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CHANCE CONSTRAINED OPTIMIZATION OF ELLIPTIC PDE SYSTEMS WITH A SMOOTHING CONVEX APPROXIMATION*

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Abstract. In this paper, we consider chance constrained optimization of elliptic partial differential equation (CCPDE) systems with random parameters and constrained state variables. We demonstrate that, under standard assumptions, CCPDE is a convex optimization problem. Since chance constrained optimization problems are generally nonsmooth and difficult to solve directly, we propose a smoothing inner-outer approximation method to generate a sequence of smooth approximate problems for the CCPDE. Thus, the optimal solution of the convex CCPDE is approximable through optimal solutions of the inner-outer approximation problems. A numerical example demonstrates the viability of the proposed approach.

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1. Introduction

Many engineering processes can be described by partial differential equations (PDEs). Examples of such processes are found in physics, biology, chemistry, fluid dynamics, thermodynamics, finance, and many other disciplines (see, for instance, [28, 44, 47, 57]). Frequently, a PDE model contains parameters, for example, describing heat capacity, diffusion, viscosity, hydraulic conductivity, pressure, permeability, etc. These parameters are usually difficult to precisely determine, *i.e.*, they vary randomly as well as spatially and, therefore, can be considered as *random fields*. In addition, such processes may be influenced from uncertain environment, *e.g.*, ambient temperature and pressure, which are modeled as random forcing terms of the PDE system. All these uncertainties may have non-negligible effects on the process. Therefore, stochastic optimization methods are needed to gain optimal as well as reliable solutions for processes governed by PDEs under uncertainty.

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When a PDE model involves random parameters and/or random forcing terms, the states of the system become random. There have been many studies on statistical simulation and uncertainty quantification of PDE systems under uncertainty (*e.g.*, [6, 7, 8, 24, 42, 45, 64]).

Recently, optimization of PDE systems with random parameters has gained increasing attention. The authors in [14, 15] studied optimization of parabolic PDEs in the context of reaction-diffusion processes with a tracking-type objective function, where diffusion and reaction coefficients were considered as random fields (see also [13, 16] for a steady-state (*i.e.*, elliptic) PDE consideration). In [59] a parabolic PDE system concerning stochastic inverse heat conduction problems was investigated, where the thermal conductivity and heat capacity coefficients were assumed to be random fields. In [48] the authors studied optimization of thermal processes described as parabolic PDEs by considering uncertainties in the thermal conductivity, convective heat transfer coefficients, initial data of the process as well as control variables. The work in [4, 49] investigated optimal shape design problems with elliptic PDE models, where uncertainties arise from material, geometrical and loading properties. In [2, 40, 41] the authors considered mean-variance risk-avers objective functions for the optimization of elliptic PDE systems with random data. Various numerical methods were also investigated for the solution of elliptic PDE optimization problems with random coefficients, *e.g.*, sparse grid stochastic collocation ([38, 39, 43, 58]), reduced basis ([19]), finite element approximation [32, 60], etc.

Despite the fact that the state variables of most practical processes are constrained, almost all of the above studies have not considered state constraints with the exception of [23] and [52]. In [23] the authors studied continuity and convexity of chance (probabilistic) constraints in infinite dimensions (see also [62] for differentiability analysis). The work [52] studies a feedback control design for a state-constrained parabolic PDE system with distributed, but bounded uncertainties. Instead of probabilistic state-constraints, feedback control was designed using a min-max (robust or worst-case) approach, *i.e.*, the state constraints are required to be satisfied for almost all realization of the uncertain variables. Nevertheless, detailed theoretical investigations as well as numerical solution methods for state-constrained optimization of PDE systems with random parameters still remain open.

In this work, optimization of elliptic PDE systems with random coefficients and random forcing terms is considered. In particular, inequality constraints on state variables of such systems are formulated as chance (probabilistic) constraints, leading to a chance constrained optimization (CCOPT) problem of elliptic PDEs (CCPDE). Considering a tracking-type objective function, we demonstrate that this kind of optimization problems is convex under standard assumptions on the uncertainties.

Since chance constrained optimization problems are in general nonsmooth and intractable, we propose smoothing inner-outer approximations for the CCPDE, where the outer-approximation problem is shown to be convex. The feasible sets of the approximating problems converge asymptotically to the feasible set of the CCPDE.

Moreover, the cluster points of the solutions of the approximating problems are also shown to be solutions of the CCPDE. Furthermore, for a convex CCPDE problem, the solutions of the smoothing outer-approximation problems converge to the solution of the CCPDE with respect to the approximation parameter. As a result, the smooth parametric PDE optimization problems are solved to obtain approximate optimal solutions to the CCPDE. The viability of the proposed approach is demonstrated through a numerical example. This work extends our recent research findings in [26, 27] with respect to finite dimensional chance constrained optimization problems to infinite dimensional CCPDE problems. We point out to our readers that recent numerical considerations in [21, 31] w.r.t. finite-dimensional CCOPT are also potential extendable for CCPDE.

The rest of the manuscript is organized as follows. Section 2 states the CCPDE problem to be investigated. Section 3 briefly summarizes the conditions for unique solvability of elliptic PDE systems with uncertainties and investigates the convexity of the chance constraints as well as that of the objective function of CCPDE. Section 4 discusses convexity and differentiability properties of the parametric function for smoothing inner-outer approximation of CCPDE. Specifically, Section 5 verifies that the optimal solution of the convex CCPDE is approximable through optimal solutions of the inner-outer approximation problems. A numerical example in Section 6 demonstrates the viability of the proposed approach. The paper concludes with Section 7 by providing conclusive remarks.

2. Problem definition

Let $(\Omega, \mathcal{S}, Pr)$ be a complete probability space with the sample space Ω, probability measure $Pr : \Omega \to [0, 1]$ with $Pr(\Omega) = 1$, and the set of events $\mathcal{S}$ representing a σ-algebra on Ω. In the following, an expression which is valid with probability 1 or almost everywhere (w.r.t. the underlying probability measure) is said to be valid almost sure, briefly, a.s.

We consider the optimization problem

$$
\begin{aligned}
(CCPDE) \qquad & \min_u \left\{ E\left[\|y - y_d\|^2_{H^1_0(D)}\right] + \frac{\rho}{2}\|u\|^2_{L^2(D)} \right\} && (2.1)\\
& \text{s.t.} - \nabla \cdot (\kappa(x, \xi)\nabla y(x)) = f(u, x, \xi), \text{ for } x \in D,\ \xi \in \Omega \text{ a.s.}, && (2.2)\\
& y(x)|_{x\in\partial D} = 0, && (2.3)\\
& Pr\{y(x) \le y_c(x)\} \ge \alpha, x \in D_c, && (2.4)\\
& u \in U, && (2.5)
\end{aligned}
$$

where $D \subset \mathbb{R}^n$ is a given bounded convex open spatial domain with boundary ∂D. The dimension $n \ge 2$ is fixed, $\rho > 0$ is a given regularization constant, D_c is a given compact subset of $\overline{D}$,[1] ∇ represents the gradient operator w.r.t. $x \in \mathbb{R}^n$ in the weak sense of Sobolev spaces. The functions y_c, y_d are given continuous functions in $H^1_0(D) = \left\{g \in W^{1,2}(D) \mid g(x) = 0, x \in \partial D\right\}$ with $H^1_0(D)$ being a closed subspace of the Sobolev space $H^1(D) = W^{1,2}(D)$.[2] With $< h, g >_{H^1_0(D)}$ and $< h, g >_{H^1(D)}$ we denote the related standard scalar products (see *e.g.* [1, 17] for more details on Sobolev spaces).

The right hand-side of the PDE system (2.2) is assumed to be a functional over some L^2 space, for any fixed u and ξ. Hence, the analysis of the existence of a solution to the PDE, for a fixed u and ξ, requires the $H^2(D)$[3] space with the related norm $\|\cdot\|_{H^2(D)}$.

In addition, the following separable and reflexive Bochner space $\mathcal{W} := L(\Omega; W(D))$ indicates the fact that the mapping from the probability space belongs to some separable and reflexive Sobolev space $W(D)$. We define

$$
L(\Omega; W(D)) := \left\{ v : \Omega \to W(D) : v \text{ is measurable, } \|v\|^2_{\mathcal{W}} := \int_\Omega \|v(\cdot, \xi)\|^2_{W(D)}\phi(\xi)\mathrm{d}\xi < +\infty \right\} \qquad (2.6)
$$

and use the separable and reflexive Bochner spaces

$$
\mathcal{L} := L\left(\Omega; L^2(D)\right), \quad \mathcal{H} := L\left(\Omega; H^1_0(D)\right), \quad \mathcal{G} := L(\Omega, H^2(D)). \qquad (2.7)
$$

Measurability here refers to strong measurability which is equivalent to weak measurability for separable spaces (see *e.g.* [30], Sect. 3.5 Cor. 2). Furthermore, the following scalar products and norms are used

$$
\begin{aligned}
\langle a, b\rangle_{\mathcal{L}} &= \int_\Omega \int_D a(x, \xi)b(x, \xi)\mathrm{d}x\phi(\xi)\mathrm{d}\xi, \quad \|a\|^2_{\mathcal{L}} =< a, a >_{\mathcal{L}}\ , && (2.8)\\
\langle a, b\rangle_{\mathcal{H}} &= \int_\Omega \int_D (\nabla a(x, \xi))^T \nabla b(x, \xi)\mathrm{d}x\phi(\xi)\mathrm{d}\xi, \quad \|a\|^2_{\mathcal{H}} =< a, a >_{\mathcal{H}}, && (2.9)
\end{aligned}
$$

respectively, for $a, b \in \mathcal{L}$ and $a, b \in \mathcal{H}$.

[1]The closure of D is denoted by $\overline{D}$.

[2]The space $H^1(D)$ is a Hilbert space with norm $\|\cdot\|_{H^1(D)}$. It is the completion of $C^1(\overline{D})$ w.r.t. the norm $\|\cdot\|_{H^1(D)}$.

[3]Note that H^1_0 is necessary for existence, but because of $f \in L^2$ the solution lies also in H^2.

That is, if $v \in \mathcal{H}$, then $v(\cdot,\xi) \in H_0^1(D)$ and $E\left[\|v(\cdot,\xi)\|^2_{H_0^1(D)}\right] < +\infty$, where the expected value operation $E[\cdot]$ is taken with respect to the probability space $(\Omega, \mathcal{S}, Pr)$ and the probability measure possesses the Radon-Nikodym derivative ϕ w.r.t. the Lebesgue measure μ, *i.e.*, $\mathrm{d}Pr(\xi) = \phi(\xi)\mathrm{d}\mu(\xi)$. Furthermore, we suppress the measure μ and write simply $\mathrm{d}Pr(\xi) = \phi(\xi)\mathrm{d}\xi$.

The probability density ϕ is assumed to be Lebesgue measurable and almost everywhere positive on Ω. Hence, the spaces $\mathcal{L}$, $\mathcal{G}$, and $\mathcal{H}$ are Hilbert spaces using the standard equivalence classes. Note also that $\mathcal{G}, \mathcal{H}, \mathcal{K}$ are dense subspaces of $\mathcal{L}$ in the topology of $\mathcal{L}$. The *variable* $u \in L^2(D)$ is a decision variable that belongs to the set of admissible decisions

$$U := \left\{u \in L^2(D) \mid u_a \leq u \leq u_b\right\},$$

where $u_a, u_b \in L^2(D)$ are given functions with $u_a \leq u_b$. Observe that equalities and inequalities of functions in the Lebesgue space $L^2(D)$ and corresponding Sobolev spaces are valid only almost everywhere on D w.r.t. the underlying measure. The term "almost every where (a.e.)" will be suppressed in the paper assuming no confusions arise. Note that U is a nonempty, convex, closed and bounded subset of $L^2(D)$.

In the elliptic PDE system (2.2)–(2.3), the random parameters in the coefficient $\kappa \in \mathcal{K}$[4] represent the effect of imprecise model parameters, while those in the *forcing term* f are random external disturbances. Here, for the sake of simplicity of presentation, the random parameters in κ and f are lumped together into a single random vector ξ. Furthermore, the forcing term is continuous w.r.t. u for fixed x and ξ a.s. and $f(u,\cdot,\cdot) \in \mathcal{L}$ for fixed u. The *state variable* y of the PDE system depends on u, the random event ξ and the space variable x. As a result, a state constraint $y \leq y_c$, over the spatial region D_c, cannot be satisfied deterministically. Hence, the expression in (2.4) defines a chance (probabilistic) constraint by stipulating the satisfaction of the inequality constraint on y with a pre-given probability (reliability) level $\alpha \in [0,1]$. Moreover, (2.4) represents a point-wise chance constraint, *i.e.*, the constraint on the state variable is required to hold with the same reliability level α at each spatial location $x \in D_c$. Such infinite number of constraints makes sense whenever in the equivalence class y is, w.r.t. x, a continuous element which is ensured by Sobolev embedding theorems in $H^2(D) \cap H_0^1(D)$, nice properties of the inhomogeneity term f and the convexity of D. Based on Sobolev's embedding theory, one can use a more general setting in $W^{1,p}$ with $p > d$ and sufficiently regular spatial domain D. Thus the convexity of D can be relaxed. We give here one instance, where it works. It is essential for our approximation approach (see, *e.g.*, Lem. 3.3), that the space for y can be continuously embedded in the space of continuous functions. Then, such a CCPDE represents a *semi-infinite* separate (single) chance constrained optimization problem.

An alternative approach (see [23]) could be

$$Pr\left\{\sup_{x\in D_c}\ [y(u,x,\xi) - y_c(x)] \leq 0\ \right\} \geq \alpha$$

which specifies the *uniform* (joint) satisfaction of the state constraint with a single reliability level α for all $x \in D_c$ and a given function y_c. In comparison, the point-wise constraint (2.4) provides a more flexible specification of reliability levels α depending on the spatial location $x \in D_c$. This consideration facilitates to extend and apply our inner-outer approximation approach [26].

For instance, at some critical spatial locations, α can be chosen near 1. Therefore, in this study, we focus on the solution of CCPDE with point-wise constraints, but considering α independent of x for simplicity of representation. Further it is not trivial to directly extend our inner-outer approximation concept (in [26]) to joint (uniform) chance constraints for infinite number of $x \in D_c$. For finite dimensional joint chance constrained optimization problems, we refer to our recent work [25].

[4]The coefficients of the elliptic operator need to be C^1 smooth for ensuring $x \mapsto y(u,x,\xi) \in H^2(D)$ (see: [29])

In this paper, we consider the weak form of the PDE system. Thus, considering the homogeneous Dirichlet boundary-value elliptic PDE system with random parameters (2.2)–(2.3) for a fixed $u \in L^2(D)$

$$-\nabla \cdot (\kappa(x,\xi)\nabla y(x,\xi) = f(u,x,\xi), \text{ on } D \times \Omega \text{ a.s.}, \tag{2.10}$$
$$y(x,\xi)|_{x\in\partial D} = 0, \quad \xi \in \Omega \text{ a.s.} \tag{2.11}$$

From equation (2.10) it follows by Green's theorem and (2.11) for each $v \in \mathcal{H}$ that

$$E\left[\int_D \kappa(x,\cdot)\nabla y(x,\cdot)\nabla v(x,\cdot)\mathrm{d}x\right] = E\left[\int_D f(u,x,\cdot)v(x,\cdot)\mathrm{d}x\right], \quad v \in \mathcal{H} \tag{2.12}$$

which is the weak form of the PDE system (2.10)–(2.11) with $f(u,\cdot,\cdot)$ in the Hilbert space $\mathcal{L}$ and solution $y(u,\cdot,\cdot) \in \mathcal{H}$.

Since $f(u,\cdot,\cdot) \in \mathcal{L}$, $\kappa \in \mathcal{K}$ and D is convex, the well-known *shift statements* (see, for instance [29], Thm. 3.30; *i.e.*, higher regularity of f is shifted to higher regularity of y) imply that the equivalence class of y contains some element $x \mapsto y(u,x,\xi) \in C(\overline{D})$. Since the continuity of $x \mapsto y(u,x,\xi)$ is required only on the subset D_c, the convexity of D is not necessary whenever $D_c \subset intD$. However, to guarantee the well-posedness of the weak form, our investigation is based-on the following standard assumptions.

Assumption 2.1.

A1 The domain D is convex, the set $D_c \subset \overline{D}$ is compact and $y_c \in C(D) \cap H_0^1(D), y_d \in H_0^1(D) \cap H^2(D)$. The random field $\kappa(\cdot,\cdot) \in \mathcal{K}$ is positive and bounded such that

$$0 < \kappa_{min} \leq \kappa(x,\xi) \leq \kappa_{max}, (x,\xi) \in D \times \Omega \ \text{ a.s.}, \tag{2.13}$$

where $\kappa_{min}, \kappa_{max}$ are finite constants.

A2 For each $u \in L^2(D)$, the random forcing term $u \mapsto f(u,\cdot,\cdot) : L^2(D) \to \mathcal{L}$ is continuous.

A2D For each $u \in L^2(D)$, the random forcing term $u \mapsto f(u,\cdot,\cdot) : L^2(D) \to \mathcal{L}$ is continuously Fréchet differentiable.

A3 The forcing term has the form $f(u,x,\xi) = u(x) + f_0(x,\xi)$, where $u \in L^2(D)$ and $f_0 \in \mathcal{L}$.

A4 The random variables $\xi^\top = (\xi_1,\dots,\xi_p)$ are independently, identically distributed with a continuous joint multivariate probability density function $\phi(\xi) = \Pi_{i=1}^p \phi_i(\xi_i)$ and the set $\Omega = \Pi_{i=1}^p \Omega_i$, where $\Omega_i \subset \mathbb{R}, i = 1,\dots,p$, such that

$$f(u,x,\xi) = u(x) + a_0(x) + \sum_{k=1}^p a_k(x)\xi_k \tag{2.14}$$

with $u, a_k \in L^2(D)$, $k = 0,1,2,\dots$

The following example demonstrates the relevance of A2D.

Example 2.2. The function f defined as

$$f(u,x,\xi) = \int_D u(t)^2 k(t,x,\xi)\mathrm{d}t,$$

is nonlinear in u, Frechét differentiable w.r.t. u and $f(u,\cdot,\cdot) \in \mathcal{L}$, where $\text{esssup}_{t\in D} |k(t,x,\xi)| \leq c(x,\xi)$ and $c \in \mathcal{L}$, since

$$\int_\Omega \int_D |f(u,x,\xi)|^2 \mathrm{d}x\phi(\xi)\mathrm{d}\xi = \int_\Omega \int_D \left|\int_D u(t)^2 k(t,x,\xi)\mathrm{d}t\right|^2 \mathrm{d}x\phi(\xi)\mathrm{d}\xi$$

$$\leq \|u\|^4_{L^2(D)} \int_\Omega \int_D |c(x,\xi)|^2 \, \mathrm{d}x\phi(\xi)\mathrm{d}\xi$$
$$= \|u\|^4_{L^2(D)} \|c\|^2_{\mathcal{L}} < \infty.$$

Consequently, $f(u,\cdot,\cdot) \in \mathcal{L}$. The Frechét derivative will be

$$h \to \int_D 2u(t)h(t)k(t,x,\xi)\mathrm{d}t.$$

Indeed (the linearity is clear)

$$f(u+h,x,\xi) - f(u,x,\xi) = \int_D 2u(t)h(t)k(t,x,\xi)\mathrm{d}t + \int_D h(t)^2 k(t,x,\xi)\mathrm{d}t,$$
$$\|\int_D 2u(t)h(t)k(t,x,\xi)\mathrm{d}t\|_{\mathcal{L}} \leq 2\|u\|_{L^2(D)}\|h\|_{L^2(D)}\|c\|_{\mathcal{L}},$$
$$\|\int_D h(t)^2 k(t,x,\xi)\mathrm{d}t\|_{\mathcal{L}} \;/\; \|h\|_{L^2(D)} = \left[\int_\Omega \int_D \left|\int_D h(t)^2 k(t,x,\xi)\mathrm{d}t\right|^2 \mathrm{d}x\phi(\xi)\mathrm{d}\xi\right]^{\frac{1}{2}} \;/\; \|h\|_{L^2(D)}$$
$$\leq \|h\|_{L^2(D)}\|c\|_{\mathcal{L}} \to 0 \text{ for } \|h\|_{L^2(D)} \to 0.$$

Obviously, A4 sharpens A3 and A3 sharpens A2. The assumption A1 on κ implies that

$$Pr\{\xi \in \Omega \;|\kappa_{min} \leq \kappa(x,\xi) \leq \kappa_{max}, x \in D \text{ a.e.}\} = 1.$$

In addition, according to A3, the forcing term considered in this work involves a linear control with an additive random disturbance. Assumption A4 is commonly called *finite dimensional noise* representation (2.14) (see *e.g.* [10, 54]). In fact, for numerical computations, it is mandatory to reduce the dimension of the uncertainties in κ and f.

3. Properties of CCPDE

3.1. Properties of the PDE system

Referring to equation (2.12) let

$$\mathcal{B}(\kappa)(y,v) := E\left[\int_D \kappa(x,\cdot)\nabla y(x,\cdot)\nabla v(x,\cdot)\mathrm{d}x\right]. \tag{3.1}$$

Then, assumptions A1 and the Cauchy-Schwartz inequality yield that

$$\begin{aligned} |\mathcal{B}(\kappa)(y,v)|^2 &\leq E\left[\int_D |\kappa(x,\cdot)|^2|\nabla y(x,\cdot)|^2\mathrm{d}x\right] E\left[\int_D |\nabla v(x,\cdot)|^2\mathrm{d}x\right] \\ &\leq \|\kappa\|^2_\infty \|y\|^2_{\mathcal{H}}\|v\|^2_{\mathcal{H}} \leq \kappa^2_{max}\|y\|^2_{\mathcal{H}}\|v\|^2_{\mathcal{H}}; \end{aligned} \tag{3.2}$$

i.e., $\mathcal{B}(\kappa) : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is a continuous bilinear operator. Moreover, since

$$\mathcal{B}(\kappa)(v,v) \geq \kappa_{min}\|v\|^2_{\mathcal{H}}, \tag{3.3}$$

it is also coercive. By the Lax-Milgram theorem and assumption A1, there is a uniquely determined linear continuous automorphism $\mathcal{A}(\kappa) : \mathcal{H} \to \mathcal{H}$ (briefly: $\mathcal{A}(\kappa) \in Auto(\mathcal{H})$) such that

$$\mathcal{B}(\kappa)(a, b) = \langle \mathcal{A}(\kappa)a, b\rangle_{\mathcal{H}}, \tag{3.4}$$

for each κ and $a, b \in \mathcal{H}$. Furthermore, we identify the dual $\mathcal{L}^*$ with $\mathcal{L}$ and set

$$F_a(b) = \langle a, b\rangle_{\mathcal{L}},$$

for $F_a \in \mathcal{L}^*$. For a linear and continuous functional $G_a \in \mathcal{H}^*$, the uniquely determined canonical isomorphism $j_{\mathcal{H}} : \mathcal{H}^* \to \mathcal{H}$ will be used with the operator norm $\|j_{\mathcal{H}}\| = 1$. Thus, the Riesz representation theorem yields that $G_a(b) = \langle j_{\mathcal{H}}(a), b\rangle_{\mathcal{H}}$ and $\|G_a\|_{\mathcal{H}^*} = \|a\|_{\mathcal{H}}$, for all $G_a \in \mathcal{H}^*$ and $b \in \mathcal{H}$.

Theorem 3.1. *Let $u \in L^2(D)$ be given and assumptions A1 and A2 hold true. Then, the system (2.12) has a unique solution $y(u, \cdot, \cdot) \in \mathcal{H} \cap \mathcal{G}$ with the properties*

1.

$$y(u, x, \xi) = [\mathcal{A}(\kappa)^{-1}(j_{\mathcal{H}}(f(u, \cdot, \cdot)))](x, \xi), \quad \text{for all } x \in D \text{ and } \xi \in \Omega \text{ a.s.} \tag{3.5}$$

2.

$$\|y(u, \cdot, \xi)\|_{H_0^1(D)} \le C\left(\frac{1}{\kappa_{min}}\right)\|f(u, \cdot, \xi)\|_{L^2(D)}, \ \text{ a.s. on } \Omega, \tag{3.6}$$

where C is a constant which is independent of ξ.

3. *For fixed u, there is some constant $c > 0$ such that*

$$\|y(u, \cdot, \cdot)\|_{\mathcal{G}} \le c\|f(u, \cdot, \cdot)\|_{\mathcal{L}} \tag{3.7}$$

which implies that the operator $\mathcal{A}(\kappa)^{-1} \circ j_{\mathcal{H}} : \mathcal{L} \to \mathcal{G}$ is continuous.

4. *The operator $\kappa \mapsto \mathcal{A}(\kappa) : L^\infty(D \times \Omega) \to Auto(\mathcal{H})$ is linear and continuous.*

For more details on the relation in equation (3.6), see [5], ([6], Lem. 1.2) and ([18], Prop. 2.3), for instance.

Remark 3.2. Since $\mathcal{A}$ is a linear operator, y depends linearly on f. Recall also that, by assumption A4, f depends on u and ξ linearly. Therefore,

(i) y depends linearly and continuously on u and linearly on those components of ξ which appear in f. Recall also that $y(u, \cdot, \cdot) \in \mathcal{H} \cap \mathcal{G}$.

(ii) y depends continuously on κ and f w.r.t. the topologies of $L^\infty(D \times \Omega)$ and $\mathcal{L}$, respectively (see, for instance, [8], Cor. 2.1, also [45], Thm. 9.31, [50], Prop. 2).

Equation (3.5) in Theorem 3.1 and the linear and continuous dependence of y on u (Rem. 3.2(i)) are useful for the Frechét differentiability of the objective function of CCPDE with respect to u. Moreover, the continuous dependence, in Remark 3.2(ii), of the solution y on κ and f provides closure properties for the feasible set of CCPDE, which is essential for guaranteeing the existence of an optimal solution. In addition, the continuous dependence implies the stability of the solution y of the PDE system to perturbations in κ and f. This issue, in fact, is classical to the study of linear elliptic PDE systems (see, for instance, [20], Rem. 1.1.3, [35], Rem. 2.6, [46], Lem. 2).

According to the discussion above, for a given $u \in U$ and a realization of the random variable $\xi \in \Omega$, there is a unique weak solution of the PDE system given by $y(u, x, \xi)$. Since $y(u, \cdot, \cdot)$ is contained in the subspace $\mathcal{G} \cap \mathcal{H}$, the problem CCPDE can be stated in a reduced form with pointwise chance constraints on D_c as follows

$$(CCPDE_{red}) \qquad \min_u \; J(u) \tag{3.8}$$

$$\text{subject to: } \; Pr\{y(u,x,\xi) \le y_c(x)\} \ge \alpha, x \in D_c, \tag{3.9}$$

$$u \in U, \tag{3.10}$$

where the objective functional is expressed as

$$J(u) := E\left[\|y(u,\cdot,\xi) - y_d(\cdot)\|^2_{H^1_0(D)}\right] + \frac{\rho}{2}\|u\|^2_{L^2(D)} \tag{3.11}$$

and the feasible set of CCPDE is defined as

$$\mathcal{P} = \{u \in U \mid p(u,x) \ge \alpha, x \in D_c\} \tag{3.12}$$

with the probability function

$$p(u,x) := Pr\{y(u,x,\xi) \le y_c(x)\}, x \in D_c. \tag{3.13}$$

In the following, the Frechét derivative of y w.r.t. u is denoted by $y'_u(u,\cdot,\cdot)$. Hence, next, we show that a solution y of the PDE system satisfies some convergence, in probability, w.r.t. sequences u_n, x_n under A2D. Similar properties also hold true for the Frechét derivative y'_u. The Lemma below will be used in Section 4.1 for verifying the continuity and differentiability of the inner and outer approximation functions which subsequently imply the continuity of p.

Lemma 3.3. *Let the map $u \mapsto f(u,\cdot,\cdot) : L^2(D) \to \mathcal{L}$ be continuous at u_0, let $y(u_0,\cdot,\cdot)$ be the corresponding weak solution of the PDE system, and let $\{u_n\}_{n\in\mathbb{N}} \subset U$ be an arbitrary sequence. Then the following statements are true.*

1. *The map $u \mapsto y(u,\cdot,\cdot) : L^2(D) \to \mathcal{H} \cap \mathcal{G}$ is continuous at u_0 w.r.t. the topology in $\mathcal{G}$.*
2. *Uniform convergence of $y(u_n,\cdot,\cdot)$ on D in probability: If $u_n \to u_0$ in $L^2(D)$, then*

$$\int_\Omega \operatorname{esssup}_{x\in D} |y(u_n,x,\xi) - y(u_0,x,\xi)|\phi(\xi)\,\mathrm{d}\xi \to 0$$

3. *Convergence of $y(u_n,x_n,\cdot)$[5] in probability: If $u_n \to u_0$ in $L^2(D)$ and $x_n \to x_0$ in D then*

$$\int_\Omega |y(u_n,x_n,\xi) - y(u_0,x_0,\xi)|\phi(\xi)\mathrm{d}\xi \to 0$$

Proof.
1. Let κ be fixed and $S := \mathcal{A}(\kappa)^{-1} j_{\mathcal{H}}$. The continuity of $S : \mathcal{L} \to \mathcal{H} \cap \mathcal{G}$ yields immediately the continuity of $u \mapsto y(u,\cdot,\cdot) : L^2(D) \to \mathcal{H} \cap \mathcal{G}$ by the representation in (3.5).
2. Because of the continuous embedding $C_0(D) \leftarrow H^2(D) \cap H^1_0(D)$ and $\|v\|_{C_0(D)} \le d\|v\|_{H^2(D)}$, from the Sobolev's embedding theorem (see, for instance, [1]) and using 1., it follows that

$$\int_\Omega \operatorname{esssup}_{x\in D} |y(u_n,x,\xi) - y(u_0,x,\xi)|^2\phi(\xi)\mathrm{d}\xi \le d^2 \int_\Omega \|y(u_n,\cdot,\xi) - y(u_0,\cdot,\xi)\|^2_{H^2(D)}\,\phi(\xi)\mathrm{d}\xi$$
$$= d^2\|y(u_n,\cdot,\cdot) - y(u_0,\cdot,\cdot)\|^2_G \to 0.$$

[5] When working with the continuity of $y(u,\cdot,\xi)$, we assume that the continuous element of the corresponding equivalent class is selected.

The convergence in $L^2(\Omega, Pr)$ with $\mathrm{d}Pr(\xi) = \phi(\xi)\mathrm{d}\xi$ implies the convergence of the sequence in probability. Thus, $\{y(u_n, x, \cdot)\}_{n\in\mathbb{N}}$ converges uniformly to $y(u_0, x, \cdot)$ in probability, for $x \in D$.
3.

$$\begin{aligned}|y(u_n, x_n, \xi) - y(u_0, x_0, \xi)| &\leq |y(u_n, x_n, \xi) - y(u_0, x_n, \xi)| + |y(u_0, x_n, \xi) - y(u_0, x_0, \xi)| \\ &\leq \sup_{x\in D} |y(u_n, x, \xi) - y(u_0, x, \xi)| + |y(u_0, x_n, \xi) - y(u_0, x_0, \xi)|.\end{aligned}$$

The first summand is convergent in probability by 2. and the second one is a.s. convergent, because of continuity in x. ☐

Lemma 3.4. *Let the map $u \mapsto f(u, \cdot, \cdot) : L^2(D) \to \mathcal{L}$ be continuously Fréchet differentiable at u_0 and let $y(u_0, \cdot, \cdot)$ be the corresponding weak solution of the PDE system. Then, the following statements are true.*

1. *The map $u \mapsto y(u, \cdot, \cdot) : L^2(D) \to \mathcal{G}$ is continuously Fréchet differentiable at u_0.*
2. *For an arbitrary zero sequence $\{t_n\}_{n\in\mathbb{N}}$ and an arbitrary $h \in L^2(D)$, the sequence $\{(y(u_0 + t_n h, x, \cdot) - y(u_0, x, \cdot))/t_n\}_{n\in\mathbb{N}}$ converges to $y'_u(u_0, x, \cdot)h$ in probability, uniformly for $x \in D$.*
3. *If $u_n \to u_0$ in $L^2(D)$, then $y'_u(u_n, x, \cdot)$ satisfies the uniform limit*

$$\lim_{n\to\infty} \sup_{\|h\|_{L^2(D)}=1} \int_\Omega \mathrm{esssup}_{x\in D} |(y'_u(u_n, x, \xi) - y'_u(u_0, x, \xi))h|^2 \phi(x)\mathrm{d}\xi\} = 0. \tag{3.14}$$

4. *If $u_n \to u_0$ in $L^2(D)$ and $x_n \to x_0$ in D, then for each $h \in L^2(D)$*

$$\| [y'_u(u_n, x_n, \xi) - y'_u(u_0, x_0, \xi)]\, h\|_{\mathcal{L}} \to 0. \tag{3.15}$$

Proof.
1. The statement is a simple consequence of the chain rule and Fréchet differentiability of continuous linear maps.
2. By 1. it follows that $\lim_{n\to\infty} \|(y(u_0 + t_n h, \cdot, \cdot) - y(u_0, \cdot, \cdot))/t_n - y'_u(u_0, \cdot, \cdot)h\|_{\mathcal{G}} = 0$. Subsequently, using similar arguments as in the proof of Lemma 3.3 2., the claim follows.
3. Here again, similar arguments as in the proof of Lemma 3.3 2. can be used. Thus,

$$\int_\Omega \mathrm{esssup}_{x\in D} |(y'_u(u_n, x, \xi) - y'_u(u_0, x, \xi))h|^2 \phi(\xi)\mathrm{d}\xi \tag{3.16}$$

$$\leq d^2 \int_\Omega \|(y'_u(u_n, \cdot, \xi) - y'_u(u_0, \cdot, \xi))h\|^2_{H^2(D)} \phi(\xi)\mathrm{d}\xi$$

$$\leq d^2 \sup_{\|h\|_{L^2(D)}=1} \int_\Omega \|(y'(u_n, \cdot, \xi) - y'_u(u_0, \cdot, \xi))h\|^2_{H^2(D)} \phi(\xi)\mathrm{d}\xi. \tag{3.17}$$

Since $u \mapsto y(u, \cdot, \cdot) : L^2(D) \to \mathcal{G}$ is continuously Fréchet differentiable at u_0, the expression in (3.17) goes to zero for $n \to \infty$. Hence, taking the supremum of (3.16) w.r.t. $\|h\|_{L^2(D)} = 1$, the statement follows.
4.

$$\begin{aligned}&\int_\Omega |\,(y'_u(u_n, x_n, \xi) - y'_u(u_0, x_0, \xi))\,h|^2 \phi(x)\mathrm{d}\xi \\ \leq &\int_\Omega |\,(y'_u(u_n, x_n, \xi) - y'_u(u_0, x_n, \xi))\,h|^2 \phi(x)\mathrm{d}\xi \\ &+ \int_\Omega |\,(y'_u(u_0, x_n, \xi) - y'_u(u_0, x_0, \xi))\,h|^2 \phi(x)\mathrm{d}\xi.\end{aligned}$$

By 3., the first summand is convergent to zero uniformly in h. However, the second summand is only weakly convergent to zero, *i.e.*, convergent for each fixed h. Indeed, $y'_u(u_0,\cdot,\cdot)h \in \mathcal{G}$ and $x \mapsto y'_u(u_0,x,\xi)h$ is continuous on D for $\xi \in \Omega$ a.s. Further, the absolute factor of the integrand is bounded by $2\,\mathrm{esssup}_{x\in D}\,|y'_u(u_0,x,\xi)h| + 1$ having finite expectation according to

$$\int_{\Omega} 2\,\mathrm{esssup}_{x\in D}\,|y'_u(u_0,x,\xi)h|\,\phi(\xi)\mathrm{d}\xi \le \|y'_u(u_0,\cdot,\cdot)h\|_{\mathcal{G}}.$$

Thus, Lebesgue's Dominated Convergence Theorem can be applied. □

3.2. Convexity properties of the chance constrained problem

Let $x \in D$ be fixed. To verify the concavity of the probability function $u \mapsto p(u,x)$, $y(\cdot,x,\cdot)$ needs to be jointly convex w.r.t. (u,ξ) along with the log-concavity of the density function $\phi(\cdot)$, considering all random variables ξ in the model (see, *e.g.*, [55, 56]). For this purpose, the following two special cases are considered for the function f satisfying assumption A4.

We first recall the log-concavity of functions.

Definition 3.5. Let E be a linear space. A function $f : E \to (0,+\infty]$ is said to be log-concave if $f(\lambda x + (1-\lambda)y) \ge [f(x)]^{\lambda}[f(y)]^{1-\lambda}$, for any $x,y \in X$ and $\lambda \in [0,1]$.

Case 1 $\kappa \equiv$ constant and f is random.
Then $(u,\xi) \mapsto y(u,x,\xi) : L^2(D) \times \Omega \to \mathbb{R}$ is linear by (3.5), hence, jointly convex w.r.t. (u,ξ), for each $x \in D$ (see Rem. 3.2(i)).
Case 2 κ is independent of the spatial variable x, concave w.r.t. ξ and $y_c(x) \ge 0$.
Then, κ is a *homogeneous random field*. Such random fields could appear in physical processes with homogenous material properties. Thus, the elliptic PDE system (2.10)–(2.11) becomes

$$-\triangle y(x,\xi) = \frac{1}{\kappa(\xi)} f(u,x,\xi), \text{ for } x \in D,\ \xi \in \Omega \text{ a.s.}, \tag{3.18}$$

$$y(x,\xi) = 0, \text{ for } x \in \partial D,\ \xi \in \Omega \text{ a.s.} \tag{3.19}$$

Since κ is independent of x, it follows $(\mathcal{A}(\kappa))^{-1} = \frac{1}{\kappa}\mathcal{A}^{-1}$ and thus

$$y(u,x,\xi) = \frac{1}{\kappa(\xi)}\mathcal{A}^{-1}\left[j_{\mathcal{H}}(u + f_0(\cdot,\xi))\right](x) \le y_c(x).$$

It is equivalent to

$$\left[\mathcal{A}^{-1} j_{\mathcal{H}}(u + f_0(\cdot,\xi))\right](x) - \kappa(\xi) y_c(x) \le 0.$$

The expression on the left hand-side of the last inequality is jointly convex w.r.t. (u,ξ) because of the linearity of f w.r.t. (u,ξ), the concavity of κ and the nonnegativity of y_c.

Since

$$Pr\left\{y(u,x,\xi) \le y_c(x)\right\} = Pr\left\{\mathcal{A}^{-1}\left[j_{\mathcal{H}}(u + f_0(\cdot,\xi))\right] - \kappa(\xi)y_c(x) \le 0\right\}. \tag{3.20}$$

is valid, ([23], Prop. 3) yields the following result.

Proposition 3.6. *Suppose A1 and A4 hold true, U is convex, the coefficient κ is either a constant or independent of x, concave w.r.t. ξ and $y_c(\cdot) \ge 0$ on D_c. Further assume $y_c \in H^2(D) \cap H^1_0(D)$ and $y(u,\cdot,\cdot) \in \mathcal{G} \cap \mathcal{H}$ for $u \in U$. If the density function $\phi(\cdot)$ is log-concave, then the probability function $p(\cdot,x)$ is concave on U, for* *each $x \in D_c$. Consequently, the feasible set $\mathcal{P}$ of CCPDE$_{red}$ is a convex set, for $\alpha \in [0, 1]$. Moreover, if $p(\cdot, x)$ is continuous on U, then $\mathcal{P}$ is a closed set and, due to convexity, it is also weakly sequentially closed.*

Remark 3.7. Nevertheless, if κ depends on both x and ξ, the analysis of the convexity of y w.r.t. ξ is generally less transparent and needs detailed investigations, which is beyond the scope of this work. The following simple considerations illustrate the problems that may occur. Also, if $(u, \xi) \mapsto f(u, x, \xi) = f_1(u)(x) + f_0(x, \xi)$ is convex in u and linear in ξ or if A2 is valid where f is jointly convex in (u, ξ), then $(u, \xi) \mapsto y(u, x, \xi)$ need not be jointly convex in (u, ξ). For instance, the solution y of $\triangle y = f, y \in H_0^1(D)$ is given for $n = 3$ by

$$\begin{aligned} y(u, x, \xi) &= y_1(u, x, \xi) - y_2(u, x, \xi) \\ y_1(u, x, \xi) &= \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f(u, \hat{x}, \xi)}{|x - \hat{x}|} \mathrm{d}\hat{x} \\ y_2(u, x, \xi) &= \int_{\partial D} G(x, \hat{x}) y_1(u, \hat{x}, \xi) \mathrm{d}S(\hat{x}), \end{aligned}$$

where the second integral is a surface integral. Since the potential is nonnegative, the first term y_1 has the required joint convexity in (u, ξ). However, in the second term, the kernel $G(x, \hat{x})$ is not necessarily smaller than zero, it strongly depends on the domain D, it is harmonic and exists for Lyapunov boundaries.

Hence, the linearity in ξ and u of f, given by A4, is the only simple assumption to guarantee the joint convexity of $y(\cdot, x, \cdot)$. As we later see, the smoothness of the inner and outer problem admits a much more general function f. Hence, our inner-outer approximation approach is still extendable to tackle chance constrained PDE problems with a nonconvex forcing term f.

Referring to the objective function of CCPDE in equation (3.11)

$$J(u) := E\left[\|y(u, \cdot, \xi) - y_d(\cdot)\|^2_{W(D)}\right] + \frac{\rho}{2}\|u\|^2_{L^2(D)}, \tag{3.21}$$

where $W(D)$ could be $L_2(D)$, $H_0^1(D)$ or some other reflexive and separable Sobolev space, the joint convexity of y w.r.t. (u, ξ) implies the following properties of J (see *e.g.* [12]).

(a) $\lim_{\|u\| \to +\infty} J(u) = +\infty$; *i.e.*, $J(\cdot)$ is super-coercive.
(b) $J(\cdot)$ is bounded on any bounded subset of $L^2(D)$.
(c) $J(\cdot)$ is uniformly convex.
(d) The level sets $\{u \in L_2(D) \mid J(\cdot) \le c\}$ are weakly compact.

If y is (continuously) Fréchet differentiable w.r.t. u, then the chain rule and the (continuous) Fréchet differentiability of the scalar product imply the (continuous) Fréchet differentiability of J.

Proposition 3.8. *Suppose the assumptions of Proposition 3.6 hold true and in (3.11) the regularization parameter $\rho > 0$. If $p(\cdot, x)$ is a continuous function, for all $x \in D_c$, then CCPDE$_{red}$ has a unique optimal solution.*

Proof. $J(\cdot)$ is strictly convex. Moreover, the feasible set $\mathcal{P}$ of CCPDE is closed, convex, and bounded (since $\mathcal{P} \subset U$). Since in a Hilbert (reflexive Banach) space, a closed, bounded, and convex set is weakly sequentially compact, the problem

$$\min_{u \in \mathcal{P}} J(u)$$

has a unique optimal solution $u^* \in \mathcal{P}$ (see, for instance, [3], Thm. 3.3.4 with Rem. 3.3.1, and also [61], Thm. 2.14). □

In Proposition 3.8, the continuity assumption on $p(\cdot, x)$ is guaranteed by Corollary 4.8 (see Sect. 4 below). However, despite the guarantee for existence of a unique optimal solution by Proposition 3.8, CCPDE is usually hard to solve directly.

Therefore the next task of this work is to solve the problem CCPDE approximatively by solving a sequence of smooth chance constrained PDE optimization problems which are generally nonconvex. For the special case of convexity, the upper approximation problem turns out to be convex, whereas the inner approximation problem is nonconvex. The results of this section will be also useful for verifying the convergence properties of the sequence of the approximate solutions to the solution of CCPDE.

4. The approximation functions and their properties

In this section, we use parametric functions to define smooth approximations to the chance constraints. This leads to a sequence of smooth parametric optimization problems. The solutions of the CCPDE$_{red}$ are shown to be approximable by the family of solutions of the inner-outer approximation problems, when the approximation parameter tends to zero. In case of convexity, the norm convergence of the approximate solutions to a solution of CCPDE$_{red}$ can be proved. For this purpose, we employ and extend our recent result from [26, 27] where inner-outer approximation methods, for finite-dimensional chance constrained optimization problems, were proposed. In the work [26, 27] convexity issues were not considered. In the following, the properties of the underlying parametric functions are briefly summarized, and their convexity properties are studied. In addition, we assume that $y(u, \cdot, \cdot) \in \mathcal{G} \cap \mathcal{H}$ and $y_c(\cdot) \in H^2(D) \cap H_0^1(D)$. Since we need a continuous representant of each equivalent class $y(u, \cdot, \cdot)$ and $y_c(\cdot)$, further discussions should be understood in this sense.

First, define the unit jump function

$$h(s) = \begin{cases} 1, & \text{if } s \geq 0, \\ 0, & \text{if } s < 0. \end{cases} \tag{4.1}$$

Assumption 4.1. Suppose there is a parametric family $\{\Theta(\tau, \cdot), \tau \in (0,1)\}$ of functions $\Theta : (0,1) \times \mathbb{R} \to \mathbb{R}_+$ which possesses the following *strict monotonicity* and *uniform limit* properties:

P1: There is a constant C with $1 < C < +\infty$ such that

$$C \geq \Theta(\tau, s) > h(s), \forall s \in \mathbb{R}, \tau \in (0,1). \tag{4.2}$$

P2: $\Theta(\cdot, s)$ is strictly increasing on $(0,1)$, for each $s \in \mathbb{R}$,
P3: $\Theta(\tau, \cdot)$ is continuously differentiable and strictly increasing on $\mathbb{R}$, for each $\tau \in (0,1)$,
P4: $\inf_{\tau \in (0,1)} \Theta(\tau, s) = h(s)$, for all $s \in \mathbb{R}$,
P5: $\lim_{\tau \searrow 0_+} \sup_{s \in (-\infty, -\varepsilon) \cup [0,\infty)} (\Theta(\tau, s) - h(s)) = 0$, for all $\varepsilon > 0$.

Remark 4.2. For $s \neq 0$, Property P1 of Assumption 4.1 implies the sandwich condition

$$1 - \Theta(\tau, s) < h(-s) < \Theta(\tau, -s) \tag{4.3}$$

which is important for the construction of inner and outer approximations to the feasible set of CCPDE$_{red}$.

The properties (P1)–(P5) were introduced in [26, 27] in the context of finite dimensional chance constrained optimization problems to characterize admissible families of parametric functions. A special type of parametric function was also introduced in [26, 27] for the purpose of inner-outer approximations of the feasible set corresponding to lower-upper approximations of the function $p(\cdot, x), x \in D_c$.

Corollary 4.3 (Geletu *et al.* [27], Prop. 3.3., and also [26]). *Let m_1, m_2 be constants with $0 < m_2 \leq m_1/(1+m_1)$. Then, the parametric family*

$$\Theta(\tau, s) = \frac{1 + m_1\tau}{1 + m_2\tau \exp\left(-\frac{s}{\tau}\right)}, \quad for \quad \tau \in (0,1), \ s \in \mathbb{R}, \tag{4.4}$$

satisfies the properties P1-P5 of Assumption 4.1.

The following assumption is essential for a tight approximation of the probability function $p(u, x)$, for each $x \in D$, using the sandwich condition (4.3).

Assumption 4.4 (Measure zero property).
(**A5**:) For the solution y of the PDE system (2.2)–(2.3) with $g(u, x, \xi) := y(u, x, \xi) - y_c(x)$,

$$Pr\{\xi \in \Omega \mid g(u, x, \xi) = 0\} = 0, \text{ for all } x \in D_c,$$

holds true.

Remark 4.5. Let $\Omega \subset \mathbb{R}^m$ be an open set. If the set $X(u, x) := \{\xi \in \Omega \mid g(u, x, \xi) \leq 0\}$ satisfies the Mangasarian-Fromowitz-constraint-qualification (MFCQ), *i.e.*, for only one constraint g that $\nabla_\xi g(u, x, \xi) \neq 0$, for all ξ with $g(u, x, \xi) = 0$), then the measure zero property is satisfied [26]. Especially, $\{\xi \mid g(u, x, \xi) = 0\}$ does not contain an open set. Another criteria is that $\{\xi \mid g(u, x, \xi) = 0\}$ is an $(m-1)$-dimensional manifold (see, for instance, [33, 34]).

Now, based on the parametric function Θ introduced in Assumption 4.1, define

$$\psi(\tau, u, x) := E\left[\Theta(\tau, g(u, x, \xi))\right] \text{ and } \varphi(\tau, u, x) := E\left[\Theta(\tau, -g(u, x, \xi))\right], \tau \in (0,1). \tag{4.5}$$

Hence, from equation (4.1), assumptions A1, A2 and A5, it follows that

$$p(u, x) = Pr\left\{g(u, x, \xi) \leq 0\right\} = E\left[h(-g(u, x, \xi))\right] = 1 - E\left[h(g(u, x, \xi))\right] \geq \alpha, \tag{4.6}$$

for $\tau \in (0,1)$ and $x \in D_c$. Consequently, property P3, inequalities (4.3) and equations (4.5), (4.6) imply the lower and upper approximations of $p(\cdot, x)$ given as

$$\varphi(\tau_2, u, x) \geq \varphi(\tau_1, u, x) \geq p(u, x) \geq 1 - \psi(\tau_1, u, x) \geq 1 - \psi(\tau_2, u, x), \tag{4.7}$$

for $0 < \tau_1 \leq \tau_2 < 1$, $u \in U$, and $x \in D_c$. Subsequently, $1 - \psi$ and φ are referred as lower and upper approximations of p, respectively.

4.1. Continuity, differentiability and convexity of the approximation functions

We begin by stating results for the parametric function $\Theta(\tau, \cdot), \tau \in (0,1)$, which are complementary to Lemma 3.3.

Lemma 4.6. *Under the assumptions of Lemma 3.3, the following statements hold true for each $\tau \in (0,1)$.*

1. *If $u_n \to u_0$ in $L^2(D)$, then*

$$\int \sup_{x \in D} \left|\Theta(\tau, y(u_n, x, \xi)) - \Theta(\tau, y(u_0, x, \xi))\right| \phi(\xi) \mathrm{d}\xi \to 0,$$
$$\int \sup_{x \in D} \left|\frac{\partial}{\partial s}\Theta(\tau, y(u_n, x, \xi)) - \frac{\partial}{\partial s}\Theta(\tau, y(u_0, x, \xi))\right| \phi(\xi) \mathrm{d}\xi \to 0.$$

2. If $u_n \to u_0$ in $L^2(D)$ and $x_n \to x_0$ in D, then

$$\int |\Theta(\tau, y(u_n, x_n, \xi)) - \Theta(\tau, y(u_0, x_0, \xi))|\, \phi(\xi)\mathrm{d}\xi \to 0,$$
$$\int \left| \frac{\partial}{\partial s}\Theta(\tau, y(u_n, x_n, \xi)) - \frac{\partial}{\partial s}\Theta(\tau, y(u_0, x_0, \xi)) \right| \phi(\xi)\mathrm{d}\xi \to 0.$$

Proof.
1. Since $\Theta(\tau, \cdot)$ and $\frac{\partial}{\partial s}\Theta(\tau, \cdot)$ are Lipschitz continuous, the claim follows from ([9], Satz 20.9).
2. Let $\eta : \mathbb{R} \to \mathbb{R}_+$ be Lipschitz continuous. Then,

$$\begin{aligned}|\eta(y(u_n, x_n, \xi)) - \eta(y(u_0, x_0, \xi))| &\le |\eta(y(u_n, x_n, \xi)) - \eta(y(u_0, x_n, \xi))| + |\eta(y(u_0, x_n, \xi)) - \eta(y(u_0, x_0, \xi))| \\ &\le \sup_{x \in D} |\eta(y(u_n, x, \xi)) - \eta(y(u_0, x, \xi))| + |\eta(y(u_0, x_n, \xi)) - \eta(y(u_0, x_0, \xi))|.\end{aligned}$$

Hence, the claim follows from 1. here and by similar arguments for the proof of Lemma 3.3 3. and 1. above using $\eta = \Theta(\tau, \cdot)$ and $\eta = \frac{\partial}{\partial s}\Theta(\tau, \cdot)$, respectively. □

Theorem 4.7. [Continuity, Tightness]
Let A1 and A2 be satisfied, then
1. the functions $\psi(\tau, \cdot, \cdot)$ and $\varphi(\tau, \cdot, \cdot)$ are jointly continuous on $L^2(D) \times D_c$, for each $\tau \in (0, 1)$.
2. For each $x \in D_c$ and each $u \in L^2(D)$, the following limits are valid

$$p(u, x) = \inf_{\tau \in (0,1)} \varphi(\tau, u, x) \tag{4.8}$$

and, under the measure zero property A5,

$$\sup_{\tau \in (0,1)} (1 - \psi(\tau, u, x)) = p(u, x). \tag{4.9}$$

Proof.
1. Let $\|u_n - u_0\|_{L^2(D)} \to 0$ and $|x_n - x_0| \to 0$ with $x_n \in D_c$, then $x_0 \in D_c$ and, by Lemma 4.6 2., the sequence $\{\Theta(\tau, g(u_n, x_n, \cdot)\}_{n \in \mathbb{N}}$ is convergent to $\Theta(\tau, g(u_0, x_0, \cdot))$ in probability. Since $0 \le \Theta(\tau, s) \le C < \infty$, Lebesgue's Dominated Convergence Theorem for stochastic convergence ([53], Satz 1, p. 166) yields the claim.
2. Let $x \in D_c$ and $u \in L^2(D)$ be fixed. For any zero sequence $\{\tau_k\}_{k \in \mathbb{N}} \subset (0, 1)$, property P1 of Assumption 4.1 guarantees that (see [27]) $0 \le \Theta(\tau_k, g(u, x, \xi)) \le C$, for $\xi \in \Omega$ a.s. and properties P2–P4 imply that $\lim_{k \to +\infty} \Theta(\tau_k, s)|_{s = g(u,x,\xi)} = h(s)|_{s = g(u,x,\xi)}$ for $\xi \in \Omega$ a.s. Consequently, Lebesgue's Dominated Convergence Theorem implies that

$$\lim_{k \to +\infty} \int_\Omega (1 - \Theta(\tau_k, g(u, x, \xi)))\phi(\xi)\mathrm{d}\xi = \int_\Omega (1 - h(g(u, x, \xi)))\phi(\xi)\mathrm{d}\xi.$$

Hence, applying the measure zero property A5, we conclude that

$$\lim_{k \to +\infty} (1 - \psi(\tau_k, u, x)) = p(u, x).$$

Similar arguments also hold true w.r.t. φ without using A5. As a result, the infimum and supremum properties (4.8) and (4.9) follow from (4.7). □

Corollary 4.8. *Let assumptions A1, A2 and A5 be satisfied. Then, the function $p(\cdot,\cdot)$ is continuous on $L^2(D) \times D_c$ and the convergences $\lim_{\tau\searrow 0^+} \psi(\tau,u,x) = 1 - p(u,x)$ and $\lim_{\tau\searrow 0^+} \varphi(\tau,u,x) = p(u,x)$ are uniform on any compact subset $W \subset L^2(D) \times D_c$.*

Proof. According to Corollary 4.3, there is at least one family of functions $\{\Theta(\tau,\cdot), \tau \in (0,1)\}$ guaranteeing the approximations. The infimum (supremum) of continuous functions is upper (lower) semi-continuous [22], Section 12.7.7. Thus, $p(\cdot,\cdot)$ is continuous. Since $p(\cdot,\cdot)$ continuous, the monotonic convergence to $p(u,x)$, for all $(u,x) \in L^2(D) \times D_c$, yields the uniform convergence on compact subsets of $L^2(D) \times D_c$ by Dini's Theorem. ☐

Theorem 4.9. *Suppose A1 holds true. Let $u_0 \in U$ be arbitrary and let the map $u \mapsto f(u,\cdot,\cdot) : L^2(D) \to \mathcal{L}$ be continuously Fréchet differentiable in some neighborhood W of u_0. Moreover, for each fixed $\tau \in (0,1)$, let $s \mapsto \frac{\partial\Theta(\tau,s)}{\partial s}$ be bounded by some constant $C(\tau) > 0$.*

Then, for each $x \in D_c$, $u \in W$ and $\tau \in (0,1)$, the maps $u \mapsto \psi(\tau,u,x)$ and $u \mapsto \varphi(\tau,u,x)$ are continuously Fréchet differentiable on W uniformly for each $x \in D_c$. Moreover, the maps $(u,x) \to \nabla_u\psi(\tau,u,x)h$ and $(u,x) \to \nabla_u\varphi(\tau,u,x)h$ are continuous on $W \times D_c$, for each $h \in L^2(D)$. The derivative can be calculated by exchanging integration and differentiation operations according to

$$\nabla_u\psi(\tau,u,x) = \int_\Omega \frac{\partial}{\partial s}\Theta(\tau,s)\Big|_{s=g(u,x,\xi)} \nabla_u g(u,x,\xi)\phi(\xi)\mathrm{d}\xi, \tag{4.10}$$

$$\nabla_u\varphi(\tau,u,x) = -\int_\Omega \frac{\partial}{\partial s}\Theta(\tau,s)\Big|_{s=-g(u,x,\xi)} \nabla_u g(u,x,\xi)\phi(\xi)\mathrm{d}\xi. \tag{4.11}$$

Proof. Let $u_0 \in W$ be given. Considering the difference quotient for the directional derivative with an arbitrary zero sequence $t_n \to 0$ and, at the same time, using the mean value theorem for functionals it follows that

$$\begin{aligned}
&\frac{\psi(\tau,u_0+t_nh,x) - \psi(\tau,u_0,x)}{t_n} \\
&= \int_\Omega \frac{[\Theta(\tau,g(u_0+t_nh,x,\xi)) - \Theta(\tau,g(u_0,x,\xi))]}{t_n}\phi(\xi)\mathrm{d}\xi \\
&= \int_\Omega \frac{\partial\Theta(\tau,g(u_0,x,\xi)+\eta_n(g(u_0+t_nh,x,\xi)-g(u_0,x,\xi)))}{\partial s}\nabla_u g(u_0+\rho_nt_nh,x,\xi)h\phi(\xi)\mathrm{d}\xi
\end{aligned}$$

with $\eta_n \in (0,1), \rho_n \in (0,1)$. Hence, Lemma 3.6 yields

$$\left|\frac{\partial\Theta(\tau,g(u_0+t_nh,x,\cdot))}{\partial s} - \frac{\partial\Theta(\tau,g(u_0,x,\cdot))}{\partial s}\right| \to 0$$

in probability and Lemma 3.4 implies that $|\nabla_u g(u_0+\rho_nt_nh,x,\xi)h - \nabla_u g(u_0,x,\xi)h| \to 0$ in probability. The continuity and boundedness of $s \to \frac{\partial\Theta(\tau,s)}{\partial s}$ by some constant $C(\tau) > 0$ together with the shown majorization in Lemma 3.4 on the neighborhood W, Lebesgue's Dominated Convergence Theorem yields

$$\begin{aligned}
\nabla_u\psi(\tau,u_0,x)h &:= \lim_{n\to\infty}\frac{\psi(\tau,u_0+t_nh,x) - \psi(\tau,u_0,x)}{t_n} \\
&= \int_\Omega \frac{\partial\Theta(\tau,g(u_0,x,\xi)}{\partial s}\nabla_u g(u_0,x,\xi)h\phi(\xi)\mathrm{d}\xi
\end{aligned} \tag{4.12}$$

which is obviously linear in h. Furthermore, it holds that

$$|\nabla_u \psi(\tau, u_0, x)h|^2 \leq C(\tau)^2 \int_\Omega |\nabla_u g(u_0, x, \xi)h|^2 \mathrm{d}\xi \leq C(\tau)^2 \|\nabla_u g(u_0, \cdot, \cdot)h\|_{\mathcal{L}}^2 \leq Q^2 \|h\|_{L^2(D)}^2$$

where Q is the finite operator norm of $\nabla_u g(u_0, \cdot, \cdot)$, which implies the continuity of $h \mapsto \nabla_u \psi(\tau, u_0)h$ w.r.t. h.

Next, we show that the continuity w.r.t. u yields continuous Fréchet differentiability. Let $\|u_n - u_0\|_{L^2(D)} \to 0$. Hence, the Cauchy-Schwartz inequality yields

$$\begin{aligned}
&|\nabla_u \psi(\tau, u_n, x)h - \nabla_u \psi(\tau, u_0, x)h| \\
\leq &\int_\Omega \left| \frac{\partial \Theta(\tau, g(u_n, x, \xi))}{\partial s} \nabla_u g(u_n, x, \xi)h - \frac{\partial \Theta(\tau, g(u_0, x, \xi))}{\partial s} \nabla_u g(u_0, x, \xi)h \right| \phi(\xi)\mathrm{d}\xi \\
\leq &\int_\Omega \left| \frac{\partial \Theta(\tau, g(u_n, x, \xi))}{\partial s} - \frac{\partial \Theta(\tau, g(u_0, x, \xi))}{\partial s} \right| |\nabla_u g(u_0, x, \xi)h| \, \phi(\xi)\mathrm{d}\xi \\
&+ \int_\Omega \left| \frac{\partial \Theta(\tau, g(u_n, x, \xi))}{\partial s} \right| |\nabla_u g(u_0, x, \xi)h - \nabla_u g(u_n, x, \xi)h| \, \phi(\xi)\mathrm{d}\xi \\
\leq &\left[\int_\Omega \left| \frac{\partial \Theta(\tau, g(u_n, x, \xi))}{\partial s} - \frac{\partial \Theta(\tau, g(u_0, x, \xi))}{\partial s} \right|^2 \phi(\xi)\mathrm{d}\xi \int_\Omega |\nabla_u g(u_0, x, \xi)h|^2 \, \phi(\xi)\mathrm{d}\xi \right]^{\frac{1}{2}} \\
&+ \left[\int_\Omega \left| \frac{\partial \Theta(\tau, g(u_n, x, \xi))}{\partial s} \right|^2 \phi(\xi)\mathrm{d}\xi \int_\Omega |\nabla_u g(u_0, x, \xi)h - \nabla_u g(u_n, x, \xi)h|^2 \, \phi(\xi)\mathrm{d}\xi \right]^{\frac{1}{2}}.
\end{aligned}$$

In the expression above, the first integral of the first summand goes to zero and is independent of h. The second integral is bounded by $Q\|h\|_{L^2(D)^2}$. Hence, the first summand goes uniformly to zero, for $\|h\|_{L^2(D)} = 1$. The first integral in the second summand is bounded by C^2. Thus,

$$\begin{aligned}
&\lim_{n\to\infty} \sup_{\|h\|_{L^2(D)}=1} \sup_{x \in D_c} |\nabla_u \psi(\tau, u_n, x)h - \nabla_u \psi(\tau, u_0, x)h| \\
\leq &C(\tau) \left[\lim_{n\to\infty} \sup_{\|h\|_{L^2(D)}=1} \int_\Omega \sup_{x \in D_c} |\nabla_u g(u_0, x, \xi)h - \nabla_u g(u_n, x, \xi)h|^2 \, \phi(\xi)\mathrm{d}\xi \right]^{\frac{1}{2}} = 0
\end{aligned}$$

by Lemma 3.4 4. Consequently, $u \mapsto \nabla_u \psi(\tau, u, x)$ is continuous at u_0 as a map from $L^2(D)$ into its dual with the operator norm. Moreover, the exchangeability of the integration and differentiation is shown in (4.12) verifying (4.10).

Similar arguments yield the continuous differentiability of $\varphi(\tau, \cdot, x)$ at u_0. □

Observe that, the boundedness assumption on $\frac{\partial \Theta(\tau,s)}{\partial s}$ in Theorem 4.9 is already satisfied for the special function $\Theta(\tau, s)$ in Corollary 4.3 (see [26, 27]). In the following, we verify some convexity properties of the upper approximation function $\varphi(\tau, \cdot)$.

Theorem 4.10. [Convexity]
If $g(u, x, \xi) = y(u, x, \xi) - y_c(x)$ is a jointly convex function w.r.t. (u, ξ) which is e.g. given under A1 and A4 and $\phi(\xi)$ is a log-concave density function, then $\varphi(\tau, \cdot, x)$ is log-concave w.r.t. u, for each fixed $\tau \in (0, 1)$ and $x \in D_c$.

Proof. Since $g(\cdot, x, \cdot)$ is a convex function w.r.t. (u, ξ) and $\frac{1}{\tau} > 0$, the function $m_2 \tau \exp\left(\frac{1}{\tau} g(u, x, \xi)\right)$ is log-convex w.r.t (u, ξ). Hence, the sum of the log-convex functions

$$1 + m_2 \tau \exp\left(\frac{1}{\tau} g(u, x, \xi)\right)$$

is a log-convex function w.r.t. (u, ξ) (see [36] and also [56], Sect. 4.3). Consequently, $\Theta(\tau, -g(u, x, \xi))$ is a log-concave function w.r.t (u, ξ). Since the density function $\phi(\xi) > 0$ is log-concave, the product $\Theta(\tau, -g(u, x, \xi))\phi(\xi)$ is a log-concave function w.r.t. (u, ξ). Consequently,

$$\varphi(\tau, u, x) = \int_\Omega \Theta(\tau, -g(u, x, \xi))\phi(\xi) \mathrm{d}\xi$$

is a log-concave function, since it is the integral of a jointly log-concave function (see [55], Thm. 6). ☐

According to the discussion in Section 3.2, the joint convexity assumption on $g(u, x, \xi)$ in Theorem 4.10 is guaranteed if either κ is a constant or κ is a concave (or log-concave) function of ξ coupled with the linearity of the forcing term f w.r.t. (u, ξ).

Remark 4.11. It should be noted that, for the special parametric function (4.4), despite the joint quasi-convexity of the function $(u, \xi) \mapsto \Theta(\tau, g(u, x, \xi))$ w.r.t. (u, ξ), for each $x \in D$, either convexity or quasi-convexity properties of the approximation function $\psi(\tau, \cdot, x)$ is, generally, not available. In fact, we are unable to generally verify (or find) a result stating that the integral of a quasi-convex function is also quasi-convex. Therefore, for a convex inner approximation of the feasible set, it will be necessary to replace the function in (4.4) by another one guaranteeing convexity or quasi-convexity properties of $\psi(\tau, \cdot, x)$.

5. The smoothing optimization problems and their properties

Using the parametric functions $\psi(\tau, \cdot)$ and $\varphi(\tau, \cdot)$, we define the following problems with an arbitrary continuous J.

(IA_τ)		(OA_τ)	
	$\min_u J(u)$		$\min_u J(u)$
s.t.		s.t.	
	$\psi(\tau, u, x) \leq 1 - \alpha, x \in D_c,$		$\varphi(\tau, u, x) \geq \alpha, x \in D_c,$
	$u \in U,$		$u \in U,$

with respective feasible sets

$$\mathcal{M}(\tau) := \{u \in U \mid \psi(\tau, u, x) \leq 1 - \alpha, x \in D_c\}, \ \mathcal{S}(\tau) := \{u \in U \mid \varphi(\tau, u, x) \geq \alpha, x \in D_c\}.$$

Note that, the feasible sets can also be written as $\mathcal{M}(\tau) = \bigcap_{x \in D_c} \mathcal{M}(\tau, x)$ and $\mathcal{S}(\tau) = \bigcap_{x \in D_c} \mathcal{S}(\tau, x)$, where $\mathcal{M}(\tau, x) := \{u \in U \mid 1 - \psi(\tau, u, x) \geq \alpha\}$ and $\mathcal{S}(\tau, x) := \{u \in U \mid \varphi(\tau, u, x) \geq \alpha\}, \ \ x \in D_c, \ \tau \in (0, 1)$. Furthermore, associated to the feasible $\mathcal{P}$ in (3.12) of CCPDE_{red}, let

$$\mathcal{P}(x) := \{u \in U \mid p(u, x) \geq \alpha\}, x \in D_c.$$

In the following, the problems IA_τ and OA_τ are referred as *inner* and *outer* smoothing approximations to CCPDE_{red}, respectively. For the inner approximation problem IA_τ the following assumption is required.

Assumption 5.1 (Regularity condition). Assume $\alpha \in [0, 1)$.
(A6):

$$\mathcal{P} = \overline{\mathcal{P}^0}, \text{ where } \mathcal{P}^0 := \{u \in U \mid p(u, x) > \alpha, x \in D_c\}. \tag{5.1}$$

Lemma 5.2. *If A6 is valid, then* $\mathcal{P} = \bigcap_{x \in D_c} \overline{\mathcal{P}(x)^0}$, *where* $\mathcal{P}(x)^0 := \{u \in U \mid p(u, x) > \alpha\}$ *for all* $x \in D_c$.

Proof. Taking into account that $\mathcal{P}(x)$ is a closed set (by Cor. 4.8), the inclusions

$$\mathcal{P} = \overline{\mathcal{P}^0} \subset \bigcap_{x \in D_c} \overline{\mathcal{P}(x)^0} \subset \bigcap_{x \in D_c} \mathcal{P}(x) = \mathcal{P}$$

hold true. Hence, the equality follows. □

Lemma 5.3. *(Extended Slater condition) If* p *is concave in* u, *jointly continuous in* (u, x) *and there is some* $\hat{u}$ *such that* $p(\hat{u}, x) \geq \gamma > \alpha$ *for all* $x \in D_c$ *then A6 is satisfied.*

Proof. The infimum of p over x in the compact set D_c is continuous w.r.t. $u \in L_2(D)$ and it is not smaller than γ at $\hat{u}$. Hence, there is a ball around $\hat{u}$ such that, for all u in this ball, the infimum over x in D_c is not smaller than α. Thus, $\hat{u}$ is in the interior of $\mathcal{P}$. It is well-known from convex analysis that, for each element u in $\mathcal{P}$, all points v on the relative interior of the straight line segment from $\hat{u}$ to u satisfy $p(v, x) > \alpha$, for all $x \in D_c$. □

Remark 5.4. The regularity condition A6 is well-known from Barrier-methods. In case of finite number of $x_i \in D_c, i = 1, 2, \ldots, q$, as used for the finite dimensional approximation of the PDGL, the corresponding condition is satisfied if the Mangasarian Fromowitz Constraint Qualification (MFCQ) is valid for the system $p(u, x_i) > \alpha, i = 1, 2, \ldots, q$ (se *e.g.* [26]). Assumption A6 excludes the so called "apple stems" of $\mathcal{P}$. Assumption A6 assures that any point on the boundary of $\mathcal{P}$ is approximable by a sequence of points $u_n \in U$ of $\mathcal{P}$ with $p(u_n, x) > \alpha$ for each $x \in D_c$. In addition, assumption A6 along with properties P1 and P4 of the approximation function $\psi(\tau, u, x)$ guarantees that, for $\alpha < 1$, the feasible set $\mathcal{M}(\tau)$ is non-empty for sufficiently small $\tau \in (0, 1)$ and $\mathcal{M}(\tau)$ converges to $\mathcal{P}$, for $\tau \searrow 0$. This is already shown in ([27], Thm. 4.2) (and also in [26], Thm. 6) for finite dimensional chance constrained optimization problems.

Lemma 5.5. *Let* $x \in D_c$ *be given, A1, A2 be satisfied and* $\mathcal{P} \neq \emptyset$. *Then,*
1. the sets $\mathcal{M}(\tau, x)$ *and* $\mathcal{S}(\tau, x)$ *are closed, for each* $\tau \in (0, 1)$.
2. For $0 < \tau_2 \leq \tau_1 < 1$, *the following inclusions hold*

$$\mathcal{M}(\tau_1, x) \subset \mathcal{M}(\tau_2, x) \subset \mathcal{P}(x) \subset \mathcal{S}(\tau_2, x) \subset \mathcal{S}(\tau_1, x). \tag{5.2}$$

3.

$$\bigcap_{\tau > 0} \mathcal{S}(\tau, x) = \mathcal{P}(x). \tag{5.3}$$

4. If in addition A6 is valid, then $\mathcal{P}(x)^0 \neq \emptyset$ *and*

$$\mathcal{P}(x)^0 \subset \bigcup_{\tau > 0} \mathcal{M}(\tau, x) \subset \mathcal{P}(x). \tag{5.4}$$

Proof.
1. The sets are closed, since U is closed and the level sets of the continuous functions φ, ψ are closed.
2. The inclusions in (5.2) follow directly from the monotonicity in (4.7).

3. Obviously, 2. implies that $\bigcap_{\tau\in(0,1)} \mathcal{S}(\tau,x) \supset \mathcal{P}(x)$. Conversely, if $u \in \bigcap_{\tau\in(0,1)} S(\tau,x)$, then $\varphi(\tau,u,x) \geq \alpha$, for every $\tau \in (0,1)$ and $u \in U$. Furthermore, Theorem 4.7.2 implies $p(u,x) = \lim_{\tau\searrow 0^+} \varphi(\tau,u,x) \geq \alpha$ and thus $u \in \mathcal{P}(x)$. Consequently, the equality $\mathcal{P}(x) = \bigcap_{\tau\in(0,1)} \mathcal{S}(\tau,x)$ holds true.
4. Note that, $\emptyset \neq \mathcal{P}^0 \subset \mathcal{P}(x)^0$ which yields the first part. Moreover, (ii) implies that $\bigcup_{\tau\in(0,1)} \mathcal{M}(\tau,x) \subset \mathcal{P}(x)$. Conversely, since $u \in \mathcal{P}(x)^0$ is nonempty, take some $u \in \mathcal{P}(x)^0$. Since $1-\psi(\tau,u,x) \to p(u,x)$, for $\tau \to 0_+$, there is some $\tau_0 > 0$ such that $1-\psi(\tau,u,x) \geq \alpha$ for all $\tau \in (0,\tau_0)$. Hence, u belongs to the union of all $\mathcal{M}(\tau,x)$, $\tau \in (0,1)$. Consequently, $\{u \in U \mid p(u,x) > \alpha\} \subset \bigcup_{\tau\in(0,1)} \mathcal{M}(\tau,x)$. ☐

Proposition 5.6. *Let A1, A2 be satisfied and $\mathcal{P} \neq \emptyset$. Then*
1. the feasible sets $\mathcal{M}(\tau)$ and $\mathcal{S}(\tau)$ are closed for $\tau \in (0,1)$.
2. For $0 < \tau_2 \leq \tau_1 < 1$, there is the inclusion

$$\mathcal{M}(\tau_1) \subset \mathcal{M}(\tau_2) \subset \mathcal{P} \subset \mathcal{S}(\tau_2) \subset \mathcal{S}(\tau_1) \subset U. \tag{5.5}$$

3.

$$\bigcap_{\tau>0} \mathcal{S}(\tau) = \mathcal{P}. \tag{5.6}$$

4.

$$\mathcal{P}^0 \subset \bigcup_{\tau>0} \mathcal{M}(\tau) \subset \mathcal{P}. \tag{5.7}$$

If A6 is valid then there is some $\tau_0 \in (0,1)$ with $\mathcal{M}(\tau_0) \neq \emptyset$ and

$$\overline{\bigcup_{\tau>0} \mathcal{M}(\tau)} = \mathcal{P}. \tag{5.8}$$

Proof. The claims in 1. and 2. follow from the definition of $\mathcal{M}(\tau)$ and $\mathcal{S}(\tau)$ by the corresponding properties in Lemma 5.5. 1., 2. The statement in 3. follows by exchanging the intersections over $\tau \in (0,1)$ and $x \in D_c$. The claim in 4. needs more discussions, since intersection and union operations cannot be simply exchanged. Lemma 5.5. 4. and simple set-algebraic operations imply that

$$\bigcup_{\tau>0} \mathcal{M}(\tau) = \bigcup_{\tau>0} \bigcap_{x\in D_c} \mathcal{M}(\tau,x) \subset \bigcap_{x\in D_c} \bigcup_{\tau>0} \mathcal{M}(\tau,x) \subset \bigcap_{x\in D_c} \mathcal{P}(x) = \mathcal{P}.$$

Hence, the claim follows if we prove that $\mathcal{P}^0 \subset \bigcup_{\tau>0} \bigcap_{x\in D_c} \mathcal{M}(\tau,x)$, since by A6 we have $\overline{\mathcal{P}^0} = \mathcal{P}$ and (5.8) will be satisfied. Let $\widehat{u}$ be in $\mathcal{P}^0$, then $p(\widehat{u},x) > \alpha$ for all $x \in D_c$. Since p is continuous in x and D_c is compact, $\min_{x\in D_c} p(\widehat{u},x) = \beta > \alpha$. By Corollary 4.8, $1-\psi(\tau,u,x)$ converges uniformly to $p(u,x)$, for $\tau \searrow 0^+$. Hence, there is some $\tau(\varepsilon) \in (0,1)$, for $0 < \varepsilon < \beta - \alpha$, such that $1-\psi(\tau(\varepsilon),\widehat{u},x) \geq \beta - \varepsilon > \alpha$, for all $x \in D_c$. Hence, $\mathcal{M}(\tau_0) \neq \emptyset$, for $\tau_0 = \tau(\varepsilon)$, and $\widehat{u} \in \bigcup_{\tau>0} \bigcap_{x\in D_c} \mathcal{M}(\tau,x)$. ☐

Because of the monotonic properties of the families $\{\mathcal{M}(\tau)\}_{\tau\in(0,1)}$ and $\{\mathcal{S}(\tau)\}_{\tau\in(0,1)}$, they are called inner and outer approximation of the feasible set $\mathcal{P}$, respectively, and Proposition 5.6 is also valid for an arbitrary sequence $\{\tau_k\}_{k\in\mathbb{N}} \subset (0,1)$ which is a non-increasing zero sequence. If $\{\mathcal{M}_k\}_{k\in\mathbb{N}}$ and $\{\mathcal{S}_k\}_{k\in\mathbb{N}}$ are additionally sequences of convex sets (see Thm. 4.10 below), then the approximation sequences can be shown to converge to $\mathcal{P}$ *in the sense of Mosco.*

Definition 5.7. A sequence of closed convex subsets $\{C_k\}_{k\in\mathbb{N}}$ of a reflexive Banach space E, with its dual Banach space E^*, is said to converge to the set $C_0 \subset E$ in the sense of Mosco if and only if the following two conditions are satisfied.

1. For every $u \in C_0$, there is a sequence $\{u_k\}_{k\in\mathbb{N}}$ such that $u_k \in C_k, k \in \mathbb{N}$, and $\lim_{k\to+\infty} \|u_k - u\| = 0$.
2. If $\{u_{n_k}\}$ is any subsequence with $u_{n_k} \in C_{n_k}$ and $v^*(u_{n_k} - u) \to 0$ for all $v \in E^*$, then $u \in C_0$.

Corollary 5.8. *Let $\{\tau_k\}_{k\in\mathbb{N}} \subset (0,1)$ with $\tau_k \searrow 0^+$. If the sequence of feasible sets $\{M_k\}_{k\in\mathbb{N}}$ of (IA$_{\tau_k}$) and $\{S_k\}_{k\in\mathbb{N}}$ (OA$_{\tau_k}$) are convex, then each sequence converges to the feasible set $\mathcal{P}$ in the sense of Mosco, where, in case of the sequence $\{M_k\}_{k\in\mathbb{N}}$, A6 is assumed to hold.*

Proof. According to (5.5), (5.6) and (5.8) in Proposition 5.6, both sequences converge monotonically to $\mathcal{P}$. Therefore, the claim follows by ([51], Lem. 1.2 & Lem. 1.3). □

Theorem 5.9. *Let Assumptions A1, A2, A5 be satisfied, $\{\tau_k\}_{k\in\mathbb{N}} \subset (0,1)$ be an arbitrary decreasing sequence with $\tau_k \searrow 0^+$, and $\mathcal{P} \neq \emptyset$. Then, the following statements hold true.*
1. For each $1 > \tau_1 \geq \tau_2 > 0$,

$$\inf_{u\in\mathcal{M}(\tau_1)} J(u) \geq \inf_{u\in\mathcal{M}(\tau_2)} J(u) \geq \inf_{u\in\mathcal{P}} J(u) \geq \inf_{u\in\mathcal{S}(\tau_2)} J(u) \geq \inf_{u\in\mathcal{S}(\tau_1)} J(u) \geq \inf_{u\in U} J(u), \tag{5.9}$$

where $\inf\emptyset := \infty$.
2. If, for each $k \in \mathbb{N}$, the problem (OA)$_{\tau_k}$ possesses a global solution u_k and $\lim_{k\to\infty} u_k = u^*$, then $u^* \in \mathcal{P}$ and

$$J(u^*) = \min_{u\in\mathcal{P}} J(u) = \lim_{k\to+\infty} \inf_{u\in\mathcal{S}(\tau_k)} J(u). \tag{5.10}$$

3. If A6 is satisfied, then

$$a := \lim_{k\to+\infty} \inf_{u\in M(\tau_k)} J(u) = \inf_{u\in\mathcal{P}} J(u) =: \tilde{J}. \tag{5.11}$$

4. If $M(\tau_k)$ and $\mathcal{S}(\tau_k)$ are convex and J is strongly convex, then the problems (IA)$_{\tau_k}$ and (OA)$_{\tau_k}$ have unique optimum solutions $u^*_{IA}(\tau_k) \in M(\tau_k)$ and $u^*_{OA}(\tau_k) \in S(\tau_k)$, respectively.

Proof.
1. The inclusions in (5.9) are obvious from Proposition 5.6.2.
2. Set $\tilde{J} := \inf_{u\in\mathcal{P}} J(u)$. To verify (5.10) for the outer approximation, note that the set $X := (\{u^*\} \cup \{u_k\}_{k\in\mathbb{N}})$ is compact in $L^2(D)$. From (5.9), it follows that $J(u_k) \leq J(u_{k'}) \leq \lim_{k\to\infty} J(u_k) = J(u^*) \leq \tilde{J}$, for all $k \leq k', k, k' \in \mathbb{N}$.
Assume that $J(u^*) < \tilde{J}$. The uniform convergence on compact sets of $L^2(D) \times D_C$ in Corollary 4.8 yields that

$$\begin{aligned}&\lim_{k\to\infty} \sup_{x\in D_c} |\varphi(\tau_k, u_k, x) - p(u^*, x)| \\ &\leq \lim_{k\to\infty} \left[\sup_{x\in D_c} \sup_{u\in X} |\varphi(\tau_k, u, x) - p(u, x)| + \sup_{x\in D_c} |p(u_k, x) - p(u^*, x)|\right] = 0\end{aligned}$$

and $\varphi(\tau_k, u_k, x) \geq \alpha$, for all $x \in D_c$. Hence, $p(u^*, x) \geq \alpha$, for all $x \in D_c$. This implies that $u^* \in \mathcal{P}$ and $\tilde{J} \leq J(u^*)$. Consequently, the assumption $J(u^*) < \tilde{J}$ leads to the contradiction that $J(u^*) < \tilde{J} \leq J(u^*)$. Therefore, $J(u^*) \geq \tilde{J}$ and, together with (5.9), $J(u^*) = \tilde{J}$.

3. To verify the equality (5.11) for the inner approximation under (**A6**), let $\tau_0 \in (0,1)$ be sufficiently small such that $M(\tau) \neq \emptyset$, for all $\tau \in (0, \tau_0)$. Let $(u_k)_{k\in\mathbb{N}} \subset \mathcal{P}$ be a minimizing sequence such that $\lim_{k\to\infty} J(u_k) =: \tilde{J}$.

Assume $a > \tilde{J}$. By the uniform convergence in Corollary 4.8 and the continuity of J, for each $k \in \mathbb{N}$, there exist some $\tau_k \in (0, \tau_0)$, $v_k \in M(\tau_k)$ and some $\delta(1/k, u_k)$ such that $\|v_k - u_k\| < \delta(1/k, u_k)$ and $|J(v_k) - J(u_k)| < 1/k$. It follows that $\tilde{J} < a \leq J(v_k) < 1/k + J(u_k)$. This leads to the contradiction $\tilde{J} < a \leq \tilde{J}$ in the limit for $k \to +\infty$.
4. Since $(\mathrm{IA})_\tau$ and $(\mathrm{OA})_\tau$ have closed and convex feasible sets and share the same strongly convex objective function as in CCPDE_{red}, the existence of unique optimum solutions $u^*_{IA}(\tau)$ and $u^*_{OA}(\tau)$ follows by the same argument as in the proof of Proposition 3.8. □

Remark 5.10. If the above convexity properties hold, then 2. and 3. are simple consequences of the Mosco convergence of the inner and outer approximations. However, if the convexity cannot be shown, which is mostly the case in applications, then 2. and 3. are important statements for the inner and outer approximations.

The result of [11] becomes useful in order to show the convergence of the solution of the approximation problems IA_k and OA_k (written for brevity instead of $(\mathrm{OA})_{\tau_k}$ and $(\mathrm{IA})_{\tau_k}$, resp.) to the solution of CCPDE.

Theorem 5.11. *(see [11]). Suppose the assumptions A1, A4, A5, A6 hold true, ϕ is a log-concave density function, the objective functional J is quadratic as given in (3.11), and let $\{\tau_k\}_{k\in\mathbb{N}} \subset (0,1)$ be an arbitrary sequence with $\tau_k \searrow 0^+$. Then, for each $k \in \mathbb{N}$, the problem OA_k has a unique solution u^*_k and the sequence $\{u^*_k\}_{k\in\mathbb{N}}$ is norm-convergent to the unique optimal solution u^* of CCPDE. If there is some $k_0 \in \mathbb{N}$ such that the sets $M(\tau_k)$ are convex, for all $k \geq k_0$, then, for each $k \geq k_0$, the problem IA_k has a unique solution $\hat{u}^*_k$ and the sequence $\{u^*_k\}_{k\in\mathbb{N},k\geq k_0}$ is norm-convergent to the unique optimal solution u^* of CCPDE.*

Proof. The convexity of the feasible sets of the outer approximation problem is guaranteed by Theorem 4.10. Furthermore, the Mosco convergence of the feasible sets $\{S(\tau_k)\}_{k\in\mathbb{N}}$ to $\mathcal{P}$ is valid by Corollary 5.8. The same also holds true for the feasible sests $\{M(\tau_k)\}_{k\in\mathbb{N}}$ of the inner approximation if each set $M(\tau_k), k \in \mathbb{N}$, is a convex set. Therefore, the claim follows from [11], Theorem 2.9, since A6 ensures the regularity condition given there. □

Remark 5.12. In Theorem 5.11, the convexity assumption on $M(\tau_k), k \in \mathbb{N}$, may not be guaranteed if $\psi(\tau, \cdot, x)$ is not convex (or quasi-convex). This is the case, for instance, if $\psi(\tau, \cdot, x)$ is defined through the special function of equation (4.4) as indicated in Remark 4.11. Note that, under the assumptions, the convex set $\mathcal{P}$ is the monotonic limit of the sequence $\{M(\tau_k)\}_{k\in\mathbb{N}}$. Therefore, one needs conditions that guarantee the convexity of $M(\tau_k)$ at least for sufficiently large k. This open issue can be addressed, for example, by designing a new parametric function $\Theta(\tau, s)$ or through the concept of eventual convexity (*e.g.*, [63]). Nevertheless, under the assumptions and for the special function (4.4), the outer approximation problem OA_τ is a smoothing convex approximation, while the inner approximation IA_τ is a smoothing approximation to the CCPDE.

The sandwich property of the inner-outer approximation provides a suitable framework for numerical considerations and error estimation. In particular, the inner approximation provides an *a priori* feasible solution, whereby the quality of the this solution is certified by the outer approximation.

6. A numerical example

Temperature distribution processes are widely used in industrial, medical, and biological applications. This section considers the optimal heating of an enclosed, thermally well-insulated, spatial domain $D \subset \mathbb{R}^d$ to a desired temperature y_d. The heat-injection is effected through a distributed stationary heat source. The heat source is assumed to be affected by uncertainties, *e.g.*, due to inaccuracies arising from heating devices and/or inlet heating processes, etc. The enclosed medium is assumed to be homogenous, so that the thermal conductivity κ is spatially constant which is not precisely known and contains uncertainties, *i.e.*, $\kappa(x, \xi) \equiv \kappa(\xi)$. Furthermore, despite the specified overall desired temperature y_d, it should be kept below a maximum allowed value y_c with a high reliability level α in a given subset D_c of D.

The following represents an exemplary chance constrained PDE optimization problem

$$(CCPDE) \qquad \min_u \left\{ E\left[\|y(u, x, \xi) - y_d(x)\|^2_{H^1_0(D)} \right] + \frac{\rho}{2}\|u\|^2_{L^2(D)} \right\} \tag{6.1}$$

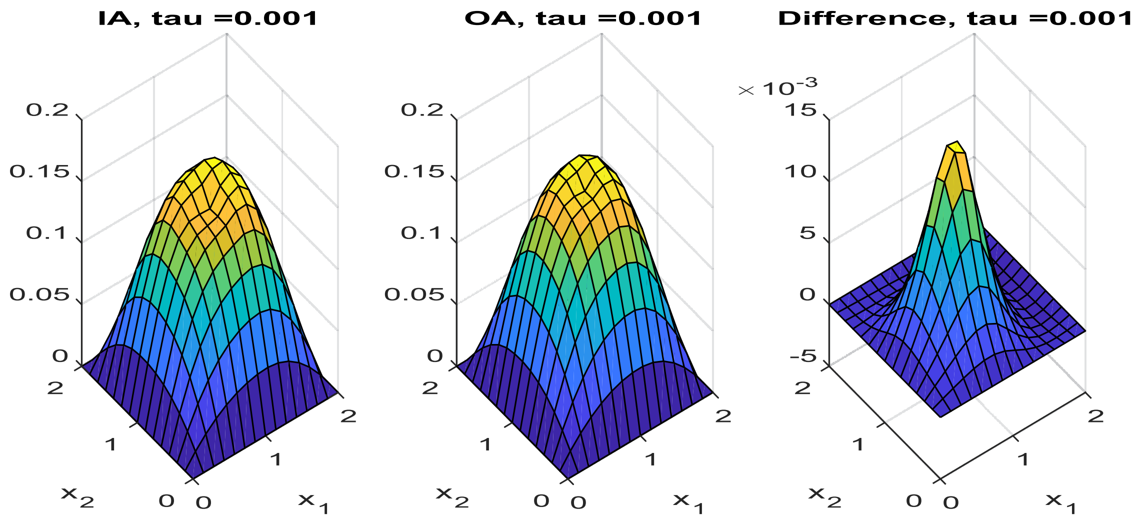

FIGURE 1. Solutions of the inner-outer approximation methods, $\tau = 0.001$.

$$\text{s.t.} - \nabla \cdot (\kappa(\xi)\nabla y) = f(u, x, \xi), \text{ for } x \in D, \ \xi \in \Omega \text{ a.s. }, \quad (6.2)$$

$$y|_{\partial D} = 0, \xi \in \Omega \text{ a.s. }, \quad (6.3)$$

$$Pr\{y(u, x, \xi) \leq y_c(x)\} \geq \alpha, \text{ on } D_c, \quad (6.4)$$

$$u \in U, \quad (6.5)$$

where $D = [0, 2] \times [0, 2], D_c = [0, 1] \times [0, 1]$, $y_d = 0.2 \cdot \left|\sin\left(\frac{\pi}{2}x\right)\right|$, regularization parameter $\rho = 10^{-3}$, $u_{\min} = -5$, $u_{\max} = 5$, $y_{\max}(x) \equiv y_c = 0.15$.

Here, $\kappa(\xi) = 0.1 \cdot \left(\exp(6) - \exp\left(\sum_{j=1}^{4} \xi_j\right)\right)$ and the forcing term is given by $f(u, x, \xi) = u + \xi_5 \sin(x_2) + \xi_6 \cos(x_1)$, where $\xi^\top = (\xi_1, \ldots, \xi_6)$ is a vector of independently, identically Beta-distributed random variables with $\xi \in [0, 1]^6$ and parameters $a = 2$ and $b = 2$. The reliability level is given to be $\alpha = 0.95$. This problem is identical to the problem (2.1)–(2.5), except that it is here defined on D_c.

Here, we verify assumptions A5 and A6, since the remaining assumptions are clear from the problem (6.1)–(6.5). By (3.20) the chance constraints (6.4) are equivalent to

$$Pr\{\beta(u)(x) + \gamma(x_2)\xi_5 + \omega(x_2)\xi_6 + 0.1[-\exp(6) + \exp(\sum_{j=1}^{4} \xi_j)] \leq 0\} \geq \alpha, \text{ on } [0, 1]^2,$$

where β is a linear and continuous operator from $L_2(D)$ in $H^2(D) \cap H_0^1(D)$, and γ, ω are continuous functions in their arguments. For each fixed $u \in L_2(D)$ and fixed given $x \in [0, 1]^2$, the given function in ξ on the left hand side of the inequality is smooth and the gradient w.r.t. ξ is different from zero for the first four coordinates. Hence, the measure zero property A5 is satisfied. Because of Proposition 3.8 there exists for each $\alpha \in (0, 1]$ a unique solution u_α. Hence, in our example with $\alpha = 0.95 < 1$ the solution $u_{0.95}$ satisfies the slater condition of Lemma 5.3 for our example. Together with Corollary 4.8 and the conclusion of Lemma 5.3 the assumption A6 is satisfied. Naturally for robust optimization, *i.e.* $\alpha = 1$, assumption A6 cannot be satisfied and the inner approximation does not exists for each $\tau \in (0, 1)$.

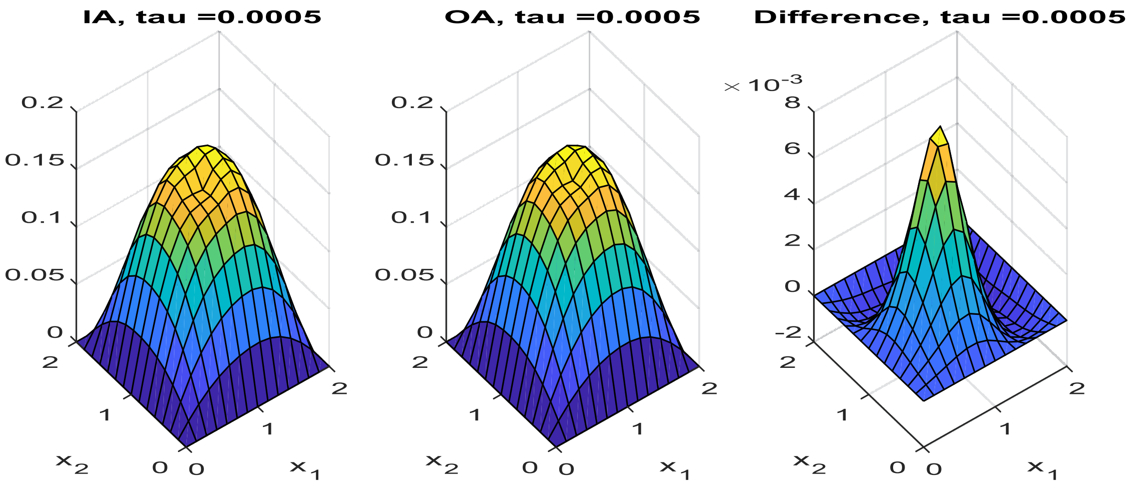

FIGURE 2. Solutions of the inner-outer approximation method, $\tau = 0.0005$.

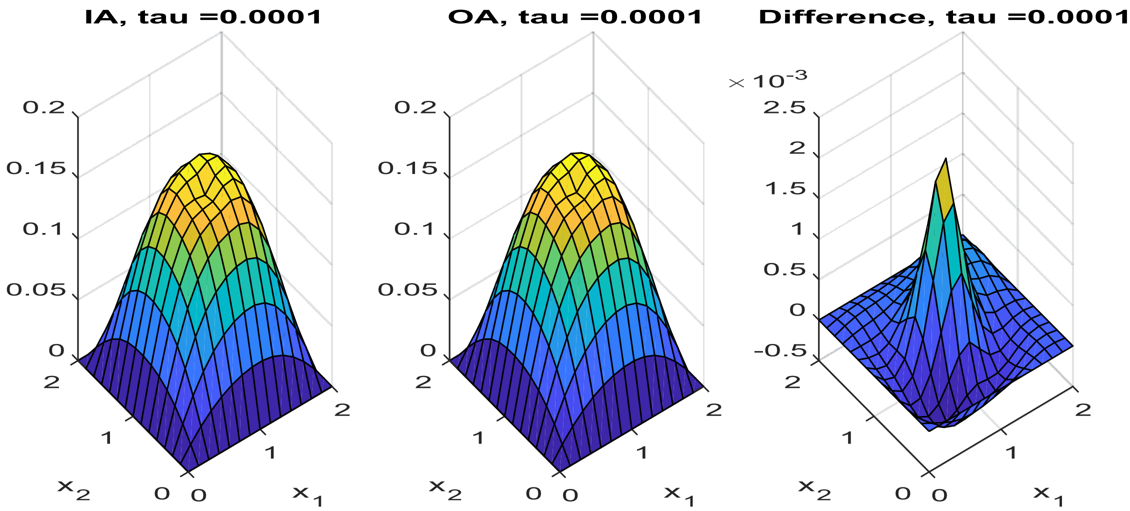

FIGURE 3. Solutions of the inner-outer approximation method, $\tau = 0.00001$.

6.1. Computation results

The problem (6.1)–(6.5) is approximately solved by using the proposed smoothing inner-outer approximation approach as described in [26] for the finite dimensional chance constrained problem which arizes after using finite difference discretization of the PDE. For the discretization we use $15 \times 15 = 225$ points along with 100 quasi-Monte Carlo samples for the beta(2,2)-distributed random numbers ξ. The infinite number of chance constraints is reduced to the corresponding chosen finite number of points belonging to D_c. The constant values $m_1 = 1.4$ and $m_2 = 1$ are used in the parametric functions $\psi(\tau, \cdot)$ and $\varphi(\tau, \cdot)$ of equation (4.5).

The computation of integrals and gradients w.r.t. u for the objective and the smoothing functions is done through Quasi-Monte Carlo sampling. The example above is only for demonstration purpose in order to show that the inner-outer approximation approach is viable and provides meaningful results. Otherwise, as τ decreases the number of QMC samples need to be adjusted. For instance, the ξ discretization can be chosen coarser far from $g(u, x, \xi) = 0$ and can be refined near to $g(u, x, \xi) = 0$ (see [37] for such considerations). In general, the parameter τ cannot be selected quite small, since the computation of the derivative of the approximations

FIGURE 4. Optimal objective function values of the inner and outer approximations for decreasing values of τ.

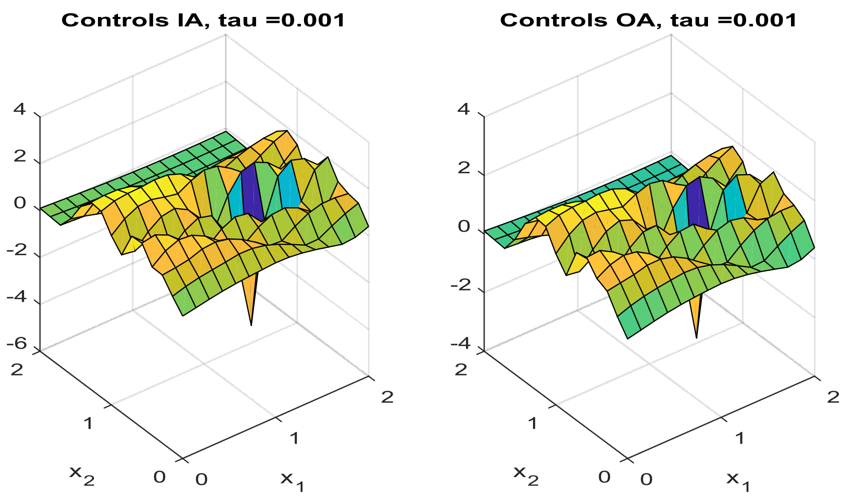

FIGURE 5. The optimal controls for the inner and outer approximations for $\tau = 0.001$.

is numerically unstable (ill-posed) whenever τ goes to zero. If the integration error caused by this instability becomes larger than the difference of the approximated values of the inner and outer approximation, then τ cannot be further reduced.

Now we illustrate the computational results for one chosen discretization of the PDE and three decreasing values of τ. For a simple illustration of the results we present the expected value of y in the Figures 5–7. The corresponding optimization problems are solved using the function `fmincon` from the Matlab Optimization Toolbox coupled with a reduced space method. All the computations were carried out on an Intel Core i7-2600 running at 3.4 HZ and with 16 GB of RAM.

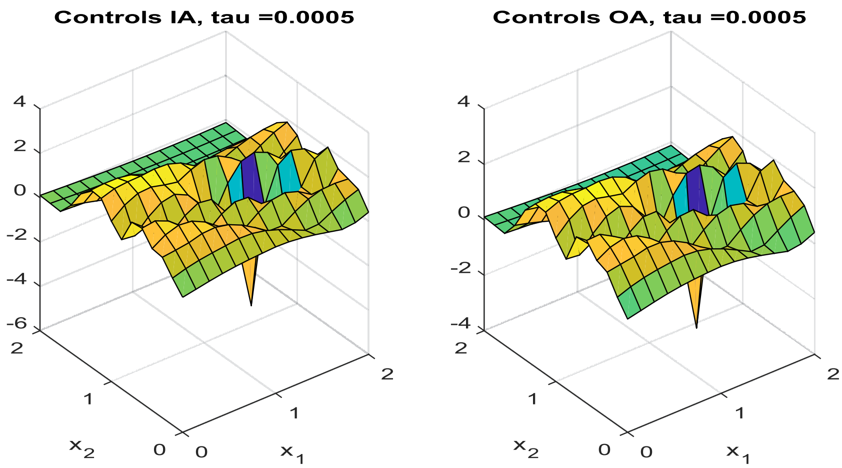

FIGURE 6. The optimal controls for the inner and outer approximations for $\tau = 0.0005$.

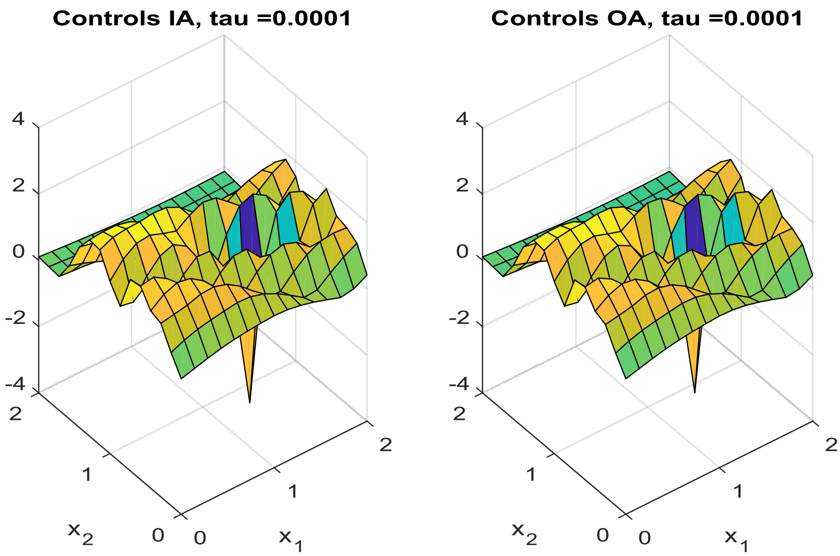

FIGURE 7. The optimal controls for the inner and outer approximations for $\tau = 0.0001$.

The left hand-side of Figure 1 depicts $E\left[y^*_{IA}(u,x,\xi)\right]$, the middle figure presents $E\left[y^*_{OA}(u,x,\xi)\right]$, while the figure on the right displays $E\left[y^*_{IA}(u,x,\xi) - y^*_{IA}(u,x,\xi)\right]$ which is the expected difference between the solutions $y^*_{IA}(u,x,\xi)$ and $y^*_{OA}(u,x,\xi)$ of the inner and outer approximations, respectively. Observe that, for $\tau = 10^{-3}$, the maximum expected error between the inner and outer approximate solutions lies below 15×10^{-3}. In Figure 2, the maximum expected error $E\left[y^*_{IA}(u,x,\xi) - y^*_{OA}(u,x,\xi)\right]$ is reduced to almost 8×10^{-3} for $\tau = 5 \cdot 10^{-3}$. For $\tau = 10^{-4}$, Figure 3 shows that the expected error $E\left[y^*_{IA}(u,x,\xi) - y^*_{OA}(u,x,\xi)\right]$ lies below 2.5×10^{-3}.

The value of the objective function for IA_τ is always greater than that of the corresponding problem OA_τ. As can be seen in Figure 4, as τ decreases, the optimal values of the objective functions of the inner and outer approximations get closer. The parameter value $\tau = 10^{-4}$ leads to the inner and outer approximations to provide an approximate solution well close to the problem (6.1)–(6.5).

Furthermore, the optimal controls are given in Figures 5–7, corresponding to the depicted state variables and the chosen τ values. Specifically, in Figure 7, the optimal controls of the inner and outer approximation are almost identical for $\tau = 10^{-4}$.

In summary, the example above in for demonstration purpose and indicates the viability of solving CCPDE problems through smoothing inner-outer approximations coupled with a finite dimensional approximation of the PDE system. In this respect, the results attained are reasonable. However, detailed numerical analysis and convergence studies of the finite-dimensional solutions to the solutions of infinite-dimensional CCPDE problems are still open.

7. Conclusions

Chance constrained optimization of elliptic PDE (CCPDE) is studied in this paper. Under standard assumptions, such problems are shown to be convex in the respective Hilbert space. Nevertheless, despite the presence of convexity, chance constrained problems are known to be very difficult to solve. In addition, the knowledge of differentiability of chance constraints generally entails little computational advantages. Therefore, this work uses an inner-outer smoothing approximation for the solution of CCPDE. The smoothing problems are defined based on a parametric function introduced in [27] and studied in more detail in [26]. Consequently, the feasible sets of the inner and the outer approximation are always subsets and supersets, respectively, of the feasible set of the CCPDE problem. Furthermore, each of these feasible sets are shown to converge to the feasible set of CCPDE with respect to the approximation parameter. In addition, for the outer approximation problem, the convergence of the feasible sets is shown to be in the sense of Mosco. The simple numerical example in Section 6 demonstrates the viability of the proposed approach.

The parametric function $\Theta(\tau, s)$ in Corollary 4.3 is capable of providing only a smoothing convex outer-approximation, while it does not automatically provide a convex inner-approximation problem. Nevertheless, the feasible sets of the inner-approximation converge to a convex set. Therefore, in a future work we will address the issue of convexity of the inner-approximation, for instance, by introducing a new parametric function as well as by extending the concept of eventual convexity. Furthermore, convergence analysis of the numerical solutions of discretized CCPDE to a solution of the origional CCPDE is an open issue which will be addressed in a future work.

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