quad (4.2)$$

where \mathbb{E}_n denotes the expectation with respect to \mathbb{P}_n .

Step II): By proving Aldous condition for the sequence $\{u_n\}$ in $W^{-1,p'}(D)$ and then applying Theorem 3.5 along with the uniform-estimate (4.2), one can establish the tightness of $\{\mathcal{L}(u_n)\}$ on $(\mathcal{Z}, \mathcal{T})$. Moreover, due to the uniform-bound (4.2), and the tightness of the family of laws $\{\mathcal{L}(N_n(dz, dt))\}$ on $M_{\mathbb{N}}(\mathbb{R} \times [0, T])$, the set $\{\mathcal{L}(u_n, N_n, U_n)\}$ is tight in \mathcal{X} . Therefore, by Corollary 2 from [19], there exist a subsequence of $\{n\}$, still we denote it by same $\{n\}$, a probability space $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$ and, on this space \mathcal{X} -valued random variables (u^*, U^*, N^*) and (u_n^*, U_n^*, N_n^*) such that

$$\text{i). } \mathcal{L}(u_n^*, U_n^*, N_n^*) = \mathcal{L}(u_n, U_n, N_n) \text{ for all } n \in \mathbb{N},$$

- ii). $(u_n^*, U_n^*, N_n^*) \rightarrow (u^*, U^*, N^*)$ in \mathcal{X} \mathbb{P}^* -a.s. ($n \rightarrow \infty$),
- iii). $N_n^*(\omega^*) = N^*(\omega^*)$ for all $\omega^* \in \Omega^*$.

The sequences $\{u_n^*\}$ and $\{U_n^*\}$ satisfy the same estimate as the original sequences $\{u_n\}$ and $\{U_n\}$ respectively. In particular,

$$\begin{cases} \sup_n \mathbb{E}^* [\|U_n^*\|_{W^{1,p}(D)}^p] \leq C, \\ \sup_n \mathbb{E}^* \left[\sup_{0 \leq t \leq T} \|u_n^*(t)\|_{L^2(D)}^2 + \int_0^T \|u_n^*(t)\|_{W_0^{1,p}(D)}^p dt \right] \leq C. \end{cases} \quad (4.3)$$

Moreover, in view of (4.1) and i) of **Step II**, one can conclude \mathbb{P}^* -a.s.,

$$\begin{aligned} u_n^*(t) &= u_0 + U_n^* + \int_0^t \operatorname{div}_x (|\nabla u_n^*(s)|^{p-2} \nabla u_n^*(s) + \vec{f}(u_n^*(s))) ds \\ &\quad + \int_0^t \int_{|z|>0} \eta(u_n^*(s); z) \tilde{N}_n^*(dz, ds). \end{aligned} \quad (4.4)$$

Step III): Let \mathbb{F}^* be the natural filtration of (u_n^*, N_n^*, u^*, N^*) . Since $N_n^*(\omega^*) = N^*(\omega^*)$ for all $\omega^* \in \Omega^*$, N_n^* and N^* are the time homogeneous Poisson random measures on \mathbb{R} over the stochastic basis $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*, \mathbb{F}^*)$. Using the similar arguments as in Lemmas 3.9-3.10 and 3.12 along with **step ii)** in Section 3.5, one can pass to the limit in (4.4) and conclude that the $W_0^{1,p}(D)$ -valued \mathbb{F}^* -predictable stochastic process u^* satisfies the following: \mathbb{P}^* -a.s. and a.e. $t \in [0, T]$,

$$\begin{aligned} (u^*(t), \phi)_{L^2(D)} &= (u_0 + U^*, \phi)_{L^2(D)} + \int_0^t \left\langle \operatorname{div}_x (|\nabla u^*(s)|^{p-2} \nabla u^*(s) + \vec{f}(u^*(s))), \phi \right\rangle ds \\ &\quad + \left\langle \int_0^t \int_{|z|>0} \eta(u^*(s); z) \tilde{N}^*(dz, ds), \phi \right\rangle \quad \forall \phi \in W_0^{1,p}(D). \end{aligned} \quad (4.5)$$

Since $u^* \in \mathbb{D}([0, T]; L_w^2(D))$, (4.5) holds true for all $t \in [0, T]$ and all $\phi \in W_0^{1,p}(D)$, and hence $\pi^* = (\Omega^*, \mathcal{F}^*, \mathbb{P}^*, \mathbb{F}^*, N^*, u^*, U^*) \in \mathcal{U}_{\text{ad}}^w(u_0; T)$. Moreover, (u^*, U^*) satisfies the estimate (2.1).

Step IV): Since $\pi^* \in \mathcal{U}_{\text{ad}}^w(u_0; T)$, obviously $\Lambda \leq \mathcal{J}(\pi^*)$. We now show that $\mathcal{J}(\pi^*) \leq \Lambda$. Note that, the mapping

$$\begin{aligned} S : L^2(D) \times W^{1,p}(D) &\mapsto [0, \infty] \\ (u, U) &\mapsto \|u - u_{\text{tar}}\|_{L^2(D)}^2 + \|U\|_{W^{1,p}(D)}^p \end{aligned}$$

is a measurable, non-negative and lower semi-continuous convex function. Thus, invoking i) – iii) of **step II** along with Fatou's lemma, we get

$$\begin{aligned} \mathcal{J}(\pi^*) &= \mathbb{E}^* \left[\int_0^T \|u^*(t) - u_{\text{tar}}(t)\|_{L^2(D)}^2 dt + \|U^*\|_{W^{1,p}(D)}^p \right] + \mathbb{E}^* [\Psi(u^*(T))] \\ &\leq \liminf_{n \rightarrow \infty} \left\{ \mathbb{E}^* \left[\int_0^T \|u_n^*(t) - u_{\text{tar}}(t)\|_{L^2(D)}^2 dt + \|U_n^*\|_{W^{1,p}(D)}^p \right] + \mathbb{E}^* [\Psi(u_n^*(T))] \right\} \\ &= \liminf_{n \rightarrow \infty} \left\{ \mathbb{E}_n \left[\int_0^T \|u_n(t) - u_{\text{tar}}(t)\|_{L^2(D)}^2 dt + \|U_n\|_{W^{1,p}(D)}^p \right] + \mathbb{E}_n [\Psi(u_n(T))] \right\} \\ &= \liminf_{n \rightarrow \infty} \mathcal{J}(\pi_n) = \Lambda. \end{aligned}$$

This implies that $\pi^* = (\Omega^*, \mathcal{F}^*, \mathbb{P}^*, \mathbb{F}^*, N^*, u^*, U^*)$ is a weak optimal solution of the control problem (1.2), and U^* is an optimal control. This finishes the proof of Theorem 2.6. \square

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