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DYNAMIC PROGRAMMING AND FEEDBACK ANALYSIS OF THE TWO DIMENSIONAL TIDAL DYNAMICS SYSTEM

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Abstract. In this work, we consider the controlled two dimensional tidal dynamics equations in bounded domains. A distributed optimal control problem is formulated as the minimization of a suitable cost functional subject to the controlled 2D tidal dynamics equations. The existence of an optimal control is shown and the dynamic programming method for the optimal control of 2D tidal dynamics system is also described. We show that the feedback control can be obtained from the solution of an infinite dimensional Hamilton-Jacobi equation. The non-differentiability and lack of smoothness of the value function forced us to use the method of viscosity solutions to obtain a solution of the infinite dimensional Hamilton-Jacobi equation. The Bellman principle of optimality for the value function is also obtained. We show that a viscosity solution to the Hamilton-Jacobi equation can be used to derive the Pontryagin maximum principle, which give us the first order necessary conditions of optimality. Finally, we characterize the optimal control using the adjoint variable.

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1. Introduction

From the earliest ages onwards, people have been fascinated by the periodic rise and fall of the sea surface. The tidal motion can be described as the oscillation of ocean waters under the influence of enchanting gravitational forces of the moon and the sun. Ocean tides have been investigated by many mathematicians and physicists, starting from great scholars like Galileo Galilei, Isaac Newton etc. (see [15, 33]). Newton's equilibrium theory of tides described the observed dominant semi-diurnal periodicity of ocean tides (see [16] for more details). The ocean tide information have been massively used in the geophysical areas such as Earth tides, the elastic properties of the Earth's crust, tidal variations of gravity, and in calculating the orbits of artificial satellites used for space exploration etc. (see [27, 28]). They are also applied, in space studies to calculate the trajectories of man-made satellites of the Earth and to interpret the results of satellite measurements. The kind of model considered in this paper goes back to the works of Laplace. He was the first one to investigate the free and forced oscillations of a thin atmosphere on a spherical planet (see [24]). Laplace rearranged the rotating shallow water equations into the system that underlies the tides and is known as the *Laplace tidal equations*. By taking the shallow water model on a rotating sphere, which is a slight generalization of the Laplace model, the tidal

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dynamics model considered in [22, 27, 28] is obtained. We refer the interested readers to [27, 28, 34] etc., for extensive study on the recent progress in this field.

In this work, we consider the controlled two dimensional tidal dynamics equations in bounded domains and establish the Pontryagin maximum principle, *via* the dynamic programing method. We now describe the controlled two dimensional tidal dynamics equations with a distributed control. Let $\Omega\subset\mathbb{R}^2$ be bounded with smooth boundary conditions. That is, Ω is a horizontal ocean basin, where tides are induced over the time interval $[0,T]$. The boundary contour $\partial\Omega$ is composed of two disconnected parts:

(1) a solid part Γ_1, coinciding with the edge of the continental and island shelves,
(2) an open boundary Γ_2.

We assume that sea water is incompressible and the vertical velocities are small compared with the horizontal velocities, and hence we are able to exclude acoustic waves. Also long waves, including tidal waves, are stood out from the family of gravitational oscillations. Moreover, in order to reduce computational difficulties, we assume that the Earth is absolutely rigid, and the gravitational field of the Earth is not affected by movements of ocean tides. Also, we ignore the effect of the atmospheric tides on the ocean tides and the effect of curvature of the surface of the Earth on horizontal turbulent friction. Under these commonly used assumptions, we consider the following controlled tidal dynamics model (see [17, 22, 26–30, 43] etc.):

$$\left\{\begin{aligned}\frac{\partial\mathbf{w}}{\partial t}+l\mathbf{k}\times\mathbf{w}+g\nabla\zeta+\frac{r}{h}|\mathbf{w}|\mathbf{w}-\kappa_h\Delta\mathbf{w}=\mathbf{g}+\mathrm{U},&\quad\text{in }\Omega\times(0,T),\\ \frac{\partial\zeta}{\partial t}+\operatorname{div}(h\mathbf{w})=0,&\quad\text{in }\Omega\times(0,T),\\ \mathbf{w}=\mathbf{w}^0,&\quad\text{on }\partial\Omega\times[0,T],\\ \mathbf{w}(0)=\mathbf{w}_0,\ \zeta(0)=\zeta_0,&\quad\text{in }\Omega,\\ \mathbf{w}^0=\mathbf{0},\ \text{on }\Gamma_1,\ \text{ and }&\int_0^T\int_{\Gamma_2}h\mathbf{w}^0\cdot\mathbf{n}\mathrm{d}\Gamma_2\mathrm{d}t=0,\end{aligned}\right.\tag{1.1}$$

where $\mathbf{w}(x,t)\in\mathbb{R}^2$, the horizontal transport vector, is the averaged integral of the velocity vector over the vertical axis, $l=2\rho\cos\theta$ is the Coriolis parameter, where ρ is the angular velocity of the Earth rotation and θ is the colatitude, $\mathbf{k}$ is a unit vector oriented vertically upward, g is the free fall acceleration, the friction coefficient r is a universal constant, κ_h is the horizontal turbulent viscosity coefficient, $\mathbf{n}$ is the unit outward normal to the boundary Γ_2. The scalar $\zeta(x,t)\in\mathbb{R}$ is the deviation of free surface with respect to the ocean bottom (*i.e.*, $\zeta=\zeta_s-\zeta_b$, where ζ_s is the surface and ζ_b is the shift of the ocean bottom), $\mathbf{g}=\gamma_L g\nabla\zeta^+$ is the known tide-generating force with γ_L is the Love factor approximately equal to 0.7 and ζ^+ is the height of the static tide. In (1.1), $h(x)$ is the vertical scale of motion, that is, the depth of the calm sea at x in the region Ω and we assume that it is a continuously differentiable function nowhere becoming zero, so that

$$0<\lambda=\min_{x\in\Omega}h(x),\quad \mu=\max_{x\in\Omega}h(x),\quad \max_{x\in\Omega}|\nabla h(x)|\leq M,\tag{1.2}$$

where M is a positive constant which equals to zero at a constant ocean depth. The quantity U appearing in the right hand side of momentum equation in (2.1) is the control, which is a distributed external force acting on the system.

Next, we discuss about the boundary conditions satisfied by the averaged velocity field. Remember that the contour $\partial\Omega$ consists of two parts, a solid part Γ_1 coinciding with the shelf edge and the open boundary Γ_2. The function $\mathbf{w}^0(x,t)$ is a known function on the boundary. This impermeability condition is given on the solid part of the boundary, *i.e.*, the restriction $\mathbf{w}^0|_{\Gamma_1}=\mathbf{0}$ is the no-slip boundary condition on the shoreline, and $\int_0^T\int_{\Gamma_2}h\mathbf{w}^0\cdot\mathbf{n}\mathrm{d}\Gamma_2\mathrm{d}t=0$, follows from the mass conservation law. The initial datum $\mathbf{w}_0(x)$ and $\zeta_0(x)$ are given. In particular, $\mathbf{w}_0$ and ζ_0 can be set equal to zero, which indicates the fact that initially the ocean is at rest. The unique global solvability results of the system (1.1) is obtained by simplifying the non-homogeneous boundary value problem to a homogeneous Dirichlet boundary value problem (see (2.1) in the next section). For more details, we refer the readers to [21, 26–29], etc.

Let us now discuss the solvability results available in the literature for the system (1.1) (or equivalently (2.1) in the next section). The existence and uniqueness of a weak solution for the tidal dynamic equations (see systems (1.1) or (2.1)) in bounded domains has been obtained in [21, 27, 28], using compactness arguments. The authors in [26] obtained similar results for the deterministic tidal dynamics system using global monotonicity property of the linear and nonlinear operators (so that the results are valid even in unbounded domains). The existence of a periodic solution for the tidal dynamics problem in two dimensional finite domains is obtained in [17]. The existence and uniqueness of weak and strong solutions of the stationary tidal dynamic equations in bounded and unbounded domains is obtained in [29]. The global solvability results for stochastic perturbations in bounded and unbounded domains, and different asymptotic behaviors have been established in [2, 20, 26, 40, 43] etc.

Optimal control theory of fluid dynamic models has been one of the important research areas of applied mathematics with enormous applications in the fields like Oceanography, Geophysics, Engineering and Technology, etc. (*cf.* [1, 4, 9, 14, 19, 25, 39, 41], etc.). Recently, the author in [30] considered several distributed control problems like and minimization of total energy and dissipation of energy of the flow, data assimilation problems in meteorology, etc. with 2D tidal dynamics equations as constraints. The existence of an optimal control as well as the first order necessary conditions of optimality for such systems is established, and the optimal control is characterized *via* adjoint variable. The Pontryagin maximum principle for the state constrained optimization problem for the 2D tidal dynamic system using Ekeland's variational principle is established in [31]. Some optimal control problems in optimizing the sea level induced by a tidal component in a small basin, tidal power generation and related problems are considered in [5, 32, 35, 36, 42], etc. The authors in [2] formulated a martingale problem of Stroock and Varadhan associated to an initial value control problem and established the existence of optimal controls.

In this work, we formulate a distributed optimal control problem as the minimization of a suitable cost functional subject to the controlled two dimensional tidal dynamics equations. The cost functional under our consideration is the sum of dissipation of energy of the flow, $\mathbb{L}^2$-energy of the elevation and the total effort by the distributed controls. We first discuss about the existence of an optimal control for such optimization problems. Then the dynamic programming method for the optimal control of 2D tidal dynamics system is described. That is, we show that the feedback control can be obtained from the solution of an infinite dimensional Hamilton-Jacobi equation. The non-differentiability and lack of smoothness of the value function imposed us to use the method of viscosity solutions to obtain a solution of the infinite dimensional Hamilton-Jacobi equation. We also establish the Bellman principle of optimality for the value function. The Pontryagin maximum principle gives the first order necessary conditions of optimality. We show that a viscosity solution to the Hamilton-Jacobi equation can be used to derive the Pontryagin maximum principle. At the end of this work, we characterize the optimal control using the adjoint variable. Of course, we are able to prove only the existence of viscosity solution and uniqueness is still open. We will address the uniqueness question in a future work.

The rest of the paper is organized as follows. In the next section, we first simplify the non-homogeneous boundary value problem (1.1) to a homogeneous Dirichlet boundary value problem. We give an abstract formulation of the model and describe the necessary functional spaces needed to obtain the unique solvability results. The existence and uniqueness of weak as well as strong solutions to the system (1.1) (see Thms. A.1, A.6), and continuous dependence results are stated in the Appendix (see Thms. A.3, 2.6 and 2.7). The existence and uniqueness of weak solutions to the linearized problem and a continuous dependence result of the same problem is also discussed in the same section (see Thm. 2.8 and Prop. 2.9). A distributed optimal control problem, subject to the controlled tidal dynamics system, is formulated in Section 3 as the minimization of a cost functional, which contains sum of dissipation of energy of the flow, $\mathbb{L}^2$-energy of the elevation and the total effort by the distributed controls. The existence of an optimal control is also discussed (see Thm. 3.4). In the same section, we derive the adjoint system formally and discuss about the unique solvability results to the adjoint system (see Thm. 3.3). In Section 4, we describe the dynamic programming method for the optimal control of two-dimensional tidal dynamics system. We define the value function as the minimum of the cost functional and show that the value function satisfies an infinite dimensional Hamilton-Jacobi equation. In the same section, we also establish the Bellman principle of optimality for the value function (see Thm. 4.4). Due to the non-differentiability and lack of smoothness of the value function, we use the method of viscosity solutions to obtain a solution of the infinite dimensional Hamilton-Jacobi equation (see Thm. 4.7). In the final section, we prove that the value function, a viscosity solution to the Hamilton-Jacobi equation can be used to derive the Pontryagin maximum principle (see Thm. 5.5). The feedback optimal control is characterized *via* adjoint variable. A verification theorem is also provided in Section 5 (see Thm. 5.6).

2. Mathematical formulation

In this section, we provide an abstract formulation of the system (1.1) and explain the necessary function spaces needed to obtain the global solvability results of (1.1). We provide the global existence and uniqueness of weak as well as strong solutions of the uncontrolled system. We also give an insight into the global solvability results of the corresponding linearized problem and adjoint system.

In order to simplify the non-homogeneous boundary value problem (1.1) to a homogeneous Dirichlet boundary value problem (for convenience, we take $\tau = 0$ in this section), we set

$$\mathbf{u}(x,t) := \mathbf{w}(x,t) - \mathbf{w}^0(x,t),$$

and

$$\xi(x,t) := \zeta(x,t) + \int_\tau^t \mathrm{div}(h(x)\mathbf{w}^0(x,s))\mathrm{d}s,$$

which are referred to as the *tidal flow* and the *elevation*. The full flow $\mathbf{w}^0$, which is given *a priori* on the boundary $\partial\Omega$, has been extended to the whole domain $\Omega \times (\tau, T]$ as a smooth function. We denote this function also by $\mathbf{w}^0$. We write the controlled tidal dynamics system (1.1) in the abstract form as:

$$\left\{\begin{aligned} \frac{\partial \mathbf{u}(t)}{\partial t} + \mathrm{A}\mathbf{u}(t) + \mathrm{B}(\mathbf{u}(t)) + \nabla\xi(t) &= \mathbf{f}(t) + \mathrm{U}(t), \ \text{ in } \ \Omega\times(\tau,T), \\ \frac{\partial \xi(t)}{\partial t} + \mathrm{div}(h\mathbf{u}(t)) &= 0, \ \text{ in } \ \Omega\times(\tau,T), \\ \mathbf{u}(t) &= \mathbf{0}, \ \text{ on } \ \partial\Omega\times(\tau,T), \\ \mathbf{u}(\tau) = \mathbf{u}_0, \ \xi(\tau) &= \xi_0, \ \text{ in } \ \Omega, \end{aligned}\right. \tag{2.1}$$

where we scaled g to unity. For $\mathrm{U} = \mathbf{0}$, we call the system (2.1) an uncontrolled tidal dynamics system. Using conservation of mass, we have

$$\int_\Omega \xi(x,t)\mathrm{d}x = \int_\Omega \xi(x,\tau)\mathrm{d}x = \int_\Omega \xi_0(x)\mathrm{d}x = |\Omega|\overline{\xi_0}, \ \text{ for all } \ t\in[\tau,T], \tag{2.2}$$

where $\overline{\xi}_0 = \frac{1}{|\Omega|}\int_\Omega \xi_0(x)\mathrm{d}x$, and $|\Omega|$ is the Lebesgue measure of Ω. Let us now describe the operators appearing in (2.1). The operator A denotes the matrix operator:

$$\mathrm{A} := \begin{pmatrix} -\alpha\Delta & -\beta \\ \beta & -\alpha\Delta \end{pmatrix}, \tag{2.3}$$

where Δ is the Laplacian operator, $\alpha := \kappa_h$, $\beta := 2\rho\cos\theta$ are positive constants. For $x \in \Omega$, B denotes the nonlinear vector operator,

$$\mathrm{B}(\mathbf{u}) := \gamma(x)|\mathbf{u}+\mathbf{w}^0|(\mathbf{u}+\mathbf{w}^0), \tag{2.4}$$

where $\mathbf{w}^0(x,t) \in \mathbb{R}^2$ is a known deterministic function on the boundary $\partial\Omega$. The function $\gamma(x) := \frac{r}{h(x)}$ is a strictly positive smooth function (note that the friction coefficient $r > 0$ is a universal constant and $h(\cdot)$ satisfies (1.2), [28], see p. 46). The function $\mathbf{f}$ and initial data $(\mathbf{u}_0, \xi_0)$ are given by

$$\begin{cases} \mathbf{f} = \mathbf{g} - \dfrac{\partial \mathbf{w}^0}{\partial t} + \nabla \displaystyle\int_\tau^t \mathrm{div}(h\mathbf{w}^0)\mathrm{d}s + \kappa_h \Delta \mathbf{w}^0 - l\mathbf{k} \times \mathbf{w}^0, \\ \mathbf{u}_0(x) = \mathbf{w}_0(x) - \mathbf{w}^0(x,0), \\ \xi_0(x) = \zeta_0(x). \end{cases} \tag{2.5}$$

The regularities of $(\mathbf{u}_0, \xi_0)$, $\mathbf{w}^0$ and $\mathbf{f}$ are discussed in the sections given below.

2.1. Functional Setting

For $p \geq 1$, we denote by $\mathrm{L}^p(\Omega) := \mathrm{L}^p(\Omega;\mathbb{R})$, the space consisting of equivalence classes of measurable real valued functions for which the p^{th} power of the absolute value is Lebesgue integrable, where the functions which agree almost everywhere are identified. We know that $\mathrm{L}^2(\Omega)$ is a Hilbert space, and the norm and inner product in $\mathrm{L}^2(\Omega)$ are denoted by $\|\cdot\|_{\mathrm{L}^2}$ and $(\cdot,\cdot)_{\mathrm{L}^2}$. We define $\mathbb{L}^p(\Omega) := \mathrm{L}^p(\Omega;\mathbb{R}^2)$ as the Banach space of Lebesgue measurable $\mathbb{R}^2$-valued, p-integrable functions on Ω with the norm:

$$\|\mathbf{u}\|_{\mathbb{L}^p} := \left(\int_\Omega |\mathbf{u}(x)|^p \mathrm{d}x\right)^{1/p}.$$

For $p = 2$, $\mathbb{L}^2(\Omega) := \mathrm{L}^2(\Omega;\mathbb{R}^2)$ is a Hilbert space equipped with the inner product given by

$$(\mathbf{u},\mathbf{v})_{\mathbb{L}^2} := \int_\Omega \mathbf{u}(x)\cdot\mathbf{v}(x)\mathrm{d}x, \ \ \mathbf{u},\mathbf{v} \in \mathbb{L}^2(\Omega).$$

Then the norm on $\mathbb{L}^2(\Omega)$ is defined by

$$\|\mathbf{u}\|_{\mathbb{L}^2} := (\mathbf{u},\mathbf{u})_{\mathbb{L}^2} = \left(\int_\Omega |\mathbf{u}(x)|^2 \mathrm{d}x\right)^{1/2}.$$

Let $\mathbb{H}^1(\Omega) := \mathrm{H}^1(\Omega;\mathbb{R}^2)$ denotes the Sobolev space $\mathbb{W}^{1,2}(\Omega) := \mathrm{W}^{1,2}(\Omega;\mathbb{R}^2)$ with the norm defined by

$$\|\mathbf{u}\|_{\mathbb{H}^1} := \left(\|\mathbf{u}\|_{\mathbb{L}^2}^2 + \|\nabla\mathbf{u}\|_{\mathbb{L}^2}^2\right)^{1/2}, \ \ \text{for all} \ \ \mathbf{u} \in \mathbb{H}^1(\Omega).$$

We also let $\mathbb{H}_0^1(\Omega) := \mathrm{H}_0^1(\Omega;\mathbb{R}^2)$ to be the closure of $\mathrm{C}_c^\infty(\Omega;\mathbb{R}^2)$ in $\mathbb{H}^1(\Omega)$ norm, where $\mathrm{C}_c^\infty(\Omega;\mathbb{R}^2)$ is the space of all infinitely differentiable functions with compact support in Ω. Then $\mathbb{H}_0^1(\Omega)$ is also a Sobolev space under the induced norm. Since Ω is a bounded domain, in view of the Poincaré inequality, that is,

$$\|\mathbf{u}\|_{\mathbb{L}^2} \leq C_\Omega \|\nabla\mathbf{u}\|_{\mathbb{L}^2},$$

the norms $\|\nabla\mathbf{u}\|_{\mathbb{L}^2(\Omega)}$ and $\|\mathbf{u}\|_{\mathbb{H}^1}$ are equivalent in $\mathbb{H}^1_0(\Omega)$. Thus $\mathbb{H}^1_0(\Omega)$ is also a Hilbert space equipped with inner product:

$$(\mathbf{u},\mathbf{v})_{\mathbb{H}^1_0}:=(\nabla\mathbf{u},\nabla\mathbf{v})_{\mathbb{L}^2}=\int_\Omega\nabla\mathbf{u}(x)\cdot\nabla\mathbf{v}(x)\mathrm{d}x,$$

and the norm:

$$\|\mathbf{u}\|_{\mathbb{H}^1_0}=\|\nabla\mathbf{u}\|_{\mathbb{L}^2}=\left(\int_\Omega|\nabla\mathbf{u}(x)|^2\mathrm{d}x\right)^{1/2}.$$

We denote the dual of $\mathbb{H}^1_0(\Omega)$ by $(\mathbb{H}^1_0(\Omega))'=\mathbb{H}^{-1}(\Omega)$. The induced duality between the spaces $\mathbb{H}^1_0(\Omega)$ and $\mathbb{H}^{-1}(\Omega)$ is denoted by $\langle\cdot,\cdot\rangle_{\mathbb{H}^{-1}\times\mathbb{H}^1_0}$. Then, we have the following continuous and dense embedding:

$$\mathbb{H}^1_0(\Omega)\subset\mathbb{L}^2(\Omega)\equiv(\mathbb{L}^2(\Omega))'\subset\mathbb{H}^{-1}(\Omega).$$

The first embedding is also compact, since Ω is bounded. Using the Gelfand triple $(\mathbb{H}^1_0(\Omega),\mathbb{L}^2(\Omega),\mathbb{H}^{-1}(\Omega))$, we may consider ∇ or Δ as a linear map from $\mathbb{L}^2(\Omega)$ or $\mathbb{H}^1_0(\Omega)$ into $\mathbb{H}^{-1}(\Omega)$ respectively. In the sequel, we use the notations $\mathbb{H}^m(\Omega):=\mathrm{H}^m(\Omega;\mathbb{R}^2)=\mathrm{W}^{m,2}(\Omega;\mathbb{R}^2)$ and $\mathrm{H}^m(\Omega):=\mathrm{H}^m(\Omega;\mathbb{R})=\mathrm{W}^{m,2}(\Omega;\mathbb{R})$ for Sobolev spaces of order m. We also use the notations,

$$\mathbb{H}=\mathbb{L}^2(\Omega),\ \mathbb{V}=\mathbb{H}^1_0(\Omega),\ \mathbb{V}'=\mathbb{H}^{-1}(\Omega)\ \text{ and }\ \mathbb{W}=\mathbb{H}^2(\Omega).$$

Let us denote by

$$\mathrm{H}=\mathrm{L}^2(\Omega),\ \mathrm{V}=\mathrm{H}^1(\Omega),\ \text{ and } \mathrm{V}'=\text{the dual of }\ \mathrm{H}^1(\Omega),$$

and the duality pairing $\langle\cdot,\cdot\rangle_{\mathrm{V}'\times\mathrm{V}}$. For every $f\in\mathrm{V}'$, we denote $\overline{f}$ the average of f over Ω, *i.e.*, $\overline{f}:=|\Omega|^{-1}\langle f,1\rangle_{\mathrm{V}'\times\mathrm{V}}$. Let us also introduce the spaces (see [13])

$$\begin{aligned}\mathrm{V}_0&=\{v\in\mathrm{V}:\ \overline{v}=0\},\\ \mathrm{V}_0'&=\{f\in\mathrm{V}':\ \overline{f}=0\},\end{aligned}$$

and the operator $\mathcal{A}:\mathrm{V}\to\mathrm{V}'$ is defined by

$$\langle\mathcal{A}u,v\rangle_{\mathrm{V}'\times\mathrm{V}}:=\int_\Omega\nabla u(x)\cdot\nabla v(x)\mathrm{d}x,\ \text{ for all }\ u,v\in\mathrm{V}.$$

Clearly $\mathcal{A}$ is linear and it maps V into V_0' and its restriction $\mathcal{B}$ of $\mathcal{A}$ to V_0 onto V_0' is an isomorphism. We know that for every $f\in\mathrm{V}_0'$, $\mathcal{B}^{-1}f$ is the unique solution with zero mean value of the *Neumann problem*:

$$\left\{\begin{array}{l}-\Delta u=f,\ \text{ in }\ \Omega,\\ \frac{\partial u}{\partial\mathbf{n}}=0,\ \text{ on }\ \partial\Omega.\end{array}\right.$$

In addition, we have

$$\langle\mathcal{A}u,\mathcal{B}^{-1}f\rangle_{\mathrm{V}'\times\mathrm{V}}=\langle f,u\rangle_{\mathrm{V}'\times\mathrm{V}},\ \text{ for all }\ u\in\mathrm{V},\ f\in\mathrm{V}_0',\tag{2.6}$$

$$\langle f,\mathcal{B}^{-1}g\rangle_{\mathrm{V}'\times\mathrm{V}}=\langle g,\mathcal{B}^{-1}f\rangle_{\mathrm{V}'\times\mathrm{V}}=\int_\Omega\nabla(\mathcal{B}^{-1}f)\cdot\nabla(\mathcal{B}^{-1}g)\mathrm{d}x,\ \text{ for all }\ f,g\in\mathrm{V}_0'.\tag{2.7}$$

Note that $\mathcal{B}$ can be viewed as an unbounded linear operator on H with domain $\mathrm{D}(\mathcal{B}) = \left\{v \in \mathrm{H}^2(\Omega) : \frac{\partial v}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega\right\}$.

Let us now give the well known inequality due to Ladyzhenskaya (see [23], Lem. 1 and 2, Chap. 1), which is used in the paper quite frequently.

Lemma 2.1 (Ladyzhenskaya inequality)**.** *For* $\mathbf{u} \in \mathrm{C}_0^\infty(\Omega;\mathbb{R}^n), n = 2, 3$, *there exists a constant* $C > 0$ *such that*

$$\|\mathbf{u}\|_{\mathbb{L}^4} \leq C^{1/4}\|\mathbf{u}\|_{\mathbb{H}}^{1-\frac{n}{4}}\|\nabla\mathbf{u}\|_{\mathbb{H}}^{\frac{n}{4}}, \text{ for } n = 2, 3, \tag{2.8}$$

where $C = 2, 4$, *for* $n = 2, 3$ *respectively.*

Thus, for $n = 2$, we have

$$\|\mathbf{u}\|_{\mathbb{L}^4} \leq 2^{1/4}\|\mathbf{u}\|_{\mathbb{H}}^{1/2}\|\nabla\mathbf{u}\|_{\mathbb{H}}^{1/2} \leq C_\Omega\|\nabla\mathbf{u}\|_{\mathbb{H}} = C_\Omega\|\mathbf{u}\|_{\mathbb{V}}, \tag{2.9}$$

where we also used the Poincaré inequality. Thus, we obtain a continuous embedding

$$\mathbb{V} \hookrightarrow \mathbb{L}^4(\Omega) \hookrightarrow \mathbb{H} \hookrightarrow \mathbb{L}^{4/3}(\Omega) \hookrightarrow \mathbb{V}'.$$

2.2. Linear operator

Let us define the non-symmetric bilinear form:

$$a(\mathbf{u},\mathbf{v}) := \alpha[(\nabla\mathbf{u}_1, \nabla\mathbf{v}_1)_{\mathbb{H}} + (\nabla\mathbf{u}_2, \nabla\mathbf{v}_2)_{\mathbb{H}}] + \beta[(\mathbf{u}_1,\mathbf{v}_2)_{\mathbb{H}} - (\mathbf{u}_2,\mathbf{v}_1)_{\mathbb{H}}],$$

where $\mathbf{u} = (\mathbf{u}_1,\mathbf{u}_2), \mathbf{v} = (\mathbf{v}_1,\mathbf{v}_2) \in \mathbb{V}$. If $\mathbf{u}$ has a smooth second order derivatives (or if $\mathbf{u} \in \mathbb{V}\cap\mathbb{W}$), then

$$a(\mathbf{u},\mathbf{v}) = (\mathrm{A}\mathbf{u},\mathbf{v})_{\mathbb{H}}, \quad \text{for all} \quad \mathbf{v} \in \mathbb{H}.$$

For $\mathbf{u},\mathbf{v} \in \mathbb{V}$, we take $a(\mathbf{u},\mathbf{v}) = \langle \mathrm{A}\mathbf{u},\mathbf{v}\rangle_{\mathbb{V}'\times\mathbb{V}}$. For $\mathbf{u},\mathbf{v} \in \mathbb{V}\cap\mathbb{W}$, we consider $\mathrm{A}\mathbf{u} = -\alpha\Delta\mathbf{u} + \beta\mathbf{k}\times\mathbf{u}$ and an integration by parts twice yields (using the fact that $\mathbf{u}\big|_{\partial\Omega} = 0$)

$$\begin{aligned}(-\alpha\Delta\mathbf{u} + \beta\mathbf{k}\times\mathbf{u},\mathbf{v})_{\mathbb{H}} &= \alpha(\nabla\mathbf{u},\nabla\mathbf{v})_{\mathbb{H}} + \beta(\mathbf{k}\times\mathbf{u},\mathbf{v})_{\mathbb{H}} \\ &= (\mathbf{u}, -\alpha\Delta\mathbf{v} - \beta\mathbf{k}\times\mathbf{v})_{\mathbb{H}},\end{aligned}$$

and hence $(\mathrm{A}\mathbf{u},\mathbf{v})_{\mathbb{H}} \neq (\mathbf{u},\mathrm{A}\mathbf{v})_{\mathbb{H}}$, so that A is not symmetric. The bilinear form $a(\cdot,\cdot)$ is continuous and coercive in $\mathbb{V}$, that is,

$$|a(\mathbf{u},\mathbf{v})| \leq C_a\|\mathbf{u}\|_{\mathbb{V}}\|\mathbf{v}\|_{\mathbb{V}}, \quad \text{for all} \quad \mathbf{u},\mathbf{v} \in \mathbb{V}, \tag{2.10}$$

$$\langle \mathrm{A}\mathbf{u},\mathbf{u}\rangle_{\mathbb{V}'\times\mathbb{V}} = a(\mathbf{u},\mathbf{u}) = \alpha\|\nabla\mathbf{u}\|_{\mathbb{H}}^2 = \alpha\|\mathbf{u}\|_{\mathbb{V}}^2, \tag{2.11}$$

for some positive constant C_a. Even though A is not a self-adjoint operator, the operator $\widehat{\mathrm{A}} := -\Delta$ is a self-operator on $\mathbb{H}$ with $\mathrm{D}(\widehat{\mathrm{A}}) = \mathbb{V}\cap\mathbb{W}$.

2.3. Properties of the nonlinear operator

In the sequel, $\mathcal{L}(\mathrm{H};\mathrm{K})$ denotes the space of all bounded linear operators from H to K.

Lemma 2.2 ([26, 29, 40])**.** *The operator* B *defined in* (2.4) *has the following properties:*
For all $\mathbf{u},\mathbf{v},\mathbf{w}^0 \in \mathbb{L}^4(\Omega)$, *we have*

(*i*) $\|\mathrm{B}(\mathbf{u})\|_{\mathbb{H}} \leq C_\gamma(\|\mathbf{u}\|_{\mathbb{L}^4} + \|\mathbf{w}^0\|_{\mathbb{L}^4})^2$, *where* $0 \leq C_\gamma = \|\gamma\|_{\mathrm{L}^\infty} \leq \frac{r}{\lambda}$ *is a constant depending on* γ,

(*ii*) $\mathrm{B}(\cdot)$ *is a nonlinear continuous operator from* $\mathbb{V}$ *into* $\mathbb{V}'$,
(*iii*) $(\mathrm{B}(\mathbf{u})-\mathrm{B}(\mathbf{v}),\mathbf{u}-\mathbf{v})_{\mathbb{H}}\geq 0$,
(*iv*) $(\mathrm{B}(\mathbf{u}),\mathbf{u})_{\mathbb{H}}\geq -\frac{r}{2\lambda}(\|\mathbf{w}^0\|_{\mathbb{L}^4}^4+\|\mathbf{u}\|_{\mathbb{H}}^2)$,
(*v*) $\|\mathrm{B}(\mathbf{u})-\mathrm{B}(\mathbf{v})\|_{\mathbb{H}}\leq C_\gamma(\|\mathbf{u}\|_{\mathbb{L}^4}+\|\mathbf{v}\|_{\mathbb{L}^4}+\|\mathbf{w}^0\|_{\mathbb{L}^4})\|\mathbf{u}-\mathbf{v}\|_{\mathbb{L}^4}$,
(*vi*) *The operator* $\mathrm{B}(\cdot)$ *is Gâteaux differentiable with the Gâteaux derivative* $\mathrm{B}'(\mathbf{u})=2\gamma|\mathbf{u}+\mathbf{w}^0|\in\mathcal{L}(\mathbb{L}^4;\mathbb{H})$ *and* $(\mathrm{B}'(\mathbf{u})\mathbf{v},\mathbf{v})_{\mathbb{H}}\geq 0$.
(*vii*) *Moreover, for* $\mathbf{u}\in\mathbb{L}^4(\Omega)$, *we have* $\mathrm{B}'(\mathbf{u})=2\gamma|\mathbf{u}+\mathbf{w}^0|\in\mathcal{L}(\mathbb{H};\mathbb{V}')$, *and*

$$\|\mathrm{B}'(\mathbf{u})\mathbf{v}\|_{\mathbb{V}'}\leq \frac{Cr}{\lambda}\|\mathbf{u}+\mathbf{w}^0\|_{\mathbb{L}^4}\|\mathbf{v}\|_{\mathbb{H}},\quad \textit{for all}\ \ \mathbf{v}\in\mathbb{H}. \tag{2.12}$$

(*viii*) *For* $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{L}^4(\Omega)$, *we have* $\|(\mathrm{B}'(\mathbf{u})-\mathrm{B}'(\mathbf{v}))\mathbf{w}\|_{\mathbb{H}}\leq\frac{2r}{\lambda}\|\mathbf{u}-\mathbf{v}\|_{\mathbb{L}^4}\|\mathbf{w}\|_{\mathbb{L}^4}$.

Proof. A proof of the Lemma (*i*)-(*v*) can be obtained from the Lemma 2.3 in [29]. We prove here (*vi*)-(*viii*) parts.

(*vi*) Using the definition, it is clear that $\mathrm{B}(\cdot)$ is Gâteaux differentiable (in fact Fréchet differentiable) with the Gâteaux derivative $\mathrm{B}'(\mathbf{u})=2\gamma|\mathbf{u}+\mathbf{w}^0|\in\mathcal{L}(\mathbb{V};\mathbb{V}')$. Also for each fixed $\mathbf{u}\in\mathbb{L}^4$, using Hölder's inequality, we have

$$\|\mathrm{B}'(\mathbf{u})\mathbf{v}\|_{\mathbb{H}}=2\|\gamma|\mathbf{u}+\mathbf{w}^0|\mathbf{v}\|_{\mathbb{H}}\leq 2\|\gamma\|_{\mathrm{L}^\infty}\|\mathbf{u}+\mathbf{w}^0\|_{\mathbb{L}^4}\|\mathbf{v}\|_{\mathbb{L}^4}\leq\frac{2r}{\lambda}(\|\mathbf{u}\|_{\mathbb{L}^4}+\|\mathbf{w}^0\|_{\mathbb{L}^4})\|\mathbf{v}\|_{\mathbb{L}^4}, \tag{2.13}$$

and hence we have $\|\mathrm{B}'(\mathbf{u})\|_{\mathcal{L}(\mathbb{L}^4;\mathbb{H})}\leq\frac{2r}{\lambda}(\|\mathbf{u}\|_{\mathbb{L}^4}+\|\mathbf{w}^0\|_{\mathbb{L}^4})$, so that $\mathrm{B}'(\cdot)\in\mathcal{L}(\mathbb{L}^4;\mathbb{H})$. Since $\gamma>0$, it can be easily seen that

$$(\mathbf{B}'(\mathbf{u})\mathbf{v},\mathbf{v})_{\mathbb{H}}=2\int_\Omega\gamma|\mathbf{u}+\mathbf{w}^0||\mathbf{v}|^2\mathrm{d}x\geq 0, \tag{2.14}$$

for all $\mathbf{u},\mathbf{v},\mathbf{w}^0\in\mathbb{L}^4(\Omega)$.

(*vii*) For any $\mathbf{w}\in\mathbb{V}$, using (2.13), Hölder's and Ladyzhenskaya's and Poincaré's inequalities, we have

$$\begin{aligned}|\langle\mathrm{B}'(\mathbf{u})\mathbf{v},\mathbf{w}\rangle_{\mathbb{V}'\times\mathbb{V}}|&=|(\mathrm{B}'(\mathbf{u})\mathbf{v},\mathbf{w})_{\mathbb{H}}|\leq\|\mathrm{B}'(\mathbf{u})\mathbf{v}\|_{\mathbb{L}^{4/3}}\|\mathbf{w}\|_{\mathbb{L}^4}\\&\leq 2^{1/4}\|2\gamma|\mathbf{u}+\mathbf{w}^0|\mathbf{v}\|_{\mathbb{L}^{4/3}}\|\mathbf{w}\|_{\mathbb{H}}^{1/2}\|\nabla\mathbf{w}\|_{\mathbb{H}}^{1/2}\\&\leq 2^{5/4}C_\Omega\|\gamma\|_{\mathrm{L}^\infty}\|\mathbf{u}+\mathbf{w}^0\|_{\mathbb{L}^4}\|\mathbf{v}\|_{\mathbb{H}}\|\nabla\mathbf{w}\|_{\mathbb{H}}\\&\leq\frac{Cr}{\lambda}\big(\|\mathbf{u}\|_{\mathbb{L}^4}+\|\mathbf{w}^0\|_{\mathbb{L}^4}\big)\|\mathbf{v}\|_{\mathbb{H}}\|\mathbf{w}\|_{\mathbb{V}}.\end{aligned} \tag{2.15}$$

Hence, we get (2.12) and since (2.12) is true for all $\mathbf{v}\in\mathbb{H}$ and $\mathbf{u}\in\mathbb{L}^4(\Omega)$, we finally obtain $\mathrm{B}'(\mathbf{u})\in\mathcal{L}(\mathbb{H};\mathbb{V}')$.

(*viii*) For $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{L}^4(\Omega)$, using Hölder's inequality, we have

$$\begin{aligned}\|(\mathrm{B}'(\mathbf{u})-\mathrm{B}'(\mathbf{v}))\mathbf{w}\|_{\mathbb{H}}&=\|2\gamma(|\mathbf{u}+\mathbf{w}^0|-|\mathbf{v}+\mathbf{w}^0|)\mathbf{w}\|_{\mathbb{H}}\\&\leq 2\|\gamma\|_{\mathrm{L}^\infty}\||\mathbf{u}+\mathbf{w}^0|-|\mathbf{v}+\mathbf{w}^0|\|_{\mathbb{L}^4}\|\mathbf{w}\|_{\mathbb{L}^4}\\&\leq\frac{2r}{\lambda}\|\mathbf{u}-\mathbf{v}\|_{\mathbb{L}^4}\|\mathbf{w}\|_{\mathbb{L}^4},\end{aligned} \tag{2.16}$$

and also $\|\mathrm{B}'(\mathbf{u})-\mathrm{B}'(\mathbf{v})\|_{\mathcal{L}(\mathbb{H};\mathbb{L}^4)}\leq\frac{2r}{\lambda}\|\mathbf{u}-\mathbf{v}\|_{\mathbb{L}^4}$. □

The estimates (2.11) and (*iii*) easily imply the following:

Lemma 2.3. *The operator* $\mathrm{F}(\mathbf{u}):=\mathrm{A}\mathbf{u}+\mathrm{B}(\mathbf{u})-\mathbf{f}$ *is a globally monotone operator, i.e,*

$$\langle\mathrm{F}(\mathbf{u})-\mathrm{F}(\mathbf{v}),\mathbf{u}-\mathbf{v}\rangle_{\mathbb{V}'\times\mathbb{V}}\geq 0,\quad \textit{for all}\ \ \mathbf{u},\mathbf{v}\in\mathbb{V}.$$

2.4. Global existence and uniqueness

Let us now give the definition of weak as well as strong solutions to the controlled system (2.1). The global solvability results have been discussed in [26–29] (see Appendix A also).

Definition 2.4. The pair

$$(\mathbf{u},\xi)\in(\mathrm{C}([\tau,T];\mathbb{H})\cap\mathrm{L}^2(\tau,T;\mathbb{V}))\times\mathrm{C}([\tau,T];\mathrm{H}),$$

with

$$(\partial_t\mathbf{u},\partial_t\xi)\in\mathrm{L}^2(\tau,T;\mathbb{V}')\times\mathrm{L}^2(\tau,T;\mathrm{H}),$$

is called a *weak solution* to the system (2.1), if for $\mathbf{f}\in\mathrm{L}^2(\tau,T;\mathbb{V}')$, $(\mathbf{u}_0,\xi_0)\in\mathbb{H}\times\mathrm{H}$ and $(\mathbf{v},\eta)\in\mathbb{V}\times\mathrm{V}$, $(\mathbf{u},\xi)$ satisfies:

$$\begin{cases}\langle\partial_t\mathbf{u}(t)+\mathrm{A}\mathbf{u}(t)+\mathrm{B}(\mathbf{u}(t))+\nabla\xi(t),\mathbf{v}\rangle_{\mathbb{V}'\times\mathbb{V}}=\langle\mathbf{f}(t)+\mathrm{U}(t),\mathbf{v}\rangle_{\mathbb{V}'\times\mathbb{V}},\\ (\partial_t\xi+div(h\mathbf{u}),\eta)_{\mathrm{H}}=0,\\ \lim\limits_{t\downarrow 0}\int_\Omega\mathbf{u}(t)\mathbf{v}\mathrm{d}x=\int_\Omega\mathbf{u}_0\mathbf{v}\mathrm{d}x,\ \lim\limits_{t\downarrow 0}\int_\Omega\xi(t)\eta\mathrm{d}x=\int_\Omega\xi_0\eta\mathrm{d}x,\end{cases}\tag{2.17}$$

and the energy equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\sqrt{h}\mathbf{u}(t)\|_{\mathbb{H}}^2+\|\xi(t)\|_{\mathrm{H}}^2\Big)+2\langle\mathrm{F}(\mathbf{u}(t)),h\mathbf{u}(t)\rangle_{\mathbb{V}'\times\mathbb{V}}=\langle\mathbf{f}(t)+\mathrm{U}(t),h\mathbf{u}(t)\rangle_{\mathbb{V}'\times\mathbb{V}}.\tag{2.18}$$

Definition 2.5. The pair

$$(\mathbf{u},\xi)\in(\mathrm{C}([\tau,T];\mathbb{V})\cap\mathrm{L}^2(\tau,T;\mathbb{W}))\times\mathrm{C}([\tau,T];\mathrm{V})$$

with

$$(\partial_t\mathbf{u},\partial_t\xi)\in\mathrm{L}^2(\tau,T;\mathbb{H})\times\mathrm{L}^2(\tau,T;\mathrm{V}),$$

is called a *strong solution* to the system (2.1), if for $\mathbf{f}\in\mathrm{L}^2(\tau,T;\mathbb{H})$, $\mathrm{U}\in\mathrm{L}^2(\tau,T;\mathbb{H})$ and $(\mathbf{v},\varrho)\in\mathbb{V}\times\mathrm{V}$, $(\mathbf{u},\xi)$ satisfies

$$\begin{cases}\partial_t\mathbf{u}(t)+\mathrm{A}\mathbf{u}(t)+\mathrm{B}(\mathbf{u}(t))+g\nabla\xi(t)=\mathbf{f}(t)+\mathrm{U}(t),\ \text{ in }\ \mathrm{L}^2(\tau,T;\mathbb{H}),\\ \partial_t\xi(t)+div(h\mathbf{u}(t))=0,\ \text{ in }\ \mathrm{L}^2(\tau,T;\mathrm{V}).\end{cases}\tag{2.19}$$

Let us now establish some continuous dependence relations, which is useful in the subsequent sections. The constant C appearing in the next theorem depends on α,μ,λ,M, $\|\mathbf{f}\|_{\mathrm{L}^2(\tau,T;\mathbb{V}')}$ and $\|\mathbf{w}^0\|_{\mathrm{L}^4(\tau,T;\mathbb{L}^4(\Omega))}$ also, and we suppress that dependence.

Theorem 2.6. *Let $(\mathbf{v}_1,\varrho_1),(\mathbf{v}_2,\varrho_2)\in\mathbb{H}\times\mathrm{H}$ be given. Then the solution of the tidal dynamics system (2.1) corresponding to the control $\mathrm{U}(\cdot)$ satisfies:*

$$\begin{aligned}&\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|_{\mathbb{V}'}^2+\|\xi_1(t)-\xi_2(t)\|_{\mathrm{V}'}^2+\alpha\int_\tau^t\|\mathbf{u}_1(r)-\mathbf{u}_2(r)\|_{\mathbb{H}}^2\mathrm{d}r\\&\leq C\big(\|\mathrm{U}\|_{\mathrm{L}^2(\tau,T;\mathbb{V}')},\|\mathbf{v}_1\|_{\mathbb{H}},\|\mathbf{v}_2\|_{\mathbb{H}},\|\varrho_1\|_{\mathrm{H}},\|\varrho_2\|_{\mathrm{H}}\big)\big(\|\mathbf{v}_1-\mathbf{v}_2\|_{\mathbb{V}'}^2+\|\varrho_1-\varrho_2\|_{\mathrm{V}'}^2\big),\end{aligned}\tag{2.20}$$

for all $t\in[\tau,T]$.

Proof. Let us set $\mathbf{z}(t)=\mathbf{u}_1(t)-\mathbf{u}_2(t)$ and $\vartheta(t)=\xi_1(t)-\xi_2(t)$, so that $\mathbf{z}(\tau)=\mathbf{v}_1-\mathbf{v}_2$ and $\vartheta(\tau)=\varrho_1-\varrho_2$. Then $(\mathbf{z},\vartheta)$ satisfies the following system:

$$\left\{\begin{aligned}\frac{\partial\mathbf{z}(t)}{\partial t}+\mathrm{A}\mathbf{z}(t)+\mathrm{B}(\mathbf{u}_1(t))-\mathrm{B}(\mathbf{u}_2(t))+\nabla\vartheta(t)&=\mathbf{0}, \ \text{ in } \ \Omega\times(\tau,T),\\ \frac{\partial\vartheta(t)}{\partial t}+\mathrm{div}(h\mathbf{z}(t))&=0, \ \text{ in } \ \Omega\times(\tau,T),\\ \mathbf{u}(t)&=\mathbf{0}, \ \text{ on } \ \partial\Omega\times(\tau,T),\\ \mathbf{u}(\tau)=\mathbf{v}_1-\mathbf{v}_2, \ \xi(\tau)&=\varrho_1-\varrho_2, \ \text{ in } \ \Omega.\end{aligned}\right. \tag{2.21}$$

From the existence and uniqueness theorem (see Thm. A.1), we know that $(\mathbf{z},\vartheta)\in(\mathrm{C}([\tau,T];\mathbb{H})\cap\mathrm{L}^2(\tau,T;\mathbb{V}))\times\mathrm{C}([\tau,T];\mathrm{H})$. Let us take inner product with $(-\Delta)^{-1}\mathbf{z}(\cdot)$ to the first equation, $\mathcal{B}^{-1}(\vartheta(\cdot)-\overline{\vartheta})$ to the second equation in (2.21) and then add them together to obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(-\Delta)^{-1/2}\mathbf{z}(t)\|_{\mathbb{H}}^2+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathcal{B}^{-1/2}(\vartheta(t)-\overline{\vartheta})\|_{\mathrm{H}}^2+\alpha\|\mathbf{z}(t)\|_{\mathbb{H}}^2\\ &\quad=-\beta(\mathbf{k}\times\mathbf{z}(t),(-\Delta)^{-1}\mathbf{z}(t))_{\mathbb{H}}-(\mathrm{B}(\mathbf{u}_1(t))-\mathrm{B}(\mathbf{u}_2(t)),(-\Delta)^{-1}\mathbf{z}(t))_{\mathbb{H}}\\ &\qquad-(\nabla\vartheta(t),(-\Delta)^{-1}\mathbf{z}(t))_{\mathbb{H}}-(\mathrm{div}(h\mathbf{z}(t)),\mathcal{B}^{-1}(\vartheta(t)-\overline{\vartheta}))_{\mathrm{H}}\\ &\quad=:\sum_{I=1}^4 I_i,\end{aligned} \tag{2.22}$$

where I_i, $1\leq i\leq 4$, are the four terms appearing in the right hand side of (2.22). Since the operator $(-\Delta)^{-1/2}$ is self-adjoint and $\mathbf{z}\big|_{\partial\Omega}=\mathbf{0}$, we find I_1 as

$$I_1=-\beta(\mathbf{k}\times\mathbf{z},(-\Delta)^{-1}\mathbf{z})_{\mathbb{H}}=\beta((-\Delta)^{-1/2}(\mathbf{k}\times\mathbf{z}),(-\Delta)^{-1/2}\mathbf{z})_{\mathbb{H}}=0. \tag{2.23}$$

Using Taylor formula (see [7], Thm. 7.9.1, [8], Chap. 6), (2.12), Cauchy-Schwarz and Young's inequalities, I_2 can be estimated as

$$\begin{aligned}|I_2|&=\left|\left(\int_0^1\mathrm{B}'(\theta\mathbf{u}_1+(1-\theta)\mathbf{u}_2)\mathbf{z}\mathrm{d}\theta,(-\Delta)^{-1}\mathbf{z}\right)_{\mathbb{H}}\right|\\ &=\left|\int_0^1((-\Delta)^{-1/2}[\mathrm{B}'(\theta\mathbf{u}_1+(1-\theta)\mathbf{u}_2)\mathbf{z}],(-\Delta)^{-1/2}\mathbf{z})_{\mathbb{H}}\mathrm{d}\theta\right|\\ &\leq\sup_{0<\theta<1}\|(-\Delta)^{-1/2}[\mathrm{B}'(\theta\mathbf{u}_1+(1-\theta)\mathbf{u}_2)\mathbf{z}]\|_{\mathbb{H}}\|(-\Delta)^{-1/2}\mathbf{z}\|_{\mathbb{H}}\\ &\leq\frac{Cr}{\lambda}\sup_{0<\theta<1}\big(\|\theta\mathbf{u}_1+(1-\theta)\mathbf{u}_2\|_{\mathbb{L}^4}+\|\mathbf{w}^0\|_{\mathbb{L}^4}\big)\|\mathbf{z}\|_{\mathbb{H}}\|(-\Delta)^{-1/2}\mathbf{z}\|_{\mathbb{H}}\\ &\leq\frac{\alpha}{6}\|\mathbf{z}\|_{\mathbb{H}}^2+\frac{3C^2r^2}{2\alpha\lambda^2}\big(\|\mathbf{u}_1\|_{\mathbb{L}^4}+\|\mathbf{u}_2\|_{\mathbb{L}^4}+\|\mathbf{w}^0\|_{\mathbb{L}^4}\big)^2\|(-\Delta)^{-1/2}\mathbf{z}\|_{\mathbb{H}}^2.\end{aligned} \tag{2.24}$$

Since $\nabla(\overline{\vartheta})=0$, using (2.6), Cauchy-Schwarz and Young's inequalities, we estimate $|I_3|$ as

$$\begin{aligned}|I_3|&=|((-\Delta)^{-1}\nabla(\vartheta-\overline{\vartheta}),\mathbf{z})_{\mathbb{H}}|\leq\|(-\Delta)^{-1}\nabla(\vartheta-\overline{\vartheta})\|_{\mathbb{H}}\|\mathbf{z}\|_{\mathbb{H}}\\ &\leq\frac{\alpha}{6}\|\mathbf{z}\|_{\mathbb{H}}^2+\frac{3}{2\alpha}\|\mathcal{B}^{-1/2}(\vartheta-\overline{\vartheta})\|_{\mathrm{H}}^2.\end{aligned} \tag{2.25}$$

Observe that $\mathcal{B}^{-1}(\vartheta-\overline{\vartheta})\in \mathrm{D}(\mathcal{B})$. Using an integration by parts and Cauchy-Schwarz and Young's inequalities, we estimate $|I_4|$ as

$$\begin{aligned}|I_4| &= |(h\mathbf{z},\nabla\mathcal{B}^{-1}(\vartheta-\overline{\vartheta}))_{\mathbb{H}}| \leq \|h\mathbf{z}\|_{\mathbb{H}}\|\nabla\mathcal{B}^{-1}(\vartheta-\overline{\vartheta})\|_{\mathbb{H}} \\ &\leq \|h\|_{\mathrm{L}^\infty}\|\mathbf{z}\|_{\mathbb{H}}\|\mathcal{B}^{-1/2}(\vartheta-\overline{\vartheta})\|_{\mathrm{H}} \\ &\leq \frac{\alpha}{6}\|\mathbf{z}\|_{\mathbb{H}}^2+\frac{3\mu^2}{2\alpha}\|\mathcal{B}^{-1/2}(\vartheta-\overline{\vartheta})\|_{\mathrm{H}}^2. \end{aligned} \tag{2.26}$$

Combining (2.23)–(2.26), using it in (2.22) and then integrating from τ to t, we obtain

$$\begin{aligned}&\|(-\Delta)^{-1/2}\mathbf{z}(t)\|_{\mathbb{H}}^2+\|\mathcal{B}^{-1/2}(\vartheta(t)-\overline{\vartheta})\|_{\mathrm{H}}^2+\alpha\int_\tau^t\|\mathbf{z}(s)\|_{\mathbb{H}}^2\mathrm{d}s \\ &\leq \|(-\Delta)^{-1/2}\mathbf{z}(\tau)\|_{\mathbb{H}}^2+\|\mathcal{B}^{-1/2}(\vartheta(\tau)-\overline{\vartheta})\|_{\mathrm{H}}^2+\frac{3}{\alpha}\big(\mu^2+1\big)\int_\tau^t\|\mathcal{B}^{-1/2}(\vartheta(s)-\overline{\vartheta})\|_{\mathrm{H}}^2\mathrm{d}s \\ &\quad+\int_\tau^t\left[\frac{3C^2r^2}{\alpha\lambda^2}\big(\|\mathbf{u}_1(s)\|_{\mathbb{L}^4}+\|\mathbf{u}_2(s)\|_{\mathbb{L}^4}+\|\mathbf{w}^0(s)\|_{\mathbb{L}^4}\big)^2\right]\|(-\Delta)^{-1/2}\mathbf{z}(s)\|_{\mathbb{H}}^2\mathrm{d}s. \end{aligned} \tag{2.27}$$

An application of Gronwall's inequality in (2.27) yields

$$\begin{aligned}\|\mathbf{z}(t)\|_{\mathbb{V}'}^2+\|\vartheta(t)\|_{\mathrm{V}'}^2 &\leq \big(\|\mathbf{z}(\tau)\|_{\mathbb{V}'}^2+\|\vartheta(\tau)\|_{\mathrm{V}'}^2\big)e^{\frac{3}{\alpha}\left(\mu^2+1\right)(t-\tau)} \\ &\quad\times\exp\left\{\frac{9C^2r^2}{\alpha\lambda^2}\int_\tau^T\big(\|\mathbf{u}_1(s)\|_{\mathbb{L}^4}^2+\|\mathbf{u}_2(s)\|_{\mathbb{L}^4}^2+\|\mathbf{w}^0(s)\|_{\mathbb{L}^4}^2\big)\mathrm{d}s\right\}. \end{aligned} \tag{2.28}$$

Since $\mathbf{u}_1,\mathbf{u}_2\in \mathrm{L}^\infty(\tau,T;\mathbb{H})\cap\mathrm{L}^2(\tau,T;\mathbb{V})$ and $\mathbf{w}^0\in\mathrm{L}^4(\tau,T;\mathbb{L}^4(\Omega))$, and the estimate (A.1) helps us to get the required result in (2.20). ☐

Theorem 2.7. *Let the Assumption A.4 hold, and let* $\mathbf{f}\in\mathrm{L}^2(\tau,T;\mathbb{H})$*, and* $(\mathbf{v}_1,\varrho_1),(\mathbf{v}_2,\varrho_2)\in\mathbb{H}\times\mathrm{H}$ *be given. Then the solution of the tidal dynamics system* (2.1) *corresponding to the control* $\mathrm{U}(\cdot)$ *satisfies:*

$$\begin{aligned}&\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|_{\mathbb{V}}^2+\|\xi_1(t)-\xi_2(t)\|_{\mathrm{V}}^2+\alpha\int_\tau^t\|\mathbf{u}_1(r)-\mathbf{u}_2(r)\|_{\mathbb{W}}^2\mathrm{d}r \\ &\leq C\big(\|\mathrm{U}\|_{\mathrm{L}^2(\tau,T;\mathbb{V}')},\|\mathbf{v}_1\|_{\mathbb{V}},\|\mathbf{v}_2\|_{\mathbb{V}},\|\varrho_1\|_{\mathrm{V}},\|\varrho_2\|_{\mathrm{V}}\big)\big(\|\mathbf{v}_1-\mathbf{v}_2\|_{\mathbb{V}}^2+\|\varrho_1-\varrho_2\|_{\mathrm{V}}^2\big), \end{aligned} \tag{2.29}$$

for all $t\in[\tau,T]$.

Proof. Let us take $\mathbf{z}(t)=\mathbf{u}_1(t)-\mathbf{u}_2(t)$ and $\vartheta(t)=\xi_1(t)-\xi_2(t)$, so that $\mathbf{z}(\tau)=\mathbf{v}_1-\mathbf{v}_2$ and $\vartheta(\tau)=\varrho_1-\varrho_2$, and $(\mathbf{z},\vartheta)$ satisfies the system (2.21). We take inner product with $\mathrm{A}(\mathbf{u}_1(\cdot)-\mathbf{u}_2(\cdot))$ to the first equation in (2.21) and $-\Delta(\xi_1(\cdot)-\xi_2(\cdot))$ to the second equation in (2.21), and then add them together to obtain (see [29], Thm. 2.11

for more details)

$$
\begin{aligned}
&\frac{\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\mathbf{z}(t)\|_{\mathbb{H}}^2+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\vartheta(t)\|_{\mathrm{H}}^2+\|\mathrm{A}\mathbf{z}(t)\|_{\mathbb{H}}^2\\
&=-(\mathrm{B}(\mathbf{u}_1(t))-\mathrm{B}(\mathbf{u}_2(t)),\mathrm{A}\mathbf{z}(t))_{\mathbb{H}}+(\nabla\vartheta(t),\mathrm{A}\mathbf{z}(t))_{\mathbb{H}}-((\mathbf{z}\cdot\nabla)(\nabla h),\nabla\vartheta)_{\mathbb{H}}\\
&\quad-((\nabla h\cdot\nabla\mathbf{z}),\nabla\vartheta)_{\mathbb{H}}-(\text{div }\mathbf{z}\nabla h,\nabla\vartheta)_{\mathbb{H}}-(h\nabla(\text{div }\mathbf{z}),\nabla\vartheta)_{\mathbb{H}}\\
&=:\sum_{i=1}^{6}I_i,
\end{aligned}\tag{2.30}
$$

where I_i, for $i=1,\dots,6$ represents the six terms in the right hand side of (2.30). We estimate I_1 using Taylor's formula, Cauchy-Schwarz, Ladyzhenskaya and Hölder's inequalities as

$$
\begin{aligned}
|I_1|&=\left|\left(\int_0^1\mathrm{B}'(\theta\mathbf{u}_1+(1-\theta)\mathbf{u}_2)\mathbf{z}\mathrm{d}\theta,\mathrm{A}\mathbf{z}\right)_{\mathbb{H}}\right|\\
&\leq\sup_{0<\theta<1}\|\mathrm{B}'(\theta\mathbf{u}_1+(1-\theta)\mathbf{u}_2)\mathbf{z}\|_{\mathbb{H}}\|\mathrm{A}\mathbf{z}\|_{\mathbb{H}}\\
&\leq 2\|\gamma\|_{\mathrm{L}^\infty}\sup_{0<\theta<1}\|\theta\mathbf{u}_1+(1-\theta)\mathbf{u}_2+\mathbf{w}^0\|_{\mathbb{L}^4}\|\mathbf{z}\|_{\mathbb{L}^4}\|\mathrm{A}\mathbf{z}\|_{\mathbb{H}}\\
&\leq\frac{1}{6}\|\mathrm{A}\mathbf{z}\|_{\mathbb{H}}^2+3\|\gamma\|_{\mathrm{L}^\infty}^2\left(\|\mathbf{u}_1\|_{\mathbb{L}^4}+\|\mathbf{u}_2\|_{\mathbb{L}^4}+\|\mathbf{w}^0\|_{\mathbb{L}^4}\right)^2\|\mathbf{z}\|_{\mathbb{L}^4}^2\\
&\leq\frac{1}{6}\|\mathrm{A}\mathbf{z}\|_{\mathbb{H}}^2+\frac{3\sqrt{2}r^2}{\lambda^2}\left(\|\mathbf{u}_1\|_{\mathbb{L}^4}+\|\mathbf{u}_2\|_{\mathbb{L}^4}+\|\mathbf{w}^0\|_{\mathbb{L}^4}\right)^2\|\mathbf{z}\|_{\mathbb{H}}\|\nabla\mathbf{z}\|_{\mathbb{H}}\\
&\leq\frac{1}{6}\|\mathrm{A}\mathbf{z}\|_{\mathbb{H}}^2+\frac{1}{2}\|\nabla\mathbf{z}\|_{\mathbb{H}}^2+\frac{243r^4}{\lambda^4}\left(\|\mathbf{u}_1\|_{\mathbb{L}^4}^4+\|\mathbf{u}_2\|_{\mathbb{L}^4}^4+\|\mathbf{w}^0\|_{\mathbb{L}^4}^4\right)\|\mathbf{z}\|_{\mathbb{H}}^2.
\end{aligned}\tag{2.31}
$$

Remember that

$$
\|\mathrm{A}\mathbf{z}\|_{\mathbb{H}}^2=\alpha^2\|\Delta\mathbf{z}\|_{\mathbb{H}}^2+\beta^2\|\mathbf{z}\|_{\mathbb{H}}^2.\tag{2.32}
$$

We estimate the rest of the terms in (2.30), using Cauchy-Schwarz, Hölder's and Young's inequalities as

$$
\begin{aligned}
|I_2|&\leq\|\nabla\vartheta\|_{\mathbb{H}}\|\mathrm{A}\mathbf{z}\|_{\mathbb{H}}\leq\frac{1}{6}\|\mathrm{A}\mathbf{z}\|_{\mathbb{H}}^2+\frac{3}{2}\|\nabla\vartheta\|_{\mathbb{H}}^2\\
|I_3|&\leq\|(\mathbf{z}\cdot\nabla)(\nabla h)\|_{\mathbb{H}}\|\nabla\vartheta\|_{\mathbb{H}}\leq\|\nabla(\nabla h)\|_{\mathbb{L}^\infty}\|\mathbf{z}\|_{\mathbb{H}}\|\nabla\vartheta\|_{\mathbb{H}}\leq\frac{1}{6}\|\nabla\vartheta\|_{\mathbb{H}}^2+\frac{3M_1^2}{2}\|\mathbf{z}\|_{\mathbb{H}}^2,\\
|I_4|&\leq\|\nabla h\cdot\nabla\mathbf{z}\|_{\mathbb{H}}\|\nabla\vartheta\|_{\mathbb{H}}\leq\|\nabla h\|_{\mathbb{L}^\infty}\|\nabla\mathbf{z}\|_{\mathbb{H}}\|\nabla\vartheta\|_{\mathbb{H}}\leq\frac{1}{6}\|\nabla\vartheta\|_{\mathbb{H}}^2+\frac{3M^2}{2}\|\nabla\mathbf{z}\|_{\mathbb{H}}^2,\\
|I_5|&\leq\|\text{div }\mathbf{z}\nabla h\|_{\mathbb{H}}\|\nabla\vartheta\|_{\mathbb{H}}\leq\|\nabla h\|_{\mathbb{L}^\infty}\|\text{div }\mathbf{z}\|_{\mathbb{H}}\|\nabla\vartheta\|_{\mathbb{H}}\leq\frac{1}{6}\|\nabla\vartheta\|_{\mathbb{H}}^2+3M^2\|\nabla\mathbf{z}\|_{\mathbb{H}}^2,\\
|I_6|&\leq\|h\nabla(\text{div }\mathbf{z})\|_{\mathbb{H}}\|\nabla\vartheta\|_{\mathbb{H}}\leq\|h\|_{\mathbb{L}^\infty}\|\nabla(\text{div }\mathbf{z})\|_{\mathbb{H}}\|\nabla\vartheta\|_{\mathbb{H}}\leq\mu\|\mathbf{z}\|_{\mathbb{H}^2}\|\nabla\vartheta\|_{\mathbb{H}}\\
&\leq\mu C\|\Delta\mathbf{z}\|_{\mathbb{H}}\|\nabla\vartheta\|_{\mathbb{H}}\leq\frac{\alpha^2}{6}\|\Delta\mathbf{z}\|_{\mathbb{H}}^2+\frac{3\mu^2C^2}{4\alpha^2}\|\nabla\vartheta\|_{\mathbb{H}}^2\leq\frac{1}{6}\|\mathrm{A}\mathbf{z}\|_{\mathbb{H}}^2+\frac{3\mu^2C^2}{4\alpha^2}\|\nabla\vartheta\|_{\mathbb{H}}^2,
\end{aligned}
$$

where in the third estimate we used the fact that $\|\text{div }\mathbf{z}\|_{\mathbb{H}}\leq\sqrt{2}\|\nabla\mathbf{z}\|_{\mathbb{H}}$ and in the final estimate we used Remark 2.10 from [29]. Combining the estimates for I_1-I_6, substituting it in (2.30) and then integrating it from

0 to t, we obtain

$$
\begin{aligned}
&\alpha\|\nabla\mathbf{z}(t)\|_{\mathbb{H}}^2+\|\nabla\vartheta(t)\|_{\mathrm{H}}^2+\int_\tau^t\|\mathrm{A}\mathbf{z}(s)\|_{\mathbb{H}}^2\mathrm{d}s\\
&\leq \alpha\|\nabla\mathbf{z}(\tau)\|_{\mathbb{H}}^2+\|\nabla\vartheta(\tau)\|_{\mathrm{H}}^2+(1+9M^2)\int_\tau^t\|\nabla\mathbf{z}(s)\|_{\mathbb{H}}^2\mathrm{d}s+\left(4+\frac{3\mu^2C^2}{2\alpha^2}\right)\int_\tau^T\|\nabla\vartheta(s)\|_{\mathbb{H}}^2\mathrm{d}s\\
&\quad+2\sup_{s\in[\tau,t]}\|\mathbf{z}(s)\|_{\mathbb{H}}^2\left[\frac{243r^4}{\lambda^4}\int_\tau^t\big(\|\mathbf{u}_1(s)\|_{\mathbb{L}^4}^4+\|\mathbf{u}_2(s)\|_{\mathbb{L}^4}^4+\|\mathbf{w}^0(s)\|_{\mathbb{L}^4}^4\big)\mathrm{d}s+3M_1^2(t-\tau)\right]. \qquad (2.33)
\end{aligned}
$$

An application of Gronwall's inequality in (2.33) yields

$$
\begin{aligned}
&\alpha\|\nabla\mathbf{z}(t)\|_{\mathbb{H}}^2+\|\nabla\vartheta(t)\|_{\mathrm{H}}^2+\int_\tau^t\|\mathrm{A}\mathbf{z}(s)\|_{\mathbb{H}}^2\mathrm{d}s\\
&\leq\left\{\alpha\|\nabla\mathbf{z}(\tau)\|_{\mathbb{H}}^2+\|\nabla\vartheta(\tau)\|_{\mathrm{H}}^2+K\sup_{s\in[\tau,t]}\|\mathbf{z}(s)\|_{\mathbb{H}}^2\right\}\\
&\quad\times\exp\left(\max\left\{\frac{1+9M^2}{\alpha},\left(4+\frac{3\mu^2C^2}{2\alpha^2}\right)\right\}(t-\tau)\right), \qquad (2.34)
\end{aligned}
$$

for all $t\in[\tau,T]$, where

$$
\begin{aligned}
K&=\frac{486r^4}{\lambda^4}\int_\tau^t\big(\|\mathbf{u}_1(s)\|_{\mathbb{L}^4}^4+\|\mathbf{u}_2(s)\|_{\mathbb{L}^4}^4+\|\mathbf{w}^0(s)\|_{\mathbb{L}^4}^4\big)\mathrm{d}s+6M_1^2(t-\tau)\\
&\leq C\big(r,\lambda,\mu,M,M_1,T,\tau,\|\mathbf{v}\|_{\mathbb{H}},\|\varrho\|_{\mathrm{H}},\|\mathbf{w}^0\|_{\mathrm{L}^4(\tau,T;\mathbb{L}^4(\Omega))},\|\mathbf{f}\|_{\mathrm{L}^2(\tau,T;\mathbb{V}')},\|\mathrm{U}\|_{\mathrm{L}^2(\tau,T;\mathbb{V}')}\big),
\end{aligned}
$$

using (A.1). Hence, the estimate (2.29) follows easily by using the continuous dependence result given in the Theorem A.3 (see (A.2)). □

2.5. The linearized system

Let us linearize the equations (2.1) around $(\widehat{\mathbf{u}},\widehat{\xi})$ which is the *unique weak solution* of system (2.1) with control term $\mathrm{U}=\mathbf{0}$ (uncontrolled system), external forcing $\widetilde{\mathbf{g}}$, and initial datum $\widehat{\mathbf{u}}_0$ and $\widehat{\xi}_0$ are such that $\widetilde{\mathbf{g}}\in\mathrm{L}^2(\tau,T;\mathbb{V}')$ and $(\mathbf{u}_0,\xi_0)\in\mathbb{H}\times\mathrm{H}$. We consider the following linearized system:

$$
\left\{
\begin{aligned}
\frac{\partial\mathfrak{w}(t)}{\partial t}+\mathrm{A}\mathfrak{w}(t)+\mathrm{B}'(\widehat{\mathbf{u}}(t))\mathfrak{w}(t)+\nabla\eta(t)&=\widehat{\mathbf{g}}(t)+\mathrm{U}(t),\quad\text{in }\ \Omega\times(\tau,T),\\
\frac{\partial\eta(t)}{\partial t}+\operatorname{div}(h\mathfrak{w}(t))&=0,\quad\text{in }\ \Omega\times(\tau,T),\\
\mathfrak{w}(t)&=\mathbf{0},\quad\text{on }\ \partial\Omega\times(\tau,T),\\
\mathfrak{w}(0)=\mathfrak{w}_0,\ \eta(0)&=\eta_0,\quad\text{in }\ \Omega,
\end{aligned}
\right. \qquad (2.35)
$$

where $\widehat{\mathbf{g}}=\mathbf{g}-\widetilde{\mathbf{g}}$, $\mathrm{B}'(\widehat{\mathbf{u}})=2\gamma|\widehat{\mathbf{u}}+\mathbf{w}^0|$ and $(\mathfrak{w}_0,\eta_0)\in\mathbb{H}\times\mathrm{H}$. Using a standard Faedo-Galerkin approximation technique, we have the following theorem.

Theorem 2.8 ([30], Thm. 2.7)**.** *Let $(\mathfrak{w}_0,\eta_0)\in\mathbb{H}\times\mathrm{H}$ be given. For $\widehat{\mathbf{g}}\in\mathrm{L}^2(\tau,T;\mathbb{V}')$ and $\mathrm{U}\in\mathrm{L}^2(\tau,T;\mathbb{V}')$, there exists a unique weak solution to the system (2.35) satisfying*

$$
(\mathfrak{w},\eta)\in(\mathrm{C}([\tau,T];\mathbb{H})\cap\mathrm{L}^2(\tau,T;\mathbb{V}))\times\mathrm{C}([\tau,T];\mathrm{H}),
$$

with

$$(\partial_t \mathfrak{w}, \partial_t \eta) \in \mathrm{L}^2(\tau, T; \mathbb{V}') \times \mathrm{L}^2(\tau, T; \mathrm{H}),$$

and

$$\begin{aligned}
&\|\mathfrak{w}(t)\|_{\mathbb{H}}^2 + \|\eta(t)\|_{\mathrm{H}}^2 + \alpha \int_\tau^t \|\nabla \mathfrak{w}(s)\|_{\mathbb{H}}^2 \mathrm{d}s \\
&\leq \left(\|\mathfrak{w}_0\|_{\mathbb{H}}^2 + \|\eta_0\|_{\mathrm{H}}^2 + \frac{4}{\alpha} \int_\tau^t \|\widehat{\mathbf{g}}(s)\|_{\mathbb{V}'}^2 \mathrm{d}s + \frac{4}{\alpha} \int_\tau^t \|\mathrm{U}(s)\|_{\mathbb{V}'}^2 \mathrm{d}s \right) e^{\left(\frac{8(\mu^2+1)}{\alpha} + M \right) t},
\end{aligned} \tag{2.36}$$

for all $t \in [\tau, T]$.

The following proposition on continuous dependence is needed in the later sections.

Proposition 2.9. *The linearized problem* (2.35) *with* $\widehat{\mathbf{g}} = \mathrm{U} = \mathbf{0}$*, where* $(\mathbf{u}, \xi)$ *solves the tidal dynamics system* (2.1) *with initial data* $(\mathbf{v}, \varrho) \in \mathbb{H} \times \mathrm{H}$*, satisfies the following estimate:*

$$\begin{aligned}
&\|\mathfrak{w}(t)\|_{\mathbb{V}'}^2 + \|\eta(t)\|_{\mathrm{V}'}^2 + \alpha \int_\tau^t \|\mathfrak{w}(r)\|_{\mathbb{H}}^2 \mathrm{d}r \\
&\leq C(\|\mathrm{U}\|_{\mathrm{L}^2(\tau,T;\mathbb{V}')}, \|\mathbf{v}\|_{\mathbb{H}}, \|\varrho\|_{\mathrm{H}}) \big(\|\mathbf{w}_0\|_{\mathbb{V}'}^2 + \|\eta_0\|_{\mathrm{V}'}^2 \big).
\end{aligned} \tag{2.37}$$

Also, the following continuous dependence estimate for $(\widetilde{\mathfrak{w}}, \widetilde{\eta}) = (\mathfrak{w}_1 - \mathfrak{w}_2, \eta_1 - \eta_2)$ *holds:*

$$\begin{aligned}
&\|\widetilde{\mathfrak{w}}(t)\|_{\mathbb{V}'}^2 + \|\widetilde{\eta}(t)\|_{\mathrm{V}'}^2 + \alpha \int_\tau^t \|\widetilde{\mathfrak{w}}(r)\|_{\mathbb{H}}^2 \mathrm{d}r \\
&\leq C(\|\mathrm{U}\|_{\mathrm{L}^2(\tau,T;\mathbb{V}')}, \|\mathbf{v}_1\|_{\mathbb{H}}, \|\mathbf{v}_2\|_{\mathbb{H}}, \|\varrho_1\|_{\mathrm{H}}, \|\varrho_2\|_{\mathrm{H}}) \big(\|\widetilde{\mathfrak{w}}_0\|_{\mathbb{V}'}^2 + \|\widetilde{\eta}_0\|_{\mathrm{V}'}^2 + \|\mathbf{v}_1 - \mathbf{v}_2\|_{\mathbb{H}}^2 + \|\varrho_1 - \varrho_2\|_{\mathrm{H}}^2 \big),
\end{aligned} \tag{2.38}$$

where $(\mathfrak{w}_i, \eta_i)$*, for* $i = 1, 2$*, solves*

$$\left\{ \begin{aligned}
\frac{\partial \mathfrak{w}_i(t)}{\partial t} + \mathrm{A}\mathfrak{w}_i(t) + \mathrm{B}'(\mathbf{u}_i(t))\mathfrak{w}_i(t) + \nabla \eta_i(t) &= \mathbf{0}, \quad \text{in} \ \ \Omega \times (\tau, T), \\
\frac{\partial \eta_i(t)}{\partial t} + \mathrm{div}(h \mathfrak{w}_i(t)) &= 0, \quad \text{in} \ \ \Omega \times (\tau, T), \\
\mathfrak{w}_i(t) &= \mathbf{0}, \quad \text{on} \ \ \partial\Omega \times (\tau, T), \\
\mathfrak{w}_i(0) = \mathfrak{w}_0^i, \ \eta_i(0) &= \eta_0^i, \quad \text{in} \ \ \Omega.
\end{aligned} \right. \tag{2.39}$$

Here $(\mathbf{u}_i, \xi_i)$ *solves tidal dynamics system* (2.1) *with the initial data* $(\mathbf{v}_i, \varrho_i)$ *and control* U*.*

Proof. Let us take inner product with $(-\Delta)^{-1}\mathfrak{w}(\cdot)$ to the first equation in (2.35) (with $\widehat{\mathbf{g}} = \mathrm{U} = \mathbf{0}$) and $\mathcal{B}^{-1}(\eta(\cdot) - \overline{\eta})$ to the second equation in (2.35) to find

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathfrak{w}(t)\|_{\mathbb{V}'}^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathcal{B}^{-1/2}(\eta(t) - \overline{\eta})\|_{\mathrm{H}}^2 + \alpha\|\mathfrak{w}(t)\|_{\mathbb{H}}^2 \\
&= -\beta(\mathbf{k} \times \mathfrak{w}(t), (-\Delta)^{-1}\mathfrak{w}(t))_{\mathbb{H}} - (\mathrm{B}'(\mathbf{u}_i(t))\mathfrak{w}_i(t), (-\Delta)^{-1}\mathfrak{w}_i(t))_{\mathbb{H}} \\
&\quad - (\nabla \eta_i(t), (-\Delta)^{-1}\mathfrak{w}_i(t))_{\mathbb{H}} - (\mathrm{div}(h\mathfrak{w}_i(t)), \mathcal{B}^{-1}(\eta(t) - \overline{\eta}))_{\mathrm{H}} \\
&= \sum_{i=5}^{8} I_i,
\end{aligned} \tag{2.40}$$

where I_i, for $5 \leq i \leq 8$, are the four terms appearing in the right-hand side of (2.40). As in the calculations of (2.23)–(2.26), we estimate I_5–I_8 as

$$
\begin{aligned}
&I_5 = 0,\\
&|I_6| \leq \frac{\alpha}{6}\|\mathfrak{w}\|_{\mathbb{H}}^2 + \frac{3C^2r^2}{2\alpha\lambda^2}\big(\|\mathbf{u}\|_{\mathbb{L}^4} + \|\mathbf{w}^0\|_{\mathbb{L}^4}\big)^2\|(-\Delta)^{-1/2}\mathfrak{w}\|_{\mathbb{H}}^2,\\
&|I_7| \leq \frac{\alpha}{6}\|\mathfrak{w}\|_{\mathbb{H}}^2 + \frac{3}{2\alpha}\|\mathcal{B}^{-1/2}(\eta-\overline{\eta})\|_{\mathrm{H}}^2,\\
&|I_8| \leq \frac{\alpha}{6}\|\mathfrak{w}\|_{\mathbb{H}}^2 + \frac{3\mu^2}{2\alpha}\|\mathcal{B}^{-1/2}(\eta-\overline{\eta})\|_{\mathrm{H}}^2.
\end{aligned}
$$

Thus, from (2.40), it is immediate that

$$
\begin{aligned}
&\|\mathfrak{w}(t)\|_{\mathbb{V}'}^2 + \|\mathcal{B}^{-1/2}(\eta(t)-\overline{\eta})\|_{\mathrm{H}}^2 + \alpha\int_\tau^t \|\mathfrak{w}(r)\|_{\mathbb{H}}^2\mathrm{d}r\\
&\leq \|\mathfrak{w}_0\|_{\mathbb{V}'}^2 + \|\mathcal{B}^{-1/2}(\eta_0-\overline{\eta})\|_{\mathrm{H}}^2 + \frac{3C^2r^2}{\alpha\lambda^2}\int_\tau^t\big(\|\mathbf{u}(r)\|_{\mathbb{L}^4} + \|\mathbf{w}^0(r)\|_{\mathbb{L}^4}\big)^2\|(-\Delta)^{-1/2}\mathfrak{w}(r)\|_{\mathbb{H}}^2\mathrm{d}r\\
&\quad + \frac{3}{\alpha}(\mu^2+1)\int_\tau^t\|\mathcal{B}^{-1/2}(\eta(r)-\overline{\eta})\|_{\mathrm{H}}^2\mathrm{d}r. \qquad (2.41)
\end{aligned}
$$

An application of Gronwall's inequality yields

$$
\begin{aligned}
&\|\mathfrak{w}(t)\|_{\mathbb{V}'}^2 + \|\eta(t)\|_{\mathrm{V}'}^2 + \alpha\int_\tau^t\|\mathfrak{w}(r)\|_{\mathbb{H}}^2\mathrm{d}r\\
&\leq \big(\|\mathfrak{w}_0\|_{\mathbb{V}'}^2 + \|\eta_0\|_{\mathrm{V}'}^2\big)e^{\frac{3}{\alpha}(\mu^2+1)t}\exp\bigg\{\frac{6C^2r^2}{\alpha\lambda^2}\int_\tau^t\big(\|\mathbf{u}(r)\|_{\mathbb{L}^4}^2 + \|\mathbf{w}^0(r)\|_{\mathbb{L}^4}^2\big)\mathrm{d}r\bigg\}. \qquad (2.42)
\end{aligned}
$$

Using the fact that $(\mathbf{u},\xi)$ is a unique weak solution of (2.1) with the initial data $(\mathbf{v},\varrho)$ and control U, we finally obtain (2.37).

The estimate (2.38) can be obtained in a similar way as of (2.37) except for the following term

$$
\begin{aligned}
&(\mathrm{B}'(\mathbf{u}_1)\mathfrak{w}_1 - \mathrm{B}'(\mathbf{u}_2)\mathfrak{w}_2, (-\Delta)^{-1}\mathfrak{w})_{\mathbb{H}}\\
&= (\mathrm{B}'(\mathbf{u}_1)(\mathfrak{w}_1-\mathfrak{w}_2), (-\Delta)^{-1}\mathfrak{w})_{\mathbb{H}} + ((\mathrm{B}'(\mathbf{u}_1)-\mathrm{B}'(\mathbf{u}_2))\mathfrak{w}_2, (-\Delta)^{-1}\mathfrak{w})_{\mathbb{H}} := I_9 + I_{10}. \qquad (2.43)
\end{aligned}
$$

We estimate I_9 similarly as in (2.24) as

$$
|I_9| \leq \frac{\alpha}{6}\|\mathfrak{w}\|_{\mathbb{H}}^2 + \frac{3C^2r^2}{2\alpha\lambda^2}\big(\|\mathbf{u}_1\|_{\mathbb{L}^4} + \|\mathbf{w}^0\|_{\mathbb{L}^4}\big)^2\|(-\Delta)^{-1/2}\mathfrak{w}\|_{\mathbb{H}}^2. \qquad (2.44)
$$

Using Cauchy-Schwarz inequality, Hölder's inequality and Lemma 2.2, I_{10} can be estimated as

$$
\begin{aligned}
|I_{10}| &= ((-\Delta)^{-1/2}[(\mathrm{B}'(\mathbf{u}_1)-\mathrm{B}'(\mathbf{u}_2))\mathfrak{w}_2], (-\Delta)^{-1/2}\mathfrak{w})_{\mathbb{H}}\\
&\leq \|(-\Delta)^{-1/2}[(\mathrm{B}'(\mathbf{u}_1)-\mathrm{B}'(\mathbf{u}_2))\mathfrak{w}_2]\|_{\mathbb{H}}\|(-\Delta)^{-1/2}\mathfrak{w}\|_{\mathbb{H}}\\
&\leq C\|(\mathrm{B}'(\mathbf{u}_1)-\mathrm{B}'(\mathbf{u}_2))\mathfrak{w}_2\|_{\mathbb{H}}\|\mathfrak{w}\|_{\mathbb{V}'}\\
&\leq \frac{2r}{\lambda}\|\mathbf{u}_1-\mathbf{u}_2\|_{\mathbb{L}^4}\|\mathfrak{w}_2\|_{\mathbb{L}^4}\|\mathfrak{w}\|_{\mathbb{V}'}\\
&\leq \|\mathbf{u}_1-\mathbf{u}_2\|_{\mathbb{L}^4}^2 + \frac{r^2}{\lambda^2}\|\mathfrak{w}_2\|_{\mathbb{L}^4}^2\|\mathfrak{w}\|_{\mathbb{V}'}^2. \qquad (2.45)
\end{aligned}
$$

A calculation similar to (2.42) yields

$$\begin{aligned}&\|\mathfrak{w}(t)\|_{\mathbb{V}'}^2+\|\eta(t)\|_{\mathrm{V}'}^2+\alpha\int_\tau^t\|\mathfrak{w}(r)\|_{\mathbb{H}}^2\mathrm{d}r\\&\leq\left(\|\mathfrak{w}_0\|_{\mathbb{V}'}^2+\|\eta_0\|_{\mathrm{V}'}^2+\int_\tau^t\|\mathbf{u}_1(r)-\mathbf{u}_2(r)\|_{\mathbb{L}^4}^2\mathrm{d}r\right)e^{\frac{3}{\alpha}(\mu^2+1)t}\\&\quad\times\exp\left\{\frac{6C^2r^2}{\alpha\lambda^2}\int_\tau^t\left(\|\mathbf{u}_1(r)\|_{\mathbb{L}^4}^2+\|\mathbf{w}^0(r)\|_{\mathbb{L}^4}^2\right)\mathrm{d}r+\frac{r^2}{\lambda^2}\int_\tau^t\|\mathfrak{w}_2(r)\|_{\mathbb{L}^4}^2\mathrm{d}r\right\}.\end{aligned}\tag{2.46}$$

Using Ladyzhenskaya's inequality, the fact that $(\mathbf{u}_1,\xi_1)$ is the unique weak solution of the system (2.1) with the initial data $(\mathbf{v}_1,\varrho_1)$ and control U, $(\mathfrak{w}_2,\eta_2)$ is the unique weak solution of (2.35) (with $\widehat{\mathbf{g}}=\mathrm{U}=\mathbf{0}$) corresponding to the trajectory $(\mathbf{u}_2,\xi_2)$, and the continuous dependence result in Theorem A.3 finally gives (2.38). □

3. Optimal control problem and existence of optimal control

In this section, we formulate a distributed optimal control problem as the minimization of a suitable cost functional subject to the controlled tidal dynamics system (2.1). For $\tau=0$, the cost functional under our consideration is given by

$$\begin{aligned}\mathcal{J}(\tau,T,\mathbf{v},\varrho,\mathbf{u},\xi,\mathrm{U}):=&\frac{1}{2}\|\mathbf{u}(T)-\mathbf{u}_M^f\|_{\mathbb{H}}^2+\frac{1}{2}\|\xi(T)-\xi_M^f\|_{\mathrm{H}}^2\\&+\frac{1}{2}\int_\tau^T\left[\|\nabla(\mathbf{u}(t)-\mathbf{u}_M(t))\|_{\mathbb{H}}^2+\|\xi(t)-\xi_M(t)\|_{\mathrm{H}}^2+\|\mathrm{U}(t)\|_{\mathbb{H}}^2\right]\mathrm{d}t,\end{aligned}\tag{3.1}$$

where $\mathbf{u}_M(\cdot)$ is the measured velocity of the fluid, $\xi_M(\cdot)$ is the measured elevation, $\mathbf{u}_M^f$ is the measured velocity at time T and ξ_M^f is the measured elevation at time T. Physically, one can think it as an optimal estimation problem, where we are trying to find an unknown external force based on measurements and the cost functional is then the difference between measurement and the tidal dynamics. For simplicity, we fix

$$\mathbf{u}_M(t)=\xi_M(t)=\mathbf{u}_M^f=\xi_M^f=0,\ \text{for all}\ \ t\in(0,T),$$

in the rest of the paper. In the subsequent sections, we need the explicit dependence of τ in the cost functional given in (3.1). Note that the cost functional given in (3.1) is the sum of dissipation of energy of the flow, $\mathbb{L}^2$-energy of the elevation and the total effort by the distributed controls. Next we provide the definition of class of admissible controls and solutions.

Definition 3.1. Let $\mathscr{U}_{\mathrm{ad}}$ be a closed and convex subset of $\mathrm{L}^2(\tau,T;\mathbb{H})$ consisting of controls U (containing $\mathbf{0}$ also).

Definition 3.2 (Admissible class)**.** The *admissible class* $\mathscr{A}_{\mathrm{ad}}$ of triples $(\mathbf{u},\xi,\mathrm{U})$ is defined as the set of states $(\mathbf{u},\xi)$ solving the system (2.1) with the control $\mathrm{U}\in\mathscr{U}_{ad}\subseteq\mathrm{L}^2(\tau,T;\mathbb{H})$. That is,

$$\mathscr{A}_{\mathrm{ad}}:=\Big\{(\mathbf{u},\xi,\mathrm{U}):(\mathbf{u},\xi)\text{ is a unique weak solution of (2.1) with the control U}\Big\}.$$

Clearly $\mathscr{U}_{\mathrm{ad}}$ is nonempty. Note also that $\mathscr{A}_{\mathrm{ad}}$ is also a nonempty set as for any $\mathrm{U}\in\mathscr{U}_{\mathrm{ad}}$, there exists a unique weak solution of the system (2.1). In view of the above definition, the optimal control problem we are considering can be formulated as:

$$\min_{(\mathbf{u},\xi,\mathrm{U})\in\mathscr{A}_{\mathrm{ad}}}\mathcal{J}(\tau,T,\mathbf{v},\varrho,\mathbf{u},\xi,\mathrm{U}).\tag{3.2}$$

A solution to the problem (3.2) is called an *optimal solution*. The optimal triplet is denoted by $(\mathbf{u}^*, \xi^*, \mathrm{U}^*)$ and the control U^* is called *optimal control*.

3.1. The adjoint system

In order to establish Pontryagin's maximum principle, first of all, we need to find the adjoint system corresponding to (2.1). Remember that the optimal control is characterized *via* adjoint variable. In this subsection, we formally derive the adjoint system corresponding to the problem (2.1). Let us first define

$$\begin{cases} \mathcal{N}_1(\mathbf{u}, \xi, \mathrm{U}) := -\mathrm{A}\mathbf{u} - \mathrm{B}(\mathbf{u}) - \nabla\xi + \mathrm{U}, \\ \mathcal{N}_2(\mathbf{u}, \xi) := -\mathrm{div}(h\mathbf{u}). \end{cases} \tag{3.3}$$

Then the tidal dynamics system (2.1) can be written as

$$\{\partial_t \mathbf{u}, \partial_t \xi\} = \{\mathcal{N}_1(\mathbf{u}, \xi, \mathrm{U}), \mathcal{N}_2(\mathbf{u}, \xi)\}.$$

Let us define the *augmented cost functional* $\widetilde{\mathcal{J}}$ as

$$\begin{aligned} \widetilde{\mathcal{J}}(\tau, T, \mathbf{v}, \varrho, \mathbf{u}, \xi, \mathrm{U}) := & \int_\tau^T \langle \mathbf{p}(t), \partial_t \mathbf{u}(t) - \mathcal{N}_1(\mathbf{u}(t), \xi(t), \mathrm{U}(t))\rangle \mathrm{d}t \\ & + \int_\tau^T \langle \varphi(t), \partial_t \xi(t) - \mathcal{N}_2(\mathbf{u}(t), \xi(t))\rangle \mathrm{d}t - \mathcal{J}(\tau, T, \mathbf{v}, \varrho, \mathbf{u}, \xi, \mathrm{U}), \end{aligned} \tag{3.4}$$

where $\mathbf{p}$ and φ denote the adjoint variables corresponding to the states $\mathbf{u}$ and ξ, respectively. Next, we derive the adjoint equations formally by differentiating the augmented cost functional $\widetilde{\mathcal{J}}$ in the Gâteaux sense with respect to each of its variables. The adjoint variables $\mathbf{p}, \varphi$ and the control U satisfy the following system:

$$\begin{cases} -\dfrac{\partial \mathbf{p}}{\partial t} - [\partial_{\mathbf{u}}\mathcal{N}_1]^*\mathbf{p} - [\partial_{\mathbf{u}}\mathcal{N}_2]^*\varphi = \mathcal{J}_{\mathbf{u}}, \\ -\dfrac{\partial \varphi}{\partial t} - [\partial_\xi\mathcal{N}_1]^*\mathbf{p} - [\partial_\xi\mathcal{N}_2]^*\varphi = \mathcal{J}_\xi, \\ -[\partial_{\mathrm{U}}\mathcal{N}_1]^*\mathbf{p} - [\partial_{\mathrm{U}}\mathcal{N}_2]^*\varphi = \mathcal{J}_{\mathrm{U}}, \\ \mathbf{p}\big|_{\partial\Omega} = \mathbf{0}, \\ \mathbf{p}(T, \cdot) = \mathbf{u}(T),\ \eta(T, \cdot) = \eta(T). \end{cases} \tag{3.5}$$

Further, we compute $[\partial_{\mathbf{u}}\mathcal{N}_1]^*\mathbf{p}$, $[\partial_{\mathbf{u}}\mathcal{N}_2]^*\varphi$, $[\partial_\xi\mathcal{N}_1]^*\mathbf{p}$, $[\partial_\xi\mathcal{N}_2]^*\varphi$ as

$$\begin{cases} [\partial_{\mathbf{u}}\mathcal{N}_1]^*\mathbf{p} = -\widetilde{\mathrm{A}}\mathbf{p} - \mathrm{B}'(\mathbf{u})\mathbf{p}, & [\partial_{\mathbf{u}}\mathcal{N}_2]^*\varphi = h\nabla\varphi, \\ [\partial_\xi\mathcal{N}_1]^*\mathbf{p} = \mathrm{div}\ \mathbf{p}, & [\partial_\xi\mathcal{N}_2]^*\varphi = 0, \end{cases} \tag{3.6}$$

where $\widetilde{\mathrm{A}}\mathbf{p} = -\alpha\Delta\mathbf{p} - \beta\mathbf{k}\times\mathbf{p}$. Since A is non-symmetric, we have $\mathrm{A}\mathbf{v} \neq \widetilde{\mathrm{A}}\mathbf{v}$. But one can easily show that $\langle \mathrm{A}\mathbf{v}, \mathbf{v}\rangle_{\mathbb{V}'\times\mathbb{V}} = \alpha\|\nabla\mathbf{v}\|_{\mathbb{H}}^2 = \langle \widetilde{\mathrm{A}}\mathbf{v}, \mathbf{v}\rangle_{\mathbb{V}'\times\mathbb{V}}$, for all $\mathbf{v} \in \mathbb{V}$.

Also note that the third condition in (3.5) gives $\mathrm{U}=-\mathbf{p}$ if we take $\mathrm{U}\in\mathscr{U}_{\mathrm{ad}}=\mathrm{L}^2(\tau,T;\mathbb{H})$. Thus from (3.5), it follows that the adjoint variables $(\mathbf{p},\eta)$ satisfy the following adjoint system:

$$\begin{cases} -\dfrac{\partial \mathbf{p}(t)}{\partial t}+\widetilde{\mathrm{A}}\mathbf{p}(t)+\mathrm{B}'(\mathbf{u}(t))\mathbf{p}(t)-h\nabla\varphi(t)=-\Delta\mathbf{u}(t), \ \text{ in } \ \Omega\times(\tau,T),\\ -\dfrac{\partial\varphi(t)}{\partial t}-\mathrm{div}\ \mathbf{p}(t)=\xi(t), \ \text{ in } \ \Omega\times(\tau,T),\\ \mathbf{p}(t)=\mathbf{0}, \ \text{ on } \ \partial\Omega\times(\tau,T),\\ \mathbf{p}(T,\cdot)=\mathbf{u}(T), \ \varphi(T,\cdot)=\xi(T) \ \text{ in } \ \Omega. \end{cases} \tag{3.7}$$

Using a Faedo-Galerkin approximation technique, one can obtain the global solvability resuslts of (3.7). The following theorem gives the global existence and uniqueness of weak solution to the system (3.7) (see [30], Thm. 4.3).

Theorem 3.3. *Let $(\mathbf{u}(T),\xi(T))\in\mathbb{H}\times\mathrm{H}$ be given. Then there exists a unique weak solution to the system (3.7) satisfying*

$$(\mathbf{p},\varphi)\in(\mathrm{C}([\tau,T];\mathbb{H})\cap\mathrm{L}^2(\tau,T;\mathbb{V}))\times\mathrm{C}([\tau,T];\mathrm{H}),$$

with

$$(\partial_t\mathbf{p},\partial_t\varphi)\in\mathrm{L}^2(\tau,T;\mathbb{V}')\times\mathrm{L}^2(\tau,T;\mathrm{H}),$$

and

$$\begin{aligned}&\|\mathbf{p}(t)\|_{\mathbb{H}}^2+\|\varphi(t)\|_{\mathrm{H}}^2+\alpha\int_t^T\|\nabla\mathbf{p}(s)\|_{\mathbb{H}}^2\mathrm{d}s\\ &\leq\left(\|\mathbf{p}_T\|_{\mathbb{H}}^2+\|\varphi_T\|_{\mathrm{H}}^2+\int_t^T\|\mathbf{u}(s)\|_{\mathbb{H}}^2\mathrm{d}s+\int_t^T\|\xi(s)\|_{\mathrm{H}}^2\mathrm{d}s\right)e^{\left[M+2\mu^2+4\left(\frac{1}{\alpha}+1\right)\right]T},\end{aligned}\tag{3.8}$$

for all $t\in[\tau,T]$, where $(\mathbf{u},\xi)$ is the unique weak solution of the system (2.1) with the initial data $(\mathbf{v},\varrho)$ and control U.

3.2. Existence of optimal control

The following result provides the existence of an optimal triplet $(\mathbf{u}^*,\xi^*,\mathrm{U}^*)$ for the optimal control problem (3.2).

Theorem 3.4 (Existence of an optimal triplet, [30], Thm. 3.3)**.** *Let $(\mathbf{v},\varrho)\in\mathbb{H}\times\mathrm{H}$ and $\mathbf{f}\in\mathrm{L}^2(\tau,T;\mathbb{V}')$ be given. Then there exists at least one triplet $(\mathbf{u}^*,\xi^*,\mathrm{U}^*)\in\mathscr{A}_{\mathrm{ad}}$ such that the functional $\mathcal{J}(\tau,T,\mathbf{v},\varrho,\mathbf{u},\xi,\mathrm{U})$ attains its minimum at $(\mathbf{u}^*,\xi^*,\mathrm{U}^*)$, where $(\mathbf{u}^*,\xi^*)$ is the unique weak solution of the system (2.1) with the control U^*.*

Theorem 3.5. *Let U^* be the optimal control obtained in the Theorem 3.4. Then, we have*

$$\|\mathrm{U}^*\|_{\mathrm{L}^2(\tau,T;\mathbb{H})}\leq C\big(\|\mathbf{u}_0\|_{\mathbb{H}},\|\xi_0\|_{\mathrm{H}},\|\mathbf{w}^0\|_{\mathrm{L}^4(\tau,T;\mathbb{L}^4(\Omega))}\|\mathbf{f}\|_{\mathrm{L}^2(\tau,T;\mathbb{V}')}\big).\tag{3.9}$$

Proof. Let us take an arbitrary fixed $\mathrm{U}_0 \in \mathbb{H}$. Then using this element as control and $(\mathbf{u}_0, \xi_0) \in \mathbb{H} \times \mathrm{H}$ as the initial data, the corresponding solution $(\mathbf{u}^0(\cdot), \xi^0(\cdot))$ satisfies the following estimate:

$$\begin{aligned}
&\|\mathbf{u}^0(t)\|_{\mathbb{H}}^2 + \|\xi^0(t)\|_{\mathrm{H}}^2 + \alpha \int_\tau^t \|\nabla \mathbf{u}^0(s)\|_{\mathbb{H}}^2 \mathrm{d}s \\
&\leq \left(\|\mathbf{u}_0\|_{\mathbb{H}}^2 + \|\xi_0\|_{\mathrm{H}}^2 + t\|\mathrm{U}_0\|_{\mathbb{H}}^2 + \frac{r}{\lambda} \int_\tau^t \|\mathbf{w}^0(s)\|_{\mathbb{L}^4}^4 \mathrm{d}s + \int_\tau^t \|\mathbf{f}(s)\|_{\mathbb{V}'}^2 \mathrm{d}s \right) e^{Kt}, \qquad (3.10)
\end{aligned}$$

for all $t \in [\tau, T]$. Since $(\mathbf{u}^*, \xi^*, \mathrm{U}^*)$ is an optimal triplet and U_0 is a non-optimal control in general, we have

$$\begin{aligned}
&\frac{1}{2}\left[\|\mathbf{u}^*(T)\|_{\mathbb{H}}^2 + \|\xi^*(T)\|_{\mathbb{H}}^2 + \int_\tau^T \left[\|\nabla \mathbf{u}^*(t)\|_{\mathbb{H}}^2 + \|\xi^*(t)\|_{\mathrm{H}}^2 + \|\mathrm{U}^*(t)\|_{\mathbb{H}}^2 \right] \mathrm{d}t \right] \\
&\leq \frac{1}{2}\left[\|\mathbf{u}^0(T)\|_{\mathbb{H}}^2 + \|\xi^0(T)\|_{\mathbb{H}}^2 + \int_\tau^T \left[\|\nabla \mathbf{u}^0(t)\|_{\mathbb{H}}^2 + t\|\xi^0(t)\|_{\mathrm{H}}^2 + \|\mathrm{U}_0\|_{\mathbb{H}}^2 \right] \mathrm{d}t \right] \\
&\leq C \sup_{\tau \leq t \leq T} \left[\|\mathbf{u}^0(t)\|_{\mathbb{H}}^2 + T\|\xi^0(t)\|_{\mathrm{H}}^2 \right] + T\|\mathrm{U}_0\|_{\mathbb{H}}^2 + \int_\tau^T \|\nabla \mathbf{u}^0(t)\|_{\mathbb{H}}^2 \mathrm{d}t, \qquad (3.11)
\end{aligned}$$

and hence the Theorem follows using (3.10). □

4. Dynamic programming and feedback analysis: the Hamilton-Jacobi equation

In this section, we describe the dynamic programming method for the optimal control of two-dimensional tidal dynamics system. We use the same method used in [11, 38, 39] for the 2D Navier-Stokes equations to establish Pontryagin's maximum principle (see [18] also). Our main aim is to show that the feedback control can be obtained from the solution of an infinite dimensional Hamilton-Jacobi equation. In this context, we consider the optimal control problem (3.2) with the control

$$\mathrm{U} \in \mathrm{L}^2(\tau, T; \mathcal{U}_R), \quad \text{where } \mathcal{U}_R := \{\mathrm{U} \in \mathbb{H} : \|\mathrm{U}\|_{\mathbb{H}} \leq R\}.$$

That is, here we take $\mathscr{U}_{\mathrm{ad}} = \mathrm{L}^2(\tau, T; \mathcal{U}_R)$. Moreover, we consider $\mathbf{f} \in \mathrm{C}([\tau, T]; \mathbb{H})$ (that is, we are also taking $\mathbf{w}^0 \in \mathrm{C}([\tau, T]; \mathbb{W}) \cap \mathrm{C}^1((\tau, T); \mathbb{H})$) and $(\mathbf{u}(\tau), \xi(\tau)) = (\mathbf{v}, \varrho) \in \mathbb{V} \cap \mathbb{W} \times \mathrm{V}$. We define the value function as

$$\mathcal{V}(\tau, \mathbf{v}, \varrho) := \min\left\{ \mathcal{J}(\tau, T, \mathbf{v}, \varrho, \mathbf{u}, \xi, \mathrm{U}) : \mathrm{U} \in \mathrm{L}^2(0, T; \mathbb{H}) \right\}, \qquad (4.1)$$

where $\mathcal{J}$ is defined in (3.1). We show that the value function $\mathcal{V}$ satisfies the Hamilton-Jacobi equation:

$$\frac{\partial \mathcal{V}}{\partial t} - \mathscr{H}\left(\mathbf{v}, \varrho, \frac{\partial \mathcal{V}}{\partial \mathbf{v}}, \frac{\partial \mathcal{V}}{\partial \varrho} \right) = 0, \quad \text{for } (\tau, \mathbf{v}, \varrho) \in (0, T) \times (\mathbb{V} \cap \mathbb{W}) \times \mathrm{V} \qquad (4.2)$$

with

$$\mathcal{V}(T, \mathbf{v}, \varrho) = \frac{1}{2}\|\mathbf{v}\|_{\mathbb{H}}^2 + \frac{1}{2}\|\varrho\|_{\mathrm{H}}^2, \ (\mathbf{v}, \varrho) \in \mathbb{H} \times \mathrm{H}. \qquad (4.3)$$

We obtain the *true-Hamiltonian* $\mathscr{H}$ from the *pseudo-Hamiltonian* $\widetilde{\mathscr{H}}$ in the following way:

$$\mathscr{H}(\mathbf{v}, \varrho, \mathbf{p}, \varphi) = \sup_{\mathrm{V} \in \mathcal{U}_R} \widetilde{\mathscr{H}}(\mathbf{v}, \varrho, \mathbf{p}, \varphi, \mathrm{V}), \qquad (4.4)$$

where

$$\widetilde{\mathscr{H}}(\mathbf{v},\varrho,\mathbf{p},\varphi,\mathrm{V})=(\mathrm{A}\mathbf{v}+\mathrm{B}(\mathbf{v})+\nabla\xi-\mathbf{f},\mathbf{p})_{\mathbb{H}}-(\mathrm{V},\mathbf{p})_{\mathbb{H}}+(\mathrm{div}(h\mathbf{v}),\varphi)_{\mathrm{H}} \\ -\frac{1}{2}\big(\|\nabla\mathbf{v}\|_{\mathbb{H}}^2+\|\varrho\|_{\mathrm{H}}^2+\|\mathrm{V}\|_{\mathbb{H}}^2\big). \tag{4.5}$$

Thus, the true-Hamiltonian is given by

$$\mathscr{H}(\mathbf{v},\varrho,\mathbf{p},\varphi)=(\mathrm{A}\mathbf{v}+\mathrm{B}(\mathbf{v})+\nabla\xi-\mathbf{f},\mathbf{p})_{\mathbb{H}}+(\mathrm{div}(h\mathbf{v}),\varphi)_{\mathrm{H}}+\Upsilon(\mathbf{p})-\frac{1}{2}\big(\|\nabla\mathbf{v}\|_{\mathbb{H}}^2+\|\varrho\|_{\mathrm{H}}^2\big), \tag{4.6}$$

where

$$\Upsilon(\mathbf{p})=\begin{cases}\frac{1}{2}\|\mathbf{p}\|_{\mathbb{H}}^2, & \text{if } \|\mathbf{p}\|_{\mathbb{H}}\leq R,\\ R\|\mathbf{p}\|_{\mathbb{H}}-\frac{1}{2}R^2, & \text{if } \|\mathbf{p}\|_{\mathbb{H}}\geq R.\end{cases} \tag{4.7}$$

Also, the optimal control is obtained as,

$$\mathrm{U}(t)=\sigma(\mathbf{p}(t)),\quad \text{with}\quad \sigma(\mathbf{p})=\begin{cases}-\mathbf{p}, & \text{if } \|\mathbf{p}\|_{\mathbb{H}}\leq R,\\ -\frac{\mathbf{p}R}{\|\mathbf{p}\|_{\mathbb{H}}}, & \text{if } \|\mathbf{p}\|_{\mathbb{H}}\geq R,\end{cases} \tag{4.8}$$

and $\mathbf{p}(t)=\partial_{\mathbf{v}}\mathcal{V}(t,\mathbf{u}(t),\xi(t))$, if the value function $\mathcal{V}$ is differentiable. However, $\mathcal{V}$ is not differentiable and the lack of smoothness tells us that the above quantities must be understood in a generalized sense. We use the method of viscosity solution to get the required result. The existence and uniqueness of strong solution to the system (2.1) is discussed in Theorem A.6 and in the following Proposition, we provide an estimate which is used in proving the smoothness property of the value function.

Proposition 4.1. *Under the assumptions of Theorem A.6, for some* $\mathrm{U}_0\in\mathbb{H}$*, the corresponding solution* $(\mathbf{u}^0,\xi^0)$ *of* (2.1) *satisfies*

$$\|\mathbf{u}^0(t)\|_{\mathbb{V}}^2+\|\xi^0(t)\|_{\mathrm{V}}^2+\int_\tau^T\|\mathbf{u}^0(t)\|_{\mathbb{W}}^2\mathrm{d}t\leq C\big(\|\mathbf{v}\|_{\mathbb{V}},\|\mathrm{U}_0\|_{\mathbb{H}},\|\mathbf{f}\|_{\mathrm{L}^\infty(\tau,T;\mathbb{H})},\|\mathbf{w}^0\|_{\mathrm{L}^\infty(\tau,T;\mathbb{L}^4(\Omega))}\big), \tag{4.9}$$

and

$$\|\mathbf{u}^0(t)-\mathbf{v}\|_{\mathbb{H}}+\|\xi^0(t)-\varrho\|_{\mathrm{H}}\leq C\big(\|\mathbf{v}\|_{\mathbb{V}},\|\varrho\|_{\mathrm{H}},\|\mathrm{U}_0\|_{\mathbb{H}},\|\mathbf{f}\|_{\mathrm{L}^\infty(\tau,T;\mathbb{H})},\|\mathbf{w}^0\|_{\mathrm{L}^\infty(\tau,T;\mathbb{L}^4(\Omega))}\big)|t-\tau|, \tag{4.10}$$

for all $t\in[\tau,T]$*. The constant* C *depends on the constants* $r,\alpha,\mu,\lambda,M,M_1,T$*.*

Proof. The existence and uniqueness of strong solution to the system (2.1) and the estimate (4.9) is established in Theorem 2.11 from [29]. In order to get (4.10), we first note that $(\mathbf{z}(t),\vartheta(t))=(\mathbf{u}^0(t)-\mathbf{v},\xi^0(t)-\varrho)$ satisfies

$$\begin{cases}\dfrac{\partial\mathbf{z}(t)}{\partial t}+\mathrm{A}\mathbf{z}(t)+\mathrm{B}(\mathbf{u}^0(t))-\mathrm{B}(\mathbf{v})+\nabla\vartheta(t)=\mathbf{f}(t)+\mathrm{U}_0-\mathrm{A}\mathbf{v}-\mathrm{B}(\mathbf{v})-\nabla\varrho, & \text{in } \Omega\times(\tau,T),\\ \dfrac{\partial\vartheta(t)}{\partial t}+\mathrm{div}(h\mathbf{z}(t))=-\mathrm{div}(h\mathbf{v}), & \text{in } \Omega\times(\tau,T),\\ \mathbf{z}(t)=\mathbf{0}, & \text{on } \partial\Omega\times(\tau,T),\\ \mathbf{z}(\tau)=\mathbf{0},\ \vartheta(\tau)=0, & \text{in } \Omega.\end{cases} \tag{4.11}$$

Let us take an inner product with $\mathbf{z}(\cdot)$ to the first equation and $\vartheta(\cdot)$ to the second equation in (4.11), and then add to obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{z}(t)\|_{\mathbb{H}}^2+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\vartheta(t)\|_{\mathbb{H}}^2+\alpha\|\nabla\mathbf{z}(t)\|_{\mathbb{H}}^2\\
&=-(\mathrm{B}(\mathbf{u}^0(t))-\mathrm{B}(\mathbf{v}),\mathbf{z}(t))_{\mathbb{H}}-\langle\nabla\vartheta(t),\mathbf{z}(t)\rangle+\langle\mathbf{f}(t)+\mathrm{U}_0-\mathrm{A}\mathbf{v}-\mathrm{B}(\mathbf{v})-\nabla\varrho,\mathbf{z}(t)\rangle\\
&\quad-(\mathrm{div}(h\mathbf{z}(t)),\vartheta(t))_{\mathrm{H}}+(\mathrm{div}(h\mathbf{v}),\vartheta(t))_{\mathrm{H}}\\
&=-(\mathrm{B}(\mathbf{u}^0(t))-\mathrm{B}(\mathbf{v}),\mathbf{z}(t))_{\mathbb{H}}+(\vartheta(t),\mathrm{div}\,\mathbf{z}(t))_{\mathbb{H}}+\langle\mathbf{f}(t)+\mathrm{U}_0-\mathrm{A}\mathbf{v}-\mathrm{B}(\mathbf{v})-\nabla\varrho,\mathbf{z}(t)\rangle\\
&\quad-(h\mathrm{div}\,\mathbf{z}(t),\vartheta(t))_{\mathrm{H}}-(\nabla h\cdot\mathbf{z}(t),\vartheta(t))_{\mathbb{H}}+(h\mathrm{div}\,\mathbf{v},\vartheta)_{\mathrm{H}}+(\nabla h\cdot\mathbf{v},\vartheta)_{\mathbb{H}}\\
&\le\|\vartheta(t)\|_{\mathrm{H}}\|\mathrm{div}\,\mathbf{z}(t)\|_{\mathrm{H}}+\|\mathbf{f}(t)\|_{\mathbb{H}}\|\mathbf{z}(t)\|_{\mathbb{H}}+\|\mathrm{U}_0\|_{\mathbb{H}}\|\mathbf{z}(t)\|_{\mathbb{H}}+\|\mathrm{A}\mathbf{v}\|_{\mathbb{V}'}\|\mathbf{z}(t)\|_{\mathbb{V}}\\
&\quad+\|\mathrm{B}(\mathbf{v})\|_{\mathbb{H}}\|\mathbf{z}(t)\|_{\mathbb{H}}+\|\varrho\|_{\mathrm{H}}\|\mathrm{div}\,\mathbf{z}(t)\|_{\mathrm{H}}+\|h\|_{\mathrm{L}^\infty}\|\mathrm{div}\,\mathbf{z}(t)\|_{\mathrm{H}}\|\vartheta(t)\|_{\mathrm{H}}\\
&\quad+\|\nabla h\|_{\mathbb{L}^\infty}\|\mathbf{z}(t)\|_{\mathbb{H}}\|\vartheta(t)\|_{\mathrm{H}}+\|h\|_{\mathrm{L}^\infty}\|\mathrm{div}\,\mathbf{v}\|_{\mathrm{H}}\|\vartheta(t)\|_{\mathrm{H}}+\|\nabla h\|_{\mathbb{L}^\infty}\|\mathbf{v}\|_{\mathbb{H}}\|\vartheta(t)\|_{\mathrm{H}}\\
&\le\frac{\alpha}{2}\|\nabla\mathbf{z}(t)\|_{\mathbb{H}}^2+\left[\frac{4}{\alpha}\big(1+2\mu^2\big)+2M^2\right]\|\vartheta(t)\|_{\mathrm{H}}^2+\|\mathbf{z}(t)\|_{\mathbb{H}}^2+\|\mathbf{f}(t)\|_{\mathbb{H}}^2+\|\mathrm{U}_0\|_{\mathbb{H}}^2\\
&\quad+\Big(1+\frac{\alpha}{8}\Big)\|\nabla\mathbf{v}\|_{\mathbb{H}}^2+\|\mathrm{B}(\mathbf{v})\|_{\mathbb{H}}^2+\frac{4}{\alpha}\|\varrho\|_{\mathrm{H}}^2+\frac{1}{4}\|\mathbf{v}\|_{\mathbb{H}}^2, \qquad (4.12)
\end{aligned}
$$

where we used the fact that $(\mathrm{B}(\mathbf{u}^0(t))-\mathrm{B}(\mathbf{v}),\mathbf{z}(t))_{\mathbb{H}}\ge 0$ (see Lem. 2.2 (*iii*)), $\|\mathrm{div}\,\mathbf{z}\|_{\mathbb{H}}\le\sqrt{2}\|\nabla\mathbf{u}\|_{\mathbb{H}}$, Cauchy-Schwarz, Young's and Hölder's inequalities. Let us now integrate the inequality (4.12) from τ to t to find

$$
\begin{aligned}
&\|\mathbf{z}(\tau)\|_{\mathbb{H}}^2+\|\vartheta(\tau)\|_{\mathrm{H}}^2+\int_\tau^t\|\nabla\mathbf{z}(s)\|_{\mathbb{H}}^2\mathrm{d}s\\
&\le\max\left\{1,\frac{4}{\alpha}\big(1+2\mu^2\big)+2M^2\right\}\int_\tau^t\big[\|\mathbf{z}(s)\|_{\mathbb{H}}^2+\|\vartheta(s)\|_{\mathrm{H}}^2\big]\mathrm{d}s+\int_\tau^t\|\mathbf{f}(s)\|_{\mathbb{H}}^2\mathrm{d}s\\
&\quad+\int_\tau^t\|\mathrm{B}(\mathbf{v})\|_{\mathbb{H}}^2\mathrm{d}s+\left(\|\mathrm{U}_0\|_{\mathbb{H}}^2+\Big(1+\frac{\alpha}{8}\Big)\|\nabla\mathbf{v}\|_{\mathbb{H}}^2+\frac{4}{\alpha}\|\varrho\|_{\mathrm{H}}^2+\frac{1}{4}\|\mathbf{v}\|_{\mathbb{H}}^2\right)(t-\tau). \qquad (4.13)
\end{aligned}
$$

Using Lemma 2.2 (i), we know that

$$
\begin{aligned}
\|\mathrm{B}(\mathbf{v})\|_{\mathbb{H}}^2&\le\frac{r^2}{\lambda^2}(\|\mathbf{v}\|_{\mathbb{L}^4}+\|\mathbf{w}^0\|_{\mathbb{L}^4})^4\le\frac{8r^2}{\lambda^2}\big(\|\mathbf{v}\|_{\mathbb{L}^4}^4+\|\mathbf{w}^0\|_{\mathbb{L}^4}^4\big)\\
&\le\frac{16r^2}{\lambda^2}\|\mathbf{v}\|_{\mathbb{H}}^2\|\nabla\mathbf{v}\|_{\mathbb{H}}^2+\frac{8r^2}{\lambda^2}\|\mathbf{w}^0\|_{\mathbb{L}^4}^4, \qquad (4.14)
\end{aligned}
$$

where we used Ladyzhenskaya's and Young's inequalities. Use (4.14) in (4.13) and an application of Gronwall's inequality in (4.13) yields

$$
\begin{aligned}
&\|\mathbf{z}(\tau)\|_{\mathbb{H}}^2+\|\vartheta(\tau)\|_{\mathrm{H}}^2+\int_\tau^t\|\nabla\mathbf{z}(s)\|_{\mathbb{H}}^2\mathrm{d}s\\
&\le\Bigg[\int_\tau^t\|\mathbf{f}(s)\|_{\mathbb{H}}^2\mathrm{d}s+\frac{8r^2}{\lambda^2}\int_\tau^t\|\mathbf{w}^0(s)\|_{\mathbb{L}^4}^4\mathrm{d}s\\
&\quad+\left(\|\mathrm{U}_0\|_{\mathbb{H}}^2+\Big(1+\frac{\alpha}{8}\Big)\|\nabla\mathbf{v}\|_{\mathbb{H}}^2+\frac{4}{\alpha}\|\varrho\|_{\mathrm{H}}^2+\frac{1}{4}\|\mathbf{v}\|_{\mathbb{H}}^2\right)(t-\tau)\Bigg]e^{\max\left\{1,\frac{4}{\alpha}\left(1+2\mu^2\right)+2M^2\right\}(t-\tau)}\\
&\le\left[\sup_{s\in[\tau,t]}\|\mathbf{f}(s)\|_{\mathbb{H}}^2+\frac{8r^2}{\lambda^2}\sup_{s\in[\tau,t]}\|\mathbf{w}^0(s)\|_{\mathbb{L}^4}^4+\|\mathrm{U}_0\|_{\mathbb{H}}^2+\left(1+\frac{\alpha}{8}+\frac{C_\Omega}{4}\right)\|\nabla\mathbf{v}\|_{\mathbb{H}}^2+\frac{4}{\alpha}\|\varrho\|_{\mathrm{H}}^2\right]
\end{aligned}
$$

$$\times (t-\tau)e^{\max\left\{1,\frac{4}{\alpha}\left(1+2\mu^2\right)+2M^2\right\}t}, \tag{4.15}$$

since $\mathbf{f}\in \mathrm{L}^{\infty}(0,T;\mathbb{H})$ and $\mathbf{w}^0\in \mathrm{L}^{\infty}(0,T;\mathbb{L}^4(\Omega))$. In (4.15), C_{Ω} is the Poincaré constant and the inequality (4.10) follows from (4.15). □

Remark 4.2. In virtue of (2.5) and the Sobolev embedding ($\mathrm{H}^s(0,T)\subset \mathrm{L}^{\infty}(0,T)$, for $s>1/2$), in order to obtain the regularity $\mathbf{f}\in \mathrm{L}^{\infty}(0,T;\mathbb{H})$, we need

$$\mathbf{w}^0\in \mathrm{L}^{\infty}(0,T;\mathbb{W})\cap \mathrm{H}^1(0,T;\mathbb{H}).$$

Using the Proposition 4.1 and Theorem 2.6, we prove the following smoothness theorem on the value function $\mathcal{V}$.

Theorem 4.3. *The value function $\mathcal{V}$ is continuous. That is, $\mathcal{V}\in \mathrm{C}([0,T]\times\mathbb{H}\times \mathrm{H})$. Furthermore, we have*

- (i) *for all $t\in[0,T]$, $\mathcal{V}(t,\cdot,\cdot):\mathbb{H}\times\mathrm{H}\to[0,\infty)$ is locally Lipschitz. That is, for all $C>0$, there exists $K=K(C)>0$ such that*

$$|\mathcal{V}(t,\mathbf{v}_1,\varrho_1)-\mathcal{V}(t,\mathbf{v}_2,\varrho_2)|\leq K[\|\mathbf{v}_1-\mathbf{v}_2\|_{\mathbb{H}}+\|\varrho_1-\varrho_2\|_{\mathrm{H}}], \tag{4.16}$$

for all $\mathbf{v}_1,\mathbf{v}_2\in\mathbb{H}$ and $\varrho_1,\varrho_2\in\mathrm{H}$ with $\|\mathbf{v}_1\|_{\mathbb{H}},\|\mathbf{v}_2\|_{\mathbb{H}},\|\varrho_1\|_{\mathrm{H}},\|\varrho\|_{\mathrm{H}}\leq C$.

- (ii) *For all $\mathbf{v}\in\mathbb{V}$ and $\varrho\in\mathrm{H}$, $\mathcal{V}(\cdot,\mathbf{v},\varrho):[0,T]\to[0,\infty)$ is locally Lipschitz. That is, for all $C>0$, there exists $K=K(C)>0$ such that*

$$|\mathcal{V}(t,\mathbf{v},\varrho)-\mathcal{V}(s,\mathbf{v},\varrho)|\leq K|t-s|, \tag{4.17}$$

for all $\mathbf{v}\in\mathbb{V}$ and $\varrho\in\mathrm{H}$ with $\|\mathbf{v}\|_{\mathbb{H}},\|\varrho\|_{\mathrm{H}}\leq C$, and $t,s\in[\tau,T]$.

Finally, we have the following continuity property in the negative norm:

- (iii) *for all $C>0$, there exists $K=K(C)>0$ such that*

$$|\mathcal{V}(t,\mathbf{v}_1,\varrho_1)-\mathcal{V}(t,\mathbf{v}_2,\varrho_2)|\leq K[\|\mathbf{v}_1-\mathbf{v}_2\|_{\mathbb{V}'}+\|\varrho_1-\varrho_2\|_{\mathrm{V}'}], \tag{4.18}$$

for all $\mathbf{v}_1,\mathbf{v}_2\in\mathbb{V}$ and $\varrho_1,\varrho_2\in\mathrm{H}$ with $\|\nabla\mathbf{v}_1\|_{\mathbb{H}},\|\nabla\mathbf{v}_2\|_{\mathbb{H}},\|\varrho_1\|_{\mathrm{H}},\|\varrho\|_{\mathrm{H}}\leq C$.

The constants appearing in the Theorem 4.3 depend on the norms of $\mathbf{f}$ and $\mathbf{w}^0$, but we suppress the dependence for notational convenience.

Proof of Theorem 4.3. (i). Let $\mathrm{U}(t)$ be the optimal control corresponding to the initial data $(\tau,\mathbf{v}_1,\varrho_1)$. Then from the Theorem 3.5, we know that

$$\|\mathrm{U}\|_{\mathrm{L}^2(0,T;\mathbb{H})}\leq C(\|\mathbf{v}_1\|_{\mathbb{H}},\|\varrho_1\|_{\mathrm{H}},\tau,T). \tag{4.19}$$

Furthermore, in general U will not be an optimal control for an another initial data $(\tau,\mathbf{v}_2,\varrho_2)$. Let us denote the solution corresponding to the initial data $(\tau,\mathbf{v}_1,\varrho_1)$ with optimal control U as $(\mathbf{u}_1(t),\xi_1(t))$ and that corresponding to the initial data $(\tau,\mathbf{v}_2,\varrho_2)$ with control U as $(\mathbf{u}_2(t),\xi_2(t))$. Then, we have

$$\begin{aligned}&\mathcal{V}(\tau,\mathbf{v}_2,\varrho_2)-\mathcal{V}(\tau,\mathbf{v}_1,\varrho_1)\\&\leq\frac{1}{2}\|\mathbf{u}_2(T)\|_{\mathbb{H}}^2+\frac{1}{2}\|\xi_2(T)\|_{\mathbb{H}}^2+\frac{1}{2}\int_{\tau}^{T}\left[\|\nabla\mathbf{u}_2(t)\|_{\mathbb{H}}^2+\|\xi_2(t)\|_{\mathrm{H}}^2\right]\mathrm{d}t\end{aligned}$$

$$-\frac{1}{2}\|\mathbf{u}_1(T)\|_{\mathbb{H}}^2-\frac{1}{2}\|\xi_1(T)\|_{\mathrm{H}}^2-\frac{1}{2}\int_\tau^T\big[\|\nabla\mathbf{u}_1(t)\|_{\mathbb{H}}^2+\|\xi_1(t)\|_{\mathrm{H}}^2\big]\mathrm{d}t. \tag{4.20}$$

Using the continuous dependence result (A.2) and the energy bound (A.1), we have

$$\begin{aligned}
&\|\mathbf{u}_2(T)\|_{\mathbb{H}}^2-\|\mathbf{u}_1(T)\|_{\mathbb{H}}^2+\|\xi_2(T)\|_{\mathrm{H}}^2-\|\xi_1(T)\|_{\mathrm{H}}^2\\
&=(\|\mathbf{u}_2(T)\|_{\mathbb{H}}-\|\mathbf{u}_1(T)\|_{\mathbb{H}})(\|\mathbf{u}_2(T)\|_{\mathbb{H}}+\|\mathbf{u}_1(T)\|_{\mathbb{H}})\\
&\quad+(\|\xi_2(T)\|_{\mathrm{H}}-\|\xi_1(T)\|_{\mathrm{H}})(\|\xi_2(T)\|_{\mathrm{H}}+\|\xi_1(T)\|_{\mathrm{H}})\\
&\leq\|\mathbf{u}_2(T)-\mathbf{u}_1(T)\|_{\mathbb{H}}(\|\mathbf{u}_2(T)\|_{\mathbb{H}}+\|\mathbf{u}_1(T)\|_{\mathbb{H}})\\
&\quad+\|\xi_2(T)-\xi_1(T)\|_{\mathrm{H}}(\|\xi_2(T)\|_{\mathrm{H}}+\|\xi_1(T)\|_{\mathrm{H}})\\
&\leq C(\|\mathbf{v}_1\|_{\mathbb{H}},\|\mathbf{v}_2\|_{\mathbb{H}},\|\varrho_1\|_{\mathrm{H}},\|\varrho_2\|_{\mathrm{H}},\tau,T)\big(\|\mathbf{v}_1-\mathbf{v}_2\|_{\mathbb{H}}^2+\|\varrho_1-\varrho_2\|_{\mathrm{H}}^2\big)^{1/2}\\
&\leq C(\|\mathbf{v}_1\|_{\mathbb{H}},\|\mathbf{v}_2\|_{\mathbb{H}},\|\varrho_1\|_{\mathrm{H}},\|\varrho_2\|_{\mathrm{H}},\tau,T)(\|\mathbf{v}_1-\mathbf{v}_2\|_{\mathbb{H}}+\|\varrho_1-\varrho_2\|_{\mathrm{H}}),
\end{aligned} \tag{4.21}$$

since $(a^2+b^2)^{1/2}\leq(a+b)$, for all $a,b\geq 0$. Using Hölder's inequality, (A.2) and (A.1), we also obtain

$$\begin{aligned}
&\int_\tau^T\big[\|\nabla\mathbf{u}_2(t)\|_{\mathbb{H}}^2-\|\nabla\mathbf{u}_1(t)\|_{\mathbb{H}}^2\big]\mathrm{d}t+\int_\tau^T\big[\|\xi_2(t)\|_{\mathrm{H}}^2-\|\xi_1(t)\|_{\mathrm{H}}^2\big]\mathrm{d}t\\
&\leq\int_\tau^T\|\nabla\mathbf{u}_2(t)-\nabla\mathbf{u}_1(t)\|_{\mathbb{H}}(\|\nabla\mathbf{u}_2(t)\|_{\mathbb{H}}+\|\nabla\mathbf{u}_1(t)\|_{\mathbb{H}})\mathrm{d}t\\
&\quad+\int_\tau^T\|\xi_2(t)-\xi_1(t)\|_{\mathrm{H}}(\|\xi_2(t)\|_{\mathrm{H}}+\|\xi_1(t)\|_{\mathrm{H}})\mathrm{d}t\\
&\leq\left(\int_\tau^T\|\nabla\mathbf{u}_2(t)-\nabla\mathbf{u}_1(t)\|_{\mathbb{H}}^2\mathrm{d}t\right)^{1/2}\left(2\int_\tau^T\big[\|\nabla\mathbf{u}_2(t)\|_{\mathbb{H}}^2+\|\nabla\mathbf{u}_1(t)\|_{\mathbb{H}}^2\big]\mathrm{d}t\right)^{1/2}\\
&\quad+(T-\tau)\sup_{t\in[\tau,T]}\|\xi_2(t)-\xi_1(t)\|_{\mathrm{H}}\sup_{t\in[\tau,T]}(\|\xi_2(t)\|_{\mathrm{H}}+\|\xi_1(t)\|_{\mathrm{H}})\\
&\leq C(\|\mathbf{v}_1\|_{\mathbb{H}},\|\mathbf{v}_2\|_{\mathbb{H}},\|\varrho_1\|_{\mathrm{H}},\|\varrho_2\|_{\mathrm{H}},\tau,T)(\|\mathbf{v}_1-\mathbf{v}_2\|_{\mathbb{H}}+\|\varrho_1-\varrho_2\|_{\mathrm{H}}).
\end{aligned} \tag{4.22}$$

Similarly, for the optimal control corresponding to $(\tau,\mathbf{v}_2,\varrho_2)$, one can get estimates similar to (4.21) and (4.22), where the roles of $(\mathbf{v}_1,\varrho_1)$ and $(\mathbf{v}_2,\varrho_2)$ are interchanged, and hence (4.16) follows easily.

(*ii*). Let us now show the continuity with respect to time, *i.e.*, (4.17). In order to establish this, we construct a special control in the following way:

$$\widetilde{\mathrm{U}}(s):=\begin{cases}\mathrm{U}_0\in\mathcal{U}_R, & \text{if } s\in(\tau,t),\\ \mathrm{U}(s), & \text{if } s\in(t,T),\end{cases} \tag{4.23}$$

with $\tau\leq t\leq T$. In (4.23), U is the optimal control corresponding to the initial data $(t,\mathbf{v},\varrho)$. Let $(\widetilde{\mathbf{u}},\widetilde{\xi})$ and $(\mathbf{u},\xi)$, respectively be the solutions corresponding to

$$\left\{\begin{aligned}
\frac{\partial\widetilde{\mathbf{u}}(r)}{\partial r}+\mathrm{A}\widetilde{\mathbf{u}}(r)+\mathrm{B}(\widetilde{\mathbf{u}}(r))+\nabla\widetilde{\xi}(r)&=\mathbf{f}(r)+\widetilde{\mathrm{U}}(r), \text{ in } \Omega\times(\tau,T),\\
\frac{\partial\widetilde{\xi}(r)}{\partial r}+\mathrm{div}(h\widetilde{\mathbf{u}}(r))&=0, \text{ in } \Omega\times(\tau,T),\\
\widetilde{\mathbf{u}}(r)&=\mathbf{0}, \text{ on } \partial\Omega\times(\tau,T),\\
\widetilde{\mathbf{u}}(\tau)&=\mathbf{v},\ \widetilde{\xi}(\tau)=\varrho, \text{ in } \Omega,
\end{aligned}\right. \tag{4.24}$$

and

$$\begin{cases} \dfrac{\partial \mathbf{u}(r)}{\partial r}+\mathrm{A}\mathbf{u}(r)+\mathrm{B}(\mathbf{u}(r))+\nabla\xi(r)=\mathbf{f}(r)+\mathrm{U}(r), \ \text{ in } \ \Omega\times(t,T),\\ \dfrac{\partial \xi(r)}{\partial r}+\operatorname{div}(h\mathbf{u}(r))=0, \ \text{ in } \ \Omega\times(t,T),\\ \mathbf{u}(r)=\mathbf{0}, \ \text{ on } \ \partial\Omega\times(t,T),\\ \mathbf{u}(t)=\mathbf{v}, \ \xi(t)=\varrho, \ \text{ in } \ \Omega, \end{cases} \tag{4.25}$$

Let us first take $(\mathbf{v},\varrho)\in\mathbb{H}\times\mathrm{H}$. Note that the trajectory $(\mathbf{u},\xi)$ is optimal and $(\widetilde{\mathbf{u}},\widetilde{\xi})$ is non-optimal. Using the definition of the value function in (4.1), we have

$$\begin{aligned} &\mathcal{V}(\tau,\mathbf{v},\varrho)-\mathcal{V}(t,\mathbf{v},\varrho)\\ &\leq \frac{1}{2}\|\widetilde{\mathbf{u}}(T)\|_{\mathbb{H}}^2+\frac{1}{2}\|\widetilde{\xi}(T)\|_{\mathbb{H}}^2+\frac{1}{2}\int_{\tau}^{T}\Big[\|\nabla\widetilde{\mathbf{u}}(s)\|_{\mathbb{H}}^2+\|\widetilde{\xi}(s)\|_{\mathrm{H}}^2+\|\widetilde{\mathrm{U}}(s)\|_{\mathbb{H}}^2\Big]\mathrm{d}s\\ &\quad-\frac{1}{2}\|\mathbf{u}(T)\|_{\mathbb{H}}^2-\frac{1}{2}\|\xi(T)\|_{\mathbb{H}}^2-\frac{1}{2}\int_{t}^{T}\big[\|\nabla\mathbf{u}(s)\|_{\mathbb{H}}^2+\|\xi(s)\|_{\mathrm{H}}^2+\|\mathrm{U}(s)\|_{\mathbb{H}}^2\big]\mathrm{d}s. \end{aligned} \tag{4.26}$$

Since there exist unique weak solutions of the systems (4.24) and (4.25), making use of the semigroup property, we have (see [3])

$$\widetilde{\mathbf{x}}(r,\tau;\mathbf{v},\varrho;\mathrm{U})=\mathbf{x}(r,t;\widetilde{\mathbf{x}}(t,\tau;\mathbf{v},\varrho;\mathrm{U}_0);\mathrm{U}), \ \text{ for } \ \tau\leq t\leq r, \tag{4.27}$$

where $\mathbf{x}=(\mathbf{u},\xi)$ and $\widetilde{\mathbf{x}}=(\widetilde{\mathbf{u}},\widetilde{\xi})$. Let us use (4.27), the Theorem A.3 (continuous dependence) and energy estimate (A.1) in (4.26) to obtain

$$\begin{aligned} &\|\widetilde{\mathbf{u}}(T)\|_{\mathbb{H}}^2-\|\mathbf{u}(T)\|_{\mathbb{H}}^2+\|\widetilde{\xi}(T)\|_{\mathbb{H}}^2-\|\xi(T)\|_{\mathbb{H}}^2\\ &\leq\|\widetilde{\mathbf{u}}(T)-\mathbf{u}(T)\|_{\mathbb{H}}(\|\widetilde{\mathbf{u}}(T)\|_{\mathbb{H}}+\|\mathbf{u}(T)\|_{\mathbb{H}})+\|\widetilde{\xi}(T)-\xi(T)\|_{\mathbb{H}}\Big(\|\widetilde{\xi}(T)\|_{\mathbb{H}}+\|\xi(T)\|_{\mathbb{H}}\Big)\\ &\leq C(\|\mathbf{v}\|_{\mathbb{H}},\|\varrho\|_{\mathrm{H}})\Big(\|\widetilde{\mathbf{u}}(t)-\mathbf{v}\|_{\mathbb{H}}+\|\widetilde{\xi}(t)-\varrho\|_{\mathrm{H}}\Big). \end{aligned} \tag{4.28}$$

Similarly, we obtain

$$\begin{aligned} &\int_t^T\Big[\|\nabla\widetilde{\mathbf{u}}(s)\|_{\mathbb{H}}^2+\|\widetilde{\xi}(s)\|_{\mathrm{H}}^2\Big]\mathrm{d}s-\int_t^T\big[\|\nabla\mathbf{u}(s)\|_{\mathbb{H}}^2+\|\xi(s)\|_{\mathrm{H}}^2\big]\mathrm{d}s\\ &\leq\left(\int_t^T\|\nabla(\widetilde{\mathbf{u}}(s)-\mathbf{u}(s))\|_{\mathbb{H}}^2\mathrm{d}s\right)^{1/2}\left(2\int_t^T\big[\|\nabla\widetilde{\mathbf{u}}(s)\|_{\mathbb{H}}^2+\|\nabla\mathbf{u}(s)\|_{\mathbb{H}}^2\big]\mathrm{d}s\right)^{1/2}\\ &\quad+\sup_{s\in[t,T]}\|\widetilde{\xi}(s)-\xi(s)\|_{\mathrm{H}}\left[\sup_{s\in[t,T]}\Big(\|\widetilde{\xi}(s)\|_{\mathrm{H}}+\|\xi(s)\|_{\mathrm{H}}\Big)\right](T-t)\\ &\leq C(\|\mathbf{v}\|_{\mathbb{H}},\|\varrho\|_{\mathrm{H}})\Big(\|\widetilde{\mathbf{u}}(t)-\mathbf{v}\|_{\mathbb{H}}+\|\widetilde{\xi}(t)-\varrho\|_{\mathrm{H}}\Big). \end{aligned} \tag{4.29}$$

Finally, we have

$$\begin{aligned}&\int_\tau^t\Big[\|\nabla\widetilde{\mathbf{u}}(s)\|_{\mathbb{H}}^2+\|\widetilde{\xi}(s)\|_{\mathrm{H}}^2+\|\mathrm{U}_0\|_{\mathbb{H}}^2\Big]\mathrm{d}s\\&\leq(t-\tau)\left[\|\mathrm{U}_0\|_{\mathbb{H}}^2+\sup_{s\in[\tau,t]}\|\widetilde{\xi}(s)\|_{\mathrm{H}}^2\right]+\int_\tau^t\|\nabla\widetilde{\mathbf{u}}(s)\|_{\mathbb{H}}^2\mathrm{d}s.\end{aligned}\tag{4.30}$$

Note that the final integral appearing in (4.30) is finite, since $(\widetilde{\mathbf{u}},\widetilde{\xi})$ is a weak solution to the system (4.25). Combining (4.28)–(4.30) and using the Proposition 4.1, it is immediate that $\mathcal{V}\in\mathrm{C}([0,T]\times\mathbb{H}\times\mathrm{H})$. Furthermore, if the initial data satisfies $(\mathbf{v},\varrho)\in\mathbb{V}\times\mathrm{V}$, using the Proposition 4.1 (see (4.9)), we have

$$\int_\tau^t\|\nabla\widetilde{\mathbf{u}}(s)\|_{\mathbb{H}}^2\mathrm{d}s\leq\sup_{s\in[\tau,t]}\|\nabla\widetilde{\mathbf{u}}(s)\|_{\mathbb{H}}^2(t-\tau).\tag{4.31}$$

Once again combining (4.28)-(4.31), and then using the Proposition 4.1 (see (4.10)), we get the local Lipschitz property of the value function in time, *cf.* (4.17).

(*iii*) Let us now establish the continuity in the negative norm. The proof is similar to that of (4.16) and we use the Theorem 2.6 to get such a result. From (4.20), we have

$$\begin{aligned}&\|\mathbf{u}_2(T)\|_{\mathbb{H}}^2-\|\mathbf{u}_1(T)\|_{\mathbb{H}}^2+\|\xi_2(T)\|_{\mathrm{H}}^2-\|\xi_1(T)\|_{\mathrm{H}}^2\\&=(\mathbf{u}_2(T)-\mathbf{u}_1(T),\mathbf{u}_2(T))_{\mathbb{H}}+(\mathbf{u}_2(T)-\mathbf{u}_1(T),\mathbf{u}_1(T))_{\mathbb{H}}\\&\quad+(\xi_2(T)-\xi_1(T),\xi_2(T))_{\mathrm{H}}+(\xi_2(T)-\xi_1(T),\xi_1(T))_{\mathrm{H}}\\&\leq\|\mathbf{u}_2(T)-\mathbf{u}_1(T)\|_{\mathbb{V}'}(\|\mathbf{u}_2(T)\|_{\mathbb{V}}+\|\mathbf{u}_1(T)\|_{\mathbb{V}})\\&\quad+\|\xi_2(T)-\xi_1(T)\|_{\mathrm{V}'}(\|\xi_1(T)\|_{\mathrm{V}}+\|\xi_2(T)\|_{\mathrm{V}})\\&\leq C(R,\|\mathbf{v}_1\|_{\mathbb{V}},\|\mathbf{v}_2\|_{\mathbb{V}},\|\varrho_1\|_{\mathrm{V}},\|\varrho_2\|_{\mathrm{V}})(\|\mathbf{v}_1-\mathbf{v}_2\|_{\mathbb{V}'}+\|\varrho_1-\varrho_2\|_{\mathrm{V}'}),\end{aligned}\tag{4.32}$$

using (2.20) and the energy estimate of the strong solution (see (4.9)). Similarly, using an integration by parts, we have

$$\begin{aligned}&\int_\tau^T[\|\nabla\mathbf{u}_2(t)\|_{\mathbb{H}}^2-\|\nabla\mathbf{u}_1(t)\|_{\mathbb{H}}^2]\mathrm{d}t+\int_\tau^T[\|\xi_2(t)\|_{\mathrm{H}}^2-\|\xi_1(t)\|_{\mathrm{H}}^2]\mathrm{d}t\\&=\int_\tau^T[(\nabla\mathbf{u}_2(t)-\nabla\mathbf{u}_1(t),\nabla\mathbf{u}_2(t))_{\mathbb{H}}+(\nabla\mathbf{u}_2(t)-\nabla\mathbf{u}_1(t),\nabla\mathbf{u}_1(t))_{\mathbb{H}}]\mathrm{d}t\\&\quad+\int_\tau^T[(\xi_2(t)-\xi_1(t),\xi_2(t))_{\mathrm{H}}+(\xi_2(t)-\xi_1(t),\xi_1(t))_{\mathrm{H}}]\mathrm{d}t\\&\leq\int_\tau^T\|\mathbf{u}_2(t)-\mathbf{u}_1(t)\|_{\mathbb{H}}(\|\mathbf{u}_1(t)\|_{\mathbb{W}}+\|\mathbf{u}_2(t)\|_{\mathbb{W}})\mathrm{d}t\\&\quad+\int_\tau^T\|\xi_2(t)-\xi_1(t)\|_{\mathrm{V}'}(\|\xi_2(t)\|_{\mathrm{V}}+\|\xi_1(t)\|_{\mathrm{V}})\mathrm{d}t\\&\leq\left(\int_\tau^T\|\mathbf{u}_2(t)-\mathbf{u}_1(t)\|_{\mathbb{H}}^2\mathrm{d}t\right)^{1/2}\left(2\int_\tau^T\big[\|\mathbf{u}_1(t)\|_{\mathbb{W}}^2+\|\mathbf{u}_2(t)\|_{\mathbb{W}}^2\big]\mathrm{d}t\right)^{1/2}\\&\quad+\sup_{t\in[\tau,T]}\|\xi_2(t)-\xi_1(t)\|_{\mathrm{V}'}\sup_{t\in[\tau,T]}(\|\xi_2(t)\|_{\mathrm{V}}+\|\xi_1(t)\|_{\mathrm{V}})(T-\tau)\\&\leq C(R,\|\mathbf{v}_1\|_{\mathbb{V}},\|\mathbf{v}_2\|_{\mathbb{V}},\|\varrho_1\|_{\mathrm{V}},\|\varrho_2\|_{\mathrm{V}})(\|\mathbf{v}_1-\mathbf{v}_2\|_{\mathbb{V}'}+\|\varrho_1-\varrho_2\|_{\mathrm{V}'}),\end{aligned}\tag{4.33}$$

using (2.20) and (4.9), which completes the proof of (*iii*). □

4.1. Bellman's principle of optimality

Our next aim is to establish the Bellman's principle of optimality for the value function (4.1).

Theorem 4.4 (Bellman's principle of optimality)**.** *For $0 \leq \tau \leq t \leq T$, the value function*

$$\begin{aligned}&\mathcal{V}(\tau,\mathbf{v},\varrho)\\&=\inf\left\{\frac{1}{2}\int_\tau^t\left[\|\nabla\mathbf{u}(s)\|_{\mathbb{H}}^2+\|\xi(s)\|_{\mathrm{H}}^2+\|\mathrm{U}(s)\|_{\mathbb{H}}^2\right]\mathrm{d}s+\mathcal{V}(t,\mathbf{u}(t),\xi(t));\ \mathrm{U}\in\mathrm{L}^2(\tau,t;\mathcal{U}_R)\right\}.\end{aligned}\tag{4.34}$$

Proof. Let $\widetilde{\mathrm{U}}$ be an optimal control in the interval (τ,T) corresponding to the initial data $(\tau,\mathbf{v})$ and $(\widetilde{\mathbf{u}},\widetilde{\xi})$ be the corresponding solution trajectory. Or in other words, $(\widetilde{\mathbf{u}},\widetilde{\xi},\widetilde{\mathrm{U}})$ is an optimal triplet. Then, from the definition of value function (see (4.1)), we have

$$\begin{aligned}\mathcal{V}(\tau,\mathbf{v},\varrho)&=\frac{1}{2}\int_\tau^T\left[\|\nabla\widetilde{\mathbf{u}}(t)\|_{\mathbb{H}}^2+\|\widetilde{\xi}(t)\|_{\mathrm{H}}^2+\|\widetilde{\mathrm{U}}(t)\|_{\mathbb{H}}^2\right]\mathrm{d}t+\frac{1}{2}\|\widetilde{\mathbf{u}}(T)\|_{\mathbb{H}}^2+\frac{1}{2}\|\widetilde{\xi}(T)\|_{\mathbb{H}}^2\\&=\frac{1}{2}\int_\tau^t\left[\|\nabla\widetilde{\mathbf{u}}(s)\|_{\mathbb{H}}^2+\|\widetilde{\xi}(s)\|_{\mathrm{H}}^2+\|\widetilde{\mathrm{U}}(s)\|_{\mathbb{H}}^2\right]\mathrm{d}s\\&\quad+\frac{1}{2}\int_t^T\left[\|\nabla\widetilde{\mathbf{u}}(s)\|_{\mathbb{H}}^2+\|\widetilde{\xi}(s)\|_{\mathrm{H}}^2+\|\widetilde{\mathrm{U}}(s)\|_{\mathbb{H}}^2\right]\mathrm{d}s+\frac{1}{2}\|\widetilde{\mathbf{u}}(T)\|_{\mathbb{H}}^2+\frac{1}{2}\|\widetilde{\xi}(T)\|_{\mathbb{H}}^2\\&\geq\frac{1}{2}\int_\tau^t\left[\|\nabla\widetilde{\mathbf{u}}(s)\|_{\mathbb{H}}^2+\|\widetilde{\xi}(s)\|_{\mathrm{H}}^2+\|\widetilde{\mathrm{U}}(s)\|_{\mathbb{H}}^2\right]\mathrm{d}s\\&\quad+\inf\left\{\frac{1}{2}\int_t^T\left[\|\nabla\mathbf{z}(s)\|_{\mathbb{H}}^2+\|\vartheta(s)\|_{\mathrm{H}}^2+\|\mathrm{U}(s)\|_{\mathbb{H}}^2\right]\mathrm{d}s\right.\\&\qquad\left.+\frac{1}{2}\|\mathbf{z}(T)\|_{\mathbb{H}}^2+\frac{1}{2}\|\vartheta(T)\|_{\mathbb{H}}^2;\mathrm{U}\in\mathrm{L}^2(0,T;\mathbb{H})\right\},\end{aligned}\tag{4.35}$$

where $(\mathbf{z},\vartheta)$ solves the tidal dynamics system (2.1) with the initial data $(\widetilde{\mathbf{u}}(t),\widetilde{\xi}(t))$ and control U. Note that the second term in the right hand side of (4.35) is the value function. Thus, from (4.35), it is immediate that

$$\mathcal{V}(\tau,\mathbf{v},\varrho)\geq\frac{1}{2}\int_\tau^t\left[\|\nabla\widetilde{\mathbf{u}}(s)\|_{\mathbb{H}}^2+\|\widetilde{\xi}(s)\|_{\mathrm{H}}^2+\|\widetilde{\mathrm{U}}(s)\|_{\mathbb{H}}^2\right]\mathrm{d}s+\mathcal{V}(t,\widetilde{\mathbf{u}}(t),\widetilde{\xi}(t)).\tag{4.36}$$

Note that for all $\mathrm{U}\in\mathrm{L}^2(0,T;\mathcal{U}_R)$, and for the corresponding solution $(\mathbf{u},\xi)$ of the tidal dynamic system (2.1), we have

$$\begin{aligned}\mathcal{V}(\tau,\mathbf{v},\varrho)&\leq\frac{1}{2}\int_\tau^t\left[\|\nabla\mathbf{u}(s)\|_{\mathbb{H}}^2+\|\xi(s)\|_{\mathrm{H}}^2+\|\mathrm{U}(s)\|_{\mathbb{H}}^2\right]\mathrm{d}s\\&\quad+\frac{1}{2}\int_t^T\left[\|\nabla\mathbf{u}(s)\|_{\mathbb{H}}^2+\|\xi(s)\|_{\mathrm{H}}^2+\|\mathrm{U}(s)\|_{\mathbb{H}}^2\right]\mathrm{d}s+\frac{1}{2}\|\mathbf{u}(T)\|_{\mathbb{H}}^2+\frac{1}{2}\|\xi(T)\|_{\mathbb{H}}^2.\end{aligned}\tag{4.37}$$

Let us now choose the part of U in $[t,T]$ to be optimal control, so that the sum of second, third and final terms in the right hand side of (4.37) become equal to the value function. Thus, from (4.37), we also have

$$\mathcal{V}(\tau,\mathbf{v},\varrho) \leq \frac{1}{2}\int_\tau^t \left[\|\nabla \mathbf{u}(s)\|_{\mathbb{H}}^2 + \|\xi(s)\|_{\mathrm{H}}^2 + \|\mathrm{U}(s)\|_{\mathbb{H}}^2\right]\mathrm{d}s + \mathcal{V}(t,\mathbf{u}(t),\xi(t)). \tag{4.38}$$

Remember that the control U in $[\tau,t)$ is arbitrary. Taking infimum over all $\mathrm{U}\in \mathrm{L}^2(\tau,t;\mathcal{U}_R)$ in (4.38), we infer that

$$\begin{aligned}&\mathcal{V}(\tau,\mathbf{v},\varrho)\\ &\leq \inf\left\{\frac{1}{2}\int_\tau^t \left[\|\nabla \mathbf{u}(s)\|_{\mathbb{H}}^2 + \|\xi(s)\|_{\mathrm{H}}^2 + \|\mathrm{U}(s)\|_{\mathbb{H}}^2\right]\mathrm{d}s + \mathcal{V}(t,\mathbf{u}(t),\xi(t));\ \mathrm{U}\in \mathrm{L}^2(\tau,t;\mathcal{U}_R)\right\}.\end{aligned} \tag{4.39}$$

Combining (4.36) and (4.39), we finally obtain (4.34). □

4.2. Viscosity solution

The next goal is to show that the value function is a viscosity solution to the Hamilton-Jacobi equation (4.2). In order to establish this, we first define a certain class of test functions and also give the definition of viscosity solution.

Definition 4.5. A class of test functions $\phi : [\tau,T]\times\mathbb{H}\times\mathrm{H}\to\mathbb{R}$ is called *test functions* of class $\mathcal{D}$ if

(*i*) $\phi(\cdot,\cdot,\cdot) : [\tau,T]\times\mathbb{H}\times\mathrm{H}\to\mathbb{R}$ is locally Lipschitz and
(*ii*) the Fréchet derivative

$$\mathrm{D}\phi = (\partial_s\phi, \partial_{\mathbf{v}}\phi, \partial_\xi\phi) : [\tau,T]\times\mathbb{V}\times\mathrm{V}\to\mathbb{R}\times\mathbb{V}'\times\mathrm{V}'$$

is locally Lipschitz.

Definition 4.6. The value function $\mathcal{V} : [0,T]\times\mathbb{H}\times\mathrm{H}\to\mathbb{R}$ is called *a viscosity solution* to the Hamilton-Jacobi equation (4.2), if for all $\phi\in\mathcal{D}$, and
if $\mathcal{V}-\phi$ attains a local maximum at $(t_0,\mathbf{v}_0,\varrho_0)\in(0,T)\times\mathbb{W}\times\mathrm{V}$, then

$$-\partial_t\phi(t_0,\mathbf{v}_0,\varrho_0) + \mathscr{H}(\mathbf{v}_0,\varrho_0,\partial_{\mathbf{v}}\phi,\partial_\varrho\phi) \leq 0, \tag{4.40}$$

and

if $\mathcal{V}-\phi$ attains attains a local minimum at $(t_0,\mathbf{v}_0,\varrho_0)\in(0,T)\times\mathbb{W}\times\mathrm{V}$, then

$$-\partial_t\phi(t_0,\mathbf{v}_0,\varrho_0) + \mathscr{H}(\mathbf{v}_0,\varrho_0,\partial_{\mathbf{v}}\phi,\partial_\varrho\phi) \geq 0. \tag{4.41}$$

Theorem 4.7. *The value function* $\mathcal{V} : [0,T]\times\mathbb{H}\times\mathrm{H}\to\mathbb{R}$ *is a viscosity solution to the Hamilton-Jacobi equation* (4.2) *in the sense of Definition 4.6.*

Proof. Let $\phi\in\mathcal{D}$ and suppose that $\mathcal{V}-\phi$ attains a local minimum at $(t_0,\mathbf{v}_0,\varrho_0)\in(0,T)\times\mathbb{W}\times\mathrm{V}$. Without loss of generality, we may assume that the minimum is zero. Let us now substitute ϕ for $\mathcal{V}$ in Bellman's principle (see (4.34)) to obtain

$$\phi(t_0,\mathbf{v}_0,\varrho_0)\geq \inf\left\{\frac{1}{2}\int_{t_0}^{t_0+\varepsilon}\left[\|\nabla\mathbf{u}(s)\|_{\mathbb{H}}^2+\|\xi(s)\|_{\mathrm{H}}^2+\|\mathrm{U}(s)\|_{\mathbb{H}}^2\right]\mathrm{d}s\right.$$
$$\left.+\,\phi(t_0+\varepsilon,\mathbf{u}(t_0+\varepsilon),\xi(t_0+\varepsilon));\ \mathrm{U}\in\mathrm{L}^2(t_0,t_0+\varepsilon;\mathcal{U}_R)\right\}. \tag{4.42}$$

Using fundamental theorem of calculus, we have

$$\begin{aligned}
&\phi(t_0,\mathbf{v}_0,\varrho_0)\\
&\geq \inf\left\{\frac{1}{2}\int_{t_0}^{t_0+\varepsilon}\left[\|\nabla\mathbf{u}(s)\|_{\mathbb{H}}^2+\|\xi(s)\|_{\mathrm{H}}^2+\|\mathrm{U}(s)\|_{\mathbb{H}}^2\right]\mathrm{d}s+\phi(t_0,\mathbf{v}_0,\varrho_0)\right.\\
&\quad+\int_{t_0}^{t_0+\varepsilon}\left[\frac{\partial\phi}{\partial t}-\left(\mathrm{A}\mathbf{u}(t)+\mathrm{B}(\mathbf{u}(t))+\nabla\xi(t)-\mathbf{f}(t)-\mathrm{U}(t),\frac{\partial\phi}{\partial\mathbf{v}}\right)_{\mathbb{H}}-\left(\mathrm{div}(h\mathbf{u}(t)),\frac{\partial\phi}{\partial\xi}\right)_{\mathrm{H}}\right]\mathrm{d}t;\\
&\quad\left.\mathrm{U}\in\mathrm{L}^2(t_0,t_0+\varepsilon;\mathcal{U}_R)\right\}.
\end{aligned}\tag{4.43}$$

Using the pseudo-Hamiltonian (see (4.5)), one can rewrite (4.43) as

$$\begin{aligned}
&\sup\left\{\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon}\left[-\frac{\partial\phi}{\partial t}(t,\mathbf{u}(t),\xi(t))+\widetilde{\mathscr{H}}\left(\mathbf{u}(t),\xi(t),\frac{\partial\phi}{\partial\mathbf{v}}(t,\mathbf{u}(t),\xi(t)),\frac{\partial\phi}{\partial\xi}(t,\mathbf{u}(t),\xi(t)),\mathrm{U}(t)\right)\right]\mathrm{d}t;\right.\\
&\quad\left.\mathrm{U}\in\mathrm{L}^2(t_0,t_0+\varepsilon;\mathcal{U}_R)\right\}\geq 0.
\end{aligned}\tag{4.44}$$

Let us now use the continuous dependence result (see Prop. 4.8 below) and the continuous differentiability of ϕ to deduce that

$$\begin{aligned}
&\sup\left\{\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon}\left|\left[-\frac{\partial\phi}{\partial t}(t,\mathbf{u}(t),\xi(t))+\widetilde{\mathscr{H}}\left(\mathbf{u}(t),\xi(t),\frac{\partial\phi}{\partial\mathbf{v}}(t,\mathbf{u}(t),\xi(t)),\frac{\partial\phi}{\partial\xi}(t,\mathbf{u}(t),\xi(t)),\mathrm{U}(t)\right)\right]\mathrm{d}t\right.\right.\\
&\quad-\left.\left[-\frac{\partial\phi}{\partial t}(t_0,\mathbf{v}_0,\varrho_0)+\widetilde{\mathscr{H}}\left(\mathbf{v}_0,\varrho_0,\frac{\partial\phi}{\partial\mathbf{v}}(t_0,\mathbf{v}_0,\varrho_0),\frac{\partial\phi}{\partial\xi}(t_0,\mathbf{v}_0,\varrho_0),\mathrm{U}(t)\right)\right]\right|\mathrm{d}t;\\
&\quad\left.\mathrm{U}\in\mathrm{L}^2(t_0,t_0+\varepsilon;\mathcal{U}_R)\right\}\leq g(\varepsilon),
\end{aligned}\tag{4.45}$$

where $g(\varepsilon)\geq 0$ and $g(\varepsilon)\to 0$ as $\varepsilon\to 0$. One can estimate (4.45) in the following way. For $\mathrm{U}\in\mathrm{L}^2(t_0,t_0+\varepsilon;\mathcal{U}_R)$, let us first consider

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon}\left|\left[\frac{\partial\phi}{\partial t}(t,\mathbf{u}(t),\xi(t))-\frac{\partial\phi}{\partial t}(t_0,\mathbf{v}_0,\varrho_0)\right]\right|\mathrm{d}t\\
&\leq\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon}L(t,\|\mathbf{u}(t)\|_{\mathbb{V}},\|\xi(t)\|_{\mathrm{V}})(|t-t_0|+\|\mathbf{u}(t)-\mathbf{v}_0\|_{\mathbb{V}}+\|\xi(t)-\varrho_0\|_{\mathrm{V}})\mathrm{d}t\\
&\leq C(R,\|\mathbf{v}_0\|_{\mathbb{W}},\|\varrho_0\|_{\mathrm{V}})\big(\varepsilon+\sqrt{\varepsilon}\big),
\end{aligned}$$

where we used the local Lipschitz property of the Fréchet derivative of ϕ (see Def. 4.5, $L(\cdot,\cdot,\cdot)$ denotes the local Lipschitz constant) and (4.53). Similarly, for $\mathrm{U}\in\mathrm{L}^2(t_0,t_0+\varepsilon;\mathcal{U}_R)$, using the definition of pseudo-Hamiltonian given in (4.5), local Lipschitz property of the Fréchet derivative of ϕ and (4.53), one can establish that

$$\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon}\bigg|\widetilde{\mathscr{H}}\bigg(\mathbf{u}(t),\xi(t),\frac{\partial\phi}{\partial\mathbf{v}}(t,\mathbf{u}(t),\xi(t)),\frac{\partial\phi}{\partial\xi}(t,\mathbf{u}(t),\xi(t)),\mathrm{U}(t)\bigg) - \widetilde{\mathscr{H}}\bigg(\mathbf{v}_0,\varrho_0,\frac{\partial\phi}{\partial\mathbf{v}}(t_0,\mathbf{v}_0,\varrho_0),\frac{\partial\phi}{\partial\xi}(t_0,\mathbf{v}_0,\varrho_0),\mathrm{U}(t)\bigg)\bigg|\mathrm{d}t \leq C(R,\|\mathbf{v}_0\|_{\mathbb{W}},\|\varrho_0\|_{\mathrm{V}})\big(\varepsilon+\sqrt{\varepsilon}\big).$$

Using (4.44) in (4.45), we find

$$\sup\bigg\{\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon}\bigg[-\frac{\partial\phi}{\partial t}(t_0,\mathbf{v}_0,\varrho_0)+\widetilde{\mathscr{H}}\bigg(\mathbf{v}_0,\varrho_0,\frac{\partial\phi}{\partial\mathbf{v}}(t_0,\mathbf{v}_0,\varrho_0),\frac{\partial\phi}{\partial\xi}(t_0,\mathbf{v}_0,\varrho_0),\mathrm{U}(t)\bigg)\bigg]\mathrm{d}t;\ \mathrm{U}\in\mathrm{L}^2(t_0,t_0+\varepsilon;\mathcal{U}_R)\bigg\}\geq -g(\varepsilon). \tag{4.46}$$

This also implies that

$$\sup_{\mathrm{U}\in\mathcal{U}_R}\bigg[-\frac{\partial\phi}{\partial t}(t_0,\mathbf{v}_0,\varrho_0)+\widetilde{\mathscr{H}}\bigg(\mathbf{v}_0,\varrho_0,\frac{\partial\phi}{\partial\mathbf{v}}(t_0,\mathbf{v}_0,\varrho_0),\frac{\partial\phi}{\partial\xi}(t_0,\mathbf{v}_0,\varrho_0),\mathrm{U}(t)\bigg)\bigg]\geq -g(\varepsilon). \tag{4.47}$$

Using the definition of true-Hamiltonian (see (4.4)), we also have

$$-\frac{\partial\phi}{\partial t}(t_0,\mathbf{v}_0,\varrho_0)+\mathscr{H}\bigg(\mathbf{v}_0,\varrho_0,\frac{\partial\phi}{\partial\mathbf{v}}(t_0,\mathbf{v}_0,\varrho_0),\frac{\partial\phi}{\partial\xi}(t_0,\mathbf{v}_0,\varrho_0)\bigg)\geq -g(\varepsilon). \tag{4.48}$$

Taking $\varepsilon\to 0$ in (4.48), we get (4.41).

Now, let us suppose that $\mathcal{V}-\varphi$ attains a local maximum at $(t_0,\mathbf{v}_0,\varrho_0)\in(0,T)\times\mathbb{W}\times\mathrm{V}$. An argument similar to above leads us to (see (4.44))

$$\sup\bigg\{\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon}\bigg[-\frac{\partial\phi}{\partial t}(t,\mathbf{u}(t),\xi(t))+\widetilde{\mathscr{H}}\bigg(\mathbf{u}(t),\xi(t),\frac{\partial\phi}{\partial\mathbf{v}}(t,\mathbf{u}(t),\xi(t)),\frac{\partial\phi}{\partial\xi}(t,\mathbf{u}(t),\xi(t)),\mathrm{U}(t)\bigg)\bigg]\mathrm{d}t;\ \mathrm{U}\in\mathrm{L}^2(t_0,t_0+\varepsilon;\mathcal{U}_R)\bigg\}\leq 0. \tag{4.49}$$

For a constant control $\mathrm{U}_0\in\mathcal{U}_R$, from (4.49), we also have

$$\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon}\bigg[-\frac{\partial\phi}{\partial t}(t,\mathbf{u}(t),\xi(t))+\widetilde{\mathscr{H}}\bigg(\mathbf{u}(t),\xi(t),\frac{\partial\phi}{\partial\mathbf{v}}(t,\mathbf{u}(t),\xi(t)),\frac{\partial\phi}{\partial\xi}(t,\mathbf{u}(t),\xi(t)),\mathrm{U}_0\bigg)\bigg]\mathrm{d}t\leq 0. \tag{4.50}$$

Using the continuous differentiability of ϕ and the continuous dependence of trajectories (see Prop. 4.8), as before (see (4.45)), we obtain

$$\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon}\bigg[-\frac{\partial\phi}{\partial t}(t_0,\mathbf{v}_0,\varrho_0)+\widetilde{\mathscr{H}}\bigg(\mathbf{v}_0,\varrho_0,\frac{\partial\phi}{\partial\mathbf{v}}(t_0,\mathbf{v}_0,\varrho_0),\frac{\partial\phi}{\partial\xi}(t_0,\mathbf{v}_0,\varrho_0),\mathrm{U}_0\bigg)\bigg]\mathrm{d}t\leq g(\varepsilon), \tag{4.51}$$

and then on taking $\varepsilon \to 0$, we get

$$-\frac{\partial \phi}{\partial t}(t_0, \mathbf{v}_0, \varrho_0) + \widetilde{\mathscr{H}}\bigg(\mathbf{v}_0, \varrho_0, \frac{\partial \phi}{\partial \mathbf{v}}(t_0, \mathbf{v}_0, \varrho_0), \frac{\partial \phi}{\partial \xi}(t_0, \mathbf{v}_0, \varrho_0), \mathrm{U}_0\bigg) \leq 0. \tag{4.52}$$

Note that one can take a specific U_0 in (4.52), in particular, $\mathrm{U}_0 = \sigma\Big(\frac{\partial \phi}{\partial \mathbf{v}}(t_0, \mathbf{v}_0, \varrho_0)\Big)$, where σ is defined in (4.8). With this choice, (4.52) becomes (4.40) and it completes the proof. ☐

Proposition 4.8. *Let* $(\mathbf{v}_0, \varrho_0) \in \mathbb{W}\cap\mathbb{V}\times \mathrm{V}$ *and* $\mathrm{U} \in \mathrm{L}^2(t_0, t_0+\varepsilon; \mathcal{U}_R)$*. Then,* $(\mathbf{z}(t), \vartheta(t)) = (\mathbf{u}(t)-\mathbf{v}_0, \xi(t)-\varrho_0)$*, where* $(\mathbf{u}(t), \xi(t))$ *solves* (2.1) *with* $(\mathbf{u}(t_0), \xi(t_0) = (\mathbf{v}_0, \varrho_0)$ *satisfies:*

$$\|\mathbf{z}(t)\|_{\mathbb{V}}^2 + \|\vartheta(t)\|_{\mathrm{V}}^2 + \alpha\int_{t_0}^t \|\mathbf{z}(r)\|_{\mathbb{W}}^2 \mathrm{d}r \leq C(R, \|\mathbf{v}_0\|_{\mathbb{W}}, \|\varrho_0\|_{\mathrm{V}})\varepsilon, \ t \in (t_0, t_0+\varepsilon). \tag{4.53}$$

A proof of the Proposition can be obtained in a similar way as in the proof of Theorem 2.7.

5. Pontryagin's maximum principle

Our next aim is to prove that the value function $\mathcal{V}$, a viscosity solution to the Hamilton-Jacobi equation (4.2) can be used to derive the Pontryagin maximum principle. Let us consider the following tidal dynamics system:

$$\left\{\begin{aligned} \frac{\partial \mathbf{u}(t)}{\partial t} + \mathrm{A}\mathbf{u}(t) + \mathrm{B}(\mathbf{u}(t)) + \nabla\xi(t) &= \mathbf{f}(t) + \widetilde{\mathrm{U}}(t), \quad \text{in } \ \Omega\times(s,T), \\ \frac{\partial \xi(t)}{\partial t} + \mathrm{div}(h\mathbf{u}(t)) &= 0, \quad \text{in } \ \Omega\times(s,T), \\ \mathbf{u}(t) &= \mathbf{0}, \quad \text{on } \ \partial\Omega\times(s,T), \\ \mathbf{u}(s) &= \mathbf{v}, \ \ \xi(s) = \varrho, \quad \text{in } \ \Omega. \end{aligned}\right. \tag{5.1}$$

where $(\mathbf{v}, \varrho) \in \mathbb{V}\cap\mathbb{W}\times\mathrm{V}$ and $\widetilde{\mathrm{U}} \in \mathrm{L}^2(\tau, T; \mathcal{U}_R)$ is the optimal control in $[\tau, T]$ for the initial data $(\tau, \mathbf{v}, \varrho)$. Remember that the trajectory $\mathbf{x}(t, s; \mathbf{v}, \varrho; \widetilde{\mathrm{U}})$, where $\mathbf{x} = (\mathbf{u}, \xi)^\top$, is an optimal trajectory only if $\tau = s$.

Remark 5.1. In order to establish the Pontryagin maximum principle, we use the regularity $(\mathbf{u}, \xi) \in (\mathrm{C}([s,T]; \mathbb{V})\cap\mathrm{L}^2(s,T;\mathbb{W}))\times\mathrm{C}([s,T];\mathrm{V})$, which indeed implies that $\widetilde{\mathrm{U}} \in \mathrm{C}([s,T];\mathbb{H})$.

Let us define a functional $\mathcal{W} : [\tau, T]\times\mathbb{H}\times\mathrm{H} \to \mathbb{R}$ as

$$\mathcal{W}(s, \mathbf{v}, \varrho) = \frac{1}{2}\|\mathbf{u}(T)\|_{\mathbb{H}}^2 + \frac{1}{2}\|\xi(T)\|_{\mathbb{H}}^2 + \frac{1}{2}\int_s^T \Big[\|\nabla\mathbf{u}(r)\|_{\mathbb{H}}^2 + \|\xi(r)\|_{\mathrm{H}}^2 + \|\widetilde{\mathrm{U}}(r)\|_{\mathbb{H}}^2\Big]\mathrm{d}r, \tag{5.2}$$

with

$$\mathcal{W}(T, \mathbf{v}, \varrho) = \frac{1}{2}\big(\|\mathbf{v}\|_{\mathbb{H}}^2 + \|\varrho\|_{\mathrm{H}}^2\big). \tag{5.3}$$

Here $(\mathbf{u}, \xi) \in (\mathrm{C}([s,T]; \mathbb{V})\cap\mathrm{L}^2(s,T;\mathbb{W}))\times\mathrm{C}([s,T];\mathrm{V})$ with the control $\widetilde{\mathrm{U}} \in \mathrm{C}([s,T];\mathbb{H})$ is the unique strong solution of the system (5.1) with the initial data $(\mathbf{u}(s), \xi(s)) = (\mathbf{v}, \varrho)$. Then, we have

Proposition 5.2. (i) *The function* $\mathcal{W} : [\tau, T]\times\mathbb{H}\times\mathrm{H} \to \mathbb{R}$ *is locally Lipschitz and for* s *almost everywhere in* (τ, T)*,*

$$\partial_s\mathcal{W}(s, \cdot, \cdot) : \mathbb{V}\times\mathrm{V} \to \mathbb{R} \tag{5.4}$$

is locally Lipschitz.

(ii) For all $s \in [\tau, T]$ and $\varrho \in \mathrm{V}$, the Fréchet derivative $\partial_{\mathbf{v}}\mathcal{W}(s, \mathbf{v}, \varrho) \in \mathbb{V}$, and for all $s \in [\tau, T]$ and $\mathbf{v} \in \mathbb{V}$, the Fréchet derivative $\partial_{\varrho}\mathcal{W}(s, \mathbf{v}, \varrho) \in \mathrm{V}$. Also, $\partial_{\mathbf{v}}\mathcal{W}(s, \cdot, \varrho) : \mathbb{V} \to \mathbb{V}$ and $\partial_{\varrho}\mathcal{W}(s, \mathbf{v}, \cdot) : \mathrm{V} \to \mathrm{V}$ are locally Lipschitz.

Remark 5.3. It should be noted that for s a.e. in $[\tau, T]$, $\mathcal{W}$ has all the remaining properties of the class $\mathcal{D}$ of test functions.

Proof of Proposition 5.2. Let us define $\Psi(\cdot), \Phi(\cdot)$ and $\Theta(\cdot)$, for some $t \in [s, T]$ as

$$\begin{cases} \Psi(t) := \partial_{\mathbf{v}}\mathbf{x}(t, s; \mathbf{v}, \varrho; \widetilde{\mathrm{U}})\gamma, \ \gamma \in \mathbb{H}, \\ \Phi(t) := \partial_{\varrho}\mathbf{x}(t, s; \mathbf{v}, \varrho; \widetilde{\mathrm{U}})\nu, \ \nu \in \mathrm{H}, \quad \text{and} \\ \mathbf{y}(t) := \partial_{s}\mathbf{x}(t, s; \mathbf{v}, \varrho; \widetilde{\mathrm{U}}), \end{cases} \tag{5.5}$$

where $\mathbf{x} = (\mathbf{u}, \xi)^{\top}$. Then one can easily show that $(\Psi, \Phi)^{\top}$ solves the following system:

$$\begin{cases} \dfrac{\partial \Psi(t)}{\partial t} + \mathrm{A}\Psi(t) + \mathrm{B}'(\mathbf{u}(t))\Psi(t) + \nabla\Phi(t) = \mathbf{0}, \ \text{ in } \ \Omega \times (s, T), \\ \dfrac{\partial \Phi(t)}{\partial t} + \operatorname{div}(h\Psi(t)) = 0, \ \text{ in } \ \Omega \times (s, T), \\ \Psi(t) = \mathbf{0}, \ \text{ on } \ \partial\Omega \times (s, T), \\ \Psi(s) = \gamma, \ \Phi(s) = \nu, \ \text{ in } \ \Omega, \end{cases} \tag{5.6}$$

Using the Theorem 2.8, for $(\gamma, \nu) \in \mathbb{H} \times \mathrm{H}$, we know that the system (5.6) has a unique weak solution $(\Psi, \Phi) \in (\mathrm{C}([s, T]; \mathbb{H}) \cap \mathrm{L}^2(s, T; \mathbb{V})) \times \mathrm{C}([s, T]; \mathrm{H})$, with $(\partial_t\Psi, \partial_t\Phi) \in \mathrm{L}^2(s, T; \mathbb{V}') \times \mathrm{L}^2(s, T; \mathrm{H})$. Note that $\mathbf{y} = (\Theta, \Pi)$ satisfies the following linear system:

$$\begin{cases} \dfrac{\partial \Theta(t)}{\partial t} + \mathrm{A}\Theta(t) + \mathrm{B}'(\mathbf{u}(t))\Theta(t) + \nabla\Pi(t) = \mathbf{0}, \ \text{ in } \ \Omega \times (s, T), \\ \dfrac{\partial \Pi(t)}{\partial t} + \operatorname{div}(h\Theta(t)) = 0, \ \text{ in } \ \Omega \times (s, T), \\ \Theta(t) = \mathbf{0}, \ \text{ on } \ \partial\Omega \times (s, T), \\ \Theta(s) = \mathrm{A}\mathbf{v} + \mathrm{B}'(\mathbf{u})\mathbf{v} + \nabla\varrho - \mathbf{f}(s) - \widetilde{\mathrm{U}}(s), \ \Pi(s) = \operatorname{div}(h\mathbf{v}), \ \text{ in } \ \Omega. \end{cases} \tag{5.7}$$

Remember that

$$\Theta(s) = \mathrm{A}\mathbf{v} + \mathrm{B}'(\mathbf{u})\mathbf{v} + \nabla\varrho - \mathbf{f}(s) - \widetilde{\mathrm{U}}(s) \in \mathbb{H}, \ \text{ and } \ \Pi(s) = \operatorname{div}(h\mathbf{v}) \in \mathrm{H}, \tag{5.8}$$

since $(\mathbf{v}, \varrho) \in \mathbb{V} \cap \mathbb{W} \times \mathrm{V}$. Once again using the Theorem 2.8, we obtain a unique weak solution of the system (5.6) such that $(\Theta(\cdot), \Pi(\cdot)) \in (\mathrm{C}([s, T]; \mathbb{H}) \cap \mathrm{L}^2(s, T; \mathbb{V})) \times \mathrm{C}([s, T]; \mathrm{H})$, and $(\partial_t\Theta, \partial_t\Pi) \in \mathrm{L}^2(s, T; \mathbb{V}') \times \mathrm{L}^2(s, T; \mathrm{H})$.

Now, for $\gamma \in \mathbb{H}$, we know that

$$(\partial_{\mathbf{v}}\mathcal{W}(s, \mathbf{v}, \varrho), \gamma)_{\mathbb{H}} = (\mathbf{u}(T), \Psi(T))_{\mathbb{H}} + \int_s^T (\nabla\mathbf{u}(r), \nabla\Psi(r))_{\mathbb{H}} \mathrm{d}r, \tag{5.9}$$

and for $\nu \in \mathrm{H}$,

$$(\partial_{\varrho}\mathcal{W}(s, \mathbf{v}, \varrho), \nu)_{\mathrm{H}} = (\xi(T), \Phi(T))_{\mathrm{H}} + \int_s^T (\xi(r), \Phi(r))_{\mathrm{H}} \mathrm{d}r. \tag{5.10}$$

Let us define the vectors $\mathrm{X} := (\partial_{\mathbf{v}}\mathcal{W}(s,\mathbf{v},\varrho), \partial_{\varrho}\mathcal{W}(s,\mathbf{v},\varrho))^{\top} \in \mathbb{X} := \mathbb{H}\times\mathrm{H}$ and $\varsigma = (\gamma,\nu)^{\top} \in \mathbb{X}$. We take the norm on the product space $\mathbb{X} = \mathbb{H}\times\mathrm{H}$ as $\|\cdot\|_{\mathbb{X}} = \|\cdot\|_{\mathbb{H}} + \|\cdot\|_{\mathrm{H}}$. Using Cauchy-Schwarz and Hölder's inequalities, we estimate $|(\mathrm{X},\varsigma))_{\mathbb{X}}|$ as

$$
\begin{aligned}
|(\mathrm{X},\varsigma)_{\mathbb{X}}| &\leq |(\mathbf{u}(T),\Psi(T))_{\mathbb{H}}| + \left|\int_s^T (\nabla\mathbf{u}(r),\nabla\Psi(r))_{\mathbb{H}}\mathrm{d}r\right| \\
&\quad + |(\xi(T),\Phi(T))_{\mathrm{H}}| + \left|\int_s^T (\xi(r),\Phi(r))_{\mathrm{H}}\mathrm{d}r\right| \\
&\leq \|\mathbf{u}(T)\|_{\mathbb{H}}\|\Psi(T)\|_{\mathbb{H}} + \int_s^T \|\nabla\mathbf{u}(r)\|_{\mathbb{H}}\|\nabla\Psi(r)\|_{\mathbb{H}}\mathrm{d}r \\
&\quad + \|\xi(T)\|_{\mathrm{H}}\|\Phi(T)\|_{\mathrm{H}} + \int_s^T \|\xi(r)\|_{\mathrm{H}}\|\Phi(r)\|_{\mathrm{H}}\mathrm{d}r \\
&\leq \|\mathbf{u}(T)\|_{\mathbb{H}}\|\Psi(T)\|_{\mathbb{H}} + \left(\int_s^T \|\nabla\mathbf{u}(r)\|_{\mathbb{H}}^2\mathrm{d}r\right)^{1/2}\left(\int_s^T \|\nabla\Psi(r)\|_{\mathbb{H}}^2\mathrm{d}r\right)^{1/2} \\
&\quad + \|\xi(T)\|_{\mathbb{H}}\|\Phi(T)\|_{\mathbb{H}} + \sup_{r\in[s,T]}\|\xi(r)\|_{\mathrm{H}}\sup_{r\in[s,T]}\|\Phi(r)\|_{\mathrm{H}}(T-s).
\end{aligned} \tag{5.11}
$$

Due to the energy estimates for $(\mathbf{u},\xi) \in (\mathrm{C}([s,T];\mathbb{H})\cap\mathrm{L}^2(s,T;\mathbb{V}))\times\mathrm{C}([s,T];\mathrm{H})$, $(\Psi,\Phi) \in (\mathrm{C}([s,T];\mathbb{H})\cap\mathrm{L}^2(s,T;\mathbb{V}))\times\mathrm{C}([s,T];\mathrm{H})$, and the fact that $\varsigma(s) = (\Psi(s),\Phi(s)) = (\gamma,\nu)\in\mathbb{H}\times\mathrm{H}$, from (5.11), it is immediate that

$$
\begin{aligned}
&\|\partial_{\mathbf{v}}\mathcal{W}(s,\mathbf{v},\varrho)\|_{\mathbb{H}} \leq C(\|\mathbf{v}\|_{\mathbb{H}},\|\varrho\|_{\mathrm{H}},R), \quad \text{and} \\
&\|\partial_{\varrho}\mathcal{W}(s,\mathbf{v},\varrho)\|_{\mathrm{H}} \leq C(\|\mathbf{v}\|_{\mathbb{H}},\|\varrho\|_{\mathrm{H}},R).
\end{aligned}
$$

Let us now consider

$$
\begin{aligned}
&|(\mathrm{X}_1-\mathrm{X}_2,\varsigma)_{\mathbb{H}}| \\
&= |(\partial_{\mathbf{v}}\mathcal{W}(s,\mathbf{v}_1,\varrho_1)-\partial_{\mathbf{v}}\mathcal{W}(s,\mathbf{v}_2,\varrho_2),\gamma)_{\mathbb{H}} + (\partial_{\varrho}\mathcal{W}(s,\mathbf{v}_1,\varrho_1)-\partial_{\varrho}\mathcal{W}(s,\mathbf{v}_2,\varrho_2),\nu)_{\mathrm{H}}| \\
&\leq \left|(\mathbf{u}_1(T),\Psi_1(T))_{\mathbb{H}} - (\mathbf{u}_2(T),\Psi_2(T))_{\mathbb{H}} + \int_s^T (\nabla\mathbf{u}_1(r),\nabla\Psi_1(r))_{\mathbb{H}} - (\nabla\mathbf{u}_2(r),\nabla\Psi_2(r))_{\mathbb{H}}\mathrm{d}r\right| \\
&\quad + \left|(\xi_1(T),\Phi_1(T))_{\mathrm{H}} - (\xi_2(T),\Phi_2(T))_{\mathrm{H}} + \int_s^T (\xi_1(r),\Phi_1(r))_{\mathrm{H}} - (\xi_2(r),\Phi_2(r))_{\mathrm{H}}\mathrm{d}r\right| \\
&\leq \|\mathbf{u}_1(T)-\mathbf{u}_2(T)\|_{\mathbb{H}}\|\Psi_1(T)\|_{\mathbb{H}} + \|\mathbf{u}_2(T)\|_{\mathbb{H}}\|\Psi_1(T)-\Psi_2(T)\|_{\mathbb{H}} \\
&\quad + \left(\int_s^T \|\nabla(\mathbf{u}_1(r)-\mathbf{u}_2(r))\|_{\mathbb{H}}^2\mathrm{d}r\right)^{1/2}\left(\int_s^T \|\nabla\Psi_1(r)\|_{\mathbb{H}}^2\mathrm{d}r\right)^{1/2} \\
&\quad + \left(\int_s^T \|\nabla\mathbf{u}_2(r)\|_{\mathbb{H}}^2\mathrm{d}r\right)^{1/2}\left(\int_s^T \|\nabla(\Psi_1(r)-\Psi_2(r))\|_{\mathbb{H}}^2\mathrm{d}r\right)^{1/2} \\
&\quad + \|\xi_1(T)-\xi_2(T)\|_{\mathrm{H}}\|\Phi_1(T)\|_{\mathrm{H}} + \|\xi_2(T)\|_{\mathrm{H}}\|\Phi_1(T)-\Phi_2(T)\|_{\mathrm{H}} \\
&\quad + \sup_{r\in[s,T]}\|\xi_1(r)-\xi_2(r)\|_{\mathrm{H}}\sup_{r\in[s,T]}\|\Phi_1(r)\|_{\mathrm{H}}(T-s) \\
&\quad + \sup_{r\in[s,T]}\|\xi_2(r)\|_{\mathrm{H}}\sup_{r\in[s,T]}\|\Phi_1(r)-\Phi_2(r)\|_{\mathrm{H}}(T-s),
\end{aligned} \tag{5.12}
$$

where we used the Cauchy-Schwarz and Hölder inequalities. Using continuous dependence theorems of $(\mathbf{u},\xi)$ and (Ψ,Φ) (see Thm. A.3 and Prop. 2.9) and the fact that $(\Psi_1(s),\Phi_1(s))=(\Psi_2(s),\Phi_2(s))=(\gamma,\nu)$, we get

$$\begin{aligned}&\|\partial_{\mathbf{v}}\mathcal{W}(s,\mathbf{v}_1,\varrho_1)-\partial_{\mathbf{v}}\mathcal{W}(s,\mathbf{v}_2,\varrho_2)\|_{\mathbb{H}}+\|\partial_{\varrho}\mathcal{W}(s,\mathbf{v}_1,\varrho_1)-\partial_{\mathbf{v}}\mathcal{W}(s,\mathbf{v}_2,\varrho_2)\|_{\mathrm{H}}\\&\leq C(R,\|\mathbf{v}_1\|_{\mathbb{H}},\|\mathbf{v}_2\|_{\mathbb{H}},\|\varrho_1\|_{\mathrm{H}},\|\varrho_2\|_{\mathrm{H}})(\|\mathbf{v}_1-\mathbf{v}_2\|_{\mathbb{H}}+\|\varrho_1-\varrho_2\|_{\mathrm{H}}).\end{aligned}\tag{5.13}$$

Let us first take $\varrho_1=\varrho_2=\varrho$ and then take $\mathbf{v}_1=\mathbf{v}_2=\mathbf{v}$ to obtain

$$\|\partial_{\mathbf{v}}\mathcal{W}(s,\mathbf{v}_1,\varrho)-\partial_{\mathbf{v}}\mathcal{W}(s,\mathbf{v}_2,\varrho)\|_{\mathbb{H}}\leq C(R,\|\mathbf{v}_1\|_{\mathbb{H}},\|\mathbf{v}_2\|_{\mathbb{H}},\|\varrho\|_{\mathrm{H}})\|\mathbf{v}_1-\mathbf{v}_2\|_{\mathbb{H}},\tag{5.14}$$
$$\|\partial_{\varrho}\mathcal{W}(s,\mathbf{v},\varrho_1)-\partial_{\varrho}\mathcal{W}(s,\mathbf{v},\varrho_2)\|_{\mathbb{H}}\leq C(R,\|\mathbf{v}\|_{\mathbb{H}},\|\varrho_1\|_{\mathrm{H}},\|\varrho_2\|_{\mathrm{H}})\|\varrho_1-\varrho_2\|_{\mathrm{H}}.\tag{5.15}$$

For a.e. $s\in[\tau,T]$, we have

$$\begin{aligned}\partial_s\mathcal{W}(s,\mathbf{v},\varrho)&=(\mathbf{u}(T),\Theta(T))_{\mathbb{H}}+(\xi(T),\Pi(T))_{\mathrm{H}}+\int_s^T(\nabla\mathbf{u}(r),\nabla\Theta(r))_{\mathbb{H}}+(\xi(r),\Pi(r))_{\mathrm{H}}\mathrm{d}r\\&\quad-\frac{1}{2}\big(\|\nabla\mathbf{v}\|_{\mathbb{H}}^2+\|\varrho\|_{\mathrm{H}}^2\big)-\frac{1}{2}\|\widetilde{\mathrm{U}}(s)\|_{\mathbb{H}}^2.\end{aligned}\tag{5.16}$$

Using the energy estimates satisfied by $(\mathbf{u},\xi)$ and (Θ,Π) (see for example, (A.1) and (2.36), and (5.8) also), we get

$$\begin{aligned}|\partial_s\mathcal{W}(s,\mathbf{v},\varrho)|&\leq\|\mathbf{u}(T)\|_{\mathbb{H}}\|\Theta(T)\|_{\mathbb{H}}+\|\xi(T)\|_{\mathrm{H}}\|\Pi(T)\|_{\mathrm{H}}\\&\quad+\left(\int_s^T\|\nabla\mathbf{u}(r)\|_{\mathbb{H}}^2\mathrm{d}r\right)^{1/2}\left(\int_s^T\|\nabla\Theta(r)\|_{\mathbb{H}}^2\mathrm{d}r\right)^{1/2}\\&\quad+\sup_{r\in[s,T]}\|\xi(r)\|_{\mathrm{H}}\sup_{r\in[s,T]}\|\Pi(r)\|_{\mathrm{H}}(T-s)+\frac{1}{2}\big(\|\nabla\mathbf{v}\|_{\mathbb{H}}^2+\|\varrho\|_{\mathrm{H}}^2\big)+\frac{1}{2}\|\widetilde{\mathrm{U}}(s)\|_{\mathbb{H}}^2\\&\leq C(\|\mathbf{v}\|_{\mathbb{W}},\|\varrho\|_{\mathrm{V}})+\frac{1}{2}\|\widetilde{\mathrm{U}}(s)\|_{\mathbb{H}}^2,\end{aligned}\tag{5.17}$$

for $\mathbf{v}\in\mathbb{V}\cap\mathbb{W}$ and $\varrho\in\mathrm{V}$. Since, $\widetilde{\mathrm{U}}\in\mathrm{L}^2(\tau,T;\mathbb{H})$, we finally have $\partial_s\mathcal{W}(s,\mathbf{v},\varrho)\in\mathrm{L}^1(\tau,T)$, for all $\mathbf{v}\in\mathbb{V}\cap\mathbb{W}$ and $\varrho\in\mathrm{V}$. Using the continuous dependence property of $(\mathbf{u},\xi)$ and (Θ,Π) (see for example (A.2) and (2.38)), we also obtain

$$\begin{aligned}&|\partial_s\mathcal{W}(s,\mathbf{v}_1,\varrho_1)-\partial_s\mathcal{W}(s,\mathbf{v}_2,\varrho_2)|\\&=(\mathbf{u}_1(T),\Theta_1(T))_{\mathbb{H}}-(\mathbf{u}_2(T),\Theta_2(T))_{\mathbb{H}}+(\xi_1(T),\Pi_1(T))_{\mathrm{H}}-(\xi_2(T),\Pi_2(T))_{\mathrm{H}}\\&\quad+\int_s^T(\nabla\mathbf{u}_1(r),\nabla\Theta_1(r))_{\mathbb{H}}-(\nabla\mathbf{u}_2(r),\nabla\Theta_2(r))_{\mathbb{H}}+(\xi_1(r),\Pi_1(r))_{\mathrm{H}}-(\xi_2(r),\Pi_2(r))_{\mathrm{H}}\mathrm{d}r\\&\quad-\frac{1}{2}\big(\|\nabla\mathbf{v}_1\|_{\mathbb{H}}^2-\|\nabla\mathbf{v}_2\|_{\mathbb{H}}^2+\|\varrho_1\|_{\mathrm{H}}^2-\|\varrho_2\|_{\mathrm{H}}^2\big)\\&\leq\|\mathbf{u}_1(T)-\mathbf{u}_2(T)\|_{\mathbb{H}}\|\Theta_1(T)\|_{\mathbb{H}}+\|\Theta_1(T)-\Theta_2(T)\|_{\mathbb{H}}\|\mathbf{u}_2(T)\|_{\mathbb{H}}\\&\quad+\|\xi_1(T)-\xi_2(T)\|_{\mathrm{H}}\|\Pi_1(T)\|_{\mathrm{H}}+\|\xi_2(T)\|_{\mathrm{H}}\|\Pi_1(T)-\Pi_2(T)\|_{\mathrm{H}}\\&\quad+\left(\int_s^T\|\nabla(\mathbf{u}_1(r)-\mathbf{u}_2(r))\|_{\mathbb{H}}^2\mathrm{d}r\right)^{1/2}\left(\int_s^T\|\nabla\Theta_1(r)\|_{\mathbb{H}}^2\mathrm{d}r\right)^{1/2}\end{aligned}$$

$$
\begin{aligned}
&+\left(\int_s^T \|\nabla\mathbf{u}_2(r)\|_{\mathbb{H}}^2\mathrm{d}r\right)^{1/2}\left(\int_s^T \|\nabla(\Theta_1(r)-\Theta_2(r))\|_{\mathbb{H}}^2\mathrm{d}r\right)^{1/2}\\
&+\sup_{r\in[\tau,T]}\|\xi_1(r)-\xi_2(r)\|_{\mathrm{H}}\sup_{r\in[\tau,T]}\|\Pi_2(r)\|_{\mathrm{H}}(T-s)\\
&+\sup_{r\in[\tau,T]}\|\xi_2(r)\|_{\mathrm{H}}\sup_{r\in[\tau,T]}\|\Pi_1(r)-\Pi_2(r)\|_{\mathrm{H}}(T-s)\\
\leq\ & C(R,\|\mathbf{v}_1\|_{\mathbb{H}},\|\mathbf{v}_2\|_{\mathbb{H}},\|\varrho_1\|_{\mathrm{H}},\|\varrho_2\|_{\mathrm{H}},\|\Theta_1(s)\|_{\mathbb{H}},\|\Theta_2(s)\|_{\mathbb{H}},\|\Pi_1(s)\|_{\mathrm{H}},\|\Pi_1(s)\|_{\mathrm{H}})\\
&\quad\times(\|\mathbf{v}_1-\mathbf{v}_2\|_{\mathbb{H}}+\|\varrho_1-\varrho_2\|_{\mathrm{H}}+\|\Theta_1(s)-\Theta_2(s)\|_{\mathbb{H}}+\|\Pi_1(s)-\Pi_2(s)\|_{\mathrm{H}})\\
\leq\ & C(R,\|\mathbf{v}_1\|_{\mathbb{W}},\|\mathbf{v}_2\|_{\mathbb{W}},\|\varrho_1\|_{\mathrm{V}},\|\varrho_2\|_{\mathrm{V}})(\|\mathbf{v}_1-\mathbf{v}_2\|_{\mathbb{W}}+\|\varrho_1-\varrho_2\|_{\mathrm{V}}),
\end{aligned}\tag{5.18}
$$

since the initial data of (Θ,Π) satisfies (5.8).

Let us now prove the continuity properties in part (ii) of the Proposition. One can write $(\partial_{\mathbf{v}}\mathcal{W}(s,\mathbf{v},\varrho),\gamma)_{\mathbb{H}}$, $(\partial_{\varrho}\mathcal{W}(s,\mathbf{v},\varrho),\nu)_{\mathrm{H}}$ and $\partial_s\mathcal{W}(s,\mathbf{v},\varrho)$ as

$$
\langle\partial_{\mathbf{v}}\mathcal{W}(s,\mathbf{v},\varrho),\gamma\rangle_{\mathbb{V}\times\mathbb{V}'}=((-\Delta)^{1/2}\mathbf{u}(T),(-\Delta)^{-1/2}\Psi(T))_{\mathbb{H}}+\int_s^T\langle(-\Delta)\mathbf{u}(r),\Psi(r)\rangle_{\mathbb{V}'\times\mathbb{V}}\mathrm{d}r,\tag{5.19}
$$

$$
\langle\partial_{\varrho}\mathcal{W}(s,\mathbf{v},\varrho),\nu\rangle_{\mathrm{V}\times\mathrm{V}'}=\langle\xi(T),\Phi(T)\rangle_{\mathrm{V}\times\mathrm{V}'}+\int_s^T\langle\xi(r),\Phi(r)\rangle_{\mathrm{V}\times\mathrm{V}'}\mathrm{d}r,\tag{5.20}
$$

$$
\begin{aligned}
\partial_s\mathcal{W}(s,\mathbf{v},\varrho)&=((-\Delta)^{1/2}\mathbf{u}(T),(-\Delta)^{-1/2}\Theta(T))_{\mathbb{H}}+\langle\xi(T),\Pi(T)\rangle_{\mathrm{V}\times\mathrm{V}'}\\
&\quad+\int_s^T\langle(-\Delta)\mathbf{u}(r),\Theta(r)\rangle_{\mathbb{V}'\times\mathbb{V}}+\langle\xi(r),\Pi(r)\rangle_{\mathrm{V}\times\mathrm{V}'}\mathrm{d}r\\
&\quad-\frac{1}{2}\left(\|\nabla\mathbf{v}\|_{\mathbb{H}}^2+\|\varrho\|_{\mathrm{H}}^2\right)-\frac{1}{2}\|\widetilde{\mathrm{U}}(s)\|_{\mathbb{H}}^2.
\end{aligned}\tag{5.21}
$$

As before, we define vectors $\mathrm{X}:=(\partial_{\mathbf{v}}\mathcal{W}(s,\mathbf{v},\varrho),\partial_{\varrho}\mathcal{W}(s,\mathbf{v},\varrho))^{\top}\in\mathbb{Y}:=\mathbb{V}\times\mathrm{V}$ and $\varsigma=(\gamma,\nu)^{\top}\in\mathbb{V}'\times\mathrm{V}'$. Using the Cauchy-Schwarz and Hölder inequalities, we estimate $|\langle\mathrm{X},\varsigma)\rangle_{\mathbb{Y}\times\mathbb{Y}'}|$ as

$$
\begin{aligned}
|\langle\mathrm{X},\varsigma)\rangle_{\mathbb{Y}\times\mathbb{Y}'}|&\leq\|(-\Delta)^{1/2}\mathbf{u}(T)\|_{\mathbb{H}}\|(-\Delta)^{-1/2}\Psi(T)\|_{\mathbb{H}}+\int_s^T\|(-\Delta)\mathbf{u}(r)\|_{\mathbb{H}}\|\Psi(r)\|_{\mathbb{H}}\mathrm{d}r\\
&\quad+\|\xi(T)\|_{\mathrm{V}}\|\Phi(T)\|_{\mathrm{V}'}+\int_s^T\|\xi(r)\|_{\mathrm{V}}\|\Phi(r)\|_{\mathrm{V}'}\mathrm{d}r\\
&\leq\left(\|\mathbf{u}(T)\|_{\mathbb{V}}^2+\int_s^T\|\mathbf{u}(r)\|_{\mathbb{W}}^2\mathrm{d}r\right)^{1/2}\left(\|\Psi(T)\|_{\mathbb{V}'}^2+\int_s^T\|\Psi(r)\|_{\mathbb{H}}^2\mathrm{d}r\right)^{1/2}\\
&\quad+\left(\|\xi(T)\|_{\mathrm{V}}^2+T\sup_{r\in[s,T]}\|\xi(r)\|_{\mathrm{V}}^2\right)^{1/2}\left(\|\Phi(T)\|_{\mathrm{V}'}^2+T\sup_{r\in[s,T]}\|\Phi(r)\|_{\mathrm{V}'}^2\right)^{1/2}\\
&\leq C(R,\|\mathbf{v}\|_{\mathbb{V}},\|\varrho\|_{\mathrm{V}})(\|\gamma\|_{\mathbb{V}'}+\|\nu\|_{\mathrm{V}'}),
\end{aligned}\tag{5.22}
$$

where we also used the fact that $(\mathbf{u},\xi)$ is a strong solution, $(ab+cd)^2\leq(a^2+c^2)^{1/2}(b^2+d^2)^{1/2}$, for $a,b,c,d\geq0$ and the energy estimates in the Proposition 2.9. Since the estimate (5.22) is true for all $\varsigma=(\gamma,\nu)^{\top}\in\mathbb{V}'\times\mathrm{V}'$ and $(\|\gamma\|_{\mathbb{V}'}+\|\nu\|_{\mathrm{V}'})$ induces a norm on the product space $\mathbb{V}'\times\mathrm{V}'$, we find

$$
\|\partial_{\mathbf{v}}\mathcal{W}(s,\mathbf{v},\varrho)\|_{\mathbb{V}}\leq C(R,\|\mathbf{v}\|_{\mathbb{V}},\|\varrho\|_{\mathrm{V}}),\quad\text{and}\tag{5.23}
$$

$$
\|\partial_{\varrho}\mathcal{W}(s,\mathbf{v},\varrho)\|_{\mathrm{V}}\leq C(R,\|\mathbf{v}\|_{\mathbb{V}},\|\varrho\|_{\mathrm{V}}).\tag{5.24}
$$

Similarly, we have

$$
\begin{aligned}
|\partial_s \mathcal{W}(s,\mathbf{v},\varrho)| \leq & \left(\|\mathbf{u}(T)\|_{\mathbb{V}}^2 + \int_s^T \|\mathbf{u}(r)\|_{\mathbb{W}}^2 \mathrm{d}r\right)^{1/2} \left(\|\Theta(T)\|_{\mathbb{V}'}^2 + \int_s^T \|\Theta(r)\|_{\mathbb{H}}^2 \mathrm{d}r\right)^{1/2} \\
& + \left(\|\xi(T)\|_{\mathrm{V}}^2 + T \sup_{r\in[s,T]} \|\xi(r)\|_{\mathrm{V}}^2\right)^{1/2} \left(\|\Pi(T)\|_{\mathrm{V}'}^2 + T \sup_{r\in[s,T]} \|\Pi(r)\|_{\mathrm{V}'}^2\right)^{1/2} \\
& + \frac{1}{2}\left(\|\nabla \mathbf{v}\|_{\mathbb{H}}^2 + \|\varrho\|_{\mathrm{H}}^2\right) + \frac{1}{2}R^2.
\end{aligned}
$$

Once again using the energy estimates and the strong solution estimates, we infer that

$$
|\partial_s \mathcal{W}(s,\mathbf{v},\varrho)| \leq C(R, \|\mathbf{v}\|_{\mathbb{V}}, \|\varrho\|_{\mathrm{V}}). \tag{5.25}
$$

Thus, from the estimates (5.23)–(5.25), it is immediate that

$$
(\partial_s \mathcal{W}(s,\cdot,\cdot), \partial_{\mathbf{v}} \mathcal{W}(s,\cdot,\cdot), \partial_\varrho \mathcal{W}(s,\cdot,\cdot)) : \mathbb{V}\times \mathrm{V} \to \mathbb{R}\times\mathbb{V}\times\mathrm{V}.
$$

A similar proof as in (5.14)–(5.18) and the continuous dependence results in (see Thm. 2.7 and Prop. 2.9) yield the following Lipschitz continuity results:

$$
\begin{aligned}
\|\partial_{\mathbf{v}} \mathcal{W}(s,\mathbf{v}_1,\varrho) - \partial_{\mathbf{v}} \mathcal{W}(s,\mathbf{v}_2,\varrho)\|_{\mathbb{V}} &\leq C(R, \|\mathbf{v}_1\|_{\mathbb{V}}, \|\mathbf{v}_2\|_{\mathbb{V}}, \|\varrho\|_{\mathrm{V}}) \|\mathbf{v}_1 - \mathbf{v}_2\|_{\mathbb{V}}, && (5.26)\\
\|\partial_\varrho \mathcal{W}(s,\mathbf{v},\varrho_1) - \partial_\varrho \mathcal{W}(s,\mathbf{v},\varrho_2)\|_{\mathbb{V}} &\leq C(R, \|\mathbf{v}\|_{\mathbb{V}}, \|\varrho_1\|_{\mathrm{V}}, \|\varrho_2\|_{\mathrm{V}}) \|\varrho_1 - \varrho_2\|_{\mathrm{V}}, && (5.27)\\
|\partial_s \mathcal{W}(s,\mathbf{v}_1,\varrho_1) - \partial_s \mathcal{W}(s,\mathbf{v}_2,\varrho_2)| &\leq C(R, \|\mathbf{v}_1\|_{\mathbb{V}}, \|\mathbf{v}_2\|_{\mathbb{V}}, \|\varrho_1\|_{\mathrm{H}}, \|\varrho_2\|_{\mathrm{H}}) \\
&\quad \times (\|\mathbf{v}_1 - \mathbf{v}_2\|_{\mathbb{V}} + \|\varrho_1 - \varrho_2\|_{\mathrm{H}}), && (5.28)
\end{aligned}
$$

which completes the proof. □

5.1. Pontryagin's maximum principle

We are now in a position to establish the Pontryagin maximum principle. The following theorem gives such a result.

Theorem 5.4 (Rademacher theorem, [12], Thm. 3.1.6)**.** *If $\mathcal{O} \subseteq \mathbb{R}^n$ is open and $\mathbf{f} : \mathcal{O} \to \mathbb{R}^m$ is Lipschitz continuous, then $\mathbf{f}$ is differentiable at a.e. $x \in \mathcal{O}$.*

Theorem 5.5. *Let $\widetilde{\mathrm{U}} \in \mathrm{L}^2(\tau, T; \mathcal{U}_R)$ be an optimal control for the initial data $(\tau,\mathbf{v},\varrho) \in [0,T]\times(\mathbb{W}\cap\mathbb{V})\times\mathrm{V}$ and $(\mathbf{u},\xi) \in (\mathrm{C}([\tau,T];\mathbb{V})\cap\mathrm{L}^2(\tau,T;\mathbb{W}))\times\mathrm{C}([\tau,T];\mathrm{V})$, be the optimal trajectory, then for a.e. $t\in[\tau,T]$, we have*

$$
\begin{aligned}
&\mathscr{H}(\mathbf{u}(t),\xi(t),\partial_{\mathbf{v}}\mathcal{W}(t,\mathbf{u}(t),\xi(t)),\partial_\varrho\mathcal{W}(t,\mathbf{u}(t),\xi(t))) \\
&= \widetilde{\mathscr{H}}\Big(\mathbf{u}(t),\xi(t),\partial_{\mathbf{v}}\mathcal{W}(t,\mathbf{u}(t),\xi(t)),\partial_\varrho\mathcal{W}(t,\mathbf{u}(t),\xi(t)),\widetilde{\mathrm{U}}(t)\Big).
\end{aligned} \tag{5.29}
$$

Moreover, if

$$
(\mathbf{p},\varphi) \in (\mathrm{C}([0,T];\mathbb{H})\cap\mathrm{L}^2(0,T;\mathbb{V}))\times\mathrm{C}([0,T];\mathrm{H}),
$$

is the unique weak solution to the adjoint problem:

$$\begin{cases} -\dfrac{\partial \mathbf{p}(t)}{\partial t}+\widetilde{\mathrm{A}}\mathbf{p}(t)+\mathrm{B}'(\mathbf{u}(t))\mathbf{p}(t)-h\nabla\varphi(t)=-\Delta\mathbf{u}(t), \quad \textit{in} \ \ \Omega\times(\tau,T),\\ -\dfrac{\partial\varphi(t)}{\partial t}-\mathrm{div}\ \mathbf{p}(t)=\xi(t), \quad \textit{in} \ \ \Omega\times(\tau,T),\\ \mathbf{p}(t)=\mathbf{0}, \quad \textit{on} \ \ \partial\Omega\times(\tau,T),\\ \mathbf{p}(T)=\mathbf{u}(T), \ \varphi(T)=\xi(T), \quad \textit{in} \ \ \Omega, \end{cases} \tag{5.30}$$

then, we have

$$\mathbf{p}(t)=\partial_{\mathbf{v}}\mathcal{W}(t,\mathbf{u}(t),\xi(t)), \ \varphi(t)=\partial_{\varrho}\mathcal{W}(t,\mathbf{u}(t),\xi(t)), \ t\in[\tau,T]. \tag{5.31}$$

Furthermore, optimal control is obtained as $\widetilde{\mathrm{U}}(t)=\sigma(\mathbf{p}(t))$*, where* $\sigma(\cdot)$ *is defined in* (4.8) *and we obtain the regularity* $\widetilde{\mathrm{U}}\in\mathrm{C}([\tau,T];\mathbb{H})$ *and*

$$(\mathbf{u},\xi)\in\mathrm{C}([\tau,T];\mathbb{V}\cap\mathbb{W})\cap\mathrm{C}^1(\tau,T;\mathbb{H})\times\mathrm{C}([\tau,T];\mathrm{V})\cap\mathrm{C}^1(\tau,T;\mathrm{V}).$$

Proof. **Step 1.** *Proof of* (5.29). Since $\widetilde{\mathrm{U}}$ is an optimal control, using the semigroup property (see (4.27))

$$\mathbf{x}(r,t;\mathbf{x}(t,s;\mathbf{v},\varrho;\widetilde{\mathrm{U}});\widetilde{\mathrm{U}})=\mathbf{x}(r,s;\mathbf{v},\varrho;\widetilde{\mathrm{U}}), \ \ s\leq t\leq r\leq T, \tag{5.32}$$

we can write $\mathcal{W}(t,\mathbf{u}(t),\xi(t))$ as

$$\mathcal{W}(t,\mathbf{u}(t),\xi(t))=\frac{1}{2}\|\mathbf{u}(T)\|_{\mathbb{H}}^2+\frac{1}{2}\|\xi(T)\|_{\mathbb{H}}^2+\frac{1}{2}\int_t^T\Big[\|\nabla\mathbf{u}(r)\|_{\mathbb{H}}^2+\|\xi(r)\|_{\mathrm{H}}^2+\|\widetilde{\mathrm{U}}(r)\|_{\mathbb{H}}^2\Big]\mathrm{d}r. \tag{5.33}$$

Also, one can write $\mathcal{W}(t,\mathbf{u}(t),\xi(t))$ as

$$\begin{aligned}\mathcal{W}(t,\mathbf{u}(t),\xi(t))&=\frac{1}{2}\|\mathbf{u}(T)\|_{\mathbb{H}}^2+\frac{1}{2}\|\xi(T)\|_{\mathbb{H}}^2+\frac{1}{2}\int_t^{t+\delta t}\Big[\|\nabla\mathbf{u}(r)\|_{\mathbb{H}}^2+\|\xi(r)\|_{\mathrm{H}}^2+\|\widetilde{\mathrm{U}}(r)\|_{\mathbb{H}}^2\Big]\mathrm{d}r\\ &\quad+\frac{1}{2}\int_{t+\delta t}^T\Big[\|\nabla\mathbf{u}(r)\|_{\mathbb{H}}^2+\|\xi(r)\|_{\mathrm{H}}^2+\|\widetilde{\mathrm{U}}(r)\|_{\mathbb{H}}^2\Big]\mathrm{d}r\\ &=\mathcal{W}(t+\delta t,\mathbf{u}(t+\delta t),\xi(t+\delta t))\\ &\quad+\frac{1}{2}\int_t^{t+\delta t}\Big[\|\nabla\mathbf{u}(r)\|_{\mathbb{H}}^2+\|\xi(r)\|_{\mathrm{H}}^2+\|\widetilde{\mathrm{U}}(r)\|_{\mathbb{H}}^2\Big]\mathrm{d}r.\end{aligned} \tag{5.34}$$

Thanks to Rademacher's theorem (see Thm. 5.4), from the above expression, using the almost everywhere differentiability of $\mathcal{W}(\cdot,\cdot,\cdot)$, we have

$$\begin{aligned}&-\frac{\mathcal{W}(t+\delta t,\mathbf{u}(t),\xi(t))-\mathcal{W}(t,\mathbf{u}(t),\xi(t))}{\delta t}-\frac{\mathcal{W}(t+\delta t,\mathbf{u}(t+\delta t),\xi(t))-\mathcal{W}(t+\delta t,\mathbf{u}(t),\xi(t))}{\delta t}\\ &\quad-\frac{\mathcal{W}(t+\delta t,\mathbf{u}(t+\delta t),\xi(t+\delta t))-\mathcal{W}(t+\delta t,\mathbf{u}(t+\delta t),\xi(t))}{\delta t}\\ &\quad-\frac{1}{2\delta t}\int_t^{t+\delta t}\Big[\|\nabla\mathbf{u}(r)\|_{\mathbb{H}}^2+\|\xi(r)\|_{\mathrm{H}}^2+\|\widetilde{\mathrm{U}}(r)\|_{\mathbb{H}}^2\Big]\mathrm{d}r=0.\end{aligned} \tag{5.35}$$

Passing $\delta \to 0$ in (5.35), we find

$$\begin{aligned} &- \partial_t \mathcal{W}(t, \mathbf{u}(t), \xi(t)) - (\partial_t \mathbf{u}(t), \partial_{\mathbf{v}} \mathcal{W}(t, \mathbf{u}(t), \xi(t))_{\mathbb{H}} - (\partial_t \xi(t), \partial_{\varrho} \mathcal{W}(t, \mathbf{u}(t), \xi(t))_{\mathbb{H}} \\ &\quad - \frac{1}{2}\Big[\|\nabla \mathbf{u}(t)\|_{\mathbb{H}}^2 + \|\xi(t)\|_{\mathrm{H}}^2 + \|\widetilde{\mathrm{U}}(t)\|_{\mathbb{H}}^2\Big] = 0, \end{aligned} \tag{5.36}$$

where we used (5.5). Thus, we have

$$\begin{aligned} &- \partial_t \mathcal{W}(t, \mathbf{u}(t), \xi(t)) + (\mathrm{A}\mathbf{u}(t) + \mathrm{B}(\mathbf{u}(t)) + \nabla \xi(t) - \mathbf{f}(t) - \widetilde{\mathrm{U}}(t), \partial_{\mathbf{v}} \mathcal{W}(t, \mathbf{u}(t), \xi(t))_{\mathbb{H}} \\ &\quad + (\mathrm{div}(h\mathbf{u}(t)), \partial_{\varrho} \mathcal{W}(t, \mathbf{u}(t), \xi(t))_{\mathbb{H}} - \frac{1}{2}\Big[\|\nabla \mathbf{u}(t)\|_{\mathbb{H}}^2 + \|\xi(t)\|_{\mathrm{H}}^2 + \|\widetilde{\mathrm{U}}(t)\|_{\mathbb{H}}^2\Big] = 0. \end{aligned} \tag{5.37}$$

Hence, using the definition of pseudo-Hamiltonian (see (4.5)), we also get

$$- \partial_t \mathcal{W}(t, \mathbf{u}(t), \xi(t)) + \widetilde{\mathscr{H}}\Big(\mathbf{u}(t), \xi(t), \partial_{\mathbf{v}} \mathcal{W}(t, \mathbf{u}(t), \xi(t), \partial_{\varrho} \mathcal{W}(t, \mathbf{u}(t), \xi(t), \widetilde{\mathrm{U}}\Big) = 0, \tag{5.38}$$

with

$$\mathcal{W}(T, \mathbf{u}(T), \xi(T)) = \frac{1}{2}\big[\|\mathbf{u}(T)\|_{\mathbb{H}}^2 + \|\xi(T)\|_{\mathrm{H}}^2\big].$$

Since $\widetilde{\mathrm{U}}(\cdot)$ is an optimal control, from the above equality, we also have

$$- \partial_t \mathcal{W}(t, \mathbf{u}(t), \xi(t)) + \sup_{\mathrm{U} \in \mathcal{U}_R} \widetilde{\mathscr{H}}(\mathbf{u}(t), \xi(t), \partial_{\mathbf{v}} \mathcal{W}(t, \mathbf{u}(t), \xi(t), \partial_{\varrho} \mathcal{W}(t, \mathbf{u}(t), \xi(t), \mathrm{V}) \geq 0.$$

Using the definition of true-Hamiltonian (see (4.4)), from the above expression, we infer that

$$- \partial_t \mathcal{W}(t, \mathbf{u}(t), \xi(t)) + \mathscr{H}(\mathbf{u}(t), \xi(t), \partial_{\mathbf{v}} \mathcal{W}(t, \mathbf{u}(t), \xi(t), \partial_{\varrho} \mathcal{W}(t, \mathbf{u}(t), \xi(t)) \geq 0. \tag{5.39}$$

It should be noted that for $s \in [\tau, T]$, and $(\mathbf{v}, \varrho) \in (\mathbb{V} \cap \mathbb{W}) \times \mathrm{V}$, we have $\mathcal{W}(s, \mathbf{v}, \varrho) \geq \mathcal{V}(s, \mathbf{v}, \varrho)$ and $\mathcal{W}(\tau, \mathbf{v}, \varrho) = \mathcal{V}(\tau, \mathbf{v}, \varrho)$. In fact, if $\widetilde{\mathbf{x}}(t) = \widetilde{\mathbf{x}}(t, \tau; \mathbf{v}, \varrho; \widetilde{\mathrm{U}})$, where $\widetilde{\mathbf{x}} = (\widetilde{\mathbf{u}}, \widetilde{\xi})^\top$ is the optimal trajectory, then we have

$$\mathcal{W}(t, \widetilde{\mathbf{u}}(t), \widetilde{\xi}(t)) = \mathcal{V}(t, \widetilde{\mathbf{u}}(t), \widetilde{\xi}(t)), \ \tau \leq t \leq T. \tag{5.40}$$

From the above identity, it is immediate that $\mathcal{V} - \mathcal{W}$ attains a maximum of zero along each point of the trajectory $(t, \widetilde{\mathbf{u}}(t), \widetilde{\xi}(t))$. Let us now use the fact that $\mathcal{V}$ is a viscosity subsolution (see Thm. 4.7, (4.40)) to deduce that

$$- \partial_t \mathcal{W}(t, \widetilde{\mathbf{u}}(t), \widetilde{\xi}(t)) + \mathscr{H}\Big(\widetilde{\mathbf{u}}(t), \widetilde{\xi}(t), \partial_{\mathbf{v}} \mathcal{W}(t, \widetilde{\mathbf{u}}(t), \widetilde{\xi}(t), \partial_{\varrho} \mathcal{W}(t, \widetilde{\mathbf{u}}(t), \widetilde{\xi}(t)\Big) \leq 0. \tag{5.41}$$

Comparing (5.39) and (5.41) at $s = \tau$, we obtain

$$- \partial_t \mathcal{W}(t, \widetilde{\mathbf{u}}(t), \widetilde{\xi}(t)) + \mathscr{H}\Big(\widetilde{\mathbf{u}}(t), \widetilde{\xi}(t), \partial_{\mathbf{v}} \mathcal{W}(t, \widetilde{\mathbf{u}}(t), \widetilde{\xi}(t), \partial_{\varrho} \mathcal{W}(t, \widetilde{\mathbf{u}}(t), \widetilde{\xi}(t)\Big) = 0. \tag{5.42}$$

Finally, we compare (5.38) with (5.42) for $s = \tau$ to deduce (5.29).

Step 2. *Characterization of optimal control.* Let us now take duality pairing with $(\Psi,\Phi)\in(\mathrm{C}([s,T];\mathbb{H})\cap \mathrm{L}^2(s,T;\mathbb{V}))\times \mathrm{C}([s,T];\mathrm{H})$, with $(\partial_t\Psi,\partial_t\Phi)\in \mathrm{L}^2(s,T;\mathbb{V}')\times \mathrm{L}^2(s,T;\mathrm{H})$ to the adjoint system (5.30) to find

$$\begin{aligned}&\int_t^T\langle -\partial_r\mathbf{p}(r)+\widetilde{\mathrm{A}}\mathbf{p}(r)+\mathrm{B}'(\widetilde{\mathbf{u}}(r))\mathbf{p}(r)-h\nabla\varphi(r),\Psi(r)\rangle_{\mathbb{V}'\times\mathbb{V}}\mathrm{d}r\\&\quad+\int_t^T(-\partial_r\varphi(r)-\mathrm{div}\ \mathbf{p}(r),\Phi(r))_{\mathrm{H}}\\&=\int_t^T\langle -\Delta\widetilde{\mathbf{u}}(r),\Psi(r)\rangle_{\mathbb{V}'\times\mathbb{V}}\mathrm{d}r+\int_t^T(\widetilde{\xi}(r),\Phi(r))_{\mathrm{H}}\mathrm{d}r,\end{aligned}\tag{5.43}$$

where $(\mathbf{p},\varphi)\in(\mathrm{C}([0,T];\mathbb{H})\cap\mathrm{L}^2(0,T;\mathbb{V}))\times\mathrm{C}([0,T];\mathrm{H})$ with $(\partial_t\mathbf{p},\partial_t\varphi)\in\mathrm{L}^2(s,T;\mathbb{V}')\times\mathrm{L}^2(s,T;\mathrm{H})$. Let us perform an integration by parts in the left hand side of the equality (5.43), use the equation satisfied by (Ψ,Φ) and the fact that $(\mathbf{p}(T),\varphi(T))=(\widetilde{\mathbf{u}}(T),\widetilde{\xi}(T))$ to find

$$\begin{aligned}&(\mathbf{p}(t),\Psi(t))_{\mathbb{H}}-(\widetilde{\mathbf{u}}(T),\Psi(T))_{\mathbb{H}}+(\varphi(t),\Phi(t))_{\mathrm{H}}-(\widetilde{\xi}(T),\Phi(T))_{\mathrm{H}}\\&=\int_t^T\langle -\Delta\widetilde{\mathbf{u}}(r),\Psi(r)\rangle_{\mathbb{V}'\times\mathbb{V}}\mathrm{d}r+\int_t^T(\widetilde{\xi}(r),\Phi(r))_{\mathrm{H}}\mathrm{d}r.\end{aligned}\tag{5.44}$$

Making use of the continuity theorem, one can show that the above equality holds for all $t\in[\tau,T]$. Let us now compare these equations with (5.9) and (5.10) for $s=t$ to obtain

$$(\partial_{\mathbf{v}}\mathcal{W}(t,\widetilde{\mathbf{u}}(t),\widetilde{\xi}(t)),\gamma)_{\mathbb{H}}-(\mathbf{p}(t),\gamma)_{\mathbb{H}}+(\partial_{\varrho}\mathcal{W}(t,\widetilde{\mathbf{u}}(t),\widetilde{\xi}(t),\nu)_{\mathrm{H}}-(\varphi(t),\nu)_{\mathrm{H}}=0.\tag{5.45}$$

Since the above equality is true for arbitrary $(\gamma,\nu)\in\mathbb{H}\times\mathrm{H}$, then for all $t\in[\tau,T]$, we get

$$\mathbf{p}(t)=\partial_{\mathbf{v}}\mathcal{W}(t,\widetilde{\mathbf{u}}(t),\widetilde{\xi}(t)),\ \ \varphi(t)=\partial_{\varrho}\mathcal{W}(t,\widetilde{\mathbf{u}}(t),\widetilde{\xi}(t)),\tag{5.46}$$

and

$$\left\{\begin{aligned}&\mathbf{p}(\cdot)=\partial_{\mathbf{v}}\mathcal{W}(\cdot,\widetilde{\mathbf{u}}(\cdot),\widetilde{\xi}(\cdot))\in\mathrm{C}([\tau,T];\mathbb{H})\cap\mathrm{L}^2(\tau,T;\mathbb{V}),\\&\varphi(\cdot)=\partial_{\varrho}\mathcal{W}(\cdot,\widetilde{\mathbf{u}}(\cdot),\widetilde{\xi}(\cdot))\in\mathrm{C}([\tau,T];\mathrm{H}).\end{aligned}\right.$$

Now, since $\mathcal{V}-\mathcal{W}$ attains a maximum at $(t,\widetilde{\mathbf{u}}(t),\widetilde{\xi}(t))$, using the definition of super-differential and (4.40), we also have

$$\partial_{\mathbf{v}}\mathcal{W}(\cdot,\widetilde{\mathbf{u}}(\cdot),\widetilde{\xi}(\cdot))\in\partial^+_{\mathbf{v}}\mathcal{V}(\cdot,\widetilde{\mathbf{u}}(\cdot),\widetilde{\xi}(\cdot)),\ \text{ and }\ \partial_{\varrho}\mathcal{W}(\cdot,\widetilde{\mathbf{u}}(\cdot),\widetilde{\xi}(\cdot))\in\partial^+_{\varrho}\mathcal{V}(\cdot,\widetilde{\mathbf{u}}(\cdot),\widetilde{\xi}(\cdot)).\tag{5.47}$$

Furthermore, the optimal control is given by

$$\widetilde{\mathrm{U}}(\cdot)=\sigma(\partial_{\mathbf{v}}\mathcal{W}(\cdot,\widetilde{\mathbf{u}}(\cdot),\widetilde{\xi}(\cdot)))\in\mathrm{C}([\tau,T];\mathbb{H}),\tag{5.48}$$

where $\sigma(\cdot)$ is defined in (4.8). We finally use this regularity on the optimal control to get the regularity of the tidal dynamic system as

$$(\widetilde{\mathbf{u}},\widetilde{\xi})\in\mathrm{C}([\tau,T];\mathbb{V}\cap\mathbb{W})\cap\mathrm{C}^1(\tau,T;\mathbb{H})\times\mathrm{C}([\tau,T];\mathrm{V})\cap\mathrm{C}^1(\tau,T;\mathrm{V}),\tag{5.49}$$

which completes the proof of the Theorem. □

Theorem 5.6 (Verification theorem)**.** *The super-differentials $\partial^+_{\mathbf{v}}\mathcal{V}(t,\mathbf{v},\varrho)$ and $\partial^+_{\varrho}\mathcal{V}(t,\mathbf{v},\varrho)$ of the value function $\mathcal{V}(t,\mathbf{v},\varrho)$ is non-empty, for all $(t,\mathbf{v},\varrho)\in([\tau,T],\mathbb{V}\cap\mathbb{W},\mathrm{V})$, and for $(\mathbf{v},\varrho)\in(\mathbb{V}\cap\mathbb{W},\mathrm{V})$,*

$$-\partial_t\mathcal{V}(t,\mathbf{v},\varrho)+\mathscr{H}(t,\mathbf{v},\varrho,\mathbf{p}(t),\varphi(t))=0,\ \ a.e.,\ \ t\in(\tau,T), \tag{5.50}$$

with

$$\partial_{\mathbf{v}}\mathcal{W}(t,\mathbf{v},\varrho)\in\partial^+_{\mathbf{v}}\mathcal{V}(t,\mathbf{v},\varrho),\quad and\ \ \partial_{\varrho}\mathcal{W}(t,\mathbf{v},\varrho)\in\partial^+_{\varrho}\mathcal{V}(t,\mathbf{v},\varrho), \tag{5.51}$$

for all $(\mathbf{v},\varrho)\in\mathbb{H}\times\mathrm{H}$.

Furthermore, the optimal control is given by the feedback relation*:*

$$\widetilde{\mathrm{U}}(\cdot)=\sigma(\mathbf{p}(\cdot)),$$

for some

$$\mathbf{p}(\cdot)\in\partial^+_{\mathbf{v}}\mathcal{V}(\cdot,\widetilde{\mathbf{u}}(\cdot),\widetilde{\xi}(\cdot))$$

where $\sigma(\cdot)$ is defined in (4.8).

Proof. Using Rademacher's theorem, we note that $\mathcal{V}(\cdot,\mathbf{v},\varrho)$ is differentiable for a.e. $t\in(\tau,T)$ and $(\mathbf{v},\varrho)\in(\mathbb{V}\cap\mathbb{W})\times\mathrm{V}$. Let $t\in(\tau,T)$ be such a point of differentiability. Let $\widetilde{\mathrm{U}}$ be the optimal control and $(\widetilde{\mathbf{u}},\widetilde{\xi})$ be the optimal trajectory corresponding to the initial data $(t,\mathbf{v},\varrho)\in[\tau,T]\times(\mathbb{V}\cap\mathbb{W})\times\mathrm{V}$. Let $(\mathbf{u},\xi)$ be the trajectory corresponding to the control $\widetilde{\mathrm{U}}$ and initial data $(t+\varepsilon,\mathbf{v},\varrho)\in[\tau,T]\times(\mathbb{V}\cap\mathbb{W})\times\mathrm{V}$. Using the definition of the value function $\mathcal{V}(\cdot,\cdot,\cdot)$, we have

$$\begin{aligned}
&\mathcal{V}(t+\varepsilon,\mathbf{v},\varrho)-\mathcal{V}(t,\mathbf{v},\varrho)\\
&\leq\frac{1}{2}\|\mathbf{u}(T)\|^2_{\mathbb{H}}+\frac{1}{2}\|\xi(T)\|^2_{\mathbb{H}}+\frac{1}{2}\int_{t+\varepsilon}^T\Big[\|\nabla\mathbf{u}(r)\|^2_{\mathbb{H}}+\|\xi(r)\|^2_{\mathrm{H}}+\|\widetilde{\mathrm{U}}(r)\|^2_{\mathbb{H}}\Big]\mathrm{d}r\\
&\quad-\frac{1}{2}\|\widetilde{\mathbf{u}}(T)\|^2_{\mathbb{H}}-\frac{1}{2}\|\widetilde{\xi}(T)\|^2_{\mathbb{H}}-\frac{1}{2}\int_t^T\Big[\|\nabla\widetilde{\mathbf{u}}(r)\|^2_{\mathbb{H}}+\|\widetilde{\xi}(r)\|^2_{\mathrm{H}}+\|\widetilde{\mathrm{U}}(r)\|^2_{\mathbb{H}}\Big]\mathrm{d}r\\
&=\frac{1}{2}\big[\|\mathbf{u}(T)\|^2_{\mathbb{H}}-\|\widetilde{\mathbf{u}}(T)\|^2_{\mathbb{H}}\big]+\frac{1}{2}\Big[\|\xi(T)\|^2_{\mathbb{H}}-\|\widetilde{\xi}(T)\|^2_{\mathbb{H}}\Big]\\
&\quad+\frac{1}{2}\int_{t+\varepsilon}^T\Big[\|\nabla\mathbf{u}(r)\|^2_{\mathbb{H}}-\|\nabla\widetilde{\mathbf{u}}(r)\|^2_{\mathbb{H}}+\|\xi(r)\|^2_{\mathrm{H}}-\|\widetilde{\xi}(r)\|^2_{\mathrm{H}}\Big]\mathrm{d}r\\
&\quad-\frac{1}{2}\int_t^{t+\varepsilon}\Big[\|\nabla\widetilde{\mathbf{u}}(r)\|^2_{\mathbb{H}}++\|\xi(r)\|^2_{\mathrm{H}}+\|\widetilde{\mathrm{U}}(r)\|^2_{\mathbb{H}}\Big]\mathrm{d}r.
\end{aligned} \tag{5.52}$$

Let us now restrict t to belong to the intersection of the following three sets of full measure in $[\tau,T]$. The three sets are

- the set in which $\mathcal{V}(\cdot,\mathbf{v},\varrho)$ is differentiable,
- the set of Lebesgue points of integrand of the final integral appearing in the right hand side of the equation (5.52), and
- the set of full measure in equation (5.29) in the Pontryagin maximum principle.

Then, on dividing by ε, taking limit as $\varepsilon \downarrow 0$ and using the definition (Θ, Π) in (5.6), we find

$$\partial_t \mathcal{V}(t, \mathbf{v}, \varrho) \leq -\Big[\|\nabla \mathbf{v}\|_{\mathbb{H}}^2 + \|\varrho\|_{\mathrm{H}}^2 + \|\widetilde{\mathrm{U}}(t)\|_{\mathbb{H}}^2\Big] + \int_t^T \Big[\langle (-\Delta)\widetilde{\mathbf{u}}(r), \Theta(r)\rangle_{\mathbb{V}'\times\mathbb{V}} + (\widetilde{\xi}(r), \Pi(r))_{\mathrm{H}}\Big]\mathrm{d}r \\ + (\widetilde{\mathbf{u}}(T), \Theta(T))_{\mathbb{H}} + (\widetilde{\xi}(T), \Pi(T))_{\mathrm{H}}. \tag{5.53}$$

Remember that the adjoint system $(\mathbf{p}, \varphi)$ satisfies:

$$\begin{cases} -\dfrac{\partial \mathbf{p}(t)}{\partial t} + \widetilde{\mathrm{A}}\mathbf{p}(t) + \mathrm{B}'(\widetilde{\mathbf{u}}(t))\mathbf{p}(t) - h\nabla\varphi(t) = -\Delta\widetilde{\mathbf{u}}(t), \ \text{ in } \ \Omega\times(\tau, T), \\ -\dfrac{\partial \varphi(t)}{\partial t} - \mathrm{div}\ \mathbf{p}(t) = \widetilde{\xi}(t), \ \text{ in } \ \Omega\times(\tau,T), \\ \mathbf{p}(t) = \mathbf{0}, \ \text{ on } \ \partial\Omega\times(\tau,T), \\ \mathbf{p}(T) = \widetilde{\mathbf{u}}(T), \ \varphi(T) = \widetilde{\xi}(T), \ \text{ in } \ \Omega. \end{cases} \tag{5.54}$$

Let us substitute (5.54) in (5.53) and perform an integration by parts to obtain

$$\partial_t \mathcal{V}(t, \mathbf{v}, \varrho) \leq -\Big[\|\nabla \mathbf{v}\|_{\mathbb{H}}^2 + \|\varrho\|_{\mathrm{H}}^2 + \|\widetilde{\mathrm{U}}(t)\|_{\mathbb{H}}^2\Big] + (\mathbf{p}(t), \Theta(t))_{\mathbb{H}} + (\varphi(t), \Pi(t))_{\mathrm{H}}. \tag{5.55}$$

Note that in (5.55), the initial data for (Θ, Π) equation is applied at t and is given by (see (5.8) also)

$$\Theta(t) = \mathrm{A}\mathbf{v} + \mathrm{B}'(\mathbf{u})\mathbf{v} + \nabla\varrho - \mathbf{f}(t) - \widetilde{\mathrm{U}}(t) \in \mathbb{H}, \ \text{ and } \ \Pi(t) = \mathrm{div}(h\mathbf{v}) \in \mathrm{H}. \tag{5.56}$$

Let us substitute (5.56) in (5.55) to see that

$$\partial_t \mathcal{V}(t, \mathbf{v}, \varrho) \leq (\mathbf{p}(t), \mathrm{A}\mathbf{v} + \mathrm{B}'(\mathbf{u})\mathbf{v} + \nabla\varrho - \mathbf{f}(t) - \widetilde{\mathrm{U}}(t))_{\mathbb{H}} + (\varphi(t), \mathrm{div}(h\mathbf{v}))_{\mathrm{H}} \\ - \Big[\|\nabla \mathbf{v}\|_{\mathbb{H}}^2 + \|\varrho\|_{\mathrm{H}}^2 + \|\widetilde{\mathrm{U}}(t)\|_{\mathbb{H}}^2\Big] \\ = \widetilde{\mathscr{H}}(t, \mathbf{v}, \varrho, \mathbf{p}(t), \varphi(t), \widetilde{\mathrm{U}}(t)). \tag{5.57}$$

Using the Pontryagin maximum principle (see (5.29)) in (5.57), we find

$$\partial_t \mathcal{V}(t, \mathbf{v}, \varrho) \leq \mathscr{H}(t, \mathbf{v}, \varrho, \mathbf{p}(t), \varphi(t)). \tag{5.58}$$

Next, let $\widetilde{\mathrm{U}}$ be the optimal control $[\tau+\varepsilon, T]$ corresponding to the initial data $(\mathbf{v}, \varrho) \in (\mathbb{V}\cap\mathbb{W})\times\mathrm{V}$ at $t+\varepsilon$ and let $(\widetilde{\mathbf{u}}, \widetilde{\xi})$ be the corresponding trajectory. Let us now set

$$\mathrm{U}(r) = \begin{cases} \widetilde{\mathrm{U}}(r) & \text{if } r \in (t+\varepsilon, T] \\ \widetilde{\mathrm{U}}(t+\varepsilon) & \text{if } r\in(t, t+\varepsilon), \end{cases}$$

and denote $(\mathbf{u}, \xi)$, the trajectory corresponding to the initial data $(t, \mathbf{v}, \varrho) \in [\tau, T]\times(\mathbb{V}\cap\mathbb{W})\times\mathrm{V}$ and control U. Then, we have

$$\mathcal{V}(t+\varepsilon, \mathbf{v}, \varrho) - \mathcal{V}(t, \mathbf{v}, \varrho) \\ \geq \frac{1}{2}\|\widetilde{\mathbf{u}}(T)\|_{\mathbb{H}}^2 + \frac{1}{2}\|\widetilde{\xi}(T)\|_{\mathbb{H}}^2 + \frac{1}{2}\int_{t+\varepsilon}^T \Big[\|\nabla\widetilde{\mathbf{u}}(r)\|_{\mathbb{H}}^2 + \|\widetilde{\xi}(r)\|_{\mathrm{H}}^2 + \|\widetilde{\mathrm{U}}(r)\|_{\mathbb{H}}^2\Big]\mathrm{d}r$$

$$
\begin{aligned}
&- \frac{1}{2}\|\mathbf{u}(T)\|_{\mathbb{H}}^2 - \frac{1}{2}\|\xi(T)\|_{\mathbb{H}}^2 - \frac{1}{2}\int_t^T \big[\|\nabla\mathbf{u}(r)\|_{\mathbb{H}}^2 + \|\xi(r)\|_{\mathrm{H}}^2 + \|\mathrm{U}(r)\|_{\mathbb{H}}^2\big]\mathrm{d}r \\
&= \left\{\left(\frac{1}{2}\|\widetilde{\mathbf{u}}(T)\|_{\mathbb{H}}^2 - \frac{1}{2}\|\mathbf{u}(T)\|_{\mathbb{H}}^2\right) + \left(\frac{1}{2}\|\widetilde{\xi}(T)\|_{\mathbb{H}}^2 - \frac{1}{2}\|\xi(T)\|_{\mathbb{H}}^2\right)\right\} \\
&+ \frac{1}{2}\int_{t+\varepsilon}^T \Big[\|\nabla\widetilde{\mathbf{u}}(r)\|_{\mathbb{H}}^2 - \|\nabla\mathbf{u}(r)\|_{\mathbb{H}}^2 + \|\widetilde{\xi}(r)\|_{\mathrm{H}}^2 - \|\xi(r)\|_{\mathrm{H}}^2\Big]\mathrm{d}r \\
&- \frac{1}{2}\int_t^{t+\varepsilon} \big[\|\nabla\mathbf{u}(r)\|_{\mathbb{H}}^2 + \|\xi(r)\|_{\mathrm{H}}^2 + \|\mathrm{U}(r)\|_{\mathbb{H}}^2\big]\mathrm{d}r.
\end{aligned} \tag{5.59}
$$

Using the arguments as in (5.53)–(5.58), we obtain

$$\partial_t\mathcal{V}(t,\mathbf{v},\varrho) \geq \mathscr{H}(t,\mathbf{v},\varrho,\mathbf{p}(t),\varphi(t)). \tag{5.60}$$

Comparing (5.58) and (5.60), we infer that

$$-\partial_t\mathcal{V}(t,\mathbf{v},\varrho) + \mathscr{H}(t,\mathbf{v},\varrho,\mathbf{p}(t),\varphi(t)) = 0, \tag{5.61}$$

with

$$\partial_{\mathbf{v}}\mathcal{W}(t,\mathbf{v},\varrho) \in \partial_{\mathbf{v}}^+\mathcal{V}(t,\mathbf{v},\varrho), \quad \text{and} \quad \partial_{\varrho}\mathcal{W}(t,\mathbf{v},\varrho) \in \partial_{\varrho}^+\mathcal{V}(t,\mathbf{v},\varrho), \tag{5.62}$$

for all $(\mathbf{v},\varrho) \in \mathbb{H}\times\mathrm{H}$. □

Remark 5.7. For the Hamilton-Jacobi equation obtained in this work, we have proved only the existence of viscosity solution and uniqueness is still open. We will address such kind of problems in a future work.

Appendix A. Global solvability of the tidal dynamics system

Let us discuss about the global solvability results available in the literature for the controlled tidal dynamics system (2.1).

Theorem A.1 (Existence and uniqueness of weak solution, [27], Chap. 2, [26], Props. 3.6, 3.7)**.** *Let* $(\mathbf{u}_0,\xi_0) = (\mathbf{v},\varrho) \in \mathbb{H}\times\mathrm{H}$ *be given. For* $\mathbf{w}^0 \in \mathrm{L}^4(\tau,T;\mathbb{L}^4(\Omega))$, $\mathbf{f} \in \mathrm{L}^2(\tau,T;\mathbb{V}')$ *and* $\mathrm{U} \in \mathrm{L}^2(\tau,T;\mathbb{V}')$, *there exists* a unique weak solution $(\mathbf{u},\xi)$ *to the system (2.1) satisfying:*

$$
\begin{aligned}
&\|\mathbf{u}(t)\|_{\mathbb{H}}^2 + \|\xi(t)\|_{\mathrm{H}}^2 + \alpha\int_\tau^t \|\nabla\mathbf{u}(s)\|_{\mathbb{H}}^2\mathrm{d}s \\
&\leq \left(\|\mathbf{v}\|_{\mathbb{H}}^2 + \|\varrho\|_{\mathrm{H}}^2 + \frac{r}{\lambda}\int_\tau^t \|\mathbf{w}^0(s)\|_{\mathbb{L}^4}^4\mathrm{d}s + \int_\tau^t \|\mathbf{f}(s)\|_{\mathbb{V}'}^2\mathrm{d}s + \int_\tau^t \|\mathrm{U}(s)\|_{\mathbb{V}'}^2\mathrm{d}s\right)e^{Kt},
\end{aligned} \tag{A.1}
$$

for all $t \in [\tau,T]$, *where* $K = \max\big\{1 + M + \frac{r}{\lambda}, \frac{2}{\alpha}(1+\mu^2) + M\big\}$.

Remark A.2. From the relation (2.5), it can be easily seen that if $\mathbf{g} \in \mathrm{L}^2(\tau,T;\mathbb{V}')$ and $\mathbf{w}^0 \in \mathrm{L}^\infty(\tau,T;\mathbb{H}) \cap \mathrm{L}^2(\tau,T;\mathbb{V})$ with $\partial_t\mathbf{w}^0 \in \mathrm{L}^2(\tau,T;\mathbb{V}')$ (which also implies $\mathbf{w}^0 \in \mathrm{C}([\tau,T];\mathbb{H})$), then $\mathbf{f} \in \mathrm{L}^2(\tau,T;\mathbb{V}')$.

Theorem A.3 ([30], Lem. 3.7)**.** *Let* $(\mathbf{u}_1,\xi_1)$ *and* $(\mathbf{u}_2,\xi_2)$ *be the unique weak solutions of the controlled system (2.1) with controls* U_1 *and* U_2, *and initial data* $(\mathbf{v}_1,\varrho_1)$ *and* $(\mathbf{v}_2,\varrho_2)$, *respectively. Then, we have*

$$\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_{\mathbb{H}}^2 + \|\xi_1(t) - \xi_2(t)\|_{\mathrm{H}}^2 + \alpha\int_\tau^t \|\nabla(\mathbf{u}_1(s) - \mathbf{u}_2(s))\|_{\mathbb{H}}^2\mathrm{d}s$$

$$\leq \left\{\|\mathbf{v}_1-\mathbf{v}_2\|_{\mathbb{H}}^2+\|\varrho_1-\varrho_2\|_{\mathrm{H}}^2+\frac{4}{\alpha}\int_\tau^t\|\mathrm{U}_1(s)-\mathrm{U}_2(s)\|_{\mathbb{V}'}^2\mathrm{d}t\right\}e^{\left[\frac{4}{\alpha}(2+\mu^2)+M\right]t}, \tag{A.2}$$

for all $t\in[\tau,T]$.

In order to obtain the strong solution and further regularity results for the system (2.1), we need the following technical assumption on h (see [29] for more details).

Assumption A.4. Let us assume that $h\in\mathrm{C}_b^2(\Omega)$ with

$$\max_{x\in\Omega}|\nabla(\nabla h(x))|\leq M_1,$$

where $M_1\geq 0$ is some non-negative constant.

Remark A.5. From (2.5), we know that for $\mathbf{f}\in\mathrm{L}^2(\tau,T;\mathbb{H})$, one needs

$$\mathbf{w}^0\in\mathrm{L}^2(\tau,T;\mathbb{W})\cap\mathrm{H}^1(\tau,T;\mathbb{H}).$$

The above regularity on $\mathbf{w}^0$ and Assumption A.4 ensure that $\Delta\mathbf{w}^0$, $\frac{\partial\mathbf{w}^0}{\partial t}$, $\nabla\int_\tau^t\mathrm{div}(h\mathbf{w}^0)\mathrm{d}s\in\mathrm{L}^2(\tau,T;\mathbb{H})$.

Theorem A.6 (Strong solution, [29], Thm. 2.11)**.** *Under the Assumption A.4, for* $\mathbf{f}\in\mathrm{L}^2(\tau,T;\mathbb{H})$*,* $\mathrm{U}\in\mathrm{L}^2(\tau,T;\mathbb{H})$ *and* $(\mathbf{v},\varrho)\in\mathbb{V}\times\mathrm{V}$*, the unique weak solution to the system (2.1) is also a strong solution in the sense of Definition 2.5 satisfying:*

$$\begin{aligned}&\sup_{\tau\leq t\leq T}\left[\|\mathbf{u}(t)\|_{\mathbb{V}}^2+\|\xi(t)\|_{\mathrm{V}}^2\right]+\int_\tau^T\|\mathbf{u}(t)\|_{\mathbb{W}}^2\mathrm{d}t\\&\leq C\big(\alpha,r,\lambda,\mu,M,M_1,\|\mathbf{v}\|_{\mathbb{V}},\|\varrho\|_{\mathrm{V}},\|\mathbf{w}^0\|_{\mathrm{L}^4(\tau,T;\mathbb{L}^4(\Omega))}\|\mathrm{U}\|_{\mathrm{L}^2(\tau,T;\mathbb{H})},\|\mathbf{f}\|_{\mathrm{L}^2(\tau,T;\mathbb{H})}\big).\end{aligned} \tag{A.3}$$

Moreover, if $\mathbf{w}^0\in\mathrm{C}([\tau,T];\mathbb{W})\cap\mathrm{C}^1(\tau,T;\mathbb{H})$*,* $\mathbf{f}\in\mathrm{C}([\tau,T];\mathbb{H})$ *and* $\mathrm{U}\in\mathrm{C}([\tau,T];\mathbb{H})$ *and* $(\mathbf{v},\varrho)\in\mathbb{V}\cap\mathbb{W}\times\mathrm{V}$*, then we have*

$$(\mathbf{u},\xi)\in(\mathrm{C}([\tau,T];\mathbb{V}\cap\mathbb{W})\times\mathrm{C}([\tau,T];\mathrm{V})$$

and $(\partial_t\mathbf{u},\partial_t\xi)\in\mathrm{C}([\tau,T];\mathbb{H})\times\mathrm{C}([\tau,T];\mathrm{V})$*.*

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