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THE MINIMAL TIME FUNCTION ASSOCIATED WITH A COLLECTION OF SETS[*,**]

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Abstract. We define the minimal time function associated with a collection of sets which is motivated by the optimal time problem for nonconvex constant dynamics. We first provide various basic properties of this new function: lower semicontinuity, principle of optimality, convexity, Lipschitz continuity, among others. We also compute and estimate proximal, Fréchet and limiting subdifferentials of the new function at points inside the target set as well as at points outside the target. An application to location problems is also given.

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1. Introduction

Let X be a normed space, and let F and Ω be two nonempty subsets of X. The minimal time function associated with the constant dynamics F to the target set Ω is defined by

$$T_{\Omega}^{F}(x) := \inf\{t \geq 0 : (x + tF) \cap \Omega \neq \emptyset\}, \quad x \in X. \tag{1.1}$$

The minimal time function T_{Ω}^{F} covers three crucial functions in variational analysis: the distance function to the set Ω when we take F the closed unit ball; the Minkowski function associated with Ω when we take $F = \{0\}$ and the indicator function associated with F when we take $\Omega = \{0\}$. By this fact, we can see the importance of the study of the minimal time function. Variational analysis and generalized differentiations of the minimal time function associated with a convex dynamics set, which contains the origin in its interior, in a Hilbert space were initially studied by Colombo and Wolenski in [8, 9]. Since then, the minimal time function has been extensively studied by many researchers in various ways and for different purposes; see, *e.g.*, [2, 3, 11, 14, 17, 18, 20, 21, 26, 27, 31]. In [17], He and Ng studied generalized differentiations of the minimal time function in Banach spaces. In [18], Jiang and He studied proximal and Fréchet suddifferentials of the function in normed spaces under calmness assumptions on the initial data and without assuming that the dynamics set contains the origin. Results presented in the above-mentioned papers were extended and improved in [20, 21] by

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Mordukhovich and Nam and then were further extended to the context of Hausdorff topological vector spaces in [2, 3] by Bounkhel. We also refer the reader to [26, 31] for the study of subdifferentials of the minimal time function without calmness assumptions. Applications of variational analysis and generalization differentiations of the minimal time function to generalized Sylvester problems and to generalized Fermat-Terricelli problems were presented in [21–25, 27, 28] and references therein. For variational analysis and applications of the function T_Ω^F when F is a subset of the unit sphere, we refer the reader to [13–15].

It is known that if the dynamics F is convex, then the minimal time function T_Ω^F coincides with the minimum time function to the target Ω for the differential inclusion

$$\begin{cases} \dot{y}(t) \in F, & \text{a.e.}\, t > 0, \\ y(0) = x \in X, \end{cases} \tag{1.2}$$

in control theory which is defined by

$$\mathcal{T}(x) := \inf\{t \geq 0 : \exists y(\cdot) \text{ satisfying (1.2) and } y(t) \in \Omega\}. \tag{1.3}$$

For the study of the minimum time function $\mathcal{T}$ for more general control systems in finite dimensional setting, we refer the reader to [4–7, 10, 12, 16, 29, 32] and the references therein. If F is not convex, then $\mathcal{T}$ and T_Ω^F do not coincide. Indeed, let us consider the following simple example. Let $X = \mathbb{R}^2$, $F = \{(1,0),(0,1)\}$ and $\Omega = \{(2,2)\}$. We can easily compute that the domain of T_Ω^F is the set $\mathrm{dom}(T_\Omega^F) = \{(x,2) \in \mathbb{R}^2 : x \leq 2\} \cup \{(2,y) \in \mathbb{R}^2 : y \leq 2\}$ and that $T_\Omega^F(x,2) = 2 - x$ if $x \leq 2$ and $T_\Omega^F(2,y) = 2 - y$ if $y \leq 2$. We can also compute that the domain of $\mathcal{T}$ in this case is the set $\mathrm{dom}(\mathcal{T}) = \{(x,y) \in \mathbb{R}^2 : x \leq 2, y \leq 2\}$ and that $\mathcal{T}(x,y) = 4 - x - y$ for all $(x,y) \in \mathrm{dom}(\mathcal{T})$. Thus, $\mathcal{T}$ and T_Ω^F are not the same. However, with some more computation, we can see that, for all $x \in \mathbb{R}^2$,

$$\mathcal{T}(x) = \inf\{t_1 + t_2 : t_1, t_2 \geq 0 \text{ and } (x + t_1 F_1 + t_2 F_2) \cap \Omega \neq \emptyset\},$$

with $F_1 = \{(1,0)\}$ and $F_2 = \{(0,1)\}$ and, of course, $F_1 \cup F_2 = F$. More general, in Section 3, we present a formula for computing the minimum time function $\mathcal{T}$ when dynamics F is a finite union of disjoint convex sets. This result motivates us to define and study the minimal time function associated to a collection of sets, which is a generalization of the usual minimal time function. The primal goal of this paper is to study basic properties (lower semicontinuity, principle of optimality, convexity, Lipschitz continuity, among others) and develop subdifferential properties of this new minimal time function. An application of this study to location problems is also presented.

The paper is organized as follows. In Section 2, we present basic notations and definitions needed in the sequel. In Section 3, we define the minimal time function associated with a collection of sets based on an equivalent formula of the minimum time function for a nonconvex constant dynamics. We then present basic properties of this function: lower semicontinuity, principle of optimality, convexity, Lipschitz continuity, among others. Section 4 is devoted to subdifferentials of the new minimal time function and an application of this generalized differentiation calculus to a location problem.

2. Preliminaries

Throughout this paper, unless stated otherwise, X is a normed space with norm $||\cdot||$ and X^* is its topological dual. We also denote by $||\cdot||$ the dual norm in X^*, and by $\langle \cdot, \cdot \rangle$ the dual pair between X^* and X. We denote by w^* the weak-star topology on X^*. We also denote by $\to$ the strong convergence and by $\xrightarrow{w^*}$ the weak-star convergence. For $x \in X$, we denote by $B(x,r)$ the open ball with center $x \in X$ and radius $r > 0$, by $\mathbb{B}$ the unit open ball and by $\mathbb{S}_X$ the unit sphere of X. For a set $A \subset X$, we denote by $\mathrm{int}A$, $\mathrm{cl}A$ and $\mathrm{bdry}A$ its topological interior, closure and boundary, respectively. We denote by $\mathrm{cone}A$ the cone generated by A, and $\mathrm{span}A$ the linear hull of A. The distance from a point x to A is $\mathrm{d}(x,A) := \inf\{||x - a|| : a \in A\}$. For $x, y \in X$, we denoted by $[x,y]$ the segment joining x and y. For a set $I \subset \mathbb{R}$, we denote by $|I|$ the Lebesgue measure of I.

Let K be a subset of X and $x \in K$. The proximal normal cone $N_K^P(x)$ to K at x is the set of all $\zeta \in X^*$ for which there exists $\sigma \geq 0$ such that

$$\langle \zeta, y - x \rangle \leq \sigma ||y - x||^2 \quad \forall y \in K.$$

The Fréchet normal cone $\widehat{N}_K(x)$ to K at x is the set of all $\zeta \in X^*$ for which, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\langle \zeta, y - x \rangle \leq \varepsilon ||y - x||, \qquad \forall\, y \in K \cap B(x, \delta).$$

If X is an Asplund space (see [19] for the definition), the limiting/Mordukhovich normal cone to K at x is

$$N_K(x) := \left\{ \zeta \in X^* : \exists x_n \xrightarrow{K} x, \zeta_n \xrightarrow{w^*} \zeta, \zeta_n \in \widehat{N}_K(x_n), \, \forall n \in \mathbb{N} \right\},$$

where $x_n \xrightarrow{K} x$ means $x_n \to x$ and $x_n \in K$ for every n.

Let $f : X \to (-\infty, \infty]$. Denote by $\mathrm{dom}(f) := \{x \in X : f(x) < \infty\}$ the domain of f and by $\mathrm{epi}(f) := \{(x, \alpha) \in X \times \mathbb{R} : f(x) \leq \alpha\}$ the epigraph of f. Let $x \in \mathrm{dom}(f)$. The proximal subdifferential $\partial^P f(x)$ of f at x is the set of all $\zeta \in X^*$ satisfying $(\zeta, -1) \in N^P_{\mathrm{epi}(f)}(x, f(x))$. Equivalently,

$$\partial^P f(x) = \{\zeta \in X^* : \exists \eta > 0, \sigma \geq 0 \text{ so that } f(y) \geq f(x) + \langle \zeta, y - x \rangle - \sigma ||y - x||^2 \ \forall y \in B(x, \eta)\}.$$

The Fréchet subdifferential $\widehat{\partial} f(x)$ of f at x is the set of all $\zeta \in X^*$ satisfying $(\zeta, -1) \in \widehat{N}_{\mathrm{epi}(f)}(x, f(x))$. Equivalently,

$$\widehat{\partial} f(x) = \left\{ \zeta \in X^* : \liminf_{y \to x} \frac{f(y) - f(x) - \langle \zeta, y - x \rangle}{||y - x||} \geq 0 \right\}.$$

We can see that $\zeta \in \widehat{\partial} f(x)$ if and only if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$f(y) - f(x) - \langle \zeta, y - x \rangle \geq -\varepsilon ||y - x||, \quad \forall y \in B(x, \delta).$$

When X is an Asplund space and f is lower semicontinuous at x, the limiting/Mordukhovich subdifferential of f at x is

$$\partial f(x) := \left\{ \zeta \in X^* : \exists x_n \xrightarrow{f} x, \zeta_n \in \widehat{\partial} f(x_n), \, \forall n \in \mathbb{N} \, \text{and } \zeta_n \xrightarrow{w^*} \zeta \right\},$$

where $x_n \xrightarrow{f} x$ means that $x_n \to x$ and $f(x_n) \to f(x)$.

For an arbitrary subset A of X, the support function of A, $\rho_A : X^* \to (-\infty, \infty]$, is defined as: for $\zeta \in X^*$

$$\rho_A(\zeta) = \sup_{x \in A} \langle \zeta, x \rangle.$$

Let A, B be two nonempty subsets of X. One can easily check that

$$\rho_{A \cup B}(\zeta) = \max\{\rho_A(\zeta), \rho_B(\zeta)\}, \quad \forall \zeta \in X^*.$$

3. Minimal time function associated with a collection of sets and its basic properties

We start this section by giving an equivalent formula for the minimum time function when the dynamics is a finite union of disjoint convex sets.

Theorem 3.1. *Let X be a Banach space and let m be a positive integer. Let $U_1, \cdots, U_m$ be nonempty closed convex and pairwise disjoint subsets of X and let Ω be a nonempty subset of X. The minimum time function $\mathcal{T}$ associated with target Ω for the differential inclusion*

$$\begin{cases} \dot{y}(t) \in U_1 \cup \cdots \cup U_m, & \text{a.e.}\, t > 0, \\ y(0) = x \in X, \end{cases} \tag{3.1}$$

can be computed as

$$\mathcal{T}(x) = \inf\{t_1 + \cdot + t_m : t_1, \cdots, t_m \geq 0 \text{ and } (x + t_1U_1 + \cdots + t_mU_m) \cap \Omega \neq \emptyset\}, \quad \forall x \in X. \tag{3.2}$$

Proof. For simplicity, we prove this theorem for the case when $m = 2$. Letting $x \in X$ and $t \geq 0$, we denote by $R(t; x)$ the reachable set at time t starting from x, *i.e.*,

$$R(t; x) = \{y(t) : y(\cdot) \text{ is a solution of } (3.1)\}.$$

Let $y(\cdot)$ be a solution of (3.1). Setting

$$I_1 = \{s \in [0, t] : \dot{y}(s) \in U_1\},$$

$$I_2 = \{s \in [0, t] : \dot{y}(s) \in U_2\},$$

and $t_1 = |I_1|, t_2 = |I_2|$, we have $t = t_1 + t_2$ and

$$y(t) = x + \int_0^t \dot{y}(s)\mathrm{d}s = x + \int_{I_1} \dot{y}(s)\mathrm{d}s + \int_{I_2} \dot{y}(s)\mathrm{d}s.$$

If $t_1, t_2 > 0$, we find from the closedness and the convexity of U_1, U_2 that

$$\begin{aligned} y(t) &= x + t_1.\frac{1}{|I_1|}\int_{I_1} \dot{y}(s)\mathrm{d}s + t_2.\frac{1}{|I_2|}\int_{I_2} \dot{y}(s)\mathrm{d}s \\ &\in x + t_1U_1 + t_2U_2. \end{aligned}$$

When $t_1 = 0$ or $t_2 = 0$, we also have

$$y(t) \in x + t_1U_1 + t_2U_2.$$

It follows that

$$R(t; x) \subset \{x + t_1u_1 + t_2u_2 : t_1, t_2 \geq 0, t_1 + t_2 = t, u_1 \in U_1, u_2 \in U_2\}.$$

Further, for any $t_1, t_2 \geq 0$ with $t_1 + t_2 = t$ and for any $u_1 \in U_1, u_2 \in U_2$,

$$y_0(s) = \begin{cases} x & \text{if } s = 0 \\ x + su_1 & \text{if } 0 < s \leq t_1 \\ x + t_1u_1 + su_2 & \text{if } t_1 < s \end{cases}$$

is a solution of (3.1) satisfying $y_0(t) = x + t_1u_1 + t_2u_2$. Therefore, we have

$$R(t; x) = \{x + t_1u_1 + t_2u_2 : t_1, t_2 \geq 0, t_1 + t_2 = t, u_1 \in U_1, u_2 \in U_2\}.$$

Now,

$$\begin{aligned} \mathcal{T}(x) &= \inf\{t \geq 0 : \exists y(\cdot) \text{ satisfying (3.1) and } y(t) \in \Omega\} \\ &= \inf\{t \geq 0 : R(t; x) \cap \Omega \neq \emptyset\} \\ &= \inf\{t_1 + t_2 : t_1, t_2 \geq 0, (x + t_1U_1 + t_2U_2) \cap \Omega \neq \emptyset\}. \end{aligned}$$

This completes the proof. □

Motivated by the above result, we define the minimal time function associated with a collection of sets.

Definition 3.2. Let m be a positive integer, and let $\mathcal{U} = \{U_1, \cdots, U_m\}$ be a collection of m nonempty subsets $U_1, \cdots, U_m$ of X. Let Ω be a nonempty subset of X. The minimal time function associated with the collection $\mathcal{U}$ to the set Ω is defined as: for $x \in X$

$$T_{\mathcal{U},\Omega}(x) := \inf\left\{t_1 + \cdots + t_m : t_1, \cdots, t_m \geq 0 \text{ and } (x + t_1U_1 + \cdots + t_mU_m) \cap \Omega \neq \emptyset\right\}.$$

In this paper, for simplicity of the presentation, we consider the minimal time function associated with a collection of two subsets of X. From now on, unless otherwise stated, $\mathcal{U} = \{U_1, U_2\}$ is a collection of two nonempty subsets U_1, U_2 of X with $U_1 \cap U_2 \subset \{0\}$ and $U_1 \cup U_2 \neq \{0\}$, $\mathbb{U} = U_1 \cup U_2$ and Ω is a nonempty subset of X. The minimal time function associated with the collection $\mathcal{U}$ to Ω is now written as:

$$T_{\mathcal{U},\Omega}(x) = \inf\left\{t_1 + t_2 : t_1, t_2 \geq 0 \text{ and } (x + t_1U_1 + t_2U_2) \cap \Omega \neq \emptyset\right\}. \tag{3.3}$$

We can see that, if $U_1 = F$ and $U_2 = \{0\}$, then $T_{\mathcal{U},\Omega}$ becomes the usual minimal time function T_Ω^F defined in (1.1). When F is a singleton, say $F = \{v\}$, we put $T_\Omega^v(x) = T_\Omega^F(x)$, for all $x \in X$.

For $t \geq 0$, we define

$$\mathcal{R}(t) := \{x \in X : T_{\mathcal{U},\Omega}(x) \leq t\},$$

and

$$\mathcal{R} := \{x \in X : T_{\mathcal{U},\Omega}(x) < \infty\}.$$

For $x \in X$, $T_{\mathcal{U},\Omega}(x) < \infty$ is the smallest time to steer x to target Ω using at most one direction in each set U_1 and U_2. If the minimum in (3.3) is attained and $u_1^* \in U_1, u_2^* \in U_2$, $t_1^* \geq 0, t_2^* \geq 0$, are such that

$$x + t_1^*u_1^* + t_2^*u_2^* \in \Omega \qquad \text{and} \qquad T_{\mathcal{U},\Omega}(x) = t_1^* + t_2^*,$$

then we call (u_1^*, u_2^*) (respectively, (t_1^*, t_2^*)) an *optimal-direction pair* (respectively, an *optimal-time pair*) for x. In this case, the function $y_x^* : [0, T_{\mathcal{U},\Omega}(x)] \to X$ defined as

$$y_x^*(s) = \begin{cases} x + su_1^* & \text{if } 0 \le s \le t_1^* \\ x + t_1^* u_1^* + (s - t_1^*) u_2^* & \text{if } t_1^* \le s \le T_{\mathcal{U},\Omega}(x) \end{cases}$$

is called an *optimal path* of x corresponding to the pair (u_1^*, u_2^*). We denote the set of all optimal-direction pairs of x by $\mathcal{U}_o(x)$, that is,

$$\mathcal{U}_o(x) = \{(u_1^*, u_2^*) \in U_1 \times U_2 : (u_1^*, u_2^*) \text{ is an optimal-direction pair for } x\}.$$

We also denote the *minimal time projection* of x on Ω by $\Pi(x)$, that is,

$$\Pi(x) = \{w \in \Omega : w = y_x^*(T_{\mathcal{U},\Omega}(x)) \text{ with } y_x^*(\cdot) \text{ is an optimal path of } x\}.$$

It is obvious that if $x \in \Omega$, then $T_{\mathcal{U},\Omega}(x) = 0$. Thus, any $(u_1^*, u_2^*) \in U_1 \times U_2$ is an optimal-direction pair for x, and the optimal-time pair is $(t_1^*, t_2^*) = (0, 0)$ and y_x^* is just the point x. In this case, $\Pi(x) = \{y_x^*(T_{\mathcal{U},\Omega}(x))\} = \{x\}$.

Example 3.3. Let $X = \mathbb{R}^2$ with the usual norm. Let $U_1 = \{(0,1), (-1,0)\}, U_2 = \{(1,0)\}$ and target $\Omega = \{(2,2)\}$. We can easily compute that

$$\mathcal{R} = \{(x,y) \in \mathbb{R}^2 : x \le 2, y \le 2\} \cup \{(x,2) \in \mathbb{R}^2 : x > 2\}.$$

For $(x,y) \in X$ with $x \le 2, y \le 2$, we have $T_{\mathcal{U},\Omega}(x,y) = 4 - x - y$. In this case, the optimal-direction pair is (u_1^*, u_2^*) with $u_1^* = (0,1)$ and $u_2^* = (1,0)$ and (t_1^*, t_2^*) with $t_1^* = 2 - y$ and $t_2^* = 2 - x$ is its corresponding optimal-time pair.

For $(x,2)$ with $x > 2$, $T_{\mathcal{U},\Omega}(x,2) = x - 2$. We find that (u_1^*, u_2^*) with $u_1^* = (-1,0)$ and $u_2^* = (1,0)$ is the optimal-direction pair and $(t_1^*, t_2^*) = (2 - x, 0)$ is its corresponding optimal-time pair.

One sees that, in the classical case, an optimal path is a segment, while in the case with a collection of more than one sets, an optimal path could be a "*zigzag*" path. It means that this new type of minimal time function can be used to model problems that the classical one cannot. We next provide a simple concrete example in which the minimal time function associated with a collection of sets can be used.

Example 3.4. (*A simple shortest path problem*) An ant is on the surface $\mathcal{S}$ of a cuboid $OABC.MNPQ$ which lies on a table. The ant wants to reach the table. What is the minimum distance that the ant needs to walk to reach the table?

Assume that the face $OABC$ is on the table's face. One can see that in order to walk to the table with the smallest distance the ant must follow the following route:

(i) if the ant is on the face $OMNA$ (respectively, $OMQC$, $CQPB$ and $BPNA$), then it must walk perpendicularly to the edge OA (respectively, OC, CB and BA);
(ii) if the ant is on the face $MNPQ$, then it first walks perpendicularly to nearest edges among MN, NP, PQ and QM and once it reaches the nearest edges, it walks as in the route (i).

If we know the length, the depth and the height of the cuboid, then we can easily compute the smallest distance that the ant can walk to reach the table.

Now, we arrange a coordinate system as in the Figure 1 and let Ω be the set consisting of four edges OA, AB, BC and CO. Set

$$U_1 = \{(0,0,-1)\},\ U_2 = \{(1,0,0), (-1,0,0), (0,1,0), (0,-1,0)\}.$$

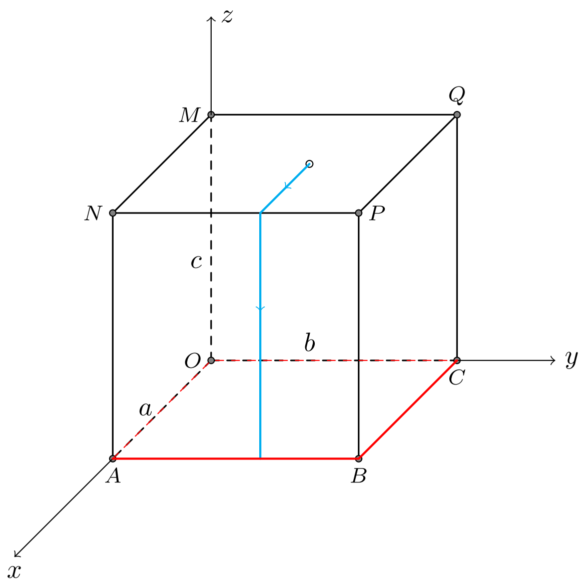

FIGURE 1. A shortest path problem.

Assume that the ant is at the position $\alpha = (x, y, z)$ on the surface of the cuboid. Then, the smallest distance that the ant needs to walk to reach the table is

$$D(\alpha) = \min\{t_1 + t_2 : t_1 \geq 0, t_2 \geq 0, (\alpha + t_1 U_1 + t_2 U_2) \cap \Omega \neq \emptyset\}.$$

Thus $D(\alpha) = T_{\mathcal{U},\Omega}(\alpha)$ - the mimimal time function associated with the collection $\mathcal{U} := \{U_1, U_2\}$ to reach the target Ω evaluating at α. A shortest path for the ant to travel to the table from position α is an optimal path for α.

From the definition, if $\mathcal{W} = \{W_1, W_2\}$ is a collection of two subsets W_1, W_2 of X with $U_1 \subset W_1$, $U_2 \subset W_2$, then

$$T_{\mathcal{U},\Omega}(x) \leq T_{\mathcal{W},\Omega}(x), \qquad \forall x \in X.$$

We are now going to present some basic properties of the minimal time function $T_{\mathcal{U},\Omega}$ by carefully extending analogous properties of the usual minimal time function which can be found in, *e.g.*, [3, 8, 14, 21, 25, 27].

Proposition 3.5. *For all $x \in X$,*

$$T_{\mathcal{U},\Omega}(x) \leq T^{\mathbb{U}}_{\Omega}(x).$$

If $\mathbb{U}$ is convex, then

$$T_{\mathcal{U},\Omega}(x) = T^{\mathbb{U}}_{\Omega}(x).$$

Proof. Let $x \in X$. If $T^{\mathbb{U}}_{\Omega}(x) = \infty$, then there is nothing to prove. Assume that $T^{\mathbb{U}}_{\Omega}(x) = t < \infty$. Then, for any $\varepsilon > 0$, there exist $u \in \mathbb{U}$ and $t_\varepsilon \in [t, t + \varepsilon)$ such that $x + t_\varepsilon u \in \Omega$. Since $u \in \mathbb{U}$, without loss of generality, we may assume that $u \in U_1$. Thus

$$(x + t_\varepsilon U_1 + 0.U_2) \cap \Omega \neq \emptyset,$$

which implies that $T_{\mathcal{U},\Omega}(x) \le t_\varepsilon + 0 < t + \varepsilon$. Letting $\varepsilon \to 0+$, we get $T_{\mathcal{U},\Omega}(x) \le t = T_\Omega^{\mathbb{U}}(x)$.

Assume now that $\mathbb{U}$ is convex. We prove that

$$T_\Omega^{\mathbb{U}}(x) \le T_{\mathcal{U},\Omega}(x), \qquad \forall x \in X. \tag{3.4}$$

Let $x \in X$. If $T_{\mathcal{U},\Omega}(x) = \infty$, then (3.4) holds. Suppose that $T_{\mathcal{U},\Omega}(x) = t < \infty$. Then, for any $\varepsilon > 0$, there exist $t_1, t_2 \ge 0$, $u_1 \in U_1, u_2 \in U_2$, such that $t \le t_1 + t_2 < t + \varepsilon$ and $x + t_1 u_1 + t_2 u_2 \in \Omega$. Thus

$$x + [t_1 + t_2]\left(\frac{t_1}{t_1+t_2}u_1 + \frac{t_2}{t_1+t_2}u_2\right) \in \Omega.$$

Since $u_1, u_2 \in \mathbb{U}$ and $\mathbb{U}$ is convex, one has

$$\frac{t_1}{t_1+t_2}u_1 + \frac{t_2}{t_1+t_2}u_2 \in \mathbb{U}.$$

Hence $(x + [t_1 + t_2]\mathbb{U}) \cap \Omega \ne \emptyset$, that is, $T_\Omega^{\mathbb{U}}(x) \le t_1 + t_2 < T_{\mathcal{U},\Omega}(x) + \varepsilon$. Letting $\varepsilon \to 0+$, we obtain the desired inequality. □

Because of the latter result, we will not consider the case when $\mathbb{U}$ is convex in this paper.

Proposition 3.6. *The following assertions hold:*

(i) If $x \in \Omega$, then $T_{\mathcal{U},\Omega}(x) = 0$.
(ii) Assume that $\mathbb{U}$ is bounded. If $T_{\mathcal{U},\Omega}(x) = 0$, then $x \in \mathrm{cl}\Omega$.
(iii) If either $0 \in \mathrm{int}U_1$ or $0 \in \mathrm{int}U_2$, then $T_{\mathcal{U},\Omega}(x) = 0$ for all $x \in \mathrm{cl}\Omega$.

Proof. (i) It is obvious.
(ii) Let $x \in X$ be such that $T_{\mathcal{U},\Omega}(x) = 0$. By the definition of $T_{\mathcal{U},\Omega}$, there exist sequences $\{t_i^n\}$ with $t_i^n \ge 0$ for all $i = 1, 2$ and for all $n \ge 0$ such that

$$t^n := t_1^n + t_2^n \to 0 \qquad \text{as} \qquad n \to \infty, \tag{3.5}$$

and for all n

$$(x + t_1^n U_1 + t_2^n U_2) \cap \Omega \ne \emptyset. \tag{3.6}$$

It follows from (3.5) that $t_i^n \to 0$ as $n \to \infty$ for all $i = 1, 2$. By (3.6), there are sequences $\{u_i^n\}$ in U_i, $i = 1, 2$ and $\{w^n\}$ in Ω such that for all n

$$x + t_1^n u_1^n + t_2^n u_2^n = w^n. \tag{3.7}$$

As $\mathbb{U}$ is bounded, we have that $\{u_i^n\}$, $i = 1, 2$, are bounded. Since $t_i^n \to 0$ for all $i = 1, 2$, one finds from (3.7) that $w^n \to x$ as $n \to \infty$. This means that $x \in \mathrm{cl}\Omega$.
(iii) We consider the case $0 \in \mathrm{int}U_1$, and the other case can be proved similarly. Let $x \in \mathrm{cl}\Omega$. For any $\varepsilon > 0$, we have $B(x, \varepsilon) \cap \Omega \ne \emptyset$. Since $0 \in \mathrm{int}U_1$, $B(x, \varepsilon) = x + B(0, \varepsilon) \subset x + \varepsilon U_1$, for all $\varepsilon > 0$ sufficiently small. Therefore, for all $\varepsilon > 0$ sufficiently small,

$$(x + \varepsilon U_1 + 0.U_2) \cap \Omega \ne \emptyset.$$

This implies $T_{\mathcal{U},\Omega}(x) \le \varepsilon$. Letting $\varepsilon \to 0^+$, we get $T_{\mathcal{U},\Omega}(x) = 0$. □

Proposition 3.7. *Assume that one of the following holds:*

(i) U_1, U_2 *are compact and* Ω *is closed.*
(ii) Ω *is compact, one of the sets* U_1, U_2 *is compact and the other is closed, bounded.*
(iii) X *is a reflexive Banach space,* U_1, U_2 *are bounded, weakly closed and* Ω *is weakly closed.*
(iv) X *is a reflexive Banach space,* U_1, U_2 *are bounded, closed, convex and* Ω *is closed, convex.*

Then the infimum in (3.3) is attained, i.e., if $x \in \mathcal{R}$*, then there exist* $t_1 \geq 0, t_2 \geq 0$ *such that*

$$T_{\mathcal{U},\Omega}(x) = t_1 + t_2 \quad \textit{and} \quad (x + t_1U_1 + t_2U_2) \cap \Omega \neq \emptyset.$$

Proof. Let $\{t^n\}$ be a sequence of real numbers such that, for all n,

$$t^n = t_1^n + t_2^n, \quad t_1^n \geq 0, t_2^n \geq 0, \quad (x + t_1^nU_1 + t_2^nU_2) \cap \Omega \neq \emptyset,$$

and $t^n \to T_{\mathcal{U},\Omega}(x)$ as $n \to \infty$.

Since $t^n \to T_{\mathcal{U},\Omega}(x)$, we have that t_i^n converges to some $t_i \geq 0$ for all $i = 1, 2$ and $T_{\mathcal{U},\Omega}(x) = t_1 + t_2$. Moreover, there exist $u_i^n \in U_i$, $i = 1, 2$ and $w \in \Omega$ such that

$$x + t_1^n u_1^n + t_2^n u_2^n = w^n. \tag{3.8}$$

(i) Since U_1, U_2 are compact, we can assume that u_i^n converges to some $u_i \in U_i$ for $i = 1, 2$. Thus $w^n \to x + t_1u_1 + t_2u_2$. Since Ω is closed, we have $x + t_1u_1 + t_2u_2 \in \Omega$.
(ii) Without loss of generality, we assume that U_2 is closed and bounded. We may assume that $u_1^n \to u_1 \in U_1$ and $w^n \to w \in \Omega$. If $t_2 = 0$, then by the boundedness of the sequence $\{u_2^n\}$ and by (3.8) we have $w = x + t_1u_1 \in \Omega$. Thus $x + t_1u_1 + t_2u_2 \in \Omega$, where u_2 is any point in U_2. Now, assume that $t_2 > 0$. Then we may assume that $t_2^n > 0$ for all n. By (3.8), one has $u_2^n = (w^n - x - t_1^n u_1^n)/t_2^n$. This implies that $u_2^n \to (w - x - t_1u_1)/t_2$ as $n \to \infty$. Since U_2 is closed, there is some $u_2 \in U_2$ such that $u_2 = (w - x - t_1u_1)/t_2$. Thus $x + t_1u_1 + t_2u_2 = w \in \Omega$.
(iii) Since $\{u_i^n\}$, $i = 1, 2$, are bounded and X is reflexive, we may assume that $\{u_i^n\}$ weakly converges to some u_i, $i = 1, 2$, in X (see, *e.g.*, Thm. 2.28(i) in [1]). By the weak closedness of U_i, we have $u_i \in U_i$, $i = 1, 2$. It follows (3.8) that $\{w^n\}$ weakly converges to $x + t_1u_1 + t_2u_2$. By the weak closedness of Ω, we have $x + t_1u_1 + t_2u_2 \in \Omega$.
(iv) Since X is reflexive, we may assume that u_i^n weakly converges to $u_i \in X$, $i = 1, 2$. By the classical Mazur theorem, a convex combination of elements from $\{u_i^n\}$ converges strongly to u_i, $i = 1, 2$. By the closedness and convexity of U_i, we have that $u_i \in U_i$, $i = 1, 2$. Since Ω is closed, convex and $\{w^n\}$ converges weakly to $x + t_1u_1 + t_2u_2$, we conclude that $x + t_1u_1 + t_2u_2 \in \Omega$. □

Proposition 3.8. *Assume that one of the conditions (i)–(iv) in Proposition 3.7 holds. Then* $T_{\mathcal{U},\Omega}$ *is lower semicontinuous on its domain* $\mathcal{R}$.

Proof. Let $x \in \mathcal{R}$ and $\{x^n\}$ be a sequence in X converging to x. We next show that

$$T_{\mathcal{U},\Omega}(x) \leq \liminf_{n \to \infty} T_{\mathcal{U},\Omega}(x^n). \tag{3.9}$$

If $\liminf_{n\to\infty} T_{\mathcal{U},\Omega}(x^n) = \infty$, then (3.9) holds. We consider the case $\liminf_{n\to\infty} T_{\mathcal{U},\Omega}(x^n) = \gamma \in [0, \infty)$. We may assume that $\lim_{n\to\infty} T_{\mathcal{U},\Omega}(x^n) = \gamma$. By the definition of $T_{\mathcal{U},\Omega}$, for each n, there exist $t_i^n \geq 0, i = 1, 2$, such that

$$T_{\mathcal{U},\Omega}(x^n) \leq t_1^n + t_2^n < T_{\mathcal{U},\Omega}(x^n) + \frac{1}{n} \quad \text{and} \quad (x + t_1^nU_1 + t_2^nU_2) \cap \Omega \neq \emptyset.$$

Thus there are $\{u_i^n\} \subset U_i, i = 1, 2$, and $\{w^n\} \subset \Omega$ such that

$$x^n + t_1^n u_1^n + t_2^n u_2^n = w^n.$$

Since $t_1^n + t_2^n \to \gamma$, we have that t_i^n converges to some $t_i \geq 0$, $i = 1, 2$, with $t_1 + t_2 = \gamma$.

Arguing as in proof of Proposition 3.7, we can show that there are $u_i \in U_i$, $i = 1, 2$ such that $x + t_1 u_1 + t_2 u_2 \in \Omega$. This implies that $T_{\mathcal{U},\Omega}(x) \leq t_1 + t_2 = \gamma$. Therefore, (3.9) holds. The proof is complete. □

Proposition 3.9. *Let Ω_1 and $\Omega_2 \subset X$ be nonempty. Assume that U_1 and U_2 are convex. Then, for all $x, y \in X$, we have*

$$T_{\mathcal{U},\Omega_1+\Omega_2}(x + y) \leq T_{\mathcal{U},\Omega_1}(x) + T_{\mathcal{U},\Omega_2}(y). \tag{3.10}$$

Proof. Let $x, y \in X$. If $T_{\mathcal{U},\Omega_1}(x) = \infty$, or $T_{\mathcal{U},\Omega_2}(y) = \infty$, then (3.10) holds. Assume now that $t := T_{\mathcal{U},\Omega_1}(x) < \infty$ and $s := T_{\mathcal{U},\Omega_2}(y) < \infty$. For any $\varepsilon > 0$, there exist $t_i \geq 0, s_i \geq 0$, $a_i, b_i \in U_i$, $i = 1, 2$, such that

$$t \leq t_1 + t_2 < t + \varepsilon,\ s \leq s_1 + s_2 < s + \varepsilon,\ x + t_1 a_1 + t_2 a_2 \in \Omega_1 \text{ and } x + s_1 b_1 + s_2 b_2 \in \Omega_2.$$

Thus

$$x + y + (t_1 a_1 + s_1 b_1) + (t_2 a_2 + s_2 b_2) \in \Omega_1 + \Omega_2. \tag{3.11}$$

Set $V = \{1 \leq i \leq 2 : t_i + s_i \neq 0\}$. Then $V \neq \emptyset$. We can rewrite (3.11) as

$$x + y + \sum_{i \in V} (t_i + s_i) \left(\frac{t_i a_i}{t_i + s_i} + \frac{s_i b_i}{t_i + s_i} \right) \in \Omega_1 + \Omega_2.$$

Since U_1, U_2 are convex, we have

$$c_i := \frac{t_i a_i}{t_i + s_i} + \frac{s_i b_i}{t_i + s_i} \in U_i, \qquad \forall i \in V.$$

Therefore,

$$x + y + \sum_{i \in V} (t_i + s_i) c_i \in \Omega_1 + \Omega_2.$$

This implies that

$$T_{\mathcal{U},\Omega_1+\Omega_2}(x + y) \leq \sum_{i \in V} (t_i + s_i) < t + s + 2\varepsilon.$$

Letting $\varepsilon \to 0^+$, we get (3.10). The proof is complete. □

Proposition 3.10 (Principle of optimality)**.** *Assume that U_1 and U_2 are convex. Then, for any $x \in X$, $a_1 \in U_1$, $a_2 \in U_2$, $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$, we have*

$$T_{\mathcal{U},\Omega}(x - \lambda_1 a_1 - \lambda_2 a_2) - \lambda_1 - \lambda_2 \leq T_{\mathcal{U},\Omega}(x) \leq T_{\mathcal{U},\Omega}(x + \lambda_1 a_1 + \lambda_2 a_2) + \lambda_1 + \lambda_2. \tag{3.12}$$

Proof. Let $x \in X$, $a_1 \in U_1$, $a_2 \in U_2$, $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$. We first prove that

$$T_{\mathcal{U},\Omega}(x) \leq T_{\mathcal{U},\Omega}(x + \lambda_1 a_1 + \lambda_2 a_2) + \lambda_1 + \lambda_2. \tag{3.13}$$

If $T_{\mathcal{U},\Omega}(x+\lambda_1 a_1+\lambda_2 a_2)=\infty$, then (3.13) holds. Suppose now that $t:=T_{\mathcal{U},\Omega}(x+\lambda_1 a_1+\lambda_2 a_2)<\infty$. Then, for any $\varepsilon>0$, there exist $t_i\geq 0, u_i\in U_i,\ i=1,2$, such that

$$t\leq t_1+t_2<t+\varepsilon \qquad \text{and} \qquad w:=x+\lambda_1 a_1+\lambda_2 a_2+t_1u_1+t_2u_2\in\Omega.$$

Setting $I=\{1\leq i\leq 2:\lambda_i+t_i\neq 0\}$, one has $I\neq\emptyset$. It follows that

$$w=x+\sum_{i\in I}[\lambda_i+t_i]\left(\frac{\lambda_i}{\lambda_i+t_i}a_i+\frac{t_i}{\lambda_i+t_i}u_i\right).$$

By the convexity of U_1,U_2, one has

$$\frac{\lambda_i}{\lambda_i+t_i}a_i+\frac{t_i}{\lambda_i+t_i}u_i\in U_i, \qquad \forall i\in I.$$

Thus,

$$T_{\mathcal{U},\Omega}(x)\leq\sum_{i\in I}(\lambda_i+t_i)<\lambda_1+\lambda_2+t+\varepsilon.$$

Letting $\varepsilon\to 0^+$, we get (3.13). Applying (3.13), we get the first inequality in (3.12). □

Corollary 3.11. *Assume that U_1 and U_2 are convex. Then, for any $x\in X$, $u\in\mathbb{U}$ and $\lambda\geq 0$, we have*

$$T_{\mathcal{U},\Omega}(x-\lambda u)-\lambda\leq T_{\mathcal{U},\Omega}(x)\leq T_{\mathcal{U},\Omega}(x+\lambda u)+\lambda.$$

Proposition 3.12. *Let $r\geq 0$ and $x\in X$ be such that $r<T_{\mathcal{U},\Omega}(x)<\infty$. If U_1 and U_2 are convex, then*

$$T_{\mathcal{U},\Omega}(x)=r+T_{\mathcal{U},\mathcal{R}(r)}(x). \tag{3.14}$$

Proof. We have $s:=T_{\mathcal{U},\mathcal{R}(r)}(x)<\infty$ as $\Omega\subset\mathcal{R}(r)$. Thus, for any $\varepsilon>0$, there exist $s_i\geq 0$, $u_i\in U_i$, $i=1,2$ and $w_1\in\mathcal{R}(r)$ such that $s\leq s_1+s_2<s+\varepsilon$ and $w_1=x+s_1u_1+s_2u_2$. Since $T_{\mathcal{U},\Omega}(w_1)\leq r$, there exist $t_i\geq 0$, $a_i\in U_i$, $i=1,2$ and $w_2\in\Omega$ such that $t_1+t_2<r+\varepsilon$ and $w_2=w_1+t_1a_1+t_2a_2$. Thus, by the convexity of U_1,U_2, one has

$$\begin{aligned} w_2&=x+s_1u_1+t_1a_1+s_2u_2+t_2a_2\\ &\in(x+s_1U_1+t_1U_1+s_2U_2+t_2U_2)\cap\Omega\\ &=[x+(s_1+t_1)U_1+(s_2+t_2)U_2]\cap\Omega, \end{aligned}$$

which implies that

$$T_{\mathcal{U},\Omega}(x)\leq s_1+t_1+s_2+t_2<T_{\mathcal{U},\mathcal{R}(r)}(x)+r+2\varepsilon.$$

Letting $\varepsilon\to 0+$, we get $T_{\mathcal{U},\Omega}(x)\leq T_{\mathcal{U},\mathcal{R}(r)}(x)+r$.

We now prove the opposite inequality in (3.14). Let $t:=T_{\mathcal{U},\Omega}(x)>r$. Then, for any $\varepsilon>0$, there exist $\gamma_i\geq 0$, $b_i\in U_i$, $i=1,2$ and $w\in\Omega$ such that $t\leq\gamma_1+\gamma_2<t+\varepsilon$ and $w=x+\gamma_1b_1+\gamma_2b_2$. Since $\gamma_1+\gamma_2>r$, we can decompose $r=r_1+r_2$ with $r_1<\gamma_1$ and $r_2\leq\gamma_2$. Set

$$w_r:=x+(\gamma_1-r_1)b_1+(\gamma_2-r_2)b_2.$$

Then $w_r + r_1 b_1 + r_2 b_2 = w \in \Omega$, that is, $w_r \in \mathcal{R}(r)$. Thus

$$T_{\mathcal{U},\mathcal{R}(r)}(x) \le (\gamma_1 - r_1) + (\gamma_2 - r_2) < t - r + \varepsilon = T_{\mathcal{U},\Omega}(x) - r + \varepsilon.$$

Letting $\varepsilon \to 0+$, we get $T_{\mathcal{U},\mathcal{R}(r)}(x) \le T_{\mathcal{U},\Omega}(x) - r$. This ends the proof. □

Proposition 3.13. *Assume that U_1 and U_2 are convex. Let $x \in \mathcal{R} \setminus \Omega$ and let $y_x(\cdot)$ be an optimal path of x. Then*

$$T_{\mathcal{U},\Omega}(x) = s + T_{\mathcal{U},\Omega}(y_x(s)), \quad \textit{for all } s \in [0, T_{\mathcal{U},\Omega}(x)]. \tag{3.15}$$

Proof. Since $y_x(\cdot)$ is an optimal path of x, one finds that $y_x(\cdot)$ has the form

$$y_x(s) = \begin{cases} x + s u_1, & \text{if } 0 \le s \le t_1, \\ x + t_1 u_1 + (s - t_1) u_2, & \text{if } t_1 \le s \le T_{\mathcal{U},\Omega}(x), \end{cases}$$

where $t_i \ge 0$, $u_i \in U_i$, $i = 1, 2$ with $t_1 + t_2 = T_{\mathcal{U},\Omega}(x)$ and $x + t_1 u_1 + t_2 u_2 \in \Omega$. Assume that $0 \le s \le t_1$. Then $y_x(s) = x + s u_1$. By Proposition 3.10, one has $T_{\mathcal{U},\Omega}(x) \le T_{\mathcal{U},\Omega}(y_x(s)) + s$. Moreover,

$$y_x(s) + (t_1 - s) u_1 + t_2 u_2 = x + t_1 u_1 + t_2 u_2 \in \Omega.$$

Thus $T_{\mathcal{U},\Omega}(y_x(s)) \le t_1 - s + t_2 = T_{\mathcal{U},\Omega}(x) - s$. Therefore, (3.15) holds. Similarly, we can prove (3.15) for the case when $t_1 \le s \le T_{\mathcal{U},\Omega}(x)$. □

Proposition 3.14. *If U_1, U_2 and Ω are convex, then $T_{\mathcal{U},\Omega}$ is convex. If $T_{\mathcal{U},\Omega}$ is convex, Ω is closed and either Ω is bounded or U_1, U_2 are bounded, then Ω is convex.*

Proof. Let $x, y \in X$ and $\lambda \in (0, 1)$. We will show that

$$T_{\mathcal{U},\Omega}(\lambda x + (1 - \lambda) y) \le \lambda T_{\mathcal{U},\Omega}(x) + (1 - \lambda) T_{\mathcal{U},\Omega}(y). \tag{3.16}$$

If $T_{\mathcal{U},\Omega}(x) = \infty$, or $T_{\mathcal{U},\Omega}(y) = \infty$, then (3.16) holds. Assume that $t := T_{\mathcal{U},\Omega}(x) < \infty$ and $s := T_{\mathcal{U},\Omega}(y) < \infty$. For any $\varepsilon > 0$, there exist $t_i \ge 0, s_i \ge 0$ and $a_i, b_i \in U_i$, $i = 1, 2$, such that

$$t \le t_1 + t_2 < t + \varepsilon,\ s \le s_1 + s_2 < s + \varepsilon,\ x + t_1 a_1 + t_2 a_2 \in \Omega \,\text{and}\, y + s_1 b_1 + s_2 b_2 \in \Omega.$$

Using the convexity of Ω, we have

$$\lambda x + (1 - \lambda) y + \sum_{i=1}^{2} (\lambda t_i a_i + (1 - \lambda) s_i b_i) \in \Omega. \tag{3.17}$$

If $\lambda t_i a_i + (1 - \lambda) s_i b_i = 0$ for all $i = 1, 2$, then $\lambda x + (1 - \lambda) y \in \Omega$. Thus $T_{\mathcal{U},\Omega}(\lambda x + (1 - \lambda) y) = 0$, *i.e.*, (3.16) holds. Assume now that

$$I = \{1 \le i \le 2 : \lambda t_i a_i + (1 - \lambda) s_i b_i \ne 0\} \ne \emptyset.$$

We can rewrite (3.17) as

$$\lambda x + (1 - \lambda) y + \sum_{i \in I} [\lambda t_i + (1 - \lambda) s_i] \left(\frac{\lambda t_i}{\lambda t_i + (1 - \lambda) s_i} a_i + \frac{(1 - \lambda) s_i}{\lambda t_i + (1 - \lambda) s_i} b_i \right) \in \Omega.$$

By the convexity of U_1, U_2, we find, for $i \in I$, that

$$c_i := \frac{\lambda t_i}{\lambda t_i + (1-\lambda)s_i} a_i + \frac{(1-\lambda)s_i}{\lambda t_i + (1-\lambda)s_i} b_i \in U_i.$$

Therefore,

$$\lambda x + (1-\lambda)y + \sum_{i\in I}[\lambda t_i + (1-\lambda)s_i]c_i \in \Omega,$$

which implies that

$$T_{\mathcal{U},\Omega}(\lambda x + (1-\lambda)y) \leq \sum_{i\in I}[\lambda t_i + (1-\lambda)s_i] < \lambda t + (1-\lambda)s + \varepsilon.$$

Letting $\varepsilon \to 0^+$, we obtain (3.16). Therefore, $T_{\mathcal{U},\Omega}$ is convex.

Now, if Ω is closed and either Ω is bounded or U_1, U_2 are bounded, then we find from Proposition 3.6 that

$$\Omega = \{x \in X : T_{\mathcal{U},\Omega}(x) \leq 0\}.$$

Thus Ω is convex if, in addition, $T_{\mathcal{U},\Omega}$ is convex. □

We next provide sufficient conditions for globally Lipschitz continuity of the minimal time function. The result extends (Prop. 4.1 in [14]) from the classical minimal time function to the minimal time function associated with a collection of sets. An earlier result of this type for the directional minimal time functions can be seen in Proposition 4.1 from [25]. The proof follows the idea of the proof of Proposition 4.1 from [14]. Recall that, for $\Omega \subset X$, its recession cone is $\Omega_\infty := \{x \in X : w + tx \in \Omega, \forall w \in \Omega\}$.

Proposition 3.15. *Assume that Ω is closed and that one of the following condition holds:*

(i) $(\text{cone}U_1 + \text{cone}U_2) \cap \text{int}\Omega_\infty \neq \emptyset$.
(ii) $\text{int}\,(\text{cone}U_1 + \text{cone}U_2) \cap \Omega_\infty \neq \emptyset$.

Then $T_{\mathcal{U},\Omega}$ is globally Lipschitz.

Proof. We first prove that there exists $\ell > 0$ such that $T_{\mathcal{U},\Omega_\infty}(x) \leq \ell||x||$ for all $x \in X$. Assume (i). Let us fix $v \in (\text{cone}U_1 + \text{cone}U_2) \cap \text{int}\Omega_\infty$ with $v \neq 0$. By Proposition 4.1 in [25], we have

$$T^v_{\Omega_\infty}(x) \leq \gamma||x||, \quad \forall x \in X, \qquad \text{with} \quad \gamma = \frac{1}{\text{d}(v, \text{bdry}\Omega_\infty)}. \tag{3.18}$$

Let $x \in X$. Since Ω_∞ is closed, we have

$$x + T^v_{\Omega_\infty}(x)v \in \Omega_\infty.$$

Since $v \in (\text{cone}U_1 + \text{cone}U_2)$, there exist $\alpha_i \geq 0$, $u_i \in U_i$, $i = 1, 2$, such that $v = \alpha_1 u_1 + \alpha_2 u_2$. Thus

$$x + \alpha_1 T^v_{\Omega_\infty}(x)u_1 + \alpha_2 T^v_{\Omega_\infty}(x)u_2 \in \Omega_\infty.$$

Therefore,

$$T_{\mathcal{U},\Omega_\infty}(x) \leq (\alpha_1 + \alpha_2)T^v_{\Omega_\infty}(x).$$

Together with (3.18), we have, for all $x \in X$, that

$$T_{\mathcal{U},\Omega_\infty}(x) \le (\alpha_1 + \alpha_2)\gamma||x||.$$

Now, assume (ii). Let $v \in \text{int}(\text{cone}U_1 + \text{cone}U_2) \cap \Omega_\infty$ be such that $v \neq 0$. There exist $\alpha_i \ge 0$ and $u_i \in U_i$, $i = 1, 2$ such that $v = \alpha_1 u_1 + \alpha_2 u_2$. As in the proof of (i), we can show that, for all $x \in X$,

$$T_{\mathcal{U},\Omega_\infty}(x) \le (\alpha_1 + \alpha_2) T^v_{\Omega_\infty}(x).$$

Now let $x \in X$. If $x \in \text{span}\{v\}$, *i.e.*, $x = \lambda v$ for some $\lambda \in \mathbb{R}$, then $x + |\lambda| v \in \Omega_\infty$. Thus

$$T^v_{\Omega_\infty}(x) \le |\lambda| = \frac{1}{||v||}||x||.$$

Hence

$$T_{\mathcal{U},\Omega_\infty}(x) \le \frac{\alpha_1 + \alpha_2}{||v||}||x||.$$

Assume now that $x \notin \text{span}\{v\}$. Let $r > 0$ be such that $B(v, r) \subset \text{cone}U_1 + \text{cone}U_2$. It follows that

$$v - \frac{r}{||x||}x \in \text{cone}U_1 + \text{cone}U_2.$$

Hence, there are $\gamma_i \ge 0, w_i \in U_i$, $i = 1, 2$ such that

$$v - \frac{r}{||x||}x = \gamma_1 w_1 + \gamma_2 w_2.$$

This implies that

$$x + \frac{||x||(\gamma_1 w_1 + \gamma_2 w_2)}{r} = \frac{||x||}{r} v \in \Omega_\infty.$$

Therefore,

$$T_{\mathcal{U},\Omega_\infty}(x) \le \frac{\gamma_1 w_1 + \gamma_2 w_2}{r}||x||.$$

Therefore, in both cases, there exists $\ell > 0$ such that

$$T_{\mathcal{U},\Omega_\infty}(x) \le \ell||x|| \qquad \text{for all } x \in X.$$

Let $x, y \in X$. By Proposition 3.9 and the fact $\Omega + \Omega_\infty = \Omega$, we have

$$T_{\mathcal{U},\Omega}(y) + T_{\mathcal{U},\Omega_\infty}(x - y) \ge T_{\mathcal{U},\Omega+\Omega_\infty}(x) = T_{\mathcal{U},\Omega}(x).$$

Thus,

$$T_{\mathcal{U},\Omega}(x) - T_{\mathcal{U},\Omega}(y) \le T_{\mathcal{U},\Omega_\infty}(x - y) \le \ell||x - y||.$$

Switching the roles of x and y in the latter inequality, we have

$$|T_{\mathcal{U},\Omega}(x) - T_{\mathcal{U},\Omega}(y)| \leq \ell||x - y||.$$

□

Corollary 3.16. *Assume that Ω is closed and either $0 \in \text{int}U_1$ or $0 \in \text{int}U_2$. Then $T_{\mathcal{U},\Omega}$ is globally Lipschitz.*

Proof. If $0 \in \text{int}U_1$ or $0 \in \text{int}U_2$, then $\text{cone}U_1 + \text{cone}U_2 = X$. Hence, $\text{int}(\text{cone}U_1 + \text{cone}U_2) \cap \Omega_\infty \neq \emptyset$. Therefore, by Proposition 3.15, $T_{\mathcal{U},\Omega}$ is globally Lipschitz. □

The following proposition is inspired by the analogous result in control theory (see, *e.g.*, [5, 32]).

Proposition 3.17. *Let U_1, U_2 be bounded and convex. The following conclusions are equivalent.*

(i) There exists $\delta > 0$ such that $T_{\mathcal{U},\Omega}$ is Lipschitz on $\Omega + \delta B$.
(ii) There exist $\sigma > 0$ and $k \geq 0$ such that

$$T_{\mathcal{U},\Omega}(x) \leq kd(x, \Omega), \qquad \forall x \in \Omega + \sigma\mathbb{B}.$$

Proof. Assume that (i) holds. For all $x, y \in \Omega + \delta\mathbb{B}$, we have

$$|T_{\mathcal{U},\Omega}(x) - T_{\mathcal{U},\Omega}(y)| \leq k||x - y||,$$

where k is the Lipschitz constant of $T_{\mathcal{U},\Omega}$. It follows that

$$T_{\mathcal{U},\Omega}(x) \leq T_{\mathcal{U},\Omega}(y) + k||x - y||.$$

Taking the infimum over $y \in \Omega$, we get $T_{\mathcal{U},\Omega}(x) \leq kd(x, \Omega)$.

Now assume that (ii) holds. Let $M = \sup\{||u|| : u \in U_1 \cup U_2\}$ and fix $0 < \varepsilon_0 < \frac{\sigma}{2M}$. We choose $\delta > 0$ such that

$$\delta < \frac{\sigma - \varepsilon_0}{kM + 1}.$$

Let $x, y \in \Omega + \delta\mathbb{B} \subset \Omega + \sigma\mathbb{B}$. We arrive at

$$t := T_{\mathcal{U},\Omega}(x) \leq kd(x, \Omega) \leq k\sigma < \infty.$$

For any $0 < \varepsilon < \varepsilon_0$, there exist $t_i \geq 0, u_i \in U_i$, $i = 1, 2$ and $w \in \Omega$ such that $t < t_1 + t_2 < t + \varepsilon$ and $x + t_1u_1 + t_2u_2 = w$. Setting $z := y + t_1u_1 + t_2u_2$, we see that

$$\begin{aligned} \text{d}(z, \Omega) &\leq ||y + t_1u_1 + t_2u_2 - y'|| \\ &\leq ||y - y'|| + (t_1 + t_2)M \\ &< ||y - y'|| + (t + \varepsilon)M, \quad \forall y' \in \Omega. \end{aligned}$$

Thus

$$\text{d}(z, \Omega) \leq \text{d}(y, \Omega) + (kd(x, \Omega) + \varepsilon)M \leq \delta + (k\delta + \varepsilon)M < \sigma.$$

By applying Proposition 3.10, we have that

$$T_{\mathcal{U},\Omega}(y) \leq T_{\mathcal{U},\Omega}(z) + t_1 + t_2 < T_{\mathcal{U},\Omega}(z) + t + \varepsilon.$$

It follows that

$$\begin{aligned} T_{\mathcal{U},\Omega}(y) - T_{\mathcal{U},\Omega}(x) &\le T_{\mathcal{U},\Omega}(z) + \varepsilon \\ &\le kd(z,\Omega) + \varepsilon \le k||z-w|| + \varepsilon \\ &= k||x-y|| + \varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0^+$, we get

$$T_{\mathcal{U},\Omega}(y) - T_{\mathcal{U},\Omega}(x) \le k||x-y||.$$

Switching the roles of x and y in the latter inequality, we conclude that $T_{\mathcal{U},\Omega}$ is Lipschitz on $\Omega + \delta\mathbb{B}$. □

4. Subgradients of minimal time functions

This section is devoted to the study of subdifferentials of the new class of minimal time functions and their applications. More precisely, we derive formulas for the proximal, Fréchet and limiting subdifferentials of the function at points in Ω as well as at points out of Ω. Finally, we give an application to a location problem. Throughout this section, we always assume that Ω is closed, U_1 and U_2 are bounded and denote by $M = \sup\{||u|| : u \in \mathbb{U}\}$, where $\mathbb{U} = U_1 \cup U_2$.

We recall that when control set F contains the origin, the formulas for computing the Fréchet (and proximal, respectively) subdifferential of the classical minimum time function T_Ω^F in Hilbert spaces at points outside the target Ω (in term of Fréchet (and proximal, respectively) normal cone to a sublevel set of T_Ω^F and a level set of the support function of F) were first given in [8, 9]. These results were then extended to the setting of Banach spaces in [17]. Note that in [17], the Fréchet (and proximal, respectively) subdifferential of T_Ω^F at points in Ω were also characterized in terms of Fréchet (and proximal, respectively) normal cone to Ω and a sublevel set of the support function of F. In [18], Jiang and He presented the same formulas for computing the Fréchet and the proximal subdifferentials of T_Ω^F in normed spaces without requiring that F contains the origin. However, a calmness condition was assumed for subifferential formulas at points outside the target. In [31], Sun and He proved the Fréchet and the proximal subdifferential formulas at points outside the target without using any calmness condition. For other paper deals with computing general differentiation of classical minimal time function, we refer the reader, to, *e.g.*, [2, 3, 14, 20, 21, 25–27].

We first adapt the formulas computing the Fréchet and the proximal subdifferentials of the classical minimal time function for the new function. Some proofs are adapted from results mentioned above. Making use of Lemmas 4.2 and 4.5, the techniques used in the proofs for difficult inclusions in Theorems 4.3 and 4.6 are different from those used in very recent paper [31] for classical one.

We first compute the proximal subdifferential of $T_{\mathcal{U},\Omega}$ at a point in the target set.

Theorem 4.1. *Let $x_0 \in \Omega$. It holds*

$$\partial^P T_{\mathcal{U},\Omega}(x_0) = N_\Omega^P(x_0) \cap \{\zeta \in X^* : \max\{\rho_{U_1}(-\zeta), \rho_{U_2}(-\zeta)\} \le 1\}. \tag{4.1}$$

Proof. Let $\zeta \in \partial^P T_{\mathcal{U},\Omega}(x_0)$. Then there are $c > 0$, and $\sigma > 0$ such that

$$T_{\mathcal{U},\Omega}(y) - \langle \zeta, y - x_0 \rangle \ge -c||y - x_0||^2 \quad \forall y \in B(x_0, \sigma). \tag{4.2}$$

Since $T_{\mathcal{U},\Omega}(x) = 0$ for all $x \in \Omega$, it follows from (4.2) that

$$\langle \zeta, y - x_0 \rangle \le c||y - x_0||^2 \quad \forall y \in \Omega \cap B(x_0, \sigma).$$

Thus $\zeta \in N_\Omega^P(x_0)$. Let $u \in \mathbb{U}$ be arbitrary and let $\lambda > 0$ be sufficiently small such that $x_0 - \lambda u \in B(x_0, \sigma)$. Since

$$(x_0 - \lambda u + \lambda \mathbb{U}) \cap \Omega \neq \emptyset,$$

we have $T_{\mathbb{U},\Omega}(x_0 - \lambda u) \leq \lambda$. By Proposition 3.5, one has

$$T_{\mathcal{U},\Omega}(x_0 - \lambda u) \leq T_\Omega^{\mathbb{U}}(x_0 - \lambda u) \leq \lambda.$$

For λ sufficiently small, we find from (4.2) that

$$\lambda \geq T_{\mathcal{U},\Omega}(x_0 - \lambda u) \geq \langle \zeta, -\lambda u \rangle - c\lambda^2 ||u||^2,$$

which implies that $\langle -\zeta, u \rangle \leq 1 + c\lambda ||u||^2$. Letting $\lambda \to 0^+$, we get $\langle -\zeta, u \rangle \leq 1$. Since u is arbitrary in $\mathbb{U}$, we have $\rho_{\mathbb{U}}(-\zeta) \leq 1$. Thus, $\max\{\rho_{U_1}(-\zeta), \rho_{U_2}(-\zeta)\} \leq 1$.

Conversely, let $\zeta \in N_\Omega^P(x_0)$ and $\max\{\rho_{U_1}(-\zeta), \rho_{U_2}(-\zeta)\} \leq 1$. Then there are $c_1 > 0$ and $\sigma_1 > 0$ such that

$$\langle \zeta, y - x_0 \rangle \leq c_1 ||y - x_0||^2, \quad \forall y \in \Omega \cap B(x_0, \sigma_1). \tag{4.3}$$

We will show that $\zeta \in \partial^P T_{\mathcal{U},\Omega}(x_0)$, *i.e.*, there exist $c_2 > 0$, and $\sigma_2 > 0$ such that

$$T_{\mathcal{U},\Omega}(y) - \langle \zeta, y - x_0 \rangle \geq -c_2 ||y - x_0||^2, \tag{4.4}$$

for all $y \in B(x_0, \sigma_2)$. Assume to the contrary that (4.4) fails. Then there exists a sequence $\{y(k)\}$ in X such that $y(k) \to x_0$ as $k \to \infty$ and

$$T_{\mathcal{U},\Omega}(y(k)) - \langle \zeta, y(k) - x_0 \rangle < -k ||y(k) - x_0||^2 \tag{4.5}$$

for all k. By (4.3), there is some k_0 such that $y(k) \notin \Omega$ for all $k > k_0$. Moreover, since $y(k)$ converges to x_0, we may choose k_0 large enough such that

$$||y(k) - x_0|| \leq \frac{\sigma_1}{2 + M||\zeta||} \qquad \forall k > k_0. \tag{4.6}$$

Set $t(k) := T_{\mathcal{U},\Omega}(y(k))$. It follows from (4.5) that

$$t(k) \leq ||\zeta||.||y(k) - x_0|| < \infty, \qquad \forall k. \tag{4.7}$$

For each $k > k_0$, let ε_k be arbitrary satisfying

$$0 < \varepsilon_k < \frac{||y(k) - x_0||}{1 + M}.$$

Since $t(k) < \infty$, there exist $t_i(k) \geq 0$, $u_i(k) \in U_i$, $i = 1, 2$ and $\omega(k) \in \Omega$ such that

$$t(k) \leq t_1(k) + t_2(k) < t(k) + \varepsilon_k \quad \text{and} \quad \omega(k) = y(k) + t_1(k)u_1(k) + t_2(k)u_2(k).$$

We have, for $k > k_0$, that

$$\begin{aligned}||\omega(k) - x_0|| &= ||y(k) + t_1(k)u_1(k) + t_2(k)u_2(k) - x_0||\\ &\leq ||y(k) - x_0|| + (t_1(k) + t_2(k))M\\ &\leq ||y(k) - x_0|| + (t(k) + \varepsilon_k)M\\ &\leq ||y(k) - x_0|| + M||\zeta||.||y(k) - x_0|| + M\frac{||y(k) - x_0||}{1+M}\\ &\leq (2 + M||\zeta||)||y(k) - x_0|| < \sigma_1. \end{aligned} \tag{4.8}$$

Thus $\omega(k) \in \Omega \cap B(x_0, \sigma_1)$. Using (4.3) and (4.8) and the fact $\max\{\rho_{U_1}(-\zeta), \rho_{U_2}(-\zeta)\} \leq 1$, we have, for all $k > k_0$, that

$$\begin{aligned}T_{\mathcal{U},\Omega}(y(k)) - \langle \zeta, y(k) - x_0\rangle &= t(k) - \langle \zeta, y(k) - \omega(k)\rangle - \langle \zeta, \omega(k) - x_0\rangle\\ &\geq t(k) - (t_1(k) + t_2(k))\langle -\zeta, u(k)\rangle - c_2||\omega(k) - x_0||^2\\ &\geq t(k) - (t_1(k) + t_2(k)) - c_2(2 + M||\zeta||)^2||y(k) - x - 0||^2\\ &> -\varepsilon_k - c_2(2 + M||\zeta||)^2||y(k) - x - 0||^2.\end{aligned}$$

In view of (4.5), we have

$$k||y(k) - x_0||^2 < \varepsilon_k + c_2(2 + M||\zeta||)^2||y(k) - x - 0||^2 \quad \forall k > k_0.$$

Fixing $k > k_0$ and letting $\varepsilon_k \to 0^+$ in both sides of the latter inequality and then divide both sides by $||y(k) - x_0||$, we get $k \leq c_2(2 + M||\zeta||)^2$. This cannot happen when k sufficiently large. The proof is complete. □

Before going to present the formula computing the proximal subdifferential of $T_{\mathcal{U},\Omega}$ at a point out of the target, we have the following technical lemma.

Lemma 4.2. *Let $x \in X$ be such that $0 < r := T_{\mathcal{U},\Omega}(x) < \infty$ and let $\{x_n\}$ be a sequence in $\mathcal{R}(r)$ converging to x. If U_1, U_2 are convex and $N^P_{\mathcal{R}(r)}(x) \cap \{\zeta \in X^* : \max\{\rho_{U_1}(-\zeta), \rho_{U_2}(-\zeta)\} = 1\}$, then $T_{\mathcal{U},\Omega}(x_n) \to T_{\mathcal{U},\Omega}(x)$ as $n \to \infty$.*

Proof. Let $\zeta \in N^P_{\mathcal{R}(r)}(x)$ be such that $\max\{\rho_{U_1}(-\zeta), \rho_{U_2}(-\zeta)\} = 1$. There exists $\sigma > 0$ such that

$$\langle \zeta, y - x\rangle \leq \sigma||y - x||^2, \qquad \forall y \in \mathcal{R}(r). \tag{4.9}$$

Let $0 < \varepsilon \ll 1$. Since $x_n \to x$ as $n \to \infty$, there exists $N > 1$ such that $||x_n - x|| < \varepsilon$ for all $n > N$. Since $\max\{\rho_{U_1}(-\zeta), \rho_{U_2}(-\zeta)\} = 1$, there exists $u \in \mathbb{U}$ such that

$$\langle -\zeta, u\rangle > 1 - \sqrt{\varepsilon}.$$

For each n, set $y_n = x_n - \sqrt{\varepsilon}u$. We claim that, for $n > N$ and ε sufficiently small, $y_n \notin \mathcal{R}(r)$. Indeed, we have

$$\begin{aligned}\langle \zeta, y_n - x\rangle &= \langle \zeta, x_n - x\rangle + \sqrt{\varepsilon}\langle -\zeta, u\rangle\\ &\geq -||\zeta||||x_n - x|| + \sqrt{\varepsilon}(1 - \sqrt{\varepsilon})\\ &> -||\zeta||\varepsilon + \sqrt{\varepsilon}(1 - \sqrt{\varepsilon}),\end{aligned}$$

and

$$\sigma||y_n - x||^2 \leq \sigma(||x_n - x|| + \sqrt{\varepsilon}||u||)^2 < \sigma(\varepsilon + M\sqrt{\varepsilon})^2.$$

For ε sufficiently small, we have $\sigma(\varepsilon + M\sqrt{\varepsilon})^2 \leq -||\zeta||\varepsilon + \sqrt{\varepsilon}(1-\sqrt{\varepsilon})$. Hence,

$$\langle \zeta, y_n - x \rangle > \sigma||y_n - x||^2.$$

In view of (4.9), we conclude that $y_n \notin \mathcal{R}(r)$. So $T_{\mathcal{U},\Omega}(y_n) > r$ and by Corollary 3.11, $T_{\mathcal{U},\Omega}(y_n) \leq T_{\mathcal{U},\Omega}(x_n) + \sqrt{\varepsilon}$ for all $n > N$. For all $n > N$, we have

$$0 \leq T_{\mathcal{U},\Omega}(x) - T_{\mathcal{U},\Omega}(x_n) = r - T_{\mathcal{U},\Omega}(x_n) < T_{\mathcal{U},\Omega}(y_n) - T_{\mathcal{U},\Omega}(x_n) \leq \sqrt{\varepsilon}.$$

This implies that $T_{\mathcal{U},\Omega}(x_n) \to T_{\mathcal{U},\Omega}(x)$ as $n \to \infty$. □

We are now ready to state the formula computing the proximal subdifferential of $T_{\mathcal{U},\Omega}$ at a point $x \notin \Omega$.

Theorem 4.3. *Let U_1, U_2 be convex and let $x_0 \in X$ be such that $0 < r := T_{\mathcal{U},\Omega}(x_0) < \infty$. Then*

$$\partial^P T_{\mathcal{U},\Omega}(x_0) = N^P_{\mathcal{R}(r)}(x_0) \cap \{\zeta \in X^* : \max\{\rho_{U_1}(-\zeta), \rho_{U_2}(-\zeta)\} = 1\}. \tag{4.10}$$

Proof. Let $\zeta \in \partial^P T_{\mathcal{U},\Omega}(x_0)$. Then there exist $c > 0$ and $\delta > 0$ such that, for all $y \in B(x_0, \delta)$,

$$T_{\mathcal{U},\Omega}(y) - r - \langle \zeta, y - x_0 \rangle \geq -c||y - x_0||^2. \tag{4.11}$$

It follows that

$$\langle \zeta, y - x_0 \rangle \leq c||y - x_0||^2, \qquad \forall y \in \mathcal{R}(r) \cap B(x_0, \delta),$$

that is, $\zeta \in N^P_{\mathcal{R}(r)}(x_0)$. Since $T_{\mathcal{U},\Omega}(x_0) = r$, for any $\varepsilon > 0$, there exist $r_i \geq 0, u_i \in U_i$, $i = 1, 2$, and $\omega \in \Omega$ such that $r \leq r_1 + r_2 < r + \varepsilon$ and $w = x_0 + r_1u_1 + r_2u_2$. Let $u \in \mathbb{U}$ and $\lambda > 0$. Without loss of generality, we may assume that $u \in U_2$. It follows that

$$\begin{aligned} \omega &\in x_0 - \lambda u + \lambda U_2 + r_1U_1 + r_2U_2 \\ &= x_0 - \lambda u + r_1U_1 + (\lambda + r_2)U_2. \end{aligned}$$

Thus,

$$T_{\mathcal{U},\Omega}(x_0 - \lambda u) \leq r_1 + r_2 + \lambda < r + \varepsilon + \lambda.$$

For λ sufficiently small, *i.e.*, $x_0 - \lambda u \in B(x_0, \delta)$, we have from (4.11) with $y := x_0 - \lambda u$, that

$$r + \varepsilon + \lambda - r - \langle \zeta, -\lambda u \rangle > -c\lambda^2||u||^2,$$

or, equivalently,

$$\lambda\langle -\zeta, u \rangle < c\lambda^2||u||^2 + \lambda + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the latter inequality yields

$$\lambda\langle -\zeta, u \rangle \leq c\lambda^2||u||^2 + \lambda.$$

Dividing both sides of the latter inequality by $\lambda > 0$ and letting $\lambda \to 0^+$, we get $\langle -\lambda, u\rangle \leq 1$. Since $u \in \mathbb{U}$ is arbitrary, $\rho_{\mathbb{U}}(-\zeta) \leq 1$. Without loss of generality, we may assume that $r_1 > 0$. Letting $0 < \varepsilon < r_1^2$, one has

$$\omega = x_0 + r_1 u_1 + r_2 u_2 = x_0 + \sqrt{\varepsilon} u_1 + (r_1 - \sqrt{\varepsilon})u_1 + r_2 u_2.$$

It follows that

$$T_{\mathcal{U},\Omega}(x_0 + \sqrt{\varepsilon} u_1) \leq r_1 + r_2 - \sqrt{\varepsilon} < r + \varepsilon - \sqrt{\varepsilon}.$$

For ε sufficiently small, taking $y := x_0 + \sqrt{\varepsilon} u_1$ in (4.11), we have

$$\varepsilon - \sqrt{\varepsilon} - \sqrt{\varepsilon}\langle \zeta, u_1\rangle > -c\varepsilon ||u_1||^2,$$

or, equivalently,

$$\langle -\zeta, u_1\rangle > -c\sqrt{\varepsilon}||u_1||^2 - \sqrt{\varepsilon} + 1.$$

Letting $\varepsilon \to 0^+$ in both sides of the latter inequality, we get $\langle -\zeta, u_1\rangle \geq 1$. Thus

$$\max\{\rho_{U_1}(-\zeta), \rho_{U_2}(-\zeta)\} \geq 1.$$

Therefore, $\max\{\rho_{U_1}(-\zeta), \rho_{U_2}(-\zeta)\} = 1$.

We are now in a position to show the opposite inclusion. Let $\zeta \in N^P_{\mathcal{R}(r)}(x_0)$ with $\max\{\rho_{U_1}(-\zeta), \rho_{U_2}(-\zeta)\} = 1$. There exists $\sigma > 0$ such that

$$\langle \zeta, y - x_0\rangle \leq \sigma ||y - x_0||^2, \qquad \forall y \in \mathcal{R}(r). \tag{4.12}$$

We want to show that $\zeta \in \partial^P T_{\mathcal{U},\Omega}(x_0)$. Assume to the contrary that $\zeta \notin \partial^P T_{\mathcal{U},\Omega}(x_0)$. Then there exists a sequence $y(k) \to x_0$ as $k \to \infty$ satisfying $y(k) \neq x_0$ and

$$T_{\mathcal{U},\Omega}(y(k)) - r - \langle \zeta, y(k) - x_0\rangle < -k||y(k) - x_0||^2 \quad \text{for all } k. \tag{4.13}$$

It follows from (4.13) that $t(k) := T_{\mathcal{U},\Omega}(y(k)) < \infty$ and

$$t(k) - r \leq ||\zeta||.||y(k) - x_0|| \quad \text{for all } k. \tag{4.14}$$

It is enough to consider three cases: (i) $t(k) = r$ for all k, (ii) $t(k) > r$ for all k, and (iii) $t(k) < r$ for all k.

Case (i). $t(k) = r$ for all k. Then $y(k) \in \mathcal{R}(r)$. By (4.13), one has

$$\langle \zeta, y(k) - x_0\rangle > k||y(k) - x_0||^2.$$

This contradicts (4.12) when k is large.

Case (ii). $t(k) > r$ for all k. For each k, let

$$0 < \varepsilon_k < \frac{||y(k) - x_0||}{1 + M}.$$

Since $t(k) < \infty$, there exits $t_i(k) \geq 0, u_i(k) \in U_i$, $i = 1, 2$, and $\omega(k) \in \Omega$ such that

$$t(k) \leq t_1(k) + t_2(k) < t(k) + \varepsilon_k, \quad \text{and} \quad \omega(k) = y(k) + t_1(k)u_1(k) + t_2(k)u_2(k).$$

Since $t_1(k) + t_2(k) > r$ for all k, we can decompose r as $r = r_1(k) + r_2(k)$ with $0 \le r_1(k) < t_1(k)$, $0 \le r_2(k) < t_1(k)$ for all k. For each k, set

$$z(k) := y(k) + (t_1(k) - r_1(k))u_1(k) + (t_2(k) - r_2(k))u_2(k).$$

Then

$$z(k) + r_1(k)u_1(k) + r_2(k)u_2(k) = y(k) + t_1(k)u_1(k) + t_2(k)u_2(k) = \omega(k) \in \Omega.$$

This implies that

$$T_{\mathcal{U},\Omega}(z(k)) \le r_1(k) + r_2(k) = r,$$

i.e., $z(k) \in \mathcal{R}(r)$. We have, for all k, that

$$\begin{aligned} ||z(k) - x_0|| &= ||y(k) + (t_1(k) - r_1(k))u_1(k) + (t_2(k) - r_2(k))u_2(k) - x|| \\ &\le ||y(k) - x_0|| + (t_1(k) - r_1(k))||u_1(k)|| + (t_2(k) - r_2(k))||u_2(k)|| \\ &< ||y(k) - x_0|| + (t(k) + \varepsilon_k - r)M \\ &< ||y(k) - x_0|| + M\frac{||y(k) - x_0||}{1 + M} + M||\zeta||||y(k) - x_0|| \\ &< (M||\zeta|| + 2)||y(k) - x_0||. \end{aligned}$$

Now using the latter estimate and the facts that $z(k) \in \mathcal{R}(r)$ and $\max\{\rho_{U_1}(-\zeta), \rho_{U_2}(-\zeta)\} = -1$, we have

$$\begin{aligned} T_{\mathcal{U},\Omega}(y(k)) - r - \langle \zeta, y(k) - x_0\rangle &= t(k) - r + \langle \zeta, z(k) - y(k)\rangle - \langle \zeta, z(k) - x_0\rangle \\ &\ge t(k) - r + \sum_{i=1}^{2}(t_i(k) - r_i(k))\langle \zeta, u_i(k)\rangle - \sigma||z(k) - x_0||^2 \\ &\ge t(k) - \sum_{i=1}^{2} t_i(k) - \sigma(M||\zeta|| + 2)^2||y(k) - x_0||^2 \\ &> -\varepsilon_k - \sigma(M||\zeta|| + 2)^2||y(k) - x_0||^2. \end{aligned}$$

This, together with (4.13), yields

$$k||y(k) - x_0||^2 < \varepsilon_k + \sigma(M||\zeta|| + 2)^2||y(k) - x_0||^2, \qquad \forall k,$$

which is a contradiction when k is large since ε_k is arbitrarily small.

Case (iii). $t(k) < r$ for all k. For each k, let $0 < \varepsilon_k < r - t(k)$. There exist $t_i(k) \ge 0, i = 1, 2$ such that

$$t(k) \le t_1(k) + t_2(k) < t(k) + \varepsilon_k, \qquad \text{and} \qquad [y(k) + t_1(k)U_1 + t_2(k)U_2] \cap \Omega \ne \emptyset.$$

Since $\rho(-\zeta) = 1$, for each k, there exists $u(k) \in \mathbb{U}$ such that

$$\langle \zeta, u(k)\rangle < -1 + \varepsilon_k. \tag{4.15}$$

Set

$$z(k) := y(k) - (r - t_1(k) - t_2(k))\, u(k). \tag{4.16}$$

Fixing k, we may assume that $u(k) \in U_2$. It follows that

$$\begin{aligned} y(k) + \sum_{i=1}^{2} t_i(k)U_i &= z(k) + (r - t_1(k) - t_2(k))\, u(k) + t_1(k)U_1 + t_2(k)U_2 \\ &\subset z(k) + t_1(k)U_1 + (r - t_1(k))\, U_2. \end{aligned}$$

Thus

$$(z(k) + t_1(k)U_1 + (r - t_1(k))\, U_2) \cap \Omega \neq \emptyset.$$

This implies that $T_{\mathcal{U},\Omega}(z(k)) \leq r$, *i.e.*, $z(k) \in \mathcal{R}(r)$. We now set $s(k) := r - t_1(k) - t_2(k)$. By (4.15), (4.16) and (4.12), we have

$$\begin{aligned} s(k) &< \langle -\zeta, u(k)\rangle s(k) + \varepsilon_k s(k) \\ &= (t_1(k) + t_2(k) - r)\, \langle \zeta, u(k)\rangle + \varepsilon_k s(k) \\ &= \langle \zeta, z(k) - y(k)\rangle + \varepsilon_k s(k) \\ &= \langle \zeta, z(k) - x_0\rangle + \langle \zeta, x_0 - y(k)\rangle + \varepsilon_k s(k) \\ &\leq \sigma ||z(k) - x_0||^2 + ||\zeta|| ||y(k) - x_0|| + \varepsilon_k s(k). \end{aligned} \tag{4.17}$$

Moreover,

$$\begin{aligned} ||z(k) - x_0|| &= ||y(k) - (r - t_1(k) - t_2(k))u(k) - x_0|| \\ &\leq ||y(k) - x_0|| + (r - t_1(k) - t_2(k))||u(k)|| \\ &\leq ||y(k) - x_0|| + Ms(k). \end{aligned} \tag{4.18}$$

From (4.17) and (4.18), we arrive at

$$\begin{aligned} s(k) &\leq \sigma(||y(k) - x_0|| + Ms(k))^2 + ||\zeta|| ||y(k) - x_0|| + \varepsilon_k s(k) \\ &\leq 2\sigma ||y(k) - x_0||^2 + 2M^2 s^2(k) + ||\zeta|| ||y(k) - x_0|| + \varepsilon_k s(k) \\ &= [2M^2 s(k) + \varepsilon_k] s(k) + [2\sigma ||y(k) - x_0|| + ||\zeta||] ||y(k) - x_0|| \\ &\leq [2M^2 (r - t(k)) + \varepsilon_k] s(k) + (2\sigma ||y(k) - x_0|| + ||\zeta||)\, ||y(k) - x_0||. \end{aligned}$$

Since $y(k) \in \mathcal{R}(r)$ and $y(k) \to x_0$ as $k \to \infty$, by Lemma 4.2, one has $t(k) \to r$ as $k \to \infty$. The,n for k sufficiently large, we can choose ε_k small enough such that $2M^2(r - t(k)) + \varepsilon_k \leq \dfrac{1}{2}$ and $||y(k) - x_0|| < 1$. Thus, for k sufficiently large, we have

$$s(k) \leq 2(2\sigma + ||\zeta||)||y(k) - x_0||.$$

For k sufficiently large, we have

$$\begin{aligned} T_{\mathcal{U},\Omega}(y(k)) - r + \langle \zeta, y(k) - x_0\rangle &= t(k) - r + \langle \zeta, z(k) - y(k)\rangle - \langle \zeta, z(k) - x_0\rangle \\ &\geq t(k) - r + (r - t_1(k) - t_2(k))\langle -\zeta, u(k)\rangle - \sigma ||z(k) - x_0||^2 \\ &\geq t(k) - r + (r - t_1(k) - t_2(k))(1 - \varepsilon_k) \\ &\quad - \sigma(||y(k) - x_0|| + Ms(k))^2 \\ &\geq t(k) - t_1(k) - t_2(k) - (r - t_1(k) - t_2(k))\varepsilon_k \end{aligned}$$

$$-\sigma[1+2M(2\sigma+||\zeta||)]^2||y(k)-x_0||^2$$
$$\geq -\varepsilon_k-(r-t(k))\varepsilon_k-\sigma[1+2M(2\sigma+||\zeta||)]^2||y(k)-x_0||^2.$$

Combining with (4.13), we have, for k sufficiently large, that

$$-\varepsilon_k-(r-t(k))\varepsilon_k-\sigma[1+2M(2\sigma+||\zeta||)]^2||y(k)-x_0||^2<-k||y(k)-x_0||^2,$$

or, equivalently,

$$k||y(k)-x_0||^2<\varepsilon_k+(r-t(k))\varepsilon_k+\sigma[1+2M(2\sigma+||\zeta||)]^2||y(k)-x_0||^2.$$

Fix k. Letting $\varepsilon_k\to 0+$, and dividing both sides by $||y(k)-x_0||^2$,we get $k\leq\sigma[1+2M(2\sigma+||\zeta||)]^2$ which is a contradiction for large k. This ends the proof. □

We are now going to present formula computing the Fréchet subdifferential of $T_{\mathcal{U},\Omega}$ at a point $x_0\in\mathcal{R}$. We first consider the case $x_0\in\Omega$.

Theorem 4.4. *For any $x_0\in\Omega$, we have*

$$\widehat{\partial}T_{\mathcal{U},\Omega}(x_0)=\widehat{N}_\Omega(x_0)\cap\{\zeta\in X^*:\max\{\rho_{U_1}(-\zeta),\rho_{U_2}(-\zeta)\}\leq 1\}. \tag{4.19}$$

Proof. Let $\zeta\in\widehat{\partial}T_{\mathcal{U},\Omega}(x_0)$. Then, for any $\sigma>0$, there exists $\delta>0$ such that

$$T_{\mathcal{U},\Omega}(x)-\langle\zeta,x-x_0\rangle\geq-\sigma||x-x_0||, \tag{4.20}$$

for all $x\in B(x_0,\delta)$. It follows from (4.20) that

$$\langle\zeta,x-x_0\rangle\leq\sigma||x-x_0||,\quad\forall x\in\Omega\cap B(x_0,\delta).$$

Thus, $\zeta\in\widehat{N}_\Omega(x_0)$. Let $u\in\mathbb{U}$ be arbitrary. Let $\lambda>0$ be sufficiently small such that $x_0-\lambda u\in B(x_0,\delta)$. Since $(x_0-\lambda u+\lambda\mathbb{U})\cap\Omega\neq\emptyset$, one has $T_\Omega^{\mathbb{U}}(x_0-\lambda u)\leq\lambda$. By Proposition 3.5, one has $T_{\mathcal{U},\Omega}(x_0-\lambda u)\leq\lambda$. For λ sufficiently small, from (4.20), we have

$$\lambda\geq T_{\mathcal{U},\Omega}(x_0-\lambda u)\geq\langle\zeta,-\lambda u\rangle-\sigma\lambda||u||,$$

which implies that $\langle-\zeta,u\rangle\leq 1+\sigma||u||$. Letting $\sigma\to 0+$, we get $\langle-\zeta,u\rangle\leq 1$. This yields

$$\max\{\rho_{U_1}(-\zeta),\rho_{U_2}(-\zeta)\}\leq 1.$$

Now let $\zeta\in\widehat{N}_\Omega(x_0)$ with $\max\{\rho_{U_1}(-\zeta),\rho_{U_2}(-\zeta)\}\leq 1$. Let $\sigma>0$. For $\sigma_0\in\left(0,\dfrac{\sigma}{1+M||\zeta||}\right)$, there exists $\delta_0>0$ such that

$$\langle\zeta,x-x_0\rangle\leq\sigma_0||x-x_0||,\qquad\forall x\in\Omega\cap B(x_0,\delta_0). \tag{4.21}$$

Let $\delta\in\left(0,\dfrac{\delta_0}{2+2||\zeta||M}\right)$. We shall show that

$$T_{\mathcal{U},\Omega}(x)-\langle\zeta,x-x_0\rangle\geq-\sigma||x-x_0||, \tag{4.22}$$

for all $x \in B(x_0, \delta)$. By (4.20), we see that (4.22) holds for all $x \in \Omega \cap B(x_0, \delta)$. We prove that (4.22) holds for all $x \in B(x_0, \delta) \setminus \Omega$. If not, there exists $y \in B(x_0, \delta) \setminus \Omega$ such that

$$T_{\mathcal{U},\Omega}(y) < \langle \zeta, y - x_0 \rangle - \sigma ||y - x_0||. \tag{4.23}$$

It follows from (4.23) that

$$T_{\mathcal{U},\Omega}(y) \le ||\zeta||.||y - x_0||. \tag{4.24}$$

Set $t := T_{\mathcal{U},\Omega}(y)$. For any $\varepsilon \in \left(0, \dfrac{\delta_0}{2M}\right)$, there exist $t_i \ge 0, u_i \in U_i, i = 1, 2$ and $w \in \Omega$ such that $t \le t_1 + t_2 < t + \varepsilon$ and $w = y + t_1 u_1 + t_2 u_2$. Using (4.24), we have

$$\begin{aligned} ||w - x_0|| &\le ||y - x_0|| + t_1 ||u_1|| + t_2 ||u_2|| \le ||y - x_0|| + (t + \varepsilon) M \\ &\le ||y - x_0|| + M ||\zeta||.||y - x_0|| + \varepsilon M < \delta + M ||\zeta|| \delta + \varepsilon M \\ &< \delta_0. \end{aligned} \tag{4.25}$$

Thus $w \in \Omega \cap B(x_0, \delta_0)$. Using (4.21), (4.25) and the fact $\max\{\rho_{U_1}(-\zeta), \rho_{U_2}(-\zeta)\} \le 1$, we have

$$\begin{aligned} T_{\mathcal{U},\Omega}(y) - \langle \zeta, y - x_0 \rangle &= t - \langle \zeta, y - w \rangle - \langle \zeta, w - x_0 \rangle \\ &\ge t - t_1 \langle -\zeta, u_1 \rangle - t_2 \langle -\zeta, u_2 \rangle - \sigma_0 ||w - x_0|| \\ &> -\varepsilon - \sigma_0 (||y - x_0|| + (t + \varepsilon) M) \\ &> -(1 + \sigma_0 M)\varepsilon - \sigma_0 (1 + M ||\zeta||) ||y - x_0|| \\ &> -(1 + \sigma_0 M)\varepsilon - \sigma ||y - x_0||. \end{aligned}$$

Letting $\varepsilon \to 0+$, we have

$$T_{\mathcal{U},\Omega}(y) - \langle \zeta, y - x_0 \rangle \ge -\sigma ||y - x_0||.$$

This contradicts (4.23). The proof is complete. □

The following lemma is not only useful for proving Theorem 4.6 but has its own interest. It provides a sufficient condition for lower semicontinuity of the minimal time function. This result is new even for the classical one.

Lemma 4.5. *Assume that U_1 and U_2 are convex. Let $x_0 \in X$ be such that $0 < r := T_{\mathcal{U},\Omega}(x_0) < \infty$. Then $T_{\mathcal{U},\Omega}$ is lower semicontinuous at x_0 provided that*

$$\widehat{N}_{\mathcal{R}(r)}(x_0) \cap \{\zeta \in X^* : \max\{\rho_{U_1}(-\zeta), \rho_{U_2}(-\zeta)\} = 1\} \ne \emptyset.$$

Proof. Let $\zeta \in \widehat{N}_{\mathcal{R}(r)}(x_0)$ be such that $\max\{\rho_{U_1}(-\zeta), \rho_{U_2}(-\zeta)\} = 1$. Fix $\gamma \in (0, 1/(1 + M))$. There exists $\delta > 0$ such that

$$\langle \zeta, y - x_0 \rangle \le \gamma ||y - x_0||, \quad \forall y \in B(x_0, \delta) \cap \mathcal{R}(r). \tag{4.26}$$

Since $\max\{\rho_{U_1}(-\zeta), \rho_{U_2}(-\zeta)\} = 1$, there exists $u \in \mathbb{U}$ such that

$$\langle -\zeta, u \rangle > 1 - \gamma.$$

Let ε be such that

$$0 < \varepsilon < \frac{\delta}{2M},$$

and σ satify

$$0 < \sigma < \min\left\{\frac{\delta}{2}, \frac{\varepsilon[1-(M+1)\gamma]}{\gamma + ||\zeta||}\right\}.$$

For any $x \in B(x_0, \sigma)$, setting $\bar{x} = x - \varepsilon u$, we have

$$||\bar{x} - x_0|| = ||x - \varepsilon u - x_0|| \leq ||x - x_0|| + \varepsilon||u|| < \sigma + \varepsilon M < \delta,$$

i.e.,

$$\bar{x} \in B(x_0, \delta). \tag{4.27}$$

Moreover,

$$\langle \zeta, \bar{x} - x_0\rangle = \langle \zeta, x - x_0\rangle + \varepsilon\langle \zeta, -u\rangle > -||\zeta||||x - x_0|| + \varepsilon(1-\gamma) > -\sigma||\zeta|| + \varepsilon(1-\gamma),$$

and

$$\gamma||\bar{x} - x_0|| \leq \gamma(||x - x_0|| + \varepsilon||u||) < \gamma(\sigma + \varepsilon M).$$

It follows that

$$\langle \zeta, \bar{x} - x_0\rangle > \gamma||\bar{x} - x_0||.$$

So, by (4.26), $\bar{x} \notin B(x_0, \delta) \cap \mathcal{R}(r)$. Hence, by (4.27), $\bar{x} \notin \mathcal{R}(r)$, *i.e.*, $T_{\mathcal{U},\Omega}(\bar{x}) > r$. By Corollary 3.11,

$$T_{\mathcal{U},\Omega}(\bar{x}) = T_{\mathcal{U},\Omega}(x - \varepsilon u) \leq T_{\mathcal{U},\Omega}(x) + \varepsilon.$$

Therefore,

$$T_{\mathcal{U},\Omega}(x_0) - T_{\mathcal{U},\Omega}(x) = r - T_{\mathcal{U},\Omega}(x) < T_{\mathcal{U},\Omega}(\bar{x}) - T_{\mathcal{U},\Omega}(x_0) \leq \varepsilon.$$

This implies that $T_{\mathcal{U},\Omega}$ is lower semicontinuous at x_0. The proof is complete. □

Theorem 4.6. *Let U_1 and U_2 be convex and let $x_0 \in X$ be such that $0 < r := T_{\mathcal{U},\Omega}(x_0) < \infty$. Then*

$$\widehat{\partial} T_{\mathcal{U},\Omega}(x_0) = \widehat{N}_{\mathcal{R}(r)}(x_0) \cap \{\zeta \in X^* : \max\{\rho_{U_1}(-\zeta), \rho_{U_2}(-\zeta)\} = 1\}. \tag{4.28}$$

Proof. Let $\zeta \in \widehat{\partial} T_{\mathcal{U},\Omega}(x_0)$. Then, for any $\sigma > 0$, there exists $\delta > 0$ such that

$$T_{\mathcal{U},\Omega}(x) - T_{\mathcal{U},\Omega}(x_0) - \langle \zeta, x - x_0\rangle \geq -\sigma||x - x_0||, \qquad \forall y \in B(x_0, \delta). \tag{4.29}$$

This implies that $\langle \zeta, x - x_0 \rangle \le \sigma ||x - x_0||$ for all $x \in \mathcal{R}(r) \cap B(x_0, \delta)$, that is, $\zeta \in \widehat{N}_{\mathcal{R}(r)}(x_0)$. Let $u \in \mathbb{U}$ and $\lambda > 0$ be arbitrary. Since $T_{\mathcal{U},\Omega}(x_0) = r$, for any $0 < \varepsilon < r^2/4$, there exist $t_i \ge 0$, $u_i \in U_i$ $i = 1, 2$ and $w \in \Omega$ such that $r \le t_1 + t_2 < r + \varepsilon$ and $w = x_0 + t_1 u_1 + t_2 u_2$. Assume first that $u \in U_1$. It follows that

$$w \in x_0 - \lambda u + \lambda U_1 + t_1 U_1 + t_2 U_2 = x_0 - \lambda u + (t_1 + \lambda) U_1 + t_2 U_2.$$

Thus, $T_{\mathcal{U},\Omega}(x_0 - \lambda u) \le t_1 + t_2 + \lambda < r + \varepsilon + \lambda$. For λ sufficiently small, *i.e.*, $x_0 - \lambda u \in B(x_0, \delta)$, we find from (4.29) with $x := x_0 - \lambda u$ that

$$r + \varepsilon + \lambda - r - \langle \zeta, -\lambda u \rangle \ge -\sigma \lambda ||u||,$$

or, equivalently,

$$\lambda \langle -\zeta, u \rangle < \varepsilon + \lambda + \lambda \sigma ||u||.$$

Letting $\varepsilon \to 0+$, we get

$$\lambda \langle -\zeta, u \rangle < \lambda + \lambda \sigma ||u||.$$

Diving both sides of the latter inequality by $\lambda > 0$ and letting $\sigma \to 0+$, we get $\langle -\zeta, u \rangle \le 1$. Similarly, if $u \in U_2$, then we can show that $\langle -\zeta, u \rangle \le 1$. Therefore, $\rho_{\mathbb{U}}(-\zeta) \le 1$. Assume that $t_1 \ge t_2 \ge 0$. Then $\varepsilon < t_1^2$. It follows that

$$w = x_0 + t_1 u_1 + t_2 u_2 = x_0 + \sqrt{\varepsilon} u_1 + (t_1 - \sqrt{\varepsilon}) u_1 + t_2 u_2.$$

Hence,

$$T_{\mathcal{U},\Omega}(x_0 + \sqrt{\varepsilon} u_1) \le t_1 + t_2 - \sqrt{\varepsilon} < r + \varepsilon - \sqrt{\varepsilon}.$$

For ε sufficiently small, taking $x := x_0 + \sqrt{\varepsilon} u_1$ in (4.29), we get

$$\varepsilon - \sqrt{\varepsilon} - \sqrt{\varepsilon} \langle \zeta, u_1 \rangle \ge -\sqrt{\varepsilon} \sigma ||u_1||,$$

or, equivalently,

$$\langle -\zeta, u_1 \rangle \ge 1 - \sqrt{\varepsilon} - \sigma ||u_1||.$$

Letting $\varepsilon \to 0+$ and then letting $\sigma \to 0+$, one has $\langle -\zeta, u_1 \rangle \ge 1$. Thus $\rho_{U_1}(-\zeta) \ge 1$. If $t_2 \ge t_1 \ge 0$, then we can also show that $\rho_{U_2}(-\zeta) \ge 1$. Therefore, we always have $\max\{\rho_{U_1}(-\zeta), \rho_{U_2}(-\zeta)\} = 1$.

We are now in a position to prove the opposite inclusion. Let $\zeta \in \widehat{N}_{\mathcal{R}(r)}(x_0)$ be such that

$$\max\{\rho_{U_1}(-\zeta), \rho_{U_2}(-\zeta)\} = 1.$$

We attempt to show that $\zeta \in \widehat{\partial} T(x_0)$, *i.e.*, for any $\sigma > 0$, there exists $\eta > 0$ such that

$$T_{\mathcal{U},\Omega}(y) - T_{\mathcal{U},\Omega}(x_0) - \langle \zeta, y - x_0 \rangle \ge -\sigma ||y - x_0||, \tag{4.30}$$

for all $y \in B(x_0, \eta)$. For $\sigma > 0$, set

$$c := \min\left\{1, \frac{1}{1+2M}, \frac{\sigma}{1+2M+2M||\zeta||}, \frac{\sigma}{2+M||\zeta||}\right\},$$

and let $\sigma_0 \in (0, c)$. Since $\zeta \in \widehat{N}_{\mathcal{R}(r)}(x_0)$, there exists $\eta_0 > 0$ such that, for any $y \in \mathcal{R}(r) \cap B(x_0, \eta_0)$,

$$\langle \zeta, y - x_0 \rangle \le \sigma_0 ||y - x_0||. \tag{4.31}$$

Take $\sigma_2 \in (0, \eta_0/(1+M))$. Since $T_{\mathcal{U},\Omega}$ is lower semicontinuous at x_0, there exists $\eta_2 > 0$ such that

$$T_{\mathcal{U},\Omega}(x_0) - T_{\mathcal{U},\Omega}(y) \le \sigma_2, \quad \forall y \in B(x_0, \eta_2). \tag{4.32}$$

Let $\eta \in (0, c_1)$ with

$$c_1 := \min\left\{\eta_0 - M\sigma_2, \eta_2, \frac{\eta_0}{2+M||\zeta||}\right\}.$$

If (4.30) does not hold, then there exists $y_0 \in B(x_0, \eta)$ such that

$$T_{\mathcal{U},\Omega}(y_0) - T_{\mathcal{U},\Omega}(x_0) - \langle \zeta, y_0 - x_0 \rangle < -\sigma ||y_0 - x_0||. \tag{4.33}$$

We have three possible cases: (i) $T_{\mathcal{U},\Omega}(y_0) = r$, (ii) $T_{\mathcal{U},\Omega}(y_0) > r$, and (iii) $T_{\mathcal{U},\Omega}(y_0) < r$.

Case (i): $T_{\mathcal{U},\Omega}(y_0) = r$. Then $y_0 \in \mathcal{R}(r) \cap B(x_0, \eta) \subset \mathcal{R}(r) \cap B(x_0, \eta_0)$. It follows from (4.31) that

$$\langle \zeta, y_0 - x_0 \rangle \le \sigma_0 ||y_0 - x_0|| \le \sigma ||y_0 - x_0||.$$

This contradicts (4.33).

Case (ii): $t := T_{\mathcal{U},\Omega}(y_0) > r$. It follows from (4.33) that

$$t - r \le ||\zeta||.||y_0 - x_0||.$$

By definition of $T_{\mathcal{U},\Omega}$, for any $0 < \varepsilon < ||y_0 - x_0||/(1+M)$, there exist $t_i \ge 0$, $u_i \in U_i$, $i = 1, 2$, $w \in \Omega$ such that $t \le t_1 + t_2 < t + \varepsilon$ and $w = y_0 + t_1 u_1 + t_2 u_2$. We decompose r as $r = r_1 + r_2$ with $0 \le r_i \le t_i$, $i = 1, 2$. Set $z := y_0 + (t_1 - r_1)u_1 + (t_2 - r_2)u_2$. Then $z + r_1 u_1 + r_2 u_2 = y_0 + t_1 u_1 + t_2 u_2 \in \Omega$. Thus $T_{\mathcal{U},\Omega}(z) \le r$, *i.e.*, $z \in \mathcal{R}(r)$. Moreover, we have

$$\begin{aligned}
||z - x_0|| &= ||y_0 + (t_1 - r_1)u_1 + (t_2 - r_2)u_2 - x_0|| && (4.34)\\
&\le ||y_0 - x_0|| + (t_1 + t_2 - r_1 - r_2)M \\
&< ||y_0 - x_0|| + M\varepsilon + (t - r)M && (4.35)\\
&\le ||y_0 - x_0|| + M\varepsilon + M||\zeta||.||y_0 - x_0|| \\
&\le (2 + M||\zeta||)||y_0 - x_0||, && (4.36)
\end{aligned}$$

since $\varepsilon < ||y_0 - x_0||/(1+M)$. More precisely,

$$||z - x_0|| < (2 + M||\zeta||)||y_0 - x_0| < (2 + M||\zeta||)\eta < \eta_0.$$

Hence $z \in \mathcal{R}(r) \cap B(x_0, \eta_0)$. It follows from (4.31) that

$$\langle \zeta, z - x_0 \rangle \leq \sigma_0 ||z - x_0||.$$

Having in mind that $\langle \zeta, u_i \rangle \geq -1$, $i = 1, 2$, we have the following estimate

$$\begin{aligned} T_{\mathcal{U},\Omega}(y_0) - T_{\mathcal{U},\Omega}(x_0) - \langle \zeta, y_0 - x_0 \rangle &= t - r - \langle \zeta, y_0 - z \rangle - \langle \zeta, z - x_0 \rangle \\ &\geq t - r + (t_1 - r_1)\langle \zeta, u_1 \rangle + (t_2 - r_2)\langle \zeta, u_2 \rangle - \sigma_0 ||z - x_0|| \\ &\geq -\varepsilon - \sigma_0 (2 + M||\zeta||)||y_0 - x_0|| \\ &\geq -\varepsilon - \sigma ||y_0 - x_0||. \end{aligned}$$

Letting $\varepsilon \to 0+$, we get

$$T_{\mathcal{U},\Omega}(y_0) - T_{\mathcal{U},\Omega}(x_0) - \langle \zeta, y_0 - x_0 \rangle \geq -\sigma ||y_0 - x_0||.$$

This contradicts (4.33).

Case (iii): $t := T_{\mathcal{U},\Omega}(y_0) < r$. By definition of $T_{\mathcal{U},\Omega}$, for any $\varepsilon \in (0, r - t)$ with $2\varepsilon < 1 - 2M\sigma_0$, there exists $t_1, t_2 \geq 0$ such that $t \leq t_1 + t_2 < t + \varepsilon$ and $\Omega \cap (y_0 + t_1 U_1 + t_2 U_2) \neq \emptyset$.

Since $\max\{\rho_{U_1}(-\zeta), \rho_{U_2}(-\zeta)\} = 1$, there is some $u \in \mathbb{U}$ such that

$$\langle \zeta, u \rangle < -1 + \varepsilon. \tag{4.37}$$

Without loss of generality, we may assume that $u \in U_1$. Set $z := y_0 - (r - t_1 - t_2)u$. By the convexity of U_1, U_2, we have

$$y_0 + t_2 U = z + (r - t_1 - t_2)u + t_1 U_1 + t_2 U_2 \subset z + (r - t_2)U_1 + t_2 U_2.$$

Since $\Omega \cap (y_0 + t_1 U_1 + t_2 U_2) \neq \emptyset$, one has $\Omega \cap (z + z + (r - t_2)U_1 + t_2 U_2) \neq \emptyset$. Thus $T_{\mathcal{U},\Omega}(z) \leq r$, *i.e.*, $z \in \mathcal{R}(r)$.

Set $t_3 := r - t_1 - t_2$. Using (4.37), we arrive at

$$t_3 \leq -\langle \zeta, u \rangle t_3 + t_3 \varepsilon = \langle \zeta, z - y_0 \rangle + t_3 \varepsilon. \tag{4.38}$$

Furthermore,

$$\begin{aligned} ||z - x_0|| &= ||y_0 - (r - t_1 - t_2)u - x_0|| \leq ||y_0 - x_0|| + M t_3 \\ &\leq ||y_0 - x_0|| + (r - t)M = ||y_0 - x_0|| + ((T_{\mathcal{U},\Omega}(x_0) - T_{\mathcal{U},\Omega}(y_0))M \\ &\leq ||y_0 - x_0|| + M\sigma_2 \qquad \text{(since (4.32))} \\ &< \eta + M\sigma_2 < \eta_0. \end{aligned} \tag{4.39}$$

Hence $z \in \mathcal{R}(r) \cap B(x_0, \eta_0)$. By (4.31), one has

$$\langle \zeta, z - x_0 \rangle \leq \sigma_0 ||z - x_0||.$$

Then we have the estimate

$$\begin{aligned} \langle \zeta, z - y_0 \rangle &= \langle \zeta, z - x_0 \rangle + \langle \zeta, x_0 - y_0 \rangle \leq \sigma_0 ||z - x_0|| + ||\zeta|| ||y_0 - x_0|| \\ &\leq \sigma_0 (||y_0 - x_0|| + M t_3) + ||\zeta| \; ||y_0 - x_0|| \qquad \text{(since 4.39))} \\ &= (\sigma_0 + ||\zeta||)||y_0 - x_0|| + M\sigma_0 t_3. \end{aligned} \tag{4.40}$$

Plugging (4.40) into (4.38) and using the assumptions on ε, we get

$$t_3 \leq (\sigma_0 + ||\zeta||)||y_0 - x_0|| + \varepsilon t_3 + M\sigma_0 t_3 < (\sigma_0 + ||\zeta||)||y_0 - x_0|| + t_3/2.$$

This implies

$$t_3 \leq 2(\sigma_0 + ||\zeta||)||y_0 - x_0|| < 2(1 + ||\zeta||)||y_0 - x_0||.$$

Finally,

$$\begin{aligned} T_{\mathcal{U},\Omega}(y_0) - T_{\mathcal{U},\Omega}(x_0) - \langle \zeta, y_0 - x_0\rangle &= t - r + \langle \zeta, z - y_0\rangle - \langle \zeta, x - x_0\rangle \\ &\geq t - r + (t_1 + t_2 - r)\langle \zeta, u\rangle - \sigma_0||z - x_0|| \\ &\geq t - r + (r - t_1 - t_2)(1 - \varepsilon) - \sigma_0(||y_0 - x_0|| + Mt_3) \\ &\geq -(1 + r)\varepsilon - \sigma_0||y_0 - x_0|| - M\sigma_0 t_3 \\ &\geq -(1 + r)\varepsilon - \sigma_0||y_0 - x_0|| - 2M\sigma_0(1 + ||\zeta||)||y_0 - x_0|| \\ &= -(1 + r)\varepsilon - \sigma_0(1 + 2M + 2M||\zeta||)||y_0 - x_0|| \\ &\geq -(1 + r)\varepsilon - \sigma||y_0 - x_0||. \end{aligned}$$

Letting $\varepsilon \to 0+$, we obtain

$$T_{\mathcal{U},\Omega}(y_0) - T_{\mathcal{U},\Omega}(x_0) - \langle \zeta, y_0 - x_0\rangle \geq -\sigma||y_0 - x_0||.$$

This contradicts (4.33). The proof is complete. □

We next present some results involving optimal paths.

Proposition 4.7. *Assume that U_1 and U_2 are convex. Let $x \in \mathcal{R}$ be such that $r := T_{\mathcal{U},\Omega}(x) > 0$ and $\Pi(x) \neq \emptyset$. Assume that $y_x(\cdot)$ is an optimal path of x. Then*

$$\widehat{N}_{\mathcal{R}(r)}(x) \subset \widehat{N}_{\mathcal{R}(r-s)}(y_x(s)), \qquad \text{for all } s \in [0, r]. \tag{4.41}$$

Proof. Since $y_x(\cdot)$ is an optimal path of x, $y_x(\cdot)$ has the form

$$y_x(s) = \begin{cases} x + su_1, & \text{if } 0 \leq s \leq t_1, \\ x + t_1u_1 + (s - t_1)u_2, & \text{if } t_1 \leq s \leq T_{\mathcal{U},\Omega}(x), \end{cases}$$

where $t_i \geq 0$, $u_i \in U_i$, $i = 1, 2$ with $t_1 + t_2 = T_{\mathcal{U},\Omega}(x)$ and $x + t_1u_1 + t_2u_2 \in \Omega$. By Proposition 3.13, $T_{\mathcal{U},\Omega}(y_x(s)) = r - s$ for all $s \in [0, r]$. We prove (4.41) for the case that $t_1 \leq s \leq r$. The case that $0 \leq s \leq t_1$ can be proved similarly. Let $\zeta \in \widehat{N}_{\mathcal{R}(r)}(x)$. Then, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\langle \zeta, y - x\rangle \leq \varepsilon||y - x||, \qquad \forall y \in B(x, \delta). \tag{4.42}$$

Let $z \in \mathcal{R}(r - s) \cap B(y_x(s), \delta)$ and set $y := z - t_1u_1 - (s - t_1)u_2$. Then

$$T_{\mathcal{U},\Omega}(y) \leq T_{\mathcal{U},\Omega}(z) + t_1 + (s - t_1) \leq r - s + s = r.$$

Moreover,

$$||y - x|| = ||(z - t_1u_1 - (s - t_1)u_2) - (y_x(s) - t_1u_1 - (s - t_1)u_2)|| = ||z - y_x(s)|| \leq \delta.$$

Thus, $y \in \mathcal{R}(r) \cap B(x,\delta)$. It follows from (4.42) that

$$\langle \zeta, z - y_x(s)\rangle = \langle \zeta, y - x\rangle \le \varepsilon||y-x|| = \varepsilon||z - y_x(s)||.$$

Hence, $\zeta \in \widehat{N}_{\mathcal{R}(r-s)}(y_x(s))$. This ends the proof. $\square$

Proposition 4.8. *Assume that U_1 and U_2 are convex. Let $x \in X$ be such that $0 < r := T_{\mathcal{U},\Omega}(x) < \infty$ and $\Pi(x) \neq \emptyset$. Then, for any optimal-direction pair (u_1,u_2) of x and its corresponding optimal path $y_x(\cdot)$, we have*

$$\widehat{\partial} T_{\mathcal{U},\Omega}(x) \subset \{\zeta \in X^* : \min\{\langle \zeta, u_1\rangle, \langle \zeta, u_2\rangle\} = -1\} \cap \widehat{N}_{\mathcal{R}(r-s)}(y_x(s)), \qquad \forall s \in [0,r]. \tag{4.43}$$

Proof. Let $(u_1,u_2) \in \mathcal{U}_o(x)$ and let $y_x(\cdot)$ be its corresponding optimal path. Let $\zeta \in \widehat{\partial} T_{\mathcal{U},\Omega}(x)$. By Theorem 4.6, $\zeta \in \widehat{N}_{\mathcal{R}(r)}(x)$. Using Proposition 4.7, $\zeta \in \widehat{N}_{\mathcal{R}(r-s)}(y_x(s))$ for all $s \in [0,r]$. Moreover, as in the first part of the proof of Theorem 4.6, we have $\langle \zeta, u_i\rangle \ge -1$, $i = 1,2$. Since $\zeta \in \widehat{\partial} T_{\mathcal{U},\Omega}(x)$, for any $\sigma > 0$, there exists $\delta > 0$ such that

$$T_{\mathcal{U},\Omega}(y) - T_{\mathcal{U},\Omega}(x) + \sigma||y-x|| \ge \langle \zeta, y - x\rangle \qquad \text{for all } y \in B(x,\delta). \tag{4.44}$$

Let $t_1, t_2 \ge 0$ and $w \in \Omega$ be such that $t_1 + t_2 = T_{\mathcal{U},\Omega}(x)$ and $w = x + t_1u_1 + t_2u_2$. Since $T_{\mathcal{U},\Omega}(x) > 0$. It happens that $t_1 > 0$ or $t_2 > 0$. Assume that $t_1 > 0$. Let $\varepsilon \in (0,t_1)$ be such that $x + \varepsilon u_1 \in B(x,\delta)$. In (4.44), taking $y := x + \varepsilon u_1$, we have

$$T_{\mathcal{U},\Omega}(x+\varepsilon u_1) - T_{\mathcal{U},\Omega}(x) - \sigma\varepsilon||u_1|| \ge \varepsilon\langle \zeta, u_1\rangle.$$

Moreover, since $x + \varepsilon u_1 + (t_1 - \varepsilon)u_1 + t_2u_2 = w \in \Omega$, $T_{\mathcal{U},\Omega}(x+\varepsilon u_1) \le t_1 - \varepsilon + t_2 = T_{\mathcal{U},\Omega}(x) - \varepsilon$. Thus,

$$\varepsilon\langle \zeta, u_1\rangle \le -\varepsilon - \varepsilon\sigma||u_1||,$$

which leads to $\langle \zeta, u_1\rangle \le -1$. Similarly, if $t_2 > 0$, then $\langle \zeta, u_2\rangle \le -1$. Therefore,

$$\min\{\langle \zeta, u_1\rangle, \langle \zeta, u_2\rangle\} = -1.$$

This ends the proof. $\square$

A result related to the inclusion in (4.43) with $s = r$ for the classical minimal time function can be found in Proposition 5.2(ii) from [14]. The following example shows that in general the opposite inclusion of (4.43) does not holds.

Example 4.9. Let $X = \mathbb{R}^2$, $U_1 = \{(0,1)\}$, $U_2 = \{(-1,0)\}$ and $\Omega = \{(0,0)\}$. One can see that the domain of the minimal time function $T_{\mathcal{U},\Omega}$ is

$$D := \operatorname{dom}(T_{\mathcal{U},\Omega}) = \{(x_1,x_2) : x_1 \ge 0, x_2 \le 0\}.$$

For $x = (x_1,x_2) \in \mathbb{R}^2$, we have

$$T_{\mathcal{U},\Omega}(x) = \begin{cases} x_1 - x_2 & \text{if } x \in D, \\ +\infty & \text{if } x \notin D. \end{cases}$$

Let $\bar{x} = (1, -1)$. It is easy to see that $\widehat{\partial} T_{\mathcal{U},\Omega}(\bar{x}) = \{(1, -1)\}$. The optimal-direction pair (u_1, u_2) of $\bar{x}$ is $u_1 = (0, 1)$, $u_2 = (-1, 0)$ and the minimal time projection $\Pi(\bar{x}) = \Omega = \{(0, 0)\} =: \{\mathbf{0}\}$. One has $\widehat{N}_\Omega(\mathbf{0}) = \mathbb{R}^2$ and

$$\begin{aligned}
&\{\zeta = (\zeta_1, \zeta_2) : \min\{\langle \zeta, u_1\rangle, \langle \zeta, u_2\rangle = -1\} \\
&= \{(\zeta_1, \zeta_2) : \min\{\langle(\zeta_1, \zeta_2)(0, 1)\rangle, \langle(\zeta_1, \zeta_2), (-1, 0)\rangle = -1\} \\
&= \{(\zeta_1, \zeta_2) : \min\{\zeta_2, -\zeta_1\} = -1\} \\
&= \{(\zeta_1, -1) : \zeta_1 \geq 1\} \cup \{(1, \zeta_2) : \zeta_2 \geq -1\}.
\end{aligned}$$

Thus,

$$\{\zeta = (\zeta_1, \zeta_2) : \min\{\langle \zeta, u_1\rangle, \langle \zeta, u_2\rangle = -1\} \cap \widehat{N}_\Omega(\mathbf{0}) = \{(\zeta_1, -1) : \zeta_1 \geq 1\} \cup \{(1, \zeta_2) : \zeta_2 \geq -1\}.$$

This means that $\widehat{\partial} T_{\mathcal{U},\Omega}(\bar{x})$ is a proper subset of $\{(\zeta_1, \zeta_2) : \min\{\langle \zeta, u_1\rangle, \langle \zeta, u_2\rangle = -1\}$. Therefore, the opposite inclusion of (4.43) does not hold.

It was proved in [30] that the analogous inclusion to (4.43) for the classical minimal time function becomes equality if the control set is a singleton. More precisely, if $v \neq \{\mathbf{0}\}$, Ω is closed, $0 < r := T^v_\Omega(x) < \infty$, then

$$\widehat{\partial} T^v_\Omega(x) = \{\zeta \in X^* : \langle \zeta, v\rangle = -1\} \cap \widehat{N}_{\mathcal{R}(\alpha r)}(\alpha x + (1 - \alpha)w),$$

for any $\alpha \in [0, 1]$ and $w \in \Pi(x)$. From the previous example, one can see that this property cannot extend to the minimal time function associated with a collection of two singleton sets.

Proposition 4.10. *Let X be an Asplund space. Let U_1 and U_2 be compact convex. For any $\bar{x} \in \mathcal{R}$ and $\zeta \in \partial T_{\mathcal{U},\Omega}(\bar{x})$, there exist an optimal-direction pair $(u_1, u_2) \in \mathcal{U}_o(\bar{x})$ and its corresponding optimal path $y_{\bar{x}}(\cdot)$ such that:*

(i) If $\bar{x} \in \Omega$, then $\min\{\langle \zeta, u_1\rangle, \langle \zeta, u_2\rangle\} \geq -1$ and $\zeta \in N_\Omega(\bar{x})$.
(ii) If $\bar{x} \notin \Omega$, then $\min\{\langle \zeta, u_1\rangle, \langle \zeta, u_2\rangle\} = -1$ and $\zeta \in N_\Omega(y_{\bar{x}}(T_{\mathcal{U},\Omega}(\bar{x})))$.

Proof. (i) If $\zeta \in \partial T_{\mathcal{U},\Omega}(\bar{x})$, then there exist $\{x_n\} \subset X$ such that $x_n \to \bar{x}$, $T_{\mathcal{U},\Omega}(x_n) \to T_{\mathcal{U},\Omega}(\bar{x}) = 0$ and $\{\zeta_n\} \subset X^*$ such that $\zeta_n \in \widehat{\partial} T_{\mathcal{U},\Omega}(x_n)$ and $\zeta_n \xrightarrow{w^*} \zeta$. If $\{x_n\}$ has a subsequence, which is still denoted by $\{x_n\}$, of elements in Ω, then $x_n \xrightarrow{\Omega} \bar{x}$. By Theorem 4.4, $\zeta_n \in \widehat{N}_\Omega(x_n)$ and $\langle \zeta_n, u\rangle \geq -1$ for all $u \in \mathbb{U}$. Thus, $\zeta \in N_\Omega(\bar{x})$ and $\langle \zeta, u\rangle \geq -1$ for all $u \in \mathbb{U}$. Assume, without loss of generality, that $x_n \notin \Omega$ for all n. There exist $t^n_i \geq 0$, $u^n_i \in U_i$, $i = 1, 2$ and $w_n \in \Omega$ such that $t^n_1 + t^n_2 = T_{\mathcal{U},\Omega}(x_n)$ and

$$x_n + t^n_1 u^n_1 + t^n_2 u^n_2 = w_n$$

for all n. Thus, by Proposition 4.8, $\zeta_n \in \widehat{N}_\Omega(w_n)$ and $\min\{\langle \zeta_n, u^n_i\rangle : i = 1, 2\} = -1$ for all n. Since $T_{\mathcal{U},\Omega}(x_n) \to 0$, we have $t^n_i \to 0$, $i = 1, 2$. By the boundedness of U_1, U_2, we have that $w_n \to \bar{x}$. Thus $\zeta \in N_\Omega(\bar{x})$. Since U_1, U_2 are compact, we may assume, after taking subsequences, that $\{u^n_i\}$ converges to some $u_i \in U_i$, $i = 1, 2$. Therefore, $\min\{\langle \zeta, u_1\rangle, \langle \zeta, u_2\rangle\rangle\} = -1$. This ends the proof for (i).

(ii) If $\zeta \in \partial T_{\mathcal{U},\Omega}(\bar{x})$, then there exist $\{x_n\} \subset X$ such that $x_n \to \bar{x}$, $T_{\mathcal{U},\Omega}(x_n) \to T_{\mathcal{U},\Omega}(\bar{x})$ and $\{\zeta_n\} \subset X^*$ such that $\zeta_n \in \widehat{\partial} T_{\mathcal{U},\Omega}(x_n)$ and $\zeta_n \xrightarrow{w^*} \zeta$. Since $\bar{x} \notin \Omega$, we may assume that $x_n \notin \Omega$ for all n. So there exist $t^n_i \geq 0$, $u^n_i \in U_i$, $i = 1, 2$ and $w_n \in \Omega$ such that $t^n_1 + t^n_2 = T_{\mathcal{U},\Omega}(x_n)$ and

$$x_n + t^n_1 u_1 + t^n_2 u_2 = w_n.$$

By Proposition 4.8, we have $\zeta_n \in \widehat{N}_\Omega(w_n)$ and $\min\{\langle \zeta_n, u_i^n\rangle : i=1,2\} = -1$ for all n. Since U_1, U_2 are compact, we may assume that $\{u_i^n\}$ converges to some $u_i \in U_i$, $i=1,2$. Since $T_{\mathcal{U},\Omega}(x_n) \to T_{\mathcal{U},\Omega}(\bar{x})$, we may also assume that $t_i^n \to t_i \geq 0$, $i=1,2$ and $t_1+t_2 = T_{\mathcal{U},\Omega}(\bar{x})$. Thus, by the closedness of Ω, $w_n \xrightarrow{\Omega} w = x + t_1u_1 + t_2u_2$, which implies that $\zeta \in N_\Omega(x+t_1u_1+t_2u_2)$. Moreover, since $\zeta_n \xrightarrow{w^*} \zeta$, we have $\min\{\langle\zeta, u_i\rangle : i=1,2\} = -1$. This ends the proof for (ii). □

To conclude this paper, we give an application of previous results to study a location problem. Given a finite number of nonempty closed targets Ω_i, $i=1,\cdots,m$ and two nonempty sets U_1, U_2 with $U_1 \cap U_2 \subset \{0\}$, $U_1 \cup U_2 \neq \{0\}$ and given a nonempty closed constraint set Ω, we consider the problem of finding a point $x \in \Omega$ such that the sum of times to reach the targets is minimal:

$$\text{minimize } f(x) := \sum_{i=1}^m T_{\mathcal{U},\Omega_i}(x) \text{ subject to } x \in \Omega. \tag{4.45}$$

Proposition 4.11. *Assume that* $\operatorname{dom}(f)\cap\Omega \neq \emptyset$, Ω_i, $i=1,\cdots,m$, Ω *are closed and* U_1, U_2 *are bounded, closed. Then the optimization problem (4.45) has an optimal solution if, in addition, one of the following conditions holds:*

(i) U_1, U_2 *and* Ω *are compact.*
(ii) U_1, U_2 *are compact and one of the sets* Ω_i, $i=1,\cdots,m$ *is compact.*
(iii) X *is a reflexive Banach space,* Ω_i, $i=1,\cdots,m$, U_1 *and* U_2 *are convex,* Ω *is compact.*
(iv) X *is a reflexive Banach space,* Ω_i, $i=1,\cdots,m$, Ω, U_1 *and* U_2 *are convex and one of the sets* Ω, Ω_i, $i=1,\cdots,m$ *is bounded.*

Proof. If one of the conditions (i) - (iv) holds, then, by Proposition 3.8, $T_{\mathcal{U},\Omega_i}$, $i=1,\cdots,m$ is lower semicontinuous on its domain. Thus, f is lower semicontinuous on its domain. (i), (iii). Since Ω is compact, by the classical Weierstrass theorem, an optimal solution exists.

(ii) We consider the infimum valued

$$\alpha := \inf_{x\in\Omega} f(x) < \infty.$$

Assume, without loss of generality, that Ω_1 is compact. Let $\{x_k\} \subset \Omega$ be a minimizing sequence. That is, $f(x_k) \to \alpha$ as $k \to \infty$. So $\{f(x_k)\}$ is a bounded sequence. Hence $\{T_{\mathcal{U},\Omega_i}(x_k)\}$, $i=1,\cdots,m$, is bounded. For eack k, set $t_k := T_{\mathcal{U},\Omega}(x_k)$. Then there exist $t_i^k \geq 0$, $u_i^k \in U_i$, $i=1,2$ and $w_k \in \Omega_1$ such that $t_k \leq t_1^k + t_2^k < t_k + 1$ and

$$w_k = x_k + t_1^k u_1^k + t_2^k u_2^k.$$

Since $\{t_1^k\}, \{t_2^k\}$ are bounded, they have convergent subsequences. Since U_1, U_2 and Ω are compact, $\{u_1^k\}$, $\{u_2^k\}$ and $\{w_k\}$ also have convergent subsequences. Thus, $\{x_k\}$ has a convergent subsequence (without relabeling) to some $\bar{x} \in \Omega$. Since f is lower semicontinuous, one has

$$f(\bar{x}) \leq \liminf_{k\to\infty} f(x_k) = \alpha.$$

Hence, $\bar{x}$ is an optimal solution of problem (4.45).

(iv) We first observe that f is convex as $T_{\mathcal{U},\Omega_i}$ is convex for all $i=1,\cdots,m$. This, together with the lower semicontinuity, implies that f is weakly lower semicontinuous. We then argue like above cases with having in mind that every closed bounded convex set in a reflexive Banach space is weakly sequentially compact. □

Proposition 4.12. *Assume, for all $i = 1, \cdots, m$, that Ω_i is closed and that either* $(\text{cone}U_1 + \text{cone}U_2) \cap \text{int}(\Omega_i)_\infty \neq \emptyset$ *or* $\text{int}(\text{cone}U_1 + \text{cone}U_2) \cap (\Omega_i)_\infty \neq \emptyset$. *Then the optimization problem (4.45) has an optimal solution, if, in addition, one of the following conditions holds:*

(i) One of the sets $\Omega_i, i = 1, \cdots, m$ and Ω is compact.
(ii) X is reflexive, $\Omega_i, i = 1, \cdots, m$, Ω are convex and one of them is bounded.

Proof. Note that the lower semicontinuity is replaced by the Lipschitz continuity. Similar to the proof of Proposition 4.11, we find the desired conclusion immediately. □

The following theorem presents a necessary optimality condition for problem (4.45). For any $x \in X$, define

$$I(x) := \{i \in \{1, \cdots, m\} : x \in \Omega_i\},$$

and

$$J(x) := \{i \in \{1, \cdots, m\} : x \notin \Omega_i\}.$$

Theorem 4.13. *Assume that X is an Asplund space, U_1, U_2 are convex, compact, Ω_i, $i = 1, \cdots, m$ and Ω are closed. Suppose that $\bar{x}$ is an optimal solution of problem (4.45) and $T_{\mathcal{U},\Omega_i}$ is locally Lipschitz at $\bar{x}$ for all $i = 1, \cdots, m$. Then there exist $\zeta_i \in X^*$ and optimal-direction pairs (u_1^i, u_2^i) with its corresponding optimal path $y_{\bar{x}}^i(\cdot)$ of $\bar{x}$ for all $i = 1, \cdots, m$ such that*

(i) $\zeta \in N_{\Omega_i}(y_{\bar{x}}^i(T_{\mathcal{U},\Omega_i}(\bar{x}))$ for $i = 1, \cdots, m$.
(ii) $\min\{\langle \zeta_i, u_1^i\rangle, \langle \zeta_i, u_2^i\rangle\} \geq -1$ for all $i \in I(\bar{x})$ and $\min\{\langle \zeta_i, u_1^i\rangle, \langle \zeta_i, u_2^i\rangle\} = -1$ for all $i \in J(\bar{x})$.
(iii) $-\sum_{i=1}^m \zeta_i \in N_\Omega(\bar{x})$.

Proof. The optimization problem (4.45) is equivalent to the following unscontrained optimization problem:

$$\text{minimize } f(x) + \delta_\Omega(x), \quad x \in X,$$

where δ_Ω is the indicator function associated with the set Ω defined by $\delta_\Omega(x) = 0$ if $x \in \Omega$ and $\delta_\Omega(x) = \infty$ if $x \notin \Omega$. Note that $T_{\mathcal{U},\Omega_i}$ is locally Lipschitz at $\bar{x}$ for all $i = 1, \cdots, m$. Applying the limiting subdifferential sum rule (see, Thm. 3.36 in [19]), we have

$$0 \in \partial(f + \delta_\Omega)(\bar{x}) \subset \partial f(\bar{x}) + N_\Omega(\bar{x}) \subset \sum_{i=1}^m \partial T_{\mathcal{U},\Omega_i}(\bar{x}) + N_\Omega(\bar{x}).$$

Thus, there exist $\zeta_i \in \partial T_{\mathcal{U},\Omega_i}(\bar{x})$, $i = 1, \cdots, m$ such that

$$-\sum_{i=1}^m \zeta_i \in N_\Omega(\bar{x}).$$

By Proposition 4.10, we obtain the desired conclusions. □

5. Conclusion

Motivated by an equivalent formula of the minimum time function for a nonconvex constant dynamics that is a finite union of convex sets and some examples, we introduced and studied some properties of the minimal time function associated with a collection of sets. In fact, various properties of this new function including, among others, lower semicontinuity, principle of optimality, convexity, Lipschitz continuity and subdifferential calculus were investigated. These properties were carefully adapted from those of the usual minimal time function.

Examples show that there are properties for the classical minimal function cannot extend to the new function and that there are situations in which the new minimal time function can be used when the classical one cannot be used. An application to location problems was also given. This is the first paper that deals with the minimal time function associated with collection of sets. For the future work, we will continue studying deeper the minimal time function associated with collection of sets. It is known that the usual minimal time function has many applications in optimization (see, *e.g.*, [13, 15, 22, 24] and references therein), we will also try to give some more applications of the new function. It is interesting to devise an algorithm to compute the minimal time function associated with a collection of sets. An other interesting perspective is the investigation of properties of the minimum time function for linear control systems with nonconvex control sets.

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