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ON A QUASILINEAR ELLIPTIC PROBLEM INVOLVING THE 1-BIHARMONIC OPERATOR AND A STRAUSS TYPE COMPACTNESS RESULT*

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Abstract. In this paper we prove the compactness of the embeddings of the space of radially symmetric functions of $BL(\mathbb{R}^N)$ into some Lebesgue spaces. In order to do so we prove a regularity result for solutions of the Poisson equation with measure data in $\mathbb{R}^N$, as well as a version of the Radial Lemma of Strauss to the space $BL(\mathbb{R}^N)$. An application is presented involving a quasilinear elliptic problem of higher-order, where variational methods are used to find the solutions.

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1. Introduction

In the last years, problems involving the 1−laplacian operator, formally defined by

$$\Delta_1 u = \operatorname{div}\left(\frac{Du}{|Du|}\right),$$

caught the attention of so many specialists in Partial Differential Equations, mainly because of its applications in image restoration and in fracture mechanics. In addition, the fact that one is led to deal with problems involving the 1−laplacian operator in the space of functions of bounded variation $BV(\Omega)$ (which by the way, is a highly complicated space to deal with), brings a lot of mathematical interest to this sort of problem. There are plenty of works dealing with this highly singular operator, among which we could cite the very first ones [2, 3], as well as [4, 14, 21–24], where an approach based on approximation by $p-$Laplacian problems have been carried out. In the monograph [5], as well as in some of the aforementioned papers, the authors give a sense to a solution of problems involving the 1−laplacian operator, which is based on a characterization of the subgradient of the

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norm in the space of functions of bounded variation. For an approach based purely on variational methods dealing with the problem itself we could cite [1, 16–18].

As far as image restoration is concerned, besides methods based on minimization of functionals involving the total variation of first order derivatives (which are related to problems involving the 1−laplacian operator), there are also methods based on the minimization of functionals involving the total variation of the laplacian, i.e., functionals defined in energy spaces where

$$\int_\Omega |\Delta u|$$

defines a norm. Indeed, in [26], the authors characterize the minimizers of the following functional

$$F_\Delta(u) = \alpha \int_\Omega |\Delta u| + \frac{1}{2}\int_\Omega (u-f)^2 \mathrm{d}x$$

and present some numerical results comparing the minimization of the functional F_Δ, with a similar one with $\int_\Omega |Du|$ playing the role of $\int_\Omega |\Delta u|$.

Related to the functional F_Δ, we have problems involving the 1−biharmonic operator

$$\Delta_1^2 u = \Delta\left(\frac{\Delta u}{|\Delta u|}\right),$$

which can be seen as the limit of the p−biharmonic operator

$$\Delta_p^2 u = \Delta\left(|\Delta u|^{p-2}\Delta u\right),$$

as $p \to 1^+$. Problems involving this operator have not been studied by so many researchers. In [27], Parini *et al.* show that

$$\Lambda_{1,1}(\Omega) = \inf_{u\in BL_0(\Omega)\setminus\{0\}} \frac{\int_\Omega |\Delta u|}{\|u\|_1}$$

is attained by a non-negative and superharmonic function v that belongs to the space $BL_0(\Omega) := \{u \in W_0^{1,1}(\Omega);\, \Delta u \in \mathcal{M}(\Omega)\}$, where $\mathcal{M}(\Omega)$ is the space of the Radon measures defined on Ω. Actually their result is even more complete, in the sense that it provides also information about the shape of the domain Ω that minimizes $\Lambda_{1,1}(\Omega)$. In [29], the same authors still deal with the 1-biharmonic operator since they study the following minimization problem

$$\Lambda_{1,1}^c(\Omega) = \inf_{u\in C_c^\infty(\Omega)\setminus\{0\}} \frac{\int_\Omega |\Delta u|}{\|u\|_1}.$$

It turns that since $C_c^\infty(\Omega)$ is not a dense subset of $BL_0(\Omega)$ in the topology of the norm, the minimizing problems above are in fact different. As in their first work [27], in [29] the authors also study the shape of the subset that minimizes the quantity $\Lambda_{1,1}^c(\Omega)$. In [28] these authors investigate some optimal constants of the Sobolev embeddings of some spaces of functions which are associated to the 1-biharmonic operator. In [9], the authors study nonlinear problems involving the 1-biharmonic operator, with different hypothesis on the nonlinearity.

In [18] the authors prove the compactness of the embeddings of the set of radially symmetric bounded variation functions of $\mathbb{R}^N$, $BV_{rad}(\mathbb{R}^N)$, into some Lebesgue spaces and present an application to study a problem involving the 1-laplacian operator in $\mathbb{R}^N$. In this paper, we exhibit a higher order counterpart of this result, showing that the same behavior appears when dealing with functions in $BL_{rad}(\mathbb{R}^N)$, the space of radially symmetric functions in $W^{1,1}(\mathbb{R}^N)$, such that their laplacian is a Radon measure.

The first theorem we prove in this paper is a compactness result which resembles that one proved in [18]. However here we need a more accurate version of the Radial Lemma of Strauss [33]. Indeed, since in this paper we deal with second order derivatives, in order to show the compactness of the embeddings of $BL_{rad}(\mathbb{R}^N)$ into some Lebesgue spaces, we need to control the growth of the gradient of the functions of $BL_{rad}(\mathbb{R}^N)$ at the infinity. Actually, we prove the following result:

Theorem 1.1. *Let* $BL_{rad}(\mathbb{R}^N) = \Big\{u \in BL(\mathbb{R}^N);\, u(x) = u(|x|)\Big\}$, *where* $N \geqslant 3$. *Then the embedding* $BL_{rad}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ *is compact for every* $r \in (1, N/(N-1))$.

In the last statement, the hypothesis $N \geqslant 3$ is justified by a technical reason, since the version of the Strauss' Radial Lemma we succeed proving use this fact in a crucial way. Also, the presence of the threshold $N/(N-1)$ rather than $N/(N-2)$ may be surprising and somewhat frustrating, but it can be explained by the fact that we are working with functions in $L^1(\mathbb{R}^N)$. Indeed, the proof of the last result follows by using a density result and working with smooth functions. However, when working with such smooth functions, it turns out that they belong to the space $W^{2,1}_\Delta(\mathbb{R}^N) = \{u \in W^{1,1}(\mathbb{R}^N);\, \Delta u \in L^1(\mathbb{R}^N)\}$, rather than to $W^{2,1}(\mathbb{R}^N)$. This gives rise to such difficulty, since up to our knowledge there is no embedding results of $W^{2,1}_\Delta(\mathbb{R}^N)$ into $L^q(\mathbb{R}^N)$ for $1 \leqslant q < N/(N-2)$, as it has for $W^{2,1}(\mathbb{R}^N)$. The interested reader can see more details about spaces like $W^{2,1}_\Delta(\mathbb{R}^N)$ in [10, 27–29] and a very interesting result in [25], where the author proves that a version of this space to bounded domains is compactly imbedded into $W^{s,1}(\Omega)$ for every $s < 2$, but not for $s = 2$.

The second part of this research deals with an application of our compactness result, in finding solutions of a quasilinear elliptic problem involving the 1-biharmonic and the 1-laplacian operators. More specifically, we study:

$$\begin{cases} \Delta_1^2 u - \Delta_1 u + \dfrac{u}{|u|} &= f(u) \quad \text{in } \mathbb{R}^N, \\ & u \in BL(\mathbb{R}^N), \end{cases} \tag{$\mathcal{P}$}$$

where the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

($\mathcal{F}_1$) f is continuous in $\mathbb{R}$;
($\mathcal{F}_2$) $f(\zeta) = o(1)$ as $\zeta \to 0$;
($\mathcal{F}_3$) There exist constants $\mathfrak{c}_1, \mathfrak{c}_2 > 0$ and $p \in (1, N/(N-1))$ such that

$$|f(\zeta)| \leqslant \mathfrak{c}_1 + \mathfrak{c}_2|\zeta|^{p-1};$$

($\mathcal{F}_4$) There exists $\kappa > 1$ such that

$$0 < \kappa F(\zeta) \leqslant f(\zeta)\zeta, \quad \text{for } \zeta \neq 0,$$

where $F(\zeta) = \int_0^\zeta f(\sigma)d\sigma$;
($\mathcal{F}_5$) f is increasing.

First of all we should say what we mean by a solution of ($\mathcal{P}$). We say that $u \in BL(\mathbb{R}^N)$ is a solution of ($\mathcal{P}$) if there exist $w \in L^\infty(\mathbb{R}^N)$, $\mathbf{z} \in L^\infty(\mathbb{R}^N, \mathbb{R}^N)$ and $\gamma \in L^\infty(\mathbb{R}^N)$, such that, $\|w\|_\infty, \|\mathbf{z}\|_\infty, \|\gamma\|_\infty \leqslant 1$ and

$$\left\{ \begin{aligned} \Delta w - \operatorname{div} \mathbf{z} + \gamma &= f(u) \quad \text{in } \mathcal{D}'(\mathbb{R}^N), \\ \int_{\mathbb{R}^N} w \Delta u &= \int_{\mathbb{R}^N} |\Delta u|, \\ \int_{\mathbb{R}^N} \mathbf{z} \cdot \nabla u \, \mathrm{d}x &= \int_{\mathbb{R}^N} |\nabla u| \, \mathrm{d}x, \\ \gamma . u &= |u| \quad \text{a.e. in } \mathbb{R}^N. \end{aligned} \right. \tag{1.1}$$

The second equation says to us that the function w plays the role of the indefinite quotient $\dfrac{\Delta u}{|\Delta u|}$, while the third and the fourth one mean that $\mathbf{z}$ and γ play the role of $\dfrac{\nabla u}{|\nabla u|}$ and $\dfrac{u}{|u|}$, respectively.

We prove the existence of a nontrivial solution for ($\mathcal{P}$) by using a variational approach and a suitable version of the Mountain Pass Theorem (see [17], Thm. 4.1). In order to prove that the Palais-Smale sequence in fact converges to a nontrivial solution of ($\mathcal{P}$), the compactness result is going to be crucial. In fact we prove the following result:

Theorem 1.2. *Assume* $(\mathcal{F}_1) - (\mathcal{F}_5)$ *to hold. Then there exists a nontrivial solution* $u \in BL(\mathbb{R}^N)$ *of* ($\mathcal{P}$)*. Moreover, this solution has the lowest energy level among all the radially symmetric and nontrivial ones.*

The last result can be view as complementary with respect to some results that has been studied before. In [18], Figueiredo and Pimenta studied the above problem without the 1-biharmonic operator, obtaining the same conclusion. While in [9], Barile and Pimenta treated the pure 1-biharmonic problem in a bounded domain. The result is closely related to the remarkable result by Strauss [33].

Beyond the last two results, another contribution of the present paper is given in Proposition 2.1, where the authors prove a regularity result of solutions of the Poisson equation in $\mathbb{R}^N$ with measure datum, which we have not been able to find in the literature.

In Section 2 we study deeper the space $BL(\mathbb{R}^N)$, where, among the results, there is a regularity result about solutions of the Poisson equation in $\mathbb{R}^N$ with measure datum, that can be interesting by itself. In Section 3 we prove the compactness result and in Section 4 we present the application of it to the problem ($\mathcal{P}$). In the Appendix we present the details of the argument that imply that the function we find in Section 4 is in fact a solution of ($\mathcal{P}$).

2. The energy space

The space we are going to deal with throughout this paper is given by

$$BL(\mathbb{R}^N) := \{u \in W^{1,1}(\mathbb{R}^N) : \Delta u \in \mathcal{M}(\mathbb{R}^N)\},$$

where $\mathcal{M}(\mathbb{R}^N)$ is the set of Radon measures in $\mathbb{R}^N$. By the Riesz Representation Theorem, it follows that $u \in W^{1,1}(\mathbb{R}^N)$ belongs to $BL(\mathbb{R}^N)$ if and only if

$$\int_{\mathbb{R}^N} |\Delta u| < +\infty,$$

where

$$\int_{\mathbb{R}^N} |\Delta u| := \sup \left\{ \int_{\mathbb{R}^N} u \Delta \varphi \, \mathrm{d}x : \varphi \in C_c^\infty(\mathbb{R}^N), \|\varphi\|_\infty \leqslant 1 \right\}.$$

Let us consider the space $BL(\mathbb{R}^N)$ endowed with the following norm

$$\|u\| = \|u\|_1 + \|\nabla u\|_1 + \int_{\mathbb{R}^N} |\Delta u|.$$

Before we prove the lower semicontinuity of the norm in $BL(\mathbb{R}^N)$ with respect to the topology of $L^q(\mathbb{R}^N)$, for $q \in [1, N/(N-1))$, let us prove a result about the regularity of the solutions of Poisson equations in $\mathbb{R}^N$ with measure datum.

Proposition 2.1. *Let $u \in L^1(\mathbb{R}^N)$ be such that $\Delta u \in \mathcal{M}(\mathbb{R}^N)$, then $u \in BL(\mathbb{R}^N)$ (i.e., $|\nabla u| \in L^1(\mathbb{R}^N)$) and there exists $C = C(N) > 0$ such that*

$$\int_{\mathbb{R}^N} |\nabla u| \mathrm{d}x \leqslant C \left(\int_{\mathbb{R}^N} |\Delta u| + \int_{\mathbb{R}^N} |u| \mathrm{d}x \right).$$

Proof. First of all let us prove that for all $y \in \mathbb{R}^N$, and $0 < R_1 < R_2$

$$\int_{B_{R_1}(y)} |\nabla u| \mathrm{d}x \leqslant C \left(\int_{B_{R_2}(y)} |\Delta u| + \int_{B_{R_2}(y)} |u| \mathrm{d}x \right), \tag{2.1}$$

where $C > 0$ does not depend on y nor on u. By density arguments it is enough to prove it for $C^\infty(\mathbb{R}^N)$ functions (see Prop. 2.4, whose proof is completely independent of this result).

Let us prove that if $R \in \left(\frac{R_1+R_2}{2}, R_2\right)$ and v is a solution of

$$\begin{cases} \Delta v &= 0 \quad \text{in } B_R(y), \\ v &= h \quad \text{on } \partial B_R(y), \end{cases} \tag{2.2}$$

for some continuous function h, then

$$\int_{B_{R_1}(y)} |\nabla v| \mathrm{d}x \leqslant C \int_{\partial B_R(y)} |h| d\mathcal{H}^{N-1}, \tag{2.3}$$

where $C = C(N, R_1, R_2)$. In fact, let us suppose without lack of generality that $y = 0$. By the Representation Formula, it follows that for all $x \in B_R$,

$$v(x) = \int_{\partial B_R} K_R(x, z) h(z) d\mathcal{H}^{N-1},$$

where K_R is the Poisson Kernel given by

$$K_R(x, z) = \frac{R^2 - |x|^2}{N\omega_N R |x - z|^N},$$

for $x \in B_R$ and $z \in \partial B_R$. Then, for all $x \in B_{R_1}$,

$$\begin{aligned} |\nabla v(x)| &\leqslant \int_{\partial B_R} |\nabla_x K_R(x, z)| |h(z)| d\mathcal{H}^{N-1} \\ &\leqslant C(N, R) \int_{\partial B_R} \left(\frac{|x|}{|x - z|^N} + \frac{(R^2 - |x|^2)}{|x - z|^{N+1}} \right) |h| d\mathcal{H}^{N-1}. \end{aligned} \tag{2.4}$$

Hence, it is easy to see that (2.4) implies (2.3), once we observe that for $x \in B_{R_1}$ and $z \in \partial B_R$, it follows that

$$\frac{R_2 - R_1}{2} \leqslant |x - z|.$$

Now, for $R \in \left(\frac{R_1+R_2}{2}, R_2\right)$, let us take v solution of (2.2), with $h = u$. Note that $(u - v)$ vanishes on $\partial B_R(y)$ and is a solution of the following Dirichlet problem in $B_R(y)$,

$$\begin{cases} \Delta w &= \Delta u \quad \text{in } B_R(y), \\ w &= 0 \quad \text{on } \partial B_R(y), \end{cases}$$

Then, from elliptic estimates (see for instance [30], Prop. 5.1), it follows that

$$\begin{aligned} \int_{B_R(y)} |\nabla(u - v)| \mathrm{d}x &\leqslant C(N) \int_{B_R(y)} |\Delta u| \mathrm{d}x \\ &\leqslant C(N) \int_{B_{R_2}(y)} |\Delta u| \mathrm{d}x, \end{aligned} \tag{2.5}$$

where the constant $C(N) > 0$ depends on R and N, but not on y.

Hence, from (2.3)

$$\begin{aligned} \int_{B_{R_1}(y)} |\nabla u| \mathrm{d}x &\leqslant \int_{B_{R_1}(y)} |\nabla(u - v)| \mathrm{d}x + \int_{B_{R_1}(y)} |\nabla v| \mathrm{d}x \\ &\leqslant \int_{B_R(y)} |\nabla(u - v)| \mathrm{d}x + C \int_{\partial B_R(y)} |u| \mathrm{d}\mathcal{H}^{N-1} \\ &\leqslant C \int_{B_{R_2}(y)} |\Delta u| \mathrm{d}x + C \int_{\partial B_R(y)} |u| \mathrm{d}\mathcal{H}^{N-1}. \end{aligned}$$

Integrating the last expression for $R \in ((R_1 + R_2)/2, R_2)$, it follows that

$$\begin{aligned} \int_{B_{R_1}(y)} |\nabla u| \mathrm{d}x &\leqslant C \int_{B_{R_2}(y)} |\Delta u| \mathrm{d}x + C \int_{B_{R_2}(y) \setminus B_{((R_1+R_2)/2)}(y)} |u| \mathrm{d}x \\ &\leqslant C \left(\int_{B_{R_2}(y)} |\Delta u| \mathrm{d}x + \int_{B_{R_2}(y)} |u| \mathrm{d}x \right), \end{aligned}$$

what proves (2.1).

Now, in order to give a global estimate to the gradient of u, let us consider a sequence $(y_n) \subset \mathbb{Z}^N$ and $0 < R_1 < R_2$ in such a way that both $\{B_{R_1}(y_n)\}_{n \in \mathbb{N}}$ and $\{B_{R_2}(y_n)\}_{n \in \mathbb{N}}$ cover $\mathbb{R}^N$ (*i.e.*, $\sqrt{N} < R_1 < R_2$), where each $x \in \mathbb{R}^N$ belongs to at most $C(N)$ elements of the cover $\{B_{R_2}(y_n)\}_{n \in \mathbb{N}}$, where $C(N)$ depends just on R_2 and N. Then, from (2.1),

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla u| \mathrm{d}x &\leqslant \sum_{n=1}^{+\infty} \int_{B_{R_1}(y_n)} |\nabla u| \mathrm{d}x \\ &\leqslant C \sum_{n=1}^{+\infty} \left(\int_{B_{R_2}(y_n)} |\Delta u| + \int_{B_{R_2}(y_n)} |u| \mathrm{d}x \right) \end{aligned}$$

$$= C \sum_{n=1}^{+\infty} \left(\int_{\mathbb{R}^N} \chi_{B_{R_2}(y_n)} |\Delta u| + \int_{\mathbb{R}^N} \chi_{B_{R_2}(y_n)} |u| \mathrm{d}x \right)$$
$$\leqslant CC(N) \left(\int_{\mathbb{R}^N} |\Delta u| + \int_{\mathbb{R}^N} |u| \mathrm{d}x \right).$$

□

The next result is crucial to find critical points of functional defined in $BL(\mathbb{R}^N)$.

Lemma 2.2. *Let $(u_n) \subset BL(\mathbb{R}^N)$ be a bounded sequence such that $u_n \to u$ in $L^q(\mathbb{R}^N)$, for some $q \in [1, N/(N-1))$. Then $u \in BL(\mathbb{R}^N)$ and*

$$\int_{\mathbb{R}^N} |\Delta u| \leqslant \liminf_{n\to\infty} \int_{\mathbb{R}^N} |\Delta u_n|. \tag{2.6}$$

Proof. Note that, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$ with $\|\varphi\|_\infty \leqslant 1$ we obtain

$$\int_{\mathbb{R}^N} u\, \Delta\varphi \,\mathrm{d}x = \lim_{n\to+\infty} \int_{\mathbb{R}^N} u_n\, \Delta\varphi\,\mathrm{d}x$$
$$= \liminf_{n\to+\infty} \int_{\mathbb{R}^N} u_n\, \Delta\varphi\,\mathrm{d}x \leqslant \liminf_{n\to\infty} \int_{\mathbb{R}^N} |\Delta u_n|.$$

Then (2.6) follows by taking the supremum over all such φ. In order to finish, note that $u \in BL(\mathbb{R}^N)$ by Proposition 2.1. □

By using the last result it is possible to show that in fact $(BL(\mathbb{R}^N), \|\cdot\|)$ is a Banach space. Moreover, as we are going to see, the space of smooth functions is not dense in $BL(\mathbb{R}^N)$ with respect to the topology of the norm. However, in [27], it is defined a sense of convergence, called strict convergence, with respect to which the space of smooth functions will be dense.

Definition 2.3. We say that a sequence $(u_n) \subset BL(\mathbb{R}^N)$ converges strictly to $u \in BL(\mathbb{R}^N)$ if both of the following conditions are satisfied

– $u_n \to u$ in $W^{1,1}(\mathbb{R}^N)$, as $n \to +\infty$,
– $\int_{\mathbb{R}^N} |\Delta u_n| \to \int_{\mathbb{R}^N} |\Delta u|$, as $n \to +\infty$.

Now we establish an important result about $BL(\mathbb{R}^N)$ spaces, namely, a version of the famous Meyers-Serrin Theorem. The proof of it can be done as in [9].

Proposition 2.4. *$C^\infty(\mathbb{R}^N) \cap BL(\mathbb{R}^N)$ is dense in $BL(\mathbb{R}^N)$ with respect to the strict convergence.*

In fact, something even stronger about the density of smooth functions in $BL(\mathbb{R}^N)$ can be proved.

Proposition 2.5. *$C_c^\infty(\mathbb{R}^N)$ is dense in $BL(\mathbb{R}^N)$ with respect to the strict convergence.*

Proof. Due to Proposition 2.4 it is sufficient to prove that $C_c^\infty(\mathbb{R}^N)$ is dense in $C^\infty(\mathbb{R}^N) \cap BL(\mathbb{R}^N)$, with respect to the strict convergence.

Let $u \in C^\infty(\mathbb{R}^N) \cap BL(\mathbb{R}^N)$. Let $\phi \in C_c^\infty(\mathbb{R}^N)$ be a cut-off function such that $\phi \equiv 1$ in $B_1(0)$, $\phi \equiv 0$ in $\mathbb{R}^N \backslash B_2(0)$ and $0 \leqslant \phi \leqslant 1$. For $m \in \mathbb{N}$, let us define $\phi_m(x) = \phi\left(\frac{x}{m}\right)$ and $u_m(x) = u(x)\phi\left(\frac{x}{m}\right)$. Then $u_m \in C_c^\infty(\mathbb{R}^N)$, $u_m \equiv u$ in $B_m(0)$ and $u_m \equiv 0$ in $\mathbb{R}^N \backslash B_{2m}(0)$.

It is easy to see that

$$\|u_m - u\|_{W^{1,1}(\mathbb{R}^N)} = o_n(1). \tag{2.7}$$

On the other hand,

$$\begin{aligned}\int_{\mathbb{R}^N} |\Delta u_m| &= \int_{\mathbb{R}^N} |\Delta(u\phi_m)| \\ &\leqslant \int_{\mathbb{R}^N} \left|\phi\Big(\frac{x}{m}\Big)\right| |\Delta u| + \frac{2}{m}\int_{\mathbb{R}^N} |\nabla u| \left|\nabla\phi\Big(\frac{x}{m}\Big)\right| \mathrm{d}x + \frac{1}{m^2}\int_{\mathbb{R}^N} |u| \left|\Delta\phi\Big(\frac{x}{m}\Big)\right| \mathrm{d}x.\end{aligned}$$

Then

$$\begin{aligned}\left|\int_{\mathbb{R}^N} |\Delta u_m|\,\mathrm{d}x - \int_{\mathbb{R}^N} |\Delta u|\right| &\leqslant \int_{\mathbb{R}^N} \left(\left|\phi\Big(\frac{x}{m}\Big)\right| - 1\right)\left||\Delta u|\,\mathrm{d}x + \frac{2}{m}\int_{\mathbb{R}^N} |\nabla u|\left|\nabla\phi\Big(\frac{x}{m}\Big)\right| \mathrm{d}x\right. \\ &\quad + \frac{1}{m^2}\int_{\mathbb{R}^N} |u|\left|\Delta\phi\Big(\frac{x}{m}\Big)\right| \mathrm{d}x \\ &= o_m(1)\end{aligned}$$

by the Lebesgue Dominated Convergence Theorem. This completes the proof. □

Now we establish the Sobolev embeddings.

Proposition 2.6. *The embedding $BL(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ is continuous for all $q \in \left[1, \frac{N}{N-1}\right]$.*

Proof. The proof follows just observing that $BL(\mathbb{R}^N)$ is continuously embedded into $W^{1,1}(\mathbb{R}^N)$ and recalling the embeddings of the later into Lebesgue spaces. □

Now we investigate the compactness of the embeddings of these spaces, as far as bounded domains are considered. For a bounded domain $\Omega \subset \mathbb{R}^N$ with Lipschitz boundary, let us denote

$$W^{2,1}_{\Delta}(\Omega) = \{u \in W^{1,1}_0(\Omega);\ \Delta u \in L^1(\Omega)\},$$

$$BL_0(\Omega) = \{u \in W^{1,1}_0(\Omega);\ \Delta u \in \mathcal{M}(\Omega)\},$$

and

$$BL(\Omega) = \{u \in W^{1,1}(\Omega);\ \Delta u \in \mathcal{M}(\Omega)\}.$$

From [11], it is known that the following embedding is continuous

$$W^{2,1}_{\Delta}(\Omega) \hookrightarrow L^q(\Omega) \quad \text{for all } 1 \leqslant q < N/(N-2). \tag{2.8}$$

Lemma 2.7. *The following embedding is compact*

$$BL_0(\Omega) \hookrightarrow L^q(\Omega) \quad \textit{for all } 1 \leqslant q < N/(N-1). \tag{2.9}$$

Proof. Let $(u_n) \subset W^{2,1}_\Delta(\Omega)$ be a bounded sequence. Note that (u_n) is also a bounded sequence in $W^{1,1}_0(\Omega)$. Then from the compactness of the embeddings of the later space in $L^q(\Omega)$, it follows that (u_n) converges in $L^q(\Omega)$ up to a subsequence. The general result now follows from the density of $C_c^\infty(\Omega)$ in $BL_0(\Omega)$.

□

Finally, let us prove another compactness result, which now does not consider any information of the functions on the boundary.

Lemma 2.8. *The following embedding is compact*

$$BL(\Omega) \hookrightarrow L^q(\Omega) \quad \text{for all } 1 \leqslant q < N/(N-1). \tag{2.10}$$

Proof. Let us denote Ω_ϵ an $\epsilon-$neighborhood of Ω, where $\epsilon > 0$. Moreover, let $\varphi_\epsilon \in C_c^\infty(\Omega_\epsilon)$ be a function such that $\varphi_\epsilon \equiv 1$ in Ω and $0 \leqslant \varphi_\epsilon \leqslant 1$ in Ω_ϵ. Then the operator $i_2 : W^{1,1}(\Omega) \to W^{1,1}_0(\Omega_\epsilon)$ given by $i_2(u) = \varphi_\epsilon \overline{u}$ is continuous, where by $\overline{u}$ we denote the image of u by the standard extension operator from $W^{1,1}(\Omega)$ into $W^{1,1}(\mathbb{R}^N)$.

Then the result follows by observing that the embedding $i : BL(\Omega) \to L^q(\Omega)$ can be written as

$$BL(\Omega) \xrightarrow{i_1} W^{1,1}(\Omega) \xrightarrow{i_2} W^{1,1}_0(\Omega_\epsilon) \xrightarrow{i_3} L^q(\Omega_\epsilon) \xrightarrow{i_4} L^q(\Omega)$$

where i_1 is just the identity operator, i_3 the usual Sobolev embedding operator, while i_4 is the standard restriction operator, all of them being continuous.

□

3. The compactness result

In this section, we prove Theorem 1.1. In order to do so, let us first denote by $C^\infty_{c,rad}(\mathbb{R}^N)$ the set of $C_c^\infty(\mathbb{R}^N)$ functions which are radially symmetric.

Arguing as in Proposition 2.5 and observing that the convolution of radially symmetric functions gives rise to a radially symmetric function, we obtain the following result.

Lemma 3.1. $C^\infty_{c,rad}(\mathbb{R}^N)$ *is dense in* $BL_{rad}(\mathbb{R}^N)$*, with respect to the strict convergence.*

The next result is crucial to prove Theorem 1.1 and it consists in a version of the Radial Lemma of Strauss to the space $BL(\mathbb{R}^N)$.

Lemma 3.2. *(Radial Lemma in* $BL(\mathbb{R}^N)$*). Let* $N \geqslant 3$ *and* $u \in BL_{rad}(\mathbb{R}^N)$*. Then*

$$|u(x)| \leqslant \frac{\omega_N^{-1}}{N-2} \frac{1}{|x|^{N-2}} \|u\| \tag{3.1}$$

and

$$|\nabla u(x)| \leqslant \omega_N^{-1} \frac{1}{|x|^{N-1}} \|u\|, \tag{3.2}$$

for a.e. $x \in \mathbb{R}^N$.

Proof. Let $u \in BL_{rad}(\mathbb{R}^N)$ and $(u_n)_{n\in\mathbb{N}} \subset C^\infty_{c,rad}(\mathbb{R}^N)$ a sequence such that

$$u_n \to u \in W^{1,1}(\mathbb{R}^N) \tag{3.3}$$

and

$$\int_{\mathbb{R}^N} |\Delta u_n| \,\mathrm{d}x \to \int_{\mathbb{R}^N} |\Delta u|, \tag{3.4}$$

as $n \to \infty$. We will use the notation $\varphi(x) = \varphi(|x|) = \varphi(r)$ when φ is a radial function in $\mathbb{R}^N$. Note that

$$\Delta u_n = u_n'' + \frac{N-1}{\rho} u_n'.$$

Thus we have that

$$\rho^{N-1} \Delta u_n(\rho) = \frac{\mathrm{d}}{\mathrm{d}\rho}(\rho^{N-1} u_n'), \tag{3.5}$$

for all $\rho > 0$.

Integrating (3.5) over $(0, r)$ we have that

$$\begin{aligned} r^{N-1}|u_n'(r)| &\leqslant \int_0^r \rho^{N-1} |(\Delta u_n)(\rho)| \,\mathrm{d}\rho \\ &\leqslant \frac{1}{\omega_N} \int_{\mathbb{R}^N} |\Delta u_n| \,\mathrm{d}x. \end{aligned} \tag{3.6}$$

Hence

$$|u_n'(r)| \leqslant \omega_N^{-1} \frac{1}{r^{N-1}} \int_{\mathbb{R}^N} |\Delta u_n| \,\mathrm{d}x. \tag{3.7}$$

By (3.3), (3.4) and (3.7) we get

$$|\nabla u(x)| \leqslant \omega_N^{-1} \frac{1}{|x|^{N-1}} \int_{\mathbb{R}^N} |\Delta u|, \text{ a.e. in } \mathbb{R}^N. \tag{3.8}$$

By (3.7) we have that

$$\begin{aligned} |u_n(r)| &\leqslant \int_r^\infty |u_n'(t)| \,\mathrm{d}t \\ &\leqslant \int_r^\infty \omega_N^{-1} \frac{1}{t^{N-1}} \left(\int_{\mathbb{R}^N} |\Delta u_n| \,\mathrm{d}x \right) \mathrm{d}t \\ &\leqslant \frac{\omega_N^{-1}}{N-2} \frac{1}{r^{N-2}} \int_{\mathbb{R}^N} |\Delta u_n| \,\mathrm{d}x. \end{aligned} \tag{3.9}$$

Again using (3.3), (3.4) and by (3.9) we have that

$$|u(x)| \leqslant \frac{\omega_N^{-1}}{N-2} \frac{1}{|x|^{N-2}} \int_{\mathbb{R}^N} |\Delta u|, \text{ a.e. in } \mathbb{R}^N. \tag{3.10}$$

□

Now let us present the proof of the main compactness result.

Proof of Theorem 1.1. Let $(u_n) \subset BL_{rad}(\mathbb{R}^N)$ be a bounded sequence and let $\mathfrak{K} > 0$ be such that

$$\|u_n\| \leqslant \mathfrak{K}, \text{ for all } n \in \mathbb{N}.$$

By Lemma 3.2 it follows that, for all $n \in \mathbb{N}$,

$$|u_n(x)| \leqslant \frac{\mathfrak{K}}{|x|^{N-2}} \text{ and } |\nabla u_n(x)| \leqslant \frac{\mathfrak{K}}{|x|^{N-1}} \quad \text{a.e. in } \mathbb{R}^N \backslash \{0\}. \tag{3.11}$$

We have that $\lim_{|x|\to+\infty} |u_n(x)| = 0$ and $\lim_{|x|\to+\infty} |\nabla u_n(x)| = 0$ uniformly with respect to n. Thus, since $r > 1$, given $\varepsilon > 0$, there exists $R > 0$ such that, for all $n \in \mathbb{N}$,

$$|u_n(x)|^r \leqslant \frac{\varepsilon}{2\mathfrak{K}}|u_n(x)|$$

for all $x \in B_R(0)^c$.

This implies that

$$\int_{B_R(0)^c} |u_n|^r \mathrm{d}x \leqslant \frac{\varepsilon}{2\mathfrak{K}} \int_{B_R(0)^c} |u_n| \mathrm{d}x \leqslant \frac{\varepsilon}{2\mathfrak{K}}\|u_n\| \leqslant \frac{\varepsilon}{2} \tag{3.12}$$

for all $n \in \mathbb{N}$.

Since (u_n) is bounded in $BL_{rad}(\mathbb{R}^N)$ and, by Lemma 2.8 the embedding $BL_{loc}(\mathbb{R}^N) \hookrightarrow L^r_{loc}(\mathbb{R}^N)$ is compact for $r \in (1, N/(N-1))$, there exists $u \in L^r_{loc}(\mathbb{R}^N)$ such that $u_n \to u$ in $L^r_{loc}(\mathbb{R}^N)$ and $u_n \to u$ a.e. in $\mathbb{R}^N$ (in particular u is radially symmetric). Moreover there exists $n_0 \in \mathbb{N}$ such that

$$\int_{B_R(0)} |u_n - u|^r \, \mathrm{d}x < \frac{\varepsilon}{2}, \quad \forall n \geqslant n_0. \tag{3.13}$$

Now let us prove that $u \in BL_{rad}(\mathbb{R}^N)$. Note that, from Fatou Lemma, it follows that $u \in L^1(\mathbb{R}^N)$. Moreover, for a given $\varrho > 0$, from the semicontinuity of the norm in $BL(B_\varrho(0))$ with respect to the $L^r(B_\varrho(0))$ convergence, we have that

$$\int_{B_\varrho(0)} |\Delta u| \leqslant \liminf_{n\to+\infty} \int_{B_\varrho(0)} |\Delta u_n| \leqslant \liminf_{n\to+\infty} \|u_n\| \leqslant \mathfrak{K}, \tag{3.14}$$

where $\mathfrak{K}$ does not depend on n or ϱ. Since (3.14) holds for every $\varrho > 0$, then $\Delta u \in \mathcal{M}(\mathbb{R}^N)$. Hence, from Proposition 2.1, it follows that $u \in BL_{rad}(\mathbb{R}^N)$. Hence, from (3.12), (3.13) and from the fact that $u \in L^r(\mathbb{R}^N)$ (which follows by interpolation), it follows that $u_n \to u$ strongly in $L^r(\mathbb{R}^N)$ as $n \to \infty$. □

4. Application

In this section we present an application of Theorem 1.1. More precisely, we prove the existence of a radial ground state solution to the following elliptic equation

$$\begin{cases} \Delta_1^2 u - \Delta_1 u + \dfrac{u}{|u|} &= f(u) \quad \text{in } \mathbb{R}^N, \\ u \in BL(\mathbb{R}^N), \end{cases} \tag{4.1}$$

where the nonlinearity f satisfies the conditions $(\mathcal{F}_1) - (\mathcal{F}_5)$.

Since (4.1) is variational, let us define an energy functional in $BL(\mathbb{R}^N)$ whose formal Euler-Lagrange equation is given by (4.1). Let $\Psi : BL(\mathbb{R}^N) \to \mathbb{R}$ be defined by

$$\Psi(u) = \int_{\mathbb{R}^N} |\Delta u| + \int_{\mathbb{R}^N} |\nabla u| \,\mathrm{d}x + \int_{\mathbb{R}^N} |u| \,\mathrm{d}x - \int_{\mathbb{R}^N} F(u) \,\mathrm{d}x$$

for every $u \in BL(\mathbb{R}^N)$. To simplify notations we will define $\Psi_0, \Psi_F : BL(\mathbb{R}^N) \to \mathbb{R}$, given by

$$\Psi_0(u) = \int_{\mathbb{R}^N} |\Delta u| + \int_{\mathbb{R}^N} |\nabla u| \,\mathrm{d}x + \int_{\mathbb{R}^N} |u| \,\mathrm{d}x \quad \text{for all } u \in BL(\mathbb{R}^N) \tag{4.2}$$

and

$$\Psi_F(u) = \int_{\mathbb{R}^N} F(u) \,\mathrm{d}x \quad \text{for all } u \in BL(\mathbb{R}^N).$$

This way Ψ can be written as

$$\Psi(u) = \Psi_0(u) - \Psi_F(u),$$

for every $u \in BL(\mathbb{R}^N)$.

Note that by Lemma 2.2, Ψ_0 is lower semicontinuous with respect to the $L^r(\mathbb{R}^N)$ topology, for $r \in [1, N/(N-1)]$. Moreover, Ψ_0 is non-smooth, even though it allows some directional derivatives. More precisely, given $u \in BL(\mathbb{R}^N)$, by ([6], Thm. 2.5) (considering $f(x,p) = |p|$ and $\mu = |\Delta u|$), for all $v \in BL(\mathbb{R}^N)$ such that $(\Delta v)^s$ is absolutely continuous with respect to $(\Delta u)^s$, $(\Delta v)^a$ vanishes a.e. in the set where $(\Delta u)^a$ vanishes, ∇v vanishes a.e. in the set where ∇u vanishes and $v \equiv 0$, a.e. in the set where u vanishes, it follows that

$$\begin{aligned}\Psi_0'(u)v = &\int_{\mathbb{R}^N} \frac{(\Delta u)^a (\Delta v)^a}{|(\Delta u)^a|} \mathrm{d}x + \int_{\mathbb{R}^N} \frac{\Delta u}{|\Delta u|}(x) \frac{\Delta v}{|\Delta v|}(x) |(\Delta v)^s| + \int_{\mathbb{R}^N} \frac{\nabla u \cdot \nabla v}{|\nabla u|} \mathrm{d}x \\ &+ \int_{\mathbb{R}^N} \operatorname{sgn}(u) v \mathrm{d}x,\end{aligned} \tag{4.3}$$

where $\operatorname{sgn}(z) = 0$ if $z = 0$ and $\operatorname{sgn}(z) = z/|z|$ if $z \neq 0$.

Taking (4.3) into account, it follows that for all $u \in BL(\mathbb{R}^N)$,

$$\Psi_0'(u)u = \Psi_0(u). \tag{4.4}$$

Across this paper, whenever we use the word *subdifferential* and the symbol $\partial\Phi(u)$, where Φ is a locally Lipschitz functional defined in a Banach space, we mean the subdifferential of Φ at u, as defined by Clarke [13] and carefully studied by Chang in [12]. Since Ψ_0 is convex and Lipschitz continuous in $BL(\mathbb{R}^N)$ and $\Psi_F \in C^1(BL(\mathbb{R})^N, \mathbb{R})$, we say that a function $u \in BL(\mathbb{R}^N)$ is a solution of $(\mathcal{P})$ if $0 \in \partial\Psi_0(u) - \Psi_F'(u)$, *i.e.*, if $\Psi_F'(u) \in \partial\Psi_0(u)$. The convexity of Ψ_0 implies, in turn, that the last condition is equivalent to

$$\Psi_0(v) - \Psi_0(u) \geqslant \Psi_F'(u)(v-u) \quad \text{for every } v \in BL(\mathbb{R}^N). \tag{4.5}$$

Every $u \in BL(\mathbb{R}^N)$ which satisfies (4.5) is going to be called a solution of (4.1).

For the next result we refer to [17] for more details.

Theorem 4.1. *(Mountain Pass Theorem) Let X be a Banach space, $\Phi = \Phi_0 - I$ where $I \in C^1(X, \mathbb{R})$ and Φ_0 is a locally Lipschitz convex functional defined in X. Suppose that Φ satisfies:*

(*i*) *There exist* $\rho > 0$, $\alpha > \Phi(0)$ *such that* $\Phi|_{\partial B_\rho(0)} \geqslant \alpha$;
(*ii*) $\Phi(e) < \Phi(0)$ *for some* $e \in X \backslash \overline{B_\rho(0)}$.

Then for all $\varepsilon > 0$ *there exists* $x_\varepsilon \in X$ *such that*

$$c - \varepsilon < \Phi(x_\varepsilon) < c + \varepsilon, \tag{4.6}$$

and

$$\Phi_0(y) - \Phi_0(x_\varepsilon) \geqslant I'(x_\varepsilon)(y - x_\varepsilon) - \varepsilon \|y - x_\varepsilon\|, \quad \forall y \in X, \tag{4.7}$$

where $c \geqslant \alpha$ *is characterized by*

$$c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} \Phi(\gamma(t)), \tag{4.8}$$

where $\Gamma = \{\gamma \in C^0([0,1], X) : \gamma(0) = 0$ *and* $\gamma(0) = e\}$.

Now we show that the functional Ψ satisfies the assumptions of Theorem 4.1.

Proposition 4.2. *The functional* $\Psi|_{BL_{rad}(\mathbb{R}^N)}$ *verifies the following properties:*

(*i*) *There exist* $\rho > 0$, $\alpha > \Psi(0)$ *such that* $\Psi|_{\partial B_\rho(0)} \geqslant \alpha$,
(*ii*) $\Psi(e) < \Psi(0)$ *for some* $e \in BL_{rad}(\mathbb{R}^N) \backslash \overline{B_\rho(0)}$,

Proof. We start by proving the first assumption. Note that from $(\mathcal{F}_2)$ and $(\mathcal{F}_3)$ it follows that for all $\varrho > 0$, there exists $C_\varrho > 0$ such that

$$|F(\zeta)| \leqslant \varrho|\zeta| + C_\varrho|\zeta|^p \tag{4.9}$$

for all $\zeta \in \mathbb{R}$.

By (4.9) and Theorem 2.6 we have that

$$\begin{aligned} \Psi(u) &= \int_{\mathbb{R}^N} |\Delta u| + \int_{\mathbb{R}^N} |\nabla u| \, dx + \int_{\mathbb{R}^N} |u| dx - \int_{\mathbb{R}^N} F(u) dx \\ &\geqslant \|u\| - \varrho|u|_1 - C_\varrho|u|_p^p \\ &\geqslant \|u\| - \varrho\|u\| - c_3\|u\|^p \\ &= \|u\| \left(1 - \varrho - c_3\|u\|^{p-1}\right) \\ &\geqslant \alpha, \end{aligned}$$

for all $u \in BL_{rad}(\mathbb{R}^N)$, such that $\|u\| = \rho$, where $0 < \varrho < 1$ is fixed, $0 < \rho < \left(\dfrac{1-\varrho}{c_3}\right)^{\frac{1}{p-1}}$ and $\alpha = \rho(1 - \varrho - c_3\rho^{p-1})$.

For the second condition, note that $(\mathcal{F}_4)$ implies that there exist constants $c_4, c_5 > 0$ such that

$$F(\zeta) \geqslant c_4|\zeta|^\kappa - c_5 \tag{4.10}$$

for all $\zeta \in \mathbb{R}$.

Let $w \in BL_{rad}(\mathbb{R}^N)$, with compact support, $w \neq 0$ and for all $t > 0$, we get

$$\Psi(tw) \leqslant t\|w\| - c_4^\kappa |w|_\kappa^\kappa + c_5|\text{supp}(w)| \to -\infty,$$

as $t \to +\infty$, since $\kappa > 1$. Then we can choose $e = tw$ such that $\Psi(e) < 0$.

□

Proof of Theorem 1.2. From the Mountain Pass Theorem (Thm. 4.1), given a sequence $\varepsilon_n \to 0$, there exists a sequence $(u_n) \subset BL_{rad}(\mathbb{R}^N)$ such that

$$\lim_{n\to\infty} \Psi(u_n) = c \tag{4.11}$$

and

$$\Psi_0(v) - \Psi_0(u_n) \geqslant \int_{\mathbb{R}^N} f(u_n)(v-u_n)\mathrm{d}x - \varepsilon_n\|v-u_n\|, \quad \forall v \in BL_{rad}(\mathbb{R}^N), \tag{4.12}$$

where c is given by

$$c = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \Phi(\gamma(t))$$

and $\Gamma = \{\gamma \in C^0([0,1], BL_{rad}(\mathbb{R}^N));\ \gamma(0) = 0 \text{ and } \gamma(1) = e\}$.

Let us show that the sequence (u_n) is bounded in $BL_{rad}(\mathbb{R}^N)$. Indeed, in (4.12), let us take as test function $v = 2u_n$ and note that

$$\|u_n\| \geqslant \int_{\mathbb{R}^N} f(u_n)u_n\mathrm{d}x - \varepsilon_n\|u_n\|,$$

and then

$$(1+\varepsilon_n)\|u_n\| \geqslant \int_{\mathbb{R}^N} f(u_n)u_n\mathrm{d}x. \tag{4.13}$$

Hence, by $(\mathcal{F}_4)$ and (4.13), we infer that

$$\begin{aligned} c + o_n(1) &\geqslant \Psi(u_n) \\ &= \|u_n\| + \int_{\mathbb{R}^N}\left(\frac{1}{\kappa}f(u_n)u_n - F(u_n)\right)\mathrm{d}x - \int_{\mathbb{R}^N}\frac{1}{\kappa}f(u_n)u_n\mathrm{d}t \\ &\geqslant \|u_n\|\left(1 - \frac{1}{\kappa} - \frac{\varepsilon_n}{\kappa}\right) \\ &\geqslant C\|u_n\|, \end{aligned}$$

for some $C > 0$ uniform in $n \in \mathbb{N}$. Then the sequence (u_n) is bounded in $BL(\mathbb{R}^N)$.

Since the sequence (u_n) is bounded in $BL_{rad}(\mathbb{R}^N)$, by Theorem 1.1, there exists $u \in BL_{rad}(\mathbb{R}^N)$ such that $u_n \to u$ in $L^r(\mathbb{R}^N)$ for all $1 < r < N/(N-1)$.

Claim: $\int_{\mathbb{R}^N} f(u_n)u_n\mathrm{d}x = \int_{\mathbb{R}^N} f(u)u\mathrm{d}x + o_n(1)$.

Indeed, by $(\mathcal{F}_2)$ and $(\mathcal{F}_3)$, for given $\xi > 0$, there exists $K_\xi > 0$ such that

$$f(\zeta)\zeta \leqslant \xi|\zeta| + K_\xi|\zeta|^p \tag{4.14}$$

for all $\zeta \in \mathbb{R}$.

Since $u_n \to u$ in $L^p(\mathbb{R}^N)$, then there exists $R > 0$ such that

$$\int_{B_R(0)^c} |u_n|^p \mathrm{d}x < \xi. \tag{4.15}$$

Since (u_n) is bounded in $L^1(\mathbb{R}^N)$, there exists $C > 0$ such that $|u_n|_1 \leqslant C$, for all $n \in \mathbb{N}$. Then, from this, (4.14), (4.15)

$$\int_{B_R(0)^c} f(u_n)u_n \,\mathrm{d}x \leqslant (C + K_\xi)\xi. \tag{4.16}$$

Since $u_n \to u$ in $L^r(B_R(0))$ for all $r \in [1, N/(N-1))$, by using the Lebesgue Dominated Convergence Theorem we get

$$\int_{B_R(0)} f(u_n)u_n \,\mathrm{d}x = \int_{B_R(0)} f(u)u \,\mathrm{d}x + o_n(1). \tag{4.17}$$

Hence, by (4.16), (4.17) and the integrability of $f(u(\cdot))u(\cdot)$ (which follows from Proposition 2.6 of and $(\mathcal{F}_3)$), we obtain the Claim.

Then, by the last claim and the lower semicontinuity of Ψ_0 with respect to the $L^r(\mathbb{R}^N)$ convergence, it follows taking the $\limsup$ both sides of (4.12) that

$$\Psi_0(v) - \Psi_0(u) \geqslant \int_{\mathbb{R}^N} f(u)(v-u)\,\mathrm{d}x, \quad \forall v \in BL_{rad}(\mathbb{R}^N). \tag{4.18}$$

Taking $v = u + tu$ in (4.18) and doing $t \to 0$ we have that $\Psi_0(u) = \int_{\mathbb{R}^N} f(u)u\,\mathrm{d}x$. Then

$$\lim_{n\to\infty} \Psi_0(u_n) = \lim_{n\to\infty} \int_{\mathbb{R}^N} f(u_n)u_n \,\mathrm{d}x = \int_{\mathbb{R}^N} f(u)u\,\mathrm{d}x = \Psi_0(u),$$

Hence, since $\Psi(u_n) = c + o_n(1)$, we get

$$\Psi(u) = c$$

and therefore $u \neq 0$.

By using a version of the Symmetric Criticality Principle of Palais proved in ([32], Thm. 4), it follows that u satisfies (4.18) for all $v \in BL(\mathbb{R}^N)$. Then, we have proved that $\Psi_F'(u) \in \partial\Psi_0(u)$. By the result presented in the Appendix, we have that u satisfies (1.1) and is in fact a solution of $(\mathcal{P})$.

Now, let us show that the solution $u \in BL(\mathbb{R}^N)$ we have got, is in fact a ground-state solution among those ones which are radially symmetric. In order to prove it, as in [16], let us define the so called Nehari set associated to Ψ, given by

$$\mathcal{N} = \left\{ u \in BL_{rad}(\mathbb{R}^N)\backslash\{0\};\ \|u\| = \int_{\mathbb{R}^N} f(u)u\mathrm{d}x \right\}.$$

Arguing as in [16] it is possible to show that $\mathcal{N}$ contains all nontrivial radially symmetric solutions of (4.1). Then, by showing that

$$\Psi(u) = \inf_{\mathcal{N}} \Psi, \tag{4.19}$$

we obtain that u would have the lowest energy level among all the radially symmetric nontrivial solutions of (4.1).

To show (4.19), in turn, it is enough to use the same arguments of Rabinowitz in [31]. More precisely, for each $u \in BL(\mathbb{R}^N)\backslash\{0\}$, by $(\mathcal{F}_1)-(\mathcal{F}_5)$, the function $h_u(t) = \Psi(tu) = t\|u\| - \int_\Omega F(tu)\mathrm{d}x$ admits a unique maximum point $t_u > 0$, such that $t_u u \in \mathcal{N}$. On the other hand, if $v \in \mathcal{N}$, then $t_v = 1$ and $\Psi(v) = \max_{t>0} \Psi(tv)$. This, in turn, implies that

$$\inf_{v \in BL(\mathbb{R}^N)\backslash\{0\}} \max_{t>0} \Psi(tv) = \inf_{v\in\mathcal{N}} \Psi(v).$$

On the other hand, it is possible to show that $u \mapsto t_u$ is a continuous function defined in $BL(\mathbb{R}^N)$. This, implies that the unitary sphere $S_1(0)$ in $BL(\mathbb{R}^N)$ is homeomorphic to $\mathcal{N}$, through the homeomorphism $u \mapsto t_u u$. This, in turn, makes possible to proceed as in as in [31] to show that

$$c = \inf_{v \in BL(\mathbb{R}^N)\backslash\{0\}} \max_{t>0} \Psi(tv) = \inf_{v\in\mathcal{N}} \Psi(v).$$

Hence the conclusion follows just by recalling that $\Psi(u) = c$.

□

Appendix A.

In this appendix we use the same notation as in Section 4 and we aim to get better information about the function u we have found in the last section.

We start by splitting the functional Ψ_0 in the following three functionals,

$$\Psi_1(u) = \int_{\mathbb{R}^N} |\Delta u|,$$

$$\Psi_2(u) = \int_{\mathbb{R}^N} |\nabla u| \mathrm{d}x,$$

and

$$\Psi_3(u) = \int_{\mathbb{R}^N} |u| \mathrm{d}x.$$

Let us assume that $u \in BL(\mathbb{R}^N)$ is a function such that $\Psi_F'(u) \in \partial\Psi_0(u)$ and let us prove that there exist $w \in L^\infty(\mathbb{R}^N)$, $\mathbf{z} \in L^\infty(\mathbb{R}^N, \mathbb{R}^N)$ and $\gamma \in L^\infty(\mathbb{R}^N)$, such that, $\|w\|_\infty, \|\mathbf{z}\|_\infty, \|\gamma\|_\infty \leqslant 1$ and

$$\begin{cases} \Delta w - \operatorname{div} \mathbf{z} + \gamma &= f(u) \quad \text{in } \mathcal{D}'(\mathbb{R}^N), \\ \displaystyle\int_{\mathbb{R}^N} w \Delta u &= \displaystyle\int_{\mathbb{R}^N} |\Delta u|, \\ \displaystyle\int_{\mathbb{R}^N} \mathbf{z} \cdot \nabla u \ \mathrm{d}x &= \displaystyle\int_{\mathbb{R}^N} |\nabla u| \ \mathrm{d}x, \\ \gamma . u &= |u| \quad \text{a.e. in } \mathbb{R}^N. \end{cases}$$

Let us consider the Banach space $X = L^1(\mathbb{R}^N) \cap L^{N/(N-1)}(\mathbb{R}^N)$ endowed with the norm $\|\cdot\|_X = \|\cdot\|_1 + \|\cdot\|_{N/(N-1)}$ and note that by the Sobolev embeddings, $BL(\mathbb{R}^N) \subset W^{1,1}(\mathbb{R}^N) \subset X$. By $(\mathcal{F}_4)$ it follows that Ψ_F

extends trivially to X. In order to save notation, let us denote by Ψ_0, Ψ_1, Ψ_2 and Ψ_3 the extension of them to X, *i.e.*,

$$\Psi_1(u) = \begin{cases} \displaystyle\int_{\mathbb{R}^N} |\Delta u|, & \text{if } u \in BL(\mathbb{R}^N), \\ +\infty, & \text{if } u \in X \backslash BL(\mathbb{R}^N), \end{cases}$$

$$\Psi_2(u) = \begin{cases} \displaystyle\int_{\mathbb{R}^N} |\nabla u| \mathrm{d}x, & \text{if } u \in W^{1,1}(\mathbb{R}^N), \\ +\infty, & \text{if } u \in X \backslash W^{1,1}(\mathbb{R}^N), \end{cases}$$

$$\Psi_3(u) = \int_{\mathbb{R}^N} |u| \mathrm{d}x, \quad \forall u \in X$$

and $\Psi_0 = \Psi_1 + \Psi_2 + \Psi_3$. It is easy to see that since $\Psi_F'(u) \in \partial\Psi_0(u)$ (where $\partial\Psi_0(u) \subset BL(\mathbb{R}^N)$), then the same is true with respect to the extended functionals, where now we should consider $\partial\Psi_0(u)$ as a subset of X'. Since Ψ_3 is continuous, convex and proper at any point of $W^{1,1}(\mathbb{R}^N)$, as well as Ψ_2 at any point of $BL(\mathbb{R}^N)$, it follows from ([7], Thm. 9.5.4) that

$$\Psi_F'(u) \in \partial\Psi_0(u) = \partial\Psi_1(u) + \partial\Psi_2(u) + \partial\Psi_3(u).$$

From the last equation, it follows that there exist $w^* \in \partial\Psi_1(u)$, $\mathbf{z}^* \in \partial\Psi_2(u)$ and $\gamma \in \partial\Psi_3(u)$, such that

$$\Psi_F'(u) = w^* + \mathbf{z}^* + \gamma. \tag{A.1}$$

In the next result, we use the same arguments from ([27], Prop. 5.2) to characterize $\partial\Psi_1(u)$.

Lemma A.1. *If $w^* \in \partial\Psi_1(u)$, then there exists $w \in L^\infty(\mathbb{R}^N)$ such that $\|w\|_\infty \leqslant 1$ and*

$$\begin{cases} w^* &= \Delta w \quad \text{in } \mathcal{D}'(\mathbb{R}^N), \\ \displaystyle\int_{\mathbb{R}^N} w\Delta u &= \displaystyle\int_{\mathbb{R}^N} |\Delta u|. \end{cases}$$

Proof. First of all, let us define

$$M^* = \{v^* \in X';\ v^* = \Delta v \text{ for some } v \in L^\infty(\mathbb{R}^N),\ \|v\|_\infty \leqslant 1\}.$$

Now let us prove that M^* is a closed set in X'. Let $(v_n^*) \subset M^*$ such that $v_n^* \to v^*$ in M^*. Let us also consider a sequence $(v_n) \subset L^\infty(\mathbb{R}^N)$ such that $\|v_n\|_\infty \leqslant 1$ and

$$v_n^* = \Delta v_n \quad \text{in } \mathcal{D}'(\mathbb{R}^N),$$

i.e., for all $\varphi \in C_c^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} v_n^* \varphi \mathrm{d}x = \int_{\mathbb{R}^N} v_n \Delta\varphi \mathrm{d}x. \tag{A.2}$$

Since (v_n) is bounded in $L^\infty(\mathbb{R}^N)$ it follows that there exists $v \in L^\infty(\mathbb{R}^N)$ such that

$$v_n \overset{*}{\rightharpoonup} v \quad \text{in } L^\infty(\mathbb{R}^N), \tag{A.3}$$

up to a subsequence. Taking the limit in (A.2) and taking into account (A.3), we have that

$$\int_{\mathbb{R}^N} v^*\varphi \mathrm{d}x = \int_{\mathbb{R}^N} v\Delta\varphi \mathrm{d}x, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^N).$$

Then $v^* = \Delta v$ in $\mathcal{D}'(\mathbb{R}^N)$. Moreover, the lower-semicontinuity of the norm with respect to the weak* convergence implies that $\|v\|_\infty \leqslant 1$. Hence $v^* \in M^*$.

Let us consider the indicator function of M^*, $I_{M^*} : X \to \mathbb{R}$ given by

$$I_{M^*}(v^*) = \begin{cases} 0, & \text{if } u \in M^*, \\ +\infty, & \text{if } u \notin M^*. \end{cases}$$

The conjugate function of I_{M^*} is given by

$$I_{M^*}^*(u) = \sup_{v^* \in X'} \left\{ \int_{\mathbb{R}^N} v^* u \mathrm{d}x - I_{M^*}(v^*) \right\} = \sup_{v^* \in M^*} \left\{ \int_{\mathbb{R}^N} v^* u \mathrm{d}x \right\}.$$

Now, let us prove that $I_{M^*}^*(u) = \Psi_1(u)$ for all $u \in X$. First, let us consider $v^* \in M^*$, $v \in L^\infty(\mathbb{R}^N)$ such that $\Delta v = v^*$ in $\mathcal{D}'(\mathbb{R}^N)$, $u \in BL(\mathbb{R}^N)$ and $(u_n) \subset C_c^\infty(\mathbb{R}^N)$ such that $u_n \to u$ in $W^{1,1}(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} |\Delta u| = \lim_{n\to\infty} \int_{\mathbb{R}^N} |\Delta u_n| \mathrm{d}x$. Then,

$$\begin{aligned} \int_{\mathbb{R}^N} v^* u \mathrm{d}x &= \int_{\mathbb{R}^N} u\Delta v \mathrm{d}x \\ &= \lim_{n\to\infty} \int_{\mathbb{R}^N} u_n \Delta v \mathrm{d}x \\ &= \lim_{n\to\infty} \int_{\mathbb{R}^N} v\Delta u_n \mathrm{d}x \\ &\leqslant \lim_{n\to\infty} \|v\|_\infty \int_{\mathbb{R}^N} |\Delta u_n| \mathrm{d}x \\ &\leqslant \int_{\mathbb{R}^N} |\Delta u| \\ &= \Psi_1(u), \end{aligned}$$

from which it follows that $I_{M^*}^*(u) \leqslant \Psi_1(u)$ for all $u \in BL(\mathbb{R}^N)$. Actually,

$$I_{M^*}^*(u) \leqslant \Psi_1(u), \quad \forall u \in X, \tag{A.4}$$

since if $u \in X \backslash BL(\mathbb{R}^N)$, then $\Psi_1(u) = +\infty$.

In order to get the opposite inequality, for $u \in X$, note that

$$\Psi_1(u) = \sup \left\{ \int_{\mathbb{R}^N} u\Delta\varphi \mathrm{d}x;\, \varphi \in C_c^\infty(\mathbb{R}^N),\, \|\varphi\|_\infty \leqslant 1 \right\}$$

$$\begin{aligned}
&\leqslant \sup\left\{\int_{\mathbb{R}^N} u\Delta v \mathrm{d}x;\ v\in L^\infty(\mathbb{R}^N),\ \Delta v\in X',\ \|v\|_\infty\leqslant 1\right\}\\
&= \sup\left\{\int_{\mathbb{R}^N} uv^*\mathrm{d}x;\ v^*\in M^*\right\}\\
&= I^*_{M^*}(u). \qquad \text{(A.5)}
\end{aligned}$$

Hence, from (A.4) and (A.5), it follows that

$$I^*_{M^*}(u) = \Psi_1(u) \quad \text{for all } u\in X. \qquad \text{(A.6)}$$

Since M^* is closed and convex, its indicator function I_{M^*} is convex and lower semicontinuous. Then, by (A.6) and ([15], Props. 3.1 and 4.1), it follows that

$$I_{M^*} = (I^*_{M^*})^* = \Psi_1^*.$$

In particular, from ([15], Prop. 5.1), $w^*\in\partial\Psi_1(u)$ if and only if

$$\int_{\mathbb{R}^N} uw^*\mathrm{d}x = \Psi_1(u)+\Psi^*(w^*) = \Psi_1(u)+I_{M^*}(w^*).$$

From the last equation we conclude that $w^*\in\partial\Psi_1(u)$ if and only if $w^*\in M^*$ and $\Psi_1(u)=\int_{\mathbb{R}^N} uw^*\mathrm{d}x$, what implies in the result. □

With the same arguments that in the last result (or by using [19], Prop. 4.23), one could characterize $\partial\Psi_2(u)$ and $\partial\Psi_3(u)$ in order to conclude that $\mathbf{z}^*\in\partial\Psi_2(u)$ and $\gamma\in\partial\Psi_3(u)$ if and only if there exist $\mathbf{z}\in L^\infty(\mathbb{R}^N,\mathbb{R}^N)$ and $\gamma\in L^\infty(\mathbb{R}^N)$ such that $\|\mathbf{z}\|_\infty\leqslant 1$, $\|\gamma\|_\infty\leqslant 1$,

$$\int_{\mathbb{R}^N} \mathbf{z}\cdot\nabla u\ \mathrm{d}x = \int_{\mathbb{R}^N} |\nabla u|\ \mathrm{d}x, \qquad \text{(A.7)}$$

and

$$\gamma.u = |u| \quad \text{a.e. in } \mathbb{R}^N. \qquad \text{(A.8)}$$

Therefore, from Lemma A.1, (A.7) and (A.8), it follows that if $u\in BL(\mathbb{R}^N)$ is such that $\Psi_F'(u)\in\partial\Psi_0(u)$, then there exist $w\in L^\infty(\mathbb{R}^N)$, $\mathbf{z}\in L^\infty(\mathbb{R}^N,\mathbb{R}^N)$ and $\gamma\in L^\infty(\mathbb{R}^N)$, such that, $\|w\|_\infty, \|\mathbf{z}\|_\infty, \|\gamma\|_\infty\leqslant 1$ and

$$\left\{\begin{aligned}
\Delta w - \operatorname{div}\mathbf{z}+\gamma &= f(u) \quad \text{in } \mathcal{D}'(\mathbb{R}^N),\\
\int_{\mathbb{R}^N} w\Delta u &= \int_{\mathbb{R}^N} |\Delta u|,\\
\int_{\mathbb{R}^N} \mathbf{z}\cdot\nabla u\ \mathrm{d}x &= \int_{\mathbb{R}^N} |\nabla u|\ \mathrm{d}x,\\
\gamma.u &= |u| \quad \text{a.e. in } \mathbb{R}^N.
\end{aligned}\right.$$

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