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ON THE OBSERVABILITY INEQUALITY OF COUPLED WAVE EQUATIONS: THE CASE WITHOUT BOUNDARY

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Abstract. In this paper, we study the observability and controllability of wave equations coupled by first or zero order terms on a compact manifold. We adopt the approach in Dehman-Lebeau's paper [B. Dehman and G. Lebeau, *SIAM J. Control Optim.* **48** (2009) 521–550.] to prove that: the weak observability inequality holds for wave equations coupled by first order terms on compact manifold without boundary if and only if a class of ordinary differential equations related to the symbol of the first order terms along the Hamiltonian flow are exactly controllable. We also compute the higher order part of the observability constant and the observation time. By duality, we obtain the controllability of the dual control system in a finite co-dimensional space. This gives the full controllability under the assumption of unique continuation of eigenfunctions. Moreover, these results can be applied to the systems of wave equations coupled by zero order terms of cascade structure after an appropriate change of unknowns and spaces. Finally, we provide some concrete examples as applications where the unique continuation property indeed holds.

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1. Introduction and main results

1.1. Coupling of order one

Let $(\mathcal{M}, g)$ be a compact connected n-dimensional Riemannian manifold without boundary. Denote Δ_g the Laplace-Beltrami operator on $\mathcal{M}$ for the metric g. We consider the observability and control problem for the system of coupled wave equations:

$$\begin{cases}\partial_t^2 V - \Delta_g V + LV = 0,\\ (V(0), \partial_t V(0)) = (V_0, V_1),\end{cases} \tag{1.1}$$

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where $V=(V^1,\cdots,V^N)^{tr}$ with $N\in\mathbb{Z}^+$ and L is a matrix of differential operator of order one on $\mathbb{R}\times\mathcal{M}$ of the form

$$L=A_0\partial_t+A_1, \tag{1.2}$$

with $A_k\in\mathcal{C}^\infty(\mathbb{R};\mathrm{Diff}^k(\mathcal{M};\mathbb{C}^{N\times N}))$, $(k=0,1)$. Here $\mathrm{Diff}^k(\mathcal{M};\mathbb{C}^{N\times N})$ is the set of matricial differential operators of order k in space with smooth coefficients.

It is known that the weak solution of the Cauchy problem of System (1.1) exists for any initial data $(V_0,V_1)\in(H^1)^N\times(L^2)^N$ (see [43]). Here and hereafter, H^s $(s\in\mathbb{R})$ denotes the Sobolev space on manifold $\mathcal{M}$ with the norm defined as follows: $\|f\|^2_{H^s}=\|\Lambda^s f\|^2_{L^2}$, where

$$\Lambda^s f:=(-\Delta+1)^{\frac{s}{2}}f=\sum_{j\in\mathbb{N}}(\kappa_j+1)^{\frac{s}{2}}(f,e_j)_{L^2}\,e_j,\quad s\in\mathbb{R} \tag{1.3}$$

$(e_j)_{j\in\mathbb{N}}$ the eigenfunctions of the Laplace-Beltrami operator associated to the eigenvalues $(\kappa_j)_{j\in\mathbb{N}}$ which forms a Hilbert basis of H^s . In this context, we are interested in the following observability problem.

Definition 1.1. We say that System (1.1) is **Exactly Observable** on $[0,T]$, if the solutions of (1.1) satisfy **Observability Inequality**

$$C^1_{\mathrm{obs}}\int_0^T\|DV(t)\|^2_{(L^2)^K}\mathrm{d}t\geq\|(V_0,V_1)\|^2_{(H^1)^N\times(L^2)^N}, \tag{1.4}$$

where $C^1_{\mathrm{obs}}>0$ is a constant independent of the initial data (V_0,V_1) and the observation operator $D\in\mathcal{C}^\infty(\mathbb{R};\mathrm{Diff}^1(\mathcal{M};\mathbb{C}^{K\times N}))$ is a matrix of differential operator of order one on $\mathbb{R}\times\mathcal{M}$ taking the form)

$$D=D_0\partial_t+D_1, \tag{1.5}$$

with $D_k\in\mathcal{C}^\infty(\mathbb{R};\mathrm{Diff}^k(\mathcal{M};\mathbb{C}^{K\times N}))$, $(k=0,1)$.

Definition 1.2. We say that System (1.1) is **Weakly Observable** on $[0,T]$, if the solutions of (1.1) satisfy **Weak Observability Inequality**

$$C^2_{\mathrm{obs}}\int_0^T\|DV(t)\|^2_{(L^2)^K}\mathrm{d}t+c_1\|(V_0,V_1)\|^2_{(H^{\frac12})^N\times(H^{-\frac12})^N}\geq\|(V_0,V_1)\|^2_{(H^1)^N\times(L^2)^N}, \tag{1.6}$$

where $C^2_{\mathrm{obs}}>0$ and c_1 are constants independent of the initial data (V_0,V_1) and the observation operator D is defined by (1.5).

Roughly speaking, the weak observability can be understood as the observability of functions with high frequency, that is, $\|(V_0,V_1)\|_{(H^1)^N\times(L^2)^N}>>\|(V_0,V_1)\|_{(H^{\frac12})^N\times(H^{-\frac12})^N}$.

We mention a few notational conventions that we will use throughout. We will use notation $\dot{X}=\frac{\mathrm{d}X}{\mathrm{d}t}$. We denote $a^*=\bar{a}^{tr}$ the adjoint matrix of a, a^{tr} the transpose of a and L^* the adjoint operator of L for the L^2 (or $(L^2)^N$) scalar product inherited from the Riemannian structure. We denote $S^*\mathcal{M}$ the cosphere bundle of $\mathcal{M}$. $\varphi_t(\rho_0)$ is the Hamiltonian flow of $|\xi|_x$ initiated at ρ_0 defined by the formula

$$\varphi_t(\rho_0)=(x(t),\xi(t)),\quad\varphi_0(\rho_0)=\rho_0. \tag{1.7}$$

Then we state our main results:

Theorem 1.3. *Solutions of System (1.1) satisfy weak observability inequality (1.6) on $[0,T]$ if and only if for any $\rho_0 \in S^*\mathcal{M}$, the finite dimensional control system*

$$\begin{cases}\dot{X}(t) = \dfrac{1}{2}a^*(t,\varphi_t(\rho_0))X(t) + \dfrac{1}{2}d^*(t,\varphi_t(\rho_0))u(t), & \text{with control } u \in L^2(0,T;\mathbb{C}^K)\\ X(0) = X_0 \in \mathbb{C}^N, \end{cases} \tag{1.8}$$

is exactly controllable on $[0,T]$. Here $X(t) = (X_1(t), \cdots, X_N(t))^{tr} \in \mathbb{C}^N$ is the state variable. The coefficients matrice a and d are defined by $a := a_0 - \frac{a_1}{i|\xi|_x}$ and $d := d_0 - \frac{d_1}{i|\xi|_x}$, where $a_k \in C^\infty(\mathbb{R}; S^k_{phg}(T^*\mathcal{M}; \mathbb{C}^{N\times N}))$ $(k = 0,1)$ is the homogenous principal symbol of A_k defined in (1.2) and $d_k \in C^\infty(\mathbb{R}; S^k_{phg}(T^*\mathcal{M}; \mathbb{C}^{K\times N}))$ $(k = 0,1)$ is the homogeneous principal symbol of D_k defined in (1.5), respectively.

As it is quite classical in control theory, see ([27], Thm. 4.1) for an abstract version, the previous result gives the observability result if some unique continuation property is fulfilled. Let us be more precise in the case of time invariant equations.

Property 1.4. *Assume A_0 and A_1 are time invariant. We say that a system satisfies the* ***Unique continuation of eigenfunctions*** *if the following property holds:*
For any $\lambda \in \mathbb{C}$, any solution $V \in (H^1)^N$ of

$$\begin{cases}-\Delta_g V + \lambda^2 V + (\lambda A_0 + A_1)V = 0,\\ \lambda D_0 V + D_1 V = 0,\end{cases} \tag{1.9}$$

is the zero solution $V \equiv 0$.

Theorem 1.5. *Assume that A_0 and A_1 are time invariant. In the setting of Theorem 1.3, the following two statements are equivalent:*

1. *System (1.1) is exactly observable according to Definition 1.1.*
2. *Property 1.4 is satisfied and for any $\rho_0 \in S^*\mathcal{M}$, System (1.8) is exactly controllable.*

Now, we will be more precise about the inequality we can obtain. In a similar way to Lebeau [33] for the stabilisation problem (see also Laurent-Léautaud [32] for scalar control and Klein [31] for systems of damped waves), it is possible to characterize the constant in the high frequency part of the weak observability estimate. Roughly speaking, we prove that the constant of the high frequency part can be exactly determined by the Gramian of the finite dimensional system (1.8). We will need more definition now.

We define the Gramian matrix of System (1.8) by the formula

$$G_{\rho_0}(T) = \frac{1}{4}\int_0^T R^*(0,t;\rho_0)d^*(t,\varphi_t(\rho_0))\mathrm{d}(t,\varphi_t(\rho_0))R(0,t;\rho_0)\mathrm{d}t \tag{1.10}$$

where $R(\cdot,\cdot;\cdot)$ is the resolvent of (1.8) (see [22], Prop. 1.5 for definition). We can also define a constant

$$\begin{aligned}\mathfrak{K}(T) &:= \min_{\rho_0\in S^*\mathcal{M}, \beta\in\mathbb{C}^N, |\beta|=1}\{\beta^* G_{\rho_0}(T)\beta\}\\ &= \min_{\rho_0\in S^*\mathcal{M}} \max\left\{s \in \mathbb{R} \,\middle|\, \beta^*(G_{\rho_0}(T) - sId_{N\times N})\beta \geq 0, \forall \beta \in \mathbb{C}^N\right\}\\ &= \min_{\rho_0\in S^*\mathcal{M}} \min\{\lambda \in \mathbb{R} \,|\, \lambda \text{ is an eigenvalue of } G_{\rho_0}(T)\}.\end{aligned} \tag{1.11}$$

The equality of the different definitions comes from the symmetry and positivity of Hermitian matrix $G_{\rho_0}(T)$. Note that $\mathfrak{K}(T) \geq 0$ and we have $\mathfrak{K}(T) > 0$ if and only if $G_{\rho_0}(T) > 0$ (in the sense of symmetric matrices) which is equivalent to the controllability of (1.8) (see [22]).

Moreover, it is very important to estimate the optimal constant of the observability inequality since it is closely related to the cost of optimal control of the dual system. The following Theorem precises Theorem 1.3 and states what is the optimal constant of the high regularity term in the weak observability inequality.

Theorem 1.6. *If* $T > T_{\mathrm{crit}} := \inf_{T_0}\{T_0 | \min_{\rho_0 \in S^*\mathcal{M}} \det(G_{\rho_0}(T_0)) > 0\}$*, then weak observability inequality* (1.6) *holds with* $C_{\mathrm{obs}}^2 = \frac{1}{2\mathfrak{K}(T)}$*. Reciprocally, if weak observability inequality* (1.6) *holds for all solutions of System* (1.1)*, then we have* $T > T_{\mathrm{crit}}$ *and* $C_{\mathrm{obs}}^2 \geq \frac{1}{2\mathfrak{K}(T)}$*, where* $G_{\rho_0}(T_0)$ *and* $\mathfrak{K}(T)$ *are defined by* (1.10) *and* (1.11)*, respectively.*

Remark 1.7. Theorem 1.6 says that the observability constant C_{obs}^2 blows up like $1/2\mathfrak{K}(T)$ as $T \to T_{\mathrm{crit}}$.

Next we introduce the adjoint system of (1.1)

$$\begin{cases} \partial_t^2 U - \Delta_g U + L^* U = D^* F, \\ (U(0), \partial_t U(0)) = (U_0, U_1). \end{cases} \tag{1.12}$$

where $U = (U^1, \cdot, U^N)^{tr}$, $F = (f_1, \cdots, f_N)^{tr} \in L^2(0,T;(H^{-1})^K)$ is control function. Clearly, the weak solution of the Cauchy problem of System (1.12) exists for any initial data $(U_0, U_1) \in (L^2)^N \times (H^{-1})^N$ and forces $F = (f_1, \cdots, f_K)^{tr} \in L^2(0,T;(L^2)^K)$ (see [43]).

Thanks to Liu-Lu-Zhang ([40], Thm. 3.2) (see also Duprez-Olive [27] for similar results for time independent systems), we obtain the following corollary concerning Finite Co-dimensional Controllability of System (1.12).

Corollary 1.8. *Assume that System* (1.8) *is exactly controllable for any* $\rho_0 \in S^*\mathcal{M}$ *on* $[0,T]$*. Then, there exists a finite dimensional subspace* H_{fin} *and a finite co-dimensional subspace* H_{cofin} *with* $H_{\mathrm{fin}} \oplus H_{\mathrm{cofin}} = (L^2)^N \times (H^{-1})^N$*, such that: for any initial data* $(U_0, U_1) \in H_{\mathrm{cofin}}$*, there exists control* $F \in L^2(0,T;(L^2)^K)$ *such that the solution of* (1.12) *satisfies* $(U(T), \partial_t U(T)) = (0,0)$*.*

Now, we want to give more qualitative properties of the *HUM* (Hilbert Uniqueness Method) control operator. We need to consider the change of variable corresponding to the half wave decomposition. More precisely, define $\Sigma(V_0, V_1) = (V_+, V_-) = (i\Lambda V_0 + V_1, -i\Lambda V_0 + V_1)$ with $\Lambda = (-\Delta_g + 1)^{1/2}$, see Section 3.1 for more precisions. Denote $\tilde{\mathcal{G}}_T := G_T + \tilde{\mathcal{R}}_T$ the Gramian operator which is defined below by (1.13). If $\tilde{\mathcal{G}}_T$ is invertible, then define $\mathcal{L}_T = (\tilde{\mathcal{G}}_T)^{-1}$ the *HUM* control operator. As a byproduct of the proof of Theorem 1.3, we obtain the following interesting characterization of $\mathcal{L}_T$ as a matricial pseudodifferential operator. This generalizes some results of Dehman-Lebeau [24] in the scalar case to systems. Note that it is also related to some trivialization along the flow that are described in Burq-Lebeau [19] in the case with boundary.

Theorem 1.9. *Let* $V_* := (V_0, V_1) \in (H^1)^N \times (L^2)^N$ *be the initial data of System* (1.1)*. Let* $T_0 > 0$*. Then for any* $T \in (0, T_0]$*, we have*

$$\int_0^T \|DV(t)\|_{(L^2)^K}^2 \mathrm{d}t = ((G_T + \tilde{\mathcal{R}}_T)\Sigma V_*, \Sigma V_*)_{(L^2)^{2N}}, \tag{1.13}$$

where $G_T \in \mathcal{C}^\infty(0, T_0; \Psi_{phg}^0(\mathcal{M}; \mathbb{C}^{2N\times 2N}))$ *and* $\tilde{\mathcal{R}}_T \in \mathcal{B}(0, T_0; \mathcal{L}((H^\sigma)^{2N}, (H^{\sigma+1})^{2N}))$ *is in a class of regularizing operators of order at least one. Moreover, the principal symbol of* G_T *can be characterized as follows:*

$$\sigma_0(G_T)(\rho_0) = \begin{pmatrix} G_{\rho_0}^+(T) & 0 \\ 0 & G_{\rho_0}^-(T) \end{pmatrix} \in \mathcal{C}^\infty(0, T_0; S_{phg}^0(\mathcal{M}; \mathbb{C}^{2N\times 2N})), \tag{1.14}$$

where $G^{\pm}_{\rho_0}(T)$ are the Gramian matrices of the control systems

$$\begin{cases} \dot{X}(t) = \dfrac{1}{2} a^*_{\pm}(t, \varphi_{\mp t}(\rho_0))X(t) + \dfrac{1}{2} d^*_{\pm}(t, \varphi_{\mp t}(\rho_0))u(t), \\ X(0) = X_0 \in \mathbb{C}^N, \end{cases} \tag{1.15}$$

where $X(t) = (X_1, \cdots, X_N)^{tr}$ is a vector having N components, $a_{\pm} = a_0 \pm \frac{a_1}{i|\xi|_x}$, $d_{\pm} = d_0 \pm \frac{d_1}{i|\xi|_x}$, $\varphi_t(\rho_0)$ is the Hamiltonian flow of $|\xi|_x$ initiated at ρ_0 and $u(t) \in L^2(0, T; \mathbb{C}^K)$ is the control.

The interest of this theorem is that at high frequency, the *HUM* operator $\mathcal{L}_T$ is a pseudolocal operator. That means that if one needs to control the initial data with a lot of oscillations localized only in some region of the phase space, the corresponding optimal *HUM* control will also present these oscillations only in the same region. We refer to the interesting numerical study of this fact in [34] the scalar case where this property is explored. Note that this would be interesting to make similar numerical study in the vectorial case we consider.

1.2. Coupling of order zero

The purpose of this section is to transfer the results we have obtained for coupling of order one to coupling of order zero. The main difference is that zero order coupling are not strong enough to transfer the information from a component to another in the natural spaces. Indeed, if we apply directly the results of the previous Section for zero order coupling, the coupling (considered as an operator of order one) will have zero principal symbol and thus, there will be no coupling at this level of regularity. So, we have to adapt the setting.

Before getting to a general result, let us study a first enlightening example: a system of two equations with cascade coupling that was completely studied in Dehman-Le Rousseau-Léautaud [25]:

$$\begin{cases} \partial_t^2 u - \Delta_g u = 1_\omega g, \\ \partial_t^2 v - \Delta_g v + a(x)u = 0. \end{cases} \tag{1.16}$$

It is clearly not possible to control both components in $H^1 \times L^2$ with a control g in $L^2(0, T; L^2))$, which is the natural regularity for scalar control. Indeed, if the initial conditions are zero for u and v and $g \in L^2(0, T; L^2))$, this will create some solutions u in $C([0, T]; H^1)$ and the source term $a(x)u$ (for v) will be in $C([0, T]; H^1)$, which will create a solution v in $C([0, T]; H^2)$. So, in that case, the natural space of control is $H^1 \times L^2$ for u and $H^2 \times H^1$ for v. Then, we see that it is necessary to classify each variable of the system according to algebraic properties of the coupling and the control operator.

Now let us move to the $N \times N$ system

$$\begin{cases} \partial_t^2 U - \Delta_g U + AU = BG, \\ (U(0), \partial_t U(0)) = (U_0, U_1). \end{cases} \quad \text{with control } G \in L^2(0, T; (L^2)^K) \tag{1.17}$$

where $A(x)$ is a matrix in $\mathbb{R}^{N\times N}$ and $B(x)$ is a matrix in $\mathbb{R}^{N\times K}$. Without loss of generality, we can assume $A(x)$ is a matrix "subdiagonal by block" in $\mathbb{R}^{N\times N}$

$$A(x) = \begin{bmatrix} A_{11} & \dots & \dots & A_{1k} \\ A_{21} & \dots & \dots & A_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & A_{k,k-1} & A_{kk} \end{bmatrix} \tag{1.18}$$

with $A_{i,j} \in \mathbb{R}^{d_i \times d_j}$ $(i = 1, \cdots, k)$ and $B(x)$ is a matrix in $\mathbb{R}^{N\times K}$ of the form

$$B(x) = [B_{11}, 0, \cdots, 0]^{tr} \tag{1.19}$$

where $B_{11} \in \mathbb{R}^{d_1 \times K}$. In fact, any $A(x), B(x)$ can be transformed into these forms simultaneously by using one algorithm detailed in Section 4. Noting the coupling of structure by blocks, one can analyze the regularity of components in blocks and easily find out the natural space for solutions of (1.17) is $\mathcal{H}^s$ as follows. We have $U \in \mathcal{H}^s$ if for every $i = 1, ..., k$, we have $U^i \in (H^{s+i-1})^{d_i}$ where d_i is the dimension of $A_{i,i}$. That is

$$\mathcal{H}^s = (H^s)^{d_1} \times (H^{s+1})^{d_2} \times \cdots \times (H^{s+k-1})^{d_k}. \tag{1.20}$$

The natural energy space is then $\mathcal{E} = \mathcal{H}^1 \times \mathcal{H}^0$ and it appears that the important terms are the subdiagonal terms of A which leads to define

$$A_{sub}(x) = \begin{bmatrix} 0 & \dots & \dots & 0 \\ A_{21} & \dots & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & A_{k,k-1} & 0 \end{bmatrix}. \tag{1.21}$$

This gives the following theorem of control.

Theorem 1.10. *Assume that $A(x)$, $A_{\text{sub}}(x)$ and $B(x)$ have some decomposition as in (1.18), (1.21) and (1.19). The System (1.17) is controllable in $\mathcal{E} = \mathcal{H}^1 \times \mathcal{H}^0$ on $[0, T]$ with control $G \in L^2(0, T; (L^2)^K)$ if and only if we have the following two properties*

- *The control system*

$$\begin{cases} \dot{X}(t) = \frac{1}{2} A_{\text{sub}}(\varphi_t(\rho_0)) X(t) + \frac{1}{2} B(\varphi_t(\rho_0)) u, \\ X(0) = X_0 \in \mathbb{R}^N. \end{cases} \tag{1.22}$$

  with control $u \in L^2(0, T; \mathbb{R}^K))$ is exactly controllable on $[0, T]$.
- *Unique continuation of eigenfunctions:*
  For any $\lambda \in \mathbb{C}$, any solution $V \in (H^1)^N$ of

$$\begin{cases} -\Delta_g V + A^*(x) V = \lambda V, \\ B^* V = 0, \end{cases} \tag{1.23}$$

  is $V = 0$.

The equivalence is true once the polarized space $\mathcal{E}$ has been chosen and the decomposition in block has been specified. Theorem 1.10 yields a necessary and sufficient condition once we fix the decomposition as in (1.18) and (1.19). This decomposition might not be unique, but each decomposition, eventually after some change of unknown, gives a different result of control, positive or negative, which has its own interest.

In Section 4.1, we will show how the assumption of subdiagonal form is actually quite general. Indeed, for any couple $A(x)$, $B(x)$, there exist some change of unknown that lead the control problem to have the subdiagonal form expected. Yet, this is not unique. For instance, the trivial decomposition with only one block always works. In that case, $A_{\text{sub}}(x) = 0$ and there is no coupling. Our result gives a necessary and sufficient condition for the control in $(H^1)^N \times (L^2)^N$. Yet, it is possible in some situation that another choice of decomposition would use better the coupling but at the cost of a loss in the space.

In the constant case $A(x) = A \in \mathbb{R}^{N \times N}$, $B(x) = B \in \mathbb{R}^{N \times K}$, the Brunovsky normal form (written in a slightly different way, see Prop. A.1) always allows to put our control system in the expected subdiagonal form with the good property. In that case, it seems to be the optimal choice that gives the best controllability result. We obtain the following theorem.

Theorem 1.11. *Let A, B constant satisfying the Kalman rank condition and ω satisfies Geometric Control Condition (*GCC *for short) [16]. Then, there exists some integer $k \leq N$ and some $d_i \in \mathbb{N}$, $i = 1, \cdots, k$, allowing to define the space $\mathcal{E} := \mathcal{H}^1 \times \mathcal{H}^0$ as in (1.20) and some matrix $Q \in GL_N(\mathbb{R})$ so that the System (1.17) is controllable in $Q\mathcal{E}$ with control $G \in L^2(0, T; (L^2)^K)$.*

The matrix Q and the integers k and d_i are strongly related to the Brunovsky normal form. Roughly speaking, this decomposition transforms the control problem in the control system with integrators

$$y_1^{(\alpha_1)} = u_1, \quad \cdots \quad , y_m^{(\alpha_m)} = u_m, \qquad \alpha_1, \cdots, \alpha_m \in \mathbb{Z}^+, \tag{1.24}$$

the state being $y_1, y_1^{(1)}, \cdots, y_1^{(\alpha_1-1)}, \cdots, y_m, y_m^{(1)}, \cdots, y_m^{(\alpha_m-1)}$ and the controls being the u_i. In that setting, k is $\max_{i=1,\cdots m} \alpha_i$, that is the stronger integrators. It is then natural that for the wave equation, the observations holds in some space H^k, that is we have integrated k times from H^1 thanks to the regularization of the wave operator with respect to a source term. Note also that it is not clear that the space $Q\mathcal{E}$ is invariant by the equation, so we should precise which kind of control we mean (control to zero, from zero...). Yet, it will be a byproduct of the proof that $Q\mathcal{E}$ is invariant by the equation. So, here, by controllability in $Q\mathcal{E}$, we mean that any state in $Q\mathcal{E}$ can be controlled to a state in $Q\mathcal{E}$.

Some previous articles (Liard-Lissy [37], Lissy-Zuazua [39]) already obtained some controllability property in this framework under the Kalman rank condition (in a more abstract and general setting). So, an improvement of our Theorem comes from the space where the controllability holds. We refer to Section 5.2 for more precisions.

1.3. Other applications

1.3.1. Other equations: parabolic and Schrödinger-like systems

Thanks to the transmutation techniques, see for instance [28, 42], all the results stated in this article might give results for the analog parabolic system and for systems of Schrödinger equations.

A lot of controllability results of parabolic system have been established and it would be impossible to give a complete view of the subject. We refer for instance to the survey paper Ammar-Khodja-Benabdallah-González-Burgos-de Teresa [12]. Under the assumption that the control domain and the coupling domain intersect each other, controllability results can be obtained under some algebraic conditions, like of Cascade type or Kalman rank condition [10, 11, 29] (see also [26]). Note also that these papers about parabolic equations often contain as a byproduct some results of unique continuation for eigenfunctions that are in the assumptions of our theorem. In the opposite direction, we also would like to refer to the interesting paper of Boyer-Olive [17] that gives several 1D counterexamples of unique continuation of eigenfunctions. It would be interesting to check if there is a link between this counterexamples and our assumption of controllability of the finite dimensional problem. Are there some cases were the unique continuation is false while the weak observability is true? or backward?

Using the transmutation method and removing the assumption of intersection of the domains of coupling and control, [5] obtains indirect controllability of parabolic system of cascade and symmetric under Geometric Control Condition (GCC).

We refer to [37, 41] for internal controllability results of systems of Schrödinger equations coupled with constant zero order terms with good algebraic structure.

1.3.2. The boundary case

The choice of using Egorov Theorem for proving our results has the advantage to be simpler and more precise. Indeed, we get a structure of the *HUM* control operator and the exact constant of high frequency. However, it has the disadvantage that it does not apply (at least up to our knowledge) to the case of domains with boundary. Most of the results presented in this paper (with the notable exception of the description of the *HUM* control operator as a pseudo-differential operator) might remain true in the case of boundary. Yet, it requires different techniques. We are therefore planning to prove similar result in the case of boundary in a forthcoming paper

[20]. The proofs will be based on the full description of microlocal defect measures of sequences of solution of wave equations as performed in [19].

1.4. Previous results

Let us discuss briefly the previous work on controllability and observability problem for wave equations. Russell [44] and Lions [38] set up the duality and proved that the exact controllability of the control system can be equivalently reduced to the observability inequality for solutions of the adjoint system. Then Bardos-Lebeau-Rauch pointed out the *GCC* is crucial to the controllability and stabilization of (scalar) wave equations [14–16]. Note that in the framework of our Theorems the *GCC* for the control of the scalar wave equation $(\partial_t^2 - \Delta_g)u = \chi_\omega(x)h$ is described in an equivalent way as the controllability for any $\rho_0 \in S^*\mathcal{M}$ of the scalar control system $\dot{x}(t) = \chi_\omega(\varphi_t(\rho_0))u(t)$ with control u on $[0, T]$.

Alabau-Boussouira [2] first studied the indirect controllability of two wave equations with constant coefficients coupled by displacements via one boundary control. The controllability result was established in a multi-level energy space similar to (1.20) and it was generalized to variable coefficients coupling under geometric control conditions on coupling and control domains in Alabau-Boussouira-Léautaud [5]. Other results for the related problem of stabilization were also formulated by the same authors [1, 4] and then by Aloui-Daoulatli [8].

In [25], Dehman-Le Roussau-Léautaud proved the controllability of two wave equations coupled by zero order terms of Cascade type on a compact manifold. Moreover, they gave the sharp controllability time and a microlocal characterization of the *HUM* control operator similar to the one of Theorem 1.9. In Section 5, we explain how our main result allows to recover some results in [25] with the study of an appropriate finite dimensional problem. The multi-speed case was also studied in [25]. As stated earlier, under Kalman rank condition and *GCC*, exact controllability of systems of wave equations with constant coefficients and general coupling structure of zero order were proved in [37, 39]. Note also the recent article of Alabau-Boussouira-Coron-Olive [6] for 1D hyperbolic systems where appears a condition on finite dimensional problems related to ours and the use of its Gramian.

There are many other control problems which are closely related to or strongly motivated by the study of controllability of systems of wave equations via less controls, for instance, the synchronization problems [35, 36], desensitizing control problems [3, 23, 38, 45] and simultaneous control problems [13, 38].

Let us mention that the above results concern only the systems with coupling of order zero. As for the systems of wave equations coupled by first order terms, we refer to [7, 21] for the stability of such systems coupled by velocities under strong geometric conditions.

Recently, Klein [31] obtained some results related to ours for the stabilisation of wave equations. He computes the best exponent for the stabilization of wave equations on compact manifolds. The coefficient he obtains is therefore solution of some ODE system of matrices. In this context, an improvement of our paper is to recognize the relation between this coefficient and the Gramian control operator of the finite dimensional system.

1.5. Plan of the paper

The plan of the paper is the following. Section 2 is devoted to provide some preliminary works. In Section 2.1 we give a proof of a System Egorov Theorem. In Section 2.2 we recall the $N \times N$ Sharp Gårding Inequality useful in our context. In Section 3, we get back to the control problem. In Section 3.1, we provide a characterization of the principal symbol of the Gramian operator. Our main results are proved in Section 3.2 and 3.3. Section 4 is about the implications of our theorem (which concerned coupling by coefficients of order 1) in the case of coupling by zero order coefficients. Section 5 is about examples of applications of our theorem, namely a cascade system and a system coupling with constant coefficients. Finally, we gathered in Appendix a reformulated Brunovsky normal form that is used in Section 4, some eigenvalue problems used in Section 5, and the proof of 1-smooth effect Lemma 3.5.

2. Preliminary works

In Subsection 2.1 and Subsection 2.2, we prove System Egorov Theorem and $N\times N$ Sharp Gårding Inequality on manifold respectively.

2.1. System Egorov Theorem

We consider the following hyperbolic system:

$$\begin{cases}\partial_t U(t)-iH(t)U(t)=0,\\ U(s)=U_0,\end{cases}\tag{2.1}$$

where

$$H(t)=c\Lambda Id_{N\times N}+iW_0(t),\tag{2.2}$$

$c\in\mathbb{R}$ and $W_0(t)\in\mathcal{C}^\infty(0,T;\Psi^0_{phg}(\mathcal{M};\mathbb{C}^{N\times N}))$ is a matrix pseudodifferential operator of order 0. Denote $w_0=\sigma_0(W_0)\in\mathcal{C}^\infty(0,T;S^0_{phg}(T^*\mathcal{M};\mathbb{C}^{N\times N}))$ the principal symbol of W_0.

We define the notation $S(t,s)$ as the solution operator associated to (2.1), that is $S(t,s)U_0=U(t)$. The main result of this section is the following variation of Egorov Theorem (see [32], Sect. A.1 for its scalar case).

Theorem 2.1 (System Egorov Theorem)**.** *For any $P_m(\cdot)\in\mathcal{C}^\infty(0,T;\Psi^m_{phg}(\mathcal{M};\mathbb{C}^{N\times N}))$, $m\in\mathbb{R}$, $p_m(s,\cdot)=\sigma_m(P_m(s))$, there exist $Q(t,s)\in\mathcal{C}^\infty((0,T)^2;\Psi^m_{phg}(\mathcal{M};\mathbb{C}^{N\times N}))$ and $R(t,s)\in\mathcal{B}((0,T)^2,\mathcal{L}((H^\sigma)^N,(H^{\sigma+1-m})^N))$ with $\partial_tR(t,s),\partial_sR(t,s)\in\mathcal{B}((0,T)^2,\mathcal{L}((H^\sigma)^N,(H^{\sigma-m})^N))$ for any $\sigma\in\mathbb{R}$, such that*

$$S(s,t)^*P_m(s)S(s,t)-Q(t,s)=R(t,s),\quad (t,s)\in(0,T)^2.\tag{2.3}$$

Moreover, the principal symbol of $Q(t,s)$ is given by $q(t,s,\cdot)$ which satisfies:

$$q(t,s,\rho)=R_1^*(t,s;\chi^c_{s,t}(\rho))p_m(s,\chi^c_{s,t}(\rho))R_1(t,s;\chi^c_{s,t}(\rho)),\tag{2.4}$$

where $\chi^c_{t,s}(\rho)$ is given by the flow of Hamiltonian vector field associated with $-c\lambda$

$$\frac{\mathrm{d}}{\mathrm{d}t}\chi^c_{t,s}=H_{-c\lambda}(\chi^c_{t,s}),\quad \chi^c_{s,s}(\rho)=\rho\in T^*\mathcal{M}\setminus\{0\}.\tag{2.5}$$

and $R_1(\tau,s;\rho)$ satisfies

$$\frac{\mathrm{d}R_1(\tau,s;\rho)}{\mathrm{d}\tau}=R_1(\tau,s;\rho)w_0(\tau,\chi^c_{\tau,s}(\rho));\quad R_1(s,s;\rho)=Id_{N\times N}.\tag{2.6}$$

Note that in fact, $\chi^c_{t,s}(\rho)=\chi^c_{t-s,0}(\rho)$ and the implicit formula (2.6) defines well R_1. We recall that λ is defined in (1.3).

The proof is inspired from [32] in the scalar case.

Proof. We firstly note that $S(t,s)$ satisfies

$$\partial_tS(t,s)-iH(t)S(t,s)=0,\quad S(s,s)=Id_{N\times N}.\tag{2.7}$$

where the time derivative is not to be taken as a derivative in a Banach space $\mathcal{L}((H^\sigma)^N,(H^\sigma)^N)$ but in the weak sense, that is the derivative when the operator is applied to a fixed function. See for instance ([32], Cor.

A.2) for more details. Since $S(t,s)S(s,t) = Id_{N\times N}$, we have

$$\begin{aligned} \partial_t S(s,t) + iS(s,t)H(t) = 0,\\ \partial_t S(t,s)^* + iS(t,s)^* H(t)^* = 0,\\ \partial_t S(s,t)^* - iH(t)^* S(s,t)^* = 0. \end{aligned} \tag{2.8}$$

with $H^*(t) = \Lambda Id_{N\times N} - iW_0^*(t)$. The well-posedness of (2.1) yields the following regularity properties $S(t,s) \in \mathcal{B}((0,T)^2; \mathcal{L}((H^\sigma)^N), \partial_t S(t,s), \partial_s S(t,s) \in \mathcal{B}((0,T)^2; \mathcal{L}((H^\sigma)^N; (H^{\sigma-1})^N)$ for all $\sigma \in \mathbb{R}$, as well as for $S(t,s)^*$.

Now, setting

$$P(t,s) = S(s,t)^* P_m(s) S(s,t), \tag{2.9}$$

and using the above equations, we have $P(s,s) = P_m(s)$ with

$$\partial_t P(t,s) = iH(t)^* P(t,s) - iP(t,s)H(t) = ic[\Lambda Id_{N\times N}, P(t,s)] + W_0(t)^* P(t,s) + P(t,s)W_0(t).$$

Here $[\cdot,\cdot]$ stands for the classic commutator. We now construct an approximate pseudodifferential solution $Q(t,s)$. Its principal symbol $q(t,s,x,\xi)$ should satisfy

$$\partial_t q(t,s,\cdot) = c\{\lambda, q(t,s,\cdot)\} + w_0^*(t,\cdot)q(t,s,\cdot) + q(t,s,\cdot)w_0(t,\cdot), \quad q(s,s,\cdot) = p_m(s,\cdot), \tag{2.10}$$

where $\{.,.\}$ stands for the Poisson bracket in the (x,ξ) variables.

We claim that $q(t,s,\rho)$ defined in (2.4) satisfies (2.10).

Indeed, since $\chi^c_{s,t} \circ \chi^c_{t,s}(\rho) = \rho$, we have $q(t,s,\chi^c_{t,s}(\rho)) = R_1^*(t,s;\rho)p_m(s,\rho)R_1(t,s;\rho)$. So

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left[q(t,s,\chi^c_{t,s}(\rho))\right] =& w_0^*(t,\chi^c_{t,s}(\rho))R_1^*(t,s;\rho)p_m(s,\rho)R_1(t,s;\rho)\\ &+ R_1^*(t,s;\rho)p_m(s,\rho)R_1(t,s;\rho)w_0(t,\chi^c_{t,s}(\rho))\\ =& w_0^*(t,\chi^c_{t,s}(\rho))q(t,s,\chi^c_{t,s}(\rho)) + q(t,s,\chi^c_{t,s}(\rho))w_0(t,\chi^c_{t,s}(\rho)). \end{aligned} \tag{2.11}$$

So, denoting $\widetilde{q}(t,s,\rho) = q(t,s,\chi^c_{t,s}(\rho))$, (2.11) can be written

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\widetilde{q}(t,s,\rho)\right] = w_0^*(t,\chi^c_{t,s}(\rho))\widetilde{q}(t,s,\rho) + \widetilde{q}(t,s,\rho)w_0(t,\chi^c_{t,s}(\rho)). \tag{2.12}$$

Therefore, since (2.5) gives $\frac{\mathrm{d}}{\mathrm{d}t}\left[\widetilde{q}(t,s,\rho)\right] = (\partial_t q)(t,s,\chi^c_{t,s}(\rho)) - c\{\lambda,q\}(t,s,\chi^c_{t,s}(\rho))$, we obtain for any $(t,s) \in (0,T)^2$ and $\rho \in T^*\mathcal{M}$

$$\partial_t q(t,s,\chi^c_{t,s}(\rho)) = w_0^*(t,\chi^c_{t,s}(\rho))q(t,s,\chi^c_{t,s}(\rho)) + q(t,s,\chi^c_{t,s}(\rho))w_0(t,\chi^c_{t,s}(\rho)) + c\{\lambda,q\}(t,s,\chi^c_{t,s}(\rho)).$$

Since for fixed $(t,s) \in (0,T)^2$, $\chi^c_{t,s}$ is a bijection of $T^*\mathcal{M}$, it gives

$$\partial_t q(t,s,\rho) = w_0^*(t,\rho)q(t,s,\rho) + q(t,s,\rho)w_0(t,\rho) + c\{\lambda,q\}(t,s,\rho).$$

We thus see that our definition (2.4) of $q(t,s,\rho)$ satisfies (2.10).

The homogeneity of λ of order one allows to keep the homogeneity of $q(t,s,\rho)$. This allows to select one $Q(t,s)$, so that

$$Q(t,s) \in \mathcal{C}^\infty((0,T)^2; \Psi^m_{phg}(\mathcal{M}; \mathbb{C}^{N\times N})) \text{ satisfies } \sigma_m(Q(t,s)) = q(t,s,.). \tag{2.13}$$

From (2.10) and pseudodifferential calculus, we now have

$$\begin{aligned}\partial_t Q(t,s) &= ic[\Lambda Id_{N\times N}, Q(t,s)] + W_0^*(t)Q(t,s) + Q(t,s)W_0(t) + \widetilde{R}(t,s)\\ &= iH(t)^*Q(t,s) - iQ(t,s)H(t) + \widetilde{R}(t,s),\end{aligned} \tag{2.14}$$

with $\widetilde{R} \in \mathcal{C}^\infty((0,T)^2; \Psi^{m-1}_{phg}(\mathcal{M};\mathbb{C}^{N\times N}))$. Next we estimate the remainder $R(t,s) = Q(t,s) - P(t,s)$. Set

$$T(t,s) = S(t,s)^*(Q(t,s) - P(t,s))S(t,s) = S(t,s)^*Q(t,s)S(t,s) - P_m(s), \tag{2.15}$$

so that we have, in view of (2.14),

$$\begin{aligned}\partial_t T(t,s) &= \partial_t\left[S(t,s)^*Q(t,s)S(t,s)\right]\\ &= S(t,s)^*\left[-iH(t)^*Q(t,s) + \partial_t Q(t,s) + iQ(t,s)H(t)\right]S(t,s)\\ &= S(t,s)^*\widetilde{R}(t,s)S(t,s).\end{aligned} \tag{2.16}$$

Thus, we obtain

$$Q(t,s) - P(t,s) = S(s,t)^*\left[Q(s,s) - P_m(s) + \int_s^t S(\tau,s)^*\widetilde{R}(\tau,s)S(\tau,s)\mathrm{d}\tau\right]S(s,t), \tag{2.17}$$

where $\widetilde{R} \in \mathcal{C}^\infty((0,T)^2; \Psi^{m-1}_{phg}(\mathcal{M};\mathbb{C}^{N\times N}))$, $Q(s,s) - P_m(s) \in \mathcal{C}^\infty((0,T); \Psi^{m-1}_{phg}(\mathcal{M};\mathbb{C}^{N\times N}))$. Therefore, it implies $Q(t,s) - P(t,s) \in \mathcal{B}((0,T)^2, \mathcal{L}((H^\sigma)^N, (H^{\sigma+1-m})^N)$ and $\partial_t(Q(t,s) - P(t,s)), \partial_s(Q(t,s) - P(t,s)) \in \mathcal{B}((0,T)^2, \mathcal{L}((H^\sigma)^N, (H^{\sigma-m})^N)$ for any $\sigma \in \mathbb{R}$. Together with the expression of Q in (2.13), we finish the proof of System Egorov Theorem. □

Remark 2.2. Note that the previous Egorov Theorem implies some propagation of microlocal defect measure, as described in Burq-Lebeau [19] in the more complicated case of domain with boundary. The equation we obtain for R_1 is actually closely related to the trivialization of the bundle that they describe in ([19], Sect. 3.2). In the present article, we will prove that Theorem 2.1 implies a link between the observability of the ODE and the observability of the PDE. Similarly, we plan to prove in the future [20] how the propagation of microlocal defect measure in [19] implies a similar link.

2.2. $N \times N$ Sharp Gårding Inequality

In this section, we state without proof some uniform $N \times N$ type Sharp Gårding Inequality on a compact manifold. This is the equivalent of the sharp Gårding inequality in $\mathbb{R}^n$ as stated in [47]. The following versions on a compact manifold can easily be obtained frome the one on $\mathbb{R}^n$ by localization in local charts. We refer for instance to ([32], Sect. A.4) for some details in the scalar case, the argument being exactly the same.

Theorem 2.3. *Assume that* $A_t \in \mathcal{C}^0([T_1,T_2]; \Psi^0_{phg}(\mathcal{M};\mathbb{C}^{N\times N}))$. *If* $\sigma_0(A_t)$ *is nonnegative Hermitian matrix of order 0 on* $[T_1,T_2] \times T^*\mathcal{M}$, *then*

$$(A_t u, u)_{(L^2)^N} \geq -C\|u\|^2_{(H^{-1/2})^N}, \quad \forall t \in [T_1,T_2], \quad u \in (H^1)^N, \tag{2.18}$$

where C *is independent with* u.

The proof of Theorem 2.3 is a direct consequence of the following theorem.

Theorem 2.4. *For $A \in \Psi^0_{phg}(\mathcal{M};\mathbb{C}^{N\times N})$, if $\sigma_0(A)$ is a nonnegative hermitian matrix of order 0 on $T^*\mathcal{M}$, then there exist $C>0$ such that*

$$(Au,u)_{(L^2)^N} \geq -C\|u\|^2_{(H^{-1/2})^N}, \quad \forall u \in (L^2)^N. \tag{2.19}$$

3. Proof of main results

In this section, we give the proof of Theorem 1.3 which is inspired from [32]. We start by writing the System (1.1) as a $2N\times 2N$ system of order 1. Then, we use a trick due to Taylor to eliminate the lower order terms. Applying System Egorov Theorem, $N\times N$ Gårding inequality and control theory of finite dimensional , we construct a connection between pseudodifferential representation and Gramian matrix of finite dimensional System (1.8).

In the following, we will be slightly more general than in Theorem 1.3, in the sense that we will allow L and D to be pseudodifferential operators in space and not only differential operators. We assume

$$L = A_0\partial_t + A_1, \tag{3.1}$$

with $A_0 \in \mathcal{C}^\infty(\mathbb{R};\Psi^0(\mathcal{M};\mathbb{C}^{N\times N}))$, $A_1 \in \mathcal{C}^\infty(\mathbb{R};\Psi^1(\mathcal{M};\mathbb{C}^{N\times N}))$. and

$$D = D_0\partial_t + D_1, \tag{3.2}$$

with $D_0 \in \mathcal{C}^\infty(\mathbb{R};\Psi^0(\mathcal{M};\mathbb{C}^{K\times N}))$, $D_1 \in \mathcal{C}^\infty(\mathbb{R};\Psi^1(\mathcal{M};\mathbb{C}^{K\times N}))$.

3.1. Gramian operator

• **Half wave decomposition**

We rewrite System (1.1) to Klein-Gordon type equations [24, 32],

$$\begin{cases}(\partial_t^2-\Delta_g)V + V + B_0\partial_t V + B_1 V = 0,\\ (V(0),\partial_t V(0)) = (V_0,V_1).\end{cases} \tag{3.3}$$

where $B_0 = A_0, B_1 = A_1 - Id_{N\times N}$.

We set

$$V_+ = (\partial_t + i\Lambda)V, \quad V_- = (\partial_t - i\Lambda)V \tag{3.4}$$

so that

$$V_0 = \frac{\Lambda^{-1}}{2i}(V_+(0)-V_-(0)), V_1 = \frac{1}{2}(V_+(0)+V_-(0)). \tag{3.5}$$

We define the map Σ:

$$\begin{aligned}\Sigma : (H^s)^N \times (H^{s-1})^N &\to (H^{s-1})^{2N},\\ (V_0,V_1) &\mapsto (V_+(0),V_-(0)).\end{aligned} \tag{3.6}$$

According to (3.5), we have:

$$\Sigma = \begin{pmatrix} i\Lambda Id_{N\times N} & Id_{N\times N}\\ -i\Lambda Id_{N\times N} & Id_{N\times N}\end{pmatrix}, \quad \Sigma^{-1} = \frac{1}{2}\begin{pmatrix} -i\Lambda^{-1}Id_{N\times N} & i\Lambda^{-1}Id_{N\times N}\\ Id_{N\times N} & Id_{N\times N}\end{pmatrix}, \tag{3.7}$$

where the operator Σ is (almost) an isometry from $(H^s)^N\times(H^{s-1})^N$ to $(H^{s-1})^{2N}$.

Note that for $(V_+(0),V_-(0))=\Sigma(V_0,V_1)$, we have

$$2\|(V_0,V_1)\|^2_{(H^s)^N\times(H^{s-1})^N}=\|(V_+(0),V_-(0))\|^2_{(H^{s-1})^{2N}}. \tag{3.8}$$

Let

$$B_+=\frac{1}{2}(B_0-iB_1\Lambda^{-1}),\quad B_-=\frac{1}{2}(B_0+iB_1\Lambda^{-1}), \tag{3.9}$$

we rewrite System (3.3) as a $2N\times 2N$ system

$$\begin{cases}(\partial_t-i\Lambda)V_++B_+V_++B_-V_-=0,\\(\partial_t+i\Lambda)V_-+B_+V_++B_-V_-=0,\end{cases} \tag{3.10}$$

since $\partial_t^2-\Delta_g+1=(\partial_t-i\Lambda)(\partial_t+i\Lambda)$. Denote

$$P=\partial_t+M_1+B,\quad M_1=\begin{pmatrix}-i\Lambda Id_{N\times N} & 0\\ 0 & i\Lambda Id_{N\times N}\end{pmatrix},\quad B=\begin{pmatrix}B_+ & B_-\\ B_+ & B_-\end{pmatrix}. \tag{3.11}$$

Then $P\mathcal{V}=0,\ \mathcal{V}=(V_+,V_-)^{tr}$. We define $\mathfrak{S}(t,s)$ as the solution operator of System (3.10). The well-posedness of Hyperbolic System (3.10) yields $\mathfrak{S}(t,s)\in\mathcal{B}((0,T)^2;\mathcal{L}(H^\sigma(\mathcal{M};\mathbb{C}^{2N})))$ and $\partial_t\mathfrak{S}(t,s),\partial_s\mathfrak{S}(t,s)\in\mathcal{B}((0,T)^2;\mathcal{L}(H^\sigma(\mathcal{M};\mathbb{C}^{2N});H^{\sigma-1}(\mathcal{M};\mathbb{C}^{2N})))$ for all $\sigma\in\mathbb{R}$.

Lemma 3.1. *Denote by $S_\pm(t,s)$ solution operator of $(\partial_t\mp i\Lambda)+B_\pm$ and let*

$$\mathcal{S}(t,s)=\begin{pmatrix}S_+(t,s) & 0\\ 0 & S_-(t,s)\end{pmatrix}. \tag{3.12}$$

The solution operator $\mathfrak{S}(t,s)$ of System (3.10) have the following decomposition

$$\mathfrak{S}(t,s)=\mathcal{S}(t,s)+\mathcal{R}(t,s), \tag{3.13}$$

where, for all $\sigma\in\mathbb{R}$,

$$\mathcal{R}(t,s)\in\mathcal{B}((0,T)^2;\mathcal{L}(H^\sigma(\mathcal{M};\mathbb{C}^{2N});H^{\sigma+1}(\mathcal{M};\mathbb{C}^{2N}))) \tag{3.14}$$

Proof. For the case of $N=1$, we refer to [32].

We use a trick to decouple the equations. More precisely, the idea is to find an operator $K\in\mathcal{C}^\infty(0,T_0;\Psi^{-1}(\mathcal{M};\mathbb{C}^{2N\times 2N}))$ so that $W=(Id_{2N\times 2N}-K)\mathcal{V}$ solves a diagonal system, up to appropriate remainders. We have on the one hand

$$(Id_{2N\times 2N}+K)W=\mathcal{V}-K^2\mathcal{V}.$$

Notice that $P\mathcal{V}=0$, then

$$\begin{aligned}(Id_{2N\times 2N}-K)P(Id_{2N\times 2N}+K)W&=(Id_{2N\times 2N}-K)P(\mathcal{V}-K^2\mathcal{V})\\&=-(Id_{2N\times 2N}-K)PK^2\mathcal{V}=R\mathcal{V}.\end{aligned}$$

Moreover, the remainder satisfies $R \in \mathfrak{R}^{-1}$, where

$$\mathfrak{R}^{-1} = \mathcal{C}^\infty(0,T;\Psi^{-1}_{phg}(\mathcal{M};\mathbb{C}^{2N\times 2N})) + \mathcal{C}^\infty(0,T;\Psi^{-2}_{phg}(\mathcal{M};\mathbb{C}^{2N\times 2N}))\partial_t$$

is the admissible class of remainders in the present context. On the other hand, we have

$$(Id_{2N\times 2N} - K)P(Id_{2N\times 2N} + K)W = PW + [P,K]W - KPKW, \tag{3.15}$$

with $KPK \in \mathfrak{R}^{-1}$. We then remark that $[\partial_t, K]W = (\partial_t K)W$ so that

$$[\partial_t, K] \in \mathcal{C}^\infty(0,T;\Psi^{-1}_{phg}(\mathcal{M};\mathbb{C}^{2N\times 2N})) \subset \mathfrak{R}^{-1}$$

and as well $[B,K] \in \mathfrak{R}^{-1}$. Hence, if we find K such that

$$\begin{pmatrix} 0 & B_- \\ B_+ & 0 \end{pmatrix} + [M_1, K] \in \mathfrak{R}^{-1}, \tag{3.16}$$

then W solves the following equation

$$P_d W = R_1 W + R_2 \mathcal{V} = R\mathcal{V}, \tag{3.17}$$

with $R_1, R_2, R \in \mathfrak{R}^{-1}$ and, with M_1 defined in (3.11),

$$P_d = \partial_t + M_1 + A_d, \qquad A_d = \begin{pmatrix} B_+ & 0 \\ 0 & B_- \end{pmatrix}. \tag{3.18}$$

Now taking

$$K := \frac{1}{2i}\begin{pmatrix} 0 & \Lambda^{-1}B_- \\ -B_+\Lambda^{-1} & 0 \end{pmatrix} \in \mathcal{C}^\infty(0,T_0;\Psi^{-1}_{phg}(\mathcal{M};\mathbb{C}^{2N\times 2N})) \tag{3.19}$$

realizes (3.16), and we are left to study $P_d W = R\mathcal{V}$, $R \in \mathfrak{R}^{-1}$, with $W = (Id_{2N\times 2N} - K)\mathcal{V}$. Note that it is crucial at this step that M_1 is diagonal so that, for instance, $\Lambda^{-1}B_-\Lambda - B_- \in \mathfrak{R}^{-1}$. $\mathcal{S}(t,s)$ defined in (3.12) is therefore the solution operator of P_d. Equation (3.17) is now solved by

$$W(t) = \mathcal{S}(t,s)W(s) + \int_s^t \mathcal{S}(t,t')R(t')\mathcal{V}(t')\mathrm{d}t', \qquad R \in \mathfrak{R}^{-1}. \tag{3.20}$$

Recalling that $W = (Id_{2N\times 2N} - K)\mathcal{V}$, and that $\mathcal{V}(t) = \mathfrak{S}(t,s)\mathcal{V}(s)$, this yields

$$\mathcal{V}(t) = \mathcal{S}(t,s)\mathcal{V}(s) + K(t)\mathfrak{S}(t,s)\mathcal{V}(s) - \mathcal{S}(t,s)K(s)\mathcal{V}(s) + \Big(\int_s^t \mathcal{S}(t,t')R(t')\mathfrak{S}(t',s)\mathrm{d}t'\Big)\mathcal{V}(s).$$

This can be rewritten as

$$\mathcal{V}(t) = \mathcal{S}(t,s)\mathcal{V}(s) + \mathcal{R}(t,s)\mathcal{V}(s),$$

with

$$\mathcal{R}(t,s) = K(t)\mathfrak{S}(t,s) - \mathcal{S}(t,s)K(s) + \Big(\int_s^t \mathcal{S}(t,t')R(t')\mathfrak{S}(t',s)\mathrm{d}t'\Big).$$

According to regularity of $\mathfrak{S}(t,s)$ and $K(s)$, one can see easily that $\mathcal{R}(t,s) \in \mathcal{B}((0,T_0)^2;\mathcal{L}(H^\sigma(\mathcal{M};\mathbb{C}^{2N});H^{\sigma+1}(\mathcal{M};\mathbb{C}^{2N})))$ and $\partial_t\mathcal{R}(t,s),\partial_s\mathcal{R}(t,s) \in \mathcal{B}((0,T_0)^2;\mathcal{L}(H^\sigma(\mathcal{M};\mathbb{C}^{2N})))$ for any $\sigma\in\mathbb{R}$. This finishes the proof of Lemma 3.1. □

Lemma 3.1 states that $\mathfrak{S}(t,s)$ can be divided into two parts, diagonal term $\mathcal{S}(t,s)$ and a more regular term $R(t,s)$. So we have a high-frequency representation formula for solutions of System (1.1).

- **Gramian Operator**

In this part, we apply System Egorov Theorem 2.1 to express the Gramian operator as a pseudodifferential operator, following [24] for the scalar case.

Theorem 3.2. *Let $V_* := (V_0,V_1)\in (H^1)^N\times(L^2)^N$ be the initial data of System (1.1). Let $T_0>0$. Then for any $T\in(0,T_0]$, we have*

$$\int_0^T \|DV(t)\|^2_{(L^2)^K}\mathrm{d}t = ((G_T+\tilde{\mathcal{R}}_T)\Sigma V_*,\Sigma V_*)_{(L^2)^N\times(L^2)^N}, \tag{3.21}$$

where $G_T\in\mathcal{C}^\infty(0,T_0;\Psi^0_{phg}(\mathcal{M};\mathbb{C}^{2N\times 2N}))$ and $\tilde{\mathcal{R}}_T\in\mathcal{B}(0,T_0;\mathcal{L}((H^\sigma)^{2N},(H^{\sigma+1})^{2N}))$ is in a class of regularizing operators of order at least one. Moreover, the principal symbol of G_T can be characterized as follows:

$$\begin{aligned}
\sigma_0(G_T) &= \begin{pmatrix} G^+_\rho(T) & 0\\ 0 & G^-_\rho(T)\end{pmatrix} \in \mathcal{C}^\infty(0,T_0;S^0_{phg}(\mathcal{M};\mathbb{C}^{2N\times 2N})),\\
G^+_\rho(T) &= \frac14\int_0^T R^*_+(0,t;\varphi_{-t}(\rho))\mathrm{d}^*_+(t,\varphi_{-t}(\rho))\mathrm{d}_+(t,\varphi_{-t}(\rho))R_+(0,t;\varphi_{-t}(\rho))\mathrm{d}t,\\
G^-_\rho(T) &= \frac14\int_0^T R^*_-(0,t;\varphi_{t}(\rho))\mathrm{d}^*_-(t,\varphi_{t}(\rho))\mathrm{d}_-(t,\varphi_{t}(\rho))R_-(0,t;\varphi_{t}(\rho))\mathrm{d}t,
\end{aligned}\tag{3.22}$$

where $R_\pm(\tau,t;\rho)$ satisfies

$$\frac{\mathrm{d}R_\pm(\tau,t;\rho)}{\mathrm{d}\tau} = R_\pm(\tau,t;\rho)b_\pm(\tau,\varphi_{\pm(t-\tau)}(\rho)),\quad R_\pm(t,t;\rho)=Id_{N\times N}, \tag{3.23}$$

with $b_\pm=\sigma_0(B_\pm)=\frac12(a_0\pm\frac{a_1}{i|\xi|_x})$, $d_\pm=d_0\pm\frac{d_1}{i|\xi|_x}$ and $\varphi_t(\rho)$ is the Hamiltonnian flow of $|\xi|_x$ initiated at ρ (see Theorem 2.1 for more precisions).

This theorem is the main step to prove Theorem 1.9. The only difference is the characterization of $G^\pm_\rho(T)$ as the Gramian matrix of appropriate control problems, which will be made in another section. The proof is a direct combination of Propositions 3.3 and 3.4 below.

Proposition 3.3. *Denote by $V_*=(V_0,V_1)\in(H^1)^N\times(L^2)^N$ the initial data of System (1.1). We have*

$$\int_0^T \|DV(t)\|^2_{(L^2)^K}\mathrm{d}t = ((\mathcal{G}_T+\mathcal{R}_T)\Sigma V_*,\Sigma V_*)_{(L^2)^{2N}}, \tag{3.24}$$

where

$$\mathcal{R}_T\in\mathcal{B}_{loc}(\mathbb{R}^+;\mathcal{L}(H^\sigma(\mathcal{M};\mathbb{C}^{2N});H^{\sigma+1}(\mathcal{M};\mathbb{C}^{2N}))),\quad \forall\sigma\in\mathbb{R}. \tag{3.25}$$

and

$$
\begin{aligned}
\mathcal{G}_T &= \int_0^T \begin{pmatrix} S(t,0)_+^* D^{11} S(t,0)_+ & S(t,0)_+^* D^{12} S(t,0)_- \\ S(t,0)_-^* D^{21} S(t,0)_+ & S(t,0)_-^* D^{22} S(t,0)_- \end{pmatrix} \mathrm{d}t, \\
D^{11} &= \frac{D_0^* D_0}{4} + \frac{\Lambda^{-1} D_1^* D_1 \Lambda^{-1}}{4} - \frac{\Lambda^{-1} D_1^* D_0}{4i} + \frac{D_0^* D_1 \Lambda^{-1}}{4i}, \\
D^{12} &= \frac{D_0^* D_0}{4} - \frac{\Lambda^{-1} D_1^* D_1 \Lambda^{-1}}{4} - \frac{\Lambda^{-1} D_1^* D_0}{4i} - \frac{D_0^* D_1 \Lambda^{-1}}{4i}, \\
D^{21} &= \frac{D_0^* D_0}{4} - \frac{\Lambda^{-1} D_1^* D_1 \Lambda^{-1}}{4} + \frac{\Lambda^{-1} D_1^* D_0}{4i} + \frac{D_0^* D_1 \Lambda^{-1}}{4i}, \\
D^{22} &= \frac{D_0^* D_0}{4} + \frac{\Lambda^{-1} D_1^* D_1 \Lambda^{-1}}{4} + \frac{\Lambda^{-1} D_1^* D_0}{4i} - \frac{D_0^* D_1 \Lambda^{-1}}{4i},
\end{aligned} \tag{3.26}
$$

where the definition of $S_\pm(t,s)$ *is given in Lemma 3.1,*

Proof. The proof of Proposition 3.3 essentially relies on some computations and an application of Lemma 3.1. According to (3.5)

$$
\begin{aligned}
\int_0^T (DV, DV)_{(L^2)^K} \mathrm{d}t = \int_0^T & \left(\frac{D_1 \Lambda^{-1}}{2i}(V_+ - V_-), \frac{D_1 \Lambda^{-1}}{2i}(V_+ - V_-) \right)_{(L^2)^K} \\
&+ \left(\frac{D_0}{2}(V_+ + V_-), \frac{D_0}{2}(V_+ + V_-) \right)_{(L^2)^K} \\
&+ \left(\frac{D_0}{2}(V_+ + V_-), \frac{D_1 \Lambda^{-1}}{2i}(V_+ - V_-) \right)_{(L^2)^K} \\
&+ \left(\frac{D_1 \Lambda^{-1}}{2i}(V_+ - V_-), \frac{D_0}{2}(V_+ + V_-) \right)_{(L^2)^K} \mathrm{d}t.
\end{aligned} \tag{3.27}
$$

Denote $\hat{D}^1 = \begin{pmatrix} D^{11} & D^{12} \\ D^{21} & D^{22} \end{pmatrix}$. Since $V_* = (V_0, V_1)'$, we have

$$
\int_0^T (DV, DV)_{(L^2)^K} \mathrm{d}t = \int_0^T \left(\hat{D}^1 \mathfrak{S}(t,0) \Sigma V_*, \mathfrak{S}(t,0) \Sigma V_* \right)_{(L^2)^{2N}}. \tag{3.28}
$$

According to Lemma 3.1,

$$
(V_+, V_-)^{tr} = \mathfrak{S}(t,0) \Sigma V_* = (\mathcal{S}(t,0) + R(t,0)) \Sigma V_*. \tag{3.29}
$$

Combining (3.29) with (3.28), we have

$$
\begin{aligned}
\int_0^T (DV, DV)_{(L^2)^K} \mathrm{d}t = \int_0^T & (\mathcal{S}^*(t,0) \hat{D}^1 \mathcal{S}(t,0) \Sigma V_*, \Sigma V_*)_{(L^2)^{2N}} \\
&+ (R^*(t,0) \hat{D}^1 \mathcal{S}(t,0) + \mathcal{S}^*(t,0) \hat{D}^1 R(t,0) \Sigma V_*, \Sigma V_*)_{(L^2)^{2N}} \\
&+ (R^*(t,0) \hat{D}^1 R(t,0) \Sigma V_*, \Sigma V_*)_{(L^2)^{2N}}.
\end{aligned}
$$

Define

$$
\mathcal{R}_T = R^*(t,0) \hat{D}^1 \mathcal{S}(t,0) + \mathcal{S}^*(t,0) \hat{D}^1 R(t,0) + R^*(t,0) \hat{D}^1 R(t,0)
$$

with

$$\mathcal{G}_T = \mathcal{S}^*(t,0)\hat{D}^1\mathcal{S}(t,0),$$

we obtain (3.24). We claim that $\mathcal{R}_T$ satisfies (3.25). Indeed, $\mathcal{S}(t,0)$ preserves the regularity thanks to (3.14) in Lemma 3.1 and $\hat{D}^1$ is a Pseudodifferential operator of order 0. □

Proposition 3.4. *$\mathcal{G}_T$ (defined in (3.26)) has a decomposition as $\mathcal{G}_T = G_T + R_T$, where R_T satisfies $R_T \in \mathcal{B}_{loc}(\mathbb{R}^+; \mathcal{L}(H^\sigma(\mathcal{M};\mathbb{C}^{2N}); H^{\sigma+1}(\mathcal{M};\mathbb{C}^{2N})))$ for all $\sigma \in \mathbb{R}$ and $G_T \in \mathcal{C}^\infty(\mathbb{R}^+; \Psi^0_{phg}(\mathcal{M};\mathbb{C}^{2N\times 2N}))$ has principal symbol*

$$\begin{aligned}\sigma_0(G_T) &= \begin{pmatrix} G^+_\rho(T) & 0 \\ 0 & G^-_\rho(T)\end{pmatrix},\\ G^\pm_\rho(T) &= \frac{1}{4}\int_0^T R^*_\pm(0,t;\varphi^\mp_t(\rho))d^*_\pm(t,\varphi^\mp_t(\rho))d_\pm(t,\varphi^\mp_t(\rho))R_\pm(0,t;\varphi^\mp_t(\rho))\mathrm{d}t,\end{aligned} \tag{3.30}$$

where $R_\pm(s,t;\rho)$ satisfies

$$\frac{\mathrm{d}R_\pm(\tau,t;\rho)}{\mathrm{d}\tau} = R_\pm(\tau,t;\rho)b_\pm(\tau,\varphi^\mp_{\tau-t}(\rho)), \quad R_\pm(t,t;\rho) = Id_{N\times N}, \tag{3.31}$$

with $b_\pm = \sigma_0(B_\pm) = \frac{1}{2}a_0 \pm \frac{1}{2}\frac{a_1}{i|\xi|_x}$, $d_\pm = d_0 \pm \frac{d_1}{i|\xi|_x}$ and $\varphi^\mp_t(\rho)$ is the Hamiltonian flow of $\mp|\xi|_x$ initiated at ρ.

The proof relies on Egorov Theorem and the following Lemma that deals with the anti-diagonal terms of the Gramian control operator whose proof is postponed to the Appendix.

Lemma 3.5. *Assume that $\mathcal{I}$ is an interval in $\mathbb{R}$, let*

$$H_\pm(t) = \pm\Lambda Id_{N\times N} + iW_0(t),$$

with $W_0 \in \mathcal{C}^\infty(0,T;\Psi^0_{phg}(\mathcal{M};\mathbb{C}^{N\times N}))$, then for any $B_0 \in \mathcal{C}^\infty(0,T;\Psi^m_{phg}(\mathcal{M};\mathbb{C}^{N\times N}))$, $m\in\mathbb{R}$, we can define

$$B(T) = \int_0^T S_\pm(t,0)^* B_0 S_\mp(t,0)\mathrm{d}t,$$

and we have $B \in \mathcal{B}_{loc}(0,T;\mathcal{L}((H^\sigma)^N,(H^{\sigma+1-m})^N)$ for all $\sigma\in\mathbb{R}$.

Proof of Proposition 3.4. Integrating the anti-diagonal terms of $\mathcal{G}_T$ in (3.26) on $[0,T]$ yields

$$\begin{aligned}&\int_0^T S(t,0)^*_+ D^{12} S(t,0)_- dt \in \mathcal{B}_{loc}(\mathcal{I};\mathcal{L}((H^\sigma)^N,(H^{\sigma+1-m})^N),\\ &\int_0^T S(t,0)^*_- D^{21} S(t,0)_+ dt \in \mathcal{B}_{loc}(\mathcal{I};\mathcal{L}((H^\sigma)^N,(H^{\sigma+1-m})^N).\end{aligned} \tag{3.32}$$

Next we claim that there exist $G^+_\rho(T), G^-_\rho(T)$ satisfying (3.30) and

$$\begin{aligned}&\int_0^T S(t,0)^*_+ D^{11} S(t,0)_+ dt - G^+_\rho(T) \in \mathcal{B}_{loc}(\mathcal{I};\mathcal{L}((H^\sigma)^N,(H^{\sigma+1-m})^N),\\ &\int_0^T S(t,0)^*_- D^{22} S(t,0)_- dt - G^-_\rho(T) \in \mathcal{B}_{loc}(\mathcal{I};\mathcal{L}((H^\sigma)^N,(H^{\sigma+1-m})^N).\end{aligned} \tag{3.33}$$

We only detail $S(t,0)_+^* D^{11} S(t,0)_+$, the other case being similar. Using Theorem 2.1 with

- $c=1$, so that $\chi^1_{\tau,s}(\rho)=\varphi^-_{\tau-s}(\rho)=\varphi_{s-\tau}(\rho)$ and $\chi^{c=1}_{t,0}=\varphi_{-t}$
- $W_0=B_+$
- $P_m=D^{11}=\frac{1}{2}(D_0+\frac{D_1\Lambda^{-1}}{i})^*\cdot\frac{1}{2}(D_0+\frac{D_1\Lambda^{-1}}{i})$

gives $S(t,0)_+^* D^{11} S(t,0)_+ = R_+^*(0,t,\varphi_{-t}(\rho))d^{11}(t,\varphi_{-t}(\rho))R_+(0,t,\varphi_{-t}(\rho))$, where R_+ solves

$$\frac{\mathrm{d}R_+(\tau,s;\rho)}{\mathrm{d}\tau}=R_+(\tau,s;\rho)b_+(\tau,\varphi^-_{\tau-s}(\rho)),\quad R_+(s,s;\rho)=Id_{N\times N} \tag{3.34}$$

and $b_+=\sigma_0(B_+)=\frac{1}{2}\sigma_0(A_0-iA_1\Lambda^{-1})=\frac{1}{2}(a_0+\frac{a_1}{i|\xi|_x}), d_+=\sigma_0(D_0+\frac{D^1\Lambda^{-1}}{i})=d_0+\frac{d_1}{i|\xi|_x}$,

The other case $S(t,0)_-^* D^{22} S(t,0)_-$ is the same with $c=-1$, $\chi^{-1}_{t,0}=\varphi_t$, $W_0=B_-$, $P_m=D^{22}$. □

- **Gramian operator and weak observability inequality** (1.6)

As a direct consequence (or verification) of Theorem 3.2, $\sigma_0(G_T)$ is a nonnegative symmetric matrix. Thanks to $N\times N$ Sharp Gårding Inequality (2.18), we can construct a connection between weak observability inequality (1.6) and Gramian control operator as follows.

Proposition 3.6. *Let $T>0$. Define*

$$\begin{aligned}\mathcal{R}_2(T)=\min\Big\{&\min_{\rho_0\in S^*\mathcal{M}}\sup\left\{s\in\mathbb{R}\,\middle|\,\beta^*(G^+_\rho(T)-sId_{N\times N})\beta\geq 0,\forall\beta\in\mathbb{C}^N\right\},\\ &\min_{\rho_0\in S^*\mathcal{M}}\sup\left\{s\in\mathbb{R}\,\middle|\,\beta^*(G^-_\rho(T)-sId_{N\times N})\beta\geq 0,\forall\beta\in\mathbb{C}^N\right\}\Big\}\end{aligned} \tag{3.35}$$

and $G^\pm_\rho(T)$ are defined in (3.22). If for all $\rho_0\in S^*\mathcal{M}$, $\sigma_0(G_T)$ is a symmetric positive matrix, then the weak observability inequality (1.6) holds for all solutions of System (1.1) with

$$C^2_{\mathrm{obs}}(T)\geq\frac{1}{2\mathcal{R}_2(T)}. \tag{3.36}$$

Proof. According to Theorem 3.2, we have

$$\begin{aligned}\int_0^T\|DV(t)\|^2_{(L^2)^K}\mathrm{d}t&=((G_T+\mathcal{R}_T)\Sigma V_*,\Sigma V_*)_{(L^2)^{2N}}\\ &\geq(G_T\Sigma V_*,\Sigma V_*)_{(L^2)^{2N}}-C^1\|\Sigma V_*\|^2_{(H^{-\frac{1}{2}})^{2N}},\end{aligned} \tag{3.37}$$

and $\sigma(G_T)$ is a symmetric positive matrix. Then $\sigma(G_T)-\mathcal{R}_2(T)Id_{2N\times 2N}$ is a nonnegative symmetric matrix (here, we are using that actually, the supremum in (3.35) is actually a maximum). By $N\times N$ Sharp Gårding Inequality (2.18) and (3.37), we obtain

$$\int_0^T\|DV(t)\|^2_{(L^2)^K}\mathrm{d}t\geq(\mathcal{R}_2(T)\Sigma V_*,\Sigma V_*)_{(L^2)^{2N}}-C^1\|\Sigma V_*\|^2_{(H^{-\frac{1}{2}})^{2N}}.$$

Combining with (3.8), we have

$$\frac{1}{\mathcal{R}_2(T)}\int_0^T\|DV(t)\|^2_{(L^2)^K}\mathrm{d}t\geq 2\|(V_0,V_1)\|^2_{(H^1)^N\times(L^2)^N}-C\|(V_0,V_1)\|^2_{(H^{\frac{1}{2}})^N\times(H^{-\frac{1}{2}})^N}. \tag{3.38}$$

So this finishes the proof of Proposition 3.6. □

- **Gramian control operator and controllability of finite dimensional system**
For fixed $\rho_0 \in S^*\mathcal{M}$, we will consider the following control system

$$\begin{cases} \dot{X}(t) = \dfrac{1}{2}a_\pm^*(t, \varphi_{\mp t}(\rho_0))X(t) + \dfrac{1}{2}d_\pm^*(t, \varphi_{\mp t}(\rho_0))u(t), \\ X(0) = X_0 \in \mathbb{C}^N, \end{cases} \tag{3.39}$$

where $X(t) = (X_1, \cdots, X_N)^{tr}$ is a vector having N components, and $a_\pm = a_0 \pm \frac{a_1}{i|\xi|_x}$ is a $N \times N$ matrix. $d_\pm = d_0 \pm \frac{d_1}{i|\xi|_x}$ is a $K \times N$ matrix. $u(t) \in L^2(0, T; \mathbb{C}^K)$ is the control.

Next we reveal connections between Gramian control operator and exact controllability of finite dimensional System (3.39) as follows.

The first step is an elementary but crucial lemma.

Lemma 3.7. *Let $\rho_0 \in S^*\mathcal{M}$. Denote $\tilde{R}_\pm(\cdot, \cdot; \rho_0)$ the resolvent of System (3.39) (see [22], Prop. 1.5 for definition). Then*

$$R_\pm(\tau, t; \rho_0) = \widetilde{R}_\pm(\tau, t; \varphi_{\pm t}(\rho_0))^*, \tag{3.40}$$

where $R_\pm(\cdot, \cdot; \rho_0)$ is defined in (3.23). Moreover, let the $G_\pm$ be the Gramian of the control System (3.39)(see [22], Def. 1.10). Then

$$G_\pm = G^\pm_{\rho_0}(T) \tag{3.41}$$

where $G^\pm_{\rho_0}(T)$ is defined in (3.22).

Proof. The equation (3.23) verified by $R_\pm$ is

$$\frac{\mathrm{d}R_\pm(\tau, t; \rho_0)}{\mathrm{d}\tau} = R_\pm(\tau, t; \rho_0) b_\pm(\tau, \varphi_{\pm(t-\tau)}(\rho_0)), \quad R_\pm(t, t; \rho_0) = Id_{N\times N}, \tag{3.42}$$

Taking the adjoint of the definition of the resolvent, see ([22], (1.10) in Prop. 1.5) applied to System (3.39) and recalling $b_\pm(\tau, \rho_0) = \frac{a_\pm(\tau, \rho_0)}{2}$ gives

$$\frac{\mathrm{d}\widetilde{R}_\pm(\tau, t; \rho_0)^*}{\mathrm{d}\tau} = \widetilde{R}_\pm(\tau, t; \rho_0)^* \frac{a_\pm(\tau, \varphi_{\mp\tau}(\rho_0))}{2} = \widetilde{R}_\pm(\tau, t; \rho_0)^* b_\pm(\tau, \varphi_{\mp\tau}(\rho_0)), \quad \widetilde{R}_\pm(t, t; \rho_0) = Id_{N\times N}.$$

Applying at the point $\varphi_{\pm t}(\rho_0)$ gives

$$\frac{\mathrm{d}\widetilde{R}_\pm(\tau, t; \varphi_{\pm t}(\rho_0))^*}{\mathrm{d}\tau} = \widetilde{R}_\pm(\tau, t; \varphi_{\pm t}(\rho_0))^* b_\pm(\tau, \varphi_{\pm(t-\tau)}(\rho_0)), \quad \widetilde{R}_\pm(t, t; \varphi_{\pm t}(\rho_0)) = Id_{N\times N}. \tag{3.43}$$

Therefore, for fixed t, the two matrices $R_\pm(\tau, t; \rho_0)$ and $\widetilde{R}_\pm(\tau, t; \varphi_{\pm t}(\rho_0))^*$ depending on τ solve the same equation with same initial data, so they are equal by the Cauchy-Lipschitz Theorem.

Concerning the second part of the Lemma, we have

$$G_\pm = \frac{1}{4}\int_0^T \tilde{R}_\pm(0, s; \rho_0) d_\pm^*(s, \varphi_{\mp s}(\rho_0)) d_\pm(s, \varphi_{\mp s}(\rho_0)) \tilde{R}_\pm^*(0, s; \rho_0) \mathrm{d}s.$$

Concerning (3.22), it can be written

$$G^\pm_{\rho_0}(T) = \frac{1}{4}\int_0^T R_\pm^*(0, t; \varphi_{\mp t}(\rho_0)) d_\pm^*(t, \varphi_{\mp t}(\rho_0)) d_\pm(t, \varphi_{\mp t}(\rho_0)) R_\pm(0, t; \varphi_{\mp t}(\rho_0)) \mathrm{d}t.$$

Now, using the obtained identity (3.40), we get the expected result

$$G^{\pm}_{\rho_0}(T) = \frac{1}{4}\int_0^T \widetilde{R}_{\pm}(0,t;\rho_0)d^*_{\pm}(t,\varphi_{\mp t}(\rho_0))d_{\pm}(t,\varphi_{\mp t}(\rho_0))\widetilde{R}^*_{\pm}(0,t;\rho_0)\mathrm{d}t.$$

□

Theorem 1.9 is now a direct consequence of Theorem 3.2 and of the previous Lemma. Another consequence is the following.

Proposition 3.8. *Let $T > 0$, for any $\rho_0 \in S^*\mathcal{M}$, we have the equivalence*

1. *Hermitian matrix $\sigma_0(G_T)$ (defined in (3.22)) is positive.*
2. *System (3.39) are exactly controllable in both cases $+$ and $-$.*

Proof. Thanks to (3.41), the result is now a direct consequence of the classical equivalence between invertibility of the Gramian and controllability, see ([22], Thm. 1.11). □

In the next proposition, we prove that if a and d have some symmetry properties, we need to check the controllability of only one system $+$ or $-$. This property will be satisfied in the two important cases

- A is a differential operator so that a_0 is even and a_1 is odd and $a_{\pm} = \frac{1}{2}\left(a_0 \pm \frac{a_1}{i|\xi|_x}\right)$ (same for d)
- $A = \Lambda A(x)$ where $A(x)$ is the operator of multiplication by a matrix $A(x)$, which will be the case for zero order coupling.

Proposition 3.9. *Assume $a_{\pm} = a_{\mp}\circ\sigma, b_{\pm} = b_{\mp}\circ\sigma, d_{\pm} = d_{\mp}\circ\sigma$, where $\sigma : T^*\mathcal{M} \to T^*\mathcal{M}$ is the involution $(x,\xi)\mapsto(x,-\xi)$. Let $T > 0$, for any $\rho_0 \in S^*\mathcal{M}$, we have the equivalence*

1. *Hermitian matrix $\sigma_0(G_T)$ (defined in (3.22)) is positive at ρ_0 and $\sigma(\rho_0)$*
2. *System (1.8) is exactly controllable in the case a_- and d_- for ρ_0 and $\sigma(\rho_0)$*
3. *System (1.8) is exactly controllable in the case a_+ and d_+ for ρ_0 and $\sigma(\rho_0)$.*

The proof is direct with Proposition 3.8 and the following Lemma at hand.

Lemma 3.10. *With the symmetry assumptions of Proposition 3.9, we have*

$$G^+_{\sigma(\rho_0)}(T) = G^-_{\rho_0}(T).$$

Proof. By using ([25], Lem. B.2), we have

$$(\varphi_t(\rho_0))\circ\sigma = \sigma\circ(\varphi_{-t}(\rho_0)), \quad (\varphi_{-t}(\rho_0))\circ\sigma = \sigma\circ(\varphi_t(\rho_0)). \tag{3.44}$$

We have, first by (3.44), then by the symmetry property of d,

$$d_+(t,\varphi_{-t}(\sigma(\rho_0))) = d_+(t,\sigma\circ\varphi_t(\rho_0)) = d_-(t,\varphi_t(\rho_0)) \tag{3.45}$$

and the same holds for the transpose. For the same reasons, we have

$$b_+(\tau,\varphi^-_{\tau-t}(\sigma(\rho_0))) = b_+(\tau,\sigma\circ\varphi^+_{\tau-t}(\rho_0)) = b_-(\tau,\varphi^+_{\tau-t}(\rho_0)), \tag{3.46}$$

In particular, we have by (3.23)

$$\frac{\mathrm{d}R_+(\tau,t;\sigma(\rho_0))}{\mathrm{d}\tau} = R_+(\tau,t;\sigma(\rho_0))b_+(\tau,\varphi^-_{\tau-t}(\sigma(\rho_0))) = R_+(\tau,t;\sigma(\rho_0))b_-(\tau,\varphi^+_{\tau-t}(\rho_0)) \tag{3.47}$$

which is the equation satisfied by $R_-(\tau, t; \rho_0)$ with same initial data, so that $R_+(\tau, t; \sigma(\rho_0)) = R_-(\tau, t; \rho_0)$. (3.44) and this symmetry of R give

$$R_+(0, t; \varphi_t^-(\sigma(\rho_0))) = R_+(0, t; \sigma \circ \varphi_t^+(\rho_0)) = R_-(0, t; \varphi_t^+(\rho_0)). \tag{3.48}$$

This gives exactly the expected result $G^+_{\sigma(\rho_0)}(T) = G^-_{\rho_0}(T)$. □

3.2. Proof of Theorem 1.3

We actually plan to prove the slightly more general theorem. Let

$$a_0 \in C^\infty(\mathbb{R}; S^0_{phg}(T^*\mathcal{M}; \mathbb{C}^{N\times N})), a_1 \in C^\infty(\mathbb{R}; S^1_{phg}(T^*\mathcal{M}; \mathbb{C}^{N\times N})), \tag{3.49}$$

are, respectively, the principal symbols of A_0 and A_1 which are defined in (3.1). Let

$$d_0 \in C^\infty(\mathbb{R}; S^0_{phg}(T^*\mathcal{M}; \mathbb{C}^{K\times N})), d_1 \in C^\infty(\mathbb{R}; S^1_{phg}(T^*\mathcal{M}; \mathbb{C}^{K\times N})) \tag{3.50}$$

are, respectively, the principal symbols of D_0 and D_1 which are defined in (3.2).

Theorem 3.11. *Solutions of System (1.1) satisfy weak observability inequality (1.6) on $[0,T]$ if and only if for any $\rho_0 \in S^*\mathcal{M}$, any initial data $X_0 \in \mathbb{C}^N$, both systems (3.39) are exactly controllable on $[0,T]$.*

Proof. **Step 1.** By Propositions 3.6 and 3.8, it is easy to show that Systems (3.39) are exactly controllable implies weak observability inequality (1.6) for all solution of System (1.1).

Step 2. We check that System (1.1) satisfy weak observability inequality (1.6) implies System (3.39) are exactly controllable. Suppose by contradiction that this part of the Theorem failed, then there exists a $\rho_0 \in S^*\mathcal{M}$ and Hamiltonian flow $\varphi_t(\rho_0)$ such that one of the System (3.39) is not controllable, let us say $-$ for fixing the ideas. Hence $G_{\rho_0}(T)$ is nonpositive. According to Proposition 3.8, we have

$$\det\left(G^-_\rho(T)\right) = 0, \tag{3.51}$$

Then there exists a vector $P \in \mathbb{C}^N, |P|_{l^2(\mathbb{C}^N)} = 1$ such that $P^* G^-_\rho(T) P = 0$. We take a local chart $x_0 \in (U_\kappa, k)$ of M so that $g_{i,j}(x_{0)} = Id$. We denote by (y_0, η_0) the coordinates of ρ_0 in this chart. We choose $\psi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ such that $supp(\psi) \subset \kappa(U_\kappa)$, and $\psi = 1$ in a neighborhood of y_0 . Next we define

$$w^k(y) = C_0 k^{\frac{n}{4}} e^{ik\varphi(y)}\psi(y), \quad \varphi(y) = y\cdot\eta_0 + i(y-y_0)^2, \quad C_0 > 0.$$

Setting now

$$v^k_- = \kappa^* w^k \in \mathcal{C}^\infty_c(\mathcal{M}), \tag{3.52}$$

We have $v^k_- \rightharpoonup 0$ and $\lim_{k\to\infty} \|v^k_-\|_{L^2} = \lim_{k\to\infty}\|Pv^k_-\|_{(L^2)^N} = 1$ for an appropriate choice of C_0, while $\lim_{k\to\infty}\|v^k_-\|_{H^{-1}} = 0$. Moreover, a classical computation on $(w^k)_{k\in\mathbb{N}}$ show that for all $A \in \Psi^0_{phg}(\mathcal{M}; \mathbb{C}^{N\times N})$, $(v^k_-)_{k\in\mathbb{N}}$ satisfies

$$\left(APv^k_-, Pv^k_-\right)_{(L^2)^N} \to \sigma_0(P^*AP)(\rho_0), \; k\to\infty. \tag{3.53}$$

Next, we set $v_+^k = 0, \quad k \in \mathbb{N}$, and $V^k = \Sigma^{-1}(0, Pv_-^k) \in (H^1)^N \times (L^2)^N$. Denoting $\mathcal{V}^k(t)$ the solution to System (1.1) with initial data V^k, Theorems 3.2 and (3.53) gives

$$\begin{aligned}\lim_{k\to\infty}\int_0^T \|D\mathcal{V}^k(t)\|^2_{(L^2)^K}\,\mathrm{d}t &= \lim_{k\to\infty}\big((G_T+\widetilde{R}_T)\Sigma V^k,\Sigma V^k\big)_{(L^2)^{2N}}\\ &= \lim_{k\to\infty}\big(G_T\Sigma V^k,\Sigma V^k\big)_{(L^2)^{2N}}\\ &= P^*G_\rho^-(T)P = 0,\end{aligned} \tag{3.54}$$

where we used that R_T is 1-smoothing, that $G_T \in \Psi^0_{phg}(\mathcal{M};\mathbb{C}^{2N\times 2N})$ has principal symbol given by (3.33), and the choice of $\rho = \rho_0$ in (3.33) . Then we obtain a contradiction and finish the proof of Theorem 3.11. ☐

Proof of Theorem 1.3. Since A and D are differential operators, Proposition 3.9 applies and the conclusion is direct from Theorem 3.11. ☐

3.3. Proof of Theorem 1.6

Proof of Theorem 1.6. By Theorem 1.3, $G_{\rho_0}(T)$ is positive for any $\rho_0 \in S^*\mathcal{M}$. Hence $T > T_{\mathrm{crit}} = \inf_{T_0}\{T_0| \min_{\rho_0\in S^*\mathcal{M},|\beta|=1}\beta^*G_{\rho_0}(T_0)\beta > 0\}$.

Using Proposition 3.9, we have

$$\mathfrak{K}(T) = \mathcal{R}_2(T). \tag{3.55}$$

Proposition 3.6 then gives that the weak observability holds with

$$C^2_{\mathrm{obs}}(T) \geq \frac{1}{2\mathfrak{K}(T)}. \tag{3.56}$$

Since $\mathcal{M}$ is a compact manifold, it suffices to show that there exists a $\rho_0 \in S^*\mathcal{M}$, such that $\sigma(G_T) - \mathcal{R}_2(T)Id_{2N\times 2N}$ is a nonpositive matrix. In view of the proof of Theorem 3.11, it is easy to obtain

$$C^2_{\mathrm{obs}}(T)\mathcal{R}_2(T) = C^2_{\mathrm{obs}}(T)\int_0^T \|D\mathcal{V}^k(t)\|^2_{(L^2)^K}\,\mathrm{d}t \geq \sum_k \|V_k\|^2_{L^2} \to 1. \tag{3.57}$$

So we finish the proof of Theorem 1.6 thanks to Corollary 1.8 is a direct consequence of ([40], Thm. 3.2) and Theorem 1.6. ☐

4. Coupling of order zero

In this section, we plan to prove that the involved systems are well-posed and prove Theorem 1.10. In a first section, we will also describe that the assumption of the matrix being in a subdiagonal form is actually quite general, up to some change of unknown.

4.1. Getting the subdiagonal form

4.1.1. Getting the subdiagonal form in the constant coupling case

We use the Brunovsky normal form as described in Proposition A.1. This gives the immediate Lemma.

Lemma 4.1. *Assume that (A, B) are constant matrices and satisfy the Kalman rank condition. Let $(\widetilde{A}, \widetilde{B})$, Q, F, M_u given by Proposition A.1. Define also the space varying matrix $\widetilde{A}_\omega(x) = \widetilde{A}_{\chi_\omega(x)} = Q^{-1}(AQ + \chi_\omega(x)BF)$ where $\chi_\omega = 1$ on ω. Then, if $\widetilde{U}$ is solution of*

$$\begin{cases} \partial_t^2 \widetilde{U} - \Delta_g \widetilde{U} + \widetilde{A}_\omega \widetilde{U} = \chi_\omega(x)\widetilde{B}\widetilde{G}, \\ (\widetilde{U}(0), \widetilde{U}_t(0)) = (\widetilde{U}_0, \widetilde{U}_1). \end{cases} \tag{4.1}$$

then, $U = Q\widetilde{U}$ is solution of the following system with control $G = -F\widetilde{U} + M_u\widetilde{G}$

$$\begin{cases} \partial_t^2 U - \Delta_g U + AU = \chi_\omega(x)BG, \\ (U(0), \partial_t U(0)) = Q(\widetilde{U}_0, \widetilde{U}_1). \end{cases} \tag{4.2}$$

In particular, if the System (4.1) is controllable in some space $\mathcal{E} = \mathcal{E}_1 \times \mathcal{E}_0$ satisfying $\mathcal{E}_1 \subset (L^2)^N$ with control $\widetilde{G} \in L^2(0, T; L^2)^K)$, then the System (4.2) is controllable in $\mathcal{E}_Q = Q\mathcal{E}$ on $[0, T]$ with control $G \in L^2(0, T; L^2)^K)$.

Moreover, we have the following properties concerning the coupling matrix.

1. *For any $x \in \mathcal{M}$, has a subdiagonal form as described in Proposition A.1.*
2. *For $x \in \omega$, $\widetilde{A}_\omega(x) = \widetilde{A}$ so that $(\widetilde{A}_{\text{sub}}, \widetilde{B}) = (\widetilde{A}, \widetilde{B})$ satisfies the Kalman rank condition, where the index sub means that we only keep the subdiagonal terms as in (1.21).*

Proof. This is just a direct computation, denote $\Box = \partial_t^2 - \Delta_g$, we have

$$\begin{aligned} \Box U + AU = Q\Box\widetilde{U} + AQ\widetilde{U} &= -Q\widetilde{A}_\omega\widetilde{U} + \chi_\omega(x)Q\widetilde{B}\widetilde{G} + AQ\widetilde{U} \\ &= -(AQ + \chi_\omega(x)BF)\widetilde{U} + \chi_\omega(x)BM_u\widetilde{G} + AQ\widetilde{U} \\ &= \chi_\omega(x)B(-F\widetilde{U} + M_u\widetilde{G}) = \chi_\omega(x)BG. \end{aligned}$$

The second property about the link between the controllability of each equation is direct. The properties of the coupling matrix are then a direct consequences of Proposition A.1. ☐

4.1.2. An algorithm to obtain a natural subdiagonal form

In this section, we describe one natural (informal) algorithm as following when considering the control System

$$\begin{cases} \partial_t^2 U - \Delta_g U + AU = BG, \\ (U(0), \partial_t U(0)) = (U_0, U_1). \end{cases} \quad \text{with control } G \in L^2(0, T; L^2)^K) \tag{4.3}$$

where $A(x)$ is a matrix in $\mathbb{R}^{N\times N}$ and $B(x)$ is a matrix in $\mathbb{R}^{N\times K}$.

We start from the subspace of $\mathbb{R}^N$ that might be reached directly by the control G (without using the coupling). Namely, we define $E_1 \subset \mathbb{R}^N$ as $E_1 = Vect\{\cup_{x\in\mathcal{M}} Range(B(x))\}$. This set of state variables might be controlled in $H^1 \times L^2$.

Next, we define the subspace that might be controlled from E_1 (if we can control in all E_1) through the coupling. This makes us to define naturally $E_2 = Vect\{\cup_{x\in\mathcal{M}} A(x)(E_1)\}$. This set of state variables might be controlled from a source term in $C([0, T]; H^1)$ so, we expect to control the states at least in $H^2 \times H^1$ (but may be better for direction that are in $E_1 \cap E_2$).

Again, we want to define the subspace that might be controlled from E_2 (if we can control in all E_2) through the coupling. The natural new space that could be reached is $E_3 = Vect\{\cup_{x\in\mathcal{M}} A(x)(E_2)\}$. This time, the new source term is in $C(0, T; H^2)$. Thus, we expect to control the states in $H^3 \times H^2$.

So, this leads to the definition of subspaces of E_i by iteration:

$$E_1 = Vect\left\{\cup_{x\in\mathcal{M}} Range(B(x))\right\}; \quad E_{i+1} = Vect\left\{\cup_{x\in\mathcal{M}} A(x)(E_i)\right\}. \tag{4.4}$$

$H_k = Vect_{i=1}^k E_i$ is clearly an increasing sequence of subspaces of $\mathbb{R}^N$ that is stationary after some steps that we call k. Morever, it satisfies the important property $A(x)(H_i) \subset H_{i+1}$ for any $x \in \mathcal{M}$. It could happen that the bigger space H_k is not equal to $\mathbb{R}^N$, but it is easy to see that the wave system is not controllable in this case. Indeed, for any control $B(x)G(t,x) \in H_k$ since H_k contains $Vect\left\{\cup_{x\in\mathcal{M}} Range(B(x))\right\}$, and $A(x)(H_k) \subset H_k$, so for any initial data in H_k, the solution remains in H_k.

We can then assume now that we can decompose $\mathbb{R}^N = \oplus_{i=1}^k F_i$ with $F_i \cap F_j = \{0\}$ if $i \neq j$ and $\oplus_{i=1}^n F_i = H_n$. In particular, in a basis according to F_i, $A(x), B(x)$ can be written as a matrix "subdiagonal by block" as (1.18) and (1.19).

Note that in the case $A(x) = A$ and $B(x) = B$, we have $E_1 = Range(B)$ and $E_i = Range(A^{i-1}B)$, so that this decomposition is related to the Kalman rank condition and the Brunovsky normal form described in Proposition A.1.

4.2. Wellposedness in multilevel spaces

Up to now and in the next section, we assume that $A(x)$ and $B(x)$ have the form described in Theorem 1.10.

The natural space for solutions of (4.3) is then then space $\mathcal{H}^s$ as follows. $U \in \mathcal{H}^s$ if for every $i = 1, ..., k$, we have $U^i \in (H^{s+i-1})^{d_i}$ where d_i is the dimension of F_i. That is

$$\mathcal{H}^s = (H^s)^{d_1} \times (H^{s+1})^{d_2} \times \cdots \times (H^{s+k-1})^{d_k}. \tag{4.5}$$

The natural energy space is then $\mathcal{E} = \mathcal{H}^1 \times \mathcal{H}^0$ and we will prove (see Thm. 4.4) that the equation

$$\begin{cases} \partial_t^2 U - \Delta_g U + AU = BG, \\ (U(0), \partial_t U(0)) = (U_0, U_1). \end{cases} \tag{4.6}$$

is well posed in $\mathcal{E}$ with source term $G \in L^2(0,T;(L^2)^K)$.

Now, it appears that the important terms are the subdiagonal terms of A as (4.7).

$$A_{\mathrm{sub}}(x) = \begin{bmatrix} 0 & \dots & \dots & 0 \\ A_{21} & \dots & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & A_{k,k-1} & 0 \end{bmatrix} \tag{4.7}$$

Note that in the previous result, the high frequency problem and the unique continuation problem, the matrix involved is not the same. We have

$$A(x) = A_{\mathrm{sub}} + A_r = \begin{bmatrix} 0 & \dots & \dots & 0 \\ A_{21} & \dots & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & A_{k,k-1} & 0 \end{bmatrix} + \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1k} \\ 0 & A_{22} & \dots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & A_{kk} \end{bmatrix} \tag{4.8}$$

The structure "subdiagonal by block" of A allows to prove the following Lemma.

Lemma 4.2. *For any $s \in \mathbb{R}$, the multiplication by*

- *$A(x)$ sends $\mathcal{H}^s$ into $\mathcal{H}^{s-1}$*
- *$A_{\mathrm{sub}}(x)$ sends $\mathcal{H}^s$ into $\mathcal{H}^{s-1}$*
- *$A_r(x)$ sends $\mathcal{H}^s$ into $\mathcal{H}^s$.*

Lemma 4.3. *Let $(U_0, U_1) \in \mathcal{E}$ and $H \in L^1(0,T;\mathcal{H}^0)$. Then, there exists a unique solution $(U, \partial_t U) \in C([0,T],\mathcal{E})$ to*

$$\begin{cases} \partial_t^2 U - \Delta_g U = H, \\ (U(0), \partial_t U(0)) = (U_0, U_1). \end{cases} \tag{4.9}$$

is well-posed in $(U, \partial_t U) \in C([0,T],\mathcal{E})$ for $(U_0, U_1) \in \mathcal{E}$ and $H \in L^1(0,T;\mathcal{H}^0)$

Proof. Since the wave operator is diagonal, we can reduce the problem to each component where the Theorem reduces to the property that the equation

$$\begin{cases} \partial_t^2 V_i - \Delta_g V_i = H_i, \\ (V_i(0), \partial_t V_i(0)) = (U_{0,i}, U_{1,i}) \end{cases}$$

is well-posed in $C([0,T],(H^i)^{d_i}) \cap C^1([0,T];(H^{i-1})^{d_i})$ with $H_i \in L^1(0,T;(H^{i-1})^{d_i})$ and $(U_{0,i}, U_{1,i}) \in (H^i)^{d_i} \times (H^{i-1})^{d_i}$. □

Theorem 4.4. *Let $(U_0, U_1) \in \mathcal{E}$ and $G \in L^1([0,T];(L^2)^K)$. Then, there exists a unique solution $(U, \partial_t U) \in C([0,T],\mathcal{E})$ to the equation*

$$\begin{cases} \partial_t^2 U - \Delta_g U + AU = BG, \\ (U(0), \partial_t U(0)) = (U_0, U_1). \end{cases}$$

Proof. The proof is direct with Lemmas 4.2 and 4.3. The source term BG is in $L^1(0,T;\mathcal{H}^0)$ because of the specific structure of B in (1.19). □

4.3. Reduction of the control problem

In this section, we will reduce the control problem, which is now with a coupling of subdiagonal form as in Section 1.2, to a coupling of order 1. This will lead to a proof of Theorem 1.10

At this stage, we notice that the matrix A_r defined in (4.8) is compact for this scale of spaces. Now, it is natural to define the following operator

$$J = \begin{bmatrix} Id & 0 & \dots & 0 \\ 0 & \Lambda & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \Lambda^{k-1} \end{bmatrix} \tag{4.10}$$

J is a natural isometry from $\mathcal{H}^s$ to $(H^s)^N$. We will need also the matricial operator P_A defined by (roughly the action of P_A on each subspace is described by $(P_A)_{i,j} = \Lambda^{i-1} A_{i,j} \Lambda^{-(j-1)}$)

$$P_A = JAJ^{-1} = \begin{bmatrix} A_{11} & A_{12}\Lambda^{-1} & A_{13}\Lambda^{-2} & \dots & \dots & A_{1k}\Lambda^{-(k-1)} \\ \Lambda A_{21} & \Lambda A_{22}\Lambda^{-1} & \Lambda A_{23}\Lambda^{-2} & \dots & \dots & \Lambda A_{2k}\Lambda^{-(k-1)} \\ 0 & \Lambda^2 A_{32}\Lambda^{-1} & \Lambda^2 A_{33}\Lambda^{-2} & \dots & \dots & \Lambda^2 A_{3k}\Lambda^{-(k-1)} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \Lambda^{k-1}A_{k,k-1}\Lambda^{-(k-2)} & \Lambda^{k-1}A_{kk}\Lambda^{-(k-1)} \end{bmatrix}$$

Therefore, we have the immediate property.

Lemma 4.5. *P_A is a pseudodifferential operator of order 1 of principal symbol $\lambda(x,\xi)A_{\text{sub}}(x)$.*

Also, the following Lemma is immediate noting that $TB = B$.

Lemma 4.6. *Let $(U_0, U_1) \in \mathcal{E}$ and $G \in L^2(0,T;(L^2)^K)$. Then, the following statements are equivalent*

1. $(U, \partial_t U) \in C([0,T],\mathcal{E})$ *is solution to the equation*

$$\begin{cases} \partial_t^2 U - \Delta_g U + AU = BG, \\ (U(0), \partial_t U(0)) = (U_0, U_1). \end{cases}$$

2. $(V, \partial_t V) = J(U, \partial_t U) \in C([0,T],(H^1)^N \times (L^2)^N)$ *is solution to the equation*

$$\begin{cases} \partial_t^2 V - \Delta_g V + P_A V = BG, \\ (V(0), \partial_t V(0)) = J(U_0, U_1). \end{cases}$$

3. $(W, \partial_t W) = \Lambda J(U, \partial_t U) \in C([0,T],(L^2)^N \times (H^{-1})^N)$ *is solution to the equation*

$$\begin{cases} \partial_t^2 W - \Delta_g W + \Lambda P_A \Lambda^{-1} W = \Lambda BG, \\ (W(0), \partial_t W(0)) = \Lambda J(U_0, U_1). \end{cases} \tag{4.11}$$

Proposition 4.7 (HUM)**.** *The following statements are equivalent*

1. *The problem (4.11) is controllable in $(L^2)^N \times (H^{-1})^N$ with control $G \in L^2(0,T;(L^2)^K)$*
2. *We have the observability estimate*

$$C^1_{\text{obs}} \int_0^T \|B^* \Lambda W\|^2_{(L^2)^K} \mathrm{d}t \geq \mathbb{E}_0(W_0, W_1),$$

   for any solution to

$$\begin{cases} \partial_t^2 W - \Delta_g W + \Lambda^{-1} P_A^* \Lambda W = 0, \\ (W(0), \partial_t W(0)) = (W_0, W_1). \end{cases}$$

3. (1.22) *is controllable and for any $\lambda \in \mathbb{C}$, any solution $W \in (H^1)^N$ of*

$$\begin{cases} -\Delta_g W + \lambda^2 W + \Lambda^{-1} P_A^* \Lambda W = 0, \\ B^* \Lambda W = 0, \end{cases}$$

   is $V = 0$.
4. (1.22) *is controllable and for any $\lambda \in \mathbb{C}$, any solution $U \in (H^1)^N$ of*

$$\begin{cases} -\Delta_g U + \lambda^2 U + A^* U = 0, \\ B^* U = 0, \end{cases}$$

   is $V = 0$.

Proof. 1⇔2 is exactly the classical *HUM* method. We refer for instance to [22].

2⇔3 is exactly Theorem 1.5 once we have noticed that $\Lambda^{-1} P_A^* \Lambda$ is a pseudodifferential of order 1 with principal symbol $\lambda(x,\xi) A_{\text{sub}}(x)^*$ as noticed in Lemma 4.5, while $B^*\Lambda$ is of symbol $\lambda(x,\xi)B(x)^*$. Note also that P_A and $\lambda(x,\xi)B(x)^*$ are not differential operators, but Theorem 3.11 is still true and we can apply Proposition 3.9 to get the same result, using that $\lambda(x,\xi)$ is even in ξ.

3⇔4 is obtained undoing the change of variable done in Lemma 4.6 in the elliptic equation (modulo some duality). More precisely, W solves the equation $-\Delta_g W + \lambda^2 W + \Lambda^{-1} P_A^* \Lambda W = 0$ if and only if $U = J\Lambda W$ solves $-\Delta_g U + \lambda^2 U + A^*(x) U = 0$. $B^* \Lambda W = 0$ is equivalent to $B^* J^{-1} U = 0$ and then $B^* U = 0$ since $J^{-1} B = B$ and so $B^* = B^* J^{-1}$. □

Theorem 1.10 follows then as a combination of Lemma 4.6 and Proposition 4.7.

5. Examples

In this section, we provide two examples as applications of Theorem 1.3. We will treat the wave equations coupled by velocities of Cascade type, and the wave equations coupled by velocities with (almost) constant coefficients. The results are not always new, but the proof we provide has the advantage to always rely on easy ODE analysis which, we believe makes it valuable and give a common feature for this systems studied in different articles.

5.1. Wave equations coupled by velocities of cascade-type

We first consider the observability problem for wave system Coupled by Velocities of cascade-type:

$$\begin{cases} \partial_t^2 u - \Delta_g u + u + \beta(t,x)\partial_t v = 0, \\ \partial_t^2 v - \Delta_g v + v = 0, \end{cases} \tag{5.1}$$

where the coupling term $\beta \in C^\infty([0,T] \times \mathcal{M})$.

Based on Theorem 1.3, we can prove the following statement. The result is mostly contained in [25] which considers the same problem with zero order coupling or coupling $\beta(t,x)\Lambda v$ for which the analysis is almost the same. Yet, we believe that the proof we present here, which mostly relies on Theorem 1.3 and ODE analysis, is interesting because it gives some ODE interpretation of some computations that were performed in [25]. We refer for example to ([25], Thm. 5.3) where the matrix of the principal symbol of the *HUM* operator is computed and corresponds to the Gramian operator of the finite dimensional control problem that we compute in Lemma 5.3 below.

Proposition 5.1. *Assume that $\alpha \in C^\infty(\mathbb{R} \times \mathcal{M})$. Then weak observability inequality*

$$\begin{aligned} \int_0^T \int_{\mathcal{M}} \alpha^2 (|\nabla u|_g^2 + |u|^2) \mathrm{d}x\mathrm{d}t + c\|(u_0,u_1,v_0,v_1)\|^2_{H^{\frac12}\times H^{-\frac12}\times H^{\frac12}\times H^{-\frac12}} \\ \geq C\|(u_0,u_1,v_0,v_1)\|^2_{H^1\times L^2\times H^1\times L^2}, \end{aligned} \tag{5.2}$$

holds if and only if α, β satisfy the following property

$$\begin{aligned} &\forall \rho_0 \in S^*\mathcal{M}, \exists 0 < t_1 < t_2 < T, \text{ such that} \\ &\alpha(t_1, \varphi_{t_1}(\rho_0)) \neq 0, \alpha(t_2, \varphi_{t_2}(\rho_0)) \neq 0, \int_{t_1}^{t_2} \beta(\tau, \varphi_\tau(\rho_0)) \neq 0 \end{aligned} \tag{5.3}$$

Here φ_t is Hamiltonnian flow of $|\xi|_x$ defined in Theorem 1.3, c, C are two positive constants independent of the initial data.

Proof of Proposition 5.1. We apply Theorem 1.3 (actually a variant) with $D(u,v) = \alpha(t,x)\Lambda u$ with $\Lambda = (-\Delta_g + 1)^{1/2}$ and $L(u,v) = (\beta(t,x)v_t, 0)$, which states that the weak observability is equivalent to the controllability of the following finite dimensional system for any $\rho_0 \in S^*M$:

$$\begin{cases} \dot{X}(t) = \dfrac{-\beta(t,\varphi_t(\rho_0))}{2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} X(t) + \dfrac{\alpha(t,\varphi_t(\rho_0))}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} g(t), \\ X(0) = X_0 \in \mathbb{R}^N. \end{cases} \tag{5.4}$$

where $g \in L^2(0,T)$ is a scalar control function. The proposition follows then directly from Lemma 5.3 below. □

Under additional assumptions, we can obtain the strong observability, as in [25].

Proposition 5.2. *With the assumptions as Proposition 5.1, let us assume furthermore that α and β only depend on x and β satisfies sign condition, i.e., $\beta \geq 0$ (or $\beta \leq 0$), then the observability inequality*

$$\int_0^T \int_{\mathcal{M}} \alpha^2(|\nabla u|_g^2 + |u|^2)\mathrm{d}x\mathrm{d}t \geq C\|(u_0,u_1,v_0,v_1)\|^2_{H^1\times L^2\times H^1\times L^2}, \tag{5.5}$$

holds if and only if $T > T_{\omega\to o\to\omega}$, where $T_{\omega\to o\to\omega}$ (cf.[25]) is defined by

$$\begin{aligned} T_{\omega\to o\to\omega} = \inf\{T > 0 \ s.t.\ \forall\varphi_0(\rho_0) = \rho_0 \in S^*\mathcal{M}, \exists 0 < t_1 < t_2 < t_3 < T, \\ such\ that\ \alpha(\varphi_{t_1}(\rho_0)), \alpha(\varphi_{t_3}(\rho_0)) \neq 0, \beta(\varphi_{t_2}(\rho_0)) \neq 0\}. \end{aligned} \tag{5.6}$$

Proof of Proposition 5.2. We apply Lemma 5.3 (the case $\beta \geq 0$) to get the equivalence for weak observability, Following Theorem 1.5, it only suffices to prove System (5.1) satisfies unique continuation. Let $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, it is easy to see that A, B satisfy Kalman rank condition and A only has eigenvalue 0. By Proposition A.2 in the appendix, we conclude the proof of unique continuation of System (5.1) and therefore of the Proposition. Note that Proposition A.2 does not take into account the case $\lambda = 0$ in (1.9). Yet, this case is trivial because we have replaced the wave equation by the Klein-Gordon equation. Indeed, (u,v) is solution of $0 = -\Delta_g u + u = -\Delta_g v + v$ and is zero. □

Lemma 5.3. *We have the following equivalence for α, $\beta \in C([0,T])$:*

1. *The following control system is controlable.*

$$\begin{cases} \dot{X} = \beta(t)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} X + \alpha(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} g, \\ X(0) = X_0 \in \mathbb{R}^2, \end{cases} \tag{5.7}$$

where $g \in L^2(0,T)$ is a scalar control function.

2. *There is no $(c,d) \in \mathbb{C}^2 \setminus (0,0)$ so that $c\alpha(t) = d\alpha(t)\int_0^t \beta(\tau)d\tau$ for all $t \in [0,T]$.*
3. *There exists $0 < t_1 < t_2 < T$, such that*

$$\alpha(t_1) \neq 0, \alpha(t_2) \neq 0, \int_{t_1}^{t_2} \beta(\tau) \neq 0.$$

Moreover, if in addition, we have $\beta(t) \geq 0$ (or $\beta(t) \leq 0$), this is also equivalent to

$$\exists 0 < t_1 < t_2 < t_3 < T, \text{ such that } \alpha(t_1) \neq 0, \alpha(t_3) \neq 0, \beta(t_2) \neq 0.$$

Proof. 1⇔2 follows from classical control theory of finite dimensional system, we omit it.

Now, we prove 3⇒2. Assume t_1, t_2 such that $0 < t_1 < t_2 < T, \alpha(t_1), \alpha(t_2) \neq 0, \int_{t_1}^{t_2} \beta(\tau) \neq 0$. Take (c, d) so that $c\alpha(t) = d\alpha(t) \int_t^0 \beta(\tau)\mathrm{d}\tau$ for all $t \in [0, T]$, we shall prove $c = d = 0$. We have then, since $\alpha(t_1) \neq 0$ and $\alpha(t_2) \neq 0$

$$c = d \int_{t_1}^{0} \beta(\tau)\mathrm{d}\tau; \quad c = d \int_{t_2}^{0} \beta(\tau)\mathrm{d}\tau \tag{5.8}$$

and by difference $0 = d \int_{t_1}^{t_2} \beta(\tau)\mathrm{d}\tau$, so $d = 0$ since $\int_{t_1}^{t_2} \beta(\tau)\mathrm{d}\tau \neq 0$. This gives $c = 0$ after (5.8).

We finish with 2⇒3.

First, 2 implies that $\alpha \not\equiv 0$ (otherwise any $(c, d) \neq 0$ works)) and there exists t_1 so that $\alpha(t_1) \neq 0$. Define the function $f(t) = \alpha(t) \int_{t_1}^{t} \beta(\tau)\mathrm{d}\tau$. We prove $f \not\equiv 0$. Indeed, if it is the case, we have $0 = \alpha(t) \int_{t_1}^{t} \beta(\tau)\mathrm{d}\tau$ for all $t \in [0, T]$. In particular, $0 = \alpha(t) \left[\int_t^0 \beta(\tau)\mathrm{d}\tau - \int_{t_1}^0 \beta(\tau)\mathrm{d}\tau\right]$ for all $t \in [0, T]$, which is impossible by assumption. So, we have proved $f \not\equiv 0$ and there exists t_2 with $f(t_2) \neq 0$, and in particular, $\alpha(t_2) \neq 0$ and $\int_{t_1}^{t_2} \beta(\tau)\mathrm{d}\tau \neq 0$, which is the expected property 3, up to exchanging the role of t_1 and t_2. Note that we have only selected $0 \leq t_1 < t_2 \leq T$, but we can impose strict inequality with the same conclusion by continuity.

The last equivalence if $\beta \geq 0$ is obvious. □

5.2. Wave equations coupled by first or zero order terms of constant coefficients

In this Section, we explain how our result allows to recover and precise some results of Liard-Lissy [37] and Lissy-Zuazua [39] which were obtained with a complete different method. In particular, it allows to precise the regularity of the directions that can be reached.

Let $A \in \mathbb{R}^{N\times N}$ and $B \in \mathbb{R}^{K\times N}$ be constant matrices. In the notations of Section 1.2, we place ourselves in the particular cases: $A(x) = A$ constant and $B(x) = B\chi_\omega$

In particular, our results precise the result in the following sense. Theorem 4.2 of [37] proves controllability in $(H^{2N-1})^N \times (H^{2N-2})^N$ with control in L^2 under the Kalman rank condition, *i.e*:

$$rank(B, AB, ..., A^{N-1}B) = N. \tag{5.9}$$

Our results proves the same result in $\mathcal{H}^1 \times \mathcal{H}^0$ which is defined by (4.5).

Two situations can be considered, coupling of order 1 or 0, that we detail in separate subsection.

5.2.1. Constant coupling of order 1

We consider the following system of wave equations on a compact manifold $(\mathcal{M}, g)$:

$$\begin{cases} (\partial_t^2 - \Delta_g + 1)V + A\partial_t V = B\chi_\omega(x)u. \\ (V(0), \partial_t V(0)) = (V_0, V_1). \end{cases} \tag{5.10}$$

where $V \in \mathbb{R}^N$, $A \in \mathbb{R}^{N\times N}$ and $B \in \mathbb{R}^{N\times K}$, $u \in L^2(0, T; (L^2)^K)$. $\chi_\omega(x)$ denotes a smooth function which satisfies

$$\chi_\omega(x) := \begin{cases} 1, & \text{if } x \in \omega; \\ 0, & \text{if } x \in \mathcal{M}\backslash\tilde{\omega} \end{cases} \tag{5.11}$$

where $\omega \subset \tilde{\omega}$. Weak solution of (5.10) exists with initial data $(V_0, V_1) \in (L^2)^N \times (H^{-1})^N$.

Proposition 5.4. *Assume A, B satisfy Kalman rank condition and ω satisfies* GCC*. Then System* (5.10) *is exactly controllable with initial data $(V_0, V_1) \in (L^2)^N \times (H^{-1})^N$.*

Proof of Proposition 5.4. Firstly, we will apply Corollary 1.8 (actually a variant) with $D^*u = B\chi_\omega(x)u$ and $L^*V = A\partial_t V$, which states that co-dimensional controllability (the weak observability of dual system) is equivalent to the controllability of the following finite dimensional system for any $\rho \in S^*M$:

$$\begin{cases} \dot{X}(t) = \frac{1}{2}AX(t) + \frac{1}{2}B\chi_\omega(\phi_t(\rho_0))u, \\ X(0) = X_0 \in \mathbb{R}^N. \end{cases} \tag{5.12}$$

Since ω satisfies *GCC*, for every $\rho_0 \in S^*\mathcal{M}$, we can find an interval $[t_1, t_2] \in \mathbb{R}$, such that $\chi_\omega(\phi_t(\rho_0)) = 1, \forall t \in [t_1, t_2]$. Hence we obtain the exact controllability of (5.12) following from classical control theory of finite dimensional system. Next we only need to show the unique continuation property of the following elliptic equations:

$$\begin{cases} (\lambda^2 - \Delta_g + 1)v - A^{tr}\lambda v = 0; \\ B^{tr}\chi_\omega v = 0 \end{cases} \quad \Rightarrow v \equiv 0. \tag{5.13}$$

Since $\omega \cap \mathcal{M} = \omega \subset \mathcal{M}$ and (A, B) satisfy Kalman rank condition, by using Proposition A.2, we conclude our Proposition 5.4. □

5.2.2. Constant coupling of order 0

We consider the controllability of the system of wave equations coupled in order zero:

$$\begin{cases} (\partial_t^2 - \Delta_g)V + AV = B\chi_\omega(x)u. \\ (V(0), \partial_t V(0)) = (V_0, V_1). \end{cases} \tag{5.14}$$

where $V \in \mathbb{R}^N$, $A \in \mathbb{R}^{N\times N}$ can be written a matrix "subdiagonal by block" as (1.18) and $B \in \mathbb{R}^{N\times K}$ can be written as (1.19), $u \in L^2(0, T; (L^2)^K)$, χ_ω satisfies (5.11).

Proposition 5.5. *Assume that A, B satisfy the Kalman rank condition and ω satisfies* GCC*. Then, with the notations of Lemma 4.1, System (4.1) is controllable in the space $\mathcal{E} = \mathcal{H}^1 \times \mathcal{H}^0$ defined by $\mathcal{H}^s = (H^s)^{d_1} \times (H^{s+1})^{d_2} \times \cdots \times H^{s+k-1}(\mathcal{M})^{d_k}$ where k and $d_i \in \mathbb{N}$, $i = 1, \cdots, k$ are given by Proposition A.1.*

Proof. We want to apply Theorem 1.10. First of all, by Item 1 of Lemma 4.1, the matrix $\widetilde{A}_\omega$ satisfies the subdiagonal condition with respect to the splitting of the variables defined by the d_i. This gives also that System (4.1) is well posed following from Theorem 4.4. Then by using Theorem 1.10, we only need to show the unique continuation of eigenfunctions and the controllability of the following finite dimensional system:

$$\begin{cases} \dot{X}(t) = \dfrac{1}{2}\widetilde{A}_\omega(\varphi_t(\rho_0))X(t) + \dfrac{1}{2}\widetilde{B}\chi_\omega(\varphi_t(\rho_0))u(t), \\ X(0) = X_0 \in \mathbb{R}^N. \end{cases} \tag{5.15}$$

Item 2 of Lemma 4.1 ensures that for $x \in \omega$, $\widetilde{A}_\omega(x), \widetilde{B}\chi_\omega(x))$ satisfy Kalman rank condition. Since ω satisfies *GCC*, this means that for any $\rho_0 \in S^*\mathcal{M}$, there exists $t \in [0, T]$ so that $\pi_x\varphi_t(\rho_0) \in \omega$ and therefore $\widetilde{A}_\omega(\varphi_t(\rho_0)), \widetilde{B}\chi_\omega(\varphi_t(\rho_0)))$ satisfy Kalman rank condition. Hence the System (5.15) is controllable.

Concerning the unique continuation of eigenfunctions, we notice that if $\widetilde{U}$ is solution to

$$\begin{cases} -\Delta_g \widetilde{U} + \widetilde{A}_\omega^* \widetilde{U} = \lambda \widetilde{U}, \\ \chi_\omega(x)\widetilde{B}^*\widetilde{U} = 0. \end{cases}$$

then, in fact $-\Delta_g \widetilde{U} + M_x^* A^* (M_x^{-1})^* \widetilde{U} = \lambda \widetilde{U}$ and $U = (M_x^{-1})^* \widetilde{U}$ is solution to

$$\begin{cases} -\Delta_g U + A^* U = \lambda U, \\ \chi_\omega(x) B^* U = 0. \end{cases}$$

for which we can apply Proposition A.2. So we finish the proof of Proposition 5.5. □

A combination of Lemma 4.1 and Proposition 5.5 simply concludes Theorem 1.11.

Appendix A.

A.1 Brunovsky normal form

The following proposition is a reformulated and precise version of the Brunovsky normal form [18] for control of ODE. We provide a proof of it because we did not find it written in this way and we needed a slight modification with a matrix $\widetilde{A}_t$ which will be useful in the change of variable of Lemma 4.1. Yet, it is quite classical in control theory, and we don't claim novelty, see for instance [46].

Proposition A.1 (Brunovsky normal form)**.** *Assume $A \in \mathbb{R}^{N\times N}, B \in \mathbb{R}^{N\times K}$ satisfy the Kalman rank condition and denote $m = rank(B)$. Then, there exist some matrices $Q \in GL_N(\mathbb{R})$, $M_u \in GL_K(\mathbb{R})$ and $F \in \mathbb{R}^{K\times N}$, and some nonincreasing sequence of integers $d_i, i = 1, \cdots, k$ (for $k \leq n$) so that*

$$\widetilde{A} = Q^{-1}(AQ + BF); \quad \widetilde{B} = Q^{-1} B M_u$$

with

$$\widetilde{A} = \begin{bmatrix} 0 & \dots & \dots & 0 \\ A_{21} & \dots & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & A_{k,k-1} & 0 \end{bmatrix}; \quad \widetilde{B} = \begin{bmatrix} Id_m & 0_{m,K-m} \\ 0_{N-m,m} & 0_{N-m,K-m} \end{bmatrix}. \tag{A.1}$$

with $A_{i+1,i} = \begin{bmatrix} Id_{d_{i+1}} & 0_{d_{i+1},d_i-d_{i+1}} \\ 0_{d_i-d_{i+1},d_{i+1}} & 0_{d_i-d_{i+1},d_i-d_{i+1}} \end{bmatrix} \in \mathbb{R}^{d_{i+1}\times d_i}$ $(i = 1, \cdots, k)$, that is $A_{i+1,i}(k,l) = \delta_{k,l}$ (recall that $d_{i+1} \leq d_i$).

Moreover, $(\widetilde{A}, \widetilde{B})$ also satisfy the Kalman rank condition.

Also, for any $t \in \mathbb{R}$, we also have the following form for $\widetilde{A}_t = Q^{-1}(AQ + tBF)$,

$$\widetilde{A}_t = \begin{bmatrix} * & \dots & \dots & * \\ A_{21} & \dots & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & A_{k,k-1} & 0 \end{bmatrix}.$$

Proof. We prove the result by iteration on the dimension. The initialization is trivial, so we prove the iteration.

There exists Q_1, $M_{u,1}$ so that $Q_1^{-1} B M_{u,1} = \begin{bmatrix} Id_m & 0_{m,K-m} \\ 0_{N-m,m} & 0_{N-m,K-m} \end{bmatrix}$. We define

$$C = Q_1^{-1} A Q_1 = \begin{bmatrix} C_{1,1} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{bmatrix},$$

where $C_{1,1} \in \mathbb{R}^{m\times m}$, $C_{2,2} \in \mathbb{R}^{(N-m)\times(N-m)}$. Since A, B satisfies Kalman rank condition, it is easy to obtain that $C, Q_1^{-1} B M_{u,1}$ satisfies Kalman rank condition. By Hautus Lemma, we check that $(C_{2,2}, C_{2,1})$ satisfies the

Kalman rank condition. Indeed,

$$rank(\lambda - C, Q_1^{-1}BM_{u,1}) = N, \quad \forall \lambda \in \mathbb{C}, \tag{A.2}$$

which is equivalent to

$$rank(\begin{bmatrix} \lambda - C_{1,1} & -C_{1,2} & Id_m & 0_{m,K-m} \\ -C_{2,1} & \lambda - C_{2,2} & 0_{N-m,m} & 0_{N-m,K-m} \end{bmatrix}) = N, \quad \forall \lambda \in \mathbb{C}, \tag{A.3}$$

so that

$$rank(\begin{bmatrix} -C_{2,1} & \lambda - C_{2,2} \end{bmatrix}) = N - m, \quad \forall \lambda \in \mathbb{C}, \tag{A.4}$$

then we obtain $(C_{2,2}, C_{2,1})$ satisfies the Kalman rank condition. By iteration, there exists $G_x \in GL_{N-m}(\mathbb{R})$ and $G_u \in GL_m(\mathbb{R})$ and $F_2 \in \mathbb{R}^{m\times(N-m)}$ so that

$$\widetilde{A}_{N-m} = G_x^{-1}(C_{2,2}G_x + C_{2,1}F_2); \quad \widetilde{B}_{N-m} = G_x^{-1}C_{2,1}G_u.$$

has the expected form. We define

$$Q_2 = \begin{bmatrix} Id_m & F_2 \\ 0_{N-m,m} & G_x \end{bmatrix}; \quad Q_2^{-1} = \begin{bmatrix} Id_m & * \\ 0_{N-m,m} & G_x^{-1} \end{bmatrix},$$

$$Q_2^{-1}CQ_2 = \begin{bmatrix} * & * \\ G_x^{-1}C_{2,1} & G_x^{-1}C_{2,1}F_2 + G_x^{-1}C_{2,2}G_x \end{bmatrix} = \begin{bmatrix} * & * \\ \widetilde{B}_{N-m}G_u^{-1} & \widetilde{A}_{N-m} \end{bmatrix}.$$

Now, we define

$$Q_3 = \begin{bmatrix} G_u & 0 \\ 0_{N-m,m} & Id_{N-m} \end{bmatrix}; \quad Q_3^{-1} = \begin{bmatrix} G_u^{-1} & 0 \\ 0_{N-m,m} & Id_{N-m} \end{bmatrix}$$

so that for $Q = Q_1Q_2Q_3 \in GL_N(\mathbb{R})$, we have

$$Q^{-1}AQ = \begin{bmatrix} T_1 & T_2 \\ \widetilde{B}_{N-m} & \widetilde{A}_{N-m} \end{bmatrix}; \quad Q^{-1}BM_{u,1} = \begin{bmatrix} G_u^{-1} & 0 \\ 0 & 0 \end{bmatrix};$$

for some matrix $T_1 \in \mathbb{R}^{m\times m}$ and $T_2 \in \mathbb{R}^{m\times(N-m)}$. So, choosing finally

$$M_u = M_{u,1}\begin{bmatrix} G_u & 0 \\ 0 & Id_{K-m} \end{bmatrix}; \quad F = -M_u\begin{bmatrix} T_1 & T_2 \\ 0_{K-m,m} & 0_{K-m,N-m} \end{bmatrix}$$

we get

$$Q^{-1}(AQ + BF) = \begin{bmatrix} 0 & 0 \\ \widetilde{B}_{N-m} & \widetilde{A}_{N-m} \end{bmatrix}; \quad Q^{-1}BM_u = \begin{bmatrix} Id_m & 0_{m,K-m} \\ 0_{N-m,m} & 0_{N-m,K-m} \end{bmatrix}.$$

This gives the result given the form of $\widetilde{B}_{N-m}$ and $\widetilde{A}_{N-m}$ given by the iteration. The fact that $(\widetilde{A}, \widetilde{B})$ also satisfy the Kalman rank condition follows by direct analysis of the associated control problem for instance.

Finally $\widetilde{A}_t = Q^{-1}(AQ + tBF) = \begin{bmatrix}(1-t)T_1 & (1-t)T_2\\ \widetilde{B}_{N-m} & \widetilde{A}_{N-m}\end{bmatrix}$ has the required form. □

A.2 Eigenvalue problem

We will show the following proposition which will be repeatedly used in Section 5.

Proposition A.2. *Assume $A \in \mathbb{R}^{N\times N}$, $B \in \mathbb{R}^{K\times N}$, α, β are smooth functions and $\omega = \{\alpha \neq 0\}$, $o = \{\beta \neq 0\}$, respectively. Then, for all $\lambda_1 \in \mathbb{C}$, $\lambda_2 \in \mathbb{C}\setminus\{0\}$, or $\lambda_1 = 1, \lambda_2 = 0$, the eigenvalue problem*

$$\begin{cases}(\lambda_1 - \Delta_g)U + \lambda_2 A\beta U = 0,\\ \alpha BU = 0.\end{cases} \qquad \forall x \in \mathcal{M}, \tag{A.5}$$

admits an unique zero solution $U \equiv 0$, if A, B and α, β satisfy one of the following assumptions

1. *(A^{tr}, B^{tr}) satisfy Kalman rank condition and $\tilde{K} = 1$ (number of distinct eigenvalues of A is 1), β satisfies a sign condition, that is, $\beta \geq 0$ (or $\beta \leq 0$).*
2. *(A^{tr}, B^{tr}) satisfy Kalman rank condition and $\omega \cap o \neq \emptyset$, β satisfies a sign condition, that is, $\beta \geq 0$ (or $\beta \leq 0$).*

Before we prove Proposition A.2, we need to recall some basic facts of linear algebra and state notations related to Jordan decomposition. For any matrix $A \in \mathbb{R}^{N\times N}$, we denote by $\{\mu_i, i = 1, \cdots, \tilde{K}\}$ distinct eigenvalues of A. l_i denotes the geometric multiplicity (the dimension of $\mathrm{Ker}(A-\mu_i)$, that is the number of jordan blocks corresponding to μ_i) of μ_i for $i = 1, \cdots, \tilde{K}$. Let $P_{ij}^1 \in \mathbb{C}^N$ be eigenvector corresponding to μ_i for $i = 1, \cdots, \tilde{K}; j = 1, ..., l_i$. We define root vectors $P_{ij}^k \in \mathbb{C}^N$ associated to each eigenvector $\{P_{ij}^1\}$, which are given by

$$\begin{cases}(A-\mu_i)P_{ij}^{k+1} = P_{ij}^k; 1 \leq k \leq l_i^j - 1\\ (A-\mu_i)P_{ij}^1 = 0,\end{cases} \tag{A.6}$$

where l_i^j denote the dimension of Jordan chain of $\{P_{ij}^1\}$ for $i = 1, \cdots, \tilde{K}; j = 1, ..., l_i$. Then by classical theory of linear algebra, we can obtain

$$\{P_{ij}^k\}, i = 1, \cdots, \tilde{K}; j = 1, ..., l_i; k = 1, \cdots, l_i^j$$

span a base of $\mathbb{C}^N$. Define a matrix

$$P := \left[P_{11}^1 | P_{11}^2 | \cdots | P_{11}^{l_1^1} | P_{12}^1 | \cdots | P_{1l_1}^{l_1^{l_1}} | P_{21}^1 | \cdots | P_{\tilde{K}l_{\tilde{K}}}^{l_{\tilde{K}}^{l_{\tilde{K}}}}\right], \tag{A.7}$$

so we have Jordan Canonical Form $\tilde{A}$ of A:

$$\tilde{A} := P^{-1}AP = diag(A_1, A_2, \cdots, A_{\tilde{K}}) \tag{A.8}$$

where

$$A_i = diag(A_{i1}, \cdots, A_{il_i}), \tag{A.9}$$

and A_{ij} is $C^{l_i^j \times l_i^j}$ jordan block corresponding to μ_i for $i = 1, \cdots, \tilde{K}; j = 1, \cdots, l_i$.

Let $\tilde{B} := BP$, then we state the proof of proposition A.2.

Proof of Proposition A.2. Case "$\lambda_1 = 1, \lambda_2 = 0$" is simple, since $1 - \Delta$ is a positive operator, then $U = 0$. So we only need to prove case "$\lambda_2 \neq 0$". Let $W := P^{-1}U$. Since $\tilde{A}$ satisfies (A.8), System (A.5) can be decoupled of $\tilde{K}$ blocks, so that we only need to consider the solution $W_{ij} = (W_{ij}^1, \cdots, W_{ij}^{l_i^j}) \in (C^\infty(\mathcal{M}))^{l_i^j}$ of the following problem:

$$(\lambda_1 - \Delta_g)W_{ij} + \lambda_2 A_{ij}\beta W_{ij} = 0, \quad \forall x \in \mathcal{M}, \tag{A.10}$$

where A_{ij} is given by (A.9) for every $i = 1, \cdots, \tilde{K}; j = 1, \cdots, l_i$. More precisely, we rewrite System (A.10) as follow,

$$\begin{cases} (\lambda_1 - \Delta_g)W_{ij}^1 + \lambda_2\mu_i\beta W_{ij}^1 + \lambda_2\beta W_{ij}^2 = 0, \\ \vdots \\ (\lambda_1 - \Delta_g)W_{ij}^{l_i^j} + \lambda_2\mu_i\beta W_{ij}^{l_i^j} = 0. \end{cases} \tag{A.11}$$

Multiplying $W_{ij}^{l_i^j-1}$-equation by $\bar{W}_{ij}^{l_i^j}$ and by integration by parts over $\mathcal{M}$, since β satisfies sign condition, we have

$$W_{ij}^{l_i^j} = 0, \quad \forall x \in o.$$

Then by unique continuation of scalar elliptic equation, we obtain

$$W_{ij}^{l_i^j} = 0, \quad \forall x \in \mathcal{M}.$$

Hence, repeating this process to each equation of $\{W_{ij}^k\}$, for $k = 2, \cdots, l_i^j$, we obtain

$$W_{ij}^k = 0, \quad \forall k = 2, \cdots, l_i^j, x \in \mathcal{M}.$$

It suffices to show that $W_{ij}^1 = 0, i = 1, \cdots, \tilde{K}, j = 1, \cdots, l_i$ under assumptions 1 or 2. Indeed, W_{ij}^1 satisfies the following equation

$$(\lambda_1 - \Delta_g)W_{ij}^1 + \lambda_2\mu_i\beta W_{ij}^1 = 0, \quad i = 1, \cdots, \tilde{K}, j = 1, \cdots, l_i. \tag{A.12}$$

Since $\tilde{B}$ can be rewritten as

$$\left[BP_{11}^1 | \cdots | BP_{\tilde{K}l_{\tilde{K}}}^{l_{\tilde{K}}^{l_{\tilde{K}}}} \right], \tag{A.13}$$

then

$$\tilde{B}W\alpha = \alpha \sum_{i,j} BP_{ij}^1 W_{ij}^1 = \alpha \sum_{i=1}^{\tilde{K}} (\sum_{j=1}^{l_i} BP_{ij}^1 W_{ij}^1) = 0. \tag{A.14}$$

If we have assumption 1, that is, $\tilde{K} = 1$ and A, B satisfy Kalman rank condition. Then we obtain that for $j = 1, \cdots, l_1, x \in \omega$, $W_{1j}^1 = 0$ following from ([9], Prop. 3.1). By unique continuation of scalar elliptic equation (A.12), we have $W_{1j}^1 = 0, x \in \mathcal{M}, \forall j = 1, \cdots, l_1$.

Next, if we have Assumption 2, that is, A, B satisfy Kalman rank condition and $\omega \cap o \neq \emptyset$, then set $\tilde{\omega} \subset \omega \cap o$, in view of (A.14), we have

$$(\lambda_1 - \Delta_g)\sum_i(\sum_j BP_{ij}^1 W_{i1}^1) = 0, \quad \forall x \in \tilde{\omega}. \tag{A.15}$$

By using (A.12), we have

$$\beta \sum_i \lambda_2 \mu_i (\sum_j BP_{ij}^1 W_{ij}^1) = 0, \quad \forall x \in \tilde{\omega}. \tag{A.16}$$

By induction, we obtain

$$\beta \sum_{i=1}^{\tilde{K}} (\lambda_2 \mu_i)^k (\sum_{j=1}^{l_i} BP_{ij}^1 W_{ij}^1) = 0, \quad \forall x \in \tilde{\omega}, k = 1, \cdots, \tilde{K}. \tag{A.17}$$

Since $\{\mu_i\}_{1,\cdots,\tilde{K}}$ are different, we have

$$\sum_{j=1}^{l_i} BP_{ij}^1 W_{ij}^1 = 0, \quad \forall x \in \tilde{\omega}, i = 1, \cdots, \tilde{K}. \tag{A.18}$$

By ([9], Prop. 3.1), we obtain

$$W_{ij}^1 = 0, \quad \forall x \in \tilde{\omega}, i = 1, \cdots, \tilde{K}, j = 1, \cdots, l_i. \tag{A.19}$$

By (A.12) and unique continuation of scalar elliptic equation, we have

$$W_{ij}^1 = 0, \quad \forall x \in \mathcal{M}, i = 1, \cdots, \tilde{K}, j = 1, \cdots, l_i. \tag{A.20}$$

So we finish the proof. □

A.3 Proof of Lemma 3.5

In the main part of the paper, we use a matrix operator type version of 1-smooth effect Lemma 3.5. A version of such a result in scalar case can be found in [32]. The proof of Lemma 3.5 relies on the following lemma.

Lemma A.3. *Let $\mathcal{I} \subset \mathbb{R}$ be an interval and let $H_\pm(t) = \pm\Lambda Id_{N\times N} + iW_0(t)$ where $W_0 \in C^\infty(\mathcal{I}; \Psi_{phg}^0(\mathcal{M}; \mathbb{C}^{N\times N}))$. Define $S_\pm(t,0)$ as the solution operator for the evolution equation $\partial_t - iH_\pm(t)$ respectively. Then, for any $A \in \Psi_{phg}^m(\mathcal{M}; \mathbb{C}^{N\times N})$, we have*

$$[A, S_\pm(t,0)] = \int_0^t S_\pm(t,s)[A, iH_\pm(s)]S_\pm(s,0)\mathrm{d}s. \tag{A.21}$$

In particular, if we take $A = \Lambda Id_{N\times N}$, then, for all $s \in \mathbb{R}$, we have

$$[\Lambda, S_\pm(t,0)], [\Lambda, S_\pm(t,0)^*] \in \mathcal{B}_{loc}(\mathcal{I}; \mathcal{L}(H^s(\mathcal{M}; \mathbb{C}^{N\times N}))). \tag{A.22}$$

Proof of Lemma A.3. Let

$$u_{\pm}(t) = [A, S_{\pm}(t,0)]u_0 = AS_{\pm}(t,0)u_0 - S_{\pm}(t,0)Au_0, \quad u_{\pm}(0) = 0. \tag{A.23}$$

solves

$$\partial_t u_{\pm}(t) = AiH_{\pm}(t)S_{\pm}(t,0)u_0 - iH_{\pm}(t)S_{\pm}(t,0)Au_0 = [A, iH_{\pm}(t)]S_{\pm}(t,0)u_0 + iH_{\pm}(t)u_{\pm}(t).$$

so that the Duhamel principal yields (A.21). We finish the proof of Lemma A.3. □

Proof of Lemma 3.5. We refer for instance to ([32], Sect. A.3) for some details in the scalar case, the proof being almost the same. So we omit it. □

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