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OPTIMAL CONTROL FOR CONTROLLABLE STOCHASTIC LINEAR SYSTEMS

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Abstract. This paper is concerned with a constrained stochastic linear-quadratic optimal control problem, in which the terminal state is fixed and the initial state is constrained to lie in a stochastic linear manifold. The controllability of stochastic linear systems is studied. Then the optimal control is explicitly obtained by considering a parameterized unconstrained backward LQ problem and an optimal parameter selection problem. A notable feature of our results is that, instead of solving an equation involving derivatives with respect to the parameter, the optimal parameter is characterized by a matrix equation.

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1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space on which a standard one-dimensional Brownian motion $W = \{W(t); 0 \leqslant t < \infty\}$ is defined, and let $\mathbb{F} = \{\mathcal{F}_t\}_{t\geqslant 0}$ be the natural filtration of W augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$. Consider the following controlled linear stochastic differential equation (SDE, for short) on a finite horizon $[t, T]$:

$$\mathrm{d}x(s) = [A(s)x(s) + B(s)u(s)]\mathrm{d}s + [C(s)x(s) + D(s)u(s)]\mathrm{d}W(s), \quad s \in [t, T], \tag{1.1}$$

where $A, C : [0, T] \to \mathbb{R}^{n\times n}$ and $B, D : [0, T] \to \mathbb{R}^{n\times m}$, called the *coefficients* of the *state equation* (1.1), are given deterministic functions. The solution $x = \{x(s); t \leqslant s \leqslant T\}$ of (1.1), which takes values in $\mathbb{R}^n$, is called a *state process*, and the process $u = \{u(s); t \leqslant s \leqslant T\}$, which takes values in $\mathbb{R}^m$ and is $\mathbb{F}$-progressively measurable, is called a *control*. For a given initial condition $x(t) = \xi$, the state process x is uniquely determined by the control u, and is often denoted by $x^{t,\xi,u}$ when it is necessary to underline the dependence on the *initial pair* (t, ξ) and the control u. In this paper we shall assume that the coefficients of the state equation (1.1) satisfy the following condition:

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(A1) $A, C : [0, T] \to \mathbb{R}^{n\times n}$ and $B, D : [0, T] \to \mathbb{R}^{n\times m}$ are bounded, Lebesgue measurable functions.

According to the standard result for SDEs (see, for example, [18], Chap. 1, Thm. 6.3), such a condition ensures that a unique pth power integrable solution exists for the SDE (1.1) whenever the *initial state* $x(t) = \xi$ and the control u are pth power integrable. We are interested in the case $p = 2$, in which the spaces of initial states, *admissible controls* and state processes are

$$\begin{aligned} L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n) &= \Big\{\xi : \Omega \to \mathbb{R}^n \;\Big|\; \xi \text{ is } \mathcal{F}_t\text{-measurable with } \mathbb{E}|\xi|^2 < \infty\Big\}, \\ L^2_{\mathbb{F}}(t, T;\mathbb{R}^m) &= \Big\{u : [t, T] \times \Omega \to \mathbb{R}^m \;\Big|\; u \text{ is } \mathbb{F}\text{-progressively measurable} \\ &\qquad \text{with } \mathbb{E}\textstyle\int_t^T |u(s)|^2 ds < \infty\Big\}, \text{ and} \\ L^2_{\mathbb{F}}(\Omega; C([t, T];\mathbb{R}^n)) &= \Big\{x : [t, T] \times \Omega \to \mathbb{R}^n \;\Big|\; x \text{ is } \mathbb{F}\text{-adapted and continuous} \\ &\qquad \text{with } \mathbb{E}\big[\sup_{t\leqslant s\leqslant T} |x(s)|^2\big] < \infty\Big\}, \end{aligned}$$

respectively.

Let $F \in \mathbb{R}^{k\times n}$ $(k \leqslant n)$ be a matrix, and let $b \in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^k)$ be a random variable. We denote by $\mathcal{H}(F, b)$ the *stochastic linear manifold*

$$\{\xi \in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n) : F\xi = b\}.$$

The problems of interest here are those for which the control u is required to drive the system (1.1) to a particular state at the end of the interval $[t, T]$ from a given stochastic linear manifold $\mathcal{H}(F, b)$ and the cost functional is of the quadratic form

$$J(t, u) = \mathbb{E}\left\{\langle Gx(t), x(t)\rangle + \int_t^T \langle Q(s)x(s), x(s)\rangle + \langle R(s)u(s), u(s)\rangle ds\right\}, \tag{1.2}$$

where the *weighting matrices* G, Q, and R are assumed to satisfy the following condition:

(A2) $G \in \mathbb{R}^{n\times n}$ is a symmetric matrix; $Q : [0, T] \to \mathbb{R}^{n\times n}$ and $R : [0, T] \to \mathbb{R}^{m\times m}$ are bounded, symmetric functions. Moreover, for some real number $\delta > 0$,

$$G \geqslant 0, \quad Q(s) \geqslant 0, \quad R(s) \geqslant \delta I_m, \quad \text{a.e. } s \in [0, T].$$

For a precise statement, we pose the following *constrained stochastic linear-quadratic (LQ, for short) optimal control problem.*

Problem (CLQ). For a given target $\eta \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$, find a control $u^* \in L^2_{\mathbb{F}}(t, T;\mathbb{R}^m)$ such that the cost functional $J(t, u)$ is minimized over $L^2_{\mathbb{F}}(t, T;\mathbb{R}^m)$, subject to the following constraints on the initial and terminal states:

$$x(t) \in \mathcal{H}(F, b), \quad x(T) = \eta. \tag{1.3}$$

A control $u^* \in L^2_{\mathbb{F}}(t, T;\mathbb{R}^m)$ that minimizes $J(t, u)$ subject to (1.3) will be called an *optimal control* with respect to the target η; the corresponding state process will be called an *optimal state process*. If an initial state $\xi \in \mathcal{H}(F, b)$ is transferred to the target η by an optimal control, we call ξ an *optimal initial state*.

If the constraint (1.3) is absent, but the initial state $x(t) = \xi$ is given, Problem (CLQ) becomes a standard stochastic LQ optimal control problem. Such kind of problems was initiated by Wonham [17] and was later investigated by many researchers; see, for example, Bismut [3], Bensoussan [2], Chen and Yong [4], Ait Rami, Moore, and Zhou [1], Tang [13], Yu [19], Sun, Li, and Yong [11], Lü, Wang, and Zhang [9], Sun, Xiong, and Yong [12], Wang, Sun, and Yong [14], and the references therein. In contrast, much less progress has been made on the constrained LQ problem for stochastic systems. This problem is particularly difficult in the stochastic setting since not only is one required to decide whether a state of the stochastic system can be transferred to another state, but in addition an optimal parameter must be evaluated.

There were some attempts in attacking the constrained stochastic LQ optimal control problem in the special case of norm optimal control; see, for instance, Gashi [5], Wang and Zhang [16], and Wang *et al.* [15]. However, in these works the state process is required to start from a *particular point*, and the optimal control is either characterized *implicitly* in terms of *coupled* forward-backward stochastic differential equations (FBSDEs, for short), which are difficult to solve, or explicitly obtained but under a *strong* assumption that the stochastic system is *exactly controllable* (which means a target can be reached from every initial state).

This paper aims to provide a complete solution to Problem (CLQ), a class of stochastic LQ optimal control problems with fixed terminal states. A distinctive feature of the problem under consideration is that the state process is allowed to start from a stochastic linear manifold $\mathcal{H}(F,b)$, instead of a fixed initial state. Clearly, our problem contains the norm optimal control as a particular case. Another feature is that the stochastic system is *not* assumed to be exactly controllable. The initial states outside the stochastic linear manifold $\mathcal{H}(F,b)$ are irrelevant to our problem, so figuring out when the target can be reached from $\mathcal{H}(F,b)$ will be enough to tackle Problem (CLQ).

The principal method adopted in the paper is combination of Lagrange multipliers and unconstrained backward LQ problems. By introducing a parameter λ, the Lagrange multiplier, Problem (CLQ) is reduced to a parameterized unconstrained backward LQ problem, whose optimal control and value function V_λ can be constructed explicitly using the solutions to a Riccati equation and a decoupled FBSDE. Then the optimal state process x^*_λ of the derived backward LQ problem is proved to be also optimal for Problem (CLQ) if the parameter λ is such that $x^*_\lambda(t)\in\mathcal{H}(F,b)$. In order to find such a parameter, called an *optimal parameter*, a first idea is to solve the equation $\frac{d}{d\lambda}V_\lambda=0$. However, this does not work well in our situation, due to the difficulty in computing the derivative of V_λ. Our approach for finding the optimal parameter is based on a refinement (Prop. 3.4) of Liu and Peng's result ([8], Thm. 2). The key is to establish an equivalence relationship between the controllability of the original system and a system involving Σ, the solution of a Riccati equation (Prop. 5.3). By observing that the controllability Gramian of the new system is exactly $\Sigma(t)$ (Prop. 5.4), we show that an optimal parameter can be obtained by solving a matrix equation (Theorem 5.1).

The rest of the paper is organized as follows. In Section 2, we collect some preliminary results. Section 3 is devoted to the study of controllability of stochastic linear systems. In Section 4, using Lagrange multipliers, we reduce the problem to a parameterized unconstrained backward LQ problem and an optimal parameter selection problem. Finally, we discuss how to find an optimal parameter and present the complete solution to Problem (CLQ) in Section 5.

2. Preliminaries

Let $\mathbb{R}^{n\times m}$ be the Euclidean space consisting of $n\times m$ real matrices, and let $\mathbb{R}^n=\mathbb{R}^{n\times 1}$. The inner product of $M,N\in\mathbb{R}^{n\times m}$, denoted by $\langle M,N\rangle$, is given by $\langle M,N\rangle=\mathrm{tr}\,(M^\top N)$, where $M^\top$ is transpose of M and $\mathrm{tr}\,(M^\top N)$ stands for the trace of $M^\top N$. This inner product induces the Frobenius norm $|M|=\sqrt{\mathrm{tr}\,(M^\top M)}$. Denote by $\mathbb{S}^n$ the space of all symmetric $n\times n$ real matrices, and by $\overline{\mathbb{S}^n_+}$ the space of all symmetric positive semidefinite $n\times n$ real matrices. For $\mathbb{S}^n$-valued functions M and N, we write $M\geqslant N$ (respectively, $M>N$) if $M-N$ is positive semidefinite (respectively, positive definite) almost everywhere. The identity matrix of size n is denoted by I_n.

We now present some lemmas that are useful in the subsequent sections. Consider the linear BSDE

$$\begin{cases} \mathrm{d}Y(s)=\big[A(s)Y(s)+C(s)Z(s)+f(s)\big]\mathrm{d}s+Z(s)\mathrm{d}W(s), \quad s\in[0,T],\\ Y(T)=\eta. \end{cases} \tag{2.1}$$

The following result, coming from the idea of proving the well-posedness of linear BSDEs (see Chap. 7, Thm. 2.2 in [18]), provides a formula for the first component of the adapted solution (Y,Z) to the BSDE (2.1).

Lemma 2.1. *Let (A1) hold, and let* $f\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$, $\eta\in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$. *Then the first component* Y *of the adapted solution to* (2.1) *has the following representation:*

$$Y(t)=\mathbb{E}\left[\Gamma(t,T)\eta-\int_t^T\Gamma(t,s)f(s)\mathrm{d}s\bigg|\mathcal{F}_t\right],\quad t\in[0,T],\tag{2.2}$$

where $\Gamma(t,s)\triangleq\Gamma(t)^{-1}\Gamma(s)$ *with* $\Gamma=\{\Gamma(s);0\leqslant s\leqslant T\}$ *being the solution to*

$$\begin{cases}\mathrm{d}\Gamma(s)=-\Gamma(s)A(s)\mathrm{d}s-\Gamma(s)C(s)\mathrm{d}W(s),\quad s\in[0,T],\\ \Gamma(0)=I_n.\end{cases}$$

Proof. Let $\theta=\Gamma(T)\eta-\int_0^T\Gamma(s)f(s)\mathrm{d}s$. By Itô's formula,

$$\begin{aligned}\mathrm{d}\Gamma Y&=-\Gamma AY\mathrm{d}s-\Gamma CY\mathrm{d}W+\Gamma(AY+CZ+f)\mathrm{d}s+\Gamma Z\mathrm{d}W-\Gamma CZ\mathrm{d}s\\&=\Gamma f\mathrm{d}s+\Gamma(Z-CY)\mathrm{d}W,\end{aligned}$$

from which it follows that

$$\begin{aligned}\Gamma(t)Y(t)&=\Gamma(T)\eta-\int_t^T\Gamma(s)f(s)\mathrm{d}s-\int_t^T\Gamma(s)\big[Z(s)-C(s)Y(s)\big]\mathrm{d}W(s)\\&=\theta+\int_0^t\Gamma(s)f(s)\mathrm{d}s-\int_t^T\Gamma(s)\big[Z(s)-C(s)Y(s)\big]\mathrm{d}W(s).\end{aligned}\tag{2.3}$$

Note that

$$\mathbb{E}\left(\int_0^T\big|\Gamma(s)[Z(s)-C(s)Y(s)]\big|^2\mathrm{d}s\right)^{\frac12}<\infty.$$

Hence, the process

$$M(t)\equiv\int_0^t\Gamma(s)\big[Z(s)-C(s)Y(s)\big]\mathrm{d}W(s),\quad 0\leqslant t\leqslant T$$

is a martingale, and by taking conditional expectations with respect to $\mathcal{F}_t$ on both sides of (2.3), we obtain

$$\Gamma(t)Y(t)=\mathbb{E}[\theta|\mathcal{F}_t]+\int_0^t\Gamma(s)f(s)\mathrm{d}s=\mathbb{E}\left[\Gamma(T)\eta-\int_t^T\Gamma(s)f(s)\mathrm{d}s\bigg|\mathcal{F}_t\right],$$

from which the desired result follows. □

We conclude this section with a simple but useful algebraic lemma.

Lemma 2.2. *Let* $A\in\mathbb{R}^{m\times n}$ *and* $B\in\overline{\mathbb{S}^n_+}$. *Then* $ABA^\top$ *and* AB *have the same range space.*

Proof. For a matrix M, let $\mathscr{R}(M)$ and $\mathscr{N}(M)$ denote the range and kernel of M, respectively. Since $\mathscr{R}(M)^\perp = \mathscr{N}(M^\top)$ for any matrix M, it is suffice to prove

$$\mathscr{N}(ABA^\top) = \mathscr{N}(BA^\top).$$

Clearly, $\mathscr{N}(BA^\top) \subseteq \mathscr{N}(ABA^\top)$. For the reverse inclusion, let $C \in \mathbb{R}^{n\times n}$ be such that $B = CC^\top$. If $x \in \mathbb{R}^m$ is such that $ABA^\top x = 0$, then

$$\left|C^\top A^\top x\right|^2 = x^\top ACC^\top A^\top x = x^\top ABA^\top x = 0.$$

Thus, $C^\top A^\top x = 0$ and hence $BA^\top x = CC^\top A^\top x = 0$. This shows that $\mathscr{N}(ABA^\top) \subseteq \mathscr{N}(BA^\top)$. □

3. Controllability of linear stochastic systems

Consider the controlled linear stochastic differential system

$$\mathrm{d}x(s) = [A(s)x(s) + B(s)u(s)]\mathrm{d}s + [C(s)x(s) + D(s)u(s)]\mathrm{d}W(s). \tag{3.1}$$

Let $(t_0, x_0) \in [0,T) \times L^2_{\mathcal{F}_{t_0}}(\Omega;\mathbb{R}^n)$ be an initial pair, and let $t_1 \in (t_0, T]$ be the terminal time. We know by the standard result for SDEs (see, for example, [18], Chap. 1, Thm. 6.3) that a solution $x^{t_0,x_0,u} \in L^2_{\mathbb{F}}(\Omega; C([t_0,t_1];\mathbb{R}^n))$ uniquely exists for every control $u \in L^2_{\mathbb{F}}(t_0,t_1;\mathbb{R}^m)$. We are now concerned with the question of finding a control such that a given target (terminal state) is reached on the terminal time.

Definition 3.1. We say that a control $u \in L^2_{\mathbb{F}}(t_0,t_1;\mathbb{R}^m)$ *transfers the state of the system* (3.1) *from* $x_0 \in L^2_{\mathcal{F}_{t_0}}(\Omega;\mathbb{R}^n)$ *at* $t = t_0$ *to* $x_1 \in L^2_{\mathcal{F}_{t_1}}(\Omega;\mathbb{R}^n)$ *at* $t = t_1$ if

$$x^{t_0,x_0,u}(t_1) = x_1$$

almost surely. We then also say that u *transfers* (t_0,x_0) *to* (t_1,x_1), or that (t_1,x_1) *can be reached from* (t_0,x_0) *by* u.

Definition 3.2. System (3.1) is called *exactly controllable on* $[t_0,t_1]$, if for every $x_0 \in L^2_{\mathcal{F}_{t_0}}(\Omega;\mathbb{R}^n)$ and every $x_1 \in L^2_{\mathcal{F}_{t_1}}(\Omega;\mathbb{R}^n)$ there exists a control $u \in L^2_{\mathbb{F}}(t_0,t_1;\mathbb{R}^m)$ transferring (t_0,x_0) to (t_1,x_1).

It was shown in [10] and [8] that system (3.1) is exactly controllable on some interval only if D has full row rank and the number of columns of D is greater than the number of rows of D (*i.e.*, $m > n$). Note that $\mathrm{rank}\,(D) = n$ means that $DD^\top$ is invertible. For technical reasons, in the sequel we shall impose, in addition to $m > n$, the following slightly stronger condition (which is usually referred to as the *nondegeneracy* condition): for some $\delta > 0$,

$$D(s)D(s)^\top \geqslant \delta I_n, \quad \forall s \in [0,T]. \tag{3.2}$$

This condition implies that we can find a bounded invertible function $M : [0,T] \to \mathbb{R}^{m\times m}$ such that

$$D(s)M(s) = (I_n, 0_{n\times(m-n)}), \quad \forall s \in [0,T]. \tag{3.3}$$

In order to study the controllability of system (3.1), we write $B(s)M(s) = (K(s), L(s))$ with $K(s)$ and $L(s)$ taking values in $\mathbb{R}^{n\times n}$ and $\mathbb{R}^{n\times(m-n)}$, respectively, and introduce the following controlled system:

$$\mathrm{d}\bar{x}(s) = \left[\bar{A}(s)\bar{x}(s) + \bar{B}(s)\bar{u}(s)\right]\mathrm{d}s + \bar{D}(s)\bar{u}(s)\mathrm{d}W(s), \tag{3.4}$$

where

$$\bar{A} = A - KC, \quad \bar{B} = BM = (K, L), \quad \bar{D} = DM = (I_n, 0_{n\times(m-n)}). \tag{3.5}$$

Note that if we write the control $\bar{u}$ as the form

$$\bar{u}(s) = \begin{bmatrix} z(s) \\ v(s) \end{bmatrix}; \quad z(s) \in \mathbb{R}^n, \ v(s) \in \mathbb{R}^{m-n},$$

the system (3.4) simplifies to

$$\mathrm{d}\bar{x}(s) = \big[\bar{A}(s)\bar{x}(s) + K(s)z(s) + L(s)v(s)\big]\mathrm{d}s + z(s)\mathrm{d}W(s). \tag{3.6}$$

The following result establishes a connection between the controllability of systems (3.1) and (3.6).

Proposition 3.3. *Let $0 \leqslant t_0 < t_1 \leqslant T$, $x_0 \in L^2_{\mathcal{F}_{t_0}}(\Omega; \mathbb{R}^n)$ and $x_1 \in L^2_{\mathcal{F}_{t_1}}(\Omega; \mathbb{R}^n)$. For system (3.6), a control $(z, v) \in L^2_{\mathbb{F}}(t_0, t_1; \mathbb{R}^n) \times L^2_{\mathbb{F}}(t_0, t_1; \mathbb{R}^{m-n})$ transfers (t_0, x_0) to (t_1, x_1) if and only if the control defined by*

$$u(s) \triangleq M(s) \begin{bmatrix} z(s) - C(s)\bar{x}(s) \\ v(s) \end{bmatrix}, \quad s \in [t_0, t_1]$$

does so for system (3.1), where $\bar{x}$ is the solution of (3.6) with initial state x_0.

Proof. We first observe that

$$\begin{aligned}
\bar{A}(s)\bar{x}(s) + K(s)z(s) + L(s)v(s) &= A(s)\bar{x}(s) + K(s)[z(s) - C(s)\bar{x}(s)] + L(s)v(s) \\
&= A(s)\bar{x}(s) + B(s)M(s) \begin{bmatrix} z(s) - C(s)\bar{x}(s) \\ v(s) \end{bmatrix} \\
&= A(s)\bar{x}(s) + B(s)u(s), \\
C(s)\bar{x}(s) + D(s)u(s) &= C(s)\bar{x}(s) + D(s)M(s) \begin{bmatrix} z(s) - C(s)\bar{x}(s) \\ v(s) \end{bmatrix} \\
&= C(s)\bar{x}(s) + (I_n, 0_{n\times(m-n)}) \begin{bmatrix} z(s) - C(s)\bar{x}(s) \\ v(s) \end{bmatrix} \\
&= z(s).
\end{aligned}$$

This means $\bar{x}$ also satisfies

$$\mathrm{d}\bar{x}(s) = [A(s)\bar{x}(s) + B(s)u(s)]\mathrm{d}s + [C(s)\bar{x}(s) + D(s)u(s)]\mathrm{d}W(s).$$

Thus, by the uniqueness of a solution, with the initial state x_0 and the control u, the solution x of system (3.1) coincides with $\bar{x}$. The result then follows immediately. ☐

From Proposition 3.3, we see that the controllability of system (3.1) is equivalent to that of system (3.6). For the controllability of system (3.6), we have the following characterization, which refines the result of Liu and Peng ([8], Thm. 2).

Proposition 3.4. *Let $0 \leqslant t_0 < t_1 \leqslant T$, $x_0 \in L^2_{\mathcal{F}_{t_0}}(\Omega; \mathbb{R}^n)$ and $x_1 \in L^2_{\mathcal{F}_{t_1}}(\Omega; \mathbb{R}^n)$. There exists a control $(z, v) \in L^2_{\mathbb{F}}(t_0, t_1; \mathbb{R}^n) \times L^2_{\mathbb{F}}(t_0, t_1; \mathbb{R}^{m-n})$ which transfers the state of system (3.6) from x_0 at $t = t_0$ to x_1 at $t = t_1$ if* *and only if* $x_0 - \mathbb{E}[\Phi(t_0,t_1)x_1|\mathcal{F}_{t_0}]$ *belongs to the range space of*

$$\Psi(t_0,t_1) \triangleq \mathbb{E}\left[\int_{t_0}^{t_1} \Phi(t_0,s)L(s)[\Phi(t_0,s)L(s)]^\top \mathrm{d}s\right] \tag{3.7}$$

almost surely, that is, there exists a $\xi \in L^2_{\mathcal{F}_{t_0}}(\Omega;\mathbb{R}^n)$ *such that*

$$x_0 - \mathbb{E}[\Phi(t_0,t_1)x_1|\mathcal{F}_{t_0}] = \Psi(t_0,t_1)\xi, \quad \text{a.s.},$$

where $\Phi(t,s) \triangleq \Phi(t)^{-1}\Phi(s)$ *with* $\Phi = \{\Phi(s); 0 \leqslant s \leqslant T\}$ *being the solution to the following SDE for* $\mathbb{R}^{n\times n}$*-valued processes:*

$$\begin{cases} \mathrm{d}\Phi(s) = -\Phi(s)\bar{A}(s)\mathrm{d}s - \Phi(s)K(s)\mathrm{d}W(s), \quad s \in [0,T], \\ \Phi(0) = I_n. \end{cases} \tag{3.8}$$

Proof. Sufficiency. Suppose that there exists a $\xi \in L^2_{\mathcal{F}_{t_0}}(\Omega;\mathbb{R}^n)$ such that

$$x_0 - \mathbb{E}[\Phi(t_0,t_1)x_1|\mathcal{F}_{t_0}] = \Psi(t_0,t_1)\xi, \quad \text{a.s.}$$

Define

$$v(s) = -[\Phi(t_0,s)L(s)]^\top\xi, \quad s \in [t_0,t_1],$$

and let (y_1,z_1) be the adapted solution to the following BSDE:

$$\begin{cases} \mathrm{d}y_1(s) = \big[\bar{A}(s)y_1(s) + K(s)z_1(s) + L(s)v(s)\big]\mathrm{d}s + z_1(s)\mathrm{d}W(s), \quad s \in [t_0,t_1], \\ y_1(t_1) = 0. \end{cases}$$

According to Lemma 2.1,

$$\begin{aligned} y_1(t_0) &= -\mathbb{E}\left[\int_{t_0}^{t_1} \Phi(t_0,s)L(s)v(s)\mathrm{d}s \,\middle|\, \mathcal{F}_{t_0}\right] \\ &= \mathbb{E}\left[\int_{t_0}^{t_1} \Phi(t_0,s)L(s)[\Phi(t_0,s)L(s)]^\top\xi\mathrm{d}s \,\middle|\, \mathcal{F}_{t_0}\right]. \end{aligned}$$

Noting that ξ is $\mathcal{F}_{t_0}$-measurable and that $\Phi(t_0,s)$ is independent of $\mathcal{F}_{t_0}$ for $s \geqslant t$, we further obtain

$$\begin{aligned} y_1(t_0) &= \mathbb{E}\left[\int_{t_0}^{t_1} \Phi(t_0,s)L(s)[\Phi(t_0,s)L(s)]^\top\mathrm{d}s\right]\xi = \Psi(t_0,t_1)\xi \\ &= x_0 - \mathbb{E}[\Phi(t_0,t_1)x_1|\mathcal{F}_{t_0}]. \end{aligned}$$

Now let (y_2,z_2) be the adapted solution to the BSDE

$$\begin{cases} \mathrm{d}y_2(s) = \big[\bar{A}(s)y_2(s) + K(s)z_2(s)\big]\mathrm{d}s + z_2(s)\mathrm{d}W(s), \quad s \in [t_0,t_1], \\ y_2(t_1) = x_1, \end{cases} \tag{3.9}$$

and define

$$\bar{x}(s) = y_1(s) + y_2(s), \quad z(s) = z_1(s) + z_2(s), \quad s \in [t_0, t_1].$$

By Lemma 2.1, $y_2(t_0) = \mathbb{E}[\Phi(t_0, t_1)x_1 | \mathcal{F}_{t_0}]$, and thus, by linearity, $(\bar{x}, z, v)$ satisfies

$$\begin{cases} \mathrm{d}\bar{x}(s) = \big[\bar{A}(s)\bar{x}(s) + K(s)z(s) + L(s)v(s)\big]\mathrm{d}s + z(s)\mathrm{d}W(s), \quad s \in [t_0, t_1], \\ \bar{x}(t_0) = x_0, \quad \bar{x}(t_1) = x_1. \end{cases}$$

This shows (t_1, x_1) can be reached from (t_0, x_0) by (z, v).

Necessity. We prove the necessity by contradiction. Suppose that (t_1, x_1) can be reached from (t_0, x_0) by some control (z, v) but there exists some $\Omega' \subseteq \Omega$ with $\mathbb{P}(\Omega') > 0$ such that $x_0(\omega) - \mathbb{E}[\Phi(t_0, t_1)x_1 | \mathcal{F}_{t_0}](\omega)$ does not lie in the range space of $\Psi(t_0, t_1)$ for every $\omega \in \Omega'$. Then we can find a $\beta \in L^2_{\mathcal{F}_{t_0}}(\Omega; \mathbb{R}^n)$ such that

$$\Psi(t_0, t_1)\beta = 0, \text{ a.s.,} \quad \text{and} \quad \mathbb{E}\big(\beta^\top \beta_0\big) > 0,$$

where $\beta_0 = x_0 - \mathbb{E}[\Phi(t_0, t_1)x_1 | \mathcal{F}_{t_0}]$. Let $\bar{x}$ be the corresponding state process. By applying the integration by parts formula to $\Phi\bar{x}$, we have

$$\Phi(t_1)x_1 - \Phi(t_0)x_0 = \int_{t_0}^{t_1} \Phi(s)L(s)v(s)\mathrm{d}s + \int_{t_0}^{t_1} \Phi(s)\big[z(s) - K(s)\bar{x}(s)\big]\mathrm{d}W(s).$$

Taking conditional expectations with respect to $\mathcal{F}_{t_0}$ on both sides of the above, we get

$$-\Phi(t_0)\beta_0 = \mathbb{E}[\Phi(t_1)x_1 | \mathcal{F}_{t_0}] - \Phi(t_0)x_0 = \mathbb{E}\left[\int_{t_0}^{t_1} \Phi(s)L(s)v(s)\mathrm{d}s \middle| \mathcal{F}_{t_0}\right],$$

from which it follows that

$$\begin{aligned} 0 < \mathbb{E}\big(\beta^\top \beta_0\big) &= -\mathbb{E}\left(\beta^\top \Phi(t_0)^{-1}\mathbb{E}\left[\int_{t_0}^{t_1} \Phi(s)L(s)v(s)\mathrm{d}s \middle| \mathcal{F}_{t_0}\right]\right) \\ &= -\mathbb{E}\left(\beta^\top \Phi(t_0)^{-1}\int_{t_0}^{t_1} \Phi(s)L(s)v(s)\mathrm{d}s\right) \\ &= -\mathbb{E}\left(\int_{t_0}^{t_1} \beta^\top \Phi(t_0, s)L(s)v(s)\mathrm{d}s\right). \end{aligned} \tag{3.10}$$

But, using the fact that $\Psi(t_0, t_1)\beta = 0$ a.s. and noting that β is independent of $\Phi(t_0, s)$ for $s \geqslant t_0$, we have

$$\begin{aligned} 0 = \mathbb{E}\big(\beta^\top \Psi(t_0, t_1)\beta\big) &= \mathbb{E}\int_{t_0}^{t_1} \beta^\top \Phi(t_0, s)L(s)[\Phi(t_0, s)L(s)]^\top \beta\mathrm{d}s \\ &= \mathbb{E}\int_{t_0}^{t_1} \big|\beta^\top \Phi(t_0, s)L(s)\big|^2\mathrm{d}s, \end{aligned}$$

which implies the vanishing of $\beta^\top \Phi(t_0, s)L(s)$ and the contradiction of (3.10). □

Remark 3.5. The symmetric positive semidefinite matrix $\Psi(t_0, t_1)$ defined by (3.7) is called the *controllability Gramian* of system (3.6) over $[t_0, t_1]$.

One could compute the controllability Gramian $\Psi(t_0,t_1)$ directly from the formula (3.7). However, this method is not convenient since it involves solving an SDE and taking expectations. The following result provides an alternative method for computing $\Psi(t_0,t_1)$.

Proposition 3.6. *Let $P:[0,t_1]\to\mathbb{S}^n$ be the solution to the linear matrix ordinary differential equation*

$$\begin{cases} \dot P(s)-\bar A(s)P(s)-P(s)\bar A(s)^\top+K(s)P(s)K(s)^\top+L(s)L(s)^\top=0, \quad s\in[0,t_1],\\ P(t_1)=0. \end{cases}$$

Then $\Psi(t_0,t_1)=P(t_0)$ for every $t_0\in[0,t_1]$.

Proof. By the integration by parts formula, we have

$$\begin{aligned} \mathrm{d}\Phi P\Phi^\top &= -\Phi\bar AP\Phi^\top\mathrm{d}s-\Phi KP\Phi^\top\mathrm{d}W+\Phi\dot P\Phi^\top\mathrm{d}s\\ &\quad -\Phi P\bar A^\top\Phi^\top\mathrm{d}s-\Phi PK^\top\Phi^\top\mathrm{d}W+\Phi KPK^\top\Phi^\top\mathrm{d}s\\ &=\Phi(\dot P-\bar AP-P\bar A^\top+KPK^\top)\Phi^\top\mathrm{d}s-\Phi(KP+PK^\top)\mathrm{d}W\\ &=-\Phi LL^\top\Phi^\top\mathrm{d}s-\Phi(KP+PK^\top)\mathrm{d}W. \end{aligned}$$

Note that for simplicity, we have suppressed the argument s in the above. Thus,

$$\Phi(t_0)P(t_0)\Phi(t_0)^\top=\int_{t_0}^{t_1}\Phi(s)L(s)[\Phi(s)L(s)]^\top\mathrm{d}s+\int_{t_0}^{t_1}\Phi(s)[K(s)P(s)+P(s)K(s)^\top]\mathrm{d}W.$$

Taking conditional expectations with respect to $\mathcal{F}_{t_0}$ on both sides yields

$$\Phi(t_0)P(t_0)\Phi(t_0)^\top=\mathbb{E}\left[\int_{t_0}^{t_1}\Phi(s)L(s)[\Phi(s)L(s)]^\top\mathrm{d}s\,\middle|\,\mathcal{F}_{t_0}\right].$$

Pre and post-multiplying this equation by $\Phi(t_0)^{-1}$ and $[\Phi(t_0)^\top]^{-1}$ respectively and then taking expectations on both sides, we obtain

$$P(t_0)=\mathbb{E}\left[\int_{t_0}^{t_1}\Phi(t_0,s)L(s)[\Phi(t_0,s)L(s)]^\top\mathrm{d}s\right]=\Psi(t_0,t_1).$$

This completes the proof. $\square$

Propositions 3.3 and 3.4 have some easy consequences which we summarize as follows.

Corollary 3.7. *Let $0\leqslant t_0<t_1\leqslant T$, and let Φ be the solution to (3.8).*

(i) *System (3.1) is exactly controllable on $[t_0,t_1]$ if and only if system (3.6) is so.*
(ii) *System (3.6) is exactly controllable on $[t_0,t_1]$ if and only if the controllability Gramian $\Psi(t_0,t_1)$ is positive definite.*
(iii) *Let $F\in\mathbb{R}^{k\times n}$ and $b\in L^2_{\mathcal{F}_{t_0}}(\Omega;\mathbb{R}^k)$. For system (3.6), there exists a point on the stochastic linear manifold*

$$\mathcal{H}(F,b)=\{\xi\in L^2_{\mathcal{F}_{t_0}}(\Omega;\mathbb{R}^n):F\xi=b\}$$

that can be transferred to $x_1 \in L^2_{\mathcal{F}_{t_1}}(\Omega;\mathbb{R}^n)$ at time $t = t_1$ if and only if there exist a $\xi \in L^2_{\mathcal{F}_{t_0}}(\Omega;\mathbb{R}^n)$ such that

$$F\Psi(t_0,t_1)\xi = b - F\mathbb{E}[\Phi(t_0,t_1)x_1|\mathcal{F}_{t_0}].$$

Proof. (i) It is a direct consequence of Proposition 3.3.

(ii) If $\Psi(t_0,t_1) > 0$, then obviously, $x_0 - \mathbb{E}[\Phi(t_0,t_1)x_1|\mathcal{F}_{t_0}]$ belongs to $\mathscr{R}(\Psi(t_0,t_1))$, the range of $\Psi(t_0,t_1)$, for all $x_0 \in L^2_{\mathcal{F}_{t_0}}(\Omega;\mathbb{R}^n)$ and all $x_1 \in L^2_{\mathcal{F}_{t_1}}(\Omega;\mathbb{R}^n)$. Thus, by Proposition 3.4, system (3.6) is exactly controllable on $[t_0,t_1]$. Conversely, if system (3.6) is exactly controllable on $[t_0,t_1]$, then for $x_1 = 0$ and any $x_0 \in \mathbb{R}^n$,

$$x_0 = x_0 - \mathbb{E}[\Phi(t_0,t_1)x_1|\mathcal{F}_{t_0}] \in \mathscr{R}(\Psi(t_0,t_1)),$$

which implies that $\Psi(t_0,t_1)$ has full rank and hence is positive definite.

(iii) By Proposition 3.4 we know that a state $x_1 \in L^2_{\mathcal{F}_{t_1}}(\Omega;\mathbb{R}^n)$ can be reached at t_1 from some $x_0 \in \mathcal{H}(F,b)$ if and only if there exists a $\xi \in L^2_{\mathcal{F}_{t_0}}(\Omega;\mathbb{R}^n)$ such that

$$\Psi(t_0,t_1)\xi = x_0 - \mathbb{E}[\Phi(t_0,t_1)x_1|\mathcal{F}_{t_0}].$$

Thus, the state x_1 can be reached from $\mathcal{H}(F,b)$ if and only if the ξ is such that

$$F\{\Psi(t_0,t_1)\xi + \mathbb{E}[\Phi(t_0,t_1)x_1|\mathcal{F}_{t_0}]\} = b.$$

The desired result then follows readily. □

The construction in the proof of Proposition 3.4 actually provides an explicit procedure for finding a control that accomplishes desired transfers. Let us recap and conclude this section.

Proposition 3.8. *Let $0 \leqslant t_0 < t_1 \leqslant T$ and $x_1 \in L^2_{\mathcal{F}_{t_1}}(\Omega;\mathbb{R}^n)$. Let $F \in \mathbb{R}^{k\times n}$ and $b \in L^2_{\mathcal{F}_{t_0}}(\Omega;\mathbb{R}^k)$. If $\xi \in L^2_{\mathcal{F}_{t_0}}(\Omega;\mathbb{R}^n)$ is such that*

$$F\Psi(t_0,t_1)\xi = b - F\mathbb{E}[\Phi(t_0,t_1)x_1|\mathcal{F}_{t_0}], \tag{3.11}$$

then with

$$v(s) \triangleq -L(s)^\top\Phi(t_0,s)^\top\xi, \quad s \in [t_0,t_1], \tag{3.12}$$

and $z = \{z(s); t_0 \leqslant s \leqslant t_1\}$ being the second component of the adapted solution to the BSDE

$$\begin{cases} \mathrm{d}y(s) = \big[\bar{A}(s)y(s) + K(s)z(s) + L(s)v(s)\big]\mathrm{d}s + z(s)\mathrm{d}W(s), \quad s \in [t_0,t_1], \\ y(t_1) = x_1, \end{cases}$$

(z,v) transfers the state of the system (3.6) from

$$x_0 = \Psi(t_0,t_1)\xi + \mathbb{E}[\Phi(t_0,t_1)x_1|\mathcal{F}_{t_0}] \in \mathcal{H}(F,b)$$

at $t = t_0$ to x_1 at $t = t_1$.

It is worth pointing out that (3.11) is simply a matrix equation. Since F, b, and x_1 are given, once the matrices $\Phi(t_0,t_1)$ and $\Psi(t_0,t_1)$ are computed, the solution ξ of (3.11) can be obtained by the usual method for solving matrix equations. Here is an example.

Example 3.9. Consider the system (3.6) with

$$\bar{A}=\begin{bmatrix}-2 & 1\\ 0 & -2\end{bmatrix},\quad K=\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix},\quad L=\begin{bmatrix}0\\ 1\end{bmatrix}.$$

In order to compute $\Phi(t_0,t_1)\triangleq\Phi(t_0)^{-1}\Phi(t_1)$, we observe first that $X\triangleq\Phi^{-1}$ satisfies the following linear matrix SDE:

$$\begin{cases}\mathrm{d}x(t)=(\bar{A}+K^2)X(t)\mathrm{d}t+KX(t)\mathrm{d}W(t), & t\in[0,T],\\ X(0)=I_2.\end{cases}\tag{3.13}$$

It is easy to see that

$$\bar{A}+K^2=\begin{bmatrix}2 & 1\\ 0 & 2\end{bmatrix},\quad e^{(\bar{A}+K^2)t}=e^{2t}\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}.$$

Thus, the solution of (3.13) is given by

$$X(t)=e^{(\bar{A}+K^2)t}e^{2W(t)-2t}=e^{2W(t)}\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix},$$

and hence

$$\Phi(t)=X(t)^{-1}=e^{-2W(t)}\begin{bmatrix}1 & -t\\ 0 & 1\end{bmatrix}.$$

Now it is not hard to see that

$$\Phi(t_0,s)\triangleq\Phi(t_0)^{-1}\Phi(s)=e^{2[W(t_0)-W(s)]}\begin{bmatrix}1 & t_0-s\\ 0 & 1\end{bmatrix}.\tag{3.14}$$

In particular,

$$\Phi(t_0,t_1)\triangleq\Phi(t_0)^{-1}\Phi(t_1)=e^{2[W(t_0)-W(t_1)]}\begin{bmatrix}1 & t_0-t_1\\ 0 & 1\end{bmatrix}.$$

Using (3.14), it is straightforward to compute

$$\begin{aligned}\mathbb{E}\Big[\Phi(t_0,s)LL^\top\Phi(t_0,s)^\top\Big]&=\mathbb{E}e^{4[W(t_0)-W(s)]}\begin{bmatrix}(t_0-s)^2 & t_0-s\\ t_0-s & 1\end{bmatrix}\\ &=e^{8(s-t_0)}\begin{bmatrix}(t_0-s)^2 & t_0-s\\ t_0-s & 1\end{bmatrix}.\end{aligned}$$

So by using the integration by parts formula, we obtain

$$\Psi(t_0,t_1)=\int_{t_0}^{t_1}\mathbb{E}\Big[\Phi(t_0,s)LL^\top\Phi(t_0,s)^\top\Big]\mathrm{d}s=\begin{bmatrix}\Psi_{11} & \Psi_{12}\\ \Psi_{12} & \Psi_{22}\end{bmatrix},$$

where

$$\Psi_{11} = \left[\frac{1}{8}(t_1 - t_0)^2 - \frac{1}{32}(t_1 - t_0) + \frac{1}{256}\right] e^{8(t_1-t_0)} - \frac{1}{256},$$
$$\Psi_{12} = -\left[\frac{1}{8}(t_1 - t_0) - \frac{1}{64}\right] e^{8(t_1-t_0)} - \frac{1}{64},$$
$$\Psi_{22} = \frac{1}{8}e^{8(t_1-t_0)} - \frac{1}{8}.$$

4. Lagrange multipliers and unconstrained backward LQ problems

We now return to Problem (CLQ). Recall that the nondegeneracy condition (3.2) is assumed so that the target η can be reached from a given stochastic linear manifold $\mathcal{H}(F,b)$. Let M be as in (3.3) and $\bar{A}$, K, L be as in (3.5). We have seen from Proposition 3.3 that systems (3.1) and (3.6) share the same controllability. So by appropriate transformations, we may assume without loss of generality that the state equation (1.1) takes the form

$$\mathrm{d}x(s) = [A(s)x(s) + K(s)z(s) + L(s)v(s)]\mathrm{d}s + z(s)\mathrm{d}W(s), \quad s \in [t,T], \tag{4.1}$$

and that the cost functional (1.2) takes the following form:

$$\begin{aligned} J(t,z,v) = \mathbb{E}\bigg\{ &\langle Gx(t), x(t)\rangle + \int_t^T \Big[\langle Q(s)x(s), x(s)\rangle \\ &+ \langle R(s)z(s), z(s)\rangle + \langle N(s)v(s), v(s)\rangle\Big]\mathrm{d}s\bigg\}. \end{aligned} \tag{4.2}$$

That is, the coefficients B and D of (1.1) are given by

$$B(s) = (K(s), L(s)), \quad D(s) = (I_n, 0_{n\times(m-n)}); \quad s \in [0,T],$$

and the control u is $\begin{bmatrix} z \\ v \end{bmatrix}$. In this case, with the given terminal state η, we may think of v alone as the control and regard (x,z) as the adapted solution to the BSDE

$$\begin{cases} \mathrm{d}x(s) = [A(s)x(s) + K(s)z(s) + L(s)v(s)]\mathrm{d}s + z(s)\mathrm{d}W(s), \quad s \in [t,T], \\ x(T) = \eta. \end{cases} \tag{4.3}$$

Further, since for given η, z is uniquely decided by v, we can simply write the cost functional (4.2) as $J(t,v)$. Therefore, solving Problem (CLQ) is equivalent to finding an optimal control v^* for the following constrained backward LQ problem.

Problem (CBLQ). For a given terminal state $\eta \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$, find a control $v^* \in L^2_{\mathbb{F}}(t,T;\mathbb{R}^{m-n})$ such that the corresponding adapted solution (x^*,z^*) of (4.3) satisfies $x^*(t) \in \mathcal{H}(F,b)$, and

$$J(t,v^*) \leqslant J(t,v), \quad \forall v \in L^2_{\mathbb{F}}(t,T;\mathbb{R}^{m-n}). \tag{4.4}$$

For this reduced problem, we impose the following assumptions that are similar to the conditions (A1) and (A2).

(H1) $A, K : [0, T] \to \mathbb{R}^{n\times n}$ and $L : [0, T] \to \mathbb{R}^{n\times(m-n)}$ are bounded measurable functions.
(H2) G is a symmetric $n \times n$ matrix; $Q, R : [0, T] \to \mathbb{R}^{n\times n}$ and $N : [0, T] \to \mathbb{R}^{(m-n)\times(m-n)}$ are bounded and symmetric. Moreover, for some $\delta > 0$,

$$G \geqslant 0, \quad Q(s) \geqslant 0, \quad R(s) \geqslant 0, \quad N(s) \geqslant \delta I_{m-n}, \quad \text{a.e. } s \in [0, T].$$

To find an optimal control for Problem (CBLQ), let $\lambda \in L^2_{\mathcal{F}_t}(\Omega; \mathbb{R}^k)$ be undetermined and define

$$\begin{aligned} J_\lambda(t, v) &\triangleq J(t, v) + 2\mathbb{E}\langle F^\top \lambda, x(t)\rangle \\ &= \mathbb{E}\bigg\{\langle Gx(t), x(t)\rangle + 2\mathbb{E}\langle F^\top\lambda, x(t)\rangle + \int_t^T \Big[\langle Q(s)x(s), x(s)\rangle \\ &\qquad + \langle R(s)z(s), z(s)\rangle + \langle N(s)v(s), v(s)\rangle\Big]\mathrm{d}s\bigg\}. \end{aligned} \tag{4.5}$$

Consider the following parameterized unconstrained backward LQ problem.

Problem (BLQ)$_\lambda$. For a given terminal state $\eta \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$, find a control $v^* \in L^2_{\mathbb{F}}(t, T; \mathbb{R}^{m-n})$ such that

$$J_\lambda(t, v^*) \leqslant J_\lambda(t, v), \quad \forall v \in L^2_{\mathbb{F}}(t, T; \mathbb{R}^{m-n}), \tag{4.6}$$

subject to the backward state equation (4.3).

If for some parameter $\lambda \in L^2_{\mathcal{F}_t}(\Omega; \mathbb{R}^k)$, the optimal control v^*_λ of Problem (BLQ)$_\lambda$ is such that the initial state of system (4.3) falls on the stochastic linear manifold $\mathcal{H}(F, b)$, then intuitively we can convince ourselves that v^*_λ is also optimal for Problem (CBLQ). In fact, we have the following result.

Proposition 4.1. *Let (H1)–(H2) hold, and let $\eta \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$ be given. If v^*_λ is an optimal control of Problem (BLQ)$_\lambda$ such that the adapted solution $(x^*_\lambda, z^*_\lambda)$ of*

$$\begin{cases} \mathrm{d}x^*_\lambda(s) = [A(s)x^*_\lambda(s) + K(s)z^*_\lambda(s) + L(s)v^*_\lambda(s)]\mathrm{d}s + z^*_\lambda(s)\mathrm{d}W(s), \quad s \in [t, T], \\ x^*_\lambda(T) = \eta, \end{cases}$$

satisfies $x^*_\lambda(t) \in \mathcal{H}(F, b)$, then v^*_λ is also optimal for Problem (CBLQ).

Proof. Since v^*_λ is optimal for Problem (BLQ)$_\lambda$, (4.6) holds. In particular, for any $v \in L^2_{\mathbb{F}}(t, T; \mathbb{R}^{m-n})$ such that the initial state of system (4.3) falls on $\mathcal{H}(F, b)$, we have

$$\begin{aligned} &J(t, v^*_\lambda) + 2\mathbb{E}\langle F^\top\lambda, x^*_\lambda(t)\rangle = J_\lambda(t, v^*_\lambda) \leqslant J_\lambda(t, v) = J(t, v) + 2\mathbb{E}\langle F^\top\lambda, x(t)\rangle, \\ &Fx(t) = Fx^*_\lambda(t) = b, \end{aligned}$$

from which it follows that

$$\begin{aligned} J(t, v^*_\lambda) &\leqslant J(t, v) + 2\mathbb{E}\langle F^\top\lambda, x(t) - x^*_\lambda(t)\rangle \\ &= J(t, v) + 2\mathbb{E}\langle \lambda, F[x(t) - x^*_\lambda(t)]\rangle = J(t, v). \end{aligned}$$

This completes the proof. □

According to Proposition 4.1, the procedure for finding the optimal control of our original Problem (CBLQ) can be divided into two steps.

Step 1. Construct the optimal control v^*_λ for the parameterized unconstrained backward LQ problem.

Step 2. Select the parameter λ such that the corresponding optimal state process x^*_λ of Problem $(\mathrm{BLQ})_\lambda$ satisfies $x^*_\lambda(t) \in \mathcal{H}(F,b)$.

For Step 1, we first present the following result, which characterizes the optimal control of Problem $(\mathrm{BLQ})_\lambda$ in terms of FBSDEs.

Theorem 4.2. *Let (H1)–(H2) hold. Let $\lambda \in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^k)$ and $\eta \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$ be given. Then a control $v^* \in L^2_{\mathbb{F}}(t,T;\mathbb{R}^{m-n})$ is optimal for Problem (BLQ)$_\lambda$ if and only if the adapted solution (x^*, z^*, y^*) to the coupled FBSDE*

$$\begin{cases} \mathrm{d}x^*(s) = (Ax^* + Kz^* + Lv^*)\mathrm{d}s + z^*\mathrm{d}W(s), \\ \mathrm{d}y^*(s) = (-A^\top y^* + Qx^*)\mathrm{d}s + (-K^\top y^* + Rz^*)\mathrm{d}W(s), \\ x^*(T) = \eta, \quad y^*(t) = Gx^*(t) + F^\top \lambda, \end{cases} \tag{4.7}$$

satisfies the following stationarity condition:

$$Nv^* - L^\top y^* = 0, \quad \text{a.e. } on\ [t,T], \ \text{a.s.} \tag{4.8}$$

Proof. First note that v^* is optimal if and only if

$$J_\lambda(t, v^* + \varepsilon v) - J_\lambda(t, v^*) \geqslant 0, \quad \forall \varepsilon \in \mathbb{R}, \ \forall v \in L^2_{\mathbb{F}}(t,T;\mathbb{R}^{m-n}). \tag{4.9}$$

For fixed but arbitrary $\varepsilon \in \mathbb{R}$ and $v \in L^2_{\mathbb{F}}(t,T;\mathbb{R}^{m-n})$, we have by linearity that the adapted solution $(x_\varepsilon, z_\varepsilon)$ to

$$\begin{cases} \mathrm{d}x_\varepsilon(s) = [Ax_\varepsilon + Kz_\varepsilon + L(v^* + \varepsilon v)]\mathrm{d}s + z_\varepsilon \mathrm{d}W(s), \quad s \in [t,T], \\ x_\varepsilon(T) = \eta, \end{cases}$$

is the sum of (x^*, z^*) and $\varepsilon(x,z)$, where (x,z) is the adapted solution to

$$\begin{cases} \mathrm{d}x(s) = (Ax + Kz + Lv)\mathrm{d}s + z\mathrm{d}W(s), \quad s \in [t,T], \\ x(T) = 0. \end{cases}$$

Then it follows by a straightforward computation that

$$\begin{aligned} J_\lambda(t, v^* + \varepsilon v) = {} & \varepsilon^2 \mathbb{E}\Big[\langle Gx(t), x(t)\rangle + \int_t^T \Big(\langle Qx, x\rangle + \langle Rz, z\rangle + \langle Nv, v\rangle\Big)\mathrm{d}s\Big] \\ & + 2\varepsilon \mathbb{E}\Big[\langle Gx^*(t) + F^\top \lambda, x(t)\rangle + \int_t^T \Big(\langle Qx^*, x\rangle + \langle Rz^*, z\rangle + \langle Nv^*, v\rangle\Big)\mathrm{d}s\Big] \\ & + J_\lambda(t, v^*). \end{aligned}$$

Thus, (4.9) in turn is equivalent to

$$\begin{aligned} & \varepsilon^2 \mathbb{E}\Big[\langle Gx(t), x(t)\rangle + \int_t^T \Big(\langle Qx, x\rangle + \langle Rz, z\rangle + \langle Nv, v\rangle\Big)\mathrm{d}s\Big] \\ & + 2\varepsilon \mathbb{E}\Big[\langle Gx^*(t) + F^\top \lambda, x(t)\rangle + \int_t^T \Big(\langle Qx^*, x\rangle + \langle Rz^*, z\rangle + \langle Nv^*, v\rangle\Big)\mathrm{d}s\Big] \geqslant 0 \end{aligned} \tag{4.10}$$

for all $\varepsilon \in \mathbb{R}$ and all $v \in L^2_{\mathbb{F}}(t, T; \mathbb{R}^{m-n})$. Since the term in the first square bracket is nonnegative by the assumption (H2), (4.10) holds for all $\varepsilon \in \mathbb{R}$ if and only if

$$\mathbb{E}\bigg[\langle Gx^*(t) + F^\top\lambda, x(t)\rangle + \int_t^T \Big(\langle Qx^*, x\rangle + \langle Rz^*, z\rangle + \langle Nv^*, v\rangle\Big)\mathrm{d}s\bigg] = 0. \tag{4.11}$$

Now by applying Itô's rule to $s \mapsto \langle y^*(s), x(s)\rangle$, we obtain

$$\mathbb{E}\langle Gx^*(t) + F^\top\lambda, x(t)\rangle = \mathbb{E}\langle y^*(t), x(t)\rangle = -\mathbb{E}\int_t^T \Big(\langle Qx^*, x\rangle + \langle L^\top y^*, v\rangle + \langle Rz^*, z\rangle\Big)\mathrm{d}s,$$

substituting which into (4.11) yields

$$\mathbb{E}\int_t^T \langle Nv^* - L^\top y^*, v\rangle \mathrm{d}s = 0.$$

Since the above has to be true for all $v \in L^2_{\mathbb{F}}(t, T; \mathbb{R}^{m-n})$, (4.8) follows. The sufficiency of (4.8) can be proved by reversing the above argument. □

We call (4.7), together with the stationarity condition (4.8), the *optimality system* for Problem $(\text{BLQ})_\lambda$. Note that from (4.8) we can represent the optimal control v^* in terms of y^* as $v^* = N^{-1}L^\top y^*$. Substituting for v^* then brings a coupling into the FBSDE (4.7). So in order to find the optimal control v^*, one actually need to solve a coupled FBSDE.

To construct an optimal control for Problem $(\text{BLQ})_\lambda$ from the optimality system (4.7)-(4.8), we now introduce the following Riccati-type equation:

$$\begin{cases} \dot{\Sigma} - \Sigma A^\top - A\Sigma - \Sigma Q\Sigma + LN^{-1}L^\top + K(I_n + \Sigma R)^{-1}\Sigma K^\top = 0, \quad s \in [0, T], \\ \Sigma(T) = 0. \end{cases} \tag{4.12}$$

It was shown in [7] (see also [6] for an alternative proof) that equation (4.12) has a unique positive semidefinite solution $\Sigma \in C([0, T]; \mathbb{S}^n)$:

$$\Sigma(s)^\top = \Sigma(s), \quad \Sigma(s) \geqslant 0; \quad \forall s \in [0, T].$$

This allows us to consider the following BSDE:

$$\begin{cases} \mathrm{d}\varphi(s) = [(A + \Sigma Q)\varphi + K(I_n + \Sigma R)^{-1}\beta]\mathrm{d}s + \beta \mathrm{d}W(s), \quad s \in [0, T], \\ \varphi(T) = -\eta, \end{cases} \tag{4.13}$$

which, by the standard result for BSDEs, admits a unique adapted solution

$$(\varphi, \beta) \in L^2_{\mathbb{F}}(\Omega; C([0, T]; \mathbb{R}^n)) \times L^2_{\mathbb{F}}(0, T; \mathbb{R}^n).$$

Consider further the following $(\varphi, \beta, \lambda)$-dependent SDE:

$$\begin{cases} \mathrm{d}y(s) = -[(A^\top + Q\Sigma)y + Q\varphi]\mathrm{d}s - (I_n + R\Sigma)^{-1}(K^\top y + R\beta)\mathrm{d}W(s), \; s \in [t, T], \\ y(t) = [I_n + G\Sigma(t)]^{-1}[F^\top\lambda - G\varphi(t)]. \end{cases} \tag{4.14}$$

Obviously, (4.14) is uniquely solvable.

Theorem 4.3. *Let (H1)–(H2) hold. Then Problem (BLQ)$_\lambda$ admits a unique optimal control which is given by*

$$v_\lambda^*(s) = N(s)^{-1}L(s)^\top y(s), \quad s \in [t,T],$$

where y is the solution to the SDE (4.14).

Proof. Let (x,z) be the adapted solution to the BSDE

$$\begin{cases} \mathrm{d}x(s) = (Ax + Kz + Lv_\lambda^*)\mathrm{d}s + z\mathrm{d}W(s), \quad s \in [t,T], \\ x(T) = \eta. \end{cases}$$

According to Theorem 4.2, it suffices to verify that the solution y of (4.14) satisfies the SDE

$$\begin{cases} \mathrm{d}y(s) = (-A^\top y + Qx)\mathrm{d}s + (-K^\top y + Rz)\mathrm{d}W(s), \quad s \in [t,T], \\ y(t) = Gx(t) + F^\top \lambda. \end{cases}$$

This can be accomplished if we are able to show that

$$x(s) = -[\Sigma(s)y(s) + \varphi(s)], \quad z(s) = [I_n + \Sigma(s)R(s)]^{-1}[\Sigma(s)K(s)^\top y(s) - \beta(s)]. \tag{4.15}$$

Indeed, if (4.15) holds, then the first relation gives

$$Gx(t) + F^\top\lambda = -G\Sigma(t)y(t) - G\varphi(t) + F^\top\lambda,$$

which, together with the initial condition in (4.14), implies that

$$\begin{aligned} y(t) &= -G\Sigma(t)y(t) + [I_n + G\Sigma(t)]y(t) = -G\Sigma(t)y(t) + F^\top\lambda - G\varphi(t) \\ &= Gx(t) + F^\top\lambda. \end{aligned}$$

Furthermore,

$$-[(A^\top + Q\Sigma)y + Q\varphi] = -A^\top y + Qx,$$

and using the second relation in (4.15) we obtain

$$\begin{aligned} K^\top y - (I_n + R\Sigma)^{-1}(K^\top y + R\beta) &= [I_n - (I_n + R\Sigma)^{-1}]K^\top y - (I_n + R\Sigma)^{-1}R\beta \\ &= (I_n + R\Sigma)^{-1}R\Sigma K^\top y - (I_n + R\Sigma)^{-1}R\beta \\ &= (I_n + R\Sigma)^{-1}R(\Sigma K^\top y - \beta) \\ &= R(I_n + \Sigma R)^{-1}(\Sigma K^\top y - \beta) \\ &= Rz, \end{aligned}$$

and hence $-(I_n + R\Sigma)^{-1}(K^\top y + R\beta) = -K^\top y + Rz$.

In order to prove (4.15), let us denote

$$\hat{x}(s) \triangleq -[\Sigma(s)y(s) + \varphi(s)], \quad \hat{z}(s) \triangleq [I_n + \Sigma(s)R(s)]^{-1}[\Sigma(s)K(s)^\top y(s) - \beta(s)].$$

Thanks to the uniqueness of an adapted solution, our proof will be complete if we can show that $(\hat{x}, \hat{z})$ satisfies the same BSDE as (x, z). To this end, we first note that $\hat{x}(T) = -[\Sigma(T)y(T) + \varphi(T)] = \eta$. Moreover, by Itô's rule,

$$\begin{aligned}
\mathrm{d}\hat{x} &= \mathrm{d}(-\Sigma y - \varphi) = -\dot{\Sigma} y \mathrm{d}s - \Sigma \mathrm{d}y - \mathrm{d}\varphi \\
&= -\dot{\Sigma} y \mathrm{d}s + \Sigma[(A^\top + Q\Sigma)y + Q\varphi]\mathrm{d}s + \Sigma(I_n + R\Sigma)^{-1}(K^\top y + R\beta)\mathrm{d}W \\
&\quad -[(A + \Sigma Q)\varphi + K(I_n + \Sigma R)^{-1}\beta]\mathrm{d}s - \beta \mathrm{d}W \\
&= [(-\dot{\Sigma} + \Sigma A^\top + \Sigma Q \Sigma)y - A\varphi - K(I_n + \Sigma R)^{-1}\beta]\mathrm{d}s \\
&\quad +\{\Sigma(I_n + R\Sigma)^{-1}K^\top y + [\Sigma(I_n + R\Sigma)^{-1}R - I_n]\beta\}\mathrm{d}W.
\end{aligned} \tag{4.16}$$

Using (4.12), we can rewrite the drift term in (4.16) as

$$\begin{aligned}
&(-\dot{\Sigma} + \Sigma A^\top + \Sigma Q \Sigma)y - A\varphi - K(I_n + \Sigma R)^{-1}\beta \\
&\quad = [-A\Sigma + LN^{-1}L^\top + K(I_n + \Sigma R)^{-1}\Sigma K^\top]y - A\varphi - K(I_n + \Sigma R)^{-1}\beta \\
&\quad = -A(\Sigma y + \varphi) + LN^{-1}L^\top y + K(I_n + \Sigma R)^{-1}(\Sigma K^\top y - \beta) \\
&\quad = A\hat{x} + Lv_\lambda^* + K\hat{z}.
\end{aligned}$$

Using the fact that

$$\Sigma(I_n + R\Sigma)^{-1} = (I_n + \Sigma R)^{-1}\Sigma, \quad \Sigma(I_n + R\Sigma)^{-1}R - I_n = -(I_n + \Sigma R)^{-1},$$

we can rewrite the diffusion term in (4.16) as

$$\begin{aligned}
&\Sigma(I_n + R\Sigma)^{-1}K^\top y + [\Sigma(I_n + R\Sigma)^{-1}R - I_n]\beta \\
&\quad = (I_n + \Sigma R)^{-1}\Sigma K^\top y - (I_n + \Sigma R)^{-1}\beta \\
&\quad = \hat{z}.
\end{aligned}$$

This shows that $(\hat{x}, \hat{z})$ satisfies the same BSDE as (x, z) and hence completes the proof. □

5. Selection of optimal parameters

In this section we show how to find a $\lambda \in L^2_{\mathcal{F}_t}(\Omega; \mathbb{R}^k)$, called an *optimal parameter*, such that the corresponding optimal state process of Problem $(\mathrm{BLQ})_\lambda$ satisfies $x_\lambda^*(t) \in \mathcal{H}(F, b)$. It is worth pointing out that the usual method of Lagrange multipliers does not work efficiently in our situation, due to the difficulty in computing the derivative of $J_\lambda(t, v_\lambda^*)$ in λ. The key of our approach is to establish an equivalence relationship between the controllability of (4.1) and a system involving Σ, the solution of the Riccati equation (4.12). It turns out that an optimal parameter exists and can be obtained by solving a matrix equation.

Recall that Σ and (φ, β) are the unique solutions to equations (4.12) and (4.13), respectively. The main result of this section can be stated as follows.

Theorem 5.1. *Let (H1)–(H2) hold. If the state of system (4.1) can be transferred to (T, η) from the stochastic linear manifold $\mathcal{H}(F, b)$, then the matrix equation*

$$\Big\{F[I_n + \Sigma(t)G]^{-1}\Sigma(t)F^\top\Big\}\lambda = -\Big\{F[I_n + \Sigma(t)G]^{-1}\varphi(t) + b\Big\} \tag{5.1}$$

has a solution. Moreover, every solution λ^* *of* (5.1) *is an optimal parameter, and the optimal controls* v^* *of Problem (CBLQ) are given by*

$$v^*(s)=N(s)^{-1}L(s)^\top y^*(s),\quad s\in[t,T],$$

where y^* *is the solution of*

$$\begin{cases}\mathrm{d}y^*(s)=-[(A^\top+Q\Sigma)y^*+Q\varphi]\mathrm{d}s-(I_n+R\Sigma)^{-1}(K^\top y^*+R\beta)\mathrm{d}W,\ s\in[t,T],\\ \ y^*(t)=[I_n+G\Sigma(t)]^{-1}[F^\top\lambda^*-G\varphi(t)].\end{cases}\tag{5.2}$$

Remark 5.2. By using the pseudoinverse of a matrix, we can give an explicit representation of the solutions to (5.1). For simplicity of notation, let

$$\Lambda=F[I_n+\Sigma(t)G]^{-1}\Sigma(t)F^\top,\quad \alpha=-F[I_n+\Sigma(t)G]^{-1}\varphi(t)-b.$$

Then the solutions (provided a solution exists) to (5.1) are given by

$$\lambda=\Lambda^\dagger\alpha+(I-\Lambda^\dagger\Lambda)\rho,$$

where $\Lambda^\dagger$ is the Moore–Penrose inverse of Λ, and $\rho\in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^k)$ is arbitrary.

In preparation for the proof of Theorem 5.1, let us consider the following system:

$$\mathrm{d}\hat{x}(s)=[\widehat{A}(s)\hat{x}(s)+\widehat{K}(s)\hat{z}(s)+\widehat{L}(s)\hat{v}(s)]\mathrm{d}s+\hat{z}(s)\mathrm{d}W(s),\tag{5.3}$$

where the coefficients are given by

$$\widehat{A}=A+\Sigma Q,\quad \widehat{K}=K(I_n+\Sigma R)^{-1},\quad \widehat{L}=(LN^{-\frac{1}{2}},\,-\Sigma Q^{\frac{1}{2}},\,K(I_n+\Sigma R)^{-1}\Sigma R^{\frac{1}{2}}).\tag{5.4}$$

The following result shows that the controllability of system (4.1) is equivalent to that of system (5.3).

Proposition 5.3. *Let (H1)–(H2) hold. Let* $0\leqslant t_0<t_1\leqslant T$, $x_0\in L^2_{\mathcal{F}_{t_0}}(\Omega;\mathbb{R}^n)$ *and* $x_1\in L^2_{\mathcal{F}_{t_1}}(\Omega;\mathbb{R}^n)$*. A control* $(z,v)\in L^2_{\mathbb{F}}(t_0,t_1;\mathbb{R}^n)\times L^2_{\mathbb{F}}(t_0,t_1;\mathbb{R}^{m-n})$ *transfers* (t_0,x_0) *to* (t_1,x_1) *for system* (4.1) *if and only if the control* $(\hat{z},\hat{v})$ *defined by*

$$\hat{z}(s)\triangleq z(s),\quad \hat{v}(s)\triangleq\begin{bmatrix}[N(s)]^{\frac{1}{2}}v(s)\\ [Q(s)]^{\frac{1}{2}}x(s)\\ [R(s)]^{\frac{1}{2}}z(s)\end{bmatrix};\quad s\in[t_0,t_1]\tag{5.5}$$

does so for system (5.3)*, where* x *is the solution of* (4.1) *with respect to the initial pair* (t_0,x_0) *and the control* (z,v)*.*

Proof. Let $(\hat{z},\hat{v})$ be defined by (5.5) and $\hat{x}$ be the solution to

$$\begin{cases}\mathrm{d}\hat{x}(s)=[\widehat{A}(s)\hat{x}(s)+\widehat{K}(s)\hat{z}(s)+\widehat{L}(s)\hat{v}(s)]\mathrm{d}s+\hat{z}(s)\mathrm{d}W(s),\quad s\in[t_0,t_1],\\ \hat{x}(t_0)=x_0.\end{cases}\tag{5.6}$$

We prove the assertion by showing $\hat{x} = x$. Substituting (5.4) and (5.5) into (5.6), we have

$$\begin{cases} \mathrm{d}\hat{x}(s) = [A\hat{x} + \Sigma Q(\hat{x} - x) + Kz + Lv]\mathrm{d}s + z\mathrm{d}W(s), \quad s \in [t_0, t_1], \\ \hat{x}(t_0) = x_0. \end{cases} \tag{5.7}$$

Clearly, x is also a solution of (5.7) and hence $x = \hat{x}$ by the uniqueness of a solution. □

Although the system (5.3) looks more complicated than (4.1), the controllability Gramian of (5.3) takes a simpler form, as shown by the following result.

Proposition 5.4. *Let (H1)–(H2) hold. Then for every $t \in [0, T]$, the controllability Gramian of system (5.3) over $[t, T]$ is $\Sigma(t)$.*

Proof. Let $\Pi = \{\Pi(s); 0 \leqslant s \leqslant T\}$ be the solution to the following SDE for $\mathbb{R}^{n\times n}$-valued processes:

$$\begin{cases} \mathrm{d}\Pi(s) = -\Pi(s)\widehat{A}(s)\mathrm{d}s - \Pi(s)\widehat{K}(s)\mathrm{d}W(s), \quad s \in [0, T], \\ \Pi(0) = I_n, \end{cases} \tag{5.8}$$

and let $\Pi(t, s) = \Pi(t)^{-1}\Pi(s)$. By Proposition 3.4, the controllability Gramian of system (5.3) over $[t, T]$ is

$$\mathbb{E}\left\{\int_t^T \Pi(t, s)\widehat{L}(s)\big[\Pi(t, s)\widehat{L}(s)\big]^\top \mathrm{d}s\right\}.$$

On the other hand, we have by Itô's rule that

$$\begin{aligned} \mathrm{d}\Big(\Pi\Sigma\Pi^\top\Big) &= -\Pi(A + \Sigma Q)\Sigma\Pi^\top \mathrm{d}s - \Pi K(I_n + \Sigma R)^{-1}\Sigma\Pi^\top \mathrm{d}W \\ &\quad + \Pi\dot{\Sigma}\Pi^\top \mathrm{d}s - \Pi\Sigma(A + \Sigma Q)^\top\Pi^\top \mathrm{d}s - \Pi\Sigma(I_n + R\Sigma)^{-1}K^\top\Pi^\top \mathrm{d}W \\ &\quad + \Pi K(I_n + \Sigma R)^{-1}\Sigma(I_n + R\Sigma)^{-1}K^\top\Pi^\top \mathrm{d}s \\ &= -\Pi\Big[(A + \Sigma Q)\Sigma - \dot{\Sigma} + \Sigma(A + \Sigma Q)^\top \\ &\quad - K(I_n + \Sigma R)^{-1}\Sigma(I_n + R\Sigma)^{-1}K^\top\Big]\Pi^\top \mathrm{d}s \\ &\quad - \Pi\Big[K(I_n + \Sigma R)^{-1}\Sigma + \Sigma(I_n + R\Sigma)^{-1}K^\top\Big]\Pi^\top \mathrm{d}W \\ &= -\Pi\Big[LN^{-1}L^\top + \Sigma Q\Sigma + K(I_n + \Sigma R)^{-1}\Sigma K^\top \\ &\quad - K(I_n + \Sigma R)^{-1}\Sigma(I_n + R\Sigma)^{-1}K^\top\Big]\Pi^\top \mathrm{d}s \\ &\quad - \Pi\Big[K(I_n + \Sigma R)^{-1}\Sigma + \Sigma(I_n + R\Sigma)^{-1}K^\top\Big]\Pi^\top \mathrm{d}W. \end{aligned}$$

Integration from t to T and then taking conditional expectations with respect to $\mathcal{F}_t$ on both sides, we obtain

$$\begin{aligned} \Pi(t)\Sigma(t)\Pi(t)^\top = \mathbb{E}\bigg\{\int_t^T \Pi\Big[&LN^{-1}L^\top + \Sigma Q\Sigma + K(I_n + \Sigma R)^{-1}\Sigma K^\top \\ &- K(I_n + \Sigma R)^{-1}\Sigma(I_n + R\Sigma)^{-1}K^\top\Big]\Pi^\top \mathrm{d}s\,\Big|\,\mathcal{F}_t\bigg\}. \end{aligned}$$

Observe that

$$
\begin{aligned}
&LN^{-1}L^\top + \Sigma Q \Sigma + K(I_n + \Sigma R)^{-1}\Sigma K^\top - K(I_n + \Sigma R)^{-1}\Sigma(I_n + R\Sigma)^{-1}K^\top \\
&\quad = LN^{-1}L^\top + \Sigma Q \Sigma + K\Big[(I_n + \Sigma R)^{-1}\Sigma R \Sigma (I_n + R\Sigma)^{-1}\Big]K^\top \\
&\quad = \widehat{L}\widehat{L}^\top,
\end{aligned}
$$

and that $\Pi(t,s)$ is independent of $\mathcal{F}_t$ for $s \geqslant t$. Then we have

$$
\begin{aligned}
\Sigma(t) &= \Pi(t)^{-1}\left\{\mathbb{E}\int_t^T \Pi(s)\widehat{L}(s)\widehat{L}(s)^\top \Pi(s)^\top \mathrm{d}s \bigg| \mathcal{F}_t\right\}\Big[\Pi(t)^{-1}\Big]^\top \\
&= \mathbb{E}\left\{\int_t^T \Pi(t,s)\widehat{L}(s)\big[\Pi(t,s)\widehat{L}(s)\big]^\top \mathrm{d}s \bigg| \mathcal{F}_t\right\} \\
&= \mathbb{E}\left\{\int_t^T \Pi(t,s)\widehat{L}(s)\big[\Pi(t,s)\widehat{L}(s)\big]^\top \mathrm{d}s\right\}.
\end{aligned}
$$

This completes the proof. □

Proof of Theorem 5.1. First note that the state of system (5.3) can also be transferred to (T,η) from the stochastic linear manifold $\mathcal{H}(F,b)$ (Prop. 5.3) and that the controllability Gramian of system (5.3) over $[t,T]$ is $\Sigma(t)$ (Prop. 5.4). Thus, by Corollary 3.7 (iii), there exists a $\xi \in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^k)$ satisfying

$$
F\Sigma(t)\xi = b - F\mathbb{E}[\Pi(t,T)\eta|\mathcal{F}_t], \tag{5.9}
$$

where $\Pi(t,T) = \Pi(t)^{-1}\Pi(T)$ with $\Pi = \{\Pi(s); 0 \leqslant s \leqslant T\}$ being the solution of (5.8). Applying Lemma 2.1 to the BSDE (4.13), we obtain

$$
\mathbb{E}[\Pi(t,T)\eta|\mathcal{F}_t] = -\varphi(t),
$$

and hence (5.9) becomes

$$
F\Sigma(t)\xi = F\varphi(t) + b.
$$

Now using the identity

$$
I_n - \Sigma(t)[I_n + G\Sigma(t)]^{-1}G = [I_n + \Sigma(t)G]^{-1},
$$

it is straightforward to verify that

$$
\lambda = [I_n + G\Sigma(t)]\xi - G\varphi(t)
$$

is a solution of

$$
\Big\{F\Sigma(t)[I_n + G\Sigma(t)]^{-1}\Big\}\lambda = F[I_n + \Sigma(t)G]^{-1}\varphi(t) + b.
$$

That is, $F[I_n + \Sigma(t)G]^{-1}\varphi(t) + b$ lies in the range of $F\Sigma(t)[I_n + G\Sigma(t)]^{-1}$. Since

$$
F\Sigma(t)[I_n + G\Sigma(t)]^{-1} = F[I_n + \Sigma(t)G]^{-1}\Sigma(t),
$$

and $F[I_n+\Sigma(t)G]^{-1}\Sigma(t)$ and $F[I_n+\Sigma(t)G]^{-1}\Sigma(t)F^\top$ have the same range (Lem. 2.2), we see that $F[I_n+\Sigma(t)G]^{-1}\varphi(t)+b$ also lies in the range of $F[I_n+\Sigma(t)G]^{-1}\Sigma(t)F^\top$, which means the matrix equation (5.1) has a solution.

For the second assertion, let $\lambda^*\in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^k)$ and y^* be the solution to the SDE (5.2). By Theorem 4.3, the process

$$v^*(s)\triangleq N(s)^{-1}L(s)^\top y^*(s),\quad s\in[t,T]$$

is the optimal control of Problem $(\text{BLQ})_{\lambda^*}$. Further, let (x^*,z^*) be the adapted solution to the BSDE

$$\begin{cases}\mathrm{d}x^*(s)=(Ax^*+Kz^*+Lv^*)\mathrm{d}s+z^*\mathrm{d}W(s),\quad s\in[t,T],\\ x^*(T)=\eta.\end{cases}$$

We see from the proof of Theorem 4.3 that (x^*,z^*) and y^* have the following relation (recalling (4.15)):

$$x^*(s)=-[\Sigma(s)y^*(s)+\varphi(s)],\quad z^*(s)=[I_n+\Sigma(s)R(s)]^{-1}[\Sigma(s)K(s)^\top y^*(s)-\beta(s)],$$

from which we obtain

$$\begin{aligned}x^*(t)&=-\Sigma(t)y^*(t)-\varphi(t)\\&=-\Sigma(t)[I_n+G\Sigma(t)]^{-1}[F^\top\lambda^*-G\varphi(t)]-\varphi(t)\\&=-[I_n+\Sigma(t)G]^{-1}\Sigma(t)[F^\top\lambda^*-G\varphi(t)]-\varphi(t)\\&=-[I_n+\Sigma(t)G]^{-1}\Sigma(t)F^\top\lambda^*+[I_n+\Sigma(t)G]^{-1}\Sigma(t)G\varphi(t)-\varphi(t)\\&=-[I_n+\Sigma(t)G]^{-1}\Sigma(t)F^\top\lambda^*-[I_n+\Sigma(t)G]^{-1}\varphi(t).\end{aligned}\tag{5.10}$$

According to Proposition 4.1, the optimal control

$$v^*(s)=N(s)^{-1}L(s)^\top y^*(s),\quad s\in[t,T]$$

of Problem $(\text{BLQ})_{\lambda^*}$ is also optimal for Problem (CBLQ) if and only if $x^*(t)\in\mathcal{H}(F,b)$. Using (5.10), we see the latter holds if and only if λ^* is a solution of (5.1). ☐

We conclude this section with a particular case of Problem (CBLQ), the norm optimal control problem, for which the solution ξ of (3.11) and the optimal parameter λ^* decided by (5.1) have a close relationship.

The norm optimal control problem is to find, for a given terminal state $\eta\in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$, the minimum norm control $v^*\in L^2_{\mathbb{F}}(t,T;\mathbb{R}^{m-n})$ such that the adapted solution (x,z) of (4.3) satisfies $x(t)\in\mathcal{H}(F,b)$. That is, the cost function is the norm of the control:

$$J(t,v)=\mathbb{E}\int_t^T|v(s)|^2\mathrm{d}s.$$

In this case, $G=0,Q=0,R=0,N=I_{m-n}$, and the equation (5.1) becomes

$$F\Sigma(t)F^\top\lambda=-[F\varphi(t)+b].\tag{5.11}$$

Corollary 5.5. *If λ^* solves (5.11) (i.e., λ^* is an optimal parameter), then $\xi\triangleq-F^\top\lambda^*$ is a solution of the matrix equation (3.11) (with $t_0=t$ and $t_1=T$), and the control defined by (3.12) is the solution of the norm optimal control problem.*

Proof. Let Φ be the solution to the linear matrix SDE (3.8) with $\bar{A}$ replaced by A. Since $G = 0, Q = 0, R = 0, N = I_{m-n}$, by Proposition 3.6, we see that the controllability Gramian $\Psi(t, T)$ of system (4.3) over $[t, T]$ is equal to $\Sigma(t)$. Moreover, by Lemma 2.1, $\varphi(t)$ is given by

$$\varphi(t) = -\mathbb{E}[\Phi(t, T)\eta|\mathcal{F}_t].$$

So $\xi \triangleq -F^\top \lambda^*$ is a solution of the matrix equation (3.11) (with $t_0 = t$ and $t_1 = T$) if and only if λ^* is a solution of (5.11). According to Theorem 5.1, in order to show that the control defined by (3.12) is the solution of the norm optimal control problem, it suffices to verify that the solution y^* of (5.2) is given by

$$y^*(s) = \Phi(t, s)^\top F^\top \lambda^*.$$

This is evident, since for the norm optimal control problem, the SDE (5.2) becomes

$$\begin{cases} \mathrm{d}y^*(s) = -A(s)^\top y^*(s)\mathrm{d}s - K(s)^\top y^*(s)\mathrm{d}W(s), \quad s \in [t, T], \\ y^*(t) = F^\top \lambda^*. \end{cases}$$

The proof is therefore completed. □

6. Concluding remarks

In this paper, we studied a constrained stochastic LQ optimal control problem and obtained an explicit representation of the optimal control by solving a parameterized unconstrained backward LQ problem and an optimal parameter selection problem. Because of the quadratic structure of the cost functional, we concerned ourselves with the L^2 controllability of the control system, that is, the state process is required to be square-integrable. In this setting, the coefficient D in the diffusion of the state equation has to be subjective so that a target can be reached from the stochastic linear manifold $\mathcal{H}(F, b)$. The nondegeneracy condition (3.2) on D plays a crucial role since it not only allows us to convert Problem (CLQ) into a parameterized unconstrained backward LQ problem and an optimal parameter selection problem but also ensures a good representation of the optimal parameter. We point out that in the degenerate case $D = 0$, the system (3.1) might still be L^2-exactly controllable (even L^p-exactly controllable ($p > 1$)) in the sense of [15], where the state process is allowed to be only integrable:

$$\mathbb{E}\Big[\sup_{0 \leqslant s \leqslant T} |x(s)|\Big] < \infty.$$

In this sense, it is not clear if the control can be selected to be square-integrable or the state process has better integrability/regularity. For such a case, more delicate results are desired so that the method developed in this paper could apply. It is also worth noting that an optimal control of Problem (CBLQ) might still exist even if the weighting matrix N in (4.2) is not positive definite. However, for the moment we are not able to give a satisfactory result like the case when (H2) holds, due to the difficulty in establishing the solvability of the Riccati equation associated with Problem $(\mathrm{BLQ})_\lambda$. In fact, even for the backward LQ problem without constraints, it is still very challenging to explicitly construct an optimal control. We hope to report some relevant results along this line in our future publications.

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References

[1] M. Ait Rami, J.B. Moore and X.Y. Zhou, Indefinite stochastic linear quadratic control and generalized differential Riccati equation. *SIAM J. Control Optim.* **40** (2002) 1296–1311.

[2] A. Bensoussan, Lectures on stochastic control, in Nonlinear Filtering and Stochastic Control, Vol. 972 of *Lecture Notes in Mathematics*. Springer-Verlag, New York (1982) 1–62.

[3] J.M. Bismut, Contrôle des systems linéaires quadratiques: applications de l'intégrale stochastique, in *Séminaire de Probabilités XII*, edited dy C. Dellacherie, P.A. Meyer and M. Weil. Vol. 649 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin (1978) 180–264.

[4] S. Chen and J. Yong, Stochastic linear quadratic optimal control problems. *Appl. Math. Optim.* **43** (2001) 21–45.

[5] B. Gashi, Stochastic minimum-energy control. *Syst. Control Lett.* **85** (2015) 70–76.

[6] X. Li, J. Sun and J. Xiong, Linear quadratic optimal control problems for mean-field backward stochastic differential equations. *Appl. Math. Optim.* **80** (2019) 223–250.

[7] A.E.B. Lim and X.Y. Zhou, Linear-quadratic control of backward stochastic differential equations. *SIAM J. Control Optim.* **40** (2001) 450–474.

[8] F. Liu and S. Peng, On controllability for stochastic control systems when the coefficient is time-variant. *J. Syst. Sci. Complex* **23** (2010) 270–278.

[9] Q. Lü, T. Wang and X. Zhang, Characterization of optimal feedback for stochastic linear quadratic control problems. *Probab. Uncertain. Quant. Risk* **2** (2017) 11.

[10] S. Peng, Backward stochastic differential equation and exact controllability of stochastic control systems. *Prog. Nat. Sci.* **4** (1994) 274–284.

[11] J. Sun, X. Li and J. Yong, Open-loop and closed-loop solvabilities for stochastic linear quadratic optimal control problems. *SIAM J. Control Optim.* **54** (2016) 2274–2308.

[12] J. Sun, J. Xiong and J. Yong, Indefnite stochastic linear-quadratic optimal control problems with random coefcients: Closed-loop representation of open-loop optimal controls. arXiv:1809.00261v2 (2018).

[13] S. Tang, General linear quadratic optimal stochastic control problems with random coefficients: Linear stochastic Hamilton systems and backward stochastic Riccati equations. *SIAM J. Control Optim.* **42** (2003) 53–75.

[14] H. Wang, J. Sun and J. Yong, Weak closed-loop solvability of stochastic linear-quadratic optimal control problems. *Discrete Contin. Dyn. Syst. Ser. A* **39** (2019) 2785–2805.

[15] Y. Wang, D. Yang, J. Yong and Z. Yu, Exact controllability of linear stochastic differential equations and related problems. *Math. Control Relat. Fields* **7** (2017) 305–345.

[16] Y. Wang and C. Zhang, The norm optimal control problem for stochastic linear control systems. *ESAIM: COCV* **21** (2015) 399–413.

[17] W.M. Wonham, On a matrix Riccati equation of stochastic control. *SIAM J. Control* **6** (1968) 681–697.

[18] J. Yong and X.Y. Zhou, Stochastic Controls: Hamiltonian Systems and HJB Equations. Springer-Verlag, New York (1999).

[19] Z. Yu, Equivalent cost functionals and stochastic linear quadratic optimal control problems. *ESAIM: COCV* **19** (2013) 78–90.