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ORIENTED DISTANCE POINT OF VIEW ON RANDOM SETS

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Abstract. Motivated by free boundary problems under uncertainties, we consider the oriented distance function as a way to define the expectation for a random compact or open set. In order to provide a law of large numbers and a central limit theorem for this notion of expectation, we also address the question of the convergence of the level sets of f_n to the level sets of f when (f_n) is a sequence of functions uniformly converging to f. We provide error estimates in term of Hausdorff convergence. We illustrate our results on a free boundary problem.

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1. Introduction

This work is motivated by the consideration of uncertainties in free boundary problems and in shape optimization problems. Roughly speaking, we consider situations where a shape is the output of a mathematical problem like the minimization of a functional or an overdetermined boundary value problem and the situation depends on some parameters that are not exactly known but only through some statistical information over their distribution. A typical problem we have in mind is the optimal design of a bridge when the applied loading (typically the weight of peoples on the bridge) is random.

For a fixed value p of the parameter, the problem is a classical shape optimization problem associated to an objective $D \mapsto J(D, p)$ where D stands for an open subset of $\mathbb{R}^d$ in a given class of domain. This question has been widely studied. However, its solution is a shape in $Argmin\, J(\cdot, p)$ and depends on p. When p is random, the object we are interested in is a random set. After the first pioneering works of Matheron and Kendall (see [19, 21, 22]), the study of random sets is receiving growing attention in the statistical and probabilistic literature. Let us first specify some general notions about random sets and what will be our point of view in this work.

We fix a dimension d and we endow $\mathbb{R}^d$ with its Euclidean distance. Let B_∞ be a given *closed* ball in $\mathbb{R}^d$ of radius R_∞, this ball will play the role of a box and we will consider only subsets of B_∞. Let $\mathcal{K}$ be the set of all compact sets contained in B_∞. We also denote the probability space by $(\Omega, \mathcal{A}, \mathbb{P})$. The space $\mathcal{K}$ endowed with the Hausdorff metric (see Sect. 2 for *a precise definition and the statement of some properties*) is a complete separable metric space (see [15], pp. 30) and is then endowed with the Borel σ-field associated with this metric. A random compact set in B_∞ is a measurable function from Ω to $\mathcal{K}$. As well-known, $\mathcal{K}$ is not a vector space

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but just a metric space: therefore the notion of expectation of a random compact set is difficult, there is no canonical definition.

The most usual choice is the so-called Aumann mean since it corresponds to the limit of the natural Minkowski empirical means. Central Limit Theorems for this case have already been derived (see [4]). This choice is not satisfactory for our purpose since the Aumann mean of a random set is convex. If this notion is perfectly *adapted when* the minimizers are known to be convex, it seems unsuited for applications to usual mechanical devices. In [11], we have explored two possibilities in the context of free boundary problems: a notion based on a parametrization of the sets and the Vorob'ev mean based on the quantiles of the coverage function of the random set (that is the probability that a given point belongs to the random set). We would like to choose a notion of mean set adapted to the shape optimization context, leading to computations that remain realistic and sufficiently rich to provide a good description of the object.

In this work, we are interested in random sets that *arise* in shape optimization. We will make the assumption, pertinent in many applications, to deal in a bounded situation: we shall consider subsets of a fixed ball B_∞ that plays the role of a box. In that context, the compactness assumption made in the probabilistic literature is unsuited: the theory of partial differential equations requires to work with domains that are open sets. Moreover, the usual existence results for optimal shapes are stated in the class of quasi open sets. However, the optimality conditions usually allow to prove more regularity of the set: typically, one can expect to deal with open sets with a piecewise smooth boundary. Therefore, an alternative appears: either, one deals with random compact sets that are closure of open sets, either one deals directly with random open sets defined as follows. The class $\mathcal{O}$ of *open subsets of* $\mathbb{R}^d$ *that are included in* B_∞ is a separable metric space endowed with the Hausdorff distance for open sets (see Sect. 2) and a random open set is a measurable function from Ω to $\mathcal{O}$ endowed with the Borel σ- field associated with the Hausdorff distance.

In the context of shape optimization, there are two important ways to parametrize domains. The first one is the Hadamard point of view leading to the shape calculus and the notion of shape derivative: the idea is to parametrize a domain as the image of a reference domain by a diffeomorphism and to use the Banach structure of diffeomorphisms to define derivatives. The second one is to use a function $f : \mathbb{R}^d \to \mathbb{R}$ to implicitly define the domain under study D as $\{x \mid f(x) < 0\}$. We will denote level sets defined by inequalities or equalities by $[f < 0]$ or $[f = 0]$. This parametrization is the ground of the level set method which coincides, in that context, *with* the shape gradient flow. The leading idea of this work is to use such an implicit parametrization of shapes to define the expectation. Therefore, we are interested in basing the notion of mean on the level set functions.

We then have to face a difficulty: the choice of f is not unique. Therefore, in order to define an expectation, for each compact set $K \in \mathcal{K}$, we have to choose a specific function f such that $K = [f \leq 0]$. The natural choice is the oriented distance function to K. For some decades, the oriented distance function to a compact set K has become a used way to describe the property of the set. It has been first used to study the motion by mean curvature on a theoretical level. Then, Osher and Sethian [23] introduced the level set method that has become the reference encoding of evolving domains. In shape optimization, it is used to implement the gradient method [2] or even the Newton method [1] and also to obtain some compactness result on some classes of domains [12] by exploiting the deep connection between the geometric properties of the boundary of K and the oriented distance function to K. Notice that only very few theoretical convergence results have been obtained for this level set method in the context of shape optimization even at the continuous level [8, 9].

Of course, the oriented distance function is a parametrization and we have to face the difficulties of parametrization based approaches to expectation. In particular, it is not intrinsic but depends on the choice of parametrization. In addition, the natural procedure to define the expectation of the random domain as the domain whose distance function is the expectation of the distance functions to the random set makes no sense. Indeed, the set of distance functions is not convex: a mean of oriented distance functions is not the oriented distance to some set (see Sect. 2.2) and we have to consider some relaxations of the definition.

This article is organized as follows. In Section 2, we recall the definitions of the Hausdorff distance and oriented distance functions and provide useful results on their use to implicitly represent domains. In Section 3, we study approximation properties. The underlying question is: how far is $[f_n \leq 0]$ from $[f \leq 0]$ when (f_n) is a sequence converging to f ? We provide some conditions to prove convergence of the level sets and also convergence rates.

Then, in Section 4, we define a notion of expectation of random compact set through oriented distance functions. It is therefore natural to consider an estimator of this expectation defined like a level set of the empirical mean of this oriented distance function. Some results of convergence are given. In the last part, an application of these tools to the Bernoulli exterior free boundary problem is presented.

Notice that many parts of this work can be found spread in the literature on shape optimization, convex analysis and statistical estimation. In particular, we discovered during this work that the notion of oriented distance based expectation was introduced by Jankowski and Stanberry [17]. However, we provide many improvements in the results and we believe to have simplified many proofs in a self contained work.

2. Implicit parametrization of sets by distance functions

In this section, we consider various implicit representations of compact sets and provide technical results that will be used in the sequel.

2.1. Distance functions and Hausdorff convergence

Let us recall the definition of the Hausdorff distance on $\mathcal{K}$. We first consider compact sets. Let K_1 and K_2 in $\mathcal{K}$, then the excess of K_1 with respect to K_2 is

$$\rho(K_1, K_2) = \sup_{x \in K_1} d(x, K_2).$$

The Hausdorff distance is then $d_{\mathcal{H}}(K_1, K_2) = \sup(\rho(K_1, K_2), \rho(K_2, K_1))$. As a consequence, a sequence (K_n) in $\mathcal{K}$ Hausdorff converges to K if $d_{\mathcal{H}}(K_n, K) \to 0$. This will be denoted by $K_n \xrightarrow{\mathcal{H}} K$.

An important property is the characterization of the Hausdorff distance by the distance functions. For $K \in \mathcal{K}$, we set $d_K : x \mapsto d(x, K)$ and then it holds

$$d_{\mathcal{H}}(K_1, K_2) = \|d_{K_1} - d_{K_2}\|_\infty.$$

Since we will only consider continuous functions defined on the whole $\mathbb{R}^d$, the uniform convergence on $\bar{B}_\infty$ provides the notion of uniform convergence on B_∞.

Lemma 2.1. *Let K_n be a sequence of compact subsets of B_∞ and let K be a compact subset of B_∞. The sequence (d_{K_n}) converges uniformly to d_K if and only if $K_n \xrightarrow{\mathcal{H}} K$.*

In particular, this lemma means that the Hausdorff distance does not see the interior of the compact set and therefore is not suited for studying properties of boundaries. Moreover, natural geometric quantities like volume and perimeter are not continuous with respect to this topology as emphasized in the Example 2.6. However, the Hausdorff convergence enjoys nice properties for monotonic sequences of sets ([15], Sect. 2.2.3.2, p. 32).

Lemma 2.2. *A decreasing sequence of nonempty compact sets Hausdorff converges to their intersection. An increasing sequence of nonempty compact sets converges to the closure of their union.*

We shall also use the following lemma.

Lemma 2.3. *Let A_1, A_2, B_1 and B_2 be compact sets such that $A_1 \subset B_i \subset A_2$. Then it holds $d_{\mathcal{H}}(B_1, B_2) \leq d_{\mathcal{H}}(A_1, A_2)$.*

Proof of Lemma 2.3. If K_1 and K_2 are two compact sets with $K_1 \subset K_2$ then $d_{K_2} \leq d_{K_1}$. Hence, $d_{A_2} \leq d_{B_i} \leq d_{A_1}$ for $i = 1, 2$ and therefore $|d_{B_1}(x) - d_{B_2}(x)| \leq |d_{A_1}(x) - d_{A_2}(x)|$ for all x and passing to the supremum we obtain $\|d_{B_1} - d_{B_2}\|_\infty \leq \|d_{A_1} - d_{A_2}\|_\infty$. □

The Hausdorff distance is extended to open subsets of B_∞ by the definition ([15], Sect. 2.2.3.2, Rem. 2.2.11, p. 32):

$$d_{\mathcal{H}}(\Omega_1, \Omega_2) = d_{\mathcal{H}}(K \setminus \Omega_1, K \setminus \Omega_2),$$

where Ω_1, Ω_2 are open subsets of B_∞ and K is any compact subset of $\mathbb{R}^d$ with $B_\infty \subset K$. It is sometimes called the Hausdorff complementary distance.

2.2. Oriented distance functions

Let us recall the main properties of oriented distance functions and their connection with the Hausdorff distance between compact sets. We first give definition and fix notations. For any non empty subset A of $\mathbb{R}^d$, the oriented distance function to A is the function b_A defined on the whole $\mathbb{R}^d$ as

$$b_A(x) = d_A(x) - d_{\mathbb{R}^d \setminus A}(x), \quad \forall x \in \mathbb{R}^d.$$

Notice that $d_A = (b_A)_+$ and $d_{\mathbb{R}^d\setminus A} = (b_A)_-$ where $(t)_+ = \max(0,t)$ is the positive part and where $(t)_- = \max(0,-t)$ is the negative part.

The oriented distance function b_A provides a nice implicit representation of open and compact sets.

Lemma 2.4 (Implicit representation of domains by oriented distance function)**.** *Let A denote a non-empty subset of $\mathbb{R}^d$.*

- *Its closure is given by $\bar{A} = [b_A \leq 0]$. In particular, if $K \in \mathcal{K}$ is a compact set then*

$$K = [b_K \leq 0] \text{ and } \partial K = [b_K = 0].$$

- *If A satisfies $\bar{\mathring{A}} = \bar{A}$ then $\mathring{A} = [b_A < 0]$.*
- *For any real number λ and any set A, $\overline{[b_A < \lambda]} \subset [b_A \leq \lambda]$. If $\lambda > 0$ and K is compact, then the following equality holds $\overline{[b_K < \lambda]} = [b_K \leq \lambda]$.*

Proof of Lemma 2.4.

Step 1: Proof of the first point. The inclusion $\bar{A} \subset [b_A \leq 0]$ is clear since A is contained in the closed set $[b_A \leq 0]$. Conversely, if x is a point where $b_A(x) \leq 0$, then $d_A(x) = 0$ and, by definition of the distance, there is a sequence (y_n) of points in A such that $y_n \to x$. The characterization of the boundary is stated in ([12], Thm. 2-1(iii), p. 338).

Step 2: Case of open sets. We now prove that if A is open then $A = [b_A < 0]$. Indeed, on the one hand, if $b_A(x) < 0$, then $d_A(x) = 0$ and $d_{A^c}(x) > 0$ hence $x \notin A^c$ that is to say $x \in A$. On the other hand if $x \in A$ then first $d_A(x) = 0$ and second there is a open ball centered in x and contained in A so that $d_{A^c}(x) > 0$.

Step 3: Proof of the second point. The interior $\mathring{A}$ is open and $\mathring{A} = [b_{\mathring{A}} < 0]$ by the second step. To conclude, it suffices to prove that $b_{\mathring{A}} = b_A$. If $\bar{\mathring{A}} = \bar{A}$, one also has $\partial\mathring{A} = \bar{\mathring{A}} \setminus \mathring{A} = \bar{A} \setminus \mathring{A} = \partial A$. Now, we use the equivalence

$$b_{B_1} = b_{B_2} \Leftrightarrow \bar{B}_1 = \bar{B}_2 \text{ and } \partial B_1 = \partial B_2$$

proved in [12], (Thm. 2-1(ii), p. 338) to deduce that $b_{\mathring{A}} = b_A$.

Step 4: Proof of the third point. The inclusion $\overline{[b_A < \lambda]} \subset [b_A \leq \lambda]$ is clear: any point $x \in \overline{[b_A < \lambda]}$ is the limit of a sequence (x_n) such that $b_A(x_n) < \lambda$ then by continuity of b_A, $b_A(x) \leq \lambda$.

We now assume $\lambda > 0$ and K compact and prove $[b_K \le \lambda] \subset \overline{[b_K < \lambda]}$. Set $x \in [b_K \le \lambda]$. If $x \in K$, then $B(x,\lambda) \subset [b_K < \lambda]$ and $x \in \overline{[b_K < \lambda]}$. If $x \notin K$, let $y \in K$ such that $\|x-y\| = d_K(x) = b_K(x) \le \lambda$. Then, for any $t \in (0,1)$, the point $x(t) = y + t(x-y)$ of the segment $[y,x]$ satisfies $\|y - x(t)\| = t\|y-x\| < \|x-y\| \le \lambda$ and $x(t) \in [b_K < \lambda]$. Since $x(t)$ converges to x when $t \to 1$, $x \in \overline{[b_K < \lambda]}$. □

We state useful properties of the parametrization by oriented distance function.

Lemma 2.5. *Let K_1 and K_2 be two compact subsets of B_∞.*

- $b_{K_2} \le b_{K_1}$ *if and only if* $K_1 \subset K_2$ *and* $\overline{(K_1^c)} \subset \overline{(K_2^c)}$.
- *The Hausdorff distance is dominated by the gap of oriented distance function:*

$$d_{\mathcal{H}}(K_1, K_2) = \|d_{K_1} - d_{K_2}\|_\infty \le \|b_{K_1} - b_{K_2}\|_\infty.$$

Proof of Lemma 2.5. The first property is stated in ([12], Thm. 2-1(ii), p. 338). To prove the second one, it suffices to check that $d_K = (b_k)_+$ and to notice that the map $t \mapsto (t)_+$ is 1 Lipschitz on $\mathbb{R}$. Hence, for any $x \in B_\infty$, we get

$$|d_{K_1}(x) - d_{K_2}(x)| \le |b_{K_1}(x) - b_{K_2}(x)|.$$

The conclusion follows by taking the supremum to the right hand side then to the left hand side. Notice that a converse inequality is not possible if $\overbrace{K_1 \cap K_2}^{\circ} \neq \emptyset$ and $K_1 \neq K_2$. □

As a consequence of the second point, the convergence of oriented distance functions to compact subsets implies the Hausdorff convergence. However, the equivalence stated in Lemma 2.1 is lost when one replaces distance functions by oriented distance functions as shown by the following example on $\mathbb{R}$.

Example 2.6. Let (x_n) be a dense sequence in $[0,1]$. Set $K_n = \{x_0, \dots, x_n\}$, then $K_n \xrightarrow{\mathcal{H}} [0,1]$ while $b_{K_n} \to d_{[0,1]} \neq b_{[0,1]}$.

The reason is that oriented distance function contains also information on the interior of the set as stated in the following result.

Proposition 2.7 (Convergence of oriented distance functions to compact sets)**.** *Let K_n be a sequence of compacts subsets of B_∞ and let K be a compact subset of B_∞. The statements*

$$(i)\ \ (b_{K_n}) \text{ uniformly converges to } b_K;$$
$$(ii)\ \ K_n \xrightarrow{\mathcal{H}} K \text{ and } \mathring{K}_n \xrightarrow{\mathcal{H}} \mathring{K};$$

are equivalent.

Proof of Proposition 2.7.

Step 1: $(i) \implies (ii)$**.** Since the positive $()_+$ part is 1-Lipschitz, the sequence (d_{K_n}) uniformly converges to d_K implying $K_n \xrightarrow{\mathcal{H}} K$. In the same manner, the negative $()_-$ part is 1-Lipschitz and the sequence $(d_{K_n^c})$ uniformly converges to d_{K^c}. Since $d_A = d_{\bar{A}}$ for any $A \subset B_\infty$, the sequence $(d_{\overline{(K_n^c)}})$ uniformly converges to $d_{\overline{(K^c)}}$ and the compact sets $\overline{(K_n^c)} \xrightarrow{\mathcal{H}} \overline{(K^c)}$ that is $\mathring{K}_n \xrightarrow{\mathcal{H}} \mathring{K}$ since for any $A \subset B_\infty$, $\overline{A^c} = (\mathring{A})^c$.

Step 2: $(ii) \implies (i)$**.** If $K_n \xrightarrow{\mathcal{H}} K$, then (d_{K_n}) uniformly converges to d_K. If $\mathring{K}_n \xrightarrow{\mathcal{H}} \mathring{K}$ that is $\overline{(K_n^c)} \xrightarrow{\mathcal{H}} \overline{(K^c)}$, then $(d_{\overline{(K_n^c)}})$ uniformly converges to $d_{\overline{(K^c)}}$ that is $(d_{K_n^c})$ uniformly converges to d_{K^c}. □

As a clear consequence, we get a similar statement for open sets.

Proposition 2.8. *Let (Ω_n) be a sequence of non empty open sets in B_∞ and let $\Omega \subset B_\infty$ be open. The statements*

$$(i)\ (b_{\Omega_n}) \text{ uniformly converges to } b_\Omega;$$
$$(ii)\ \Omega_n \xrightarrow{\mathcal{H}} \Omega \text{ and } \bar{\Omega}_n \xrightarrow{\mathcal{H}} \bar{\Omega};$$

are equivalent.

Let us denote by $\mathcal{D}$ the set of oriented distance functions to compact sets that is $f \in \mathcal{D}$ if and only if there is a compact *set* K in B_∞ such that $f = b_K$. The main properties of functions of $\mathcal{D}$ are:

Lemma 2.9 (Properties of $\mathcal{D}$)**.** *One has:*

- $\mathcal{D} \subset Lip(\bar{B}_\infty, 1)$ *the space of* 1*-Lipschitz functions on* B_∞*,*
- $\mathcal{D}$ *is not convex.*

The first point is ([12], Thm. 2-1,(vi), p. 338). The second point is just a calculus on $\mathbb{R}$: the function $(b_{\{0\}} + b_{\{1\}})/2$ is not an oriented distance function.

Remark 2.10. It would be more natural to work in the sequel within the convex hull of $\mathcal{D}$ than in the space $Lip(\bar{B}_\infty, 1)$. However, we did not manage to characterize this convex hull. It is not the whole $Lip(\bar{B}_\infty, 1)$.

We shall work within the space $Lip(\bar{B}_\infty, 1)$. Let us recall its main properties that we shall use.

Lemma 2.11. *The space $Lip(\bar{B}_\infty, 1)$ is endowed with the norm*

$$\|f\|_\infty = \sup_{x \in B_\infty} |f(x)|.$$

It is a convex closed set. Moreover, if (f_n) is a pointwise convergent sequence of functions in $Lip(\bar{B}_\infty, 1)$, then the convergence is uniform in $\bar{B}_\infty$.

3. Sequences of continuous functions parametrizing sets

In this section, we consider continuous functions defined on B_∞ with values in $\mathbb{R}$. We say that a continuous function f is a parametrization of K if K is the 0 sublevel set of $\{x \mid f(x) \leq 0\}$ that we will denote by $[f \leq 0]$. Of course, such a parametrization is not unique since for any increasing function ϕ with $\phi(0) = 0$ one has $[f \leq 0] = [\phi \circ f \leq 0]$.

3.1. Convergence of approximated level set

We first discuss the following general question posed here in a rough way: *if the sequence of functions (f_n) converges to some function f, does the sequence of level-sets $[f_n \leq 0]$ converge in the Hausdorff sense to the level-set $[f \leq 0]$?* In general, the answer is negative as shown by the following examples in dimension one.

Example 3.1. On $B_\infty = [-10, 10]$, the sequence of functions $f_n = \inf(b_{[0,2]}, d_{\{3\}} + 1/n)$ converges to $f = \inf(b_{[0,2]}, d_{\{3\}})$. One has $\|f_n - f\|_\infty = 1/n \to 0$ while

$$[f_n \leq 0] = [0, 2] \xrightarrow{\mathcal{H}} [0, 2] \neq [f \leq 0] = [0, 2] \cup \{3\}.$$

Example 3.2. On $B_\infty = [-10, 10]$, let f be the piecewise linear function such that f is linear on the intervals between $-10, -1, 0, 1, 1.5, 2, 10$ with $f(-10) = f(-1) = f(3) = f(10) = 1$, $f(0) = f(1) = f(2) = 0$ and $f(1.5) =$

-0.5. Set $f_n = f + \phi/n$ where ϕ is a continuous function supported in $]0,1[$ with $\phi(x) > 0$ on $]0,1[$. Then $\|f_n - f\|_\infty = \|\phi\|_\infty/n \to 0$ while

$$[f_n \le 0] = \{0\} \cup [1,2] \xrightarrow{\mathcal{H}} \{0\} \cup [1,2] \neq [f \le 0] = [0,2].$$

We nevertheless obtain such a result under an additional assumption of topological nature of the limit set. Notice that we then obtain also convergence of the 0-level-set.

Theorem 3.3. *Consider a sequence of continuous functions (f_n) and a function f defined on B_∞ with values in $\mathbb{R}$. Assume that the sequence (f_n) is uniformly converging to f in B_∞ such that*

$$[f \le 0] \neq \emptyset.$$

If the following regularity from the inside condition

$$(\mathbf{A}_<) \quad \overline{[f < 0]} = [f \le 0]$$

holds then the compact level sets converge

$$[f_n \le 0] \xrightarrow{\mathcal{H}} [f \le 0]. \tag{3.1}$$

If the following regularity from the outside condition

$$(\mathbf{A}_>) \quad \overbrace{[f \le 0]}^{\circ} = [f < 0]$$

holds, then the open level sets converge

$$[f_n < 0] \xrightarrow{\mathcal{H}} [f < 0]. \tag{3.2}$$

If both conditions $(\mathbf{A}_<)$ and $(\mathbf{A}_>)$ hold, then the boundaries converge

$$[f_n = 0] \xrightarrow{\mathcal{H}} [f = 0]. \tag{3.3}$$

We propose an elementary proof of this theorem. It relies on a pinching lemma for the Hausdorff convergence.

Lemma 3.4. *Let (A_n), (B_n) and (C_n) be three sequences of compact sets (resp. open sets) such that for all n, $A_n \subset B_n \subset C_n$. Let K be a compact set (resp. open set). If (A_n) and (C_n) Hausdorff converge to K then (B_n) Hausdorff converges to K.*

Proof of Lemma 3.4. It suffices to prove the lemma for compact sets. Indeed the case of open sets is obtained by passing to the complement. One checks that

$$\rho(B_n, K) = \sup_{x \in B_n} d_K(x) \le \sup_{x \in C_n} d_K(x) = \rho(C_n, K) \le d_{\mathcal{H}}(C_n, K),$$

and

$$\rho(K, B_n) = \sup_{x \in K} d_{B_n}(x) \le \sup_{x \in K} d_{A_n}(x) = \rho(K, A_n) \le d_{\mathcal{H}}(A_n, K).$$

Then, $0 \le d_{\mathcal{H}}(B_n, K) \le \sup(d_{\mathcal{H}}(A_n, K), d_{\mathcal{H}}(C_n, K)) \to 0$. □

Proof of Theorem 3.3.

Step 1: Construction of barrier domains. Set $\epsilon_n = \sup_{k\geq n} \|f - f_k\|_\infty$. Since the sequence (f_n) uniformly converges to f, the sequence ϵ_n is decreasing and converges to 0. For each integer n, we define compact sets by

$$A_n = [f \leq -\epsilon_n], B_n = [f_n \leq 0] \text{ and } C_n = [f \leq \epsilon_n].$$

The triangle inequality

$$f(x) - \epsilon_n \leq f(x) - \|f_n - f\|_\infty \leq f_n(x) \leq f(x) + \|f_n - f\|_\infty \leq f(x) + \epsilon_n,$$

valid for each $x \in B_\infty$ translates to inclusions for level sets

$$\forall n, \quad A_n \subset B_n \subset C_n.$$

Notice that the sets A_n *are* not empty for n large enough since $[f < 0] \neq \emptyset$. In a similar way, set

$$\mathcal{A}_n = [f < -\epsilon_n], \ \mathcal{B}_n = [f_n < 0] \text{ and } \mathcal{C}_n = [f < \epsilon_n]$$

so that $\mathcal{A}_n \subset \mathcal{B}_n \subset \mathcal{C}_n$. Notice that the monotonicity of the sequence (ϵ_n) implies the monotonicity of the sequences (A_n), (C_n), $(\mathcal{A}_n)$ and $(\mathcal{C}_n)$.

Step 2: Proof of (3.1). Now, we study the limits of (A_n) and (C_n). The sequence (C_n) is a decreasing sequence of non empty compact sets, hence it converges in the sense of Hausdorff to its intersection by Lemma 2.2

$$C_n \xrightarrow{\mathcal{H}} \bigcap_{n\geq 0} C_n = \bigcap_{n\geq 0} [f \leq \epsilon_n] = [f \leq 0].$$

The sequence (A_n) is a increasing sequence of compacts set, hence it converges in the sense of Hausdorff to the closure of its union by Lemma 2.2:

$$A_n \xrightarrow{\mathcal{H}} \overline{\bigcup_{n\geq 0} A_n} = \overline{\bigcup_{n\geq 0} [f \leq -\epsilon_n]} = \overline{[f < 0]}.$$

We conclude thanks to the comparison principle (Lem. 3.4).

Step 3: Proof of (3.2). It is a straightforward adaptation of the previous step: $(\mathcal{A}_n)$ is an increasing sequence of open sets, then it converges in the sense of Hausdorff to its union that is $[f < 0]$. Then, $(\mathcal{C}_n)$ is a decreasing sequence of open sets. Therefore it converges to the interior of the intersection of all the open sets: namely the interior of $[f \leq 0]$ which is exactly $[f < 0]$ since $(\mathbf{A}_>)$ holds.

Step 4: Proof of (3.3). Set $\Gamma = [f = 0]$ and $\Gamma_n = [f_n = 0]$. These are compact sets.

We first prove that $\rho(\Gamma_n, \Gamma)$ the excess from Γ_n to Γ tends to 0. For all n, there exists $x_n \in \Gamma_n$ such that $d(x_n, \Gamma) = \rho(\Gamma_n, \Gamma)$. Since the sequence (x_n) stays in the compact B_∞, there are accumulation points. Let $\bar{x} = \lim x_{n_k}$ be such an accumulation point, then $f(\bar{x}) = \lim f_{n_k}(x_{n_k}) = 0$ since (f_n) converges uniformly to f and $\bar{x} \in \Gamma$. Then $d_\Gamma(x_{n_k}) \leq \|x_{n_k} - \bar{x}\| \to 0$.

We then prove that $\rho(\Gamma, \Gamma_n)$ the excess from Γ to Γ_n tends to 0. For all n, there exists $x_n \in \Gamma$ such that $d_{\Gamma_n}(x_n) = \rho(\Gamma, \Gamma_n)$. By *reductio ad absurdum*, assume that there exists $\eta > 0$ such that $d_{\Gamma_n}(x_n) \geq \eta$. Since the sequence (x_n) stays in the compact Γ, there are accumulation points. Let $\bar{x} = \lim x_{n_k}$ be such an accumulation point in Γ. Hence there is a rank k_0 such that for $k \geq k_0$, $\|\bar{x} - x_{n_k}\| \leq \eta/2$ then $d_{\Gamma_{n_k}}(\bar{x}) \geq \eta/2$.

Now, we use $(\mathbf{A}_<)$ since $\bar{x} \in \partial[f < 0]$, there exists a point y with $\|y - \bar{x}\| \leq \eta/4$ such that $f(y) < 0$. Since $f_n(y) \to f(y)$, there exists a rank $k_1 > k_0$ such that for $k \geq k_1$, $f_{n_k}(y) < 0$. Using $(\mathbf{A}_>)$, $\bar{x} \in \partial[f > 0]$, there exists a point z with $\|z - \bar{x}\| \leq \eta/4$ such that $f(z) > 0$. Since $f_n(z) \to f(z)$, there exists a rank $k_2 > k_1$ such that for $k \geq k_2$, $f_{n_k}(z) > 0$.

Since the ball $B(\bar{x}, \eta/4)$ is convex, it contains the whole segment $[y, z]$. For $k \geq k_2$, f_{n_k} is continuous on $[y, z]$ with $f_{n_k}(y) f_{n_k}(z) < 0$ then by the intermediate value theorem, there is a point $t \in [y, z] \subset B(\bar{x}, \eta/4)$ with $f_{n_k}(t) = 0$. Then, $d_{\Gamma_{n_k}}(\bar{x}) \leq \|\bar{x} - t\| \leq \eta/4$ that negates $d_{\Gamma_{n_k}}(\bar{x}) \geq \eta/2$

□

Remark 3.5. Notice that the implication,

$$\text{If } \sup_{B_\infty} |f_n(x) - f(x)| \to 0 \text{ and } \overline{[f < 0]} = [f \leq 0], \text{ then } [f_n \leq 0] \xrightarrow{\mathcal{H}} [f \leq 0]$$

remains true under the weaker assumption that the functions f and f_n are only lower semi-continuous. In the same spirit,

$$\text{If } \sup_{B_\infty} |f_n(x) - f(x)| \to 0 \text{ and } \overbrace{\overline{[f \leq 0]}}^{\circ} = [f < 0], \text{ then } [f_n < 0] \xrightarrow{\mathcal{H}} [f < 0],$$

when the functions are upper semi-continuous.

Let us comment the meaning of the topological assumption $(\mathbf{A}) = (\mathbf{A}_<)$ and $(\mathbf{A}_>)$. On the one hand, it prohibits the 0 level-set $[f = 0]$ to be fat. In general, the two boundaries $\partial[f < 0]$ and $\partial[f > 0]$ do not coincide with the 0 level-set that is the space delimited by these boundaries. But it also prohibits isolated points and more generally parts of the boundary of codimension bigger than two.

The topological assumption $(\mathbf{A})$ is sharp as stated by the following proposition.

Proposition 3.6. *Let f be a continuous function such that*

$$[f = 0] \not\subset \overline{[f < 0]}.$$

There exists a sequence of continuous functions (f_n) that converges uniformly to f and such that $[f_n \leq 0]$ Hausdorff converges to a compact set that is not $[f \leq 0]$.

Proof of Proposition 3.6. Since

$$[f = 0] \not\subset \overline{[f < 0]},$$

there exist a real numbers a with $f(a) = 0$ and a real number $\delta > 0$ such that $d(a, \overline{[f < 0]}) \geq \delta$ that is to say $f(x) \geq 0$ on $B(a, \delta)$. Let χ be a continuous function on $[0, +\infty)$ with $\chi(t) > 0$ on $[0, 1)$ and $\chi(x) = 0$ for $t > 1$. Set

$$f_n(x) \;=\; f(x) \;+\; \frac{1}{n}\,\chi\left(\frac{\|x - a\|}{\delta}\right).$$

By construction, $[f_n \leq 0] = [f \leq 0] \setminus B(a, \delta)$ is a constant sequence of compact sets then

$$[f_n \leq 0] \xrightarrow{\mathcal{H}} [f \leq 0] \setminus B(a, \delta) \neq [f \leq 0],$$

while $\|f_n - f\|_\infty \leq \|\chi\|_\infty/n \to 0$. □

3.2. Convergence rate

A natural question is to evaluate the rate of convergence of $[f_n \leq 0]$ to $[f \leq 0]$ stated in Theorem 3.3. A first elementary remark is that this problem reduces to the convergence rate of the sublevel sets of f reducing the problem to the study of the mapping:

$$\Psi : \lambda \mapsto [f \leq \lambda].$$

Notice that f is continuous from the compact set B_∞ with values in a compact set and hence Ψ maps a real number to a compact set of $\mathbb{R}^d$.

Proposition 3.7. *One has:*

$$d_{\mathcal{H}}([f \leq 0],[f_n \leq 0]) \leq d_{\mathcal{H}}([f < -\|f - f_n\|_\infty]),[f \leq 0]) + d_{\mathcal{H}}([f \leq 0],[f \leq \|f - f_n\|_\infty]).$$

Proof of Proposition 3.7. The basic estimation

$$f(x) - \|f - f_n\|_\infty \leq f_n(x) \leq f(x) + \|f - f_n\|_\infty,$$

translates into the pinching of $[f \leq 0]$:

$$[f \leq -\|f - f_n\|_\infty] \subset [f_n \leq 0] \subset [f \leq \|f - f_n\|_\infty].$$

Since one of course also has

$$[f \leq -\|f - f_n\|_\infty] \subset [f \leq 0] \subset [f \leq \|f - f_n\|_\infty],$$

then by Lemma 2.3:

$$d_{\mathcal{H}}([f \leq 0],[f_n \leq 0]) \leq d_{\mathcal{H}}([f \leq -\|f - f_n\|_\infty],[f \leq \|f - f_n\|_\infty]).$$

One concludes by the triangle inequality. □

We now prove that Ψ is a Lipschitz map. To that end, we require stronger regularity. A sufficient condition for $(\mathbf{A}_<)$ and $(\mathbf{A}_>)$ is that the 0 level set is a (at least Lipschitz) manifold of codimension 1. By the implicit function theorem, this is true if the strong regularity assumption S is made

($\mathbf{S}$) there are constants η and $c > 0$ such that f is continuously differentiable on Γ_η and

$$\|\nabla f(x)\| \geq c \text{ for } x \in \Gamma_\eta,$$

where $\Gamma_\eta = [d_\Gamma \leq \eta]$ is the η tubular neighborhood of $\Gamma = [f = 0]$.

Proposition 3.8. *Assume that* ($\mathbf{S}$) *holds, let* (ϵ_n) *be a sequence of non negative real numbers converging to* 0 *then there exist a non negative real number* c *depending only on* f *and a rank* n_0 *depending on* (ϵ_n) *such that for* $n \geq n_0$

$$d_{\mathcal{H}}([f \leq 0],[f \leq \pm\epsilon_n]) \leq c\epsilon_n.$$

Proof of Proposition 3.8.

Step 1: Preliminary remarks. Under (**S**), using the continuity of the gradient ∇f on the compact set Γ_η, there exists $\delta > 0$ such that for any x, y in Γ_η

$$\|x - y\| \leq \delta \implies \nabla f(x) \cdot \nabla f(y) \geq \frac{1}{2}|\nabla f(y)|^2.$$

Set $\bar{\epsilon} = \inf(\eta, \delta)$.

Since the sequence of functions $g_n^{\pm} = f \pm \epsilon_n$ uniformly converges to f, Theorem 3.3 insures that the level sets $[f \leq \pm\epsilon_n] = [g_n^{\pm} \leq 0]$ converge in the Hausdorff sense to $[f \leq 0]$ and then there is a rank n_0 such that $[f \leq \pm\epsilon_n] \subset \Gamma_{\bar{\epsilon}}$ for all $n \geq n_0$. In particular, by the local inverse theorem, the level sets $[f = 0] = \partial[f \leq 0]$ and $[f = \epsilon_n] = \partial[f \leq \epsilon_n]$ are smooth for $n \geq n_0$. In the sequel, we assume $n \geq n_0$.

Step 2: An upper bound to $\mathbf{d}_{\mathcal{H}}\boldsymbol{([f \leq 0]), [f \leq \epsilon_n])}$.

Since the level sets of f are nested: $[f \leq 0] \subset [f \leq \epsilon_n]$, it suffices to dominate the excess of $[f \leq \epsilon_n]$ to $[f \leq 0]$ since

$$d_{\mathcal{H}}([f \leq \epsilon_n], [f \leq 0]) = \rho([f \leq \epsilon_n], [f \leq 0]).$$

Let $y \in [f \leq \epsilon_n]$ such that $d_{[f\leq 0]}(y) = \rho([f \leq \epsilon_n], [f \leq 0]) > 0$. In particular, $y \notin [f \leq 0]$ and $d_{[f\leq 0]}(y) = d_\Gamma(y)$, we therefore get

$$d_{\mathcal{H}}([f \leq \epsilon_n], [f \leq 0]) = d_\Gamma(y).$$

Notice that the function

$$\varphi(x) = \left|\frac{f(x)}{b_{[f\leq 0]}(x)}\right| = \frac{|f(x)|}{d_\Gamma(x)}$$

is continuous on $B_\infty \setminus \Gamma$.

We claim it can be extended by continuity on the whole B_∞ by setting $\varphi(x) = \partial_{\boldsymbol{n}} f(x)$ for any $x \in \Gamma$. Indeed, the lower bound of the gradient (**S**) implies that Γ is $\mathcal{C}^1$ and hence $b_{[f\leq 0]}$ is differentiable for x in $\Gamma = [f = 0]$ (see [12], Thm. 8-2, pp. 366). Moreover, the gradient $\nabla f(x)$ and $\boldsymbol{n}(x)$ the outer normal to $\Gamma = \partial[f < 0]$ are colinear both pointing from $[f < 0]$ to $[f > 0]$. Then, Taylor's formulae for f and b_Γ provides:

$$f(x + \boldsymbol{h}) = \nabla f(x) \cdot \boldsymbol{h} + o(\|\boldsymbol{h}\|) = \partial_{\boldsymbol{n}} f(x)\boldsymbol{n}(x) \cdot \boldsymbol{h} + o(\|\boldsymbol{h}\|) \text{ and } b_{[f\leq 0]}(x + \boldsymbol{h}) = \boldsymbol{n}(x) \cdot \boldsymbol{h} + o(\|\boldsymbol{h}\|),$$

where $\partial_{\boldsymbol{n}} f(x)$ denote the normal derivative of f. We get for x in Γ:

$$\frac{f(x + \boldsymbol{h})}{b_{[f\leq 0]}(x + \boldsymbol{h})} = \frac{\partial_n f(x)\ \boldsymbol{n}(x) \cdot \boldsymbol{h} + o(\|\boldsymbol{h}\|)}{\boldsymbol{n}(x) \cdot \boldsymbol{h} + o(\|\boldsymbol{h}\|)} \xrightarrow[\|\boldsymbol{h}\| \to 0]{} \partial_n f(x) > 0.$$

We conclude by taking the absolute value.

By construction, the function φ takes positive values. In particular, by compactness of B_∞ there is a constant $c_2 > 0$ such that for all $x \in B_\infty$

$$c_2 d_\Gamma(x) \leq |f(x)|.$$

We conclude

$$d_\Gamma(y) \le \frac{|f(y)|}{c_2} \le \frac{\epsilon_n}{c_2}.$$

Step 3: An upper bound to $\mathbf{d_{\mathcal{H}}([f \le 0], [f \le -\epsilon_n])}$. We deduce from the inclusion $[f \le -\epsilon_n]) \subset [f \le 0]$ that it holds:

$$d_{\mathcal{H}}([f \le -\epsilon_n], [f \le 0]) = \rho([f \le 0], [f \le -\epsilon_n]).$$

Let $x \in [f \le 0]$ and $y \in [f \le -\epsilon_n]$ such that

$$\|x - y\| = d(x, [f \le -\epsilon_n]) = \rho([f \le 0], [f \le -\epsilon_n]).$$

By construction, $x \in [f = 0]$ and $y \in [f = -\epsilon_n]$ and these level sets are smooth. As y minimizes the distance to x over $[f \le -\epsilon_n]$, the Euler-Lagrange equation implies that

$$y - x = -\frac{\|x - y\|}{\|\nabla f(y)\|}\nabla f(y),$$

and we get

$$\begin{aligned}
\epsilon_n = f(x) - f(y) &= -\int_0^1 \nabla f(x + t(y - x)) \cdot (y - x)\, \mathrm{d}t \\
&= \int_0^1 \frac{\|x - y\|}{\|\nabla f(y)\|}\, \nabla f(x + t(y - x)) \cdot \nabla f(y)\, \mathrm{d}t \\
&\ge \frac{1}{2}\int_0^1 \|\nabla f(y)\| \|x - y\| \mathrm{d}t \\
&\ge \frac{c}{2}\|x - y\| \; = \; \frac{c}{2}\, d_{\mathcal{H}}([f \le -\epsilon_n], [f \le 0]).
\end{aligned}$$

□

Let us notice that $(\mathbf{S})$ is not mandatory, one can get convergence when it is not satisfied. Condition $(\mathbf{S})$ means that the tangent space to the surface $y = f(x)$ is never horizontal in the neighborhood of $[f = 0]$. When this condition does not hold, the idea is to reach a positive slope. To that end, we reparametrize the set $[f = 0]$ with a new function $g = f \circ \varphi$ such that $[f = 0] = [g = 0]$ and that the tangent space to the surface $y = g(x)$ is never horizontal in the neighborhood of $[f = 0]$. This corresponds to the redistancing procedure that appears in the numerical implementation of the level set method.

For a given continuous function φ on $\mathbb{R}$, $\mathcal{C}^1$ on $\mathbb{R}^*$ with $\varphi'(t) > 0$ for $t \neq 0$, we say that the boundary Γ has the property $(\mathbf{S}_\varphi)$ if there are constants η and c such that $\varphi \circ f$ is continuously differentiable on Γ_η and

$$\|\nabla(\varphi \circ f)(x)\| \ge c \text{ for } x \in \Gamma_\eta.$$

It is a generalization of $(\mathbf{S})$ since the first condition corresponds to the case $\varphi_0(t) = t$ *i.e.* $(\mathbf{S}) = (\mathbf{S}_{\varphi_\mathbf{0}})$. Notice that φ is strictly increasing and $[f \le \pm\epsilon_n] = [\phi \circ f \le \pm\phi(\epsilon_n)]$, then one deduces from Proposition 3.8 that

Theorem 3.9. *Assume that $(\mathbf{S}_\varphi)$ holds, let (ϵ_n) be a sequence of non negative real numbers converging to 0 then there exist a non negative real number c depending only on f and a rank n_0 depending on (f_n) such that* *for* $n \geq n_0$

$$d_{\mathcal{H}}([f \leq 0], [f \leq \pm\epsilon_n]) \leq c\varphi(\epsilon_n).$$

Remark 3.10. Let us mention that the inequality in condition $(\mathbf{S}_\varphi)$ is a particular case of the Kurdyka-Lojasiewicz inequality. A nice reference on the topic is [7] see in particular Theorem 2 and its corollaries for statements similar to Proposition 3.9 where it is shown that the Lipschitz property of Ψ from $\mathbb{R}$ endowed with the metric $d(s,t) = |\varphi(t) - \varphi(s)|$ is equivalent to $(\mathbf{S}_\varphi)$. Here Ψ maps a real number t to the sublevel set $[f < t]$.

4. Oriented distance expectation of random sets

4.1. Definitions

We follow the usual strategy of a parametrization based expectation as done in [17, 18].

Definition 4.1 (Oriented distance based expectation of a random compact set)**.** Let $\boldsymbol{K}$ be a random compact set. Let $b_{\boldsymbol{K}}$ be the random process of its oriented distance function. If this process has an expectation $\mathbb{E}[b_{\boldsymbol{K}}]$, then the oriented distance expectation of $\boldsymbol{K}$ is defined as the compact set $\mathbb{E}[\boldsymbol{K}] = [\mathbb{E}[b_{\boldsymbol{K}}] \leq 0]$. The oriented distance expectation of the boundary of $\boldsymbol{K}$ is defined as the compact set $\mathbb{E}[\partial\boldsymbol{K}] = [\mathbb{E}[b_{\boldsymbol{K}}] = 0]$.

In order to validate this definition, one has to observe that each realization of $b_{\boldsymbol{K}}$ is a 1-Lipschitz function. As the set $Lip(\bar{B}_\infty, 1)$ of 1-Lipschitz functions is a closed convex set, its expectation $\mathbb{E}[b_{\boldsymbol{K}}]$ is also a 1-Lipschitz function. On the contrary, the set of oriented distance function is not convex and the expection $\mathbb{E}[b_{\boldsymbol{K}}]$ may not be a oriented distance function. Of course, in order to obtain a computable notion, we need to use samples of the random compact set. We define an oriented distance empirical mean as follows

Definition 4.2 (Oriented distance empirical mean of a random compact set)**.** Let $\boldsymbol{K}_1, \ldots,\ \boldsymbol{K}_n$ be n random compact sets independent identically distributed. The empirical mean of the oriented distance mean is defined as

$$\hat{b}_n = \frac{1}{n}\sum_{i=1}^{n} b_{\boldsymbol{K}_i}.$$

The oriented distance empirical mean $\hat{\boldsymbol{K}}_n$ is defined as the compact set $[\hat{b}_n \leq 0]$. The oriented distance empirical mean of the boundary $\partial\hat{\boldsymbol{K}}_n$ is defined as the compact set $[\hat{b}_n = 0]$.

This definition corresponds to a plug-in estimator of the random sets based on the oriented distance parametrization.

In the very same way, let $\boldsymbol{O}$ be a random open set and $b_{\boldsymbol{O}}$ be the process of its oriented distance function. If this process as an expectation $\mathbb{E}[b_{\boldsymbol{O}}]$, we define the oriented distance expectation of $\boldsymbol{O}$ as $\mathbb{E}[\boldsymbol{O}] = [\mathbb{E}[b_{\boldsymbol{O}}] < 0]$ and the oriented distance expectation of the boundary as $\mathbb{E}[\partial\boldsymbol{O}] = [\mathbb{E}[b_{\boldsymbol{O}}] = 0]$. We also define the oriented distance empirical means as $\hat{\boldsymbol{O}}_n = [\hat{b}_n < 0]$ and $\partial\hat{\boldsymbol{O}}_n = [\hat{b}_n = 0]$ where $\hat{b}_n$ is th empirical mean of the process $b_{\boldsymbol{O}}$.

4.2. Comparison with other notions of expectation of random domains

Let us illustrate by some examples the main differences between this notion of expectation and the Aumann and Vorobe'v expectations. As a first example, we consider a random disk given as:

$$\boldsymbol{K}(\omega) = \beta D((a,0),1) + (1-\beta)D((-a,0),1),$$

where β follows a Bernoulli law of parameter $1/2$ and $D(x,1)$ is the closed disk of center x and radius 1. Let us first assume that $a < 1$. In order to determine the Vorobe'v expectation of K, that is a quantile of $\mathbb{P}[x \in K]$

with area $\mathbb{E}[|K|]$, we compute first the coverage function

$$\mathbb{P}[x \in K] = \begin{cases} 1 \text{ if } x \in D((a,0),1) \cap D((-a,0),1), \\ 1/2 \text{ if } x \in D((a,0),1) \Delta D((-a,0),1), \\ 0 \text{ else.} \end{cases}$$

Since $\mathbb{E}[|\boldsymbol{K}|] = \pi$, one easily sees that any compact set C such that

$$D((a,0),1) \cap D((-a,0),1) \subset C \subset D((a,0),1) \Delta D((-a,0),1),$$

with an area of π is a Vorobe'v expectation of $\boldsymbol{K}$. This example emphasizes the main flaw of the notion: it leads to nonuniqueness issue. In the same time, the Aumann expectation is computed through $h_{\boldsymbol{K}}$ the trace on the unit circle of the support function of $\boldsymbol{K}$:

$$h_{\boldsymbol{K}}(\theta) = (\beta a + (1-\beta)a) + cos(\theta) \text{ so that } \mathbb{E}[h_{\boldsymbol{K}}] = \cos(\theta),$$

the Aumann expectation of $\boldsymbol{K}$ is the unit close disk. In particular it does not depends on a. On the converse, the oriented distance function to $\boldsymbol{K}$ is

$$b_{\boldsymbol{K}}(x) = \beta \left| x - (a,0) \right| + (1-\beta) \left| x - (-a,0) \right| - 1,$$

so that

$$\mathbb{E}[b_{\boldsymbol{K}}](x) = \frac{1}{2}\left(|x - (a,0)| + |x - (-a,0)| \right) - 1,$$

and $\mathbb{E}[\boldsymbol{K}]$ is the the convex hull of the ellipse of foci the points $(\pm a, 0)$ and such that the sum of the distance to the foci is 2.

4.3. Limits of the oriented distance empirical mean and application for the random compact sets

Law of large numbers. We first establish the consistency of the oriented distance empirical mean estimators. This result is crucial as it provides a way to numerically access to the object we just defined.

Proposition 4.3 (Consistency of the empirical estimators)**.** *Let $\boldsymbol{K}$ be a random compact set such that $\mathbb{E}[b_K]$ exists. It holds:*

$$\lim_{n \to +\infty} \|\hat{b}_n - \mathbb{E}[b_{\boldsymbol{K}}]\|_\infty = 0 \text{ almost surely.}$$

Then, if moreover the function $\mathbb{E}[b_{\boldsymbol{K}}]$ has the property (A), that is both $(A_<)$ and $(A_>)$,

$$\hat{\boldsymbol{K}}_n \xrightarrow{\mathcal{H}} \mathbb{E}[\boldsymbol{K}] \text{ and } \partial\hat{\boldsymbol{K}}_n \xrightarrow{\mathcal{H}} \mathbb{E}[\partial \boldsymbol{K}] \text{ almost surely.}$$

The same properties hold also for random open sets.

Proof of Proposition 4.3. The usual law of large numbers implies pointwise convergence. As all the functions $\hat{b}_n$ and $\mathbb{E}[b_{\boldsymbol{K}}]$ are 1-Lipschitz functions, we also obtain uniform convergence over compact sets. Finally, the Hausdorff convergences are direct applications of Theorem 3.3.

Central limit theorem. We now quote the central limit theorem obtained by Jankowski and Stanberry in Theorem 2-6 and Proposition 2-7 of [18].

Proposition 4.4 (Central limit theorem for the ODF)**.** *There is a centered Gaussian random field Z with covariance*

$$cov[Z(x), Z(y)] = \mathbb{E}[b_{\boldsymbol{K}}(x) b_{\boldsymbol{K}}(y)] \;-\; \mathbb{E}[b_{\boldsymbol{K}}](x)\ \mathbb{E}[b_{\boldsymbol{K}}](y),$$

for any $x, y \in B_\infty$ such that

$$Z_n = \sqrt{n}\ \left(\hat{b}_n - \mathbb{E}[b_{\boldsymbol{K}}]\right) \xrightarrow{\mathcal{D}} Z,$$

in the space $\mathcal{C}(B_\infty, \mathbb{R})$ of continuous functions on $\bar{B}_\infty$ endowed with the uniform convergence topology. Moreover,

$$var[Z(x) - Z(y)] \le |x-y|^2,$$

and the sample paths of the process Z are α-Hölder for any $\alpha \in (0,1)$.

It is a direct application of the following general central limit theorem of uniformly Lipschitz processes that relies on two classical arguments: first, an estimate of the moment based on a combinatorial argument and second the Garsia, Rademich and Rumsay inequality ([10], Thm. B.1.5) applied on the regular domain B_∞.

Theorem 4.5. *Let L and M be two non negative real numbers and d an integer. Let K be a regular compact set $K \subset \mathbb{R}^d$. Let f be a random process defined on K such that for all $x, y \in K$*

$$|f(x) - f(y)| \le L|x-y| \text{ a.s. and } |f(x)| \le M \text{ a.s.}.$$

And let $f_1, \ldots, f_n$ be n independent and identically distributed random processes defined on K with the same distribution as f.

Set $\hat{f}_n = (f_1 + \ldots f_n)/n$ the empirical mean. Then, there exists a centered Gaussian random field Z with covariance

$$cov[Z(x), Z(y)] \;=\; \mathbb{E}[f(x)f(y)] \;-\; \mathbb{E}[f](x)\ \mathbb{E}[f](y),$$

for any $x, y \in K$ such that

$$Z_n \;=\; \sqrt{n}\ \left(\hat{f}_n - \mathbb{E}[f]\right) \xrightarrow{\mathcal{D}} Z,$$

in the space $\mathcal{C}(K, \mathbb{R})$ of continuous functions on K endowed with the uniform convergence topology.

Proof of Theorem 4.5. We first prove a moment estimate, then derive tightness and finally prove the convergence of the finite dimensional laws. Let us introduce some notations: we define independent centered processes on K^2 by

$$g_i(x,y) = (f_i(x) - \mathbb{E}[f](x)) - (f_i(y) - \mathbb{E}[f](y)).$$

One checks that $|g_i(x,y)| \le 2L|x-y|$ a.s. and therefore for any k

$$\mathbb{E}[g_i^k](x,y) \le (2L)^k |x-y|^k.$$

Let $N \geq 2$ be an integer and $\mathfrak{C}_N$ be the set

$$\mathfrak{C}_N = \{\mathfrak{c} \in (\mathbb{N} \setminus \{1\})^N \mid \mathfrak{c}_1 \geq \mathfrak{c}_2 \geq \cdots \geq \mathfrak{c}_N \text{ and } \mathfrak{c}_1 + \mathfrak{c}_2 + \cdots + \mathfrak{c}_N = N\},$$

that is the set of the ways to write N as a sum of integers distinct from 1. For $\mathfrak{c} \in \mathfrak{C}_N$, let $l(\mathfrak{c}) = \max\{k \mid \mathfrak{c}_k \neq 0\}$ be the number of non-zero terms of the sum. Since $\mathfrak{c}_i \neq 0$ implies $\mathfrak{c}_i \geq 2$, it holds $l(\mathfrak{c}) \leq N/2$.

Step 1: Estimating the moments. We compute

$$\begin{aligned}\mathbb{E}[|Z_n(x) - Z_n(y)|^N] &= \frac{1}{n^{N/2}} \sum_{i_1,\dots i_N=1}^{n} \mathbb{E}[g_{i_1} \dots g_{i_N}](x,y),\\ &= \frac{1}{n^{N/2}} \sum_{\mathfrak{c}\in\mathfrak{C}_N} \sum_{\substack{i_1,\dots,i_{l(\mathfrak{c})}=1\\ i_j\neq i_k \text{if} j\neq k}}^{n} \prod_{j=1}^{l(\mathfrak{c})} \mathbb{E}[g_{i_j}^{\mathfrak{c}_j}],\end{aligned}$$

since the processes g_i are independent and centered. We check

$$\alpha_{\mathfrak{c},n} = n^{-N/2} \sum_{\substack{i_1,\dots,i_{l(\mathfrak{c})}=1\\ i_j\neq i_k \text{if} j\neq k}}^{n} 1 = n^{-N/2} \prod_{i=0}^{l(\mathfrak{c})-1} (n-i) \underset{n\to+\infty}{\sim} n^{l(\mathfrak{c})-N/2},$$

by a basic combinatorial argument. Since for any $\mathfrak{c} \in \mathfrak{C}$ one has

$$\prod_{j=1}^{l(\mathfrak{c})} \mathbb{E}[g_{i_j}^{\mathfrak{c}_j}] \leq (2L)^N |x-y|^N,$$

we deduce

$$\mathbb{E}[|Z_n(x) - Z_n(y)|^N] \leq \sum_{\mathfrak{c}\in\mathfrak{C}_N} \alpha_{\mathfrak{c},n} (2L)^N |x-y|^N.$$

Since $l(\mathfrak{c}) - N/2 \leq 0$, all the sequences $\alpha_{\mathfrak{c},n}$ are bounded and there exists a constant (depending on N) such that:

$$\mathbb{E}[|Z_n(x) - Z_n(y)|^N] \leq C_1(N)|x-y|^N.$$

In the very same way, we first check:

$$\mathbb{E}[|Z_n(x)|^N] = \frac{1}{n^{N/2}} \sum_{\mathfrak{c}\in\mathfrak{C}_N} \sum_{\substack{i_1,\dots,i_{l(\mathfrak{c})}=1\\ i_j\neq i_k \text{if} j\neq k}}^{n} \prod_{j=1}^{l(\mathfrak{c})} \mathbb{E}[\tilde{f}_{i_j}^{\mathfrak{c}_j}] \leq \sum_{\mathfrak{c}\in\mathfrak{C}_N} \alpha_{\mathfrak{c},n} M^N,$$

where $\tilde{f}_i = f_i - \mathbb{E}[f]$ denotes the centered version of f_i. Then, we obtain the existence of a constant such that

$$\mathbb{E}[|Z_n(x)|^N] \leq C_2(N).$$

Step 2: Obtaining tightness. Under these bounds on the moments, one can directly use the Kolmogorov tightness criterion for continuous random fields (see [20], Thm. 1.4.7). For the sake of completeness we provide the proof. In order to derive the tightness, we want to apply Ascoli's Theorem.

To that end, we first study the equicontinuity properties and use Garsia, Rademich and Rumsay inequality. For arbitrary δ and $\beta > 2d$, there exists a universal constant $c > 0$ such that

$$|Z_n(\omega, x) - Z_n(\omega, y)| \ \leq \ c|x-y|^{(\beta-2d)/\delta} \ \left(\int_K \int_K \frac{|Z_n(\omega,\eta) - Z_n(\omega,\xi)|^\delta}{|\eta-\xi|^\beta} \, d\eta d\xi \right)^{1/\delta},$$

since Z_n is continuous. We set

$$C(\omega) = c^\delta \int_K \int_K \frac{|Z_n(\eta) - Z_n(\xi)|^\delta}{|\eta-\xi|^\beta} \, d\eta d\xi.$$

We specify the properties of the map $\omega \mapsto C(\omega)$. By Fubini's Theorem for positive functions,

$$\mathbb{E}\left[C\right] = \int_K \int_K \frac{\mathbb{E}[|Z_n(\eta) - Z_n(\xi)|^\delta]}{|\eta-\xi|^\beta} \, d\eta d\xi \leq C_1(\delta) \int_K \int_K |\eta - \xi|^{\delta-\beta} \eta d\xi,$$

for any integer $\delta > d$. Therefore, now if $\delta > \beta$ then the right hand side is finite and therefore the random variable $C(\omega)$ has a finite expectation $\mathbb{E}\left[C\right] < +\infty$. We fix $\varepsilon > 0$. For $\delta > \beta > 2d$, Markov inequality implies that

$$\mathbb{P}\left[C > \frac{\mathbb{E}[C]}{\varepsilon} \right] \leq \varepsilon.$$

We then study the equiboundedness properties of Z_n. We fix a point x_0 in K and apply Markov inequality

$$\mathbb{P}\left[|Z_n(x_0)| \geq \left(\frac{C_2(\delta)}{\varepsilon} \right)^{1/\delta} \right] = \mathbb{P}\left[|Z_n(x_0)|^\delta \geq \frac{C_2(\delta)}{\varepsilon} \right] \leq \mathbb{P}\left[|Z_n(x_0)|^\delta \geq \frac{\mathbb{E}[|Z_n(x_0)|^\delta]}{\varepsilon} \right] \leq \varepsilon.$$

Noticing that if the process Z_n satisfies the condition

$$(H) \quad |Z_n(x_0)| \leq \left(\frac{C_2(\delta)}{\varepsilon} \right)^{1/\delta} \text{ and } C < \frac{\mathbb{E}[C]}{\varepsilon},$$

then for all $x \in K$,

$$|Z_n(x)| \leq \left(\frac{C_2(\delta)}{\varepsilon} \right)^{1/\delta} + \frac{\mathbb{E}[C]}{\varepsilon} \, (diam(K))^{(\beta-2d)/\delta} = \tilde{M}.$$

Now we consider the set $\mathcal{K}_\varepsilon$ of continuous functions φ defined on K such that

$$|\varphi(x) - \varphi(y)| \leq \left(\frac{\mathbb{E}[C]}{\varepsilon} \right)^{1/\delta} |x-y|^{(\beta-2d)/\delta} \text{ and } |\varphi(x)| \leq \tilde{M}.$$

Since $(\beta - 2d)/\delta > 0$, the set $\mathcal{K}_\varepsilon$ is compact in $\mathcal{C}(K, \mathbb{R})$ by Ascoli's theorem. Since $Z_n(\omega, .) \in \mathcal{K}_\varepsilon$ if the process Z_n satisfies the condition (H), we have proven that $\mathbb{P}[Z_n \in \mathcal{K}_\varepsilon] \geq 1 - 2\epsilon$ that is the sequence (Z_n) is tight in $\mathcal{C}(K, \mathbb{R})$.

Step 3: Conclusion. The convergence of the finite-dimensional distributions is guaranteed by the classical central limit theorem. As (Z_n) is tight in $\mathcal{C}(K, \mathbb{R})$, we deduce that Z_n converges in distribution to Z in the space $\mathcal{C}(K, \mathbb{R})$ (see for instance [6]).

□

4.4. On confidence neighborhood

In practical computations, the limit sets $\mathbb{E}[\boldsymbol{K}]$ and $\mathbb{E}[\partial \boldsymbol{K}]$ are approximated through a numerical realization of a term of the sequences $\hat{\boldsymbol{K}}_n$ and $\partial \hat{\boldsymbol{K}}_n$ obtained by a sampling method like Monte Carlo. Two distinct sources of errors then appear: the first one is a deterministic one connected to the computation of each of the realizations of the random sets $\boldsymbol{K}_i$, the second one is a stochastic one due to the Monte-Carlo method. Since there are usually no convergence estimates for shape optimization problem, we shall only consider the second source of error and assume for a while that the sets are exactly known. Given a threshold $p_{\min} \in (0,1)$, we aim at compute sets V such that $\mathbb{P}[\hat{\boldsymbol{K}}_n \ \subset V] \geq p_{min}$.

A basic error estimate is given by the following remark: the triangle inequality

$$\hat{b}_n(x) - \|\hat{b}_n - \mathbb{E}[b_{\boldsymbol{K}}]\|_\infty \leq \mathbb{E}[b_{\boldsymbol{K}}](x) \leq \hat{b}_n(x) + \|\hat{b}_n - \mathbb{E}[b_{\boldsymbol{K}}]\|_\infty,$$

valid for each $x \in B_\infty$ translates to inclusions for level sets

$$\forall n, \quad [\hat{b}_n \leq -\|\hat{b}_n - \mathbb{E}[b_{\boldsymbol{K}}]\|_\infty] \subset \mathbb{E}[\boldsymbol{K}] \subset [\hat{b}_n \leq \|\hat{b}_n - \mathbb{E}[b_{\boldsymbol{K}}]\|_\infty].$$

which pinch the target set $\mathbb{E}[\boldsymbol{K}]$ between two sublevel sets $[\hat{b}_n \pm \|\hat{b}_n - \mathbb{E}[b_{\boldsymbol{K}}]\|_\infty]$ where the function $\hat{b}_n$ is computable. Assuming that $\mathbb{E}[\boldsymbol{K}]$ satisfies $(\mathbf{S}_\varphi)$ holds for some function φ, the inclusions

$$[\mathbb{E}[b_{\boldsymbol{K}}] \leq -2\|\hat{b}_n - \mathbb{E}[b_{\boldsymbol{K}}]\|_\infty] \subset [\hat{b}_n \leq -\|\hat{b}_n - \mathbb{E}[b_{\boldsymbol{K}}]\|_\infty]$$

$$[\hat{b}_n \leq \|\hat{b}_n - \mathbb{E}[b_{\boldsymbol{K}}]\|_\infty] \subset [\mathbb{E}[b_{\boldsymbol{K}}] \leq 2\|\hat{b}_n - \mathbb{E}[b_{\boldsymbol{K}}]\|_\infty]$$

provide, by the pinching Lemma 2.3 and Theorem 3.9, the estimate

$$d_{\mathcal{H}}(\hat{\boldsymbol{K}}_n,\ \mathbb{E}[\boldsymbol{K}]) \ \leq \ C\ \varphi(\|\hat{b}_n - \mathbb{E}[b_{\boldsymbol{K}}]\|_\infty).$$

This allows to translate an information of the residual $\|\hat{b}_n - \mathbb{E}[b_{\boldsymbol{K}}]\|_\infty$ into an information of the set. Of course, we have no direct access to this quantity but the central limit theorem provides the asymptotic approximation $Z/\sqrt{n}$ where Z is a zero mean Gaussian field of variance $var(b_{\boldsymbol{K}})$. In particular, Z is a continuous process by Proposition 4.4, hence the random variable $SZ = \sup_{x \in B_\infty} |Z(x)|$ is well defined and we obtain the trivial statement:

Proposition 4.6 (Convergence rate for the plug-in estimator). *If $\mathbb{E}[\boldsymbol{K}]$ satisfies $(\mathbf{S}_\varphi)$ holds for some function φ and $\mathbb{P}(SZ \leq M) \geq \alpha$ then*

$$\mathbb{P}\left(d_{\mathcal{H}}(\ \hat{\boldsymbol{K}}_n,\ \mathbb{E}[\boldsymbol{K}]) \ \leq \ C\ \varphi\left(\frac{M}{\sqrt{n}}\right)\right) \geq \alpha.$$

In [18], the authors define a confidence neighborhood based on the law of sup. However, it is very rare to be able to analytically determine this law and extremely difficulty to numerically determine its quantiles.

5. Application to the Bernoulli free boundary problem

As an application of the tools built in the previous section, we consider the exterior Bernoulli free boundary problem.

5.1. Presentation of the Bernoulli free boundary problem

We consider K a compact regular domain in $\mathbb{R}^N$ and λ a positive real number. The problem of finding a domain D and a function u such that the capacitary potential of K in D defined as the solution of

$$\begin{cases} -\Delta u = 0 & \text{in } D \setminus K \\ u = 1 & \text{on } K \\ u = 0 & \text{on } \partial D \end{cases} \tag{5.1}$$

satisfies the overdetermination condition

$$|\nabla u| = \lambda \text{ on } \partial D, \tag{5.2}$$

is called the exterior Bernoulli free boundary problem. Notice that this overdetermination condition means that for any $x \in \partial D$

$$\lim_{y \in \Omega, y \to x} |\nabla u(y)| = \lambda.$$

Existence of a solution has been established by Beurling [5] in the case of dimension two more than fifty years ago. Contemporary existence proofs are based on variational methods in the sense of shape optimization, the regularity of the free boundary has been studied by Alt and Caffarelli in [3]. If the inner boundary is convex, uniqueness of the solution has been shown by Tepper [24].

For general compact sets K, there may exist more than one solution to the free boundary value problem. However, there are classes of compact sets K such that the free boundary problem has a unique solution. The largest one is the star shaped domains. Tepper [24, 25] has also shown that if the inner boundary is starlike, then so is the outer boundary. Another interesting class is the one of convex domains. In Theorem 2-1 of [16], Henrot and Shahgholian proved that, if K is convex, the free boundary problem admits exactly one solution (D, u) and moreover this D is convex. A nice introduction lecture is [13].

5.2. Properties of the exterior Bernoulli free boundary problem

Consider $\mathcal{S}_c$ the space of compact domains in $\mathbb{R}^N$ that are star-shaped with respect to the origin and $\mathcal{S}_o$ the space of open domains that are star-shaped with respect to the origin. We define the map $\mathcal{B} : \mathcal{S}_c \times (0, +\infty) \to \mathcal{S}_o$ that maps a convex compact set K to the unique domain Ω solution of the exterior Bernoulli free boundary problem (5.1). We shall describe now the properties of the application $\mathcal{B}$ gathering results of Theorem 2-2, Theorem 3-1 of [14].

Proposition 5.1 (Properties of the map). *The map $\mathcal{B}$ is continuous and moreover:*

- *non decreasing with respect to inclusion: let K_1 and K_2 be two compact sets with $K_1 \subset K_2$ then for any $\lambda > 0$, $\mathcal{B}(K_1, \lambda) \subset \mathcal{B}(K_2, \lambda)$;*
- *decreasing with respect to the constant: let λ_1 and λ_2 be two non negative real numbers with $\lambda_1 < \lambda_2$, then for any K, $\mathcal{B}(K, \lambda_2) \subset \mathcal{B}(K, \lambda_1)$ with $\mathcal{B}(K, \lambda_2) \neq \mathcal{B}(K, \lambda_1)$.*

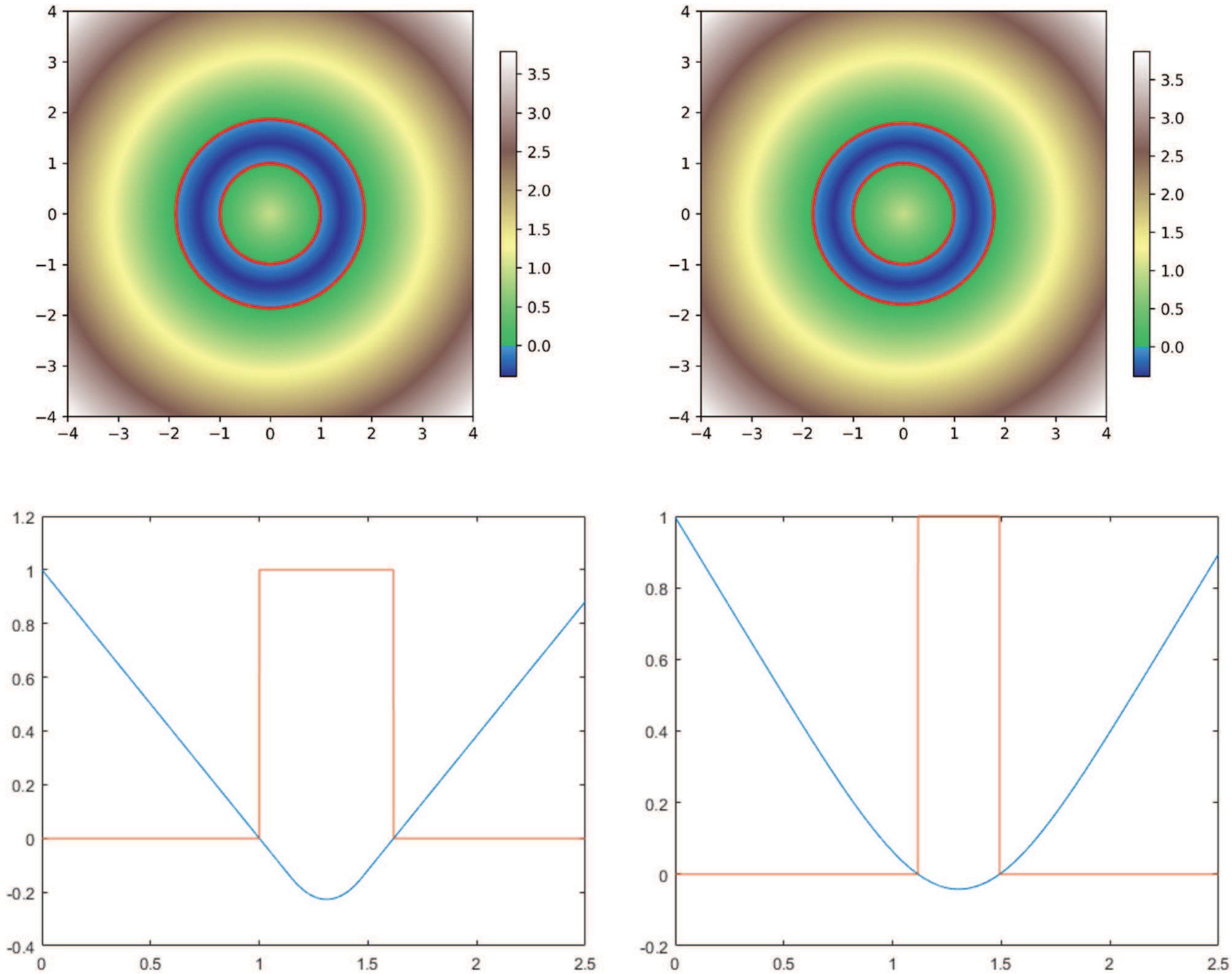

FIGURE 1. The expected distance function and expected domain: centered uniform (*left*), centered Gaussian of variance 0.3 (*right*).

As a consequence, the map $\mathcal{B}$ is measurable and given a random compact set K, one defines a random open set as $D = \mathcal{B}(K, \lambda)$. We have already considered the notion of expectation based on parametrization and the Vorobe'v expectation for this free boundary problem in [11].

5.3. Numerical illustration: an semi-analytic case

Let us consider the case where K is a random disk of radius r centered at the origin. Then, for each realization, the domain $D \setminus K$ is *an annulus* centered at the origin and of outer radius $f_d(r)$ where the function f_d is defined as:

$$f_2(r) = \frac{1}{\lambda W\left(\dfrac{1}{\lambda r}\right)} \text{ while } f_3(r) = \frac{\lambda r + \sqrt{\lambda^2 r^2 + 4\lambda r}}{2\lambda},$$

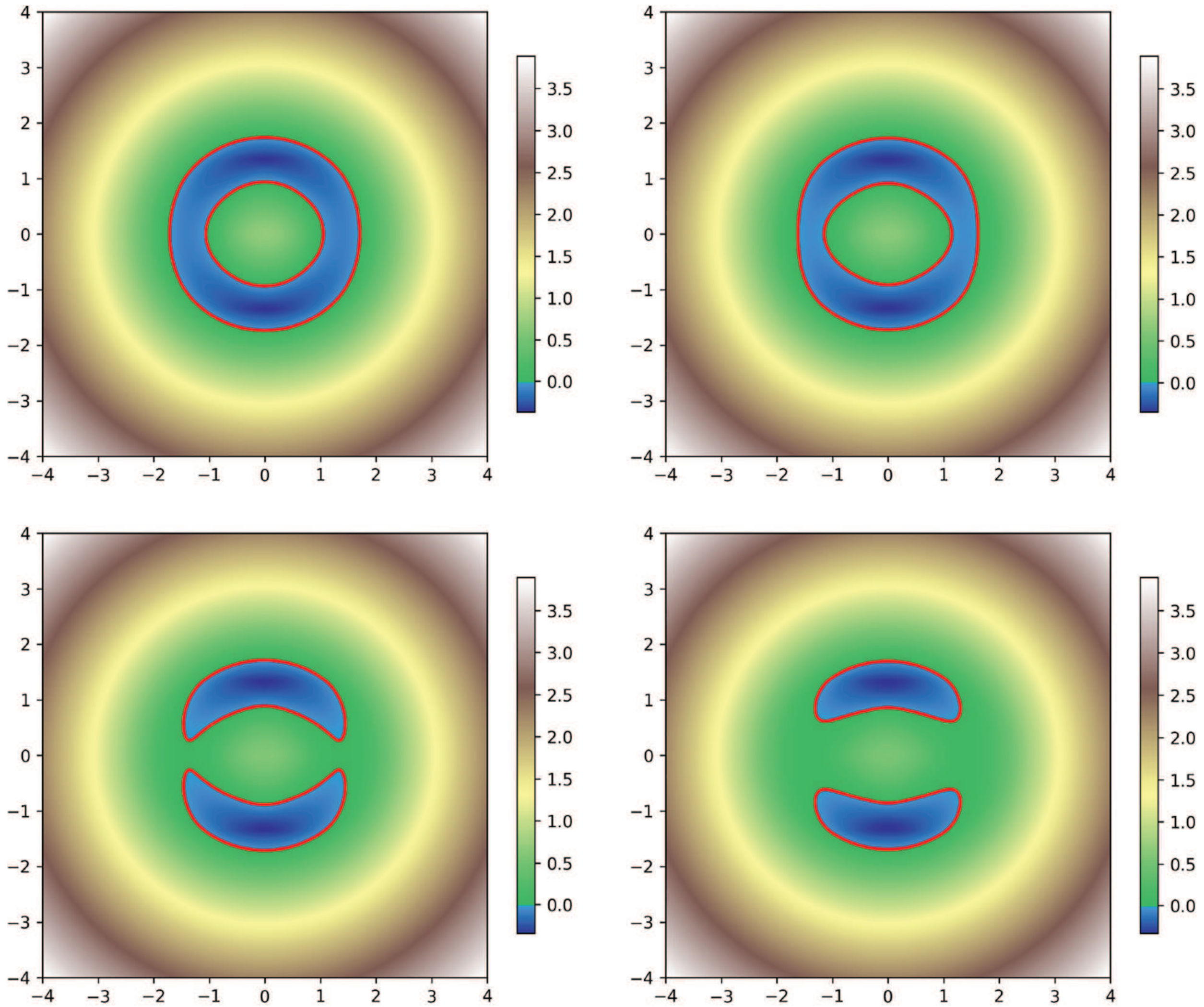

Figure 2. The expected distance function and expected domain: $a = 0.6$ (*top left*), $a = 0.7$ (*top right*), $a = 0.8$ (*bottom left*), $a = 0.9$ (*bottom right*).

where W is the inverse of $x \mapsto xe^x$. The oriented distance function to $D \setminus K$ is then explicitely known:

$$b_{D\setminus K}(x) = \begin{cases} |x| - f_d(r) \text{ if } |x| > (r + f_d(r))/2; \\ r - |x| \text{ else.} \end{cases}$$

In general, one cannot provide an explicit analytic expression of the expectation of the distance function. In order to go further, we have to do simulations even for simple cases. In Figure 1, we present the simulation obtained with 10 000 samples with the inner radius $r = 1 + 0.3\alpha$ where α is a random variable following the uniform law on $[-1/2, 1/2]$ and a centered Gaussian law of variance 0.3. The center of the inner disk is the origin. In the first line, we present the expectation of the distance function, the red line stands for the boundary of the expected domain. In the second line, we present a radial cut. In red, we present the indicator or characteristic function of the empirical mean of the domain and in blue the empirical mean of the oriented distance function.

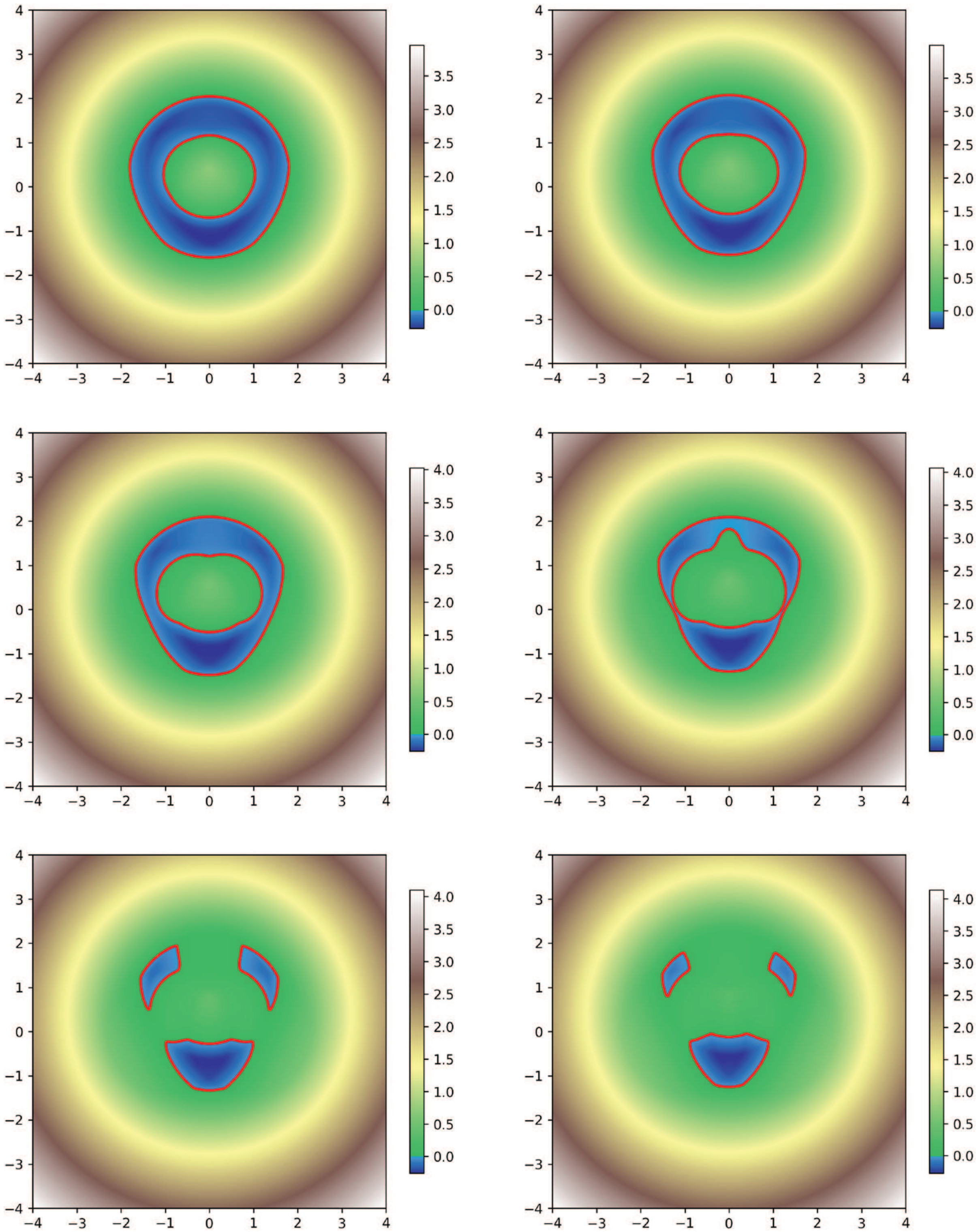

FIGURE 3. The expected distance function and expected domain: $a = 0.5$ to $a = 1$ form top left to bottom right.

Let us turn to a second example. The inner disk has a random radius following a uniform law on $[1/2, 3/2]$ while its center is chosen independently with a uniform law on the segment $[(-a, 0), (a, 0)]$. In Figure 2, we illustrate the possibility of a change of topology while the parameter a varies.

Let us turn to a third example. The inner disk has a random radius following a uniform law on $[1/2, 3/2]$ while its center is chosen independently as follows: $(-a, 0)$ with probability $1/4$, $(a, 0)$ with probability $1/4$ and $(0, a\sqrt{2} = 3)$ with probability $1/2$. In Figure 3, we illustrate the possibility of a change of topology while the parameter a varies.

6. Conclusion

For more general inner boundary, one has no analytic expression for the outer boundary and its distance function. Dedicated numerical schemes are mandatory. This requires precise analysis and is out of scope of this paper, it will be presented in a work in preparation.

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