ESAIM: COCV 26 (2020) 94
https://doi.org/10.1051/cocv/2020016

**ESAIM: Control, Optimisation and Calculus of Variations**
www.esaim-cocv.org

# NEAR-OPTIMAL CONTROL PROBLEMS FOR FORWARD-BACKWARD REGIME-SWITCHING SYSTEMS*

Min Li and Zhen Wu**

**Abstract.** This paper investigates the near-optimality for a class of forward-backward stochastic differential equations (FBSDEs) with continuous-time finite state Markov chains. The control domains are not necessarily convex and the control variables do not enter forward diffusion term. Some new estimates for state and adjoint processes arise naturally when we consider the near-optimal control problem in the framework of regime-switching. Inspired by Ekeland's variational principle and a spike variational technique, the necessary conditions are derived, which imply the near-minimum condition of the Hamiltonian function in an integral sense. Meanwhile, some certain convexity conditions and the near-minimum condition are sufficient for the near-optimal controls with order $\varepsilon^{\frac{1}{2}}$. A recursive utility investment consumption problem is discussed to illustrate the effectiveness of our theoretical results.

**Mathematics Subject Classification.** 93E20, 60H10, 60J27.

Received September 11, 2018. Accepted April 4, 2020.

## 1. Introduction

Recently, regime-switching model has attracted consistent and intensive research attentions from both the academic community and the market practitioner. In the real world, many systems are extremely complicated in which continuous dynamics and discrete events exist at the same time. The regime-switching model originates from the need of more realistic models, which can better reflect the uncertainty in random market environment. For example, some models in financial market whose key parameters, like bank interest rate, stock appreciation rate, and volatility rate, are modulated by continuous-time finite state Markov chains. The most salient feature of regime-switching model is that it switches among the finite number states. As a consequence, it can describe complex systems and capture the inherent uncertainty in random environment. The pioneering consideration for regime-switching can be traced to [8], where the Markov-switching autoregressive time series were studied. In problem of option pricing, [15] discussed the hedging strategy for regime-switching European option. Using Latice method and risk-neutral theory, [3, 4] explored the valuing problem of American option. Beyond that, regime-switching model has also been widely applied to other fields. [25] used this model to forecast optimal stock selling rule in financial market, [26] applied it to portfolio management problem, and [30] investigated a continuous Markowitz's mean-variance portfolio selection problem with regime-switching.

*This work was supported by the National Natural Science Foundation of China (11831010,61961160732), and Shandong Provincial Natural Science Foundation (ZR2019ZD42).

*Keywords and phrases:* Forward-backward stochastic differential equation, regime-switching, near-optimality, Ekeland's variational principle, necessary conditions, sufficient conditions.

School of Mathematics, Shandong University, Jinan 250100, P.R. China.

** Corresponding author: wuzhen@sdu.edu.cn


© EDP Sciences, SMAI 2020

As a powerful tool, there exists rich literature on backward stochastic differential equation (BSDE). Linear case was first introduced by [2] in the study of stochastic optimal control problems. General nonlinear BSDE and corresponding well-posedness of solution were formulated in [16]. Independently, [6] introduced BSDEs in the context of economy, and presented a stochastic differential utility, which was the extension of standard additive utility. Some important properties of BSDE and applications in finance were given in [12]. As its name suggests, a BSDE coupled with a forward stochastic differential equation (SDE) constitutes a forward-backward stochastic differential equation (FBSDE). The celebrated Hamiltonian system arising from maximum principle for stochastic optimal control problem is a kind of FBSDEs. Forward-backward control systems have wide applications in various fields, such as finance, mathematical economic and stochastic control, *etc.* When it comes to the solvability of FBSDEs, there exist three fundamental methods: *contraction mapping method* [17], *four-step scheme* [14], and *continuation method* [9, 19]. Along the line of maximum principle, [18] first established the results for a class of FBSDEs when the control domains were convex. The general maximum principle for forward-backward system was an open problem until [22] successfully solved it. Some recent literature studies may include [10, 24], where time delay and Poisson jump were considered in the framework of FBSDE. The interested readers can refer [13, 23] for a systematic introduction of FBSDEs.

The objective of this paper is to study near-optimal controls for systems driven by FBSDEs with regime-switching. To authors' knowledge, it is an unexplored field. Due to their nice structure and properties, near-optimal controls, as the alternative to exact optimal ones, have received considerable research interests in a variety of fields. In fact, near-optimal controls always exist while exact optimal ones not. Moreover, there are many candidates for near-optimal controls, so it is advisable to choose a proper one to reduce the difficulties and costs of implementation. Compared to exact optimal controls, near-optimal controls are more available and flexible in practice. In the past time, a lot of literatures were acquired on this topic. As for the deterministic dynamic system governed by an ordinary differential equation (ODE), [28] established both necessary and sufficient conditions for near-optimal controls. The first attempt for stochastic near-optimal controls was in [29], where the state was controlled by a SDE. Near-optimal control problems for forward-backward stochastic systems have been treated in the literature, see [1, 11, 27] and the relevant research has been made continuously in progress. However, these listed results were only focused on the systems without regime-switching.

In this paper, near-optimal control problems for forward-backward stochastic regime-switching system are considered. The control domains are not necessarily convex and the control variables do not enter forward diffusion term. It should be noted that the *difficulties* we are facing. On one hand, the classical methods dealing with exact optimal controls cannot be used to treat near-optimal ones on the fact that a maximum (minimum) point of a function implies zero derivative while this is no longer the case for near-optimality. On the other hand, the conventional estimates results for near-optimal control problems cannot be directly used since the additional martingale terms deduced from the Markov chain, which bring more technical difficulties to obtain the desired conclusions. Inspired by Ekeland's variational principle and a spike variational technique, we derive both necessary and sufficient conditions via some new estimates for the state and adjoint processes. More specifically, we will show that any $\varepsilon$-optimal controls near-minimize the Hamiltonian function in an integral sense. Meanwhile, some certain convexity conditions and near-minimum condition are sufficient to gain the near-optimal controls with order $\varepsilon^{\frac{1}{2}}$. The main *novelties* of this paper are characterized by the following features. Firstly, compared with works in [1, 11, 27], the system under consideration is assumed to be modulated by a continuous-time finite state Markov chain. Due to the presence of both continuous dynamics and discrete events, the results in this paper are more general in uncertainty environment. Secondly, compared with [20], near-optimal control problems are first concerned within this background and the control domains are not necessarily convex. Last but not least, the appearance of Markov chain brings about some additional martingale terms, which should be the part of solution quadruple. Based on the fact, the estimates for the state and adjoint processes are new.

The rest of this paper is organized as follows. In Section 2, some basic notations and preliminaries are reviewed. Near-optimal control problems are also formulated. The main results of this paper are given in Sections 3 and 4, including both necessary and sufficient conditions of near-optimality. An example in finance and numerical simulation results are presented in Section 5. Section 6 gives a conclusion for this paper.

## 2. Preliminaries and problem formulation

For a fixed time $T$, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, \mathbb{P})$ be a completed filtered probability space. The natural filtration $\mathcal{F}_t$ is generated by $\{B(s), \alpha(s); 0 \le s \le t\}$, where $\{B(t), 0 \le t \le T\}$ is a standard Brownian motion defined on this space, $\{\alpha(t), 0 \le t \le T\}$ is a continuous-time finite state Markov chain with state space $I = \{1, 2, \dots k\}$. The transition intensities are $\lambda(i,j)$ for $i \ne j$ with $\lambda(i,j)$ nonnegative and bounded. $\lambda(i,i) = -\sum_{j\in I\setminus\{i\}} \lambda(i,j)$.

Define $V$ as the integer-valued random measure on $([0,T] \times I, \mathcal{B}[0,T] \otimes \mathcal{B}_I)$. $V_t(j)$ counts the jumps from $\alpha$ to state $j$ between the time interval 0 and $t$. The compensated measure of $V_t(j)$ is defined as $\mathrm{d}\tilde{V}_t(j) := \mathrm{d}V_t(j) - \mathbf{1}_{\{\alpha(t)\ne j\}}\lambda(\alpha(t), j)\,\mathrm{d}t$, where $\mathbf{1}_{\{\alpha(t)\ne j\}}\lambda(\alpha(t), j)\ \mathrm{d}t$ is the compensator. The canonical special semimartingale representation for $\alpha(t)$ is given by

$$\mathrm{d}\alpha(t) = \sum_{j\in I}\lambda(\alpha(t), j)(j - \alpha(t))\,\mathrm{d}t + \sum_{j\in I}(j - \alpha(t-))\,\mathrm{d}\tilde{V}_t(j).$$

Throughout this paper, let $\mathbb{R}$ be 1-dimensional Euclid space, $|\cdot|$ be its usual normal. We denote $n_t(j) = \mathbf{1}_{\{\alpha(t)\ne j\}}\lambda(\alpha(t), j)$. For simplicity of representation, we introduce the following basic notation.

1) $\mathcal{M}_\rho = \{v(\cdot) : (I, \mathcal{B}_I, \rho) \to \mathbb{R} \mid v(\cdot)$ is a measurable function such that $|v|_t = \sum_{j\in I}[v(j)^2 n_t(j)]^{\frac{1}{2}} \le \infty\}$.

2) $\mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}) = \{f(\cdot) : [0,T] \times \Omega \to \mathbb{R} \mid f(\cdot)$ is an $\mathcal{F}_t$-adapted process such that $\mathbb{E}\big[\sup_{0\le t\le T}|f(t)|^2\big] < \infty\}$.

3) $L^2_{\mathcal{F}}(0,T;\mathbb{R}) = \{f(\cdot) : [0,T] \times \Omega \to \mathbb{R} \mid f(\cdot)$ is an $\mathcal{F}_t$-adapted process such that $\mathbb{E}\big[\int_0^T |f(t)|^2\mathrm{d}t\big] < \infty\}$.

4) $H^2_{\mathcal{V}}(0,T;\mathbb{R}) = \{V(\cdot) : [0,T] \times \Omega \to \mathbb{R} \mid V(\cdot)$ is a $\tilde{\mathbb{P}}$ measurable function such that $\mathbb{E}[\int_0^T \sum_{j\in I} V_t(j)^2\ n_t(j)\mathrm{d}t] < \infty\}$, where $\tilde{\mathbb{P}} = \mathcal{P} \bigotimes \mathcal{B}_I$ and $\mathcal{P}$ is the predictable sigma algebra on $\Omega \times$ T.

Consider the following system, which is governed by a FBSDE with a continuous-time finite state Markov chain of the type

$$\begin{cases} \mathrm{d}x(t) = b(t, \alpha(t), x(t), u(t))\,\mathrm{d}t + \sigma(t, \alpha(t), x(t))\,dB(t), \\ \mathrm{d}y(t) = -f(t, \alpha(t), x(t), y(t), z(t), W_t n_t, u(t))\,\mathrm{d}t \\ \qquad\qquad + z(t)\,dB(t) + \sum_{j\in I} W_t(j)\,\mathrm{d}\tilde{V}_t(j), \\ x(0) = x_0,\ y(T) = g(x(T)), \end{cases} \tag{2.1}$$

where $b : [0,T] \times I \times \mathbb{R} \times U \to \mathbb{R}$, $\sigma : [0,T] \times I \times \mathbb{R} \to \mathbb{R}$ and $f : [0,T] \times I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^k \times U \to \mathbb{R}$, and $g : \mathbb{R} \to \mathbb{R}$ are deterministic bounded measurable functions. For convenience, $W_t n_t$ denotes $(W_t(1)n_t(1), W_t(2)n_t(2), \dots, W_t(k)n_t(k))$ here.

Let $U \subseteq \mathbb{R}$ be a nonempty closed set. The admissible control set is defined as $\mathcal{U}_{ad}[0,T] = \{u(\cdot) : [0,T] \times \Omega \to \mathbb{R} \mid u(\cdot)$ is an $\mathcal{F}_t$-adapted process such that $\mathbb{E}[\int_0^T |u(t)|^2\mathrm{d}t] < \infty\}$.

Then the cost functional is assumed to be

$$J(0, x_0; u(\cdot)) = \mathbb{E}\Big\{\int_0^T l(t, \alpha(t), x(t), y(t), z(t), W_t n_t, u(t))\,\mathrm{d}t + h(x(T)) + \gamma(y(0))\Big\}, \tag{2.2}$$

where $l : [0,T] \times I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^k \times U \to \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}$ and $\gamma : \mathbb{R} \to \mathbb{R}$ are deterministic bounded measurable functions.

Now, we formulate the optimal control problem of FBSDEs with regime-switching as follows.

**Problem (OC)** Find $\bar{u}(\cdot) \in \mathcal{U}_{ad}[0,T]$ such that

$$J(0, x_0; \bar{u}(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}_{ad}[0,T]} J(0, x_0; u(\cdot)).$$

For further study, we impose the following assumptions.
**Assumption 1:** $b, \sigma, f, g, l, h, \gamma$ are continuously differentiable with respect to $(x, y, z, m)$,
i) when $\psi = b, \sigma, b_x, \sigma_x$,

$$|\psi(t, i, x, u) - \psi(t, i, x', u)| \leq C|x - x'|,$$

ii) when $\phi = f, l, f_x, f_y, f_z, f_w, l_x, l_y, l_z, l_w$,

$$|\phi(t, i, x, y, z, m, u) - \phi(t, i, x', y', z', m', u)| \leq C(|x - x'| + |y - y'| + |z - z'| + |m - m'|),$$

iii) when $\varphi = h, g, h_x, g_x$,

$$|\varphi(x) - \varphi(x')| \leq C|x - x'|,$$

iv) when $\lambda = \gamma, \gamma_y$,

$$|\lambda(y) - \lambda(y')| \leq C|y - y'|,$$

hold uniformly for any $(t, i, x, y, z, m, u)$, $(t, i, x', y', z', m', u) \in [0, T] \times I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^k \times U$, where $C$ is a positive constant.
**Assumption 2:**
i) For all $(t, i) \in [0, T] \times I$,

$$\mathbb{E} \int_0^T |f(t, i, 0, 0, 0, 0, 0)|^2 \mathrm{d}t < \infty.$$

ii)

$$|b(t, i, x, u)| + |\sigma(t, i, x)| \leq C(1 + |x|),\ |h(x)| \leq C(1 + |x|^2),\ |\gamma(y)| \leq C(1 + |y|^2),$$

$$|l(t, i, x, y, z, m, u)| \leq C(1 + |x|^2 + |y|^2 + |z|^2 + |m|^2 + |u|^2),$$

uniformly for all $(t, i, x, y, z, m, u) \in [0, T] \times I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^k \times U$, where $C$ is a positive constant.
**Assumption 3**: The derivatives of $b, \sigma, f, g$ are all bounded.

**Remark 2.1.** Under the above Assumptions, for any $u(\cdot) \in \mathcal{U}_{ad}[0, T]$, FBSDEs (2.1) admit a unique $\mathcal{F}_t$-adapted solution denoted by quadruple $(x(\cdot), y(\cdot), z(\cdot), W(\cdot)) \in \mathcal{L}^2_{\mathcal{F}}(0, T; \mathbb{R}) \times \mathcal{L}^2_{\mathcal{F}}(0, T; \mathbb{R}) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}) \times H^2_{\mathcal{V}}(0, T; \mathbb{R})$, which can refer to [5, 21] for more details.

**Remark 2.2.** Noting that the predictable covariation of $\tilde{V}_t(j)$ is

$$\mathrm{d}\langle \tilde{V}_t(j), \tilde{V}_t(j)\rangle_t = n_t(j)\,\mathrm{d}t.$$

For any $u(\cdot) \in \mathcal{U}_{ad}[0, T]$, cost functional (2.2) is well-defined according to the above Assumptions and Remark 2.1. Thus, we can define the following function

$$V(0, x_0) = \inf_{u(\cdot) \in \mathcal{U}_{ad}[0, T]} J(0, x_0; u(\cdot)),$$

which is called the value function of Problem (OC).

In this paper, we are devoted to investigating the near-optimal control problems instead of exact optimal ones for the forward-backward regime-switching system. Then some precise definitions of near-optimality are needed, which can be seen in [29].

**Definition 2.3.** For a given $\varepsilon > 0$, an admissible control $u^\varepsilon(\cdot)$ is called $\varepsilon$-optimal if

$$|J(0, x_0, u^\varepsilon(\cdot)) - V(0, x_0)| \leq \varepsilon.$$

**Definition 2.4.** For a given $\varepsilon > 0$, an admissible control $u^\varepsilon(\cdot)$ is called near-optimal if

$$|J(0, x_0, u^\varepsilon(\cdot)) - V(0, x_0)| \leq r(\varepsilon)$$

holds for sufficiently small $\varepsilon$, where $r$ is a function of $\varepsilon$, satisfying $r(\varepsilon) \to 0$ as $\varepsilon \to 0$. The estimate $r(\varepsilon)$ is called an error bound. If $r(\varepsilon) = c\varepsilon^\delta$ for some $\delta > 0$ independent of the constant $c$, then $u^\varepsilon(\cdot)$ is called near-optimal with order $\varepsilon^\delta$.

Next, let us introduce the associated adjoint process $(p(\cdot), q(\cdot), k(\cdot), M(\cdot))$ satisfying

$$\begin{cases} \mathrm{d}p(t) = \big[f_y(t, \alpha(t), x(t), y(t), z(t), W_t n_t, u(t))\, p(t) \\ \qquad - l_y(t, \alpha(t), x(t), y(t), z(t), W_t n_t, u(t))\big]\mathrm{d}t \\ \qquad + \big[f_z(t, \alpha(t), x(t), y(t), z(t), W_t n_t, u(t))\, p(t) \\ \qquad - l_z(t, \alpha(t), x(t), y(t), z(t), W_t n_t, u(t))\big] dB(t) \\ \qquad + \sum_{j\in I}\big[f_{w(j)}(t, \alpha(t), x(t), y(t), z(t), W_t n_t, u(t))\, p(t) \\ \qquad - l_{w(j)}(t, \alpha(t), x(t), y(t), z(t), W_t n_t, u(t))\big]\mathrm{d}\tilde{V}_t(j), \\ -\mathrm{d}q(t) = \big[b_x(t, \alpha(t), x(t), u(t))\, q(t) + \sigma_x(t, \alpha(t), x(t))\, k(t) \\ \qquad - f_x(t, \alpha(t), x(t), y(t), z(t), W_t n_t, u(t))\, p(t) \\ \qquad + l_x(t, \alpha(t), x(t), y(t), z(t), W_t n_t, u(t))\big]\mathrm{d}t \\ \qquad - k(t)\, dB(t) - \sum_{j\in I} M_t(j)\, \mathrm{d}\tilde{V}_t(j), \\ p(0) = -\gamma_y(y(0)), q(T) = -g_x(x(T))\, p(T) + h_x(x(T)), \end{cases} \tag{2.3}$$

where $u(\cdot) \in \mathcal{U}_{ad}[0, T]$ and $(x(\cdot), y(\cdot), z(\cdot), W(\cdot))$ is the solution of FBSDE (2.1).

**Lemma 2.5.** *Let the Assumptions 1, 2 and 3 hold. The adjoint equation (2.3) admits a unique solution* $(p(\cdot), q(\cdot), k(\cdot), M(\cdot)) \in \mathcal{L}^2_{\mathcal{F}}(0, T; \mathbb{R}) \times \mathcal{L}^2_{\mathcal{F}}(0, T; \mathbb{R}) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}) \times H^2_{\mathcal{V}}(0, T; \mathbb{R})$, *and there exists a positive constant $C$ such that*

$$\mathbb{E}\left\{\sup_{0\leq t\leq T}|p(t)|^2 + \int_0^T |q(t)|^2\mathrm{d}t + \int_0^T |k(t)|^2\mathrm{d}t + \int_0^T \sum_{j\in I}|M_t(j)|^2 n_t\mathrm{d}t\right\} \leq C.$$

*Proof.* Its proof can be found in [5], so we omit the procedure for simplicity. □

## 3. Necessary conditions of near-optimality

In this section, we will show that $\varepsilon$-optimal control near-minimize Hamiltonian function in an integral sense, which are the necessary conditions of near-optimality for FBSDEs with regime-switching. To do this, the estimates for state and adjoint processes are the key ingredients. Compared to the similar problem without regime-switching, it is natural to consider the estimates for the additional martingale terms deduced from

Markov chain. Thereafter, by virtue of Ekeland's variational principle and a spike variational technique, we can obtain the main results in this paper.

For avoid of heavy notation, we denote $\Lambda(t)=(x(t),y(t),z(t)), \Lambda^{\varepsilon}(t)=(x^{\varepsilon}(t),y^{\varepsilon}(t),z^{\varepsilon}(t)), \tilde{\Lambda}^{\varepsilon}(t)=(\tilde{x}^{\varepsilon}(t), \tilde{y}^{\varepsilon}(t),\tilde{z}^{\varepsilon}(t))$, $\Lambda'(t)=(x'(t),y'(t),z'(t))$, $\Theta(t)=(p(t),q(t),k(t)), \Theta^{\varepsilon}(t)=(p^{\varepsilon}(t),q^{\varepsilon}(t),k^{\varepsilon}(t))$. Next, we first state the main result in this section, and its proof will be given later.

**Theorem 3.1.** *For any $\delta\in[0,\frac{1}{3}]$, $\varepsilon>0$, suppose $u^{\varepsilon}(\cdot)$ is $\varepsilon$-optimal control of Problem (OC), then there exists a positive constant $C$ such that*

$$\begin{aligned}\mathbb{E}\int_0^T\Big\{&[b(t,\alpha(t),x^{\varepsilon}(t),u(t))-b(t,\alpha(t),x^{\varepsilon}(t),u^{\varepsilon}(t))]p^{\varepsilon}(t)\\&-[f(t,\alpha(t),\Lambda^{\varepsilon}(t),W_t^{\varepsilon}n_t,u(t))-f(t,\alpha(t),\Lambda^{\varepsilon}(t),W_t^{\varepsilon}n_t,u^{\varepsilon}(t))]q^{\varepsilon}(t)\\&+[l(t,\alpha(t),\Lambda^{\varepsilon}(t),W_t^{\varepsilon}n_t,u(t))-l(t,\alpha(t),\Lambda^{\varepsilon}(t),W_t^{\varepsilon}n_t,u^{\varepsilon}(t))]\Big\}\mathrm{d}t\geq -C\varepsilon^{\delta},\end{aligned}\tag{3.1}$$

*for all $u(\cdot)\in\mathcal{U}_{ad}[0,T]$, where $(\Lambda^{\varepsilon}(\cdot),W^{\varepsilon}(\cdot))$ is the solution of state equation (2.1), and $(\Theta^{\varepsilon}(\cdot),M^{\varepsilon}(\cdot))$ denotes the solution of corresponding adjoint equation with respect to $u^{\varepsilon}(\cdot)$.*

The Hamiltonian function is defined by

$$\begin{aligned}H(t,\alpha(t),\Lambda(t)),W_tn_t,u(t),\Theta(t))=&-f(t,\alpha(t),\Lambda(t),W_tn_t,u(t))\,p(t)+b(t,\alpha(t),x(t),u(t))\,q(t)\\&+\sigma(t,\alpha(t),x(t))\,k(t)+l(t,\alpha(t)),\Lambda(t),W_tn_t,u(t)),\end{aligned}\tag{3.2}$$

where $H:[0,T]\times I\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^k\times U\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$.

Noting that the form of Hamiltonian function and Theorem 3.1, it follows:

**Corollary 3.2.** *Under the Assumptions of Theorem 3.1, for any $u(\cdot)\in\mathcal{U}_{ad}[0,T]$, it holds that*

$$\begin{aligned}&\inf_{u(\cdot)\in\mathcal{U}_{ad}[0,T]}\mathbb{E}\int_0^T H(t,\alpha(t),\Lambda^{\varepsilon}(t),W_t^{\varepsilon}n_t,u(t),\Theta^{\varepsilon}(t))\,\mathrm{d}t\\&\geq\mathbb{E}\int_0^T H(t,\alpha(t),\Lambda^{\varepsilon}(t),W_t^{\varepsilon}n_t,u^{\varepsilon}(t),\Theta^{\varepsilon}(t))\,\mathrm{d}t-C\varepsilon^{\delta}.\end{aligned}\tag{3.3}$$

In order to complete the whole proof of Theorem 3.1, we need several lemmas which give the estimates for state and adjoint processes. To start with, we introduce a metric on $\mathcal{U}_{ad}[0,T]$.

For any $u(\cdot),u'(\cdot)\in\mathcal{U}_{ad}[0,T]$, we define

$$\mathrm{d}(u(\cdot),u'(\cdot))=\hat{\mathbb{P}}\{(t,\omega)\in[0,T]\times\Omega:u(t,\omega)\neq u'(t,\omega)\},$$

where $\hat{\mathbb{P}}$ is the product measures of the lebesgue measure d$t$ and probability measure $\mathbb{P}$. Since $U$ is closed, we can verify that $(\mathcal{U}_{ad}[0,T],d)$ is a complete metric space similar as [29].

**Lemma 3.3.** *For any $\theta>0$ and $\beta>0$ satisfying $\theta\beta<1$, there is a positive constant $C$ such that for any $u(\cdot)$, $u'(\cdot)\in\mathcal{U}_{ad}[0,T]$, along with corresponding solution quadruple $(x(\cdot),y(\cdot),z(\cdot),W(\cdot))$, $(x'(\cdot),y'(\cdot),z'(\cdot),W'(\cdot))$ of FBSDE (2.1), it holds that*

$$
\begin{aligned}
&\mathbb{E}\big[\sup_{0\le t\le T}|x(t)-x'(t)|^{2\beta}\big]\le Cd\left(u(\cdot),u'(\cdot)\right)^{\theta\beta},\\
&\mathbb{E}\big[\sup_{0\le t\le T}|y(t)-y'(t)|^{2\beta}\big]+\mathbb{E}\int_0^T|z(t)-z'(t)|^{2\beta}\mathrm{d}t\\
&\qquad+\mathbb{E}\int_0^T\sum_{j\in I}|W_t(j)-W_t'(j)|^{2\beta}n_t\mathrm{d}t\le Cd\left(u(\cdot),u'(\cdot)\right)^{\theta\beta}.
\end{aligned}
$$

*Proof.* The estimate for $x(\cdot),x'(\cdot)$ without regime-switching is a well-known result in [29]. Since the proof is similar to this, we omit it for simplicity. As for the estimates between $y(\cdot),y'(\cdot)$, $z(\cdot),z'(\cdot)$ and $W(\cdot),W'(\cdot)$, we consider the following BSDE with regime-switching

$$
\begin{cases}
\mathrm{d}\left(y(t)-y'(t)\right)=[f(t,\alpha(t),\Lambda(t),W_tn_t,u(t))-f(t,\alpha(t),\Lambda'(t),W_t'n_t,u'(t))]\mathrm{d}t\\
\qquad\qquad+[z(t)-z'(t)]\,\mathrm{d}B(t)+\sum_{j\in I}[W_t(j)-W_t'(j)]\mathrm{d}\tilde{V}_t(j),\\
y(T)-y'(T)=g(x(T))-g(x'(T)).
\end{cases}
$$

Upon taking integrations on both sides, we obtain

$$
\begin{aligned}
&y(t)-y'(t)+\int_t^T[z(s)-z'(s)]\,\mathrm{d}B(s)+\int_t^T\sum_{j\in I}[W_s(j)-W_s'(j)]\mathrm{d}\tilde{V}_s(j)\\
&=g(x(T))-g(x'(T))-\int_t^T[f(s,\alpha(s),\Lambda(s),W_sn_s,u(s))-f(s,\alpha(s),\Lambda'(s),W_s'n_s,u'(s))]\mathrm{d}s.
\end{aligned}
$$

For any $\beta>0$, the following inequality holds

$$
\begin{aligned}
&\mathbb{E}\big[\sup_{0\le t\le T}|y(t)-y'(t)|^{2\beta}\big]+\mathbb{E}\int_0^T|z(t)-z'(t)|^{2\beta}\mathrm{d}t+\mathbb{E}\int_0^T\sum_{j\in I}|W_t(j)-W_t'(j)|^{2\beta}n_t\mathrm{d}t\\
&\le C\mathbb{E}\int_0^T|f(t,\alpha(t),\Lambda(t),W_tn_t,u(t))-f(t,\alpha(t),\Lambda'(t),W_t'n_t,u'(t))|^{2\beta}\mathrm{d}t\\
&\quad+C\mathbb{E}\big[\left|g(x(T))-g(x'(T))\right|^{2\beta}\big].
\end{aligned}
\tag{3.4}
$$

According to the Assumption 1 and the estimate for $x(\cdot),x'(\cdot)$, we have

$$
\mathbb{E}\big[|g(x(T))-g(x'(T))|^{2\beta}\big]\le C\mathbb{E}\big[|x(T)-x'(T)|^{2\beta}\big]\le Cd\left(u(\cdot),u'(\cdot)\right)^{\theta\beta}.
\tag{3.5}
$$

On the fact that $f$ is Lipschitz continuous with respect to $(x,y,z,m)$, the first term of (3.4) can be estimated in the following way:

$$\begin{aligned}
&\mathbb{E}\int_0^T |f(t,\alpha(t),\Lambda(t),W_t n_t,u(t)) - f(t,\alpha(t),\Lambda'(t),W_t' n_t,u'(t))|^{2\beta}\mathrm{d}t\\
&\le C\mathbb{E}\int_0^T |f(t,\alpha(t),\Lambda(t),W_t n_t,u(t)) - f(t,\alpha(t),\Lambda'(t),W_t' n_t,u(t))|^{2\beta}\mathrm{d}t\\
&\quad + C\mathbb{E}\int_0^T |f(t,\alpha(t),\Lambda'(t),W_t' n_t,u(t)) - f(t,\alpha(t),\Lambda'(t),W_t' n_t,u'(t))|^{2\beta}\chi_{u(t,\omega)\neq u'(t,\omega)}\mathrm{d}t.\\
&\le C\mathbb{E}\int_0^T \left\{|x(t)-x'(t)|^{2\beta} + |y(t)-y'(t)|^{2\beta} + |z(t)-z'(t)|^{2\beta} + |W_t - W_t'|^{2\beta} n_t\right\}\mathrm{d}t\\
&\quad + C\mathbb{E}\int_0^T |f(t,\alpha(t),\Lambda'(t),W_t' n_t,u(t)) - f(t,\alpha(t),\Lambda'(t),W_t' n_t,u'(t))|^{2\beta}\chi_{u(t,\omega)\neq u'(t,\omega)}\mathrm{d}t.
\end{aligned} \tag{3.6}$$

Since Hölder's inequality, we have

$$\begin{aligned}
&\mathbb{E}\int_0^T |f(t,\alpha(t),\Lambda'(t),W_t' n_t,u(t)) - f(t,\alpha(t),\Lambda'(t),W_t' n_t,u'(t))|^{2\beta}\chi_{u(t,\omega)\neq u'(t,\omega)}\mathrm{d}t\\
&\le \left\{\mathbb{E}\int_0^T |f(t,\alpha(t),\Lambda'(t),W_t' n_t,u(t)) - f(t,\alpha(t),\Lambda'(t),W_t' n_t,u'(t))|^{\frac{2\beta}{1-\theta\beta}}\mathrm{d}t\right\}^{1-\theta\beta}\\
&\quad\times\left\{\mathbb{E}\int_0^T \chi_{u(t,\omega)\neq u'(t,\omega)}\mathrm{d}t\right\}^{\theta\beta}\\
&\le Cd(u(\cdot),u'(\cdot))^{\theta\beta}.
\end{aligned} \tag{3.7}$$

Combining (3.4)–(3.7) and Gronwall's inequality, we get the desired result. □

**Lemma 3.4.** *For any $0<\theta<1, 1<\beta<2$ satisfying $(1+\theta)\beta<2$, and for any $u(\cdot),u'(\cdot)\in\mathcal{U}_{ad}[0,T]$, $(p(\cdot),q(\cdot),k(\cdot),M(\cdot))$, $(p'(\cdot),q'(\cdot),k'(\cdot),M'(\cdot))$ are the solutions of adjoint equations with respect to $u(\cdot),u'(\cdot)$, there exists a positive constant $C$ such that*

$$\mathbb{E}\big[\sup_{0\le t\le T}|p(t)-p'(t)|^{\beta}\big] \le Cd(u(\cdot),u'(\cdot))^{\frac{\theta\beta}{2}},$$

*and*

$$\mathbb{E}\int_0^T (|q(t)-q'(t)|^{\beta} + |k(t)-k'(t)|^{\beta} + \sum_{j\in I}|M_t(j)-M_t'(j)|^{\beta} n_t)\mathrm{d}t \le Cd(u(\cdot),u'(\cdot))^{\frac{\theta\beta}{2}}.$$

*Proof.* Denote $\bar{p}(t) = p(t) - p'(t)$, it is easy to see that $\bar{p}(t)$ satisfies the following equation

$$\begin{cases}
\mathrm{d}\bar{p}(t) = [f_y(t,\alpha(t),\Lambda(t),W_t n_t,u(t))\bar{p}(t) + G_y(t)]\mathrm{d}t + [f_z(t,\alpha(t),\Lambda(t),W_t n_t,u(t))\bar{p}(t)\\
\qquad\qquad + G_z(t)]\mathrm{d}B(t) + \sum_{j\in I}[f_{w(j)}(t,\alpha(t),\Lambda(t),W_t n_t,u(t))\bar{p}(t) + G_{w(j)}(t)]\mathrm{d}\tilde{V}_t(j),\\
\bar{p}(0) = \gamma_y(y'(0)) - \gamma_y(y(0)),
\end{cases} \tag{3.8}$$

where

$$\begin{aligned}
G_y(t) &= [f_y(t,\alpha(t),\Lambda(t),W_tn_t,u(t)) - f_y(t,\alpha(t),\Lambda'(t),W_t'n_t,u'(t))]p'(t)\\
&\quad + l_y(t,\alpha(t),\Lambda'(t),W_t'n_t,u'(t)) - l_y(t,\alpha(t),\Lambda(t),W_tn_t,u(t)),\\
G_z(t) &= [f_z(t,\alpha(t),\Lambda(t),W_tn_t,u(t)) - f_z(t,\alpha(t),\Lambda'(t),W_t'n_t,u'(t))]p'(t)\\
&\quad + l_z(t,\alpha(t),\Lambda'(t),W_t'n_t,u'(t)) - l_z(t,\alpha(t),\Lambda(t),W_tn_t,u(t)),\\
G_{w(j)}(t) &= [f_{w(j)}(t,\alpha(t),\Lambda(t),W_tn_t,u(t)) - f_{w(j)}(t,\alpha(t),\Lambda'(t),W_t'n_t,u'(t))]p'(t)\\
&\quad + l_{w(j)}(t,\alpha(t),\Lambda'(t),W_t'n_t,u'(t)) - l_{w(j)}(t,\alpha(t),\Lambda(t),W_tn_t,u(t)).
\end{aligned}$$

From the system (3.8), it follows that

$$\begin{aligned}
\bar p(t) &= \gamma_y(y'(0)) - \gamma_y(y(0)) + \int_0^t [f_y(s,\alpha(s),\Lambda(s),W_sn_s,u(s))\bar p(s) + G_y(s)]\,\mathrm{d}s\\
&\quad + \int_0^t [f_z(s,\alpha(s),\Lambda(s),W_sn_s,u(s))\bar p(s) + G_z(s)]\,\mathrm{d}B(s)\\
&\quad + \int_0^t \left[f_{w(j)}(s,\alpha(s),\Lambda(s),W_sn_s,u(s))\bar p(s) + G_{w(j)}(s)\right]\mathrm{d}\tilde V_s(j).
\end{aligned}$$

Before dealing with the estimate for $\bar p(t)$ with order of $\beta$, where $1<\beta<2$, we first assume $\beta=2$. Note that Remark 2.2 and Burkholder-Davis-Gundy inequality, we have

$$\begin{aligned}
\mathbb{E}\big[\sup_{0\le t\le T}|\bar p(t)|^2\big] \le & C\Big\{\mathbb{E}\left[|y(0)-y'(0)|^2\right] + \mathbb{E}\int_0^T |\bar p(t)|^2\mathrm{d}t\\
&+ \mathbb{E}\int_0^T \left[|G_y(t)|^2 + |G_z(t)|^2 + |G_{w(j)}(t)|^2\right]\mathrm{d}t\Big\}.
\end{aligned} \tag{3.9}$$

Since the estimate for $y(t), y'(t)$ in Lemma 3.3, we can check with $\beta=1$

$$\mathbb{E}[|y(0)-y'(0)|^2] \le \mathrm{d}(u(\cdot),u'(\cdot))^\theta. \tag{3.10}$$

In the remaining part, we pay attention to estimating the term $\mathbb{E}\int_0^T[|G_y(t)|^2+|G_z(t)|^2+|G_{w(j)}(t)|^2]\mathrm{d}t$. By virtue of Assumption 1 and Lemma 3.3, we can get

$$\begin{aligned}
&\mathbb{E}\int_0^T |G_y(t)|^2\mathrm{d}t\\
&\le CE\int_0^T |f_y(t,\alpha(t),\Lambda(t),W_tn_t,u(t)) - f_y(t,\alpha(t),\Lambda'(t),W_t'n_t,u'(t))|^2|p'(t)|^2\mathrm{d}t\\
&\quad +C\mathbb{E}\int_0^T |l_y(t,\alpha(t),\Lambda(t),W_tn_t,u(t)) - l_y(t,\alpha(t),\Lambda'(t),W_t'n_t,u'(t))|^2\mathrm{d}t\\
&\le C\mathbb{E}\int_0^T |f_y(t,\alpha(t),\Lambda(t),W_tn_t,u(t)) - f_y(t,\alpha(t),\Lambda'(t),W_t'n_t,u(t))|^2|p'(t)|^2\mathrm{d}t\\
&\quad +C\mathbb{E}\int_0^T |f_y(t,\alpha(t),\Lambda'(t),W_t'n_t,u(t)) - f_y(t,\alpha(t),\Lambda'(t),W_t'n_t,u'(t))|^2|p'(t)|^2\mathrm{d}t\\
&\quad +C\mathbb{E}\int_0^T |l_y(t,\alpha(t),\Lambda(t),W_tn_t,u(t)) - l_y(t,\alpha(t),\Lambda'(t),W_t'n_t,u(t))|^2\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&+C\mathbb{E}\int_0^T |l_y\left(t,\alpha\left(t\right),\Lambda'(t),W_t'n_t,u\left(t\right)\right)-l_y\left(t,\alpha\left(t\right),\Lambda'(t),W_t'n_t,u'\left(t\right)\right)|^2\mathrm{d}t\\
\le\ & C\mathbb{E}\int_0^T(|x\left(t\right)-x'\left(t\right)|^2+|y\left(t\right)-y'\left(t\right)|^2+|z\left(t\right)-z'\left(t\right)|^2+|W_t-W_t'|^2n_t)\mathrm{d}t\\
&+C\big\{\mathbb{E}\int_0^T \mathrm{d}t\big\}^{1-\theta}d(u(\cdot),u'(\cdot))^\theta\\
\le\ & Cd(u(\cdot),u'(\cdot))^\theta.
\end{aligned}\tag{3.11}
$$

In the same way, we have

$$
\mathbb{E}\int_0^T|G_z\left(t\right)|^2\mathrm{d}t\le Cd(u(\cdot),u'(\cdot))^\theta,\ \mathbb{E}\int_0^T|G_{w(j)}\left(t\right)|^2\mathrm{d}t\le Cd(u(\cdot),u'(\cdot))^\theta.\tag{3.12}
$$

Combining the (3.9)–(3.12) and Gronwall's inequality, it indicates that

$$
\mathbb{E}[\sup_{0\le t\le T}|\bar{p}(t)|^2]\le Cd(u(\cdot),u'(\cdot))^\theta.
$$

The desired case for $1<\beta<2$ can be inferred from Hölder's inequality, which completes the whole proof of first estimate.

Next, we address with the another estimate. Similarly, we denote $\bar{q}\left(t\right)=q\left(t\right)-q'\left(t\right)$, $\bar{k}\left(t\right)=k\left(t\right)-k'\left(t\right)$, $\bar{M}_t\left(j\right)=M_t\left(j\right)-M_t'\left(j\right)$, which satisfies the following BSDE with regime-switching

$$
\begin{cases}
\mathrm{d}\bar{q}\left(t\right)=-\big\{b_x\left(t,\alpha\left(t\right),x\left(t\right),u\left(t\right)\right)\bar{q}\left(t\right)+\sigma_x\left(t,\alpha\left(t\right),x\left(t\right)\right)\bar{k}\left(t\right)\\
\qquad\qquad -f_x\left(t,\alpha\left(t\right),\Lambda(t),W_tn_t,u\left(t\right)\right)\bar{p}\left(t\right)+Q\left(t\right)\big\}\mathrm{d}t\\
\qquad\qquad +\bar{k}\left(t\right)\mathrm{d}B\left(t\right)+\sum_{j\in I}\bar{M}_t\left(j\right)\mathrm{d}\tilde{V}_t\left(j\right),\\
\bar{q}\left(T\right)=-g_x\left(x\left(T\right)\right)p\left(T\right)+g_x\left(x'\left(T\right)\right)p'\left(T\right)+h_x(x(T))-h_x(x'(T))
\end{cases}
$$

where

$$
\begin{aligned}
Q\left(t\right)=\ &[f_x\left(t,\alpha\left(t\right),\Lambda'(t),W_t'n_{t,}u'\left(t\right)\right)-f_x\left(t,\alpha\left(t\right),\Lambda(t),W_tn_t,u\left(t\right)\right)]p'\left(t\right)\\
&+l_x\left(t,\alpha\left(t\right),\Lambda(t),W_tn_t,u\left(t\right)\right)-l_x\left(t,\alpha\left(t\right),\Lambda'(t),W_t'n_t,u'\left(t\right)\right)\\
&+\left[b_x\left(t,\alpha\left(t\right),x\left(t\right),u\left(t\right)\right)-b_x\left(t,\alpha\left(t\right),x'\left(t\right),u'\left(t\right)\right)\right]q'\left(t\right)\\
&+\left[\sigma_x\left(t,\alpha\left(t\right),x\left(t\right)\right)-\sigma_x\left(t,\alpha\left(t\right),x'\left(t\right)\right)\right]k'\left(t\right).
\end{aligned}
$$

For further processing, we introduce a linear SDE as follows.

$$
\begin{cases}
\mathrm{d}\varphi\left(t\right)=\left[b_x\left(t,\alpha\left(t\right),x\left(t\right),u\left(t\right)\right)\varphi\left(t\right)+|\bar{q}\left(t\right)|^{\beta-1}sgn\left(\bar{q}\left(t\right)\right)\right]\mathrm{d}t\\
\qquad\qquad +\left[\sigma_x\left(t,\alpha\left(t\right),x\left(t\right)\right)\varphi\left(t\right)+|\bar{k}\left(t\right)|^{\beta-1}sgn\left(\bar{k}\left(t\right)\right)\right]\mathrm{d}B\left(t\right)\\
\qquad\qquad +\sum_{j\in I}|\bar{M}_t\left(j\right)|^{\beta-1}sgn(\sum_{j\in I}\bar{M}_t\left(j\right))\mathrm{d}\tilde{V}_t\left(j\right),\\
\varphi\left(0\right)=0.
\end{cases}
$$

By Cauchy-Schwarz inequality, we can easily verify

$$\begin{aligned}\mathbb{E}[\sup_{0\le t\le T}|\varphi(t)|^r] &\le C\mathbb{E}\int_0^T (|\bar{q}(t)|^{r(\beta-1)} + |\bar{k}(t)|^{r(\beta-1)} + \sum_{j\in I}|\bar{M}_t(j)|^{r(\beta-1)} n_t)\mathrm{d}t \\ &= C\mathbb{E}\int_0^T (|\bar{q}(t)|^{\beta} + |\bar{k}(t)|^{\beta} + \sum_{j\in I}|\bar{M}_t(j)|^{\beta} n_t)\mathrm{d}t \end{aligned} \tag{3.13}$$

where $1<\beta<2$, $\frac{1}{r}+\frac{1}{\beta}=1$.

Applying Itô's formula to $\bar{q}(t)\varphi(t)$, we derive

$$\begin{aligned}&\mathbb{E}\int_0^T (|\bar{q}(t)|^{\beta} + |\bar{k}(t)|^{\beta} + \sum_{j\in I}|\bar{M}_t(j)|^{\beta} n_t)\mathrm{d}t \\ &= -\mathbb{E}\int_0^T \Big\{\varphi(t)Q(t) - f_x(t,\alpha(t),\Lambda(t),W_t n_t,u(t))\bar{p}(t)\varphi(t)\Big\}\mathrm{d}t \\ &\quad +\mathbb{E}\Big[(-g_x(x(T))p(T) + g_x(x'(T))p'(T) + h_x(x(T)) - h_x(x'(T)))\varphi(T)\Big] \\ &\le C\Big\{\mathbb{E}\int_0^T |Q(t) - f_x(t,\alpha(t),\Lambda(t),W_t n_{t,} u(t))\bar{p}(t)|^{\beta}\mathrm{d}t\Big\}^{\frac{1}{\beta}}\Big\{\mathbb{E}\int_0^T|\varphi(t)|^r\mathrm{d}t\Big\}^{\frac{1}{r}} \\ &\quad +C\Big\{\mathbb{E}[-g_x(x(T))p(T) + g_x(x'(T))p'(T) + h_x(x(T) - h_x(x'(T)))]^{\beta}\Big\}^{\frac{1}{\beta}}\Big\{\mathbb{E}|\varphi(T)|\Big\}^{\frac{1}{r}}.\end{aligned}$$

Substituting (3.13) into the above inequality, we have

$$\begin{aligned}&\mathbb{E}\int_0^T (|\bar{q}(t)|^{\beta} + |\bar{k}(t)|^{\beta} + \sum_{j\in I}|\bar{M}_t(j)|^{\beta} n_t)\mathrm{d}t \\ &\le C\mathbb{E}\int_0^T |Q(t) - f_x(t,\alpha(t),\Lambda(t),W_t n_t,u(t))\bar{p}(t)|^{\beta}\mathrm{d}t \\ &\quad +C\mathbb{E}|-g_x(x(T))p(T) + g_x(x'(T))p'(T) + h_x(x(T)) - h_x(x'(T))|^{\beta} \\ &\le C\mathbb{E}\int_0^T[|\bar{p}(t)|^{\beta} + |Q(t)|^{\beta}]\mathrm{d}t + C\mathbb{E}\{|g_x(x'(T))p'(T) - g_x(x'(T))p(T)|^{\beta} \\ &\quad +|g_x(x'(T))p(T) - g_x(x(T))p(T)|^{\beta} + |h_x(x(T)) - h_x(x'(T))|^{\beta}\} \\ &\le C\mathbb{E}\int_0^T[|\bar{p}(t)|^{\beta} + |Q(t)|^{\beta}]\mathrm{d}t + C\mathbb{E}\{|\bar{p}(t)|^{\beta} + |g_x(x(T)) - g_x(x'(T))|^{\beta} \\ &\quad +|h_x(x(T)) - h_x(x'(T))|^{\beta}\},\end{aligned} \tag{3.14}$$

where the last two inequalities are deduced from Assumption 3.

By Assumption 1 and Lemma 3.3, it implies that

$$\mathbb{E}[|g_x(x(T)) - g_x(x'(T))|^{\beta}] \le CE[|x(T) - x'(T)|^{\beta}] \le Cd(u(\cdot),u'(\cdot))^{\frac{\theta\beta}{2}}. \tag{3.15}$$

Similarly,

$$\mathbb{E}[|h_x(x(T)) - h_x(x'(T))|^{\beta}] \le Cd(u(\cdot),u'(\cdot))^{\frac{\theta\beta}{2}}. \tag{3.16}$$

Moreover, we can get

$$\begin{aligned}\mathbb{E}\int_0^T|Q(t)|^\beta\mathrm{d}t\leq C\mathbb{E}\int_0^T&|[b_x(t,\alpha(t),x(t),u(t))-b_x(t,\alpha(t),x'(t),u'(t))]q'(t)|^\beta\mathrm{d}t\\&+C\mathbb{E}\int_0^T|[\sigma_x(t,\alpha(t),x(t))-\sigma_x(t,\alpha(t),x'(t))]k'(t)|^\beta\mathrm{d}t\\&+C\mathbb{E}\int_0^T|[f_x(t,\alpha(t),\Lambda'(t),W_t'n_t,u'(t))-f_x(t,\alpha(t),\Lambda(t),W_tn_t,u(t))]p'(t)|^\beta\mathrm{d}t\\&+C\mathbb{E}\int_0^T|l_x(t,\alpha(t),\Lambda(t),W_tn_t,u(t))-l_x(t,\alpha(t),\Lambda'(t),W_t'n_t,u'(t))|^\beta\mathrm{d}t.\end{aligned}$$

As for the first term of the above inequality, we have

$$\begin{aligned}&\mathbb{E}\int_0^T|[b_x(t,\alpha(t),x(t),u(t))-b_x(t,\alpha(t),x'(t),u'(t))]q'(t)|^\beta\mathrm{d}t\\&\leq C\mathbb{E}\int_0^T\{|[b_x(t,\alpha(t),x(t),u(t))-b_x(t,\alpha(t),x(t),u'(t))]q'(t)|^\beta\\&\quad+|[b_x(t,\alpha(t),x(t),u'(t))-b_x(t,\alpha(t),x'(t),u'(t))]q'(t)|^\beta\}\mathrm{d}t\\&\leq C\mathbb{E}\int_0^T|\chi_{u(t,\omega)\neq u'(t,\omega)}q'(t)|^\beta\mathrm{d}t+C\mathbb{E}\int_0^T|x(t)-x'(t)|^\beta\mathrm{d}t\\&\leq C\Big\{\mathbb{E}\int_0^T|q'(t)|^{\beta\cdot\frac{2}{\beta}}\mathrm{d}t\Big\}^{\frac{\beta}{2}}d(u(\cdot),u'(\cdot))^{1-\frac{\beta}{2}}+C\mathbb{E}\int_0^T|x(t)-x'(t)|^\beta\mathrm{d}t\\&\leq Cd(u(\cdot),u'(\cdot))^{\frac{\theta\beta}{2}}.\end{aligned}$$

The other terms can be handled with analogously. Hence,

$$\mathbb{E}\int_0^T|Q(t)|^\beta\mathrm{d}t\leq Cd(u(\cdot),u'(\cdot))^{\frac{\theta\beta}{2}}. \tag{3.17}$$

Substituting (3.15)–(3.17) into (3.14) and since the estimate of $\bar{p}(t)$, we can get

$$\mathbb{E}\int_0^T(|\bar{q}(t)|^\beta+|\bar{k}(t)|^\beta+\sum_{j\in I}|\bar{M}_t(j)|^\beta n_t)\mathrm{d}t\leq Cd(u(\cdot),u'(\cdot))^{\frac{\theta\beta}{2}}.$$

□

Based on the above results, we will complete the proof of Theorem 3.1.

*Proof.* Recalling that the definition of $\varepsilon$-optimal control, we have

$$J(0,x_0;u^\varepsilon(\cdot))\leq\inf_{u(\cdot)\in\mathcal{U}_{ad}[0,T]}J(0,x_0;u(\cdot))+\varepsilon.$$

As a result, due to Ekeland's variational principle in [7], there exists $\tilde{u}^\varepsilon(\cdot)\in\mathcal{U}_{ad}[0,T]$, $\lambda=\varepsilon^{\frac{2}{3}}$ such that

$$d(u^\varepsilon(\cdot),\tilde{u}^\varepsilon(\cdot))\leq\varepsilon^{\frac{2}{3}}, \tag{3.18}$$

and

$$J\left(0,x_0;\tilde{u}^{\varepsilon}\left(\cdot\right)\right)\leq J\left(0,x_0;u\left(\cdot\right)\right)+\varepsilon^{\frac{1}{3}}d\left(u\left(\cdot\right),\tilde{u}^{\varepsilon}\left(\cdot\right)\right),$$

for any $u(\cdot)\in\mathcal{U}_{ad}[0,T]$.

The above inequality is equivalent to

$$\tilde{J}\left(0,x_0;\tilde{u}^{\varepsilon}\left(\cdot\right)\right)\leq \tilde{J}\left(0,x_0;u\left(\cdot\right)\right),$$

if we provide the definition

$$\tilde{J}\left(0,x_0;u\left(\cdot\right)\right)=J\left(0,x_0;u\left(\cdot\right)\right)+\varepsilon^{\frac{1}{3}}d\left(u\left(\cdot\right),\tilde{u}^{\varepsilon}\left(\cdot\right)\right).$$

In other words, $\tilde{u}^{\varepsilon}\left(\cdot\right)$ is the optimal control under the new cost function $\tilde{J}$. On account of the fact that the control domains are not necessarily convex, we introduce a spike variational technique as follows.

$$u^{\rho}\left(t\right)=\begin{cases} u\left(t\right), & \forall t\in\left[\tau,\tau+\rho\right], \quad \text{for fixed } \tau\in\left[0,T\right] \\ \tilde{u}^{\varepsilon}\left(t\right), & \text{otherwise.}\end{cases}$$

Since the exact optimality of $\tilde{u}^{\varepsilon}\left(t\right)$, we can obtain

$$\tilde{J}\left(0,x_0;u^{\rho}\left(t\right)\right)\geq \tilde{J}\left(0,x_0;\tilde{u}^{\varepsilon}\left(t\right)\right),$$

which indicates

$$J\left(0,x_0;u^{\rho}\left(t\right)\right)+\varepsilon^{\frac{1}{3}}d\left(u^{\rho}\left(t\right),\tilde{u}^{\varepsilon}\left(t\right)\right)\geq J\left(0,x_0;\tilde{u}^{\varepsilon}\left(t\right)\right).$$

Therefore,

$$J\left(0,x_0;u^{\rho}\left(t\right)\right)-J\left(0,x_0;\tilde{u}^{\varepsilon}\left(t\right)\right)\geq -\varepsilon^{\frac{1}{3}}\rho.$$

With the aid of variational and duality methods, we derive

$$\begin{aligned}\mathbb{E}\int_{\tau}^{\tau+\rho}&\Big\{\left[b\left(t,\alpha\left(t\right),\tilde{x}^{\varepsilon}\left(t\right),u\left(t\right)\right)-b\left(t,\alpha\left(t\right),\tilde{x}^{\varepsilon}\left(t\right),\tilde{u}^{\varepsilon}\left(t\right)\right)\right]\tilde{p}^{\varepsilon}\left(t\right)\\ &-\big[f(t,\alpha\left(t\right),\tilde{\Lambda}^{\varepsilon}(t),\tilde{W}_t^{\varepsilon}n_t,u\left(t\right))-f(t,\alpha\left(t\right),\tilde{\Lambda}^{\varepsilon}(t),\tilde{W}_t^{\varepsilon}n_t,\tilde{u}^{\varepsilon}\left(t\right))\big]\tilde{q}^{\varepsilon}\left(t\right)\\ &+l(t,\alpha\left(t\right),\tilde{\Lambda}^{\varepsilon}(t),\tilde{W}_t^{\varepsilon}n_t,u\left(t\right))-l(t,\alpha\left(t\right),\tilde{\Lambda}^{\varepsilon}(t),\tilde{W}_t^{\varepsilon}n_t,\tilde{u}^{\varepsilon}\left(t\right))\Big\}\mathrm{d}t+o\left(\rho\right)\geq -\varepsilon^{\frac{1}{3}}\rho.\end{aligned}$$

Dividing by $\rho$ on both side and sending $\rho\to 0$, it implies that

$$\begin{aligned}\mathbb{E}\Big\{&\left[b\left(t,\alpha\left(t\right),\tilde{x}^{\varepsilon}\left(t\right),u\left(t\right)\right)-b\left(t,\alpha\left(t\right),\tilde{x}^{\varepsilon}\left(t\right),\tilde{u}^{\varepsilon}\left(t\right)\right)\right]\tilde{p}^{\varepsilon}\left(t\right)\\ &-[f(t,\alpha\left(t\right),\tilde{\Lambda}^{\varepsilon}(t),\tilde{W}_t^{\varepsilon}n_t,u\left(t\right))-f(t,\alpha\left(t\right),\tilde{\Lambda}^{\varepsilon}(t),\tilde{W}_t^{\varepsilon}n_t,\tilde{u}^{\varepsilon}\left(t\right))]\tilde{q}^{\varepsilon}\left(t\right)\\ &+l(t,\alpha\left(t\right),\tilde{\Lambda}^{\varepsilon}(t),\tilde{W}_t^{\varepsilon}n_t,u\left(t\right))-l(t,\alpha\left(t\right),\tilde{\Lambda}^{\varepsilon}(t),\tilde{W}_t^{\varepsilon}n_t,\tilde{u}^{\varepsilon}\left(t\right))\Big\}\geq -\varepsilon^{\frac{1}{3}}.\end{aligned}$$

To proceed, it is necessary to state some comparisons. We find that the above inequality is close to our target result with all the $(\tilde{x}^{\varepsilon}\left(t\right),\tilde{y}^{\varepsilon}\left(t\right),\tilde{z}^{\varepsilon}\left(t\right),\tilde{W}_t^{\varepsilon}n_t,\tilde{u}^{\varepsilon}\left(t\right),\tilde{p}^{\varepsilon}\left(t\right),\tilde{q}^{\varepsilon}\left(t\right),\tilde{k}^{\varepsilon}\left(t\right))$ replaced by $(x^{\varepsilon}\left(t\right),y^{\varepsilon}\left(t\right),z^{\varepsilon}\left(t\right),$ $W_t^{\varepsilon}n_t,u^{\varepsilon}\left(t\right),p^{\varepsilon}\left(t\right),q^{\varepsilon}\left(t\right),k^{\varepsilon}\left(t\right))$.

In order to get the desired inequality, we deal with the following estimate.

$$\begin{aligned}&\mathbb{E}\int_0^T \big\{\tilde{q}^\varepsilon(t)[f(t,\alpha(t),\tilde{\Lambda}^\varepsilon(t),\tilde{W}_t^\varepsilon n_t,u(t))-f(t,\alpha(t),\tilde{\Lambda}^\varepsilon(t),\tilde{W}_t^\varepsilon n_t,\tilde{u}^\varepsilon(t))]\\&\qquad -q^\varepsilon(t)[f(t,\alpha(t),\Lambda^\varepsilon(t),W_t^\varepsilon n_t,u(t))-f(t,\alpha(t),\Lambda^\varepsilon(t),W_t^\varepsilon n_t,u^\varepsilon(t))]\big\}\mathrm{d}t\\&=I_1+I_2+I_3,\end{aligned}\tag{3.19}$$

where

$$\begin{aligned}I_1&=\mathbb{E}\int_0^T(\tilde{q}^\varepsilon(t)-q^\varepsilon(t))[f(t,\alpha(t),\tilde{\Lambda}^\varepsilon(t),\tilde{W}_t^\varepsilon n_t,u(t))-f(t,\alpha(t),\tilde{\Lambda}^\varepsilon(t),\tilde{W}_t^\varepsilon n_t,\tilde{u}^\varepsilon(t))]\mathrm{d}t,\\I_2&=\mathbb{E}\int_0^T q^\varepsilon(t)[f(t,\alpha(t),\tilde{\Lambda}^\varepsilon(t),\tilde{W}_t^\varepsilon n_t,u(t))-f(t,\alpha(t),\Lambda^\varepsilon(t)),W_t^\varepsilon n_t,u(t))]\mathrm{d}t,\\I_3&=-\mathbb{E}\int_0^T q^\varepsilon(t)[f(t,\alpha(t),\tilde{\Lambda}^\varepsilon(t),\tilde{W}_t^\varepsilon n_t,\tilde{u}^\varepsilon(t))-f(t,\alpha(t),\Lambda^\varepsilon(t),W_t^\varepsilon n_t,u^\varepsilon(t))]\mathrm{d}t.\end{aligned}$$

For any $\delta\in\left[0,\frac{1}{3}\right]$, let $\theta=3\delta$. Since Assumptions 1, 3 and the previous estimate results, we have

$$\begin{aligned}I_1&\le\Big\{\mathbb{E}\int_0^T|f(t,\alpha(t),\tilde{\Lambda}^\varepsilon(t),\tilde{W}_t^\varepsilon n_t,u(t))-f(t,\alpha(t),\tilde{\Lambda}^\varepsilon(t),\tilde{W}_t^\varepsilon n_t,\tilde{u}^\varepsilon(t))|^r\mathrm{d}t\Big\}^{\frac{1}{r}}\\&\quad\times\Big\{\mathbb{E}\int_0^T|\tilde{q}^\varepsilon(t)-q^\varepsilon(t)|^\beta\mathrm{d}t\Big\}^{\frac{1}{\beta}}\\&\le C\Big\{d(u^\varepsilon(\cdot),\tilde{u}^\varepsilon(\cdot))^{\frac{\theta\beta}{2}}\Big\}^{\frac{1}{\beta}}\le C\varepsilon^{\frac{\theta}{3}}=C\varepsilon^\delta,\end{aligned}\tag{3.20}$$

$$\begin{aligned}I_2&\le\Big\{\mathbb{E}\int_0^T|f(t,\alpha(t),\tilde{\Lambda}^\varepsilon(t),\tilde{W}_t^\varepsilon n_t,u(t))-f(t,\alpha(t),\Lambda^\varepsilon(t),W_t^\varepsilon n_t,u(t))|^2\mathrm{d}t\Big\}^{\frac{1}{2}}\Big\{\mathbb{E}\int_0^T|q^\varepsilon(t)|^2\mathrm{d}t\Big\}^{\frac{1}{2}}\\&\le C\Big\{\mathbb{E}\int_0^T[|\tilde{x}^\varepsilon(t)-x^\varepsilon(t)|^2+|\tilde{y}^\varepsilon(t)-y^\varepsilon(t)|^2+|\tilde{z}^\varepsilon(t)-z^\varepsilon(t)|^2+|\tilde{W}_t^\varepsilon-W_t^\varepsilon|^2n_t]\mathrm{d}t\Big\}^{\frac{1}{2}}\\&\le Cd(u^\varepsilon(\cdot),\tilde{u}^\varepsilon(\cdot))^{\frac{\theta}{2}}\\&\le C\{\varepsilon^{\frac{2\theta}{3}}\}^{\frac{1}{2}}\\&=C\varepsilon^\delta,\end{aligned}\tag{3.21}$$

$$\begin{aligned}I_3&=-\mathbb{E}\int_0^T q^\varepsilon(t)[f(t,\alpha(t),\tilde{\Lambda}^\varepsilon(t),\tilde{W}_t^\varepsilon n_t,\tilde{u}^\varepsilon(t))-f(t,\alpha(t),\tilde{\Lambda}^\varepsilon(t),\tilde{W}_t^\varepsilon n_t,u^\varepsilon(t))]\mathrm{d}t\\&\quad-\mathbb{E}\int_0^T q^\varepsilon(t)[f(t,\alpha(t),\tilde{\Lambda}^\varepsilon(t),\tilde{W}_t^\varepsilon n_t,u^\varepsilon(t))-f(t,\alpha(t),\Lambda^\varepsilon(t),W_t^\varepsilon n_t,u^\varepsilon(t))]\mathrm{d}t\\&\le\Big\{\mathbb{E}\int_0^T|f(t,\alpha(t),\tilde{\Lambda}^\varepsilon(t),\tilde{W}_t^\varepsilon n_t,\tilde{u}^\varepsilon(t)-f(t,\alpha(t),\tilde{\Lambda}^\varepsilon(t),\tilde{W}_t^\varepsilon n_t,u^\varepsilon(t))|^2\chi_{\tilde{u}^\varepsilon(t,\omega)\neq u^\varepsilon(t,\omega)}\mathrm{d}t\Big\}^{\frac{1}{2}}\\&\quad\times\Big\{\mathbb{E}\int_0^T|q^\varepsilon(t)|^2\mathrm{d}t\Big\}^{\frac{1}{2}}+C\varepsilon^\delta\end{aligned}$$

$$
\begin{aligned}
&\leq C\Big\{\mathbb{E}\int_0^T |f(t,\alpha(t),\tilde{\Lambda}^\varepsilon(t),\tilde{W}_t^\varepsilon n_t,\tilde{u}^\varepsilon(t)) - f(t,\alpha(t),\tilde{\Lambda}^\varepsilon(t),\tilde{W}_t^\varepsilon n_t,u^\varepsilon(t))|^4 \mathrm{d}t\Big\}^{\frac{1}{4}} \\
&\quad \times\Big\{\mathbb{E}\int_0^T \chi_{\tilde{u}^\varepsilon(t,\omega)\neq u^\varepsilon(t,\omega)}\mathrm{d}t\Big\}^{\frac{1}{4}} \\
&\leq Cd\left(u^\varepsilon(\cdot),\tilde{u}^\varepsilon(\cdot)\right)^{\frac{1}{6}} \\
&\leq C\varepsilon^\delta, \qquad (3.22)
\end{aligned}
$$

where the second term of $I_3$ can be estimated similarly as $I_2$.

Substituting (3.20)–(3.22) to (3.19), it follows that

$$
\begin{aligned}
\mathbb{E}\int_0^T &\Big\{\tilde{q}^\varepsilon(t)\left[f(t,\alpha(t),\tilde{\Lambda}^\varepsilon(t),\tilde{W}_t^\varepsilon n_t,u(t)) - f(t,\alpha(t),\tilde{\Lambda}^\varepsilon(t),\tilde{W}_t^\varepsilon n_t,\tilde{u}^\varepsilon(t))\right] \\
&-q^\varepsilon(t)\left[f\left(t,\alpha(t),\Lambda^\varepsilon(t),W_t^\varepsilon n_t,u(t)\right) - f\left(t,\alpha(t),\Lambda^\varepsilon(t),W_t^\varepsilon n_t,u^\varepsilon(t)\right)\right]\Big\}\mathrm{d}t \leq C\varepsilon^\delta. \qquad (3.23)
\end{aligned}
$$

Similarly, we can prove

$$
\begin{aligned}
\mathbb{E}\int_0^T &\Big\{\left[b\left(t,\alpha(t),\tilde{x}^\varepsilon(t),u(t)\right) - b\left(t,\alpha(t),\tilde{x}^\varepsilon(t),\tilde{u}^\varepsilon(t)\right)\right]\tilde{p}^\varepsilon(t) \\
&- \left[b\left(t,\alpha(t),x^\varepsilon(t),u(t)\right) - b\left(t,\alpha(t),x^\varepsilon(t),u^\varepsilon(t)\right)\right]p^\varepsilon(t)\Big\} \\
&+\Big\{\left[l(t,\alpha(t),\tilde{\Lambda}^\varepsilon(t),\tilde{W}_t^\varepsilon n_t,u(t)) - l(t,\alpha(t),\tilde{\Lambda}^\varepsilon(t),\tilde{W}_t^\varepsilon n_t,\tilde{u}^\varepsilon(t))\right] \\
&-\left[l\left(t,\alpha(t),\Lambda^\varepsilon(t),W_t^\varepsilon n_t,u(t)\right) - l\left(t,\alpha(t),\Lambda^\varepsilon(t),W_t^\varepsilon n_t,u^\varepsilon(t)\right)\right]\Big\}\mathrm{d}t \leq C\varepsilon^\delta. \qquad (3.24)
\end{aligned}
$$

The objective inequality is the direct result of (3.19), (3.23) and (3.24). □

## 4. Sufficient conditions of near-optimality

This section, we obtain sufficient conditions of near-optimal control problems for forward-backward stochastic system with regime-switching under additional assumption. The convexity conditions and the near-minimum condition of the Hamiltonian function in an integral sense are required to prove near-optimal controls with order $\varepsilon^{\frac{1}{2}}$. Before giving the main results, we first present the following additional assumption.

**Assumption 4:** For any $(t,i,x,y,z,m) \in [0,T]\times I\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^k$, $u,u'\in U$, the functions $b,f,l$ are differentiable in $u$, and there exists a positive constant $C$ such that

$$
|b_u\left(t,i,x,u\right) - b_u\left(t,i,x,u'\right)| \leq C|u-u'|,
$$

$$
|\varrho\left(t,i,x,y,z,m,u\right) - \varrho(t,i,x,y,z,m,u')| \leq C|u-u'|,
$$

for $\varrho = f_u, l_u$.

**Theorem 4.1.** *Suppose the Assumptions 1–4 hold. Let $(\Lambda^\varepsilon(\cdot),W^\varepsilon(\cdot))$ be the admissible state quadruple, and $(\Theta^\varepsilon(\cdot),M^\varepsilon(\cdot))$ be the solution of corresponding adjoint equation with to admissible control $u^\varepsilon(\cdot)$. Moreover, assume $H(t,\alpha(t),\cdot,\cdot,\cdot,\cdot,\cdot,\Theta^\varepsilon(t)$ is convex a.e., a.s. $t\in[0,T]$, $h(\cdot)$ and $\gamma(\cdot)$ are convex, respectively. If for any*

$\varepsilon > 0$, *it holds*

$$\begin{aligned}&\inf_{u(\cdot)\in\mathcal{U}_{ad}[0,T]}\mathbb{E}\int_0^T H\left(t,\alpha\left(t\right),\Lambda^{\varepsilon}(t),W_t^{\varepsilon}n_t,u\left(t\right),\Theta^{\varepsilon}(t)\right)\mathrm{d}t\\&\quad\geq\mathbb{E}\int_0^T H\left(t,\alpha\left(t\right),\Lambda^{\varepsilon}(t),W_t^{\varepsilon}n_t,u^{\varepsilon}\left(t\right),\Theta^{\varepsilon}(t)\right)\mathrm{d}t-\varepsilon,\end{aligned}\tag{4.1}$$

*then*

$$J\left(0,x_0;u^{\varepsilon}\left(\cdot\right)\right)\leq\inf_{u(\cdot)\in\mathcal{U}_{ad}[0,T]}J\left(0,x_0;u\left(\cdot\right)\right)+\varepsilon^{\frac{1}{2}}.$$

*Proof.* Firstly, for any $u(\cdot),u'(\cdot)\in\mathcal{U}_{ad}[0,T]$, we introduce a new functional and a new metric on $\mathcal{U}_{ad}[0,T]$ as follows.

$$\begin{aligned}F\left(u(\cdot)\right)&=\mathbb{E}\int_0^T H(t,\alpha\left(t\right),\Lambda^{\varepsilon}(t),W_t^{\varepsilon}n_t,u(\cdot),\Theta^{\varepsilon}(t))\mathrm{d}t,\\\tilde{d}\left(u\left(\cdot\right),u'\left(\cdot\right)\right)&=\mathbb{E}\int_0^T\left(1+|p^{\varepsilon}\left(t\right)|+|q^{\varepsilon}\left(t\right)|+|k^{\varepsilon}\left(t\right)|\right)|u\left(t\right)-u'\left(t\right)|\mathrm{d}t.\end{aligned}$$

It is easy to verify that $\mathcal{U}_{ad}[0,T]$ is a complete metric space under $\tilde{d}$ and $F\left(u(\cdot)\right)$ is a continuous functional with respect to any admissible control $u\left(\cdot\right)$.

Noting that near-minimum condition of the Hamiltonian function in an integral sense and the definition of $F$, we have

$$F\left(u^{\varepsilon}\left(\cdot\right)\right)\leq\inf_{u(\cdot)\in\mathcal{U}_{ad}[0,T]}F\left(u\left(\cdot\right)\right)+\varepsilon.$$

According to Ekeland's variational principle, there exists $\tilde{u}^{\varepsilon}\left(t\right)\in\mathcal{U}_{ad}[0,T]$ such that

$$\tilde{d}\left(\tilde{u}^{\varepsilon}\left(\cdot\right),u^{\varepsilon}\left(\cdot\right)\right)\leq\varepsilon^{\frac{1}{2}},\tag{4.2}$$

and

$$F\left(\tilde{u}^{\varepsilon}\left(\cdot\right)\right)\leq F\left(u\left(\cdot\right)\right)+\varepsilon^{\frac{1}{2}}\ \tilde{d}\left(u\left(\cdot\right),\tilde{u}^{\varepsilon}\left(\cdot\right)\right),$$

for any $u\left(\cdot\right)\in\mathcal{U}_{ad}[0,T]$.

Equivalently, it implies that

$$\begin{aligned}\mathbb{E}\int_0^T H\left(t,\alpha\left(t\right),\Lambda^{\varepsilon}(t),W_t^{\varepsilon}n_t,\tilde{u}^{\varepsilon}\left(t\right),\Theta^{\varepsilon}(t)\right)\mathrm{d}t&\leq\mathbb{E}\int_0^T H\left(t,\alpha\left(t\right),\Lambda^{\varepsilon}(t),W_t^{\varepsilon}n_t,u\left(t\right),\Theta^{\varepsilon}(t)\right)\mathrm{d}t\\&\quad+\mathbb{E}\int_0^T\varepsilon^{\frac{1}{2}}\left(1+|p^{\varepsilon}\left(t\right)|+|q^{\varepsilon}\left(t\right)|+|k^{\varepsilon}\left(t\right)|\right)|u\left(t\right)-\tilde{u}^{\varepsilon}\left(t\right)|\mathrm{d}t.\end{aligned}$$

Meanwhile, if we introduce

$$\begin{aligned}\tilde{H}\left(t,\alpha\left(t\right),\Lambda^{\varepsilon}(t),W_t^{\varepsilon}n_t,u\left(t\right),\Theta^{\varepsilon}(t)\right)&=H\left(t,\alpha\left(t\right),\Lambda^{\varepsilon}(t),W_t^{\varepsilon}n_t,u\left(t\right),\Theta^{\varepsilon}(t)\right)\\&\quad+\varepsilon^{\frac{1}{2}}\left(1+|p^{\varepsilon}\left(t\right)|+|q^{\varepsilon}\left(t\right)|+|k^{\varepsilon}\left(t\right)|\right)|u\left(t\right)-\tilde{u}^{\varepsilon}\left(t\right)|,\end{aligned}$$

then we can see that

$$\tilde{H}\left(t,\alpha\left(t\right),\Lambda^{\varepsilon}(t),W_{t}^{\varepsilon}n_{t},\tilde{u}^{\varepsilon}\left(t\right),\Theta^{\varepsilon}(t)\right)=\min_{u(\cdot)\in\mathcal{U}_{ad}[0,T]}\tilde{H}\left(t,\alpha\left(t\right),\Lambda^{\varepsilon}(t),W_{t}^{\varepsilon}n_{t},u\left(t\right),\Theta^{\varepsilon}(t)\right).$$

Recalling the Lemma 2.3 of Chapter 3 in [23], we have

$$\begin{aligned}&0\in\partial_{u}\tilde{H}\left(t,\alpha\left(t\right),\Lambda^{\varepsilon}(t),W_{t}^{\varepsilon}n_{t},\tilde{u}^{\varepsilon}\left(t\right),\Theta^{\varepsilon}(t)\right)\subset\partial_{u}H\left(t,\alpha\left(t\right),\Lambda^{\varepsilon}(t),W_{t}^{\varepsilon}n_{t},\tilde{u}^{\varepsilon}\left(t\right),\Theta^{\varepsilon}(t)\right)\\&\quad+[-\varepsilon^{\frac{1}{2}}\left(1+|p^{\varepsilon}\left(t\right)|+|q^{\varepsilon}\left(t\right)|+|k^{\varepsilon}\left(t\right)|\right),\varepsilon^{\frac{1}{2}}\left(1+|p^{\varepsilon}\left(t\right)|+|q^{\varepsilon}\left(t\right)|+|k^{\varepsilon}\left(t\right)|\right)].\end{aligned}$$

Hence, there exists $\eta^{\varepsilon}\left(t\right)$ satisfies

$$H_{u}\left(t,\alpha\left(t\right),\Lambda^{\varepsilon}(t),W_{t}^{\varepsilon}n_{t},\tilde{u}^{\varepsilon}\left(t\right),\Theta^{\varepsilon}(t)\right)+\eta^{\varepsilon}\left(t\right)=0. \tag{4.3}$$

Now, we will discuss the estimate of $H_{u}\left(t,\alpha\left(t\right),\Lambda^{\varepsilon}(t),W_{t}^{\varepsilon}n_{t},u^{\varepsilon}\left(t\right),\Theta^{\varepsilon}(t)\right)$, which is the key ingredient of remaining proof. From the Assumption 4 and (4.2), (4.3), we can get

$$\begin{aligned}&|H_{u}\left(t,\alpha\left(t\right),\Lambda^{\varepsilon}(t),W_{t}^{\varepsilon}n_{t},u^{\varepsilon}\left(t\right),\Theta^{\varepsilon}(t)\right)|\\&\leq|H_{u}\left(t,\alpha\left(t\right),\Lambda^{\varepsilon}(t),W_{t}^{\varepsilon}n_{t},u^{\varepsilon}\left(t\right),\Theta^{\varepsilon}(t)\right)-H_{u}\left(t,\alpha\left(t\right),\Lambda^{\varepsilon}(t),W_{t}^{\varepsilon}n_{t},\tilde{u}^{\varepsilon}\left(t\right),\Theta^{\varepsilon}(t)\right))|\\&\quad+|H_{u}\left(t,\alpha\left(t\right),\Lambda^{\varepsilon}(t),W_{t}^{\varepsilon}n_{t},\tilde{u}^{\varepsilon}\left(t\right),\Theta^{\varepsilon}(t)\right)|\\&\leq|\eta^{\varepsilon}\left(t\right)|+|[b_{u}\left(t,\alpha\left(t\right),x^{\varepsilon}\left(t\right),u^{\varepsilon}\left(t\right)\right)-b_{u}\left(t,\alpha\left(t\right),x^{\varepsilon}\left(t\right),\tilde{u}^{\varepsilon}\left(t\right)\right)]q^{\varepsilon}\left(t\right)\\&\quad+[f_{u}\left(t,\alpha\left(t\right),\Lambda^{\varepsilon}(t),W_{t}^{\varepsilon}n_{t},\tilde{u}^{\varepsilon}\left(t\right)\right)-f_{u}\left(t,\alpha\left(t\right),\Lambda^{\varepsilon}(t),W_{t}^{\varepsilon}n_{t},u^{\varepsilon}\left(t\right)\right)]p^{\varepsilon}\left(t\right)\\&\quad+l_{u}\left(t,\alpha\left(t\right),\Lambda^{\varepsilon}(t),W_{t}^{\varepsilon}n_{t},u^{\varepsilon}\left(t\right)\right)-l_{u}\left(t,\alpha\left(t\right),\Lambda^{\varepsilon}(t),W_{t}^{\varepsilon}n_{t},\tilde{u}^{\varepsilon}\left(t\right)\right)|\\&\leq C\left(1+|p^{\varepsilon}\left(t\right)|+|q^{\varepsilon}\left(t\right)|+|k^{\varepsilon}\left(t\right)|\right)|u^{\varepsilon}\left(t\right)-\tilde{u}^{\varepsilon}\left(t\right)|\\&\quad+\varepsilon^{\frac{1}{2}}\left(1+|p^{\varepsilon}\left(t\right)|+|q^{\varepsilon}\left(t\right)|+|k^{\varepsilon}\left(t\right)|\right)\\&\leq C\varepsilon^{\frac{1}{2}}.\end{aligned} \tag{4.4}$$

Based on the definition of cost functional, the difference can be divided into four parts as:

$$J\left(0,x_{0};u^{\varepsilon}\left(\cdot\right)\right)-J\left(0,x_{0};u\left(\cdot\right)\right)=\bar{I}_{1}+\bar{I}_{2}+\bar{I}_{3}+\bar{I}_{4}, \tag{4.5}$$

where

$$\begin{aligned}\bar{I}_{1}=&-\mathbb{E}\int_{0}^{T}[f\left(t,\alpha\left(t\right),\Lambda(t),W_{t}n_{t},u\left(t\right)\right)-f\left(t,\alpha\left(t\right),\Lambda^{\varepsilon}(t),W_{t}^{\varepsilon}n_{t},u^{\varepsilon}\left(t\right)\right)]p^{\varepsilon}\left(t\right)\mathrm{d}t\\&-\mathbb{E}\int_{0}^{T}[b\left(t,\alpha\left(t\right),x^{\varepsilon}\left(t\right),u^{\varepsilon}\left(t\right)\right)-b\left(t,\alpha\left(t\right),x\left(t\right),u\left(t\right)\right)]q^{\varepsilon}\left(t\right)\mathrm{d}t\\&-\mathbb{E}\int_{0}^{T}\left[\sigma\left(t,\alpha\left(t\right),x^{\varepsilon}\left(t\right)\right)-\sigma\left(t,\alpha\left(t\right),x\left(t\right)\right)\right]k^{\varepsilon}\left(t\right)\mathrm{d}t,\\\bar{I}_{2}=&\,\mathbb{E}\int_{0}^{T}[H\left(t,\alpha\left(t\right),\Lambda^{\varepsilon}(t),W_{t}^{\varepsilon}n_{t},u^{\varepsilon}\left(t\right),\Theta^{\varepsilon}(t)\right)-H\left(t,\alpha\left(t\right),\Lambda(t),W_{t}n_{t},u\left(t\right),\Theta^{\varepsilon}(t)\right)]\mathrm{d}t,\\\bar{I}_{3}=&\,\mathbb{E}\left[h\left(x^{\varepsilon}\left(T\right)\right)-h\left(x\left(T\right)\right)\right],\quad\bar{I}_{4}=E\left[\gamma\left(y^{\varepsilon}\left(0\right)\right)-\gamma\left(y\left(0\right)\right)\right].\end{aligned}$$

In order to get the target inequality, some analysis for the estimates of $\bar{I}_2, \bar{I}_3, \bar{I}_4$ is needed. We can deduce the following results from the convexity conditions given in the hypothesis.

$$
\begin{aligned}
\bar{I}_2 \leq & \mathbb{E}\int_0^T H_x\left(t, \alpha\left(t\right), \Lambda^{\varepsilon}(t), W_t^{\varepsilon} n_t, u^{\varepsilon}\left(t\right), \Theta^{\varepsilon}(t)\right)\left(x^{\varepsilon}\left(t\right) - x\left(t\right)\right) \mathrm{d}t \\
& + \mathbb{E}\int_0^T H_y(t, \alpha\left(t\right), \Lambda^{\varepsilon}(t), W_t^{\varepsilon} n_t, u^{\varepsilon}\left(t\right), \Theta^{\varepsilon}(t))\left(y^{\varepsilon}\left(t\right) - y\left(t\right)\right) \mathrm{d}t \\
& + \mathbb{E}\int_0^T H_z(t, \alpha\left(t\right), \Lambda^{\varepsilon}(t), W_t^{\varepsilon} n_t, u^{\varepsilon}\left(t\right), \Theta^{\varepsilon}(t))\left(z^{\varepsilon}\left(t\right) - z\left(t\right)\right) \mathrm{d}t \\
& + \mathbb{E}\int_0^T H_u(t, \alpha\left(t\right), \Lambda^{\varepsilon}(t), W_t^{\varepsilon} n_t, u^{\varepsilon}\left(t\right), \Theta^{\varepsilon}(t))(u^{\varepsilon}\left(t\right) - u\left(t\right)) \mathrm{d}t \\
& + \mathbb{E}\int_0^T \sum_{j\in I} H_{w(j)}(t, \alpha\left(t\right), \Lambda^{\varepsilon}(t), W_t^{\varepsilon} n_t, u^{\varepsilon}\left(t\right), \Theta^{\varepsilon}(t))\left(W_t^{\varepsilon}\left(j\right) - W_t\left(j\right)\right) n_t \mathrm{d}t, \\
\bar{I}_3 \leq & \mathbb{E}\left[h_x\left(x^{\varepsilon}\left(T\right)\right)\left(x^{\varepsilon}\left(T\right) - x\left(T\right)\right)\right], \quad \bar{I}_4 \leq \mathbb{E}\left[\gamma_y\left(y^{\varepsilon}\left(0\right)\right)\left(y^{\varepsilon}\left(0\right) - y\left(0\right)\right)\right].
\end{aligned} \tag{4.6}
$$

Applying Itô's formula to $q^{\varepsilon}\left(t\right)\left(x^{\varepsilon}\left(t\right) - x\left(t\right)\right)$ and $p^{\varepsilon}\left(t\right)\left(y^{\varepsilon}\left(t\right) - y\left(t\right)\right)$, we can get

$$
\begin{aligned}
\mathbb{E}\left[h_x\left(x^{\varepsilon}\left(T\right)\right)\left(x^{\varepsilon}\left(T\right) - x\left(T\right)\right)\right] = & - \mathbb{E}\int_0^T H_x(t, \alpha\left(t\right), \Lambda^{\varepsilon}(t), W_t^{\varepsilon} n_t, u^{\varepsilon}\left(t\right), \Theta^{\varepsilon}(t)\left(x^{\varepsilon}\left(t\right) - x\left(t\right)\right) \mathrm{d}t \\
& + \mathbb{E}\int_0^T [(b(t, \alpha\left(t\right), x^{\varepsilon}\left(t\right), u^{\varepsilon}\left(t\right)) - b(t, \alpha\left(t\right), x\left(t\right), u\left(t\right))]q^{\varepsilon}\left(t\right) \mathrm{d}t \\
& + \mathbb{E}\int_0^T [\sigma(t, \alpha\left(t\right), x^{\varepsilon}\left(t\right) - \sigma\left(t, \alpha\left(t\right), x\left(t\right)\right)]k^{\varepsilon}\left(t\right) \mathrm{d}t \\
& + \mathbb{E}\big[g_x\left(x^{\varepsilon}\left(T\right)\right) p^{\varepsilon}\left(T\right)\left(x^{\varepsilon}\left(T\right) - x\left(T\right)\right)\big],
\end{aligned} \tag{4.7}
$$

$$
\begin{aligned}
\mathbb{E}\left[\gamma_y\left(y^{\varepsilon}\left(0\right)\right)\left(y^{\varepsilon}\left(0\right) - y\left(0\right)\right)\right] \leq & - \mathbb{E}\int_0^T H_y\left(t, \alpha\left(t\right), \Lambda^{\varepsilon}(t), W_t^{\varepsilon} n_t, u^{\varepsilon}\left(t\right), \Theta^{\varepsilon}(t)\right)\left(y^{\varepsilon}\left(t\right) - y\left(t\right)\right) \mathrm{d}t \\
& - \mathbb{E}\int_0^T H_z(t, \alpha\left(t\right), \Lambda^{\varepsilon}(t), W_t^{\varepsilon} n_t, u^{\varepsilon}\left(t\right), \Theta^{\varepsilon}(t))\left(z^{\varepsilon}\left(t\right) - z\left(t\right)\right) \mathrm{d}t \\
& - \mathbb{E}\int_0^T \sum_{j\in I} H_{w(j)}(t, \alpha\left(t\right), \Lambda^{\varepsilon}(t), W_t^{\varepsilon} n_t, u^{\varepsilon}\left(t\right), \Theta^{\varepsilon}(t))\left(W_t^{\varepsilon}\left(j\right) - W_t\left(j\right)\right) n_t \mathrm{d}t \\
& + \mathbb{E}\int_0^T [f(t, \alpha\left(t\right), \Lambda(t), W_t n_t, u\left(t\right)) - f(t, \alpha\left(t\right), \Lambda^{\varepsilon}(t), W_t^{\varepsilon} n_t, u^{\varepsilon}\left(t\right))]p^{\varepsilon}\left(t\right) \mathrm{d}t \\
& + \mathbb{E}\big[g_x\left(x^{\varepsilon}\left(T\right)\right) p^{\varepsilon}\left(T\right)\left(x\left(T\right) - x^{\varepsilon}\left(T\right)\right)\big].
\end{aligned} \tag{4.8}
$$

Combining (4.5)–(4.8), we can gain

$$
J\left(0, x_0; u^{\varepsilon}\left(\cdot\right)\right) - J\left(0, x_0; u\left(\cdot\right)\right) \leq \mathbb{E}\int_0^T H_u\left(t, \alpha\left(t\right), \Lambda^{\varepsilon}(t), W_t^{\varepsilon} n_t, u^{\varepsilon}\left(t\right), \Theta^{\varepsilon}(t)\right)\left(u^{\varepsilon}\left(t\right) - u\left(t\right)\right) \mathrm{d}t.
$$

On the basis of (4.4), we have

$$
J\left(0, x_0; u^{\varepsilon}\left(\cdot\right)\right) \leq J\left(0, x_0; u\left(\cdot\right)\right) + C\varepsilon^{\frac{1}{2}}.
$$

Since $u(\cdot) \in \mathcal{U}_{ad}[0,T]$ is arbitrary, the desired result is clear. □

## 5. Application in recursive utility investment consumption problem

In this section, we study an investment and consumption model under the stochastic recursive utility in financial market. Stochastic recursive utility was introduced originally by [6], which was a generalization of standard additive utility with an instantaneous utility depending not only on an instantaneous consumption rate but also on a future utility. As it puts in [12], the stochastic recursive utility process can be represented by the solution of a special BSDE. Therefore, a control system with cost functional described by a solution of a BSDE is the so-called the recursive optimal control problem. Actually, due to the limitation of internal and external factors in market, the exact optimal consumption strategy can not be achieved in some specific cases. Applying necessary and sufficient conditions derived in Sections 3 and 4, we obtain near-optimal consumption processes below a certain "tolerance level".

In financial market, we assume that there are two types of securities which can be invested. One of them is risk-free asset like putting money in the bank account, buying fixed-income bonds and other similar products. Suppose the price process of risk-free asset subjects to

$$\mathrm{d}S^0(t) = r(t)S^0(t)\mathrm{d}t,\ S^0(0) > 0.$$

Another alternative is to hold stock, whose price $S^1(t)$ is given by a classical geometric Brownian motion

$$\mathrm{d}S^1(t) = \mu(t)S^1(t)\mathrm{d}t + \sigma(t)S^1(t)\mathrm{d}B(t),\ S^1(0) > 0.$$

It is natural that investing in risky asset will have more scope for efficiency gains, but must take on more risk. Therefore, we have $\mu(t) > r(t)$, $\sigma(t) > 0$. In addition, $r(t), \mu(t)$ and $\sigma(t)$ are all deterministic bounded functions.

In our investment consumption problem, we denote $\pi(t)$ the proportion invested in stock, which is a fixed deterministic bounded function with respect to $t$. $c(t)$ represents the consumption process. Then the evolvement of wealth value in uncertainty environment can be described by

$$\begin{cases} \mathrm{d}x(t) = \big\{r(t)x(t) + [\mu(t) - r(t)]\pi(t)x(t) - c(t)\big\}\mathrm{d}t \\ \qquad\qquad + \sigma(t)\pi(t)x(t)\mathrm{d}B(t), \\ x(0) = x. \end{cases}$$

However, if we take macroeconomic conditions into account in this model. The dynamic of wealth process suffered from one major drawback: it lacks the flexibility to capture the uncertainty in investment environment. To overcome this difficulty, we introduce a continuous-time finite state Markov chain in our model. So the wealth value satisfies a switching diffusion process as follows.

$$\begin{cases} \mathrm{d}x(t) = \big\{r(t,\alpha(t))x(t) + [\mu(t,\alpha(t)) - r(t,\alpha(t))]\pi(t)x(t) \\ \qquad\qquad - c(t)\big\}\mathrm{d}t + \sigma(t,\alpha(t))\pi(t)x(t)\mathrm{d}B(t), \\ x(0) = x. \end{cases} \tag{5.1}$$

Let $U$ be a nonnegative nonempty subset of $\mathbb{R}$. Consumption process $c(t)$ is called an admissible control if $c(t) \in \mathcal{U}_{ad}$, where $\mathcal{U}_{ad} = \{c(\cdot) : [0,T] \times \Omega \to U \mid c(\cdot)$ is an $\mathcal{F}_t$-adapted process, $\mathbb{E}\int_0^T |c(t)|^2\mathrm{d}t < \infty\ \}$. Here, $\mathcal{F}_t$ is the natural filtration generated by Brownian motion and Markov chain augmented by all $\mathbb{P}$-null sets.

Considering the following stochastic recursive utility, where is formulated by a BSDE with the Markov chain $\alpha(t)$:

$$\begin{cases} \mathrm{d}y(t) = -\,[A(t,\alpha(t))x(t) + B(t,\alpha(t))y(t) + C(t,\alpha(t))z(t) \\ \qquad\qquad - c(t)]\mathrm{d}t + z(t)\mathrm{d}B(t) + \sum_{j\in I} W_t(j)\mathrm{d}\tilde{V}_t(j), \\ y(T) = x(T). \end{cases} \tag{5.2}$$

The relevant utility cost functional is given by

$$J(c(\cdot)) = \mathbb{E}\Big\{ \int_0^T L e^{-\beta t} \frac{(c(t))^{1-R}}{1-R} \mathrm{d}t + Dy(0) \Big\}, \tag{5.3}$$

where $L, D$ are positive constants, $\beta \in (0,1)$ is a discount factor, and $R \in (0,1)$, $(1-R)$ denotes the relative risk aversion of investors.

The optimal control problem of our recursive utility investment consumption model is to maximize cost function defined as above, and to find the optimal consumption process $\bar{c}(t)$ such that

$$J(\bar{c}(\cdot)) = \max_{c(\cdot)\in\mathcal{U}_{ad}} J(c(\cdot)).$$

However, due to the limitation of internal and external factors in market, the exact optimal consumption strategy can not be achieved in some specific cases. The main target in this example is to choose near-optimal consumption processes $c^{\varepsilon}(t)$, which are more reasonable and tractable in practice.

For further study, we introduce the following adjoint equation

$$\begin{cases} \mathrm{d}p(t) = B(t,\alpha(t))p(t)\mathrm{d}t + C(t,\alpha(t))p(t)\mathrm{d}B(t), \\ \mathrm{d}q(t) = -\,\{[r(t,\alpha(t)) + (\mu(t,\alpha(t)) - r(t,\alpha(t))\pi(t))]q(t) \\ \qquad\qquad + \sigma(t,\alpha(t))\pi(t)k(t) - A(t,\alpha(t))p(t)\}\mathrm{d}t \\ \qquad\qquad - k(t)\mathrm{d}B(t) - \sum_{j\in I} M_t(j)\mathrm{d}\tilde{V}_t(j), \\ p(0) = -\,D,\ q(T) = -p(T). \end{cases} \tag{5.4}$$

The Hamiltonian function corresponding to this model has the following form

$$\begin{aligned} &H(t,\alpha(t),x(t),y(t),z(t),W_t n_t,c(t),p(t),q(t),k(t)) \\ =&-\,[A(t,\alpha(t))x(t) + B(t,\alpha(t))y(t) + C(t,\alpha(t))z(t) - c(t)]p(t) \\ &+ [r(t,\alpha(t))x(t) + (\mu(t,\alpha(t)) - r(t,\alpha(t)))\pi(t)x(t) - c(t)]q(t) \\ &+ \sigma(t,\alpha(t))\pi(t)x(t)k(t) + Le^{-\beta t}\frac{(c(t))^{1-R}}{1-R}. \end{aligned}$$

From (5.4), we can obtain that

$$\begin{aligned} p(t) &= -D\exp\Big\{ \int_0^t [B(s,\alpha(s)) - \frac{1}{2}C^2(s,\alpha(s))]\mathrm{d}s + \int_0^t C(s,\alpha(s))\mathrm{d}B(s) \Big\}, \\ q(t) &= \mathbb{E}\Big\{ -p(T)\Lambda_T - \int_t^T A(s,\alpha(s))p(s)\Lambda_s \mathrm{d}s | \mathcal{F}_t \Big\}, \end{aligned}$$

where

$$\Lambda_s = \exp\Big\{ \int_t^s [r(\tau,\alpha(\tau))(1-\pi(\tau)) + \mu(\tau,\alpha(\tau)\pi(\tau)) - \frac{1}{2}\sigma^2(\tau,\alpha(\tau))\pi^2(\tau)]d\tau + \int_t^s \sigma(\tau,\alpha(\tau))\pi(\tau)\mathrm{d}B(\tau)\Big\}.$$

For any sufficient small $\varepsilon > 0$ any $\delta \in [0, \frac{1}{3}]$, if $u^\varepsilon(t)$ is $\varepsilon$-optimal then it follows from Theorem 3.1 that

$$\mathbb{E}\int_0^T H(t,\alpha(t),x^\varepsilon(t),y^\varepsilon(t),z^\varepsilon(t),W_t^\varepsilon n_t,c^\varepsilon(t),p^\varepsilon(t),q^\varepsilon(t),k^\varepsilon(t))\mathrm{d}t \le \inf_{c(t)\in\mathcal{U}_{ad}} \mathbb{E}\int_0^T H(t,\alpha(t),x^\varepsilon(t),y^\varepsilon(t),z^\varepsilon(t),W_t^\varepsilon n_t,c(t),p^\varepsilon(t),q^\varepsilon(t),k^\varepsilon(t))\mathrm{d}t,$$

which implies

$$\mathbb{E}\int_0^T [c^\varepsilon(t)(p^\varepsilon(t)-q^\varepsilon(t)) + Le^{-\beta t}\frac{(c^\varepsilon(t))^{1-R}}{1-R}]\mathrm{d}t \le \mathbb{E}\int_0^T \frac{R}{1-R}(q^\varepsilon(t)-p^\varepsilon(t))^{1-\frac{1}{R}}L^{\frac{1}{R}}e^{\frac{-\beta t}{R}}\mathrm{d}t + C\varepsilon^\delta.$$

Here, $(x^\varepsilon(\cdot), y^\varepsilon(\cdot), z^\varepsilon(\cdot), W^\varepsilon(\cdot))$ is the state process and $(p^\varepsilon(\cdot), q^\varepsilon(\cdot), k^\varepsilon(\cdot), M^\varepsilon(\cdot))$ is the adjoint process with respect to any admissible consumption process $c^\varepsilon(\cdot)$.

For example, if we let

$$c^\varepsilon(t) = \left(\frac{L(1-\varepsilon^R)}{q^\varepsilon(t)-p^\varepsilon(t)}\right)^{\frac{1}{R}} e^{\frac{-\beta t}{R}},$$

then $c^\varepsilon(t)$ is a candidate $\varepsilon$-optimal consumption process. Since Theorem 4.1, it is easy to verify that $c^\varepsilon(t)$ is indeed the near-optimal controls for any $\varepsilon > 0$.

Next, we give some numerical simulation results in the framework of our model. For simplicity, we consider a continuous-time Markov chain $\alpha(t)$, which has two states $\{1, 2\}$. To better illustrate the economic explanation, we suppose $\alpha(t) = 1$ indicates bull market and $\alpha(t) = 2$ bear market. More specially, in bull market, we have $\mu(1) > 0$ and $\mu(2) < 0$ in bear market.

Let common parameters set as $\{\varepsilon, \beta, R, L, D, \pi\} = \{0.01, 0.5, 0.2, 2, 0.1, 0.5\}$, $T = 1$ yr.

To be more intuitive, we give a figure to illustrate the changes of near-optimal consumption process $c^\varepsilon(t)$ with respect to the Markov chain. The parameters are assigned as $r(1) = 0.02$, $r(2) = 0.03$, $A(1) = 0.4$, $A(2) = 0.6$, $B(1) = 0.3$, $B(2) = 0.5$, $C(1) = 0.6$, $C(2) = 0.8$, $\sigma(1) = 0.6$, $\sigma(2) = 0.5$, $\mu(1) = 0.05$, $\mu(2) = -0.05$. Due to the random nature of the Markov chain, we cannot accurately predict the moment when the jump occurs. In fact, it is not hard to verify that these moments are stopping times. In the following, we give a simulation result provided that the initial state is bull market and the regime-switching times subject to 0.15 yr, 0.5 yr and 0.8 yr. As shown in Figure 1, the near-optimal consumption process $c^\varepsilon(t)$ switches back and forth between the two markets. Moreover, it increases in bull market while decreases in bear market. This figure supposes that there exists three given regime-switching times in one year. The more general case can be addressed without essential difficulties.

In the following, some analysis about the relationship between the near-optimal consumption process $c^\varepsilon(t)$ and the return rate of risk-free asset $r(t)$, the near-optimal consumption process $c^\varepsilon(t)$ and the return rate of

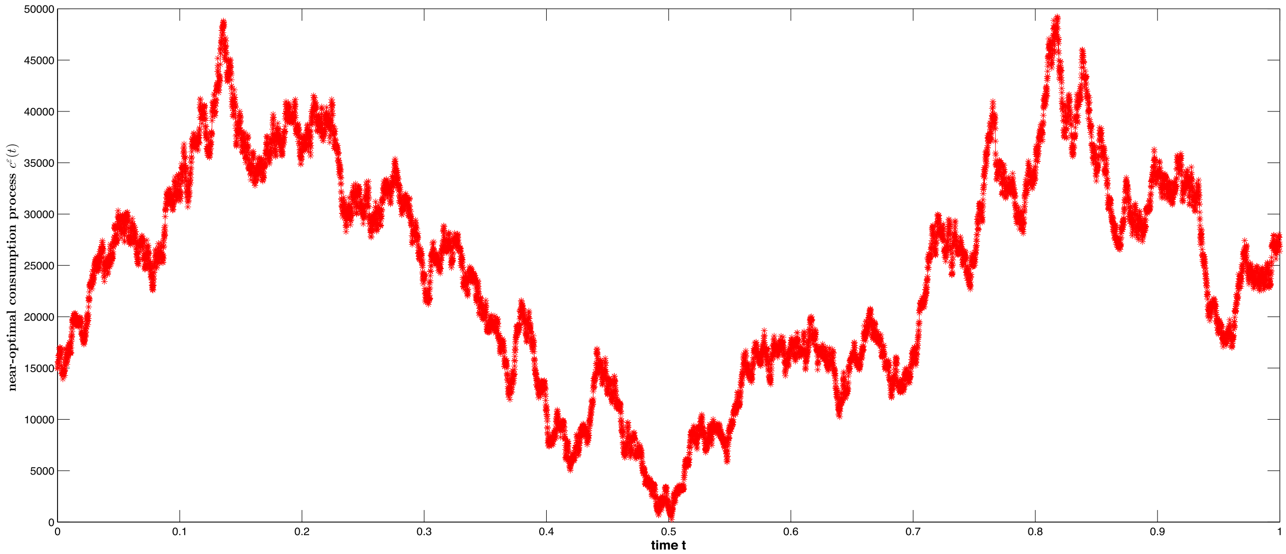


FIGURE 1. $c^{\varepsilon}(t)$ changes according to the evolution of $\alpha(t)$.

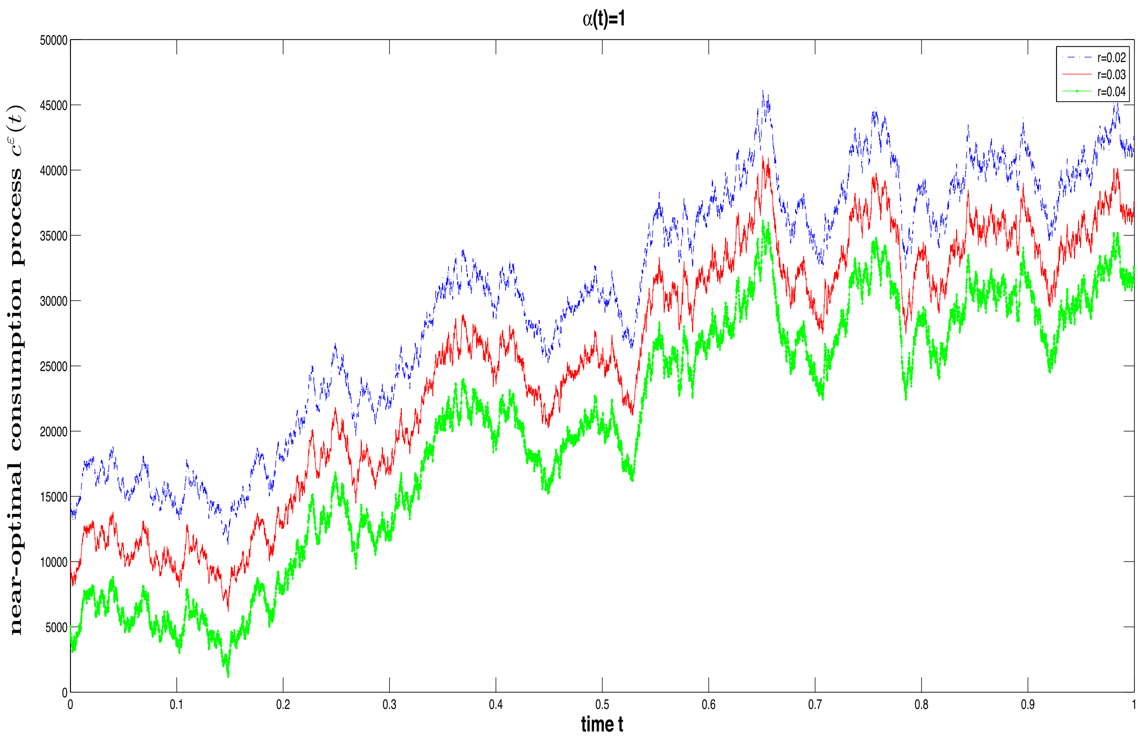


FIGURE 2. Relationship between $c^{\varepsilon}(t)$ and $r(t)$ in a bull market.

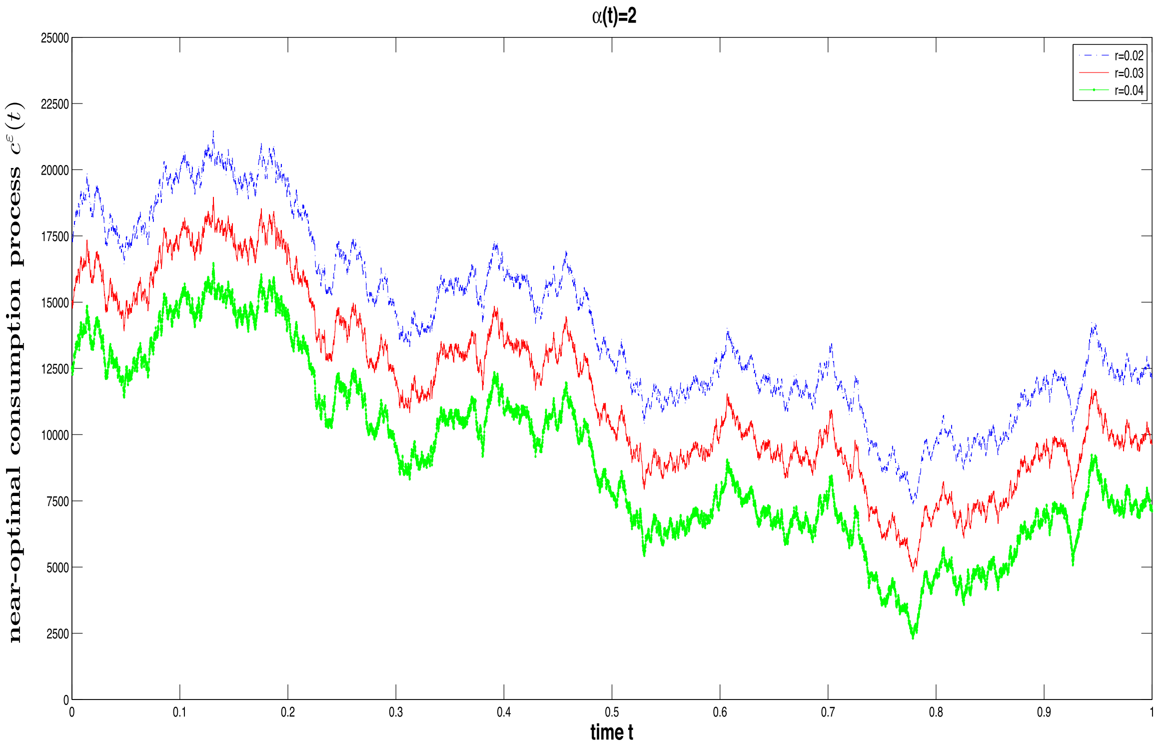


FIGURE 3. Relationship between $c^{\varepsilon}(t)$ and $r(t)$ in a bear market.

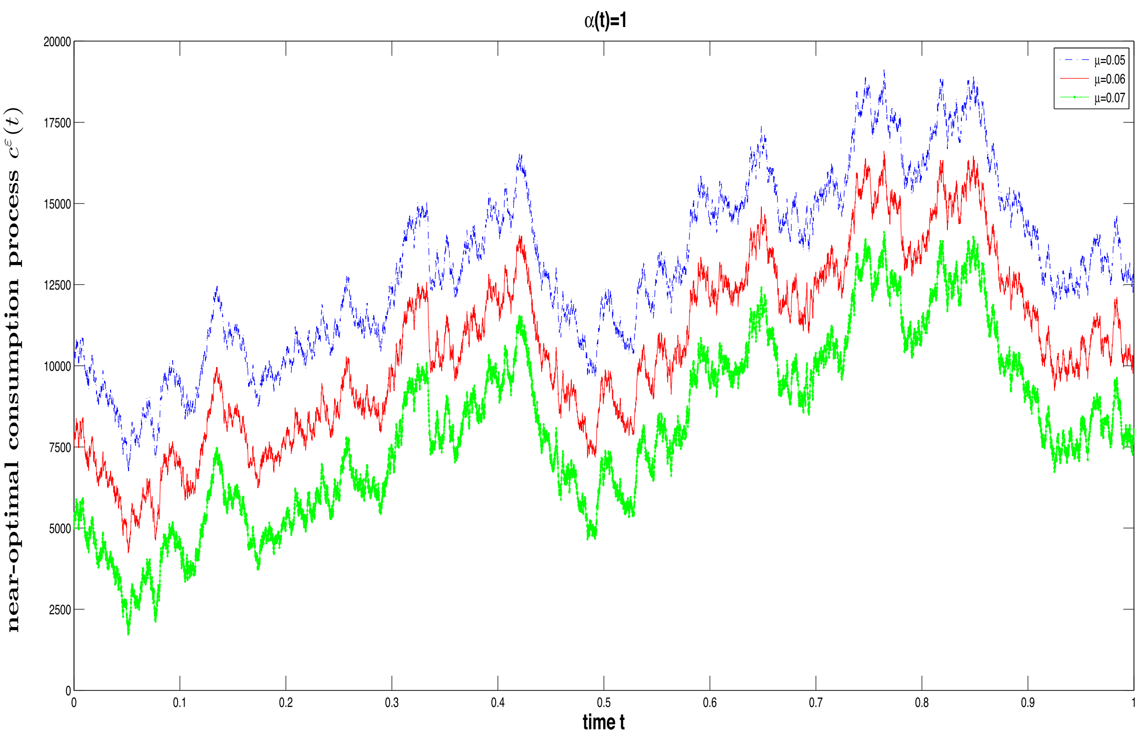


FIGURE 4. Relationship between $c^{\varepsilon}(t)$ and $\mu(t)$ in a bull market.

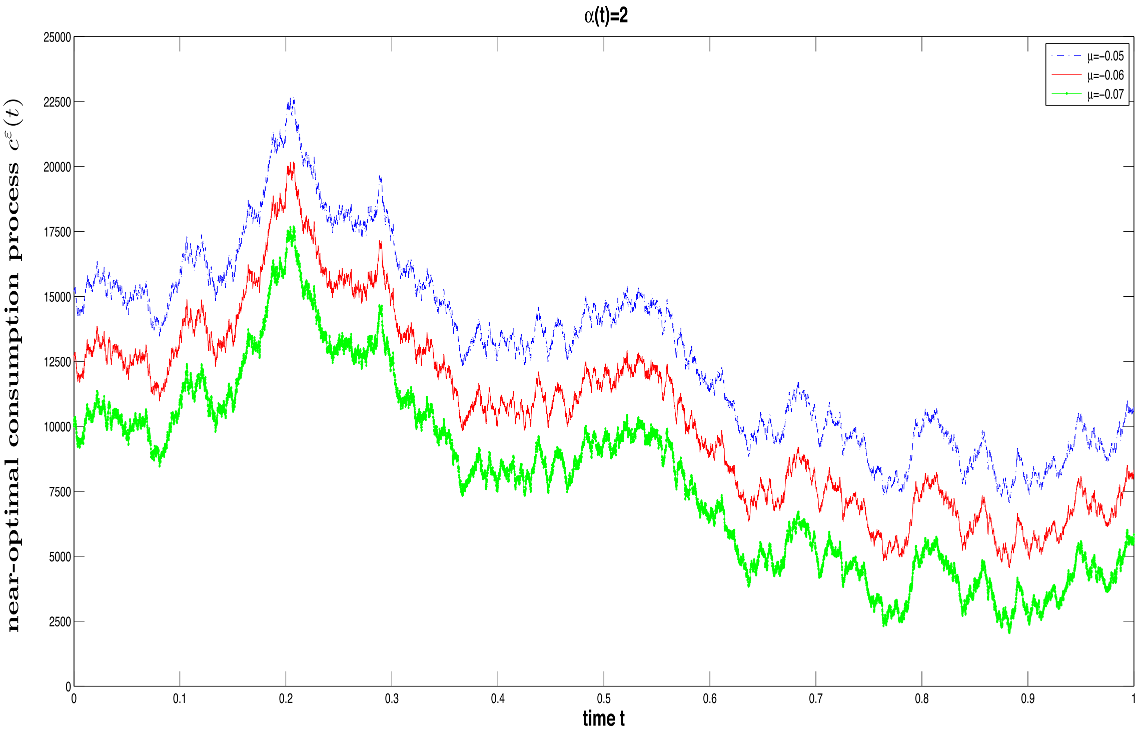


FIGURE 5. Relationship between $c^{\varepsilon}(t)$ and $\mu(t)$ in a bear market.

risk asset $\mu(t)$ in bull and bear markets are presented.

**Relationship between $c^{\varepsilon}(t)$ and $r(t)$:**

When $\alpha(t)=1$, we choose $\{r_1, r_2, r_3, A(\alpha(t)), B(\alpha(t)), C(\alpha(t)), \sigma(\alpha(t)), \mu(\alpha(t))\} = \{0.02, 0.03, 0.04, 0.5.0.4, 0.6, 0.05, 0.05\}$.

From Figure 2, we can see, the higher the $r(t)$ is, the lower the near-optimal consumption process $c^{\varepsilon}(t)$ is. This phenomenon is consistent with the practice: when the deposit rate rise, people prefer to put the money in the bank to make more profits instead of consuming. It reflects that as the stock price goes up, the consumer demand is growing, which is owing to people's good expectations about the economy.

When $\alpha(t)=2$, we choose $\{r_1, r_2, r_3, A(\alpha(t)), B(\alpha(t)), C(\alpha(t)), \sigma(\alpha(t)), \mu(\alpha(t))\} = \{0.02, 0.03, 0.04, -0.45, 0.5, -0.3, 0.05, -0.05\}$.

Figure 3 shows that the money for consumption cut down sharply as time goes on. As a matter of fact, in bear market, where share prices remain severely depressed, many people worry concerning the future. Hence the falling of stock price results in this reduction. In addition, the near-optimal consumption $c^{\varepsilon}(t)$ of higher risk-free rate is inferior to the lower which accounts for the same reason with Figure 2.

**Relationship between $c^{\varepsilon}(t)$ and $\mu(t)$:**

When $\alpha(t)=1$, we choose $\{\mu_1, \mu_2, \mu_3, A(\alpha(t)), B(\alpha(t)), C(\alpha(t)), \sigma(\alpha(t)), r(\alpha(t))\} = \{0.05, 0.06, 0.07, 0.25, 0.5, 0.4, 0.05, 0.02\}$.

Figure 4 presents the relationship between the quantity of near-optimal consumption process $c^{\varepsilon}(t)$ and the risk rate $\mu(t)$ in bull market. We find that the whole tendency is identical to Figure 2. In other words, the increasing risk rate will spur more people to stand stock instead of spending.

When $\alpha(t)=2$, we choose $\{\mu_1, \mu_2, \mu_3, A(\alpha(t)), B(\alpha(t)), C(\alpha(t)), \sigma(\alpha(t)), r(\alpha(t))\} = \{-0.05, -0.06, -0.07, 0.65, -0.35, 0.45, 0.05, 0.02\}$.

As Figure 5 shows, for every one percent drop in stocks, the quantity of near-optimal consumption strategy $c^{\varepsilon}(t)$ falls by a certain amount. The three curves depict that the near-optimal consumption strategy $c^{\varepsilon}(t)$ with three different risk rate: $\mu_1 = -0.05, \mu_2 = -0.06, \mu_3 = -0.07$, and we can see the lower risk rate is, the lower consumption is. It is obviously that the sluggish economy holding back people's desire to consume.

Up to now, we have discussed the relationship between $c^{\varepsilon}(t)$ and $r(t)$, $c^{\varepsilon}(t)$ and $\mu(t)$. From these figures, we can draw conclusions: In bull market, *i.e.* when the price of stock goes up, many people are over confident, and more optimistic about the financial market, therefore the extension of consumption is due to the good expectation about future. While in bear market, *i.e.* the decline in stock, it has a negative impact on consumption. The sluggish economy is a barrier to people's consumption. After careful observation, we notice that the overall upward curve has a local drop while the overall download curve has a local rise. The results have proven to be well matched with actual financial market, where there exist share sell-off in a bull market and also stock investment in a bear market.

## 6. Conclusions

In this paper, the near-optimality for a class of forward-backward regime-switching systems has been studied. The control domains have not been necessarily convex and the control variables have not entered forward diffusion term. Based on some new estimates for state and adjoint processes, the necessary conditions have been obtained by Ekeland's variational principle and a spike variational technique. Distinguished from the classical maximum principle for optimal control problems, the near-minimum condition has been proven to be of the Hamiltonian function in an integral sense. Under certain convexity conditions, the sufficient conditions have been derived. To illustrate the application of our theoretical results, an investment and consumption model under stochastic recursive utility has been considered. In the future, we hope to focus on the general case for the near optimal control problems of forward-backward regime-switching systems which means that the diffusion term contains control variables and explore the desired results.

*Acknowledgements.* The authors would like to thank the two anonymous referees for their insightful and constructive comments, which help us to improve the quality of this paper.

## References

[1] K. Bahlali, N. Khelfallah and B. Mezerdi, Necessary and sufficient conditions for near-optimality in stochastic control of FBSDEs. *Syst. Control Lett.* **58** (2009) 857–864.
[2] J.M. Bismut, An introductory approach to duality in optimal stochastic control. *SIAM Rev.* **20** (1978) 62–78.
[3] N.P.B. Bollen, Valuing options in regime-switching models. *J. Deriv.* **6** (1998) 38–50.
[4] J. Buffington and R.J. Elliott, American options with regime switching. *Theor. Appl. Finance* **5** (2002) 497–514.
[5] S. Crépey and A. Matoussi, Reflected and doubly reflected BSDEs with jumps: a priori estimates and comparison. *Ann. Appl. Probab.* **18** (2008) 2041–2069.
[6] D. Duffie and L.G. Epstein, Stochastic differential utility. *Econometrica* **60** (1992) 353–394.
[7] I. Ekeland, On the variational principle. *J. Math. Anal. Appl.* **47** (1974) 324–353.
[8] J.D. Hamilton, A new approach to the economic analysis of nonstationary time series and the business cycle. *Econometrica* **57** (1989) 357–384.
[9] Y. Hu and S. Peng, Solution of forward-backward stochastic differential equations. *Probab. Theory Relat. Field* **103** (1995) 273–283.
[10] J. Huang and J. Shi, Maximum principle for optimal control of fully coupled forward-backward stochastic differential delayed equations. *ESAIM: COCV* **18** (2012) 1073–1096.
[11] J. Huang, X. Li and G. Wang, Near-optimal control problems for linear forward-backward stochastic systems. *Automatica* **46** (2010) 397–404.
[12] N.E. Karoui, S. Peng and M.C. Quenez, Backward stochastic differential equations in finance. *Math. Finance* **7** (1997) 1–71.
[13] J. Ma and J. Yong, Forward-backward stochastic differential equations and their applications. Springer-Verlag, Berlin (1999).
[14] J. Ma, P. Protter, and J. Yong, Solving forward-backward stochastic differential equations explicitly − a four step scheme. *Probab. Theory Relat. Fields* **98** (1994) 339–359.
[15] G.B.Di. Masi, Y.M. Kabanov and W.J. Runggaldier, Mean-variance hedging of options on stocks with Markov volatilities. *Theory Prob. App.* **39** (1995) 172–182.
[16] E. Pardoux and S. Peng, Adapted solution of a backward stochastic differential equation. *Syst. Control Lett.* **14** (1990) 55–61.
[17] E. Pardoux and S. Tang, Forward-backward stochastic differential equations and quasilinear parabolic PDEs. *Probab. Theory Relat. Fields.* **114** (1999) 123–150.
[18] S. Peng, Backward stochastic differential equations and applications to optimal control. *Appl. Math. Optim.* **27** (1993) 125–144.

[19] S. Peng and Z. Wu, Fully coupled forward-backward stochastic differential equations and applications to optimal control. *SIAM J. Control Optim.* **37** (1999) 825–843.

[20] R. Tao and Z. Wu, Maximum principle for optimal control problems of forward-backward regime-switching system and applications. *Syst. Control Lett.* **61** (2012) 911–917.

[21] R. Tao, Z. Wu and Q. Zhang, BSDEs with regime switching: Weak convergence and applications. *J. Math. Anal. Appl.* **407** (2013) 97–111.

[22] Z. Wu, A general maximum principle for optimal control of forward–backward stochastic systems. *Automatica* **49** (2013) 1473–1480.

[23] J. Yong and X. Zhou, Stochastic controls: Hamiltonian systems and HJB equations. Springer-Verlag, New York (1999).

[24] Z. Yu, Infinite horizon jump-diffusion forward-backward stochastic differential equations and their application to backward linear-quadratic problems. *ESAIM: COCV* **23** (2017) 1331–1359.

[25] Q. Zhang, Stock trading: an optimal selling rule. *SIAM J. Control Optim.* **40** (2001) 64–87.

[26] Q. Zhang and G. Yin, Nearly-optimal asset allocation in hybrid stock investment models. *J. Optim. Theory Appl.* **121** (2004) 419–444.

[27] L. Zhang, J. Huang and X. Li, Necessary condition for near optimal control of linear forward-backward stochastic differential equations. *Int. J. Control* **88** (2015) 1594–1608.

[28] X. Zhou, Deterministic near-optimal control, Part I: necessary and sufficient conditions for near-optimality. *J. Optim. Theory Appl.* **85** (1995) 473–488.

[29] X. Zhou, Stochastic near-optimal controls: necessary and sufficient conditions for near-optimality. *SIAM J. Control Optim.* **36** (1998) 929–947.

[30] X. Zhou and G. Yin, Markowitz's mean-variance portfolio selection with regime switching: A continuous-time model. *SIAM J. Control Optim.* **42** (2003) 1466–1482.