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SPECTRA OF OPERATOR PENCILS WITH SMALL $\mathcal{PT}$-SYMMETRIC PERIODIC PERTURBATION

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Abstract. We study the spectrum of a quadratic operator pencil with a small $\mathcal{PT}$-symmetric periodic potential and a fixed localized potential. We show that the continuous spectrum has a band structure with bands on the imaginary axis separated by usual gaps, while on the real axis, there are no gaps but at certain points, the bands bifurcate into small parabolas in the complex plane. We study the isolated eigenvalues converging to the continuous spectrum. We show that they can emerge only in the aforementioned gaps or in the vicinities of the small parabolas, at most two isolated eigenvalues in each case. We establish sufficient conditions for the existence and absence of such eigenvalues. In the case of the existence, we prove that these eigenvalues depend analytically on a small parameter and we find the leading terms of their Taylor expansions. It is shown that the mechanism of the eigenvalue emergence is different from that for small localized perturbations studied in many previous works.

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1. Introduction

It is well-known that adding a small localized perturbation to a differential operator can generate isolated eigenvalues emerging from the edges of the continuous spectrum. This phenomenon was studied in all details for many models and cases, including very simple ones like Schrödinger operators perturbed by localized potentials, see, for instance, [23, 25, 36, 41] rather complicated models describing various perturbations of quantum waveguides like in [3, 7, 12, 17, 18, 20, 22, 29–31] and even perturbed non-self-adjoint operators [5, 6, 9, 10, 15]. The main features of these models is that the localized perturbation keeps the continuous spectrum unchanged and can generate only isolated eigenvalues (or resonances) near its edges. A case of a small periodic perturbation was much less studied. Of course, there is a simple case of a Schrödinger operator with a small periodic potential and with no other potential. The continuous spectrum of such model is calculated easily via the standard Floquet-Bloch theory since the latter leads one to a regular perturbation of an appropriate operator on a periodicity cell. This was done, for instance, in [4]; a more complicated case of a quasi-periodic potential was successfully

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treated in [32]. A more interesting problem is to consider operators with a small periodic potential and a fixed localized potential, for instance,

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+V+\varepsilon\gamma \quad \text{in} \quad L_2(\mathbb{R}), \tag{1.1}$$

where V is a localized potential, γ is periodic, and ε is a small parameter. The continuous spectrum of this operator is independent of V and in general, it has small gaps. The localized potential can generate isolated eigenvalues in these gaps and it is an interesting issue to study the existence and the behavior of such eigenvalues. To the best of the authors' knowledge, this was done only in [4] in the case when the periodic potential $\varepsilon\gamma$ is replaced by a fast oscillating potential. It was shown that such operator had small gaps in the continuous spectrum running to infinity and in certain cases, there could be isolated eigenvalues inside these gaps appearing exactly due to the localized potential.

Our interest to the operators with a small periodic potentials is stimulated by numerous studies of linear and nonlinear $\mathcal{PT}$-symmetric equations motivated in particular by interesting applications in modern optics and recent studies of non-self-adjoint linear operators, see, for instance, [1, 9, 11, 13–16, 19, 21, 26–28, 33–35, 37, 40]. In particular, papers [9, 15, 33, 34] were devoted to studying a perturbed one-dimensional Klein-Gordon equation

$$u_{tt}-u_{xx}+\varepsilon\gamma(x)u_t+F(u)=0, \qquad t>0, \quad x\in\mathbb{R}. \tag{1.2}$$

The perturbative viscous-like term $\varepsilon\gamma u_t$ describes loss and gain in the system and these gain and loss are balanced in the sense that the function γ is odd. The system itself is parity-time-symmetric, namely, the change of variables $x\mapsto -x$, $t\mapsto -t$ keeps equation (1.2) unchanged. The function F describes a non-linearity in the equation and it is assumed that the equation

$$-v_{xx}+F(v)=0, \qquad x\in\mathbb{R},$$

has a static kink solution $v_{kink}=v_{kink}(x)$ obeying the identity $\lim\limits_{x\to\pm\infty}F'(v_{kink}(x))=\varkappa$, where $\varkappa\in\mathbb{R}$ is some constant. The issue on stability of this kink, namely, on considering a small dynamical perturbation in the form $u(x,t)=v_{kink}(x)+e^{i\lambda t}\psi(x)$, $\lambda\in\mathbb{C}$, and linearizing then equation (1.2) for small ψ, gives rise to the operator pencil

$$-\psi_{xx}+V+\varkappa+(i\varepsilon\lambda\gamma-\lambda^2)\psi=0, \qquad x\in\mathbb{R}, \tag{1.3}$$

where potential V is defined as $V:=F'(v_{kink})-\varkappa$ and it is localized in the sense that it vanishes at infinity. This rises a natural question on the structure of the spectrum of the latter operator pencil. The case of a localized function γ was studied analytically in [15] and in [9] for a two-dimensional operator, when the second derivative was replaced by the Laplacian. The case, when γ is a periodic function, was not studied rigorously. At the same time, the numerical studies for equation (1.2) with a periodic function γ indicated many interesting phenomena and this is a good motivation for studying the structure of the spectrum of the operator pencil in (1.3).

The operator pencil in (1.3) probably looks quite specific and if we are interested in the main phenomena we could simplify the problem replacing it by studying the spectrum of the operator

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+V+i\varepsilon\gamma \quad \text{in} \quad L_2(\mathbb{R}). \tag{1.4}$$

In a pure periodic case, that is, as $V=0$, such operators were studied, for instance, in [1, 11, 35, 38, 39]. It was shown that the continuous spectrum can bifurcate into the complex plane [35]. The mechanism of the appearance of the complex-valued spectrum was studied quite well [38, 39]. The cited works also provided

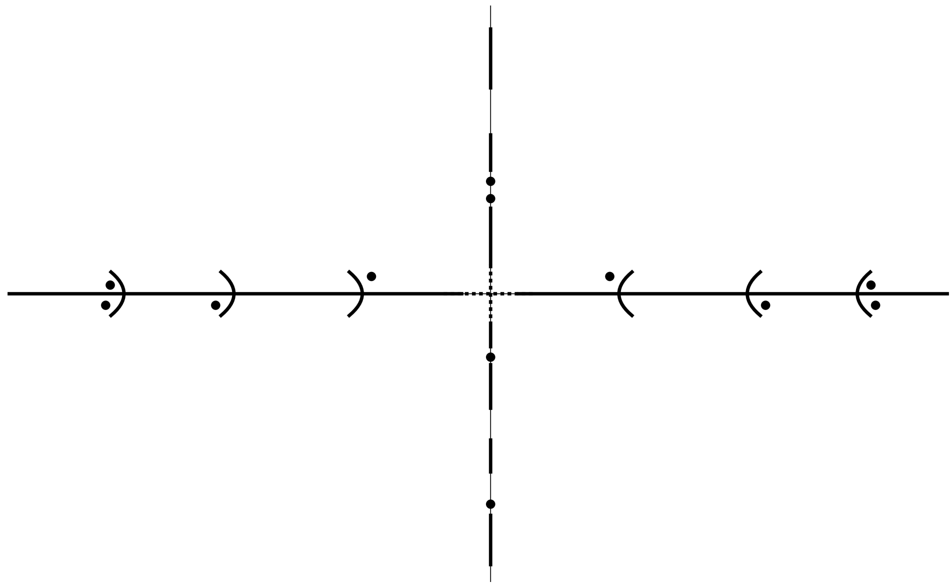

FIGURE 1. Approximate shape of the continuous spectrum and possible isolated eigenvalues converging to the continuous spectrum. The dotted lines at zero indicate that here the structure of the spectrum requires further studies not made in the present paper.

conditions ensuring the reality of a considered zone in the continuous spectrum. To the best of the authors' knowledge, operator (1.4) was not studied before for $V \neq 0$.

The above overview of the current state-of-art motivates the study of spectra of Schrödinger operators with a small periodic potential and a fixed non-zero localized potential. As an appropriate model, we can choose one of operators (1.1), (1.4) or the operator pencil in (1.3). In this paper our choice is the operator pencil in (1.3); there are several reasons for this. The first is that this operator pencil arose in recent studies in nonlinear $\mathcal{PT}$-symmetric equations. The second reason is that a small $\mathcal{PT}$-symmetric potential does make the continuous spectrum to bifurcate into the complex plane. And in contrast to the usual spectral problem for operator (1.4), the continuous spectrum of the operator pencil in (1.3) has a richer structure. The third reason is that our technique can be easily adapted also for studying operators (1.1), (1.4). So, it is natural to study the problem with most interesting and richest results, that is, the spectral problem for operator pencil in (1.3); let us denote it by $\mathcal{H}_\varepsilon(\lambda)$.

Our main results are as follows. We study in details the spectrum of $\mathcal{H}_\varepsilon(\lambda)$. We show that the residual spectrum is empty and there is only the continuous and point components. We prove that the spectrum of $\mathcal{H}_\varepsilon(\lambda)$ converge to that of the limiting operator pencil $\mathcal{H}_0(\lambda)$ obtained by letting $\varepsilon = 0$. The continuous spectrum of the unperturbed operator pencil is the union of two rays on the real line and it can also contain a finite segment on the imaginary axis. We find that the perturbation by $i\varepsilon\gamma$ creates small usual gaps but on the imaginary axis, while on the real axis, the continuous spectrum bifurcates into the complex plane as small parabolas opening no gaps, see Figure 1. The most part of the paper is devoted to the isolated eigenvalues converging to the continuous spectrum as $\varepsilon \to +0$. We show that these eigenvalue can emerge only in the aforementioned small gaps or near small parabolas. We provide sufficient conditions ensuring the existence of such eigenvalues as well as conditions guaranteeing the absence. It turns out that there can be at most two eigenvalues in each gap and in the vicinity of each small parabola, see Figure 1. Their existence is determined by the scattering data of the operator $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V$ at certain points and by the potential γ in a highly non-trivial way. The emerging eigenvalues are shown to be *holomorphic* on ε and we calculate explicitly the leading terms in their Taylor expansions.

In conclusion we also note that the considered phenomena are in a sense one-dimensional. Namely, for multi-dimensional operator the bands in the spectrum usually overlap and it is rather exceptional to have two bands with a common end point. A small periodic perturbation can open a small gap only at such end points. Once such points are absent, we have no small gaps and no isolated eigenvalues in these gaps. Rare examples of multi-dimensional operators, in whose continuous spectra small periodic perturbation can open small gaps, are operators in waveguide structures, see, for instance [8]. But the mechanism of the gaps opening in these models is again one-dimensional. The mechanism of the eigenvalues emergence also employs the features of the one-dimensional scattering. This is one more motivation for an independent study of the one-dimensional case presented in this paper.

2. Problem and main results

We introduce the self-adjoint Schrödinger operator

$$\mathcal{H}_0 := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V,$$

in $L_2(\mathbb{R})$ on domain $H^2(\mathbb{R})$. Here $V = V(x)$, $x \in \mathbb{R}$, is a real piecewise continuous function decaying at infinity: $|V(x)| \leqslant Ce^{-\vartheta|x|}$, $x \in \mathbb{R}$, where C, ϑ are some positive constants independent of x. The main object of our study is the quadratic operator pencil

$$\mathcal{H}_\varepsilon(\lambda) := \mathcal{H}_0 + i\varepsilon\lambda\gamma + \varkappa - \lambda^2 \quad \text{in} \quad L_2(\mathbb{R}).$$

Here $\gamma = \gamma(x)$, $x \in \mathbb{R}$, is a real odd 2π-periodic piecewise continuous function, $\varkappa \in \mathbb{R}$ is a fixed constant. The domain of $\mathcal{H}_\varepsilon(\lambda)$ is the Sobolev space $H^2(\mathbb{R})$. In this paper we are interested in the behavior of the spectrum of operator pencil $\mathcal{H}_\varepsilon(\lambda)$ as $\varepsilon \to +0$.

We define the resolvent set of $\mathcal{H}_\varepsilon(\lambda)$ as the set of λ such that there exists a bounded inverse operator $\mathcal{H}_\varepsilon^{-1}(\lambda)$ in $L_2(\mathbb{R})$. The spectrum $\sigma(\mathcal{H}_\varepsilon)$ of $\mathcal{H}_\varepsilon(\lambda)$ is the complement to the resolvent set. The point spectrum $\sigma_{\mathrm{p}}(\mathcal{H}_\varepsilon)$ is introduced as the set of eigenvalues, and an eigenvalue of $\mathcal{H}_\varepsilon(\lambda)$ is a number $\lambda \in \mathbb{C}$ such that the equation $\mathcal{H}_\varepsilon(\lambda)\psi = 0$ has a non-trivial solution called an eigenfunction. The continuous spectrum $\sigma_{\mathrm{c}}(\mathcal{H}_\varepsilon)$ is defined in terms of the characteristic sequences. Namely, $\lambda \in \sigma_{\mathrm{c}}(\mathcal{H}_\varepsilon)$ if there exists a sequence $\psi_n \in H^2(\mathbb{R})$, which is bounded and non-compact in $L_2(\mathbb{R})$ and $\mathcal{H}_\varepsilon(\lambda)\psi_n \to 0$ in $L_2(\mathbb{R})$ as $n \to \infty$. The residual spectrum $\sigma_{\mathrm{r}}(\mathcal{H}_\varepsilon)$ is defined as

$$\sigma_{\mathrm{r}}(\mathcal{H}_\varepsilon) := \sigma(\mathcal{H}_\varepsilon) \setminus \big(\sigma_{\mathrm{c}}(\mathcal{H}_\varepsilon) \cup \sigma_{\mathrm{p}}(\mathcal{H}_\varepsilon)\big).$$

Let $\mathcal{T}$ be the operator of complex conjugation in $L_2(\mathbb{R})$: $\mathcal{T}u = \overline{u}$. Then it is obvious that the operator pencil $\mathcal{H}_\varepsilon(\lambda)$ is $\mathcal{T}$-self-adjoint in the following sense:

$$(\mathcal{H}_\varepsilon(\lambda))^* = \mathcal{H}_\varepsilon(-\overline{\lambda}), \quad \mathcal{T}(\mathcal{H}_\varepsilon(\lambda))^* = \mathcal{H}_\varepsilon(\overline{\lambda})\mathcal{T}.$$

In particular, it implies immediately that $\big(\mathcal{H}_\varepsilon(-\overline{\lambda})\big)^{-1} = \mathcal{T}\big(\mathcal{H}_\varepsilon(\lambda)\big)^{-1}\mathcal{T}$ and

$$\big(\sigma(\mathcal{H}_\varepsilon)\big)^\dagger = -\sigma(\mathcal{H}_\varepsilon), \quad \big(\sigma_{\mathrm{p}}(\mathcal{H}_\varepsilon)\big)^\dagger = -\sigma_{\mathrm{p}}(\mathcal{H}_\varepsilon), \quad \big(\sigma_{\mathrm{c}}(\mathcal{H}_\varepsilon)\big)^\dagger = -\sigma_{\mathrm{c}}(\mathcal{H}_\varepsilon), \tag{2.1}$$

where the superscript $\dagger$ denotes the complex conjugation of a set: $M^\dagger := \{\overline{\lambda} :\ \lambda \in M\}$, and we also denote $-M := \{-\lambda :\ \lambda \in M\}$. If, in addition, the function V is even, then

$$\mathcal{P}(\mathcal{H}_\varepsilon(\lambda))^* = \mathcal{H}_\varepsilon(\overline{\lambda})\mathcal{P}, \quad \mathcal{P}\mathcal{T}\mathcal{H}_\varepsilon(\lambda) = \mathcal{H}_\varepsilon(\overline{\lambda})\mathcal{P}\mathcal{T}, \quad \big(\mathcal{H}_\varepsilon(\overline{\lambda})\big)^{-1} = \mathcal{P}\mathcal{T}\big(\mathcal{H}_\varepsilon(\lambda)\big)^{-1}\mathcal{P}\mathcal{T},$$

where $(\mathcal{P}u) := u(-x)$, and in this case

$$\big(\sigma(\mathcal{H}_\varepsilon)\big)^\dagger = \sigma(\mathcal{H}_\varepsilon), \quad \big(\sigma_{\mathrm{p}}(\mathcal{H}_\varepsilon)\big)^\dagger = \sigma_{\mathrm{p}}(\mathcal{H}_\varepsilon), \quad \big(\sigma_{\mathrm{c}}(\mathcal{H}_\varepsilon)\big)^\dagger = \sigma_{\mathrm{c}}(\mathcal{H}_\varepsilon). \tag{2.2}$$

We stress that we do not assume that V is even and our results are true for a general function V obeying the aforementioned conditions.

To formulate our main results, we first describe the spectrum of the operator $\mathcal{H}_0$. Thanks to the assumptions for V, this is a self-adjoint lower semi-bounded operator in $L_2(\mathbb{R})$. Its residual spectrum is empty, the continuous spectrum is $\sigma_{\mathrm{c}}(\mathcal{H}_0) = [0, +\infty)$ since the potential $V(x)$ tends to zero as $x \to \pm\infty$. Since the potential V decays exponentially fast at infinity, the discrete spectrum $\sigma_{\mathrm{p}}(\mathcal{H}_0)$ consists of finitely many simple discrete negative eigenvalues. Hence, the residual spectrum of the operator pencil $\mathcal{H}_0 + \varkappa - \lambda^2$ is empty and the discrete spectrum consists of finitely many simple isolated eigenvalues.

The continuous spectrum of $\mathcal{H}_0 + \varkappa - \lambda^2$ is a pair of curves $\pm\sqrt{\varkappa + t}$, $t \in [0, +\infty)$. If $\varkappa \geqslant 0$, these are just two half-lines $\sigma_{\mathrm{c}}(\mathcal{H}_0 + \varkappa - \lambda^2) = (-\infty, -\sqrt{\varkappa}] \cup [\sqrt{\varkappa}, +\infty)$. As $\varkappa < 0$, the continuous spectrum is a cross in the complex plane:

$$\sigma_{\mathrm{c}}(\mathcal{H}_0 + \varkappa - \lambda^2) = \Big\{\lambda = it : \ -\sqrt{|\varkappa|} \leqslant t \leqslant \sqrt{|\varkappa|}\Big\} \cup \mathbb{R}.$$

Now we are in position to formulate our first main result.

Theorem 2.1. *The residual spectrum of $\mathcal{H}_\varepsilon(\lambda)$ is empty. The spectrum of $\mathcal{H}_\varepsilon(\lambda)$ satisfies the relation*

$$\sigma(\mathcal{H}_\varepsilon) \subseteq \left\{\lambda \in \mathbb{C} : \ \operatorname{dist}(\lambda^2 - \varkappa, \sigma(\mathcal{H}_0)) \leqslant \varepsilon|\lambda| \sup_{[-\pi,\pi]} |\gamma| \right\}. \tag{2.3}$$

By $B_r(a)$ we denote the ball $B_r(a) := \{\lambda : |\lambda - a| < r\}$ and we let $\lambda_0^{(m)} := \sqrt{\frac{m^2}{4} + \varkappa}\operatorname{sign} m$, $m \in \mathbb{Z} \setminus \{0\}$. Our next result describes the continuous spectrum of $\mathcal{H}_\varepsilon(\lambda)$.

Theorem 2.2. *For each $R > 0$ there exists $\varepsilon_0 > 0$ and $c > 0$ such that for all $0 < \varepsilon \leqslant \varepsilon_0$ the continuous spectrum of $\mathcal{H}_\varepsilon(\lambda)$ located inside the ball $B_R(0)$ is as follows. Outside the balls $B_{c\varepsilon}(\lambda_0^{(m)})$, $m \in \mathbb{Z}_+$, and $B_{c\varepsilon}(0)$, the continuous spectrum of $\mathcal{H}_\varepsilon(\lambda)$ coincides with that of $\mathcal{H}_0 + \varkappa - \lambda^2$. Inside the balls $B_{c\varepsilon}(\lambda_0^{(m)})$, $\lambda_0^{(m)} \neq 0$, $\lambda_0^{(m)} \in \mathbb{R}$, the spectrum is a continuous arc in a complex plane with the following approximate parametric description:*

$$\begin{aligned}
\lambda = \lambda_0^{(m)} &\pm \frac{i\varepsilon}{2\lambda_0^{(m)}}\sqrt{(\lambda_0^{(m)})^2\alpha_0^2(m) - s^2m^2 + O(\varepsilon^2)} \\
&+ \frac{\varepsilon^2}{4\lambda_0^{(m)}}\left(\frac{2\varkappa s^2}{(\lambda_0^{(m)})^2} - \frac{\alpha_0^2(m)}{2} + (\lambda_0^{(m)})^2\alpha_1(m)\right) + O(\varepsilon^3),
\end{aligned} \tag{2.4}$$

$$\alpha_0(m) := \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} \gamma(x)\sin mx \,\mathrm{d}x,$$

$$\alpha_1(m) := \frac{1}{\sqrt{\pi}}\int\limits_{-\pi}^{\pi} \gamma(x)\left(u_+(x,m)\cos\frac{mx}{2} + u_-(x,m)\sin\frac{mx}{2}\right)\mathrm{d}x,$$

where $m\in\mathbb{Z}_+$ and $u_\pm$ are the solutions to the boundary value problems

$$\begin{aligned}&\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}-\frac{m^2}{4}\right)u_+(x,m)=\frac{1}{\sqrt{\pi}}\left(\gamma(x)\cos\frac{m}{2}x-\alpha_0(m)\sin\frac{m}{2}x\right),\quad x\in(-\pi,\pi),\\&\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}-\frac{m^2}{4}\right)u_-(x,m)=\frac{1}{\sqrt{\pi}}\left(\gamma(x)\sin\frac{m}{2}x-\alpha_0(m)\cos\frac{m}{2}x\right),\quad x\in(-\pi,\pi),\\&u_\pm(-\pi,m)=(-1)^m u_\pm(\pi,m),\qquad \frac{\mathrm{d}u_\pm}{\mathrm{d}x}(-\pi,m)=(-1)^m\frac{\mathrm{d}u_\pm}{\mathrm{d}x}(\pi,m),\end{aligned}\tag{2.5}$$

orthogonal to $\cos\frac{m}{2}x$ and $\sin\frac{m}{2}x$ in $L_2(-\pi,\pi)$. Inside the balls $B_{c\varepsilon}(\lambda_0^{(m)})$ with a non-zero pure imaginary $\lambda_0^{(m)}\neq 0$, the continuous spectrum of $\mathcal{H}_\varepsilon(\lambda)$ is pure imaginary. If $\alpha_0(m)\neq 0$, this part of the continuous spectrum contains a gap $\big(\beta_m^-(\varepsilon),\beta_m^+(\varepsilon)\big)$ with the end-points obeying the asymptotics:

$$\beta_m^\pm(\varepsilon)=\lambda_0^{(m)}\pm\frac{i\varepsilon|\alpha_0(m)|}{2}-\frac{\varepsilon^2}{4\lambda_0^{(m)}}\left(\frac{\alpha_0^2(m)}{2}+|\lambda_0^{(m)}|^2\alpha_1(m)\right)+O(\varepsilon^3).\tag{2.6}$$

The operator pencil $\mathcal{H}_\varepsilon(\lambda)$ can also have eigenvalues converging to the continuous spectrum as $\varepsilon\to+0$. These eigenvalues emerge only as $\varepsilon>0$ and in what follows they are referred to as emerging. Our next main result describes such eigenvalues. In order to formulate this result, we first introduce additional notations.

Given $k\in\mathbb{C}\setminus\{0\}$ with $|\operatorname{Im}k|<\frac{\vartheta}{6}$, we define the Jost functions $Y_1=Y_1(x,k)$, $Y_2=Y_2(x,k)$, $x\in\mathbb{R}$, as the solutions to the equation

$$-Y''+VY-k^2Y=0,\quad x\in\mathbb{R},\qquad Y_\pm(x,k)=e^{\pm ikx}+O\big(e^{-\frac{\vartheta}{3}|x|}\big),\quad x\to\pm\infty.\tag{2.7}$$

The functions Y_1, Y_2 are related by the identities ([2], Chap. 2, Sect. 2.6.5, Eqs. (6.61), (6.67”)):

$$Y_\pm(x,k)=\mp b(\mp k)Y_\mp(x,k)+a(k)Y_\mp(x,-k),\tag{2.8}$$

where $a(k)$, $b(k)$ are the transmission and reflection coefficients, respectively. As $k\in\mathbb{R}\setminus\{0\}$, these coefficients satisfy the identities

$$a(-k)=\overline{a(k)},\qquad b(-k)=\overline{b(k)},\qquad |a(k)|^2-|b(k)|^2=1.\tag{2.9}$$

For real k we represent $a(k)$ as $a(k)=|a(k)|e^{i\theta(k)}$, $0\leqslant\theta(k)<2\pi$, and we denote $a_{\mathrm{r}}(k):=\operatorname{Re}a(k)$, $a_{\mathrm{i}}(k):=\operatorname{Im}a(k)$, $b_{\mathrm{r}}(k):=\operatorname{Re}b(k)$, $b_{\mathrm{i}}(k):=\operatorname{Im}b(k)$.

We fix $m\in\mathbb{Z}\setminus\{0\}$ and by $\zeta_\pm$ we denote the roots of the equation

$$\left|a\left(\tfrac{m}{2}\right)\right|\cos\left(2\zeta+\theta\left(\tfrac{m}{2}\right)\right)-b_r\left(\tfrac{m}{2}\right)=0,\quad \zeta_\pm=-\frac{1}{2}\left(\theta\left(\frac{m}{2}\right)\pm\arccos\frac{b_{\mathrm{r}}\left(\frac{m}{2}\right)}{\left|a\left(\frac{m}{2}\right)\right|}\right).\tag{2.10}$$

We introduce the functions

$$X_\pm(x,m):=e^{i\zeta_\pm}Y_+\left(x,\frac{m}{2}\right)+e^{-i\zeta_\pm}Y_+\left(x,-\frac{m}{2}\right).\tag{2.11}$$

It is easy to confirm that these functions solve equation (2.7) with $k=\frac{m}{2}$ and behave at infinity as

$$\begin{aligned} &X_\pm(x,m)=2\cos\Big(\frac{m}{2}x+\zeta_\pm\Big)+O\big(e^{-\frac{\vartheta}{3}|x|}\big), && x\to+\infty,\\ &X_\pm(x,m)=-2\frac{\Upsilon_\pm}{\tilde\Upsilon_\pm}\sin\Big(\frac{m}{2}x-\zeta_\pm\Big)+O\big(e^{-\frac{\vartheta}{3}|x|}\big), && x\to-\infty,\end{aligned} \tag{2.12}$$

$$\Upsilon_\pm:=\frac{1}{4m}+b_{\mathrm{i}}\left(\frac{m}{2}\right)\tilde\Upsilon_\pm,\qquad \tilde\Upsilon_\pm:=\frac{1}{4m|a\left(\frac{m}{2}\right)|\sin\left(2\zeta_\pm+\theta\left(\frac{m}{2}\right)\right)}. \tag{2.13}$$

We denote

$$\kappa(\lambda)=\sqrt{\lambda^2-\varkappa},\qquad \rho_0(k):=\int\limits_0^{2\pi}\gamma(x)\frac{\sin k(\pi-x)}{k}\,\mathrm{d}t,$$

$$\hat\rho:=\int\limits_0^{2\pi}\mathrm{d}x\int\limits_0^{x}\gamma(x)\gamma(t)\sin m(x-t)\,\mathrm{d}t, \tag{2.14}$$

$$\begin{aligned} K_\pm:=&\frac{m\rho_0(m)\sin 2\zeta_\pm}{2\pi}\lim_{N\to+\infty}\left(-\int\limits_{-4\pi N}^{4\pi N}X_\pm^2(x,m)\,\mathrm{d}x+8\pi N\left(1+\frac{\Upsilon_\pm^2}{\tilde\Upsilon_\pm^2}\right)\right)\\ &+\lim_{N\to+\infty}\left((-1)^m\int\limits_{-4\pi N}^{4\pi N}\gamma(x)X_\pm^2(x,m)\,\mathrm{d}x+4N\rho_0(m)\sin 2\zeta_\pm\left(\frac{\Upsilon_\pm^2}{\tilde\Upsilon_\pm^2}-1\right)\right)\\ &+\frac{\rho_0(m)\sin^2 2\zeta_\pm}{\pi}\left(1-\frac{\Upsilon_\pm^2}{\tilde\Upsilon_\pm^2}\right),\end{aligned} \tag{2.15}$$

$$\begin{aligned} L_\pm:=&\frac{1}{4\pi}\left(\frac{\hat\rho}{m}+\frac{m}{\pi}\rho_0(m)\rho_0'(m)\sin^2 2\zeta_\pm\right)\left(4mb_{\mathrm{i}}\left(\frac{m}{2}\right)\tilde\Upsilon_\pm\cos^2 2\zeta_\pm+1\right)\\ &-\frac{m}{2\pi}\rho_0(m)\Upsilon_\pm K_\pm\cos 2\zeta_\pm-\frac{\rho_0^2(m)}{32\pi^2}\left(4-\frac{m^2}{\big(\lambda_0^{(m)}\big)^2}\right)\sin^2 2\zeta_\pm,\end{aligned} \tag{2.16}$$

$$\begin{aligned} M_\pm:=&\frac{4}{m}\tilde\Upsilon_\pm b_{\mathrm{i}}\left(\frac{m}{2}\right)\hat\rho\sin 4\zeta_\pm-4\rho_0(m)\Upsilon_\pm K_\pm\sin 2\zeta_\pm\\ &-\frac{\rho_0(m)\rho_0'(m)}{\pi}\left(4mb_{\mathrm{i}}\left(\frac{m}{2}\right)\tilde\Upsilon_\pm\sin^2 2\zeta_\pm-1\right)\sin 4\zeta_\pm,\end{aligned} \tag{2.17}$$

where the branch of the square root in the definition of $\kappa(\lambda)$ is fixed by the condition $\sqrt{1}=1$. We observe that all just introduced constants except for $\kappa(\lambda)$ are real and

$$\rho_0(m)=\frac{1}{m}\int\limits_0^{2\pi}\gamma(t)\sin m(\pi-t)\,\mathrm{d}t=\frac{(-1)^{m+1}}{m}\int\limits_0^{2\pi}\gamma(t)\sin mt\,\mathrm{d}t=\frac{2(-1)^{m+1}\pi}{m}\alpha_0(m).$$

Now we are in position to formulate our next main result.

Theorem 2.3. *All isolated eigenvalues of the operator pencil $\mathcal{H}_\varepsilon(\lambda)$ converging to the continuous spectrum converge to $\lambda_0^{(m)}$, $m\in\mathbb{Z}$. Namely, for each compact set Q in the complex plane obeying $Q\cap\sigma_{\mathrm{p}}(\mathcal{H}_0+\varkappa-\lambda^2)=\emptyset$, there exists a function $\varrho(\varepsilon)\to+0$ as $\varepsilon\to 0$ such that for sufficiently small ε the set*

$\Big\{\lambda\in Q:\, |\lambda-\lambda_0^{(m)}|\geqslant \varrho(\varepsilon),\ m\in\mathbb{Z}\setminus\{0\},\ |\lambda\pm\sqrt{\varkappa}|\geqslant\varrho(\varepsilon)\Big\}$ *contains no isolated eigenvalues of the operator pencil* $\mathcal{H}_\varepsilon$.

Assume that $\lambda_0^{(m)}\neq 0$, $\rho_0(m)\neq 0$ *for some* $m\in\mathbb{Z}\setminus\{0\}$. *Then the operator* $\mathcal{H}_\varepsilon(\lambda)$ *can have at most two eigenvalues* $\lambda_\pm(\varepsilon)$ *(counting multiplicities) converging to* $\lambda_0^{(m)}$ *as* $\varepsilon\to+0$. *If such eigenvalue exists, it is holomorphic in* ε *and the leading terms of its Taylor expansions are*

$$\lambda_\pm(\varepsilon)=\lambda_0^{(m)}+i\varepsilon\frac{\alpha_0(m)\sin 2\zeta_\pm}{2}+\varepsilon^2\lambda_0^{(m)}L_\pm+O(\varepsilon^3) \tag{2.18}$$

for one of $\zeta_\pm$ *with constants* $L_\pm$ *defined by (2.16). The associated eigenfunction can be chosen so that it becomes holomorphic in* ε *in the norm of the space* $H^2(-r,r)$ *for each fixed* $r>0$ *and the leading term of its Taylor expansions reads as* $\psi_\pm(\cdot,\varepsilon)=X_\pm+O(\varepsilon)$.

Fix one of $\zeta_\pm$. *Under the conditions*

$$\begin{aligned} &M_\pm<0 && \text{if } \lambda_0^{(m)} \text{ is real}\\ &(-1)^{m+1}\operatorname{Im}\lambda_0^{(m)}\rho_0(m)\cos 2\zeta_\pm<0 \quad\text{or} && \\ &\cos 2\zeta_\pm=0 \quad\text{and}\quad \rho_0(m)\Upsilon_\pm\tilde{K}_\pm\sin 2\zeta_\pm<0 && \text{if } \lambda_0^{(m)} \text{ is pure imaginary}\end{aligned} \tag{2.19}$$

the operator pencil $\mathcal{H}_\varepsilon$ *has the corresponding eigenvalue* $\lambda_\pm(\varepsilon)$ *satisfying (2.18). Under the conditions*

$$\begin{aligned} &M_\pm>0 && \text{if } \lambda_0^{(m)} \text{ is real}\\ &(-1)^{m+1}\operatorname{Im}\lambda_0^{(m)}\rho_0(m)\cos 2\zeta_\pm>0 \quad\text{or} && \\ &\cos 2\zeta_\pm=0 \quad\text{and}\quad \rho_0(m)\Upsilon_\pm\tilde{K}_\pm\sin 2\zeta_\pm>0 && \text{if } \lambda_0^{(m)} \text{ is pure imaginary}\end{aligned} \tag{2.20}$$

the operator pencil $\mathcal{H}_\varepsilon$ *has no corresponding eigenvalue* $\lambda_\pm(\varepsilon)$ *satisfying (2.18).*

Let us discuss briefly the main results of the work. Theorem 2.1 states the convergence of the spectrum $\sigma(\mathcal{H}_\varepsilon)$ to the spectrum of the unperturbed operator pencil $\mathcal{H}_0+\varkappa-\lambda^2$. Relation (2.3) provides explicitly a domain, in which the spectrum $\sigma(\mathcal{H}_\varepsilon)$ is located.

According Theorem 2.2, outside small balls centered at $\lambda_0^{(m)}$, $m\neq 0$, and at zero, the continuous spectrum of $\mathcal{H}_\varepsilon(\lambda)$ remains unchanged and coincides with the continuous spectrum of $\mathcal{H}_0(\lambda)$. In the vicinities of $\lambda_0^{(m)}$, $m\neq 0$, the continuous spectrum feels the perturbation. In the imaginary axis small gaps are opened with the end-points described approximately in (2.6). As $\lambda_0^{(m)}$ is real, the continuous spectrum looks even more interesting. Provided $\alpha_0(m)\neq 0$, it bifurcates into the complex plane as an additional continuous curve. The shape of this curve is approximately a parabola described in (2.4), see Figure 1. If $\alpha_0(m)=0$, the leading terms in asymptotics (2.4) turn out to be real but this still does not guarantee that λ is real.

The last theorem, Theorem 2.3, is devoted to the emerging eigenvalues. Such eigenvalues can exist only in the vicinities of points $\lambda_0^{(m)}$ or of zero. In the paper we consider the eigenvalues in the vicinities of $\lambda_0^{(m)}$ with $m\neq 0$. There can be at most two eigenvalues and these eigenvalues are holomorphic in ε. Each of these eigenvalues is associated with the roots $\zeta_\pm$ of equation (2.10) in the sense of asymptotics (2.18). To check the existence or absence of such eigenvalues, for each of roots $\zeta_\pm$ we just need to check conditions (2.19) or (2.20). If condition (2.19) is satisfied for a fixed root, then the operator pencil has the eigenvalue associated with this root in the sense of (2.18). If opposite condition (2.20) holds, then the operator pencil has no the associated eigenvalue. Hence, the possible options are no eigenvalues, one simple eigenvalue or two simple eigenvalues/one double eigenvalue. The case of a double eigenvalue is surely excluded if $\sin 2\zeta_+\neq\sin 2\zeta_-$ or if $\sin 2\zeta_+=\sin 2\zeta_-$ but $L_+\neq L_-$. As $\lambda_0^{(m)}$ is pure imaginary, all such eigenvalues are also pure imaginary and they are located inside the above discussed gaps in the imaginary axis, cf. (2.6) and (2.18). For real $\lambda_0^{(m)}$, these eigenvalues are located in the vicinities of the aforementioned small curves, see Figure 1. Formulae (2.13), (2.16), (2.17) for constants $\Upsilon_\pm$, $M_\pm$, $K_\pm$, $L_\pm$ determining the existence and the asymptotics of the emerging eigenvalues are very explicit and we only need to know the scattering data at $m/2$ and the functions $X_\pm$. We stress that these formulae are not easily predictable in comparison with the known results on the emerging eigenvalues and the same concerns equation (2.10) and functions (2.11). The main reason is that here the eigenvalues emerge as a result of interaction between the scattering at the localized potential and a small periodic potential $\varepsilon\gamma$. The mechanism of the eigenvalues emergence in our case is rather different in comparison with the case of the eigenvalues emergence due to a small localized perturbation. While the eigenvalues emerging in the gaps on the imaginary axis still resemble the eigenvalues created in spectral gaps by small localized perturbations like in [5, 6, 41], the eigenvalues near the real line emerge from interior points in the continuous spectrum. The latter phenomenon in a sense similar to a phenomenon recently observed numerically in [40] for the spectra of $\mathcal{PT}$-symmetric operators.

We also observe the following interesting fact. As $\varepsilon = 0$, the eigenvalue equation for the operator pencil $\mathcal{H}_\varepsilon(\lambda)$ becomes equation (2.7) with k replaced by λ. Its solutions, the Jost functions, determine constants $\zeta_\pm$ via (2.10). And these constants appear only in the term of order ε in asymptotics (2.18) and not in the leading term. The potential γ is involved only in formulae for $L_\pm$, that is, this potential influences only the term of order ε^2 in asymptotics (2.18). This is quite unusual since one could expect that solutions to equation (2.7) should determine the leading term in asymptotics for the emerging eigenvalue and the potential γ should be involved in the formula for the coefficient at ε. Instead of this, we see a “shift” of the expected situation by one power of ε.

3. Examples

In this section we discuss two examples of the potentials V and γ and apply to them our main results. We let $\varkappa := -1$ and we mostly consider the simplest example of step-like potentials:

$$\gamma(x) = \begin{cases} 1, & x \in (0,\pi], \\ 0, & x = 0, \\ -1, & x \in [-\pi, 0), \end{cases} \qquad V(x) = \begin{cases} 1, & x \in [-\pi,\pi], \\ 0, & \text{otherwise.} \end{cases}$$

As $m = \pm 1$, the functions $u_+(x,\pm 1)$ and $u_-(x,\pm 1)$ can be found explicitly and we have

$$\lambda_0 = \pm\frac{i\sqrt{3}}{2}, \qquad \alpha_0(\pm 1) = \pm\frac{2}{\pi}, \qquad \alpha_1(\pm 1) = \frac{4(\pi^2+2)}{\pi^2}.$$

Formulae (2.6) become

$$\beta_1^\pm(\varepsilon) = \frac{i\sqrt{3}}{2} \pm \frac{i\varepsilon}{\pi} + \frac{i\varepsilon^2}{2\sqrt{3}}\frac{3\pi^2+8}{\pi^2} + O(\varepsilon^3), \qquad \beta_{-1}^\pm(\varepsilon) = -\frac{i\sqrt{3}}{2} \pm \frac{i\varepsilon}{\pi} - \frac{i\varepsilon^2}{2\sqrt{3}}\frac{3\pi^2+8}{\pi^2} + O(\varepsilon^3). \tag{3.1}$$

We skip the values $m = \pm 2$, since $\lambda_0^{(\pm 2)} = 0$ and consider $m = \pm 3, \pm 4, \pm 5, \ldots$ Then

$$\lambda_0^{(m)} = \sqrt{\frac{m^2}{4} - 1}\,\operatorname{sign} m, \qquad \alpha_0(m) = -\frac{2}{\pi m} \quad \text{for even } m, \qquad \alpha_0(m) = 0 \quad \text{for odd } m.$$

We also have $\alpha_1(m)=0$ for all m and this identity is based on explicit calculations of the functions $u_\pm(x,m)$. Formulae (2.4) cast into the form:

$$\begin{aligned}
&\lambda=\lambda_0^{(m)}\pm\frac{i\varepsilon}{2\lambda_0^{(m)}}\sqrt{\frac{m^2-4}{\pi^2m^2}-s^2m^2+O(\varepsilon^2)}-\frac{\varepsilon^2}{2\lambda_0^{(m)}}\left(\frac{4s^2}{m^2+4}+\frac{1}{\pi^2m^2}\right)+O(\varepsilon^3)\quad\text{for even } m,\\
&\lambda=\lambda_0^{(m)}\mp\frac{\varepsilon}{2\lambda_0^{(m)}}\sqrt{s^2m^2+O(\varepsilon^2)}-\frac{\varepsilon^2s^2}{2(\lambda_0^{(m)})^3}+O(\varepsilon^3)\quad\text{for odd } m.
\end{aligned}$$

We proceed to the emerging eigenvalues. We first choose $m=1$. It is straightforward to check that

$$Y_+\left(x,\pm\frac12\right)=\begin{cases} e^{\pm\frac{i}{2}x}, & x>\pi,\\ \pm\, i\cosh\frac{\sqrt3}{2}(x-\pi)-\frac{1}{\sqrt3}\sinh\frac{\sqrt3}{2}(x-\pi), & |x|\leqslant\pi,\\ \left(\frac{1}{\sqrt3}\sinh\sqrt3\pi\pm i\cosh\sqrt3\pi\right)\cos\frac{x+\pi}{2} & \\ -\left(\cosh\sqrt3\pi\pm i\sqrt3\sinh\sqrt3\pi\right)\sin\frac{x+\pi}{2}, & x<-\pi,\end{cases}$$

$$Y_-\left(x,\frac12\right)=Y_+\left(-x,\frac12\right),\qquad a\left(\frac12\right)=\cosh\sqrt3\pi+\frac{i}{\sqrt3}\sinh\sqrt3\pi,\qquad b\left(\frac12\right)=-\frac{2i}{\sqrt3}\sinh\sqrt3\pi,$$

$$\left|a\left(\frac12\right)\right|=\frac{(4\cosh^2\sqrt3\pi-1)^{\frac12}}{\sqrt3},\qquad \theta\left(\frac12\right)=\arctan\frac{\tanh\sqrt3\pi}{\sqrt3},\qquad \zeta_\pm=-\frac12\arctan\frac{\tanh\sqrt3\pi}{\sqrt3}\mp\frac{\pi}{4}.$$

Then we get:

$$\rho_0(1)=4,\quad \rho_0'(1)=-4,\quad \hat\rho=2\pi,\quad \Upsilon_\pm=\frac14\pm\frac{\sinh\sqrt3\pi}{2\sqrt3(4\cosh^2\sqrt3\pi-1)^{\frac12}},\quad \tilde\Upsilon_\pm=\mp\frac{\sqrt3}{4(4\cosh^2\sqrt3\pi-1)^{\frac12}}.$$

We see easily that

$$\cos(2\zeta_\pm)=\mp\frac{\sqrt2\sinh(\sqrt3\pi)}{\sqrt{1+5\cosh(2\sqrt3\pi)}}$$

and condition (2.19) is satisfied for ζ_+, while condition (2.20) holds for ζ_-. Hence, we have just one eigenvalue near $\lambda_+(\varepsilon)$ near $\frac{i\sqrt3}{2}$. The formulae for constants K_+, L_+ here become too lengthy and this is why we find them numerically:

$$K_+\cong 151653.7844,\qquad L_+\cong -6030.514687.$$

The asymptotics of $\lambda(\varepsilon)$ is given by (2.18):

$$\lambda_+(\varepsilon)=i\frac{\sqrt3}{2}-\frac{i\varepsilon}{\pi}\frac{\sqrt{3\cosh2\sqrt3\pi+3}}{\sqrt{4\cosh2\sqrt3\pi+2}}+\frac{i\varepsilon^2\sqrt3L_+}{2}+O(\varepsilon^3). \tag{3.2}$$

Since the spectrum is symmetric with respect to the real axis, we also have the only eigenvalue converging to $-i\frac{\sqrt3}{2}$ and its asymptotics is obtained by complex conjugation of (3.2).

We proceed to the case $m = \pm 4, \pm 6, \ldots$. The case of odd m is not considered since then $\rho_0(m) = 0$ and Theorem 2.3 is not applicable. For each even $m = 2n$ we denote $q_n := \sqrt{n^2 - 1}$ and we have:

$$Y_+(x,n) = \begin{cases} e^{inx}, & x > \pi, \\ (-1)^n \left(\cos q_n(x-\pi) + \dfrac{in}{q_n} \sin q_n(x-\pi) \right), & |x| \leqslant \pi, \\ (-1)^n \left(\cos 2\pi q_n - \dfrac{in}{q_n} \sin 2\pi q_n \right) \cos n(x+\pi) & \\ \quad + (-1)^n \left(\dfrac{q_n}{n} \sin 2\pi q_n + i \cos 2\pi q_n \right) \sin n(x+\pi), & x < -\pi, \end{cases}$$

$$Y_-(x,n) = Y_+(-x,n), \qquad a(n) = \cos 2\pi q_n - i \frac{2n^2-1}{2nq_n} \sin 2\pi q_n, \qquad b(n) = -\frac{i}{2nq_n} \sin 2\pi q_n,$$

$$\theta(n) = \pi - \arctan \frac{2n^2-1}{2nq_n} \tan 2\pi q_n, \qquad \zeta_\pm = \frac{1}{2} \arctan \frac{2n^2-1}{2nq_n} \tan 2\pi q_n - \frac{\pi}{2} \mp \frac{\pi}{4},$$

$$\rho_0(2n) = \frac{1}{n^2}, \qquad \rho_0'(2n) = \hat{\rho} = 0, \qquad \tilde{\Upsilon}_\pm = \mp \frac{1}{2n|a(n)|}, \qquad \Upsilon_\pm = \frac{1}{2n} \pm \frac{\sin 2\pi q_n}{4n^2 q_n |a(n)|}.$$

To check conditions (2.19), (2.20), now we have to calculate all constants $K_\pm$, $M_\pm$ and the formulae for these constants turn out to be lengthy. This is why we restrict ourselves by the case $m = 4$ and we find the needed constants numerically:

$$K_+ \cong 0.3303, \quad L_+ \cong 0.0132, \quad M_+ \cong 0.0079, \quad K_- \cong -0.9560, \quad L_- \cong -0.0382, \quad M_- \cong 0.0305.$$

In both cases $M_\pm > 0$ and there is no eigenvalues converging to $\lambda_0^{(4)}$ as $\varepsilon \to +0$. Since the spectrum is symmetric with respect to the imaginary axis, we also conclude that there is no eigenvalues converging to $\lambda_0^{(-4)}$.

The above analytic results are demonstrated in Figures 2. Here we show the picture generated by the above obtained asymptotic formulae, namely, by their leading terms neglecting the error terms. In the figures, we choose $\varepsilon = 10^{-6}$ and the explanation of such choice is that this is the largest scale, for which the value of the asymptotics in (3.2) is located between similar values in (3.1). The left figure provides the general picture of two gaps on the imaginary axis with the eigenvalues inside and two parabolas corresponding to $m = \pm 4$. These gaps and parabolas turn out to be too small and are not properly visible. This is why on the right figure we magnify their vicinities and show a fine structure of the gaps and parabolas and the location of the isolated eigenvalues. We have also tried to find the spectrum numerically and this has produced the same picture.

The constants $M_\pm$ are very sensible to the choice of the function γ. Namely, keeping the same function V as above, let us replace the function γ by the following one: $\gamma(x) := \sin 4x$, $x \in (-\pi, \pi)$. Then the constants $K_\pm$, $L_\pm$, $M_\pm$ for $m = 4$ read as:

$$K_+ \cong 5.5981, \quad L_+ \cong -0.7294, \quad M_+ \cong -0.4032, \quad K_- \cong -7.6822, \quad L_- \cong 0.8659, \quad M_- \cong -0.7415.$$

Here in both cases $M_\pm < 0$ and there exist two eigenvalues converging to $\lambda_0^{(4)} = \sqrt{3}$ as $\varepsilon \to +0$; their asymptotics read as

$$\lambda_\pm(\varepsilon) = \sqrt{3} \pm \varepsilon \frac{3i}{\sqrt{144 + 147 \tan^2 2\sqrt{3}\pi}} + \varepsilon^2 L_\pm + O(\varepsilon^3).$$

And since the spectrum is symmetric with respect to the imaginary axis, the same situation holds in the vicinity of $\lambda_0^{(-4)} = -\sqrt{3}$. Figure 3 demonstrates the structure of the essential spectrum and the location of the isolated

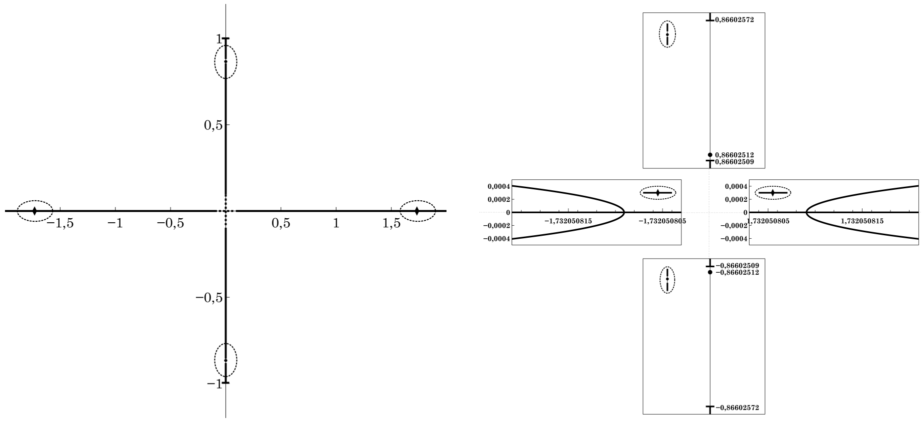

FIGURE 2. Left figure shows the general picture of the spectrum for step-like potentials. Right figure demonstrates at a small scale the pieces of the spectrum spectrum inside dotted ellipses. In both figures we choose $\varepsilon = 10^{-6}$. Black lines represent the essential spectrum, small rhombus show the location of small parabolas, while the circles indicate the locations locations of the isolated eigenvalues.

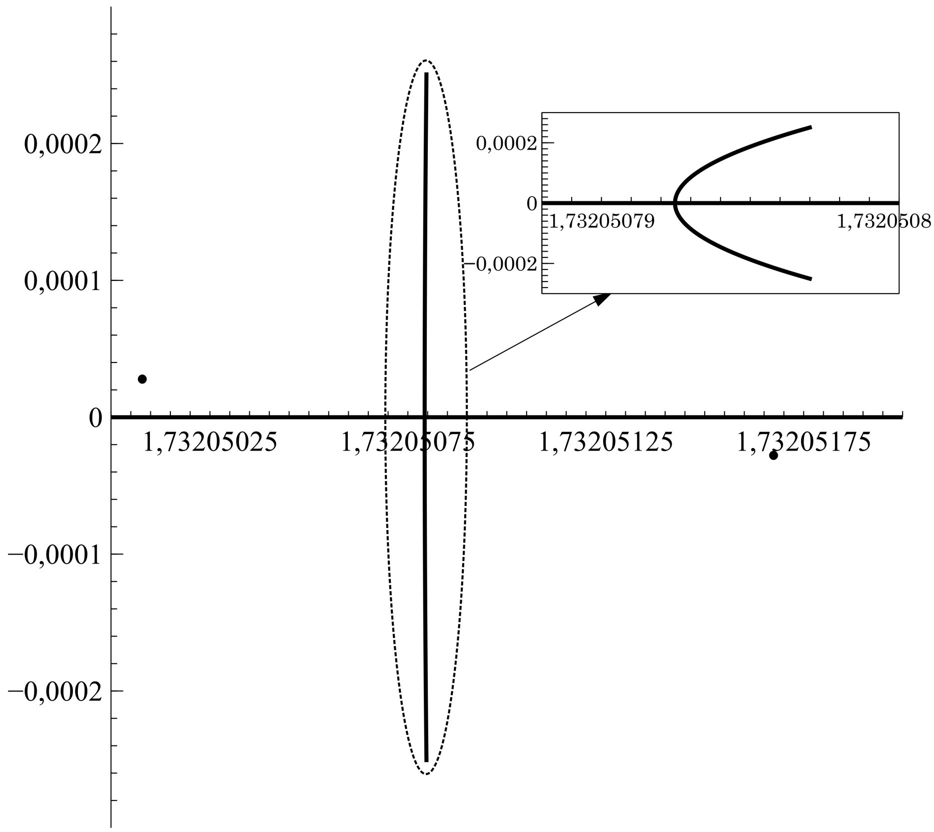

FIGURE 3. Spectrum for $\gamma(x) = \sin 4x$ and the step-like potential V as $\varepsilon = 10^{-3}$ in the vicinity of $\lambda_0^{(4)}$. Black curves represent the essential spectrum, while the black circles indicate the isolated eigenvalues.

eigenvalues in the vicinity of $\lambda_0^{(4)} = \sqrt{3}$ as $\varepsilon = 10^{-3}$. The magnified graph inside a boxed panel shows the fine structure of the essential spectrum at a small scale.

4. Convergence of spectrum

In this section we prove Theorem 2.1. We begin with the absence of the residual spectrum. As in ([24], Chap. I, Sect. 1.1), we get $\sigma_{\rm r}(\mathcal{H}_\varepsilon(\lambda)) = -\,\sigma_{\rm p}(\mathcal{H}_\varepsilon^*)^\dagger \setminus \sigma_{\rm p}(\mathcal{H}_\varepsilon)$ and by (2.1) we conclude that $\sigma_{\rm r}(\mathcal{H}_\varepsilon)$ is empty.

We proceed to proving (2.3). Since the operator $\mathcal{H}_0$ is self-adjoint, we have

$$\|(\mathcal{H}_0 + \varkappa - \lambda^2)^{-1}\|_{L_2(\mathbb{R})\to L_2(\mathbb{R})} = \frac{1}{\operatorname{dist}(\lambda^2 - \varkappa^2, \sigma(\mathcal{H}_0))}, \tag{4.1}$$

where $\|\cdot\|_{L_2(\mathbb{R})\to L_2(\mathbb{R})}$ stands for the norm of a bounded operator in $L_2(\mathbb{R})$. Thanks to the identity

$$\big(\mathcal{H}_\varepsilon(\lambda)\big)^{-1} = (\mathcal{H}_0 + \varkappa + i\varepsilon\lambda\gamma - \lambda^2)^{-1} = (\mathcal{H}_0 + \varkappa - \lambda^2)^{-1}\left(\mathcal{I} + i\varepsilon\lambda\gamma(\mathcal{H}_0 + \varkappa - \lambda^2)^{-1}\right)^{-1},$$

the inverse operator for $\mathcal{H}_\varepsilon(\lambda)$ is well-defined as a bounded operator in $L_2(\mathbb{R})$ provided

$$\varepsilon|\lambda|\|\gamma(\mathcal{H}_0 + \varkappa - \lambda^2)^{-1}\|_{L_2(\mathbb{R})\to L_2(\mathbb{R})} < 1.$$

In view of (4.1), this inequality holds true if $\varepsilon|\lambda| \sup\limits_{[-\pi,\pi]} |\gamma| < \operatorname{dist}(\lambda^2 - \varkappa, \sigma(\mathcal{H}_0))$, i.e., such λ are in the resolvent set of operator pencil $\mathcal{H}_\varepsilon(\lambda)$. The opposite inequality leads us to (2.3).

5. Continuous spectrum

In this section we prove Theorem 2.2. In order to describe the continuous spectrum of $\mathcal{H}_\varepsilon$, we employ the Floquet-Bloch theory. Namely, we introduce an auxiliary operator pencil:

$$\mathcal{Q}_\varepsilon^\tau(\lambda) := \mathcal{Q}_0(\tau) + \varkappa + i\varepsilon\lambda\gamma - \lambda^2,$$

where

$$\mathcal{Q}_0(\tau) := \left(i\frac{\mathrm{d}}{\mathrm{d}x} + \tau\right)^2, \quad \tau \in \mathbb{R},$$

is a self-adjoint operator in $L_2(-\pi,\pi)$ subject to periodic boundary conditions. Its domain is a subspace in $H^2(-\pi,\pi)$ formed by the functions satisfying the periodic boundary conditions. For future purposes, it is convenient to choose the Brillouin zone as follows: $\tau \in [-\frac{1}{4}, \frac{3}{4})$.

Lemma 5.1. *For each ε and τ the continuous and residual spectra of $\mathcal{Q}_\varepsilon^\tau(\lambda)$ are empty, i.e., $\sigma(\mathcal{Q}_\varepsilon^\tau) = \sigma_{\rm p}(\mathcal{Q}_\varepsilon^\tau)$. The continuous spectrum of operator pencil $\mathcal{H}_\varepsilon(\lambda)$ is symmetric both w.r.t. the real and imaginary axes. It is given by the formula*

$$\sigma_{\rm c}(\mathcal{H}_\varepsilon) = \bigcup_{\tau\in[-\frac{1}{4},\frac{3}{4})} \sigma_{\rm p}(\mathcal{Q}_\varepsilon^\tau). \tag{5.1}$$

Proof. In the same way as in the proof of Theorem 2.1, we check easily that the residual spectrum of the operator pencil $\mathcal{Q}_\varepsilon^\tau(\lambda)$ is empty for each fixed ε and τ. The continuous spectrum of $\mathcal{Q}_\varepsilon^\tau$ is also empty. Indeed, if

$\lambda \in \sigma_{\mathrm{c}}(\mathcal{Q}_\varepsilon^\tau)$ and u_p is an associated characteristic sequence, then

$$\|iu_p' + \tau u_p\|^2_{L_2(-\pi,\pi)} + i\varepsilon\lambda(\gamma u_p, u_p)_{L_2(-\pi,\pi)} - \lambda^2 \|u_p\|^2_{L_2(-\pi,\pi)} \to 0, \qquad p \to \infty.$$

Thus, u_p is also bounded in $W_2^1(-\pi,\pi)$ and by the compact embedding of $W_2^1(-\pi,\pi)$ into $L_2(-\pi,\pi)$, the sequence u_p is compact in $L_2(-\pi,\pi)$. This contradicts the definition of the continuous spectrum. Hence,

$$\sigma(\mathcal{Q}_\varepsilon^\tau) = \sigma_{\mathrm{p}}(\mathcal{Q}_\varepsilon^\tau). \tag{5.2}$$

We introduce an auxiliary operator pencil:

$$\mathring{\mathcal{H}}_\varepsilon(\lambda) := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + i\varepsilon\lambda\gamma + \varkappa - \lambda^2 \quad \text{in} \quad L_2(\mathbb{R}),$$

which is the operator pencil $\mathcal{H}_\varepsilon(\lambda)$ in the case $V \equiv 0$. Let us show that

$$\sigma_{\mathrm{c}}(\mathcal{H}_\varepsilon) = \sigma_{\mathrm{c}}(\mathring{\mathcal{H}}_\varepsilon) = \bigcup_{\tau \in [-\frac{1}{4}, \frac{3}{4})} \sigma_{\mathrm{p}}(\mathcal{Q}_\varepsilon^\tau). \tag{5.3}$$

The proof of the first identity reproduces literally the proof of Lemma 2.3 in [5]. To prove the second identity, we observe that $\mathring{\mathcal{H}}_\varepsilon(\lambda)$ is a periodic operator. Then we employ Floquet-Bloch theory to see that the operator $\mathring{\mathcal{H}}_\varepsilon(\lambda)$ has a bounded inverse in $L_2(\mathbb{R})$ provided λ is not in the spectrum of $\mathcal{Q}_\varepsilon^\tau$, or, thanks to (5.2), λ is not an eigenvalue of the operator pencil $\mathcal{Q}_\varepsilon^\tau(\lambda)$ for some $\tau \in [-\frac{1}{4}, \frac{3}{4})$. This proves identity (5.1) in the case $V \equiv 0$. Due to the proven identity, for each $\lambda \in \sigma(\mathring{\mathcal{H}}_\varepsilon)$, there exists $\tau \in [-\frac{1}{4}, \frac{3}{4})$ and a function $\phi \in W_2^2(-\pi,\pi)$ satisfying periodic boundary conditions such that λ and ϕ are an eigenvalue and an associated eigenfunction of $\mathcal{Q}_\varepsilon^\tau(\lambda)$. We continue the function ϕ periodically on the entire axis. Multiplying then the function $e^{i\tau x}\phi(x)$ by an appropriate sequence of cut-off functions, we construct easily a characteristic sequence for $\mathring{\mathcal{H}}_\varepsilon$ associated with λ and this proves the second identity in (5.3). ☐

It is obvious that if λ is an eigenvalue of $\mathcal{Q}_\varepsilon^\tau$ and $\phi(x)$ is an associated eigenfunction, then $\overline{\lambda}$ is also an eigenvalue of $\mathcal{Q}_\varepsilon^\tau$ with the associated eigenfunction $\overline{\phi(-x)}$. We also observe that $-\overline{\lambda}$ and $\overline{\phi(x)}$ are an eigenvalue and an associated eigenfunction of $\mathcal{Q}_\varepsilon^{-\tau}$.

We proceed to studying the structure of the continuous spectrum. Reproducing almost literally the proof of Theorem 2.1, one can check easily that the eigenvalues of the operator pencil $\mathcal{Q}_\varepsilon^\tau$ converge to those of $\mathcal{Q}_0^\tau$ as $\varepsilon \to 0$.

Let $\lambda(\tau,\varepsilon)$ be an eigenvalue of $\mathcal{Q}_\varepsilon^\tau$ located close enough to the imaginary axis and separated from the origin by a distance $C\varepsilon$ with a sufficient large positive constant C. We normalize an associated eigenfunction ϕ_ε in $L_2(-\pi,\pi)$. Then we multiply eigenvalue equation (5.5) for $\lambda(\tau,\varepsilon)$ by $\overline{\phi}_\varepsilon$, integrate by parts over $(-\pi,\pi)$ and calculate the imaginary part of the obtained identity:

$$\big(\varepsilon(\gamma\phi_\varepsilon, \phi_\varepsilon)_{L_2(-\pi,\pi)} - 2\operatorname{Im}\lambda(\tau,\varepsilon)\big)\operatorname{Re}\lambda(\tau,\varepsilon) = 0.$$

Since $\operatorname{Im}\lambda(\tau,\varepsilon)$ is close to some non-zero constant for sufficiently small $\varepsilon \to 0$, the expression in the brackets is non-zero for sufficiently small ε. Hence, $\operatorname{Re}\lambda(\tau,\varepsilon) = 0$ and $\lambda(\tau,\varepsilon)$ is pure imaginary.

We choose $n \in \mathbb{Z} \setminus \{0\}$ and we take the eigenvalues and the associated eigenfunctions

$$\begin{aligned} &\lambda_+(\tau) := \sqrt{(n-\tau)^2 + \varkappa}\,\operatorname{sign} n, && \lambda_-(\tau) := \sqrt{(n+\tau)^2 + \varkappa}\,\operatorname{sign} n, \\ &\phi_+(x) := \frac{1}{\sqrt{2\pi}} e^{inx}, && \phi_-(x) := \frac{1}{\sqrt{2\pi}} e^{-inx}, \end{aligned} \tag{5.4}$$

of the operator pencil $\mathcal{Q}_0(\tau)$, where $\tau\in[-\frac{1}{4},\frac{1}{4}]$. The functions $\lambda_\pm(\tau)$ range in a same segment on the real line and we assume that zero does not belong to this segment.

We consider the eigenvalue equation

$$\big(\mathcal{Q}_0(\tau)+\varkappa+i\varepsilon\lambda\gamma-\lambda^2\big)\phi_\varepsilon=0, \tag{5.5}$$

for sufficiently small ε and λ in the complex plane close to the range of functions $\lambda_\pm(\tau)$. For such values of λ, the resolvent of $\mathcal{Q}_0(\tau)+\varkappa$ satisfies the representation:

$$\big(\mathcal{Q}_0(\tau)+\varkappa-\lambda^2\big)^{-1}f=\frac{(f,\phi_+)_{L_2(-\pi,\pi)}}{\lambda_+^2-\lambda^2}\phi_+ +\frac{(f,\phi_-)_{L_2(-\pi,\pi)}}{\lambda_-^2-\lambda^2}\phi_- +\mathcal{L}_1(\lambda^2,\tau)f, \tag{5.6}$$

where $\mathcal{L}_1$ is a bounded operator from $L_2(-\pi,\pi)$ into $W_2^2(-\pi,\pi)$ jointly holomorphic in the considered values of λ^2 and τ. We apply the resolvent $\big(\mathcal{Q}_0(\tau)+\varkappa-\lambda^2\big)^{-1}$ to equation (5.5) and use representation (5.6). This leads us to the equation

$$\phi_\varepsilon+\frac{i\varepsilon\lambda(\gamma\phi_\varepsilon,\phi_+)_{L_2(-\pi,\pi)}}{\lambda_+^2(\tau)-\lambda^2}\phi_+ +\frac{i\varepsilon\lambda(\gamma\phi_\varepsilon,\phi_-)_{L_2(-\pi,\pi)}}{\lambda_-^2(\tau)-\lambda^2}\phi_- +i\varepsilon\lambda\mathcal{L}_1(\lambda^2,\tau)\gamma\phi_\varepsilon=0. \tag{5.7}$$

Thanks to the aforementioned properties of the operator $\mathcal{L}_1$, the operator $\mathcal{L}_2(\lambda,\tau,\varepsilon):=\big(\mathcal{I}+i\varepsilon\lambda\mathcal{L}_1(\lambda^2,\tau)\gamma\big)^{-1}$, where $\mathcal{I}$ stands for the identity mapping, is well-defined as a bounded operator in $L_2(-\pi,\pi)$ and it is jointly holomorphic in λ, τ, and ε. We apply this operator to equation (5.7):

$$\phi_\varepsilon+\frac{i\varepsilon\lambda(\gamma\phi_\varepsilon,\phi_+)_{L_2(-\pi,\pi)}}{\lambda_+^2(\tau)-\lambda^2}\mathcal{L}_2(\lambda,\tau,\varepsilon)\phi_+ +\frac{i\varepsilon\lambda(\gamma\phi_\varepsilon,\phi_-)_{L_2(-\pi,\pi)}}{\lambda_-^2(\tau)-\lambda^2}\mathcal{L}_2(\lambda,\tau,\varepsilon)\phi_-=0. \tag{5.8}$$

We multiply this equation by γ and calculate the scalar products with $\phi_\pm$ in $L_2(-\pi,\pi)$. This gives a system of linear equations for $(\gamma\phi_\varepsilon,\phi_\pm)_{L_2(-\pi,\pi)}$:

$$\begin{aligned}(\gamma\phi_\varepsilon,\phi_\pm)_{L_2(-\pi,\pi)}&+\frac{i\varepsilon\lambda(\gamma\mathcal{L}_2(\lambda,\tau,\varepsilon)\phi_+,\phi_\pm)_{L_2(-\pi,\pi)}}{\lambda_+^2(\tau)-\lambda^2}(\gamma\phi_\varepsilon,\phi_+)_{L_2(-\pi,\pi)}\\&+\frac{i\varepsilon\lambda(\gamma\mathcal{L}_2(\lambda,\tau,\varepsilon)\phi_-,\phi_\pm)_{L_2(-\pi,\pi)}}{\lambda_-^2(\tau)-\lambda^2}(\gamma\phi_\varepsilon,\phi_-)_{L_2(-\pi,\pi)}=0.\end{aligned}$$

The operator $\mathcal{L}_2$ satisfies the representation

$$\begin{aligned}\mathcal{L}_2(\lambda,\tau,\varepsilon)&=\mathcal{I}-i\varepsilon\lambda\mathcal{L}_1(\lambda^2,\tau)\gamma\mathcal{L}_2(\lambda,\tau,\varepsilon)\\&=\mathcal{I}-i\varepsilon\lambda\mathcal{L}_1(\lambda^2,\tau)\gamma-\varepsilon^2\lambda^2\mathcal{L}_1(\lambda^2,\tau)\gamma\mathcal{L}_1(\lambda^2,\tau)\gamma\mathcal{L}_2(\lambda,\tau,\varepsilon),\end{aligned} \tag{5.9}$$

and thanks to the parity of γ and the definition of $\phi_\pm$, we have

$$\begin{aligned}&(\gamma\mathcal{L}_2(\lambda,\tau,\varepsilon)\phi_\pm,\phi_\pm)_{L_2(-\pi,\pi)}=-i\varepsilon\lambda S^{\pm\pm}(\lambda,\tau,\varepsilon),\\&S^{\pm\pm}(\lambda,\tau,\varepsilon):=\big(\gamma\mathcal{L}_1(\lambda^2,\tau)\gamma\mathcal{L}_2(\lambda,\tau,\varepsilon)\phi_\pm,\phi_\pm\big)_{L_2(-\pi,\pi)},\end{aligned} \tag{5.10}$$

where $S^{\pm\pm}$ are jointly holomorphic in λ, τ, ε. We also denote

$$S^{\pm\mp}(\lambda,\tau,\varepsilon):=\big(\gamma\mathcal{L}_2(\lambda,\tau,\varepsilon)\phi_\pm,\phi_\mp\big)_{L_2(-\pi,\pi)} \tag{5.11}$$

and these functions are jointly holomorphic in λ, τ, ε.

In view of equation (5.8), the scalar products $(\gamma\phi_\varepsilon,\phi_\pm)_{L_2(-\pi,\pi)}$ do not vanish simultaneously since otherwise we immediately get $\phi_\varepsilon=0$ and this can not be an eigenfunction. By Cramer's rule this gives an equation for the sought eigenvalues of $\mathcal{Q}_\varepsilon^\tau$:

$$\det\begin{pmatrix}\lambda^2-\lambda_+^2(\tau)-\varepsilon^2\lambda^2 S^{++}(\lambda,\tau,\varepsilon) & -i\varepsilon\lambda S^{-+}(\lambda,\tau,\varepsilon)\\ -i\varepsilon\lambda S^{+-}(\lambda,\tau,\varepsilon) & \lambda^2-\lambda_-^2(\tau)-\varepsilon^2\lambda^2 S^{--}(\lambda,\tau,\varepsilon)\end{pmatrix}=0. \tag{5.12}$$

We calculate the determinant and solve the obtained equation as a bi-square one with respect to λ. This leads us to an equivalent pair of equations:

$$\begin{aligned}&\lambda^2=F_\pm(\lambda,\tau,\varepsilon), \qquad (5.13)\\ F_\pm:=&\frac{(1-\varepsilon^2S^{--})\lambda_+^2+(1-\varepsilon^2S^{++})\lambda_-^2-\varepsilon^2S^{+-}S^{-+}\pm\sqrt{F_*}}{2(1-\varepsilon^2S^{--})(1-\varepsilon^2S^{++})},\\ F_*:=&\big((1-\varepsilon^2S^{--})\lambda_+^2-(1-\varepsilon^2S^{++})\lambda_-^2\big)^2\\ &-2\varepsilon^2\big((1-\varepsilon^2S^{--})\lambda_+^2+(1-\varepsilon^2S^{++})\lambda_-^2\big)S^{+-}S^{-+}+\varepsilon^4\big(S^{+-}S^{-+}\big)^2,\end{aligned}$$

where the branch of the square root is fixed by the condition $\sqrt{1}=1$. As it follows from the definition of the function F_*, the function $(\lambda,\tau,\varepsilon)\mapsto\sqrt{F_*(\lambda,\tau,\varepsilon)}$ is jointly holomorphic in λ, τ, ε as $|\lambda_+^2(\tau)-\lambda_-^2(\tau)|\geqslant C\varepsilon^2$ for a sufficiently large fixed C and all sufficiently small ε. The latter inequality is true provided $C\varepsilon\leqslant|\tau|\leqslant\frac14$ and for such values of τ the left hand side of equation (5.13) are jointly holomorphic in λ, τ, ε. Hence, since $\lambda_\pm(\tau)\neq0$, by the inverse function theorem, each of equations (5.13) possesses exactly one root $\Lambda_\pm(\tau,\varepsilon)$ converging to $\lambda_\pm(\tau)$ as $\varepsilon\to0$ and this root is jointly holomorphic in τ and ε. Thanks to the above discussed symmetry properties of the eigenvalues of $\mathcal{Q}_\varepsilon^\tau$, the same is true for $\Lambda_\pm(\tau,\varepsilon)$. And as it was shown above, if $\lambda_\pm(\tau)$ is pure imaginary, the eigenvalues $\Lambda_\pm(\tau,\varepsilon)$ are also pure imaginary.

We proceed to the case of small τ. Namely, we consider $|\tau|\leqslant C\varepsilon$ for sufficiently large C. We rescale τ as $\tau=\varepsilon s$, $|s|\leqslant C$ and we are going to study the solvability of equations (5.13) in this case. We observe that $\lambda_+(0)=\lambda_-(0)=\lambda_0^{(m)}$ with $m:=2n$.

We first prove the solvability of equation (5.12) in the considered case.

Lemma 5.2. *In the vicinity of $\lambda_0^{(m)}$, equation (5.12) has exactly two roots $\Lambda_\pm=\Lambda_\pm(s,\varepsilon)$ counting multiplicities. These roots converge to $\lambda_0^{(m)}$ as $\varepsilon\to+0$ uniformly in s and they are jointly continuous in ε and s.*

Proof. In view of the definition of $\lambda_\pm$ and the properties of the functions $S^{\pm\pm}$, $S^{\pm\mp}$, we can rewrite equation (5.12) as

$$\big(\lambda^2-(\lambda_0^{(m)})^2\big)^2=\varepsilon h(\varepsilon,s,\lambda), \tag{5.14}$$

where h is a holomorphic function of its variables. In the vicinity of λ_0, the function $\lambda\mapsto(\lambda_2^2-\lambda_0^2)^2$ has exactly one double root $\lambda=\lambda_0$. Since the right hand side in (5.14) is multiplied by the small parameter, by the Rouché theorem we obtain that for each positive $\delta>0$, equation (5.14) has exactly two roots (counting multiplicity) inside the circle $|\lambda-\lambda_0|<\delta$ for all values of s provided ε is small enough. This proves the lemma. □

Let us find the asymptotics for the roots of equation (5.12). In order to do this, we again rewrite (5.12) as (5.13) and study the behavior of the function $F_\pm$ as $\varepsilon\to0$.

First of all we stress that apriori we do not know which of the equations in (5.13) is solved by each of the roots $\Lambda_\pm(s,\varepsilon)$ or probably both these roots solve one of the equations in (5.13). This is why we denote by $\Lambda(s,\varepsilon)$ one of the roots $\Lambda_\pm(s,\varepsilon)$ assuming that it solves one of the equations in (5.13).

Since in the considered case $\lambda_\pm^2(\varepsilon s) = \big(\lambda_0^{(m)}\big)^2 \pm \varepsilon s m + \varepsilon^2 s^2$, it follows immediately from (5.13) that $\Lambda(s,\varepsilon) - \lambda_0^{(m)} = O(\varepsilon)$ uniformly in s. The definition of the operator $\mathcal{L}_1(\lambda^2,\tau)$ and the identities $(\gamma\phi_\pm,\phi_\mp)_{L_2(-\pi,\pi)} = \pm i\alpha_0(m)$ imply that the functions $\mathcal{L}_1(\lambda_0^2,0)\gamma\phi_\pm$ are orthogonal to ϕ_+ and ϕ_- in $L_2(-\pi,\pi)$ and solve the equations

$$\left(\mathcal{Q}_0(0) - \frac{m^2}{4}\right)\mathcal{L}_1(\lambda_0^2,0)\gamma\phi_\pm = \frac{1}{\sqrt{2\pi}}\left(\gamma\cos\frac{m}{2}x - \alpha_0(m)\sin\frac{m}{2}x\right) \pm \frac{i}{\sqrt{2\pi}}\left(\gamma\sin\frac{m}{2}x - \alpha_0(m)\cos\frac{m}{2}x\right).$$

Hence, $\mathcal{L}_1(\lambda_0^2,0)\gamma\phi_\pm = \frac{1}{\sqrt{2}}(u_+ \pm i u_-)$, where $u_\pm$ were introduced in (2.5). The function u_+ is odd in x and u_+ is even. Therefore, by (5.9), (5.10),

$$S^{\pm\pm}(\Lambda,\varepsilon s,\varepsilon) = \frac{\alpha_1(m)}{2} + O(\varepsilon), \tag{5.15}$$

uniformly in s. In the same way by employing (5.9), (5.11) we obtain:

$$S^{\pm\mp}(\Lambda,\varepsilon s,\varepsilon) = \pm i\alpha_0(m) - i\varepsilon\frac{\lambda_0^{(m)}}{2\sqrt{\pi}}\int\limits_{-\pi}^{\pi}\gamma(x)\left(u_+(x,m)\cos\frac{m}{2}x - u_-(x,m)\sin\frac{m}{2}x\right)\,\mathrm{d}x + O(\varepsilon^2)$$

uniformly in s. We substitute these formulae and (5.15), (5.4) into (5.13) and we get:

$$\Lambda^2 = \big(\lambda_0^{(m)}\big)^2 \pm i\varepsilon\sqrt{\big(\lambda_0^{(m)}\big)^2\alpha_0^2 - m^2 s^2 + O(\varepsilon^2)} + \varepsilon^2\left(s^2 + \frac{(\lambda_0^{(m)})^2\alpha_1(m)}{2} - \frac{\alpha_0^2(m)}{2}\right) + O(\varepsilon^3).$$

This leads us to asymptotics (2.4) for the roots $\Lambda_\pm$. As identities (2.2) show, the roots $\Lambda_\pm(s,\varepsilon)$ are complex conjugate. Hence, the sign '$\pm$' in (2.4) corresponds to different roots $\Lambda_\pm(s,\varepsilon)$ and this proves (2.4).

If $\lambda_0^{(m)}$ is pure imaginary, the roots $\Lambda_\pm(s,\varepsilon)$ are pure imaginary and $\big(\lambda_0^{(m)}\big)^2\alpha_0^2(m) - s^2m^2 < 0$ for all values of s provided $\alpha_0(m)\neq 0$. Asymptotics (2.4) can be simplified and we arrive at

$$\Lambda_\pm(s,\varepsilon) = \lambda_0^{(m)} \mp \frac{\varepsilon}{2\lambda_0^{(m)}}\sqrt{m^2s^2 + |\lambda_0^{(m)}|^2\alpha_0^2(m)} - \frac{\varepsilon^2}{4\lambda_0^{(m)}}\left(\frac{2\varkappa s^2}{|\lambda_0^{(m)}|^2} + \frac{\alpha_0^2(m)}{2} + |\lambda_0^{(m)}|^2\alpha_1(m)\right) + O(\varepsilon^3).$$

Since s ranges in a fixed interval, the extrema of the leading terms in the above asymptotics are attained at $s=0$. This implies the existence of the gap $\big(\beta_m^-(\varepsilon),\beta_m^+(\varepsilon)\big)$ in the spectrum around the point $\lambda_0^{(m)}$ as well as asymptotics (2.6).

The case $\left|\tau - \frac{1}{2}\right| \leqslant \frac{1}{4}$ can be studied in the same way; we only need to let $m = 2n-1$ and to replace formulae (5.4) by

$$\lambda_+(\tau) := \sqrt{(n-\tau)^2+\varkappa}\,\operatorname{sign} n, \qquad \lambda_-(\tau) := \sqrt{(n-1+\tau)^2+\varkappa}\,\operatorname{sign} n,$$
$$\phi_+(x) := \frac{1}{\sqrt{2\pi}}e^{inx}, \qquad \phi_-(x) := \frac{1}{\sqrt{2\pi}}e^{-i(n-1)x}.$$

As $n=0$ and $|\tau|\leqslant\frac{1}{4}$, the formulae in (5.4) are to be replaced by the following ones:

$$\lambda_+(\tau) := \sqrt{\tau^2+\varkappa}, \qquad \lambda_-(\tau) := -\sqrt{\tau^2+\varkappa}, \qquad \phi_+(x) = \phi_-(x) := \frac{1}{\sqrt{\pi}}.$$

If the range of such functions $\lambda_\pm(\tau)$ does not contains 0, we have $\lambda_-(\tau)\neq\lambda_+(\tau)$ for all $|\tau|\leqslant\frac{1}{4}$. Proceeding then as above, one can check easily that there exist one eigenvalue of $\mathcal{Q}_\varepsilon^\tau$ converging to $\lambda_-(\tau)$ and one eigenvalue converging to $\lambda_+(\tau)$. If $\lambda_\pm(\tau)$ are real, the same is true for the corresponding perturbed eigenvalues. And if $\lambda_\pm(\tau)$ are pure imaginary, the perturbed eigenvalues are pure imaginary as well.

6. Convergence of emerging eigenvalues

In this section we provide the first part of the proof of Theorem 2.3. Namely, first we reduce the eigenvalue problem to an equation on a finite segment. Then we prove the first statement of the theorem on the convergence of the emerging eigenvalues. We begin with the periodic equation corresponding to the case $V=0$.

6.1. Floquet theory

In this subsection we study the fundamental solutions of the periodic equations

$$-u''+(i\varepsilon\lambda\gamma-\kappa^2(\lambda))u=0,\quad x\in\mathbb{R}, \tag{6.1}$$

where λ is assumed to range in a compact set in the complex plane.

By $W[u,v](x)$ we denote the Wronskian of functions $u=u(x)$, $v=v(x)$:

$$W[u,v](x)=\det\begin{pmatrix} u(x) & v(x)\\ u'(x) & v'(x)\end{pmatrix}.$$

According the Floquet theory for ordinary differential equations, we consider solutions $\Phi_\varepsilon=\Phi_\varepsilon(x,\lambda)$, $\Psi_\varepsilon=\Psi_\varepsilon(x,\lambda)$ to equation (6.1) with $f=0$ satisfying the initial conditions

$$\Phi_\varepsilon(0,\lambda)=1,\quad \Phi'_\varepsilon(0,\lambda)=0,\quad \Psi_\varepsilon(0,\lambda)=0,\quad \Psi'_\varepsilon(0,\lambda)=1.$$

We introduce the matrices

$$\mathrm{A}^\varepsilon(\lambda)=\begin{pmatrix}\mathrm{A}^\varepsilon_{11}(\lambda) & \mathrm{A}^\varepsilon_{12}(\lambda)\\ \mathrm{A}^\varepsilon_{21}(\lambda) & \mathrm{A}^\varepsilon_{22}(\lambda)\end{pmatrix}:=\mathrm{P}^\varepsilon(2\pi,\lambda),\quad \mathrm{P}^\varepsilon(x,\lambda):=\begin{pmatrix}\Phi_\varepsilon(x,\lambda) & \Psi_\varepsilon(x,\lambda)\\ \Phi'_\varepsilon(x,\lambda) & \Psi'_\varepsilon(x,\lambda)\end{pmatrix}$$

and we observe that

$$\det\mathrm{P}^\varepsilon(x,\lambda)\equiv 1,\qquad \det\mathrm{A}^\varepsilon(\lambda)=1. \tag{6.2}$$

Employing the latter identity, we find the eigenvalues of the matrix $\mathrm{A}^\varepsilon(\lambda)$:

$$\mu^\varepsilon_\pm(\lambda)=\frac{1}{2}\left(\mathrm{A}^\varepsilon_{11}(\lambda)+\mathrm{A}^\varepsilon_{22}(\lambda)\pm z\sqrt{(\mathrm{A}^\varepsilon_{11}(\lambda)+\mathrm{A}^\varepsilon_{22}(\lambda))^2-4}\right), \tag{6.3}$$

$$\mu^\varepsilon_+\mu^\varepsilon_-=1,\quad \mu^\varepsilon_++\mu^\varepsilon_-=\mathrm{A}^\varepsilon_{11}+\mathrm{A}^\varepsilon_{22},\quad \mu^\varepsilon_--\mu^\varepsilon_+=-z\sqrt{(\mathrm{A}^\varepsilon_{11}+\mathrm{A}^\varepsilon_{22})^2-4}, \tag{6.4}$$

where the branch of the square root is fixed by the condition $\sqrt{1}=1$ and $z=+1$ or $z=-1$. Here z is an auxiliary parameter describing which branch of the square root is chosen in (6.3). We could hide this sign inside the square root but for future calculations, it is more convenient to fix the branch of the square root as above and to move the sign into the parameter z.

We choose the eigenvectors of the matrix A^ε as

$$\mathrm{e}^\varepsilon_+(\lambda)=\begin{pmatrix}\mathrm{e}^\varepsilon_{+,1}(\lambda)\\ \mathrm{e}^\varepsilon_{+,2}(\lambda)\end{pmatrix}:=\begin{pmatrix} i\big(A^\varepsilon_{22}(\lambda)-\mu^\varepsilon_+(\lambda)\big)-\kappa(\lambda)\mathrm{A}^\varepsilon_{12}(\lambda)\\ \kappa(\lambda)\big(\mathrm{A}^\varepsilon_{11}(\lambda)-\mu^\varepsilon_+(\lambda)\big)-iA^\varepsilon_{21}(\lambda)\end{pmatrix},$$
$$\mathrm{e}^\varepsilon_-(\lambda)=\begin{pmatrix}\mathrm{e}^\varepsilon_{-,1}(\lambda)\\ \mathrm{e}^\varepsilon_{-,2}(\lambda)\end{pmatrix}:=\begin{pmatrix} \kappa(\lambda)A^\varepsilon_{12}(\lambda)-i\big(\mathrm{A}^\varepsilon_{22}(\lambda)-\mu^\varepsilon_-(\lambda)\big)\\ i\mathrm{A}^\varepsilon_{21}(\lambda)-\kappa(\lambda)\big(\mathrm{A}^\varepsilon_{11}(\lambda)-\mu^\varepsilon_-(\lambda)\big)\end{pmatrix},$$

and by the well-known theorem on fundamental solutions for periodic ordinary differential equations, we can choose fundamental solutions for equation (6.1) as

$$\psi^\varepsilon_\pm(x,\lambda):=\mathrm{e}^\varepsilon_{\pm,1}(\lambda)\Phi_\varepsilon(x,\lambda)+\mathrm{e}^\varepsilon_{\pm,2}(\lambda)\Psi_\varepsilon(x,\lambda),\quad \psi^\varepsilon_\pm(x,\lambda)=e^{x\ln\mu^\varepsilon_\pm(\lambda)}\psi^\varepsilon_{\pm,per}(x,\lambda), \tag{6.5}$$

where $\psi^\varepsilon_{\pm,per}$ are 2π-periodic functions in x.

In what follows, we shall make use of several properties of the functions Φ_ε, Ψ_ε and of the matrix $\mathrm{A}^\varepsilon(\lambda)$. The first property is the holomorphic dependence on ε and λ. It is clear that the functions Φ_ε, Ψ_ε are jointly holomorphic in ε and λ in the sense of the norm in $H^2[b_1,b_2]$ for each bounded segment $[b_1,b_2]\subset\mathbb{R}$. Their Taylor expansions are

$$\Phi_\varepsilon(x,\lambda)=\sum_{j=0}^{\infty}(i\varepsilon\lambda)^j\Phi_j(x,\kappa(\lambda)),\quad \Psi_\varepsilon(x,\lambda)=\sum_{j=0}^{\infty}(i\varepsilon\lambda)^j\Psi_j(x,\kappa(\lambda)), \tag{6.6}$$

where Φ_i, Ψ_i are the solutions to the Cauchy problems

$$\begin{aligned}&\Phi_0''+\kappa^2\Phi_0=0,\quad \Psi_0''+\kappa^2\Psi_0=0,\quad \Phi_i''+\kappa^2\Phi_i=\gamma\Phi_{i-1},\quad \Psi_i''+\kappa^2\Psi_i=\gamma\Psi_{i-1},\quad x\in\mathbb{R},\\ &\Phi_0(0,\kappa)=1,\quad \Phi_0'(0,\kappa)=0,\quad \Psi_0(0,\lambda)=0,\quad \Psi_0'(0,\lambda)=1,\\ &\Phi_i(0,\kappa)=\Phi_i'(0,\kappa)=0,\quad \Psi_i(0,\kappa)=\Psi_i'(0,\kappa)=0.\end{aligned}$$

These problems have explicit solutions:

$$\Phi_0(x,\kappa)=\cos\kappa x,\qquad \Psi_0(x,\kappa)=\frac{\sin\kappa x}{\kappa}, \tag{6.7}$$
$$\Phi_i(x,\kappa)=\int_0^x\gamma(t)\Phi_{i-1}(t,\kappa)\frac{\sin\kappa(x-t)}{\kappa}\,\mathrm{d}t,\quad \Psi_i(x,\lambda)=\int_0^x\gamma(t)\Psi_{i-1}(t,\kappa)\frac{\sin\kappa(x-t)}{\kappa}\,\mathrm{d}t. \tag{6.8}$$

As $\kappa=0$, the above formulae should be understood in the sense of the limit as $\kappa\to0$; for instance, $\Psi_0(x,0)=x$.

Since the function γ is 2π-periodic and odd, we have:

$$\begin{aligned}&\Phi_1(2\pi,\kappa)=\int_0^{2\pi}\gamma(t)\frac{\sin2\pi\kappa+\sin2\kappa(\pi-t)}{2\kappa}\,\mathrm{d}t=\rho_0(2\kappa),\\ &\Phi_1'(2\pi,\kappa)=\int_0^{2\pi}\gamma(t)\cos\kappa t\cos\kappa(2\pi-t)\,\mathrm{d}t=0,\\ &\Psi_1(2\pi,\kappa)=0,\quad \Psi_1'(2\pi,\kappa)=-\rho_0(2\kappa).\end{aligned}$$

Hence,

$$\begin{aligned}
&\mathrm{A}^{\varepsilon}_{11}(\lambda)=\cos 2\pi\kappa(\lambda)+i\varepsilon\lambda\rho_0(2\kappa(\lambda))-\varepsilon^2\lambda^2\rho^{(2)}_{11}(\kappa(\lambda))+O(\varepsilon^3),\\
&\mathrm{A}^{\varepsilon}_{12}(\lambda)=\frac{\sin 2\pi\kappa(\lambda)}{\kappa(\lambda)}-\varepsilon^2\lambda^2\rho^{(2)}_{12}(\kappa(\lambda))+O(\varepsilon^3),\\
&\mathrm{A}^{\varepsilon}_{21}(\lambda)=-\kappa(\lambda)\sin 2\pi\kappa(\lambda)-\varepsilon^2\lambda^2\rho^{(2)}_{21}(\kappa(\lambda))+O(\varepsilon^3),\\
&\mathrm{A}^{\varepsilon}_{22}(\lambda)=\cos 2\pi\kappa(\lambda)-i\varepsilon\lambda\rho_0(2\kappa(\lambda))-\varepsilon^2\lambda^2\rho^{(2)}_{22}(\kappa(\lambda))+O(\varepsilon^3),
\end{aligned}\tag{6.9}$$

$$\rho^{(j)}_{11}(\kappa):=\Phi_j(2\pi,\kappa),\quad \rho^{(j)}_{12}(\kappa):=\Psi_j(2\pi,\kappa),\quad \rho^{(j)}_{21}(\kappa):=\Phi_j'(2\pi,\kappa),\quad \rho^{(j)}_{22}(\kappa):=\Psi_j'(2\pi,\kappa).\tag{6.10}$$

The second property we shall need is an estimate for the growth rate of Φ_ε and Ψ_ε at infinity. Thank to the obvious identities

$$\mathrm{A}^\varepsilon(\lambda)=\big(\mathrm{e}^\varepsilon_-(\lambda)\ \mathrm{e}^\varepsilon_+(\lambda)\big)\,\mathrm{diag}\,\big(\mu^\varepsilon_-(\lambda),\mu^\varepsilon_+(\lambda)\big)\big(\mathrm{e}^\varepsilon_-(\lambda)\ \mathrm{e}^\varepsilon_+(\lambda)\big)^{-1},\quad \mathrm{P}_\varepsilon(x+2n\pi,\lambda)=\big(\mathrm{A}^\varepsilon(\lambda)\big)^n\mathrm{P}^\varepsilon(x,\lambda)$$

as $x\in\mathbb{R}$, $n\in\mathbb{Z}$, and (6.4) it is straightforward to check that for sufficiently small ε we have the estimate:

$$\begin{aligned}
&|\Phi_\varepsilon(x,\lambda)|+|\Phi_\varepsilon'(x,\lambda)|+|\Psi_\varepsilon(x,\lambda)|+|\Psi_\varepsilon'(x,\lambda)|\leqslant c\mu_\varepsilon(\lambda)|x|\mu_\varepsilon^{|x|}(\lambda),\qquad |x|\geqslant 1,\\
&\mu_\varepsilon(\lambda):=\max\{|\mu^\varepsilon_-(\lambda)|,|\mu^\varepsilon_+(\lambda)|\},
\end{aligned}\tag{6.11}$$

where c is a constant independent of x, λ, ε.

6.2. Reduction of eigenvalue equation

In this subsection we reduce the eigenvalue equation for the operator pencil $\mathcal{H}_\varepsilon$ to some differential equation on a finite interval with Robin boundary conditions.

Since the potential V decays exponentially fast at infinity, there are fundamental solutions of the equation

$$-\psi''+V\psi+(i\varepsilon\lambda\gamma-\kappa^2(\lambda))\psi=0\tag{6.12}$$

on $x\in\mathbb{R}$ coinciding at infinity with $\psi^\pm_\varepsilon(x,\lambda)$ up to an exponentially small term. Let us study the properties of such solutions.

We fix arbitrarily $C>0$ and we let $Q_\delta:=\{\lambda\in\mathbb{C}:\ |\lambda|<C,\ |\operatorname{Im}\kappa(\lambda)|\leqslant\delta\}$ for some $\delta\in(0,\frac{\vartheta}{6})$. In what follows we assume that $\lambda\in Q_\delta$. For each positive $r>0$, we introduce the Banach spaces

$$S_{\pm r}:=\left\{u\in C^1(J_{\pm r}):\ \|u\|_{S_{\pm r}}:=\sup_{J_{\pm r}}e^{-\frac{\vartheta}{5}|x|}(|u(x)|+|u'(x)|)<\infty\right\},$$

where $J_{-r}:=(-\infty,-r]$, $J_r:=[r,+\infty)$. On these space we define the integral operators

$$\big(\mathcal{L}_3^\pm(\lambda,\varepsilon)u\big)(x):=\int\limits_{\pm\infty}^{x}\big(\Psi_\varepsilon(x,\lambda)\Phi_\varepsilon(y,\lambda)-\Phi_\varepsilon(x,\lambda)\Psi_\varepsilon(y,\lambda)\big)V(y)u(y)\,\mathrm{d}y.\tag{6.13}$$

Thanks to estimates (6.11), it is straightforward to check that for a large enough and independent of ε, the operators $\mathcal{L}_3^\pm$ are well-defined and contracting in the spaces $S_{\pm a}$. Thanks to the holomorphic dependence on ε and λ of the functions Φ_ε and Ψ_ε discussed in the previous section, the operators $\mathcal{L}_3^\pm$ are also holomorphic in ε and λ.

In view of the definition of the operators $\mathcal{L}_3^{\pm}$ and (6.5), we see easily that the unique solutions $Z_\pm = Z_\pm(x,\lambda,\varepsilon)$ of the equations

$$Z_\pm - \mathcal{L}_3^\pm(\lambda,\varepsilon) Z_\pm = \psi_\pm^\varepsilon, \qquad Z_\pm = \big(\mathcal{I} - \mathcal{L}_3^\pm(\lambda,\varepsilon)\big)^{-1}\psi_\pm^\varepsilon, \tag{6.14}$$

are exactly the sought fundamental solutions of equation (6.12) on the intervals $J_{\pm r}$ coinciding asymptotically with $\psi_\pm^\varepsilon$ as $x\to\pm\infty$:

$$Z_\pm(x,\lambda,\varepsilon) := \psi_\pm^\varepsilon(x,\lambda) + O(e^{-\frac{\vartheta}{3}|x|}), \quad x\to\pm\infty. \tag{6.15}$$

We substitute expansions (6.6) and definition (6.13) into (6.14) and we see that

$$\begin{aligned}
&\big(\mathcal{I} - \mathcal{L}_3^\pm(\lambda,\varepsilon)\big)^{-1}\Phi_\varepsilon = \big(\mathcal{I} - \mathcal{L}_3^\pm(\lambda,0)\big)^{-1}\Phi_0 + O(\varepsilon\lambda),\\
&\big(\mathcal{I} - \mathcal{L}_3^\pm(\lambda,\varepsilon)\big)^{-1}\Psi_\varepsilon = \big(\mathcal{I} - \mathcal{L}_3^\pm(\lambda,0)\big)^{-1}\Psi_0 + O(\varepsilon\lambda).
\end{aligned} \tag{6.16}$$

Employing identities (6.7) and the definition of the operator $\mathcal{L}_3^\pm(\lambda,0)$, it is straightforward to check that

$$\begin{aligned}
&\Big(\big(\mathcal{I} - \mathcal{L}_3^\pm(\lambda,0)\big)^{-1}\Phi_0\Big)(x,\lambda) = \frac{Y_\pm(x,\kappa(\lambda)) + Y_\pm(x,-\kappa(\lambda))}{2},\\
&\Big(\big(\mathcal{I} - \mathcal{L}_3^\pm(\lambda,0)\big)^{-1}\Psi_0\Big)(x,\lambda) = \pm\frac{Y_\pm(x,\kappa(\lambda)) - Y_\pm(x,-\kappa(\lambda))}{2i\kappa(\lambda)}.
\end{aligned} \tag{6.17}$$

Let λ be an eigenvalue of the operator pencil $\mathcal{H}_\varepsilon$. We choose z in formulae (6.3) so that $\operatorname{Re}\ln\mu_-^\varepsilon(\lambda) = -\operatorname{Re}\ln\mu_+^\varepsilon(\lambda) > 0$. Then an associated eigenfunction ψ solves differential equation (6.12) coinciding with $Z_\pm$ up to a multiplicative constant as $x\to\pm\infty$. The latter condition is equivalently written as the boundary conditions

$$\big(Z_\pm\psi' - Z_\pm'\psi\big)\Big|_{x=\pm r} = W[Z_\pm,\psi](\pm r) = 0. \tag{6.18}$$

This is why in what follows, instead of dealing with the eigenvalue equation, we seek non-trivial solutions to problem (6.12), (6.18) on $(-r,r)$. While studying this problem, one of the key ingredient is the following auxiliary lemmata.

Lemma 6.1. *For each $C>0$ there exists $\delta>0$ such that for $k\in\mathbb{C}\setminus\{0\}$ obeying $|k|<C$, $|\operatorname{Im} k|<\delta$ the boundary value problem*

$$-u'' + Vu - k^2 u = f, \quad x\in(-r,r), \tag{6.19}$$
$$W[Y_\pm(\cdot,k),u](\pm r) = f_\pm, \tag{6.20}$$

is uniquely solvable. The solution is represented as

$$u = \mathcal{L}_4(k)f + \mathcal{L}_5(k)(f_-,f_+), \tag{6.21}$$

where $\mathcal{L}_4(k): L_2(-r,r)\to H^2(-r,r)$, $\mathcal{L}_5(k):\mathbb{C}\to H^2(-r,r)$ are bounded operators holomorphic in k.

Proof. The functions $Y_\pm$ are holomorphic in k in the norm in $H^2[-r,r]$. Thanks to the embedding $H^2[-r,r]\subset C^1[-r,r]$, they are also holomorphic in the norm in $C^1[-r,r]$. The Wronskian of the functions $Y_\pm$ is equal to

$2ika(k)$; the function $a(k)$ is also holomorphic. Thanks to identity (2.9), the function $a(k)$ is non-zero provided δ is small enough. Now the statement of the lemma follows the explicit formula (6.21), where

$$\big(\mathcal{L}_4(k)f\big)(x,k)=\frac{1}{2ika(k)}\left(\int\limits_{-r}^{x} Y_+(x,k)Y_-(y,k)f(y)\,\mathrm{d}y+\int\limits_{x}^{r} Y_-(x,k)Y_+(y,k)f(y)\,\mathrm{d}y\right),$$

$$\big(\mathcal{L}_5(k)(f_-,f_+)\big)(x,k)=\frac{Y_+(x,k)f_- - Y_-(x,k)f_+}{2ika(k)}.$$

□

Given $m\in\mathbb{Z}\setminus\{0\}$, we denote

$$U_\pm(x,\zeta,m):=e^{i\zeta}Y_\pm\left(x,\frac{m}{2}\right)\pm e^{-i\zeta}Y_\pm\left(x,-\frac{m}{2}\right),$$

where ζ is a complex parameter, $|\operatorname{Re}\zeta|\leqslant\pi$, $|\operatorname{Im}\zeta|\leqslant C$, $C=const$.

Lemma 6.2. *The boundary value problem for equation (6.19) as $k=\frac{m}{2}$, $m\in\mathbb{Z}\setminus\{0\}$, with the boundary condition*

$$W[U_\pm(\cdot,\zeta,m),u](\pm r)=f_\pm \tag{6.22}$$

is uniquely solvable for all $\zeta\neq\zeta_-$, $\zeta\neq\zeta_+$, where $\zeta_\pm$ are defined in (2.10). The solution is represented as

$$u=\mathcal{L}_6(\zeta)f+\mathcal{L}_7(\zeta)(f_-,f_+), \tag{6.23}$$

where $\mathcal{L}_6(\zeta):L_2(-r,r)\to H^2(-r,r)$, $\mathcal{L}_7(\zeta):\mathbb{C}^2\to H^2(-r,r)$ are bounded operators holomorphic in $\zeta\neq\zeta_\pm$. At $\zeta=\zeta_\pm$, these operators have simple poles

$$\mathcal{L}_6(\zeta)=\frac{X_\pm}{\zeta-\zeta_\pm}\mathcal{L}_8^\pm+\mathcal{L}_9^\pm(\zeta),\qquad \mathcal{L}_8^\pm f:=\Upsilon_\pm\int\limits_{-r}^{r} f(x)X_\pm(x,m)\,\mathrm{d}x, \tag{6.24}$$

$$\mathcal{L}_7(\zeta)=\frac{X_\pm}{\zeta-\zeta_\pm}\mathcal{L}_{10}^\pm+\mathcal{L}_{11}^\pm(\zeta),\qquad \mathcal{L}_{10}^\pm(f_-,f_+):=i\tilde{\Upsilon}_\pm f_-+\Upsilon_\pm f_+, \tag{6.25}$$

where $\mathcal{L}_9^\pm(\zeta):L_2(-r,r)\to H^2(-r,r)$, $\mathcal{L}_{11}^\pm(\zeta):\mathbb{C}^2\to H^2(-r,r)$ are bounded operators holomorphic in ζ close to $\zeta_\pm$.

Proof. Employing identities (2.9), it is straightforward to check that the Wronskian of the functions U_+ and U_- reads as

$$W[U_+,U_-](x,\zeta,m)=G(\zeta,m):=-2im\left(\left|a\left(\tfrac{m}{2}\right)\right|\cos\left(2\zeta+\theta\left(\tfrac{m}{2}\right)\right)-b_r\left(\tfrac{m}{2}\right)\right).$$

The solution to problem (6.19), (6.22) is given explicitly by formula (6.23), where

$$\mathcal{L}_6(\zeta)f=\frac{1}{G(\zeta,m)}\left(\int\limits_{-r}^{x} U_+(x,\zeta,m)U_-(y,\zeta,m)f(y)\,\mathrm{d}y+\int\limits_{x}^{r} U_-(x,\zeta,m)U_+(y,\zeta,m)f(y)\,\mathrm{d}y\right), \tag{6.26}$$

$$\mathcal{L}_7(\zeta)(f_-,f_+):=\frac{f_+U_-(x,\zeta,m)-f_-U_+(x,\zeta,m)}{G(\zeta,m)}. \tag{6.27}$$

We see immediately that these are bounded operators $\mathcal{L}_6(\zeta) : L_2(-r,r) \to H^2(-r,r)$, $\mathcal{L}_7(\zeta) : \mathbb{C}^2 \to H^2(-r,r)$ meromorphic in ζ with poles at the zeroes of $W(\zeta,m)$.

The zeroes of the function $W(\cdot,m)$ can be found explicitly and they are given by (2.10). Thanks to identity (2.9), we have $\frac{|b_{\mathrm{r}}(n)|}{|a(n)|} < 1$ and therefore, the zeroes of W satisfy the inequalities $2\zeta_\pm + \theta(n) \neq 0$, $2\zeta_\pm + \theta(n) \neq \pi$. This yields

$$G'(\zeta_\pm,m) = 4im\left|a\left(\tfrac{m}{2}\right)\right|\sin\left(2\zeta_\pm + \theta\left(\tfrac{m}{2}\right)\right) \neq 0 \tag{6.28}$$

and therefore, $\zeta_\pm$ are simple zeroes. Hence, the function $G^{-1}(\cdot,m)$ has a simple pole at $\zeta_\pm$.

Since $G(\zeta_\pm,m)=0$, the functions $U_+(x,\zeta_\pm,m)$ and $U_-(x,\zeta_\pm,m)$ are linearly dependent and we also have

$$\left|a\left(\tfrac{m}{2}\right)\right|e^{i\left(2\zeta_\pm+\theta\left(\frac{m}{2}\right)\right)} - \overline{b\left(\tfrac{m}{2}\right)} = i\left(\left|a\left(\tfrac{m}{2}\right)\right|\sin\left(2\zeta_\pm + \theta\left(\tfrac{m}{2}\right)\right) + b_{\mathrm{i}}\left(\tfrac{m}{2}\right)\right).$$

Employing this identity and (2.8) and comparing the behavior of $U_+(x,\zeta_\pm,m)$ and $U_-(x,\zeta_\pm,m)$ as $x\to+\infty$, we obtain that

$$U_-(x,\zeta_\pm,m) = i\left(\left|a\left(\tfrac{m}{2}\right)\right|\sin\left(2\zeta_\pm + \theta\left(\tfrac{m}{2}\right)\right) + b_{\mathrm{i}}\left(\tfrac{m}{2}\right)\right)U_+(x,\zeta_\pm,m). \tag{6.29}$$

Expanding now formula (6.26) as $\zeta\to\zeta_\pm$ and employing then the above identity, (6.28) and $X_\pm(x,m) = U_+(x,\zeta_\pm,m)$, we arrive at (6.24), (6.25). ☐

6.3. Convergence

In this subsection we prove that all isolated eigenvalues near the continuous spectrum are localized in ε-neighbourhoods of the points $\lambda_0^{(m)}$, $m\in\mathbb{Z}\setminus\{0\}$ or converge to $\pm\sqrt{\varkappa}$ as $\varepsilon\to+0$. By Theorems 2.1, 2.2 we know that the distance of all possible eigenvalues of $\mathcal{H}_\varepsilon$ to the continuous spectrum of $\mathcal{H}_0+\varkappa-\lambda^2$.

We choose a constant $C>0$ and by Lemma 6.1 we find a corresponding δ. Then we choose a sufficiently small constant $C_1>0$. Let Q^ε be a set of $\lambda\in\mathbb{C}$ defined by the conditions

$$|\kappa(\lambda)|<C, \quad |\operatorname{Im}\kappa(\lambda)|<\delta, \quad \operatorname{dist}\left(\kappa(\lambda),\frac{m}{2}\right)\geqslant\frac{\varepsilon}{C_1}, \quad m\in\mathbb{Z}\setminus\{0\}, \quad \operatorname{dist}\left(\kappa(\lambda),0\right)\geqslant C_1.$$

For $\lambda\in Q^\varepsilon$, we fix z in (6.3) by the condition $z\sqrt{-\sin^2 2\pi\kappa(\lambda)} = i\sin 2\pi\kappa(\lambda)$ and then due to (6.9) we have

$$\mu_\pm^\varepsilon(\lambda) = e^{2\pi i\kappa(\lambda)} + O(\varepsilon), \qquad A_{11}^\varepsilon(\lambda)-\mu_+^\varepsilon(\lambda) = \mu_-^\varepsilon(\lambda) - A_{22}^\varepsilon(\lambda) = -i\sin 2\pi\kappa(\lambda) + O(\varepsilon). \tag{6.30}$$

Since by the definition $\operatorname{Im}\kappa(\lambda)>0$, thanks to (6.14) and the above identities we conclude that the functions $Z_\pm$ decays exponentially as $x\to\pm\infty$ and boundary condition (6.18) corresponds to the eigenfunctions of $\mathcal{H}_\varepsilon$. Identities (6.16), (6.17), (6.30) allow us to describe the behavior of the functions $Z_\pm$ as $\varepsilon\to0$:

$$Z_\pm(\cdot,\lambda,\varepsilon) = 2\sin 2\pi\kappa(\lambda)\left(Y_\pm(x,\kappa(\lambda)) + h_\pm(\cdot,\lambda,\varepsilon)\right), \qquad \|h_\pm+\|_{C^2[-r,r]}\leqslant cC_1^2,$$

where c is a constant independent of ε, λ, C_1. Hence, we can rewrite boundary condition (6.18) as (6.20) with $u=\psi$ and

$$f_\pm := \tilde{h}_\pm\psi(\pm r), \qquad |\tilde{h}_\pm|\leqslant c\,C_1^2, \tag{6.31}$$

where c is a constant independent of ε, λ and C_1. Thanks to Lemma 6.1, we can rewrite problem (6.12), (6.18) on $(-r,r)$ as the following operator equation in $H^2(-r,r)$:

$$\psi + i\varepsilon\lambda\mathcal{L}_4(\kappa(\lambda))\gamma\psi + \mathcal{L}_5(\kappa(\lambda))\big(\tilde{h}_-\psi_-(-r), \tilde{h}_+\psi_+(r)\big) = 0.$$

In view of the holomorphic dependence of the operators $\mathcal{L}_4$, $\mathcal{L}_5$ on $\kappa(\lambda)$ and the estimate for $\tilde{h}_\pm$ in (6.31), we conclude immediately that for sufficiently small ε and c_1 the above equation has the trivial solution only. This proves the desired statement.

7. Existence and asymptotics of emerging eigenvalues

This section is devoted to the proof of the rest of Theorem 2.3. Here we study the eigenvalues of $\mathcal{H}_\varepsilon$ in the vicinities of the points $\lambda_0^{(m)}$, $m\in\mathbb{Z}\setminus\{0\}$. We choose one of such points and redenote it by λ_0. We observe that λ_0 can be either real (if $\frac{m^2}{4}+\varkappa\geqslant 0$) or pure imaginary (if $\frac{m^2}{4}+\varkappa<0$). Throughout the subsection we assume that $\lambda_0\neq 0$, $\rho_0(m)\neq 0$.

7.1. Equation for eigenvalues

In accordance with the results of Section 6.3, possible eigenvalues of $\mathcal{H}_\varepsilon(\lambda)$ in the vicinity of λ_0 satisfy the inequality $|\kappa(\lambda)-\frac{m}{2}|\leqslant C\varepsilon$ for some sufficiently large C and this is why they can be represented as

$$\kappa(\lambda) = \frac{m}{2} + i\varepsilon\frac{(-1)^m\lambda_0\rho_0(m)\xi}{2\pi}, \quad m\in\mathbb{Z}\setminus\{0\}, \tag{7.1}$$

where $\xi\in\mathbb{C}$ is a new complex parameter parameter possibly on ε. We assume that $|\xi|<C$ for some sufficiently large C. We introduce such representation for $\kappa(\lambda)$ since this simplifies significantly many technical details below.

The spectral parameter λ is expressed via ξ as

$$\lambda = \lambda_0\sqrt{1 + (-1)^{m+1}\frac{i\varepsilon\rho_0(m)}{2\pi\lambda_0}\xi - \frac{\varepsilon^2\rho_0^2(m)}{4\pi^2}\xi^2}. \tag{7.2}$$

We introduce new functions $\eta=\eta(\xi,\varepsilon)$, $\varsigma=\varsigma(\xi,\varepsilon)$ as

$$\begin{aligned} &\eta(\xi,\varepsilon) := \frac{\sqrt{\big(A_{11}^\varepsilon(\lambda)+A_{22}^\varepsilon(\lambda)\big)^2-4}}{\big(A_{11}^\varepsilon(\lambda)-A_{22}^\varepsilon(\lambda)\big)\omega(\xi,\varepsilon)}, \quad \varsigma(\xi,\varepsilon) := \frac{\xi}{\omega(\xi,\varepsilon)}, \\ &\omega(\xi,\varepsilon) := \sqrt{1 + \frac{4A_{12}^\varepsilon(\lambda)A_{21}^\varepsilon(\lambda)}{\big(A_{11}^\varepsilon(\lambda)-A_{22}^\varepsilon(\lambda)\big)^2} + \xi^2}, \end{aligned} \tag{7.3}$$

where λ is given by (7.2). Thanks to relations (6.9) and the holomorphic dependence of A_{ij}^ε on ε and λ, it is easy to confirm that the functions ς and ω are holomorphic in ε and ξ, the function η is bounded and $\omega(\xi,\varepsilon)=1+O(\varepsilon)$. The latter identity, the definition of ζ and the inverse function theory imply the ξ can be uniquely recovered from ς for sufficiently small ε and ξ is holomorphic in ς and ε.

The identities (6.4) imply that $\eta^2+\varsigma^2=1$ and this is why we can parametrize η and ς as

$$\eta = \cos 2\zeta, \qquad \varsigma = \sin 2\zeta, \qquad -\frac{\pi}{2}\leqslant |\operatorname{Re}\varsigma| < \frac{\pi}{2}, \qquad |\operatorname{Im}\varsigma|\leqslant C, \tag{7.4}$$

where ζ is a new complex parameter, and C is a sufficiently large fixed constant. In formulae (6.3) for $\mu_\pm^\varepsilon$ we let $z=1$ since then the square root is replaced by $(A_{11}^\varepsilon(\lambda)-A_{22}^\varepsilon(\lambda))\omega(\xi,\varepsilon)\cos 2\zeta$ and the choice of the branch of the square root is controlled now by $\cos 2\zeta$. We also observe that thanks to (7.1), (7.2) we have the representations:

$$\kappa^2(\lambda)=\frac{m^2}{4}+\varepsilon\tilde{\kappa}(\zeta,\varepsilon),\quad \lambda=\varepsilon\tilde{\lambda}(\zeta,\varepsilon),\quad \tilde{\kappa}(\zeta,0)=i\frac{(-1)^m m\lambda_0\rho_0(m)\sin 2\zeta}{2\pi},\quad \tilde{\lambda}(\zeta,0)=\lambda_0, \tag{7.5}$$

where $\tilde{\kappa}$ and $\tilde{\lambda}$ are jointly holomorphic in ζ and ε for sufficiently small ε.

The definition of the functions η and ζ, formulae (6.5), (6.9), (6.14), (6.16), (6.17), (7.3), (7.1), (7.2) imply that for sufficiently small ε the functions $Z_\pm$ can be represented as

$$Z_\pm(x,\lambda,\varepsilon)=\varepsilon\lambda_0\rho_0(m)e^{\mp i\zeta}\big(U_+(x,\zeta,\varepsilon)+\varepsilon\tilde{Z}_+(x,\zeta,\varepsilon)\big), \tag{7.6}$$

where $\tilde{Z}_\pm$ are jointly holomorphic in ζ and ε in the norm of $C^2[-r,r]$.

We substitute (7.6), (7.5) into (6.12), (6.18) and for $u=\psi$ we obtain equation (6.19) with $k=\frac{m}{2}$, $f=-\varepsilon\big(\tilde{\kappa}+i(\lambda_0+\varepsilon\tilde{\lambda}_0)\gamma\big)\psi$ and boundary conditions (6.22) with $f_\pm=-\varepsilon W[\tilde{Z}_\pm,\psi](\pm r,\zeta,\varepsilon)$. The solution to this problem is given by (6.23) and this allows us to rewrite the obtained problem as an operator equation in $H^2(-r,r)$:

$$\begin{gathered}\psi+\varepsilon\mathcal{L}_{12}(\zeta,\varepsilon)\psi=0, \qquad (7.7)\\ \mathcal{L}_{12}(\zeta,\varepsilon)\psi:=\mathcal{L}_6(\zeta)\big(i\tilde{\lambda}_0\gamma-\tilde{\kappa}\big)\psi+\mathcal{L}_7(\zeta)\mathcal{L}_{13}(\zeta,\varepsilon)\psi,\\ \mathcal{L}_{13}(\zeta,\varepsilon)\psi:=\Big(W[\tilde{Z}_\pm,\psi](-r,\zeta,\varepsilon),\,W[\tilde{Z}_\pm,\psi](r,\zeta,\varepsilon)\Big)\in\mathbb{C}^2.\end{gathered}$$

Here the values $\psi(\pm r)$, $\psi'(\pm r)$ are well-defined thanks to the embedding $H^2(-r,r)\subset C^1[-r,r]$.

As ζ is separated from $\zeta_\pm$ by a fixed positive distance, the operator $\mathcal{L}_{12}(\zeta,\varepsilon)$ is bounded in $H^2(-r,r)$ and jointly holomorphic in ζ and ε. Hence, for such values of ζ equation (7.7) has the trivial solution only. Therefore, the values of ζ, for which equation (7.7) has a non-trivial solution, converge to ζ_- or to ζ_+ as $\varepsilon\to+0$. For ζ close to $\zeta_\pm$, we employ the representations (6.24), (6.25) and we rewrite equation (7.7) as

$$\psi+\frac{\varepsilon\mathcal{L}_{14}^\pm(\varepsilon)\psi}{\zeta-\zeta_\pm}+\varepsilon\mathcal{L}_{15}^\pm(\zeta,\varepsilon)\psi=0,\qquad \mathcal{L}_{14}^\pm(,\varepsilon)\psi:=\mathcal{L}_8^\pm\big(i\varepsilon\tilde{\lambda}(\zeta_\pm,\varepsilon)\gamma-\tilde{\kappa}(\zeta_\pm,\varepsilon)\big)\psi+\mathcal{L}_{10}^\pm\mathcal{L}_{13}(\zeta_\pm,\varepsilon)\psi, \tag{7.8}$$

where $\mathcal{L}_{15}^\pm$ are bounded operators in $H^2(-r,r)$ jointly holomorphic in ζ close to $\zeta_\pm$ and in sufficiently small ε. Then the operator $\mathcal{L}_{16}^\pm(\zeta,\varepsilon):=\big(\mathcal{I}+\mathcal{L}_{15}^\pm(\zeta,\varepsilon)\big)^{-1}$ is well-defined, bounded in $H^2(-r,r)$ and jointly holomorphic in ζ close to $\zeta_\pm$ and in sufficiently small ε. Hence, for nontrivial solutions of equation (7.8) we necessarily have $\mathcal{L}_{14}^\pm(\zeta,\varepsilon)\psi\neq 0$. Bearing this in mind, we apply the operator $\mathcal{L}_{16}^\pm(\zeta,\varepsilon)$ to equation (7.8) and then we apply the functional $\mathcal{L}_{14}^\pm$ to the obtained identity. As a result, we see that equation (7.8) has a nontrivial solution for ζ satisfying

$$\zeta-\zeta_\pm+\varepsilon\mathcal{L}_{14}^\pm(\zeta,\varepsilon)\mathcal{L}_{16}^\pm(\zeta,\varepsilon)X_\pm=0 \tag{7.9}$$

and this nontrivial solution is

$$\psi_\varepsilon=\mathcal{L}_{16}^\pm(\zeta,\varepsilon)X_\pm. \tag{7.10}$$

By the inverse function theorem we immediately conclude that equation (7.9) possesses the unique root $\Xi_\pm(\varepsilon)$ holomorphic in ε. The leading terms of its Taylor series are calculated straightforwardly by equation (7.9):

$$\Xi_\pm(\varepsilon)=\zeta_\pm-\varepsilon\tilde{K}_\pm+O(\varepsilon^2), \tag{7.11}$$

$$\begin{aligned}\tilde{K}_\pm:=&\mathcal{L}_{14}^\pm(0)X_\pm=i\lambda_0\Upsilon_\pm\int\limits_{-r}^{r}\left(\gamma-\frac{(-1)^m m\rho_0(m)\sin 2\zeta_\pm}{2\pi}\right)X_\pm^2(x)\,\mathrm{d}x\\&+i\tilde{\Upsilon}_\pm W[\tilde{Z}_-(\cdot,\zeta_\pm,0),X_\pm]\big|_{x=-r}+\Upsilon_\pm W[\tilde{Z}_+(\cdot,\zeta_\pm,0),X_\pm]\big|_{x=r},\end{aligned} \tag{7.12}$$

where we have employed formulae (7.5) for $\tilde{\kappa}(\zeta,0)$ and $\tilde{\lambda}(\zeta,0)$, the identity

$$\mathcal{L}_{16}^\pm(\zeta,\varepsilon)=\mathcal{I}+O(\varepsilon) \tag{7.13}$$

and the definition of the operators $\mathcal{L}_8^\pm$, $\mathcal{L}_{10}^\pm$. Our next step is to calculate the second term in (7.11) and this will be done in the following subsection.

7.2. Asymptotics

We begin with calculating $\tilde{Z}_\pm(x,\zeta_\pm,0)$. Expansions (6.6) and definition (6.13) of the operator $\mathcal{L}_3^\pm(\lambda,\varepsilon)$ yield that

$$\mathcal{L}_3^\pm(\lambda,\varepsilon)=\mathcal{L}_3^\pm(\lambda,0)+i\varepsilon\lambda\mathcal{L}_{17}^\pm(\lambda)+O(\varepsilon^2\lambda^2)$$

uniformly in λ close to λ_0. Here $\mathcal{L}_{17}^\pm(\lambda)$ is the integral operator in $S_{\pm r}$:

$$\big(\mathcal{L}_{17}^\pm u\big)(x):=\int\limits_{\pm\infty}^{x}\big(\Psi_1(x)\Phi_0(y)+\Psi_0(x)\Phi_1(y)-\Psi_1(y)\Phi_0(x)-\Psi_0(y)\Phi_1(x)\big)V(y)u(y)\,\mathrm{d}y.$$

Hence, thanks to (6.17),

$$\begin{aligned}\big(\mathcal{I}-\mathcal{L}_3^\pm(\lambda,\varepsilon)\big)^{-1}\Phi_\varepsilon=&\frac{Y_\pm(x,\kappa(\lambda))+Y_\pm(x,-\kappa(\lambda))}{2}\\&+i\varepsilon\lambda\frac{\hat{Y}_\pm(x,\kappa(\lambda))+\hat{Y}_\pm(x,-\kappa(\lambda))}{2}+O(\varepsilon^2\lambda^2),\end{aligned} \tag{7.14}$$

$$\hat{Y}_\pm\big(\cdot,\kappa(\lambda)\big)=\big(\mathcal{I}-\mathcal{L}_3^\pm(\lambda,0)\big)^{-1}\left(\mathcal{L}_{17}^\pm(\lambda)Y_\pm\big(\cdot,\kappa(\lambda)\big)+\Theta_\pm(\cdot,\kappa(\lambda))\right),$$

$$\Theta_\pm(x,\kappa):=\int\limits_0^x\gamma(t)e^{\pm i\kappa t}\frac{\sin\kappa(x-t)}{\kappa}\,\mathrm{d}t,\qquad \Theta_+=\Phi_1+i\kappa\Psi_1,\qquad \Theta_-=\Phi_1-i\kappa\Psi_1.$$

We substitute expansions (6.6) into the Wronskian of Φ_ε and Ψ_ε and take into consideration that this Wronskian is identically 1. This yields:

$$\Psi_1'\Phi_0+\Psi_0'\Phi_1-\Psi_1\Phi_0'-\Psi_0\Phi_1'\equiv 0.$$

Thanks to this identity and formulae (6.7), (6.8) we immediately get that

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\kappa^2(\lambda)\right)\left(\mathcal{L}_{17}^{\pm}u\right)(x)=\gamma(x)\int\limits_{\pm\infty}^{x}\big(\Psi_0(x)\Phi_0(y)-\Psi_0(y)\Phi_0(x)\big)V(y)u(y)\,\mathrm{d}y.$$

Now, using (6.17), it is straightforward to check that $\hat{Y}_{\pm}(x,k)$ solve the problems:

$$-\hat{Y}_{\pm}''-k^2\hat{Y}_{\pm}+V\hat{Y}_{\pm}=-\gamma Y_{\pm},\quad \pm x>r,\qquad \hat{Y}_{\pm}(x,\kappa)=\Theta_{\pm}(x,\kappa)+O(e^{-\frac{\vartheta}{3}|x|}),\quad x\to\pm\infty. \tag{7.15}$$

In the same way we obtain

$$\left(\mathcal{I}-\mathcal{L}_3^{\pm}(\lambda,\varepsilon)\right)^{-1}\Psi_0=\pm\left(\frac{Y_{\pm}(x,\kappa(\lambda))-Y_{\pm}(x,-\kappa(\lambda))}{2i\kappa(\lambda)}+\varepsilon\lambda\frac{\hat{Y}_{\pm}(x,\kappa(\lambda))-\hat{Y}_{\pm}(x,-\kappa(\lambda))}{2\kappa(\lambda)}\right)+O(\varepsilon^2\lambda^2). \tag{7.16}$$

In what follows we shall need an auxiliary lemma.

Lemma 7.1. *The identities hold true:*

$$\rho_{11}^{(2)}\left(\frac{m}{2}\right)=\rho_{22}^{(2)}\left(\frac{m}{2}\right)=(-1)^m\frac{\rho_0^2(m)}{2},\qquad \rho_{21}^{(2)}\left(\frac{m}{2}\right)-\frac{m^2}{4}\rho_{12}^{(2)}\left(\frac{m}{2}\right)=\frac{(-1)^m\hat{\rho}}{m},$$

where $\hat{\rho}$ is defined in (2.14).

Proof. We substitute formulae (6.9) into the last identity in (6.2) with $\kappa(\lambda)=\frac{m}{2}$, $\lambda=\lambda_0$ and equate the coefficient at the ε^2. This yields:

$$\rho_{11}^{(2)}\left(\frac{m}{2}\right)+\rho_{22}^{(2)}\left(\frac{m}{2}\right)=(-1)^m\rho_0^2(m). \tag{7.17}$$

It follows from (6.7), (6.10) that

$$\begin{aligned}\rho_{11}^{(2)}\left(\frac{m}{2}\right)-\rho_{22}^{(2)}\left(\frac{m}{2}\right)=&\frac{4(-1)^{m+1}}{m^2}\int\limits_0^{2\pi}\mathrm{d}x\int\limits_0^{x}\gamma(x)\gamma(t)\left(\sin\frac{m}{2}x\cos\frac{m}{2}t+\cos\frac{m}{2}x\sin\frac{m}{2}t\right)\sin\frac{m}{2}(x-t)\,\mathrm{d}t\\ =&\frac{4(-1)^{m+1}}{m^2}\int\limits_0^{2\pi}\mathrm{d}x\int\limits_0^{x}\gamma(x)\gamma(t)\left(\cos^2\frac{m}{2}t-\cos^2\frac{m}{2}x\right)\,\mathrm{d}t.\end{aligned}$$

Making the change of variables $x\mapsto 2\pi-x$, $t\mapsto 2\pi-t$ and employing the oddness of γ, we arrive at the identity $\rho_{11}^{(2)}(\frac{m}{2})-\rho_{22}^{(2)}(\frac{m}{2})=0$. This identity and two previous ones lead us to first required identity. The second required identity can be checked by straightforward calculations. ☐

The proven lemma and formulae (6.9), (7.1), (7.3), (7.4) imply that

$$\begin{aligned}&\omega(\xi,\varepsilon)=1+i\varepsilon q(\xi)+O(\varepsilon^2),\qquad \xi=\sin 2\zeta+i\varepsilon q(\sin 2\zeta)+O(\varepsilon^2),\\ &q(\xi):=(-1)^m\lambda_0\xi\left(\frac{\xi^2\rho_0'(m)}{\pi}+\frac{\hat{\rho}}{m^2\rho_0(m)}\right).\end{aligned} \tag{7.18}$$

Employing these identities, (6.9), (7.1), (7.2), (7.3), (7.4) and Lemma 7.1 we obtain that

$$
\begin{aligned}
&\mathrm{e}^{\varepsilon}_{+,1}(\lambda)=2\varepsilon\lambda_0\rho_0(m)e^{-i\zeta}\big(\cos\zeta+\varepsilon T_{+,1}+O(\varepsilon^2)\big), &&\frac{\mathrm{e}^{\varepsilon}_{+,2}(\lambda)}{\kappa(\lambda)}=-2\varepsilon\lambda_0\rho_0(m)e^{-i\zeta}\big(\sin\zeta+\varepsilon T_{+,2}+O(\varepsilon^2)\big),\\
&\mathrm{e}^{\varepsilon}_{-,1}(\lambda)=2i\varepsilon\lambda_0\rho_0(m)e^{i\zeta}\big(\sin\zeta+\varepsilon T_{-,1}+O(\varepsilon^2)\big), &&\frac{\mathrm{e}^{\varepsilon}_{-,2}(\lambda)}{\kappa(\lambda)}=-2i\varepsilon\lambda_0\rho_0(m)e^{i\zeta}\big(\cos\zeta+\varepsilon T_{-,2}+O(\varepsilon^2)\big),
\end{aligned}
$$

where $T_{\pm,j}=T_{\pm,j}(\zeta)$ are some explicitly calculated functions satisfying the identity:

$$
(T_{+,1}+T_{-,2})\sin\zeta_\pm-(T_{-,1}+T_{+,2})\cos\zeta_\pm=i(-1)^m\lambda_0\left(\frac{\rho_0'(m)}{\pi}\sin^2 2\zeta_\pm+\frac{\hat\rho}{m^2\rho_0(m)}\right)\cos 2\zeta_\pm. \tag{7.19}
$$

These relations and (7.14), (7.16), (7.1), (7.18) yield the formula for $\tilde Z_\pm(x,\zeta_\pm,0)$:

$$
\begin{aligned}
\tilde Z_+(x,\zeta_\pm,0)=&\,(T_{+,1}+iT_{+,2})\,Y_+\left(x,\frac{m}{2}\right)+(T_{+,1}-iT_{+,2})\,Y_+\left(x,-\frac{m}{2}\right)\\
&+i\lambda_0\hat U_+(x,\zeta_\pm,m)+\frac{i(-1)^m\lambda_0\rho_0\sin 2\zeta_\pm}{2\pi}\dot U_+(x,\zeta_\pm,m),
\end{aligned}\tag{7.20}
$$

$$
\begin{aligned}
\tilde Z_-(x,\zeta_\pm,0)=&\,(iT_{-,1}+T_{-,2})\,Y_-\left(x,\frac{m}{2}\right)+(iT_{-,1}-T_{-,2})\,Y_-\left(x,-\frac{m}{2}\right)\\
&+i\lambda_0\hat U_-(x,\zeta_\pm,m)+\frac{i(-1)^m\lambda_0\rho_0\sin 2\zeta_\pm}{2\pi}\dot U_-(x,\zeta_\pm,m),
\end{aligned}\tag{7.21}
$$

$$
\begin{aligned}
&\hat U_\pm(x,\zeta,m):=e^{i\zeta}\hat Y_\pm\left(x,\frac{m}{2}\right)\pm e^{-i\zeta}\hat Y_\pm\left(x,-\frac{m}{2}\right),\\
&\dot U_\pm(x,\zeta,m):=e^{i\zeta}\frac{\partial Y_\pm}{\partial\kappa}\left(x,\frac{m}{2}\right)\mp e^{-i\zeta}\frac{\partial Y_\pm}{\partial\kappa}\left(x,-\frac{m}{2}\right).
\end{aligned}
$$

In view of the equations in (7.15), (2.7), the functions $\tilde Z_\pm(x,\zeta_-,0)$ and $\tilde Z_\pm(x,\zeta_+,0)$ solve the equations

$$
-\tilde Z_\pm''(x,\zeta,0)-V\tilde Z_\pm(x,\zeta,0)-\frac{m^2}{4}\tilde Z_\pm(x,\zeta,0)=-i\lambda_0\left(\gamma+\frac{(-1)^m m\rho_0(m)\sin 2\zeta}{2\pi}\right)U_\pm(x,\zeta),\quad \pm x>r,
$$

where $\zeta=\zeta_-$ or $\zeta=\zeta_+$. We multiply these equations by $U_\pm(x,\zeta)$ and integrate by parts over $(-\tilde r,-r)$ or over $(r,\tilde r)$, respectively, for some $\tilde r>r$. Then we add the result to formula (7.12) for $K_\pm$ and we see that $K_\pm$ is independent on the choice of r in its definition. Therefore, we can pass to the limit as $r\to+\infty$ and this will give us the final formula for $K_\pm$.

Let us calculate this limit in details. By (6.29) and (2.12), the functions $\dot U_\pm$ behave at infinity as

$$
\begin{aligned}
&\dot U_+(x,\zeta_\pm,m)=-2x\sin\left(\frac{m}{2}x+\zeta_\pm\right)+O\big(|x|e^{-\frac{|\vartheta|}{3}|x|}\big), && x\to+\infty,\\
&\dot U_-(x,\zeta_\pm,m)=-2ix\cos\left(\frac{m}{2}x-\zeta_\pm\right)+O\big(|x|e^{-\frac{|\vartheta|}{3}|x|}\big), && x\to-\infty,
\end{aligned}
$$

Hence, the following limit is finite:

$$\begin{aligned}&\lim_{N\to+\infty}\Bigg(-\int\limits_{-4\pi N}^{4\pi N} X_\pm^2(x,m)\,\mathrm{d}x+\frac{1}{m}W\big[\dot U_+(\cdot,\zeta_\pm,m),X_\pm\big]\Big|_{x=4\pi N}\\&\qquad+\frac{i\Upsilon_\pm}{m\tilde\Upsilon_\pm}W\big[\dot U_-(\cdot,\zeta_\pm,m),X_\pm\big]\Big|_{x=-4\pi N}\Bigg)\\&=\lim_{N\to+\infty}\Bigg(-\int\limits_{-4\pi N}^{4\pi N} X_\pm^2(x,m)\,\mathrm{d}x+8\pi N\left(1+\frac{\Upsilon_\pm^2}{\tilde\Upsilon_\pm^2}\right)\Bigg)+\frac{2}{m}\left(1-\frac{\Upsilon_\pm^2}{\tilde\Upsilon_\pm^2}\right)\sin 2\zeta_\pm.\end{aligned}\tag{7.22}$$

It is straightforward to check that the functions $\hat U$ have the differentiable asymptotics

$$\begin{aligned}&\hat U_+(x,\zeta_\pm,m)=\frac{4}{m}\int\limits_0^x\gamma(t)\cos\left(\frac{m}{2}t+\zeta_\pm\right)\sin\frac{m}{2}(x-t)\,\mathrm{d}t+O\big(e^{-\frac{\vartheta}{3}|x|}\big), && x\to+\infty,\\&\hat U_-(x,\zeta_\pm,m)=-\frac{4i}{m}\int\limits_0^x\gamma(t)\sin\left(\frac{m}{2}t-\zeta_\pm\right)\sin\frac{m}{2}(x-t)\,\mathrm{d}t+O\big(e^{-\frac{\vartheta}{3}|x|}\big), && x\to+\infty,\end{aligned}$$

and by (2.12) we obtain:

$$\begin{aligned}&W[\hat U_+(\cdot,\zeta_\pm,m),X_\pm]=-4\int\limits_0^x\gamma(t)\cos^2\left(\frac{m}{2}t+\zeta_\pm\right)\,\mathrm{d}t+O\big(e^{-\frac{\vartheta}{3}|x|}\big), && x\to+\infty,\\&W[\hat U_-(\cdot,\zeta_\pm,m),X_\pm]=-\frac{4i\Upsilon_\pm}{\tilde\Upsilon_\pm}\int\limits_0^x\gamma(t)\sin^2\left(\frac{m}{2}t-\zeta_\pm\right)\,\mathrm{d}t+O\big(e^{-\frac{\vartheta}{3}|x|}\big), && x\to-\infty.\end{aligned}$$

Hence, in view of (6.29), the following limit is finite:

$$\begin{aligned}&\lim_{N\to+\infty}\Bigg(\int\limits_{-4\pi N}^{4\pi N}\gamma(x)X_\pm^2(x,m)\,\mathrm{d}x+W[\hat U_+(\cdot,\zeta_\pm,m),X_\pm]\big|_{x=4\pi N}+\frac{i\Upsilon_\pm}{\tilde\Upsilon_\pm}W[\hat U_-(\cdot,\zeta_\pm,m),X_\pm]\big|_{x=-4\pi N}\Bigg)\\&=\lim_{N\to+\infty}\Bigg(\int\limits_{-4\pi N}^{4\pi N}\gamma(x)X_\pm^2(x,m)\,\mathrm{d}x+4N(-1)^m\rho_0(m)\sin 2\zeta_\pm\left(\frac{\Upsilon_\pm^2}{\tilde\Upsilon_\pm^2}-1\right)\Bigg).\end{aligned}$$

We substitute the above formula, (7.22), (2.12), (7.21), (7.20), (7.19) into (7.12) and we arrive at the final formula for $\tilde K_\pm$:

$$\tilde K_\pm=i(-1)^m\lambda_0\Upsilon_\pm K_\pm+2i(-1)^{m+1}m\lambda_0\Upsilon_\pm\left(\frac{\rho_0'(m)}{\pi}\sin^2 2\zeta_\pm+\frac{\hat\rho}{m^2\rho_0(m)}\right)\cos 2\zeta_\pm,$$

where $K_\pm$ is defined in (2.15).

7.3. Existence

According the results of two previous subsections, equation (6.12) on $(-r,r)$ subject to boundary conditions (6.18) has a non-trivial solution for λ determined by (7.2), (7.3), (7.4) with $\zeta=\Xi(\varepsilon)$ being the root of equation (7.9) with asymptotics (7.11). The associated non-trivial solution is given by formula (7.10). We continue this function outside the interval $(-r,r)$ as

$$\psi_\varepsilon(x)=C_\pm(\varepsilon)Z_\pm\big(x,\tilde\lambda(\Xi_\pm(\varepsilon),\varepsilon),\varepsilon\big),\quad \pm x>r,\qquad C_\pm(\varepsilon):=\frac{\psi_\varepsilon(\pm r)}{Z_\pm\big(\pm r,\tilde\lambda(\Xi_\pm(\varepsilon),\varepsilon),\varepsilon\big)},$$

and this function is a non-trivial solution to equation (6.12) on the entire line. Thanks to (6.15), the function ψ_ε behaves at infinity as

$$\psi_\varepsilon(x)=C_\pm(\varepsilon)\psi_\pm^\varepsilon\big(x,\tilde\lambda(\Xi_\pm(\varepsilon),\varepsilon)\big)+O(e^{-\frac{\vartheta}{3}|x|}),\quad x\to\pm\infty. \tag{7.23}$$

Let us prove that the constants $C_\pm$ are well-defined and non-zero. By (7.10), (7.13) we get immediately that

$$\psi_\varepsilon=X_\pm+O(\varepsilon) \tag{7.24}$$

in the norm of the space $H^2(-r,r)$ for each fixed r. Definition (2.11) of the functions $X_\pm$ and asymptotics for $Y_\pm$ in (2.7) imply that there exists a sufficiently large $r_\pm>0$ such that $|X_\pm(r_\pm)|\geqslant c_0$, $|X_\pm(-r_\pm)|\geqslant c_0$ for some fixed $c_0>0$. By (7.24), (7.6), (6.15) this yields that $|\psi_\varepsilon(r_\pm)|\geqslant\frac{1}{6}$,

$$\begin{aligned}
&|Z_+\big(r_\pm,\tilde\lambda(\Xi_\pm(\varepsilon),\varepsilon)\big)|\geqslant c_0>0, &&|Z_-\big(-r_\pm,\tilde\lambda(\Xi_\pm(\varepsilon),\varepsilon)\big)|\geqslant c_0>0,\\
&|\psi_+^\varepsilon\big(r_\pm,\tilde\lambda(\Xi_\pm(\varepsilon),\varepsilon)\big)|\geqslant c_0>0, &&|\psi_-^\varepsilon\big(-r_\pm,\tilde\lambda(\Xi_\pm(\varepsilon),\varepsilon)\big)|\geqslant c_0>0
\end{aligned}$$

for some fixed c_0 independent of ε. Hence, the constants $C_\pm(\varepsilon)$ are well-defined and are non-zero for sufficiently small ε.

Due to (7.23) and the second identity in (6.5), the function ψ_ε belongs to $H^2(\mathbb{R})$ if and only if $\operatorname{Re}\ln\mu_+^\varepsilon=-\operatorname{Re}\ln\mu_-^\varepsilon=\ln|\mu_+^\varepsilon|<0$, which is equivalent to $|\mu_+^\varepsilon|<1$. Let us find out the conditions ensuring or breaking this inequality.

Formulae (6.3), (7.3), (7.4), (7.17) yield that

$$\mu_+^\varepsilon(\lambda)=\frac{A_{11}^\varepsilon(\lambda)+A_{22}^\varepsilon(\lambda)}{2}+\frac{A_{11}^\varepsilon(\lambda)-A_{22}^\varepsilon(\lambda)}{2}\omega(\xi,\varepsilon)\cos 2\zeta,$$

where $\lambda=\tilde\lambda(\Xi_\pm(\varepsilon),\varepsilon)$ and ξ is expressed via $\zeta=\Xi_\pm(\varepsilon)$. Thanks to (6.9), (7.1), (7.2), (7.3), (7.4), (7.11), (7.12), we can calculate the asymptotics of $\mu_\varepsilon^+(\Xi_\pm(\varepsilon))$ as $\varepsilon\to+0$:

$$\mu_\varepsilon^+(\Xi_\pm(\varepsilon),\varepsilon)=(-1)^m+i\varepsilon\lambda_0\rho_0(m)\cos 2\zeta_\pm+\varepsilon^2\frac{(-1)^m\lambda_0^2}{2}(M_\pm-\rho_0^2(m)\cos^2 2\zeta_\pm)+O(\varepsilon^3),$$

where the constants $M_\pm$ are defined by (2.19). Hence,

$$|\mu_\varepsilon^+(\Xi_\pm(\varepsilon),\varepsilon)|^2=1+\varepsilon^2\lambda_0^2M_\pm+O(\varepsilon^3)$$

if λ_0 is real and

$$|\mu_\varepsilon^+(\Xi_\pm(\varepsilon),\varepsilon)|^2=1+2(-1)^{m+1}\operatorname{Im}\lambda_0\rho_0(m)\cos 2\zeta_\pm+\varepsilon^2(\operatorname{Im}\lambda_0)^2(2\rho_0^2(m)\cos^2 2\zeta_\pm-M_\pm)+O(\varepsilon^3)$$

if λ_0 is pure imaginary. The obtained asymptotics imply that the operator pencil $\mathcal{H}_\varepsilon$ has no eigenvalue associated with $\zeta_\pm$ under conditions (2.20) and it has an isolated eigenvalue given by the formula $\lambda_\varepsilon = \tilde{\lambda}(\Xi_\pm(\varepsilon), \varepsilon)$ under conditions (2.19). In the latter case the asymptotics of this eigenvalue is calculated via formulae (7.2), (7.3), (7.4), (7.11) and this leads us to (2.18). This completes the proof of Theorem 2.3.

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