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ON L^∞ STABILIZATION OF DIAGONAL SEMILINEAR HYPERBOLIC SYSTEMS BY SATURATED BOUNDARY CONTROL*

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Abstract. This paper considers a diagonal semilinear system of hyperbolic partial differential equations with positive and constant velocities. The boundary condition is composed of an unstable linear term and a saturated feedback control. Weak solutions with initial data in $L^2([0,1])$ are considered and well-posedness of the system is proven using nonlinear semigroup techniques. Local L^∞ exponential stability is tackled by a Lyapunov analysis and convergence of semigroups. Moreover, an explicit estimation of the region of attraction is given.

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1. Introduction

1.1. Literature review

The problem of saturated control has gained interest in the last few years because of the increasing need of precision for modeling real actuators. Physical controllers cannot provide infinite energy and sometimes, they saturate rendering classical unsaturated models obsolete. To avoid such situations, engineers choose controllers powerful enough to avoid saturation when the system operates in standard conditions. However, over-dimensioning actuators is not optimal in term of mass and cost of operation for many sophisticated systems as satellites for example. Moreover, in some exceptional configurations, actuators could saturate and lead to very dangerous situations; unpredictable via linear theory.

In fact, disregarding nonlinearities coming from saturation in the input can be source of undesirable and even catastrophic behaviors for the closed-loop system. In [7], authors showed that in presence of magnitude saturation, the closed-loop system can become unstable if the initial data is too "large" for a certain norm. As a consequence, it is important to determine a precise estimation of the region of attraction.

In this article, we are interested in infinite-dimensional systems involving hyperbolic 1-D partial differential equations (PDEs). To the best of our knowledge, the first work analyzing the effect of saturation in infinite-dimensional systems is [18]. In particular, in [18] the author focuses on the case of compact and bounded control operators, with an a priori constraint. Recently, in [17] the case of distributed saturating control has been

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considered. The results in [13] suggest the use of an observability condition for the analysis of systems modeled by PDEs controlled via closed-loop saturating controllers. In particular, the contraction semigroup obtained from the saturating closed-loop system is compared with the corresponding saturation-free semigroup.

In this paper, we focus on feedback boundary control of a diagonal system of semilinear hyperbolic PDEs. The literature on unsaturated linear boundary control for semilinear hyperbolic systems is rich; see, *e.g.*, [3] or [12], just to mention a few. However, when input saturation comes into play, the inherent nonlinear nature of the problem renders the analysis much harder. As such, only a few papers focused on saturated boundary control of systems modeled via PDEs. For example, in [14], the authors prove that two-dimensional quasilinear hyperbolic systems with opposite velocities are stabilizable with bounded C^1 boundary control inputs. Nevertheless, the method mainly relies on characteristics and does not seem to be generalizable to system with a larger number of PDEs. The results in [17] are tailored to the wave equation $z_{tt} = z_{xx}$ subject to a nonlinear saturated boundary condition; which is a special second-order hyperbolic PDE. In particular, inspired by [1], the authors in [17] relies on the theory of nonlinear semigroups to prove well-posedness and global H^1 exponential stability for the wave equation, in the presence of distributed or boundary saturated controllers. The main idea consists of using a sector bounded approach inspired by the literature of finite dimensional systems [19], to ensure exponential decay of an H^1-Lyapunov functional.

In this manuscript, as opposed to [14, 17], we directly consider the following system of semilinear hyperbolic PDEs of arbitrary dimension $d \in \mathbb{N}$

$$\left\{ \begin{array}{lcl} R_t + \Lambda R_x & = & g \circ R \\ R(0,t) & = & HR(1,t) + Bu(t) \\ R(.,0) & = & R_0 \end{array} \right.$$

where Λ is a diagonal positive definite matrix, H and B are $d \times d$ real matrices, and $g \in C^1(\mathbb{R}^d)$ is a globally Lipschitz function, with Lipschitz constant L_g, such that $g(0) = 0$. Moreover, g is diagonal in the sense that for all $R \in \mathbb{R}^d,\ g_i(R) = g_i(R_i)$.

The open-loop system may turn out to be unstable if the matrix H is too "large" (in the sense of a certain norm). The source term g has also its impact on the stability. According to its form, it could make the open-loop system more or less stable. In [3] chapter 1, typical examples of systems modeled by hyperbolic PDEs with feedback boundary conditions are cited; the telegrapher equations for electrical lines, the shallow water (Saint-Venant) equations for open channels, the isothermal Euler equations for gas flow in pipelines or even the Aw-Rascle equations for road traffic.

In [4], authors found a sufficient condition on matrix $K \in M_d(\mathbb{R})$ such that the linear control $u(t) = KR(1,t)$ ensures C^1 exponential stability of the following "unsaturated" closed-loop system:

$$\left\{ \begin{array}{ll} R_t + \Lambda R_x & = 0 \\ R(0,.) & = (H + BK)R(1,.) \\ R(.,0) & = R_0 \in C^1([0,1]). \end{array} \right. \tag{1.1}$$

In particular, defining for all matrices $M \in M_d(\mathbb{R})$, $\mathcal{R}_\infty(M) = \max_{i=1,\ldots,d} \sum_{j=1}^d |M_{i,j}|$, ([4], Thm. 3.3) established that if:

$$\rho_\infty(H + BK) := \inf_{\Delta \in D_d^+(\mathbb{R})} \mathcal{R}_\infty\left(\Delta (H + BK) \Delta^{-1}\right) < 1$$

then the unsaturated system (1.1) is C^1 exponentially stable for the canonical norm of $C^1([0,1])$. Note that ([4], Thm. 3.3) was proven for small initial data and for quasilinear systems.

1.2. Definition of the system and contribution

In this paper, it is assumed that there exists a matrix gain K such that $\rho_\infty(H+BK)<1$ and we will study the L^∞ stability of the saturated closed-loop system:

$$\begin{cases} R_t+\Lambda R_x = g\circ R \\ R(0,.) = HR(1,.)+B\sigma(KR(1,.)) \\ R(.,0) = R_0\in L^\infty([0,1]) \end{cases} \tag{1.2}$$

with σ defined as a saturation by component *i.e.* there exists a $\sigma_s>0$ such that for all $i\in[\![1,d]\!],\ x\in\mathbb{R}$,

$$\begin{cases} \sigma_i(x)=x & \text{if } |x|\le\sigma_s \\ \sigma_i(x)=\operatorname{sign}(x)\sigma_s & \text{otherwise.} \end{cases}$$

Remark 1.1. For simplicity, we take σ such that the value of the saturation level σ_s is identical for each component. This is not a restriction with respect to the general case

$$\begin{cases} \sigma_i(x)=x & \text{if } |x|\le\sigma_{s,i} \\ \sigma_i(x)=\operatorname{sign}(x)\sigma_{s,i} & \text{otherwise} \end{cases}$$

where $(\sigma_{s,i})_{i\in[\![1,d]\!]}\in(\mathbb{R}^+)^d$, under a suitable change of variables.

The initial data R_0 being in $L^\infty([0,1])$, solutions of (1.2) has to be understood in a weak sense. The main contribution of this paper is to answer the two following problems:

Problem 1.2. Define the sense of a weak solution to system (1.2) and prove a well-posedness theorem.

Problem 1.3. Prove the L^∞ local exponential stability of this system with an estimation of the region of attraction.

Problem 1.2 will be solved using a smooth approximation of the system (1.2) coming from a smoothed sequence of saturations. Convergence of semigroups allows to define weak solutions to (1.2) and prove the well-posedness. Problem 1.3 will be tackled using an approximation of the $L^\infty([0,1])$ norm by $L^p([0,1])$ norms ($p\in\mathbb{N}$).

The rest of this paper is organized as follows. In Section 2, all main results are formulated by two theorems; the first one states the well-posedness and the other, the exponential stability. In the same section, an estimation of the region of attraction is given. In Section 3, the estimated region of attraction for systems taken from the literature is compared with the region of non-saturation. Some concluding remarks and further orientations are given in Section 4.

Notation: For any integers n and m, the set $[\![m,n]\!]:=\{m,m+1,\dots,n\}$. Unless specified, spaces of vector valued functions in $L^p([0,1]),C^p([0,1])$ ($p\in[\![1,\infty]\!]$) are equipped, respectively, with the canonical norms $||\cdot||_{L^p([0,1])}$ and $||\cdot||_{C^p([0,1])}$. The symbol $D_d^+(\mathbb{R})$ designates the set of $d\times d$ positive definite diagonal matrices. Let $R=(R_1,R_2,\dots,R_d)\in\mathbb{R}^d$, $|R|$ denotes the Euclidean norm of R while $|R|_{\max}=\max_{i\in[\![1,d]\!]}|R_i|$. Given $M\in M_d(\mathbb{R})$, we denote $|M|=\sup_{|R|\in\mathbb{R}^d,\ |R|=1}|MR|$, $\mathcal{R}_\infty(M)=\max_{i=1..d}\left(\sum_{j=1}^d|M_{i,j}|\right)$, $\rho_\infty(M)=\inf_{\Delta\in D_d^+(\mathbb{R})}\mathcal{R}_\infty\left(\Delta M\Delta^{-1}\right)$. Given a function $(x,t)\mapsto f(x,t)$, f_t and f_x denote, respectively, the partial derivative of f with respect to t and x. When unspecified, T stands for an arbitrary positive real used to define spaces of the form $C^q([0,T],X)$ where $q\in\mathbb{N}$ and X is a Banach space.

2. Main results

In this section, results for well-posedness and exponential stability are stated.

2.1. Problem 1.2

To properly define a weak solution to system (1.2), we need to give a precise sense to the trace of this solution on the lines $s \mapsto (x = 0, t = s)$ and $s \mapsto (x = 1, t = s)$. To do so, smoothed solutions subject to smoothed saturations are used. Such smoothed saturations approximate σ in the sense of Definition 2.1.

Definition 2.1. $(\sigma_n)_n$ is a smooth approximation of σ if it is in $C^1(\mathbb{R})$ and converges uniformly to σ on $\mathbb{R}$.

Remark 2.2. An example of smoothed saturation $(\sigma_n)_n$ (with n an integer) approximation of σ is defined by:

$$\begin{cases} \sigma'_{n,i}(x) & = 1 \text{ if } x \in [-\sigma_s, \sigma_s] \\ \sigma'_{n,i}(x) & = \frac{1}{2} + \frac{\cos(n(x-\sigma_s))}{2} \ \text{ if } x \in [\sigma_s, \sigma_s + \pi/n] \\ \sigma'_{n,i}(x) & = \frac{1}{2} + \frac{\cos(n(x+\sigma_s))}{2} \ \text{ if } x \in [-\sigma_s, -\sigma_s - \pi/n] \\ \sigma'_{n,i}(x) & = 0 \text{ otherwise} \end{cases}$$

and σ_n which is a primitive of σ'_n, is chosen as:

$$\begin{cases} \sigma_{n,i}(x) & = x \text{ if } x \in [-\sigma_s, \sigma_s] \\ \sigma_{n,i}(x) & = \frac{x+\sigma_s}{2} + \frac{\sin(n(x-\sigma_s))}{2n} \ \text{ if } x \in [\sigma_s, \sigma_s + \pi/n] \\ \sigma_{n,i}(x) & = \frac{x-\sigma_s}{2} + \frac{\sin(n(x+\sigma_s))}{2n} \ \text{ if } x \in [-\sigma_s, -\sigma_s - \pi/n] \\ \sigma_{n,i}(x) & = \sigma_s + \frac{\pi}{n} \text{ otherwise .} \end{cases}$$

It is easy to show that $(\sigma_n)_n$ tends uniformly towards σ on $\mathbb{R}$. We represent both sequences $(\sigma_n)_n$ and $(\sigma'_n)_n$ in Figure 1.

Remark 2.3. In this paper and for every approximation of σ, the maximum of σ_n will be denoted $\sigma_{s,n}$ for all integers n.

Now, taking a smooth approximation $(\sigma_n)_n$ of σ and an integer n, we define another system "smoother" than (1.2). The system subject to the saturation σ_n is defined by:

$$\begin{cases} R_{n,t} + \Lambda R_{n,x} = g \circ R_n \\ \quad R_n(0,.) \quad = HR_n(1,.) + B\sigma_n(KR_n(1,.)) \\ \quad R_n(.,0) \quad = R_{0,n} \in H^2([0,1]). \end{cases} \tag{2.1}$$

Thanks to Theorem A.1 given in Appendix, we will show that if the initial data is $H^2([0,1])$ and satisfies compatibility conditions of order 1 (2.2) then the previous system of PDEs has a unique solution in $C^0([0,T], H^1([0,1])) \cap C^1([0,T], L^2([0,1]))$ for any $T > 0$. Hence, traces of this unique solution $R_n(0,.)$ and $R_n(1,.)$ are well-defined on almost everywhere sense.

Remark 2.4. Note that compatibility conditions of order 1 depends on the chosen σ_n. They are expressed as follows

$$\begin{cases} R_0(0) & = HR_0(1) + B\sigma_n(KR_0(1)) \\ R_{0,x}(0) & = \Lambda^{-1}\Big([H + B\sigma'_n(R_0(1))](\Lambda R_{0,x}(1) - g(R_0(1))) + g(R_0(0))\Big). \end{cases} \tag{2.2}$$

The definition of weak solutions is given here.

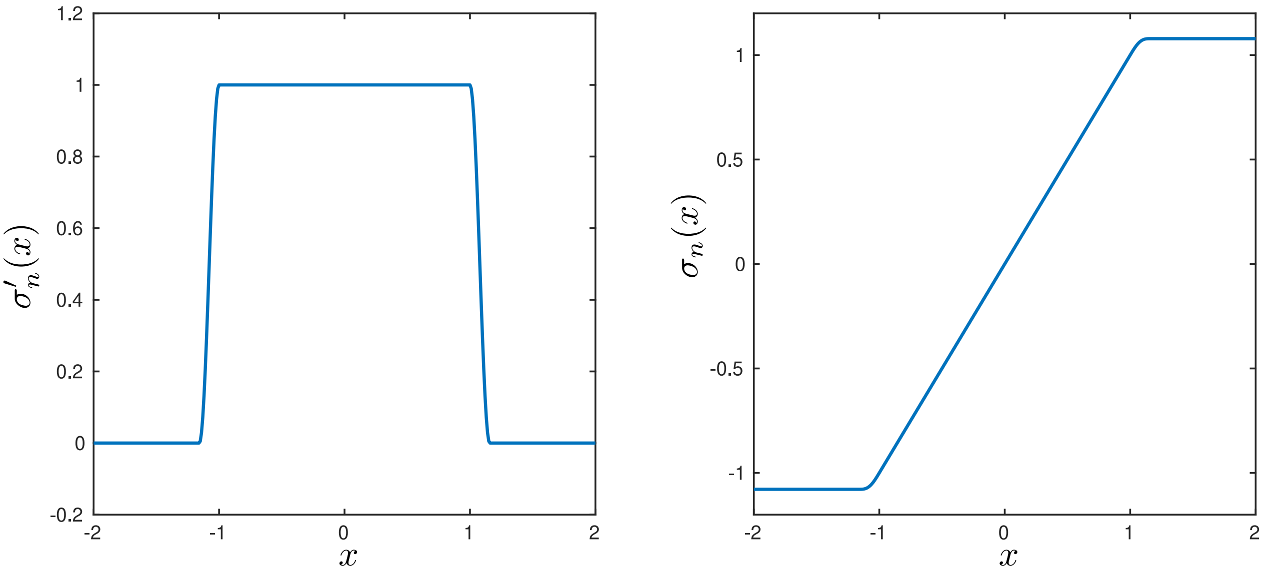

FIGURE 1. Functions σ'_n and σ_n for $n = 20$ and $\sigma_s = 1$.

Definition 2.5. For all $T > 0$, $R \in C^0([0,T], L^2([0,1]))$ is a weak solution to the problem:

$$\left\{\begin{array}{ll} R_t + \Lambda R_x & = g \circ R \\ R(0,.) & = HR(1,.) + B\sigma(KR(1,.)) \\ R(.,0) & = R_0 \in L^2([0,1]), \end{array}\right. \tag{2.3}$$

if there exist a sequence $(\sigma_n)_n$, smooth approximation of σ, and a sequence $(R_{0,n})_n$ in $H^2([0,1])$ satisfying the compatibility conditions of order 1 (2.2) (which depend on the saturation σ_n chosen) tending towards R_0 in $L^2([0,1])$, such that the sequence of solutions $(R_n)_n$ to (2.1) converges towards R in $C^0([0,T], L^2([0,1]))$.

Remark 2.6. This definition is different from the common definition of a weak L^2 solution ([3], Def. A.3). The adjoint (for the usual $L^2([0,T])$ scalar product) of the boundary operator $f \mapsto Hf + B\sigma(Kf)$ may not exist. As a consequence, it is impossible to define a boundary condition on test functions. Therefore, we cannot use the common notion of weak solutions. In [2], authors proved the well-posedness of quasilinear scalar problems subject to L^∞ boundary conditions. They used the method of vanishing viscosity to prove the existence and the uniqueness of the weak solution. The method consists of using a regularized system with additional viscosity and pass to the limit in the weak formulation of the PDE considered. Here, we use the same idea: a regularized system is considered and by a passage to the limit, the weak solution is defined.

It turns out that this problem is well-posed in the sense of Hadamard:

Theorem 2.7 (Well-Posedness)**.** *There exists a unique weak solution to problem (2.3). Moreover, the flow operator defined by:*

$$U_T : \left\{\begin{array}{ccc} L^2([0,1]) & \to & C^0([0,T], L^2([0,1])) \\ R_0 & \mapsto & R \end{array}\right.$$

is continuous for all $T > 0$.

This theorem is proven in Appendix A.

Remark 2.8. Theorem 2.7 holds for whatever σ bounded and continuous such that there exists a smooth approximation of σ in the sense of Definition 2.1. Moreover, Theorem 2.7 is also valid for a nondiagonal (but Lipschitz) source term.

2.2. Problem 1.3

For exponential stability, we introduce the Lipschitz constants $L_{g,i}$ such that

$$\forall R \in \mathbb{R}^d,\ i \in [\![1,d]\!],\ |g_i(R_i)| \leq L_{g,i}|R_i|. \tag{2.4}$$

Hence, for all integers i, the scalar function $R \mapsto g_i(R)$ is a scalar function from $\mathbb{R}$ to $\mathbb{R}$ which is $L_{g,i}$ Lipschitz. Then defining

$$\forall f \in L^\infty([0,1]),\ V(f) := \max_{i\in[\![1,d]\!]} |\delta_i f_i e^{-\mu x}|_{L^\infty([0,1])} \tag{2.5}$$

where $\Delta = \mathrm{diag}(\delta_i)$ is selected such that $\mathcal{R}_\infty(\Delta(H+BK)\Delta^{-1}) \leq 1$ (possible because $\rho_\infty(H+BK) < 1$ is assumed all along this article). One gets the following result:

Theorem 2.9 (Exponential Stability)**.** *Suppose $\rho_\infty(H+BK) < 1$. For all $\Delta \in D_d^+(\mathbb{R})$ satisfying $\mathcal{R}_\infty(\Delta(H+BK)\Delta^{-1}) < 1$, all positive $\mu < -\log(\mathcal{R}_\infty(\Delta(H+BK)\Delta^{-1}))$ and for all initial data $R_0 \in L^\infty([0,1])$, if*

$$\mu\lambda_{\min} - L_{g,\max} \geq 0$$

where $L_{g,\max} := \max_{i\in[\![1,d]\!]} L_{g,i}$,
and

$$V(R_0) < e^{-\mu} \frac{\mathcal{R}_\infty(\Delta B\Delta^{-1})\sigma_s}{|\mathcal{R}_\infty(\Delta(H+BK)\Delta^{-1}) + \mathcal{R}_\infty(\Delta B\Delta^{-1})\mathcal{R}_\infty(\Delta K\Delta^{-1}) - e^{-\mu}|}, \tag{2.6}$$

then the weak solution to (2.3) verifies:

$$\forall t \geq 0,\ V(R(.,t)) \leq e^{-(\mu\lambda_{\min} - L_{g,\max})t} V(R_0).$$

This theorem is proven in Appendix B.

Remark 2.10. As

$$\forall R \in L^\infty([0,1]),\ \min_i \{\delta_i\}\, e^{-\mu} ||R||_{L^\infty([0,1])} \leq V(R) \leq \max_i \{\delta_i\}\, ||R||_{L^\infty([0,1])},$$

Theorem 2.9 gives the classical local exponential stability of system (2.3) with respect to the usual $L^\infty([0,1])$ norm.

Remark 2.11. Concerning the estimation of the region of attraction, we will see in the proof of Theorem 2.9 (Rem. B.5) that such an estimation (2.6) comes from a sector bounded condition imposed on the dead-zone function; the difference between the linear and the saturated control.

3. Numerical example

In this section, we analyze a typical example of diagonal semilinear systems taken from [16]; an article considering the same kind of systems plus a disturbance; discarded here for our purposes. Matrices are defined as:

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{pmatrix},\ H = \begin{pmatrix} 0 & 1.1 \\ 1 & 0 \end{pmatrix},\ B = I_2.$$

In [16], authors consider a system of transport PDEs with positive velocities and without source term. They give a method to find a gain matrix K such that the equivalent linear system subject to the control $R(0,t) = (H + BK)R(1,t)$ be L^2 exponentially stable and robust. Three gain matrices K were compared with an increasing rate (or at least an estimation of this rate) of exponential decay for the L^2 norm:

$$K_1 = \begin{pmatrix} 0 & -0.1050 \\ -0.1045 & 0 \end{pmatrix}$$

$$K_2 = \begin{pmatrix} 0 & -0.4777 \\ -0.4651 & 0 \end{pmatrix}.$$

The last gain is taken from [10]:

$$K_3 = \begin{pmatrix} 0 & -0.7 \\ -1 & 0 \end{pmatrix}.$$

For all these matrix gains, we evaluate the region of attraction thanks to the estimation given in Theorem 2.9 and evaluate if it is larger than the domain where the saturation does not apply.

We take $\mu = 0$ in (2.6) and approximate the stability region of Theorem 2.9 by:

$$V(R_0) < \frac{\mathcal{R}_\infty(\Delta B \Delta^{-1})\sigma_s}{|\mathcal{R}_\infty(\Delta(H+BK)\Delta^{-1}) + \mathcal{R}_\infty(\Delta B \Delta^{-1})\mathcal{R}_\infty(\Delta K \Delta^{-1}) - 1|} \tag{3.1}$$

equivalent to:

$$\forall i \in [\![1, d]\!],\ |R_{0,i}|_{L^\infty([0,1])} < |R_{0,i}|_{L^\infty,\lim}$$

where $|R_{0,i}|_{L^\infty,\lim} := \frac{1}{\delta_i} \frac{\mathcal{R}_\infty(\Delta B \Delta^{-1})\sigma_s}{|\mathcal{R}_\infty(\Delta(H+BK)\Delta^{-1}) + \mathcal{R}_\infty(\Delta B \Delta^{-1})\mathcal{R}_\infty(\Delta K \Delta^{-1}) - 1|}$.

Taking $\Delta \in D_d^+(\mathbb{R})$ minimizing $\mathcal{R}_\infty(\Delta(H+BK)\Delta^{-1})$ and a saturation such that $\sigma_s = 1$, one gets the numerical results from Table 1 for the estimation of the region of attraction:

Table 1. The estimation of the region of attraction.

K	K_1	K_2	K_3
$\rho_\infty(H+BK)$	0.95	0.58	$\simeq 0$
$\lvert R_{0,1}\rvert_{L^\infty,\lim}$	17.5	11.0	$\simeq 0$
$\lvert R_{0,2}\rvert_{L^\infty,\lim}$	16.6	10.1	$\simeq 0$
$\lvert (KR_0)_1\rvert_{\lim}$	1.84	5.61	$\simeq 0$
$\lvert (KR_0)_2\rvert_{\lim}$	1.93	5.91	$\simeq 0$

where $|(KR_0)_1|_{\lim} := \sup\Big\{|(KR)_1| \mid |R_i| < |R_{0,i}|_{L^\infty,\lim},\ \forall i \in [\![1,2]\!]\Big\}$ and $|(KR_0)_2|_{\lim} := \sup\Big\{|(KR)_2| \mid |R_i| < |R_{0,i}|_{L^\infty,\lim},\ \forall i \in [\![1,2]\!]\Big\}$.

We added a row giving the values of $\rho_\infty(H+BK)$ as it gives an estimation of the rate of convergence $\mu\lambda_{\min} - L_{g,\max}$ of the L^∞ norm of the solution R. This can be seen from the condition "$\mu < -\log(\mathcal{R}_\infty(\Delta(H+BK)\Delta^{-1}))$" of Theorem 2.9.

We remark that for K_1 and K_2, $|(KR_0)_1|_{\lim}$ and $|(KR_0)_2|_{\lim}$ are both larger than the saturation $\sigma_s = 1$ and hence, the estimated region of attraction is larger than the linear unsaturated region. Then, we also remark that there exists a balance between the rate of convergence of the saturated system estimated by $\rho_\infty(H+BK)$ and the region of attraction. Keep in mind that the smaller $\rho_\infty(H+BK)$, the larger the estimation of the rate of exponential convergence of the saturated system. From this and results presented in Table 1, if one wants a vast region of attraction, the estimation of the rate of convergence will not be important. On the contrary, if one wants a strong rate of convergence, then the region of attraction will be limited.

Another comment has to be made on the case $K = K_3$. Here the matrix gain K_3 is chosen such that $\rho_\infty(H+BK) \simeq 0$ which means that the system is exponentially stable with a very large rate of convergence. From Table 1, the estimation of the region of attraction gives bad results. This is mainly because $\mathcal{R}_\infty(\Delta B^+K^+\Delta^{-1}) \simeq +\infty$. This last analysis tends to confirm the link between the estimation of the rate of convergence and the estimation of the region of attraction underlined earlier.

4. Conclusion

The well-posedness and the local L^∞ exponential stability of a wide class of diagonal semilinear systems was established. The PDEs under consideration resulted from a transport with constant velocities coupled with a nonlinear source term and a nonlinear boundary condition. The saturated control was applied at the boundary in order to stabilize the open-loop system. The well-posedness was tackled using nonlinear semigroup techniques. The stability has been proven using convergence of semigroups and Lyapunov theory. This work let some questions open. The case of mixed positive and negative velocities is not treated; the method of [9] which differentiates the Lyapunov functional for components with positive and negative velocities seems to be a promising idea. The case of space varying velocities (with constant sign) is also interesting and already solved in ([3], Chap. 3.5) for unsaturated systems. The generalization to non-diagonal source terms would be an important improvement. The article [11], where a specific space dependent Lyapunov functional is introduced, would be a good starting point to tackle the problem. Finally, the L^∞ (or even L^p) stability for systems of nonlinear d scalar conservation laws remains an open question even for unsaturated controllers. In [5], a feedback control was found for a single scalar conservation law whose flux is either convex or concave. Additionally, authors of [8] study the stabilization of a nonlocal one-dimensional conservation law. Starting from a linearized system, they find a sufficient condition for stability and adapt the proof to the full nonlinear PDE. However, to our knowledge, nothing seems to be generalizable to conservation laws of arbitrary dimension.

Appendix A. Proof of Theorem 2.7

Let $X = L^2([0,1])$ be the base space; the scalar product on X was introduced by [9] and is defined by:

$$\forall u, v \in X,\ (u,v) := \int_0^1 u^T \overline{v} e^{\nu(x-1)} \mathrm{d}x. \tag{A.1}$$

A.1 Existence and uniqueness of solution with a smoothed saturation

Take an arbitrary smooth approximation $(\sigma_n)_n$ of σ in the sense of Definition 2.1. For all integers n, we define the operator A_n by:

$$\left\{ \begin{array}{ll} A_n R & = -\Lambda R_x \\ D(A_n) & = \left\{ R \in H^1([0,1]); R(0) = HR(1) + B\sigma_n(KR(1)) \right\}. \end{array} \right.$$

Moreover, the operator G can be defined as follows:

$$\left\{ \begin{array}{ll} GR & = g\circ R \\ D(G) & = L^2([0,1]) \end{array} \right.$$

The following theorem states the well-posedness for the closed-loop system whose control is smoothly saturated.

Theorem A.1. *There exists $\zeta > 0$ dependent on σ_s, H, B and K such that for all integers n, the operator $A_n + G$ is ζ dissipative. Moreover, $A_n + G$ generates a semigroup T_n of type ζ and for all $R_{0,n} \in D(A_n)$, $T_n(.)R_{0,n}$ is the $C^0([0,T], L^2([0,1]))$ solution to the Cauchy problem:*

$$\left\{ \begin{array}{cl} R_{n,t} & = -\Lambda R_{n,x} + g\circ R_n \\ R_n(t=0) & = R_{0,n} \\ R_n(t) & \in D(A_n) \end{array} \right. \tag{A.2}$$

where $R_{n,t}$ is defined as the Fréchet derivative with respect to t in the L^2 space:

$$\left|\left| \frac{R_n(.,t+\mathrm{d}t) - R_n(.,t)}{\mathrm{d}t} - R_{n,t}(.,t) \right|\right|_{L^2([0,1])} \rightarrow_{\mathrm{d}t\rightarrow 0} 0.$$

Finally, if $R_{0,n}$ is $H^2([0,1])$ and satisfies compatibility conditions of order 1 (2.2) then the solution R_n belongs to $C^1([0,T], L^2([0,1])) \cap C^0([0,T], H^1([0,1]))$.

Remark A.2. A definition of the ζ dissipativity can be found in [15]. Moreover, in the following proof, we do not use the form of σ_n but the fact that it is bounded in C^1 (n fixed). Hence, conclusions of Theorem A.1 are valid for whatever bounded $\sigma_n \in C^1(\mathbb{R})$.

Let n be an integer. To get the conclusions of Theorem A.1, we need to prove some properties on the operators $A_n + G$. They are listed below:

- $A_n + G$ is ζ dissipative with ζ independent on n.
- It satisfies the range condition: $\mathrm{Rg}(I - \rho(A_n + G)) \supset D(A_n + G)$ for all positive ρ sufficiently small.
- $A_n + G$ is closed.

Having done that, we use ([15], Thm. 5.12) to prove that $A_n + G$ generates the expected semigroup.

Proof. Let n be an integer fixed all along the proof. Constants may depend on n but, in the following, this dependence is skipped when it is useless.

(1) $\mathbf{A_n + G}$ is ζ dissipative:

Let u and v in $D(A_n + G) = D(A_n)$. Recalling the definition of the scalar product (A.1), one has

$$((A_n + G)u - (A_n + G)v, u - v) = -\int_0^1 (u_x - v_x)^T \Lambda(\overline{u-v}) e^{\nu(x-1)}\mathrm{d}x + \int_0^1 (g\circ u - g\circ v)^T(\overline{u-v})e^{\nu(x-1)}\mathrm{d}x. \tag{A.3}$$

By an integration by parts, one has:

$$((A_n + G)u - (A_n + G)v, u - v) = -\left[(u-v)^T\Lambda(\overline{u-v})e^{\nu(x-1)}\right]_0^1 + \int_0^1 (u-v)^T\Lambda(\overline{u_x - v_x})e^{\nu(x-1)}\mathrm{d}x$$

$$+\nu\int_0^1 (u-v)^T\Lambda(\overline{u-v})e^{\nu(x-1)}\mathrm{d}x+\int_0^1 (g\circ u-g\circ v)^T(\overline{u-v})e^{\nu(x-1)}\mathrm{d}x.$$

From (A.3), one gets:

$$\begin{aligned}((A_n+G)u-(A_n+G)v,u-v)=&-[(u-v)^T\Lambda(\overline{u-v})e^{\nu(x-1)}]_0^1-\overline{((A_n+G)u-(A_n+G)v,u-v)}\\&+\nu\int_0^1 (u-v)^T\Lambda(\overline{u-v})e^{\nu(x-1)}\mathrm{d}x+2\mathrm{Re}\int_0^1 (g\circ u-g\circ v)^T(\overline{u-v})e^{\nu(x-1)}\mathrm{d}x.\end{aligned}$$

It implies necessarily that:

$$\begin{aligned}2\mathrm{Re}((A_n+G)u-(A_n+G)v,u-v)=&-[(u-v)^T\Lambda(\overline{u-v})e^{\nu(x-1)}]_0^1+\nu\int_0^1 (u-v)^T\Lambda(\overline{u-v})e^{\nu(x-1)}\mathrm{d}x\\&+2\mathrm{Re}\int_0^1 (g\circ u-g\circ v)^T(\overline{u-v})e^{\nu(x-1)}\mathrm{d}x\end{aligned}$$

and taking the real part in last equation:

$$\begin{aligned}2\mathrm{Re}((A_n+G)u-(A_n+G)v,u-v)=&-\mathrm{Re}\left([(u-v)^T\Lambda(\overline{u-v})e^{\nu(x-1)}]_0^1\right)+\nu\ \mathrm{Re}\int_0^1 (u-v)^T\Lambda(\overline{u-v})e^{\nu(x-1)}\mathrm{d}x\\&+2\mathrm{Re}\int_0^1 (g\circ u-g\circ v)^T(\overline{u-v})e^{\nu(x-1)}\mathrm{d}x.\end{aligned}$$

Using the fact that velocities are bounded from above,

$$\begin{aligned}2\mathrm{Re}((A_n+G)u-(A_n+G)v,u-v)\leq&-\mathrm{Re}\left([(u-v)^T\Lambda(\overline{u-v})e^{\nu(x-1)}]_0^1\right)+\nu\lambda_{\max}\mathrm{Re}(u-v,u-v)\\&+2\mathrm{Re}\int_0^1 (g\circ u-g\circ v)^T(\overline{u-v})e^{\nu(x-1)}\mathrm{d}x.\end{aligned}$$

Using the fact that g is Lipschitz, there exists a constant $\varsigma>0$ depending on L_g such that:

$$2\mathrm{Re}((A_n+G)u-(A_n+G)v,u-v)\leq-\mathrm{Re}[(u-v)^T\Lambda(\overline{u-v})e^{\nu(x-1)}]_0^1+(\nu\lambda_{\max}+\varsigma)\mathrm{Re}(u-v,u-v). \quad \text{(A.4)}$$

To simplify the notation, $u(1)$ and $v(1)$ will be denoted, respectively, by u_1 and v_1 in following computations. Boundary terms can be rewritten as follows:

$$\begin{aligned}\mathrm{Re}\left([(u-v)^T\Lambda(\overline{u-v})e^{\nu(x-1)}]_0^1\right)=&\ \mathrm{Re}(u_1-v_1)^T\Lambda\mathrm{Re}(u_1-v_1)\\&-e^{-\nu}\mathrm{Re}[H(u_1-v_1)+B(\sigma_n(Ku_1)-\sigma_n(Kv_1))]^T\Lambda\mathrm{Re}[H(u_1-v_1)\\&+B(\sigma_n(Ku_1)-\sigma_n(Kv_1))]+\mathrm{Im}(u_1-v_1)^T\Lambda\mathrm{Im}(u_1-v_1)\\&-e^{-\nu}\mathrm{Im}[H(u_1-v_1)+B(\sigma_n(Ku_1)-\sigma_n(Kv_1))]^T\Lambda\mathrm{Im}[H(u_1-v_1)\\&+B(\sigma_n(Ku_1)-\sigma_n(Kv_1))].\end{aligned}$$

For $\nu=\nu_n$ large enough compared to the norm of H,B and $\sigma_{s,n}$, we deduce that

$$\mathrm{Re}\left([(u-v)^T\Lambda(\overline{u-v})e^{\nu_n(x-1)}]_0^1\right)\geq\frac{1}{2}(\mathrm{Re}(u_1-v_1)^T\Lambda\mathrm{Re}(u_1-v_1)+\mathrm{Im}(u_1-v_1)^T\Lambda\mathrm{Im}(u_1-v_1))\geq 0$$

which implies with (A.4) that:

$$2\mathrm{Re}\left((A_n+G-\frac{\lambda_{\max}\nu_n+\varsigma}{2}I)u-(A_n+G-\frac{\lambda_{\max}\nu_n+\varsigma}{2}I)v,u-v\right)\leq -\mathrm{Re}[(u-v)^T\Lambda(\overline{u-v})e^{\nu_n(x-1)}]_0^1\leq 0$$

and therefore A_n+G is $\frac{\lambda_{\max}\nu_n+\varsigma}{2}$ dissipative.

Recall that, ν_n was taken large enough compared to the norms of H, B and $\sigma_{s,n}$. As $(\sigma_{s,n})_n$ is a bounded sequence, we can take a sequence $(\nu_n)_n$ bounded from above by a positive real ν. As A_n+G is $\frac{\nu_n\lambda_{\max}+\varsigma}{2}$ dissipative and $\frac{\nu_n\lambda_{\max}+\varsigma}{2}\leq\frac{\nu\lambda_{\max}+\varsigma}{2}$, A_n+G is $\frac{\nu\lambda_{\max}+\varsigma}{2}=\zeta$ dissipative with ζ independent on n.

(2) $\mathbf{A_n+G}$ satisfies the range condition:

Let us now prove the following range condition:

$$\exists\rho_{sup}>0;\ \forall\rho\in(0,\rho_{sup}),\ \mathrm{Rg}(I-\rho(A_n+G))\supset D(A_n). \tag{A.5}$$

It is equivalent to prove that for all v in $D(A_n)$, there exists an element u in $D(A_n)$ such that:

$$\begin{cases} u+\rho\Lambda u_x-\rho g(u)=v \\ u(0) \quad\quad = Hu(1)+B\sigma_n(Ku(1)). \end{cases}$$

This property is the most difficult to prove. It consists of proving the existence of a solution to a nonlinear ODE with a nonlinear boundary condition. To prove the existence, first, we will deal with the nonlinear boundary condition and then, using a fixed point theorem, the nonlinear source term will be taken into account; the method being inspired from [17, 20].

(2.1) Taking into account the nonlinear boundary condition

Let us now prove the following range condition:

$$\forall(v_1,v_2)\in D(A_n+G)=D(A_n),\ v_1+\rho Gv_2\in\mathrm{Rg}(I-\rho A_n). \tag{A.6}$$

To do so, take v_1, v_2 both in $C^0([0,1])$. To prove assertion (A.6), we have to find an element u in $D(A_n)$ solution of:

$$\begin{cases} u+\rho\Lambda u_x = v_1+\rho(g\circ v_2) \\ u(0) \quad = Hu(1)+B\sigma_n(Ku(1)) \end{cases}$$

equivalent to:

$$\begin{cases} u_x+\frac{\Lambda^{-1}}{\rho}u = \Lambda^{-1}\left(\frac{v_1}{\rho}+g\circ v_2\right) \\ u(0) \quad = Hu(1)+B\sigma_n(Ku(1)). \end{cases} \tag{A.7}$$

We define $\mathcal{T}: C^0([0,1])\to C^1([0,1])$

$$\begin{aligned} \mathcal{T}: C^0([0,1]) &\to C^1([0,1]) \\ y &\mapsto \text{u solution of the following system} \end{aligned}$$

$$\begin{cases} u_x + \frac{\Lambda^{-1}}{\rho} u = \Lambda^{-1}\left(\frac{v_1}{\rho} + g \circ v_2\right) \\ \quad u(0) \quad = Hu(1) + B\sigma_n(Ky(1)). \end{cases} \tag{A.8}$$

If we prove that $\mathcal{T}$ is well-defined and admits a fixed point in $D(A_n)$, then assertion (A.6) is proven.

For all y in $C^0([0,1])$, solutions u to the ODE in (A.8) are $C^1([0,1])$ (because $v_1, g(v_2) \in C^0([0,1])$) and can be expressed as follow:

$$\forall x \in [0,1],\ u(x) = e^{-\frac{\Lambda^{-1}x}{\rho}} Z(y) + \int_0^x e^{-\frac{\Lambda^{-1}}{\rho}(x-s)} \times \Lambda^{-1}\left(\frac{v_1}{\rho} + g \circ v_2\right) \mathrm{d}s$$

where $Z(y)$ is a constant of $\mathbb{R}^d$. Thus, for all y in $C^0([0,1])$, u is a solution to system (A.8) if and only if $Z(y)$ satisfies the following equation:

$$Z(y) = He^{-\frac{\Lambda^{-1}}{\rho}} Z(y) + H\int_0^1 e^{-\frac{\Lambda^{-1}}{\rho}(1-s)} \times \Lambda^{-1}\left(\frac{v_1}{\rho} + g \circ v_2\right) \mathrm{d}s + B\sigma_n(Ky(1))$$

equivalent to:

$$(I_d - He^{-\frac{\Lambda^{-1}}{\rho}})Z(y) = H\int_0^1 e^{-\frac{\Lambda^{-1}}{\rho}(1-s)} \times \Lambda^{-1}\left(\frac{v_1}{\rho} + g \circ v_2\right) \mathrm{d}s + B\sigma_n(Ky(1))$$

and for ρ sufficiently small, say $\rho < \delta(H,\Lambda)$ with $\delta(H,\Lambda) > 0$, one can invert the last relation:

$$\forall y \in C^0([0,1]),\ Z(y) = (I_d - He^{-\frac{\Lambda^{-1}}{\rho}})^{-1}\left(H\int_0^1 e^{-\frac{\Lambda^{-1}}{\rho}(1-s)} \times \Lambda^{-1}\left(\frac{v_1}{\rho} + g \circ v_2\right) \mathrm{d}s + B\sigma_n(Ky(1))\right). \tag{A.9}$$

Last equation ensures that for $\rho < \delta(H,\Lambda)$, $\mathcal{T}$ is well defined and

$$\forall x \in [0,1],\ \mathcal{T}(y)(x) = e^{-\frac{\Lambda^{-1}x}{\rho}} Z(y) + \int_0^x e^{-\frac{\Lambda^{-1}}{\rho}(x-s)} \times \Lambda^{-1}\left(\frac{v_1}{\rho} + g \circ v_2\right) \mathrm{d}s$$

with $Z(y)$ defined in (A.9).

The operator $\mathcal{T}$ being well-defined, we can focus on the fixed point argument. As σ_n is bounded, $Z(y)$ is bounded when y scans $C^0([0,1])$. Hence, $\mathcal{T}(C^0([0,1]))$ is a bounded set of $C^1([0,1])$ and there exists a set $K = \{w \in C^1([0,1]); ||w||_{C^0([0,1])} \leq M \text{ and } ||w'||_{C^0([0,1])} \leq M\}$ where M is a constant such that $\mathcal{T}(C^0([0,1])) \subset K$. As K is bounded in $C^1([0,1])$, K is compact in $C^0([0,1])$ by Ascoli-Arzela's theorem. Moreover, $\mathcal{T}$ is continuous and K is closed and convex allowing to use Schauder fixed point theorem to conclude that:

$$\exists u \in K : \mathcal{T}(u) = u.$$

Hence, $u \in D(A_n) = D(A_n + G)$ and the assertion (A.6) is proven.

(2.2) Taking into account the source term

Let v be in $D(A_n+G)=D(A_n)$, $\rho\leq\delta(H,\Lambda)$ and $\mathcal{H}: C^0([0,1])\mapsto C^0([0,1])$ be such that for all w in $C^0([0,1])$, $\mathcal{H}(w)$ is solution of:

$$\begin{cases} u_x+\frac{\Lambda^{-1}}{\rho}u &= \Lambda^{-1}(\frac{v}{\rho}+g\circ w)\\ u(0) &= Hu(1)+B\sigma_n(Ku(1)).\end{cases}$$

By assertion (A.6), $\mathcal{H}$ is well defined (take $v_1\leftarrow v$ and $v_2\leftarrow w$). We will prove that $\mathcal{H}$ has a fixed point in $D(A_n)$ which implies that the range condition (A.5) is verified.

To do so, we will prove that there exists a ball of C^0; B_r of radius r such that B_r is invariant under $\mathcal{H}$ and $\mathcal{H}(B_r)$ precompact in $C^0([0,1])$.

(2.2.1) There exists a ball of $C^0([0,1])$, $\mathbf{B_r}$, invariant under $\mathcal{H}$

Let w be in $C^0([0,1])$ and let recall the definition of the usual $C^0([0,1])$ norm:

$$||w||_{C^0([0,1])}:=\max_{x\in[0,1]}|w(x)|=\max_{x\in[0,1]}\sqrt{\sum_{i=1}^d w_i(x)^2}. \tag{A.10}$$

The continuous function $\mathcal{H}(w)$ can be expressed as:

$$\forall w\in C^0([0,1]),\ \forall x\in[0,1],\ \mathcal{H}(w)(x)=e^{-\frac{\Lambda^{-1}x}{\rho}}Z(w)+\int_0^x e^{-\frac{\Lambda^{-1}}{\rho}(x-s)}\times\Lambda^{-1}\left(\frac{v}{\rho}+g\circ w\right)\mathrm{d}s \tag{A.11}$$

where:

$$\forall w\in C^0([0,1]),\ Z(w)=(I_d-He^{-\frac{\Lambda^{-1}}{\rho}})^{-1}\left(H\int_0^1 e^{-\frac{\Lambda^{-1}}{\rho}(1-s)}\times\Lambda^{-1}\left(\frac{v}{\rho}+g\circ w\right)\mathrm{d}s+B\sigma_n(K\mathcal{H}(w)(1))\right). \tag{A.12}$$

Fixing an w in $C^0([0,1])$, we focus on a term present in both expressions of $Z(w)$ and of $\mathcal{H}(w)$:

$$\forall x\in[0,1],\ \int_0^x e^{-\frac{\Lambda^{-1}}{\rho}(x-s)}\times\frac{\Lambda^{-1}}{\rho}v\ \mathrm{d}s=v(x)-e^{-\frac{\Lambda^{-1}}{\rho}x}v(0)-\int_0^x e^{-\frac{\Lambda^{-1}}{\rho}(x-s)}v_x\ \mathrm{d}s.$$

Hence,

$$\begin{aligned}\forall x\in[0,1],\ \left|\int_0^x e^{-\frac{\Lambda^{-1}}{\rho}(x-s)}\times\frac{\Lambda^{-1}}{\rho}v\ \mathrm{d}s\right| &=\sqrt{\sum_{i=1}^d\left(\int_0^x e^{-\frac{\lambda_i^{-1}}{\rho}(x-s)}\times\frac{\lambda_i^{-1}}{\rho}v_i\ \mathrm{d}s\right)^2}\\ &=\sqrt{\sum_{i=1}^d\left(v_i(x)-e^{-\frac{\lambda_i^{-1}}{\rho}x}v_i(0)-\int_0^x e^{-\frac{\lambda_i^{-1}}{\rho}(x-s)}v_{i,x}\ \mathrm{d}s\right)^2}\\ &\leq\sqrt{\sum_{i=1}^d v_i(x)^2}+\sqrt{\sum_{i=1}^d e^{-\frac{2\lambda_i^{-1}}{\rho}x}v_i^2(0)}+\sqrt{\sum_{i=1}^d\left(\int_0^x e^{-\frac{\lambda_i^{-1}}{\rho}(x-s)}v_{i,x}\ \mathrm{d}s\right)^2}\end{aligned}$$

and because $e^{-\frac{\Lambda^{-1}}{\rho}(x-s)} \leq I_d$ when $s \leq x$:

$$\forall x \in [0,1], \ \left| \int_0^x e^{-\frac{\Lambda^{-1}}{\rho}(x-s)} \times \frac{\Lambda^{-1}}{\rho} v \ \mathrm{d}s \right| \leq 2||v||_{C^0([0,1])} + ||v||_{H^1([0,1])} =: C(v). \tag{A.13}$$

Remark that $C(v)$ is independent on ρ, w.
The second term to study is:

$$\begin{aligned}
\forall x \in [0,1], \ \left| \int_0^x e^{-\frac{\Lambda^{-1}}{\rho}(x-s)} \times \Lambda^{-1}(g \circ w) \ \mathrm{d}s \right|^2 &= \sum_{i=1}^d \left(\int_0^x e^{-\frac{\lambda_i^{-1}}{\rho}(x-s)} \times \lambda_i^{-1}(g_i \circ w) \ \mathrm{d}s \right)^2 \\
&\leq \sum_{i=1}^d |g_i \circ w|^2_{L^2([0,1])} \frac{\rho}{2\lambda_i}(1 - e^{-2\frac{\lambda_i^{-1}}{\rho}x}) \\
&\leq \sum_{i=1}^d \frac{\rho}{2\lambda_i} |g_i \circ w|^2_{L^2([0,1])} \leq \frac{\rho}{2\min_i \lambda_i} ||g \circ w||^2_{C^0([0,1])}
\end{aligned}$$

where we have used Cauchy-Schwarz inequality to get first inequality.

Using the fact that $g(0) = 0$ and g Lipschitz as a function from $\mathbb{R}^d \mapsto \mathbb{R}^d$ for the canonical $\mathbb{R}^d$ norm, one gets:

$$\forall w \in C^0([0,1]), ||g \circ w||_{C^0([0,1])} \leq \max_{x \in [0,1]} |(g \circ w)(x)) - g(0)| + |g(0)| \leq L_g \max_{x \in [0,1]} |w(x)|.$$

Hence,

$$\forall w \in C^0([0,1]), ||g \circ w||_{C^0([0,1])} \leq L_g ||w||_{C^0([0,1])}$$

and as a consequence:

$$\forall w \in C^0([0,1]), x \in [0,1], \ \left| \int_0^x e^{-\frac{\Lambda^{-1}}{\rho}(x-s)} \times \Lambda^{-1}(g \circ w) \ \mathrm{d}s \right| \leq \sqrt{\frac{\rho}{2\min_i \lambda_i}} L_g ||w||_{C^0([0,1])}. \tag{A.14}$$

Finally, using the boundedness of σ_n, one gets the existence of a constant C_3 (only dependent on σ_s, H, B) such that $\forall \rho > 0$:

$$|(I_d - He^{-\frac{\Lambda^{-1}}{\rho}})^{-1} B\sigma_n(K\mathcal{H}(w)(1))| \leq C_3. \tag{A.15}$$

Injecting inequalities (A.13), (A.14) and (A.15) in the definition of $Z(w)$ (A.12), one gets the existence of $C_1(v), C_2, C_3$ independent on ρ, w such that:

$$\forall w \in C^0([0,1]), |Z(w)| \leq C_1(v) + C_2\sqrt{\rho}||w||_{C^0([0,1])} + C_3. \tag{A.16}$$

Putting all together inequalities (A.13), (A.14) and (A.16) in the expression of $\mathcal{H}(w)$ (A.11), there exist three positive constants $\tilde{C_1}(v)$, $\tilde{C_2}$, $\tilde{C_3}$ independent on ρ, w such that:

$$\forall w \in C^0([0,1]), ||\mathcal{H}(w)||_{C^0([0,1])} \leq \tilde{C_1}(v) + \tilde{C_2}\sqrt{\rho}||w||_{C^0([0,1])} + \tilde{C_3}. \tag{A.17}$$

Taking $\rho < \frac{1}{\tilde{C_2}^2}$ (note that this bound is independent on v), it is possible to choose a strict positive r such that $\tilde{C_1}(v) + \tilde{C_2}\sqrt{\rho}r + \tilde{C_3} \leq r$ and:

$$\forall w \in B_r,\ ||\mathcal{H}(w)||_{C^0([0,1])} \leq r.$$

which is no more than B_r is invariant under $\mathcal{H}$.

(2.2.2) The set $\mathcal{H}(B_r)$ is precompact in $\mathbf{C^0}([\mathbf{0},\mathbf{1}])$

The following claim allows to get compactness:

Claim A.3. *The set $\mathcal{H}(B_r)$ is uniformly bounded in $C^1([0,1])$.*

Proof of Claim A.3. Let $w \in B_r$. Denoting $u = \mathcal{H}(w)$, we have

$$u_x = -\frac{\Lambda^{-1}}{\rho}u + \Lambda^{-1}\Big(\frac{v}{\rho} + g \circ w\Big).$$

As u, v, w are all continuous, u_x is also continuous. By previous section, $u \in B_r$ and as a consequence

$$||u_x||_{C^0([0,1])} \leq C$$

where C may depend on $(r, \rho, \lambda, L_g, v)$ but not on w.

This ends the proof of Claim A.3. □

To conclude, using Ascoli-Arzela's theorem, $\mathcal{H}(B_r)$ is relatively compact in $C^0([0,1])$ and by Schauder fixed point theorem, $\mathcal{H}$ admits a fixed point which ends the proof of the range condition (A.5).

(3) $\mathbf{A_n + G}$ is a closed operator:

To prove this, we first prove that A_n is closed. Then, using the continuity of G, we conclude on the closedness of the operator $A_n + G$.

(3.1) $\mathbf{A_n}$ is a closed operator:

Take a sequence $(u_k)_k$ of elements of $D(A_n)$ such that $\lim\limits_{k\to\infty} u_k =: u$ in $L^2([0,1])$ and $\lim\limits_{k\to\infty} A_n u_k =: \tilde{u}$ in $L^2([0,1])$. We have to show that u belongs to $D(A_n)$ and that $\tilde{u} = A_n u$. Let us define for all integers k, $\tilde{u}_k := A_n u_k \to_{L^2([0,1])} \tilde{u}$ which can also be written as:

$$\forall k \in \mathbb{N},\ u_{k,x} = -\Lambda^{-1}\tilde{u}_k \to_{L^2([0,1])} -\Lambda^{-1}\tilde{u}. \tag{A.18}$$

As a consequence,

$$\forall k \in \mathbb{N}, \forall x \in [0,1], u_k(x) = u_k(0) - \int_0^x \Lambda^{-1}\tilde{u}_k(s)\mathrm{d}s = Hu_k(1) + B\sigma_n(Ku_k(1)) - \int_0^x \Lambda^{-1}\tilde{u}_k(s)\mathrm{d}s.$$

Let x be in $[0,1]$, we have $|\int_0^x \Lambda^{-1}(\tilde{u}_k(s) - \tilde{u}(s))\mathrm{d}s| \leq C \times ||\tilde{u}_k - \tilde{u}||_{L^2([0,1])}$ which tends to zero as k tends towards infinity. Hence,

$$\forall x \in [0,1],\ \int_0^x \Lambda^{-1}\tilde{u}_k(s)\mathrm{d}s \xrightarrow[k\to\infty]{} \int_0^x \Lambda^{-1}\tilde{u}(s)\mathrm{d}s. \tag{A.19}$$

Then as $H^1([0,1]) \subset C^0([0,1])$ continuously:

$$\forall k, m \in \mathbb{N},\ |u_k(1) - u_m(1)| \leq C \times ||u_k - u_m||_{H^1([0,1])}$$

where C is the constant of the continuous injection $H^1([0,1]) \subset C^0([0,1])$.

Moreover, by (A.18), $||u_k - u_m||^2_{H^1([0,1])} = ||u_k - u_m||^2_{L^2([0,1])} + ||\Lambda^{-1}\tilde{u}_k - \Lambda^{-1}\tilde{u}_m||^2_{L^2([0,1])}$ which tends to zero as (k,m) tends towards infinity. Hence, $(u_k(1))_k$ is Cauchy and tends towards a real u_1. From this and the pointwise convergence stated in (A.19), $Hu_k(1) + B\sigma_n(Ku_k(1)) - \int_0^x \Lambda^{-1}\tilde{u}_k(s)\mathrm{d}s$ converges pointwise towards a function denoted w (which can be different from u because the convergence of w is just pointwise) and:

$$\forall x \in [0,1],\ w(x) = Hu_1 + B\sigma_n(Ku_1) - \int_0^x \Lambda^{-1}\tilde{u}(s)\mathrm{d}s. \tag{A.20}$$

Obviously, w is in $H^1([0,1])$ and $w_x = \Lambda^{-1}\tilde{u}$. The convergence of $(u_k)_k$ towards w is also in $L^2([0,1])$:

$$\begin{aligned}
\int_0^1 (w(x) - u_k(x))^2 \mathrm{d}x &= \int_0^1 \left\{ H(u_1 - u_k(1)) + B\sigma_n(Ku_1) - B\sigma_n(Ku_k(1)) + \int_0^x \Lambda^{-1}(\tilde{u}(s) - \tilde{u}_k(s))\mathrm{d}s \right\}^2 \mathrm{d}x \\
&\leq 2 \times (H(u_1 - u_k(1)) + B\sigma_n(Ku_1) - B\sigma_n(Ku_k(1)))^2 \\
&\quad + 2 \times \int_0^1 \left(\int_0^x \Lambda^{-1}(\tilde{u}(s) - \tilde{u}_k(s))\mathrm{d}s \right)^2 \mathrm{d}x \\
&\leq C \times (|u_k(1) - u_1|^2 + ||\tilde{u}_k - \tilde{u}||^2_{L^2([0,1])})
\end{aligned}$$

where we have used the identity $(a+b)^2 \leq 2a^2 + 2b^2$ to get first inequality and C is a constant depending on $H, B, \sigma_{s,n}$ and Λ.

The right-hand side of last equation tending towards zero as k tends to infinity, we have proven that $(u_k)_k$ tends towards w in $L^2([0,1])$ and by the uniqueness of the limit $w = u$ in $L^2([0,1])$. Note that as w is continuous, we can take $w = u$ in the sense of $C^0([0,1])$. Moreover, as $u_k(1)$ tends towards $w(1) = u(1)$ (because $(u_k)_k$ tends towards w pointwise) and towards u_1 (by definition of u_1) at the same time, we have that $u(1) = u_1$. Injecting this last equality in (A.20), we have:

$$\forall x \in [0,1],\ u(x) = Hu(1) + B\sigma_n(Ku(1)) - \int_0^x \Lambda^{-1}\tilde{u}(s)\mathrm{d}s.$$

Thus, $u \in D(A_n)$ and $\tilde{u} = A_n u$.

(3.2) $\mathbf{A_n + G}$ is a closed operator

As A_n is closed by previous paragraph and G is continuous as an operator from $L^2([0,1])$ into $L^2([0,1])$ (because g is Lipschitz), $A_n + G$ is closed.

(4) First conclusions on the proof of Theorem A.1

As $A_n + G$ satisfies the range condition and is ζ dissipative; by ([15], Thm. 5.12), $A_n + G$ generates a unique semigroup T_n of type ζ. By Remark 2 p. 148 (and Thm. 4.10 (ii)) of the same book, the additional facts that $A_n + G$ is closed and $L^2([0,1])$ is reflexive; if $R_{0,n} \in D(A_n)$ then $t \mapsto T_n(t)R_{0,n}$ is the unique solution of the Cauchy problem:

$$\begin{cases} R_{n,t} & = (A_n+G)R_n \\ R_n(t=0) & = R_{0,n} \\ R_n(t) & \in D(A_n) \end{cases}$$

in the sense that for almost every $t \geq 0$, $T_n(t)R_{0,n}$ is in $D(A_n)$, time-Fréchet differentiable in $L^2([0,1])$ and verifies the system presented just above.

As T_n is a semigroup of type ζ in $L^2([0,1])$, we have:

$$t \mapsto T_n(t)R_{0,n} \in C^0([0,T], L^2([0,1])).$$

(5) Regularity of the solution

Here we use the C^1 regularity assumptions on g and σ_n. The augmented system satisfied by $U_n := (R_n, R_{n,x})$ writes

$$\begin{cases} U_{n,t} + \mathrm{diag}(\Lambda,\Lambda)U_{n,x} & = g_2 \circ U \\ R_n(0) & = HR_n(1) + B\sigma_n(KR_n(1)) \\ R_{n,x}(0) & = \Lambda^{-1}\Big([H + B\sigma_n'(R_n(1))](\Lambda R_{n,x}(1) - g(R_n(1))) + g(R_n(0))\Big) \end{cases} \tag{A.21}$$

where

$$\begin{cases} g_2 : \mathbb{R}^{2d} & \mapsto & \mathbb{R}^{2d} \\ U := (U_1,U_2) & \to & (g(U_1), g'(U_1)U_2). \end{cases}$$

In substance, this augmented system is very similar to system (A.2). The transport part is identical and the source term is Lipschitz (g' is bounded by assumption). For the boundary condition of $R_{n,x}$, it is linear in the variable $R_{n,x}$. Knowing this, we can easily adapt the reasoning used for the system (A.2) for the variable R_n alone to the system (A.21), the unbounded operator being now defined as

$$\begin{cases} D(A_{2,n}) = \Big\{(U_1,U_2) \in H^1([0,1])^2 \quad | \quad \begin{matrix} U_1(0) & = HU_1(1) + B\sigma_n(KU_1(1)), \\ U_2(0) & = \Lambda^{-1}\Big([H + B\sigma_n'(U_1(1))](\Lambda U_2(1) - g(U_1(1))) + g(U_1(0))\Big) \end{matrix} \Big\} \\ A_{2,n}(U_1,U_2) = -(\Lambda U_{1,x}, \Lambda U_{2,x}) \end{cases}$$

and the source term

$$\begin{cases} D(G_2) = (L^2([0,1]))^2 \\ G_2(U_1,U_2) = g_2(U_1,U_2). \end{cases}$$

Recall that to apply the theory of nonlinear semigroups from [15], we need to prove the three following statements; $A_{2,n} + G_2$ is ζ dissipative for some $\zeta \in \mathbb{R}$, satisfies the range condition and is closed. As these proofs are very similar to what was done in the parts (1) (2) and (3) for the operator $A_n + G$, we will just give a sketch of the proof and insist on crucial hypothesis.

- For ζ dissipativity, it is a comparison between boundary terms. The fact that σ_n' is bounded (n fixed) is the key hypothesis to show this ζ dissipativity.

- For range condition, we need to solve $(I-\rho(A_{2,n}+G_2))(u_1,u_2)=(v_1,v_2)$ where (u_1,u_2) is the unknown, $(v_1,v_2)\in D(A_{2,n})$ and ρ belonging to $[0,\rho_0]$ with ρ_0 independent on (v_1,v_2) to determine. We find u_1 using the fact that A_n+G satisfies the range condition. For u_2, it suffices to remark that the ODE to solve is linear in u_2 (even for the boundary condition) with a bounded linear source term (g' is bounded as g is Lispchitz).

- Finally, for the closedness one can proceed using same techniques as in previous sections and the closedness of A_n+G.

Hence $(R_n,R_{n,x})$ is $C^0([0,T],L^2([0,1]))$. As a consequence, $R_n\in C^0([0,T],H^1([0,1]))$ and as $R_{n,t}=g(R_n)-\Lambda R_{n,x}$, we also have $R_{n,t}\in C^0([0,T],L^2([0,1]))$. To conclude, $R_n\in C^0([0,T],H^1([0,1]))\cap C^1([0,T],L^2([0,1]))$.

All points of Theorem A.1 are now proven. □

A.2 Convergence of solutions with smoothed saturations in $L^2([0,1])$

We define the operator A by:

$$\left\{\begin{array}{ll} AR & =-\Lambda R_x \\ D(A) & =\left\{R\in H^1([0,1]); R(0)=HR(1)+B\sigma(KR(1))\right\}. \end{array}\right.$$

Note that $\overline{D(A)}=L^2([0,1])$ because $C_c^\infty((0,1))\subseteq D(A)$ and $C_c^\infty((0,1))$ is a dense subset of $L^2([0,1])$ by ([6], Cor. 4.23).[1] The following lemma will be useful to prove the convergence of semigroups $(T_n)_n$.

Lemma A.4. *It holds $A\subset\lim_{n\to\infty}A_n$ which means that every element of the graph of A is the limit of a sequence $\{(x_n,A_nx_n)\}_n$ (in $L^2([0,1])\times L^2([0,1])$) where $x_n\in D(A_n)$.*

Proof. Let $R\in D(A)$. Let us define the sequence $(R_n)_n$ in $H^1([0,1])$ by:

$$\forall n\in\mathbb{N},\forall x\in[0,1],\ R_n(x):=(1-x^2)B[\sigma_n(KR(1))-\sigma(KR(1))]+R(x).$$

For all integers n, we have $R_n(0)=HR(1)+B\sigma_n(KR(1))$ (use the fact that $R\in D(A)$) and $R_n(1)=R(1)$. Hence, for all integers n, $R_n\in D(A_n)$ and by the pointwise convergence of $(\sigma_n)_n$, we have $R_n\to_{H^1([0,1])}R$ which is equivalent to $(R_n,A_nR_n)\to_{L^2([0,1])\times L^2([0,1])}(R,AR)$. □

The convergence of semigroups $(T_n)_n$ is given by the following theorem:

Theorem A.5. *The operator $A+G$ generates a semigroup T of type ζ. Moreover, for all initial data R_0 in $\overline{D(A)}=L^2([0,1])$ and sequence $(R_{0,n})_n$ such that $R_{0,n}\in D(A_n)$ for all integers n and converging to R_0 in $L^2([0,1])$:*

$$\forall T>0,\ \lim_{n\to\infty}T_n(\cdot)R_{0,n}=T(\cdot)R_0\in C^0([0,T],L^2([0,1])). \tag{A.22}$$

Proof. After Lemma A.1, all the A_n+G are ζ dissipative. Moreover, $A+G$ satisfies the range condition, the proof being identical to the proof of "A_n+G satisfies the range condition (A.5)" (because the argument only uses the boundedness and the continuity of the saturation operator). Hence, it satisfies the distance condition:

$$\forall Q\in\overline{D(A)},\ \lim\inf_{\rho\to 0^+}\rho^{-1}d(\mathrm{Rg}(I-\rho(A+G)),Q)=0. \tag{A.23}$$

[1] In fact, it is shown that $C_c^\infty(\Omega)$ is dense in $L^2(\Omega)$ for Ω open of $\mathbb{R}^N$ $(N\in\mathbb{N}))$ but the proof can be easily adapted to our context.

Finally, $A \subset \lim_{n\to\infty} A_n$ by previous lemma. As a consequence, by the continuity of G as an operator from $L^2([0,1])$ to $L^2([0,1])$, $A+G \subset \lim_{n\to\infty} A_n + G$.

Take $R_0 \in \overline{D(A)}$ and a sequence $(R_{0,n})_n$ of elements from $D(A_n)$ for all n converging to R_0 in $L^2([0,1])$. From ([15], Thm. 6.8), $A+G$ generates a semigroup T and

$$\lim_{n\to+\infty} T_n(t)R_{0,n} = T(t)R_0 \text{ in } L^2([0,1])$$

for all time $t \geq 0$.

Moreover, the equality above holds uniformly on every bounded interval of $[0,\infty)$ which proves Theorem A.5.

□

A.3 Conclusion of the proof of Theorem 2.7

(1) Existence

Let R_0 be the initial data in $\overline{D(A)} = L^2([0,1])$. Take a smooth approximation $(\sigma_n)_n$ of σ and $(R_{0,n})_n$ a sequence of $C^\infty((0,1))$ satisfying compatibility conditions of order 1 converging towards R_0 in $L^2([0,1])$. Note that there exists at least one sequence $(R_{0,n})_n$ of such initial data. Indeed, $C_c^\infty((0,1))$ is dense in $L^2([0,1])$ and functions of $C_c^\infty((0,1))$ obviously satisfy compatibility conditions of order 1.

By Theorem A.1, for all integers n, the operator $A_n + G$ generates a semigroup T_n and $R_n := T_n(.)R_0$ is the unique solution to (2.1).

After Theorem A.5, $(R_n)_n$ converges in $C^0([0,T], L^2([0,1]))$ towards an element $R \in C^0([0,T], L^2([0,1]))$. R satisfies all requirements for being a weak solution to system (2.3).

(2) Uniqueness

The limit R neither depends on the smooth approximation $(\sigma_n)_n$ nor on the sequence of initial data $(R_{0,n})_n$ chosen. To prove this, we take an arbitrary sequence of smooth approximations $(\tilde{\sigma}_n)_n$ and define the corresponding sequence of operators $(\tilde{A}_n)_n$. Then, take an arbitrary sequence $(\tilde{R}_{0,n})_n$ of elements of $D(\tilde{A}_n) \cap H^2([0,1])$ converging to R_0 in $L^2([0,1])$. For all integers n, we define $\tilde{R}_n$ as the solution of:

$$\begin{cases} R_{n,t} + \Lambda R_{n,x} &= g \circ R_n \\ \quad R_n(0,.) &= HR_n(1,.) + B\tilde{\sigma}_n(KR_n(1,.)) \\ \quad R_n(.,0) &= \tilde{R}_{0,n}. \end{cases}$$

We note $\tilde{R}$, the limit in $C^0([0,T], L^2([0,1]))$ of $(\tilde{R}_n)_n$. We define $(\overline{\sigma}_n)_n$ as equal to σ_n if n is even and equal to $\tilde{\sigma}_n$ otherwise. In the same way, we define $(\overline{R}_{0,n})_n$ as equal to $R_{0,n}$ if n is even and equal to $\tilde{R}_{0,n}$ otherwise. Obviously, $(\overline{\sigma}_n)_n$ is a smooth approximation of σ and $(\overline{R}_{0,n})_n$ is a sequence of H^2 function satisfying compatibility conditions of order 1 and converging to R_0 in $L^2([0,1])$. Hence, we can apply what was done before to prove that there exists a limit $\overline{R}$ in $C^0([0,T], L^2([0,1]))$ of the sequence $(\overline{R_n})_n$. As a consequence $R = \overline{R} = \tilde{R}$ and the limit is unique.

(3) Continuity of $\mathbf{U_T}$

Let $R_0, \tilde{R}_0$ be two initial data in $L^2([0,1])$ and $R, \tilde{R}$ the associated weak solutions.

Take an arbitrary smooth approximation $(\sigma_n)_n$ of σ and sequences of approximation $(R_{0,n})_n$ and $(\tilde{R}_{0,n})_n$ of R_0 and $\tilde{R}_0$ respectively. Using the ζ dissipativity of the $A_n + G$ proven in Theorem A.1 with ζ independent on n, one gets:

$$\forall n \in \mathbb{N}, \forall 0 \leq t \leq T, \ ||T_n(t)R_{0,n} - T_n(t)\tilde{R}_{0,n}||_{L^2([0,1])} \leq e^{\zeta t}||R_{0,n} - \tilde{R}_{0,n}||_{L^2([0,1])}.$$

Taking the limit for all time $t \leq T$ (see Theorem A.5):

$$\forall 0 \leq t \leq T,\ ||T(t)R_0 - T(t)\tilde{R}_0||_{L^2([0,1])} \leq e^{\zeta t}||R_0 - \tilde{R}_0||_{L^2([0,1])}$$

which implies that:

$$\forall 0 \leq t \leq T,\ ||T(t)R_0 - T(t)\tilde{R}_0||_{L^2([0,1])} \leq e^{\zeta T}||R_0 - \tilde{R}_0||_{L^2([0,1])}.$$

It shows that U_T is Lipschitz and hence continuous. This concludes the proof of Theorem 2.7.

Appendix B. Proof of Theorem 2.9

Take a $\Delta \in D_d^+(\mathbb{R})$ satisfying $\mathcal{R}_\infty(\Delta(H+BK)\Delta^{-1}) < 1$ and a positive $\mu < -\log(\mathcal{R}_\infty(\Delta(H+BK)\Delta^{-1}))$.

Take the smooth approximation $(\sigma_n)_n$ of σ from Remark 2.2. Notation $A_n, D(A_n)$... are the same as in Appendix A.1.

Let R_0 be an initial data in $\overline{D(A)} = L^2([0,1])$ (for the moment we do not suppose that R_0 is in $L^\infty([0,1])$). As $C_c^\infty((0,1))$ is dense in $L^2([0,1])$ (see [6]) and as $C_c^\infty((0,1)) \subset D(A_n)$ for all integers n, one can construct a sequence $(R_{0,n})_n$ belonging to $C_c^\infty((0,1)) \subset D(A_n)$ converging to R_0 in $L^2([0,1])$. In what follows, we will consider such a sequence of initial data.

B.1 Local L^∞ stability of $C^1([0,T], L^2([0,1])) \cap C^0([0,T], H^1([0,1]))$ solutions

All along section B.1, n is a **fixed** integer.

R_n is the solution to the smooth problem:

$$\begin{cases} R_{n,t} + \Lambda R_{n,x} = g \circ R_n \\ R_n(0,.) = HR_n(1,.) + B\sigma_n(KR_n(1,.)) \\ R_n(.,0) = R_{n,0} \in D(A_n) \cap H^2([0,1]). \end{cases}$$

As the initial data is $H^2([0,1])$, satisfies compatibility conditions of order 1, the solution belongs to $C^1([0,T], L^2([0,1])) \cap C^0([0,T], H^1([0,1]))$ for all $T > 0$ by Theorem A.1.

Recall the definition of $V(.)$,

$$\forall f \in L^\infty([0,1]),\ V(f) := \max_{i \in [\![1,d]\!]} |\delta_i f_i e^{-\mu x}|_{L^\infty([0,1])} \tag{B.1}$$

where $\mathrm{diag}_{i\in[\![1,d]\!]}(\delta_i) = \Delta$.

For all integers p, the functional V_{2p} is defined by:

$$\forall f \in L^\infty([0,1]),\ V_{2p}(f) := \left(\sum_{i=1}^d \int_0^1 q_{p,i}^{2p} f_i^{2p} e^{-2p\mu x} \mathrm{d}x\right)^{1/2p}$$

where $Q_p = \mathrm{diag}_{i\in[\![1,d]\!]}(q_{p,i})$ will be chosen wisely later.

Claim B.1. *We can differentiate $V_{2p}(R_n(.,t))$ with respect to time and*

$$\forall t \geq 0,\ \frac{dV_{2p}^{2p}(R_n(.,t))}{\mathrm{d}t} = 2p\sum_{i=1}^d \int_0^1 q_{p,i}^{2p} R_{n,i,t} R_{n,i}^{2p-1} e^{-2p\mu x}\mathrm{d}x. \tag{B.2}$$

Proof of Claim B.1. Take $t \geq 0$, $\mathrm{d}t > 0$ and $T > t$.

$$\begin{aligned}&\left|\tfrac{V_{2p}^{2p}(R_n(.,t+\mathrm{d}t))-V_{2p}^{2p}(R_n(.,t))}{\mathrm{d}t} - 2p\textstyle\sum_{i=1}^d \int_0^1 q_{p,i}^{2p} R_{n,i,t} R_{n,i}^{2p-1} e^{-2p\mu x}\mathrm{d}x\right| \\ &\leq \textstyle\sum_{i=1}^d \int_0^1 q_{p,i}^{2p} \left|\tfrac{R_{n,i}^{2p}(x,t+\mathrm{d}t)-R_{n,i}^{2p}(x,t)}{\mathrm{d}t} - 2pR_{n,i,t}(x,t)R_{n,i}^{2p-1}(x,t)\right| e^{-2p\mu x}\mathrm{d}x.\end{aligned}$$

Thus it is sufficient to prove that at the limit when $\mathrm{d}t$ tends to zero

$$\forall i \in [\![1,d]\!], \quad \left|\frac{R_{n,i}^{2p}(.,t+\mathrm{d}t)-R_{n,i}^{2p}(.,t)}{\mathrm{d}t} - 2pR_{n,i,t}(.,t)R_{n,i}^{2p-1}(.,t)\right| \to_{L^1([0,1])} 0.$$

Let i be in $[\![1,d]\!]$,

$$\begin{aligned}\textstyle\int_0^1 \left|\tfrac{R_{n,i}^{2p}(.,t+\mathrm{d}t)-R_{n,i}^{2p}(.,t)}{\mathrm{d}t} - 2pR_{n,i,t}(.,t)R_{n,i}^{2p-1}(.,t)\right|\mathrm{d}x \;&= \textstyle\int_0^1 \Big|\tfrac{R_{n,i}(.,t+\mathrm{d}t)-R_{n,i}(.,t)}{\mathrm{d}t}(R_{n,i}(.,t+\mathrm{d}t)^{2p-1} + \cdots \\ &\quad + R_{n,i}(.,t)^{2p-1}) - 2pR_{n,i,t}(.,t)R_{n,i}^{2p-1}(.,t)\Big|\mathrm{d}x \\ &\leq ||\tfrac{R_{n,i}(.,t+\mathrm{d}t)-R_{n,i}(.,t)}{\mathrm{d}t} - R_{n,i,t}||_{L^2([0,1])} \\ &\quad \times\sqrt{\textstyle\int_0^1 \left|R_{n,i}(.,t+\mathrm{d}t)^{2p-1} + \cdots + R_{n,i}(.,t)^{2p-1}\right|^2\mathrm{d}x} \\ &\quad + \textstyle\int_0^1 |R_{n,i,t}|\big|R_{n,i}(.,t+\mathrm{d}t)^{2p-1} + \cdots + R_{n,i}(.,t)^{2p-1} \\ &\quad - 2pR_{n,i}(.,t)^{2p-1}\big|\mathrm{d}x.\end{aligned}$$

The term $\left|\left|\frac{R_{n,i}(.,t+\mathrm{d}t)-R_{n,i}(.,t)}{\mathrm{d}t} - R_{n,t}\right|\right|_{L^2([0,1])}$ converges towards zero because $R_{n,i}$ is in $C^1([0,T],L^2([0,1]))$ while the term $\int_0^1 \left|R_{n,i}(.,t+\mathrm{d}t)^{2p-1} + \cdots + R_{n,i}(.,t)^{2p-1}\right|^2\mathrm{d}x$ is bounded because $R_{n,i} \in C^0([0,T],H^1([0,1])) \subseteq C^0([0,T]\times[0,1])$. Finally, the term $\int_0^1 \left|R_{n,i,t}\right|\left|R_{n,i}(.,t+\mathrm{d}t)^{2p-1} + \cdots + R_{n,i}(.,t)^{2p-1} - 2pR_{n,i}(.,t)^{2p-1}\right|\mathrm{d}x$ is bounded by $||R_{n,i,t}||_{L^2([0,1])}||R_{n,i}(.,t+\mathrm{d}t)^{2p-1} + \cdots + R_{n,i}(.,t)^{2p-1} - 2pR_{n,i}(.,t)^{2p-1}||_{L^2([0,1])}$. The term $||R_{n,i,t}||_{L^2([0,1])}$ is bounded as R_n is in $C^1([0,T],L^2([0,1]))$ whereas the other term converges towards zero by the dominated convergence theorem. Therefore, Claim B.1 is proven. ☐

Next, as R_n verifies $R_{n,t} + \Lambda R_{n,x} = g\circ R_n$, one gets:

$$\frac{dV_{2p}^{2p}(R_n(.,t))}{\mathrm{d}t} = 2p\sum_{i=1}^d \int_0^1 q_{p,i}^{2p}\left(-\lambda_i R_{n,i,x} + g_i\circ R\right)R_{n,i}^{2p-1}e^{-2p\mu x}\mathrm{d}x. \tag{B.3}$$

There are two terms in (B.3) whose origins are different:

– the transport term:

$$W_{2p}(R_n(.,t)) := -2p\sum_{i=1}^d \int_0^1 q_{p,i}^{2p}\lambda_i R_{n,i,x}R_{n,i}^{2p-1}e^{-2p\mu x}\mathrm{d}x \tag{B.4}$$

– and the source term:

$$Z_{2p}(R_n(.,t)) := 2p \sum_{i=1}^{d} \int_0^1 q_{p,i}^{2p} (g_i \circ R) R_{n,i}^{2p-1} e^{-2p\mu x} \mathrm{d}x \tag{B.5}$$

so that:

$$\frac{dV_{2p}^{2p}(R_n(.,t))}{\mathrm{d}t} = W_{2p}(R_n(.,t)) + Z_{2p}(R_n(.,t)). \tag{B.6}$$

Define:

$$\forall t \geq 0,\ \xi_n(t) := \Delta R_n(1,t). \tag{B.7}$$

In the rest of the proof, we will sometimes denote $\xi_n(t)$ as ξ_n for readability.

Study of the term transport term W_{2p}

Lemma B.2. *For all $\mu < -\log(\mathcal{R}_\infty(\Delta(H+BK)\Delta^{-1}))$ and $\varepsilon > 0$ there exists $\tilde{p} = \tilde{p}(\mu, H, B, K, \Delta, \varepsilon)$ such that for all initial data $R_{0,n} \in H^2([0,1])$ satisfying compatibility conditions of order 1, the solution $R_n = T_n(.)R_{0,n}$ to (2.1) verifies the following assertion.*
If at time $\bar{t} \geq 0$,

$$|\xi_n(\bar{t})|_{\max} < \frac{\mathcal{R}_\infty(\Delta B \Delta^{-1})\sigma_{s,n}}{|\mathcal{R}_\infty(\Delta(H+BK)\Delta^{-1}) + \mathcal{R}_\infty(\Delta B\Delta^{-1})\mathcal{R}_\infty(\Delta K\Delta^{-1}) - e^{-\mu-\varepsilon}|}. \tag{B.8}$$

Then:

$$\forall p > \tilde{p},\ W_{2p}(R_n(.,\bar{t})) \leq -2p\mu\lambda_{\min} V_{2p}^{2p}(R_n(.,\bar{t})). \tag{B.9}$$

Remark B.3. The positive real ε is introduced to prevent an eventual dependence of $\tilde{p}$ with respect to $\bar{t}$. Indeed, further in the paper, we integrate (B.9) with respect to time. This would not be possible if $\tilde{p} = \tilde{p}(\bar{t})$.

Proof. In this proof we will sometimes drop the time dependence notation $\xi(\bar{t})$ or $R_n(\bar{t})$ for readability.
Using an integration by parts in (B.4), we compute:

$$W_{2p}(R_n(.,\bar{t})) = -\sum_{i=1}^{d} q_{p,i}^{2p}\lambda_i \left[R_{n,i}^{2p} e^{-2p\mu x}\right]_0^1 - 2p\mu \sum_{i=1}^{d} \int_0^1 q_{p,i}^{2p} \lambda_i R_{n,i}^{2p} e^{-2p\mu x} \mathrm{d}x$$

and therefore

$$W_{2p}(R_n(.,\bar{t})) \leq -W_{2p,1}(R_n(.,\bar{t})) - 2p\mu\lambda_{\min} V_{2p}^{2p}(R_n(.,\bar{t})) \tag{B.10}$$

where

$$W_{2p,1}(R_n(.,\bar{t})) := \sum_{i=1}^{d} q_{p,i}^{2p}\lambda_i \left[R_{n,i}^{2p}e^{-2p\mu x}\right]_0^1.$$

Using the fact that $R_n(.,\bar{t})$ is in $D(A_n)$, one gets:

$$W_{2p,1}(R_n(.,\bar{t})) = \sum_{i=1}^{d} q_{p,i}^{2p}\lambda_i R_{n,i}^{2p}(1,\bar{t})e^{-2p\mu} - q_{p,i}^{2p}\lambda_i \left\{HR_n(1,\bar{t}) + B\sigma_n(KR_n(1,\bar{t}))\right\}_i^{2p}.$$

We have,

$$W_{2p,1}(R_n(.,t)) = W_{2p,11} + W_{2p,12} \tag{B.11}$$

with:

$$W_{2p,11} := \sum_{i=1}^{d} q_{p,i}^{2p}\lambda_i R_{n,i}^{2p}(1,\bar{t})e^{-2p\mu} - q_{p,i}^{2p}\lambda_i \left\{(H+BK)R_n(1,\bar{t})\right\}_i^{2p} \tag{B.12}$$

and

$$W_{2p,12} := \sum_{i=1}^{d} q_{p,i}^{2p}\lambda_i \left\{(H+BK)R_n(1,\bar{t})\right\}_i^{2p} - q_{p,i}^{2p}\lambda_i \left\{HR_n(1,\bar{t}) + B(\sigma_n(KR_n(1,\bar{t})))\right\}_i^{2p}. \tag{B.13}$$

Note that if $\xi_n = 0$, $W_{2p,1}$ is zero and the conclusion of Lemma B.2 holds. In what follows, we suppose that $\xi_n \neq 0$.

Study of $\mathbf{W_{2p,11}}$

For what follows, we choose Q_p such that:

$$\boxed{\forall i \in [\![1,d]\!], \forall p \in \mathbb{N},\ q_{p,i} = \lambda_i^{-1/2p}\delta_i} \tag{B.14}$$

Inspired by ([3], p. 123), we get the following estimates for $W_{2p,11}$

$$\begin{aligned}
W_{2p,11} &= \textstyle\sum_{i=1}^{d} q_{p,i}^{2p}\lambda_i R_{n,i}^{2p}(1,\bar{t})e^{-2p\mu} - q_{p,i}^{2p}\lambda_i \left\{(H+BK)R_n(1,\bar{t})\right\}_i^{2p} \\
&\geq |\xi_n|_{\max}^{2p}e^{-2p\mu} - \textstyle\sum_{i=1}^{d} q_{p,i}^{2p}\lambda_i \left\{(H+BK)R_n(1,\bar{t})\right\}_i^{2p} \\
&= |\xi_n|_{\max}^{2p}e^{-2p\mu} - \textstyle\sum_{i=1}^{d} \delta_i^{2p} \left\{\sum_{j=1}^{d}(H+BK)_{i,j}\frac{\xi_{n,j}}{\delta_j}\right\}^{2p} \\
&\geq |\xi_n|_{\max}^{2p}\big(e^{-2p\mu} - d \times \mathcal{R}_\infty(\Delta(H+BK)\Delta^{-1})^{2p}\big)
\end{aligned}$$

where we have used the equations $q_{p,i}^{2p}\lambda_i = \delta_i^{2p}$, $|\xi_n|_{\max}^{2p} \leq \sum_{i=1}^{d} |\xi_i|^{2p}$, (B.7) and the definition of $\mathcal{R}_\infty$.

For $\mu < -\log(\mathcal{R}_\infty(\Delta(H+BK)\Delta^{-1}))$ and $p > -\frac{\log(d)+\log(2)}{2(\mu+\log(\mathcal{R}_\infty(\Delta(H+BK)\Delta^{-1})))}$, one has:

$$\frac{W_{2p,11}}{|\xi_n|_{\max}^{2p}} \geq \frac{e^{-2p\mu}}{2} \geq 0. \tag{B.15}$$

Study of $\mathbf{W_{2p,12}}$:

The term $W_{2p,12}$ is deeply related to the effect of saturation; it corresponds to the difference between the action of a linear controller and a saturated one. We recall the definition of $W_{2p,12}$ in (B.13):

$$W_{2p,12} = \sum_{i=1}^{d} q_{p,i}^{2p}\lambda_i(\left\{(H+BK)R_n(1,\bar{t})\right\}_i^{2p} - \left\{HR_n(1,\bar{t}) + B(\sigma_n(KR_n(1,\bar{t})))\right\}_i^{2p})$$

equivalent to:

$$W_{2p,12} = \sum_{i=1}^{d} \delta_i^{2p}(\chi_i^{2p} - \upsilon_i^{2p}) \tag{B.16}$$

with:

$$\chi = (H+BK)R_n(1,\bar{t}) \text{ and } \upsilon = HR_n(1,\bar{t}) + B\sigma_n(KR_n(1,\bar{t})).$$

The following claim allows to neglect the term $W_{2p,12}$ relatively to $W_{2p,11}$ at the limit when p tends to infinity:

Claim B.4. *Under the conditions of Lemma B.2 and if $\xi_n \neq 0$, then there exists a $\tilde{p} = \tilde{p}(\mu, H, B, K, \Delta, \varepsilon)$ such that:*

$$\forall p > \tilde{p},\ |W_{2p,12}| \leq \frac{e^{-2p\mu}|\xi_n|_{\max}^{2p}}{4}. \tag{B.17}$$

Proof of Claim B.4. Let i be in $[\![1,d]\!]$:

$$\delta_i|\chi_i| = |\sum_{j=1}^{d}(H+BK)_{i,j}\delta_i R_{n,j}(1,\bar{t})| = |\sum_{j=1}^{d}(H+BK)_{i,j}\frac{\delta_i}{\delta_j}\xi_{n,j}| \leq \mathcal{R}_\infty(\Delta(H+BK)\Delta^{-1})|\xi_n|_{\max} \tag{B.18}$$

Then, denoting $R_{n,j}(1,\bar{t})$ by $R_{n,j}$ ($j \in [\![1,d]\!]$) and Sat_i the set

$$\mathrm{Sat}_i = \left\{j \in [\![1,d]\!] \mid |(KR_n)_j| > \sigma_{s,n} \text{ and } B_{i,j} \neq 0\right\},$$

one has

$$\delta_i|\upsilon_i| = \delta_i\left|\sum_{j=1}^{d}(H+BK)_{i,j}R_{n,j} + \sum_{j=1}^{d}B_{i,j}\left(\sigma_{n,j}([KR_n]_j) - [KR_n]_j\right)\right|$$

$$\begin{aligned}
&=\delta_i \left| \sum_{j=1}^{d} (H+BK)_{i,j} R_{n,j} + \sum_{j\in \mathrm{Sat}_i} B_{i,j} \left(\sigma_{n,j}([KR_n]_j) - [KR_n]_j\right) \right| \\
&\le \delta_i \sum_{j=1}^{d} |(H+BK)_{i,j} R_{n,j}| + \delta_i \sum_{j\in \mathrm{Sat}_i} \left| B_{i,j} \left(\sigma_{n,j}([KR_n]_j) - \sum_{k=1}^{d} K_{j,k} R_{n,k}\right)\right| \\
&\le \delta_i \sum_{j=1}^{d} |(H+BK)_{i,j} R_{n,j}| + \delta_i \sum_{j\in \mathrm{Sat}_i} |B_{i,j}| \left(|\sum_{k=1}^{d} K_{j,k} R_{n,k}| - \sigma_{s,n}\right) \\
&\le \mathcal{R}_\infty(\Delta(H+BK)\Delta^{-1})|\xi_n|_{\max} + \mathcal{R}_\infty(\Delta B\Delta^{-1})(\mathcal{R}_\infty(\Delta K\Delta^{-1})|\xi_n|_{\max} - \sigma_{s,n}).
\end{aligned}$$

If Sat_i is empty *ie* if the saturation does not act on the ith coordinate, then $\upsilon_i = \chi_i$ and

$$\delta_i|\upsilon_i| \ \le \mathcal{R}_\infty(\Delta(H+BK)\Delta^{-1})|\xi_n|_{\max} \tag{B.19}$$

by (B.18).

Otherwise, Sat_i is non empty and

$$\delta_i|\upsilon_i| \le \mathcal{R}_\infty(\Delta(H+BK)\Delta^{-1})|\xi_n|_{\max} + \mathcal{R}_\infty(\Delta B\Delta^{-1})(\mathcal{R}_\infty(\Delta K\Delta^{-1})|\xi_n|_{\max} - \sigma_{s,n}). \tag{B.20}$$

Moreover as $\mu < -\log(\mathcal{R}_\infty(\Delta(H+BK)\Delta^{-1}))$ by hypothesis, there exists $\alpha \in (0,1)$ such that

$$\mathcal{R}_\infty(\Delta(H+BK)\Delta^{-1}) = \alpha e^{-\mu}.$$

Additionally by (B.8),

$$\mathcal{R}_\infty(\Delta(H+BK)\Delta^{-1})|\xi_n|_{\max} + \mathcal{R}_\infty(\Delta B\Delta^{-1})(\mathcal{R}_\infty(\Delta K\Delta^{-1})|\xi_n|_{\max} - \sigma_{s,n}) < e^{-\mu-\varepsilon}|\xi_n|_{\max}.$$

Injecting the last two equations in (B.18)-(B.20),

$$\forall i \in [\![1,d]\!], \ \begin{cases} \delta_i|\chi_i| \ \le \alpha e^{-\mu}|\xi_n|_{\max} \\ \delta_i|\upsilon_i| \ \le \max(\alpha e^{-\mu}, e^{-\mu-\varepsilon})|\xi_n|_{\max}. \end{cases} \tag{B.21}$$

Finally, from (B.21) and (B.16), we have

$$\forall p \in \mathbb{N}, \ \frac{W_{2p,12}}{|\xi_n|_{\max}^{2p}} \le 2d \times \max\left\{e^{-2p\varepsilon}, \alpha^{2p}\right\} e^{-2p\mu}.$$

Hence, there exists an integer $\tilde{p} > -\frac{\log(d)+\log(2)}{2(\mu+\log(\mathcal{R}_\infty(\Delta(H+BK)\Delta^{-1})))}$ (depending on (ε,α) but not on $|\xi_n|_{\max}$) such that

$$\forall p \ge \tilde{p}, \ \left|\frac{W_{2p,12}}{|\xi_n|^{2p}}\right| \le \frac{e^{-2p\mu}}{4}.$$

This ends the proof of Claim B.4. $\square$

As a consequence, by (B.15), (B.17) and for $p > \tilde{p}$, $W_{2p,1} = W_{2p,11} + W_{2p,22} \geq \frac{|\xi_n|^2_{\max} e^{-2p\mu}}{4} \geq 0$. Injecting last statement in (B.10), one gets:

$$W_{2p}(R_n(.,\bar{t})) \leq -2p\mu\lambda_{\min} V_{2p}^{2p}(R_n(.,\bar{t})).$$

This finishes the proof of Lemma B.2. □

Remark B.5. In the proof of the above result, we enforce $W_{2p,12} =_{p\to+\infty} o(|\xi_n|^{2p}_{\max} e^{-2p\mu})$. This condition recalls a local sector bounded condition ([19], Sect. 1.7.2). Indeed, the term $W_{2p,12}$ is induced by the difference between the linear control law and the its saturated version, i.e., this term arises from a deadzone nonlinearity. The term on the right-hand side of (B.22), $e^{-\mu}|\xi_n|_{\max}$, represents the state of the system. Thus formally speaking, the condition

$$W_{2p,12} =_{p\to+\infty} o(|\xi_n|^{2p}_{\max} e^{-2p\mu}), \tag{B.22}$$

is somewhat equivalent to a regional sector condition ([19], Sect. 1.7.2).

Remark B.6. Conditions (B.8) are less restrictive when the saturation $\sigma_{s,n}$ is weaker and when the exponential decay rate μ decreases.

Analysis of the term Z_{2p}

Lemma B.7. *For all integers p and for all time $t \geq 0$,*

$$|Z_{2p}(R_n(.,t))| \leq 2pL_{g,\max} V_{2p}^{2p}(R_n(.,t)).$$

Proof. Recall the definition of $Z_{2p}(R_n(.,t))$ (B.5):

$$Z_{2p}(R_n(.,t)) = 2p\sum_{i=1}^{d}\int_0^1 q_{p,i}^{2p}(g_i \circ R_n) R_{n,i}^{2p-1} e^{-2p\mu x}\mathrm{d}x.$$

Using the hypothesis on g in (2.4):

$$|Z_{2p}(R_n(.,t))| \leq 2p\sum_{i=1}^{d}\int_0^1 q_{p,i}^{2p} L_{g,i} R_{n,i}^{2p} e^{-2p\mu x}\mathrm{d}x \leq 2pL_{g,\max} V_{2p}^{2p}(R_n(.,t)).$$

□

Conclusion on the L^∞ stability of regular solutions

Lemma B.8. *Under conditions of Lemma B.2, the solution $R_n = T_n(.)R_{0,n}$ to problem (2.1) satisfies the following statement.*

If at a time $\bar{t} \geq 0$, condition (B.8) is satisfied then:

$$\frac{\mathrm{d}V_{2p}(R_n(.,\bar{t}))}{\mathrm{d}t} \leq -(\mu\lambda_{\min} - L_{g,\max}) V_{2p}(R_n(.,\bar{t}))$$

for all $p > \tilde{p}$ where $\tilde{p}$ defined in Lemma B.2.

Proof. Let $p > \tilde{p}$. By (B.6), we have:

$$\frac{\mathrm{d}V_{2p}^{2p}}{\mathrm{d}t} = 2p\frac{\mathrm{d}V_{2p}}{\mathrm{d}t}V_{2p}^{2p-1} = W_{2p} + Z_{2p}. \tag{B.23}$$

After Lemmas B.2 and B.7,

$$W_{2p}(R_n(.,\bar{t})) \leq -2p\mu\lambda_{\min}V_{2p}^{2p}(R_n(.,\bar{t}))$$

and

$$|Z_{2p}(R_n(.,\bar{t}))| \leq 2pL_{g,\max}V_{2p}^{2p}(R_n(.,\bar{t})).$$

Summing previous inequalities and dividing by $2pV_{2p}^{2p-1}$ in (B.23), one gets:

$$\frac{\mathrm{d}V_{2p}}{\mathrm{d}t} \leq -(\mu\lambda_{\min} - L_{g,\max})V_{2p}.$$

□

Before going further into the proof, we need the following lemma which may be useful in future works:

Lemma B.9.

$$\forall R \in L^\infty([0,1]), V_{2p}(R) \to_{p\to+\infty} V(R).$$

Moreover, the convergence is uniform on all bounded sets of $H^1([0,1])$.

Proof. Let R be in $L^\infty([0,1])$

$$\forall p \in \mathbb{N},\ V_{2p}(R) = \left(\sum_{i=1}^{d}\int_0^1 \delta_i^{2p}\lambda_i R_i^{2p}e^{-2p\mu x}\mathrm{d}x\right)^{1/2p}.$$

As a consequence,

$$\lambda_{\min}^{1/2p}||x \mapsto \Delta R(x)e^{-\mu x}||_{L^{2p}([0,1])} \leq V_{2p}(R) \leq \lambda_{\max}^{1/2p}||x \mapsto \Delta R(x)e^{-\mu x}||_{L^{2p}([0,1])}.$$

By a classic result of analysis $||Q||_{L^{2p}([0,1])}$ converges towards $||Q||_{L^\infty([0,1])}$ for all $Q \in L^\infty([0,1])$. Hence, passing to the limit in last inequalities:

$$||x \mapsto \Delta R(x)e^{-\mu x}||_{L^\infty([0,1])} \leq \lim_{p\to+\infty} V_{2p}(R) \leq ||x \mapsto \Delta R(x)e^{-\mu x}||_{L^\infty([0,1])}$$

and:

$$\lim_{p\to+\infty} V_{2p}(R) = ||x \mapsto \Delta R(x)e^{-\mu x}||_{L^\infty([0,1])} = V(R).$$

We also need to prove the uniform convergence for all bounded sets of $H^1([0,1])$. We will prove it for the case of scalar functions; the case of vector valued function being similar.

Take an $r > 0$. Let B_r be the ball of radius r in $H^1([0,1])$.

Define for $\omega > 0$ and f in B_r, $S_{f,\omega} := \{x \in [0,1];\ |f(x)| \geq |f|_{L^\infty([0,1])} - \omega\}$.

For all $\omega > 0$ and f in B_r, there exists an x in $S_{f,\omega/2}$. For all y in $[0,1]$, we have:

$$f(y) = f(x) + \int_x^y f'(z)\mathrm{d}z.$$

Using Cauchy-Schwartz inequality for the L^2 canonical scalar product, one gets

$$|f(y)| \geq |f(x)| - r\sqrt{|x-y|}.$$

As x is in $S_{\omega/2}$:

$$|f(y)| \geq |f|_{L^\infty([0,1])} - \frac{\omega}{2} - r\sqrt{|x-y|}$$

Thus, for all y in $[0,1]$ such that $|x-y| \leq (\frac{\omega}{2r})^2$:

$$|f(y)| \geq |f|_{L^\infty([0,1])} - \omega$$

which implies that:

$$\forall y \in [0,1];\ |x-y| \leq \left(\frac{\omega}{2r}\right)^2 \implies y \in S_{f,\omega}.$$

As a consequence:

$$\forall f \in B_r,\ \forall 0 < \omega < r,\ \mu(S_{f,\omega}) \geq \frac{\omega^2}{2r^2} \tag{B.24}$$

where μ designates the usual Lebesgue measure.
Now using the definition of $S_{f,\omega}$ and (B.24), one has:

$$\begin{aligned} \forall f \in B_r,\ \forall \omega > 0, \forall p \in \mathbb{N},\ 1/2^{2p}\left(\tfrac{\omega}{r}\right)^{1/p}(|f|_{L^\infty([0,1])} - \omega) &\leq \mu(S_{f,\omega})^{1/2p}(|f|_{L^\infty([0,1])} - \omega) \\ &= \left(\textstyle\int_{S_{f,\omega}}(|f|_{L^\infty([0,1])} - \omega)^{2p}\mathrm{d}x\right)^{1/2p} \\ &\leq |f|_{L^{2p}([0,1])}. \end{aligned}$$

Moreover,

$$\forall f \in L^\infty([0,1]),\ |f|_{L^p([0,1])} \leq |f|_{L^\infty([0,1])}.$$

Hence:

$$\forall f \in B_r,\ \forall \omega > 0, \forall p \in \mathbb{N},\ 1/2^{2p}\left(\frac{\omega}{r}\right)^{1/p}(|f|_{L^\infty([0,1])} - \omega) \leq |f|_{L^{2p}([0,1])} \leq |f|_{L^\infty([0,1])}$$

which implies:

$$\forall f \in B_r,\ \forall \omega > 0, \forall p \in \mathbb{N},\ 0 \leq \left||f|_{L^{2p}([0,1])} - |f|_{L^\infty([0,1])}\right| \leq \left|1/2^{2p}\left(\frac{\omega}{r}\right)^{1/p}(|f|_{L^\infty([0,1])} - \omega) - |f|_{L^\infty([0,1])}\right|.$$

Now take an $0<\varepsilon<1$, f in B_r and pose $\omega=\frac{\varepsilon}{2}$;

$$\begin{aligned}\left|1/2^{2p}\left(\frac{\omega}{r}\right)^{1/p}(|f|_{L^\infty([0,1])}-\omega)-|f|_{L^\infty([0,1])}\right| &\leq \left|1/2^{2p}\left(\frac{\omega}{r}\right)^{1/p}-1\right|\left||f|_{L^\infty([0,1])}-\omega\right|\\ &\quad+\left||f|_{L^\infty([0,1])}-\omega-|f|_{L^\infty([0,1])}\right|\\ &\leq \left|1/2^{2p}\left(\frac{\omega}{r}\right)^{1/p}-1\right|(r+\omega)+\omega\\ &=\left|1/2^{2p}\left(\frac{\varepsilon}{2r}\right)^{1/p}-1\right|(r+\frac{\varepsilon}{2})+\frac{\varepsilon}{2}\\ &\leq\left|1/2^{2p}\left(\frac{\varepsilon}{2r}\right)^{1/p}-1\right|(r+1)+\frac{\varepsilon}{2}.\end{aligned}$$

There exists a $P(r,\varepsilon)$ (independent on f) such that for $p>P(r,\varepsilon)$:

$$\left|1/2^{2p}\left(\frac{\varepsilon}{2r}\right)^{1/p}-1\right|(r+1)\leq\frac{\varepsilon}{2}$$

so that:

$$\forall f\in B_r,\ \left||f|_{L^{2p}([0,1])}-|f|_{L^\infty([0,1])}\right|\leq\varepsilon.$$

This finishes the proof of Lemma B.9. ∎

Lemma B.10. *For all initial data $R_{0,n}\in H^2([0,1])$ satisfying compatibility conditions of order 1 and $\mu<-\log(\mathcal{R}_\infty(\Delta(H+BK)\Delta^{-1}))$, if*

$$\begin{cases}\mu\lambda_{\min}-L_{g,\max} & \geq 0\\ V(R_{0,n}) & < e^{-\mu}\dfrac{\mathcal{R}_\infty(\Delta B\Delta^{-1})\sigma_{s,n}}{|\mathcal{R}_\infty(\Delta(H+BK)\Delta^{-1})+\mathcal{R}_\infty(\Delta B\Delta^{-1})\mathcal{R}_\infty(\Delta K\Delta^{-1})-e^{-\mu}|}\end{cases}$$

then, the solution $R_n=T_n(.)R_{0,n}$ to (2.1) verifies:

$$\forall t\geq 0,\ V(R_n(.,t))\leq e^{-(\mu\lambda_{\min}-L_{g,\max})t}V(R_{0,n}).$$

Proof. Using a continuity argument, we can take a $\varepsilon>0$ such that

$$V(R_{0,n})<e^{-\mu}\frac{\mathcal{R}_\infty(\Delta B\Delta^{-1})\sigma_{s,n}}{|\mathcal{R}_\infty(\Delta(H+BK)\Delta^{-1})+\mathcal{R}_\infty(\Delta B\Delta^{-1})\mathcal{R}_\infty(\Delta K\Delta^{-1})-e^{-\mu-\varepsilon}|}.\tag{B.25}$$

Define for all integers p, $T_p:=\sup\left\{T\geq 0;\ \forall t\in[0,T],\ V_{2p}(R_n(.,t))\leq e^{-(\mu\lambda_{\min}-L_{g,\max})t}V_{2p}(R_{0,n})\right\}$.

(1) $(T_p)_p$ is not a bounded sequence

We will prove that the sequence $(T_p)_p$ is not bounded by contradiction. Suppose $(T_p)_p$ bounded and take a subsequence still denoted $(T_p)_p$ converging towards a limit denoted T_∞.

(1.1) T_∞ is strictly positive For all integers i in $[\![1,d]\!]$

$$\xi_{n,i}(0)e^{-\mu} = \delta_i R_{n,i}(1,0)e^{-\mu} \le V(R_n(.,0)) = V(R_{0,n}).$$

By (B.25),

$$|\xi_n|_{\max}(0) < \frac{\mathcal{R}_\infty(\Delta B\Delta^{-1})\sigma_{s,n}}{|\mathcal{R}_\infty(\Delta(H+BK)\Delta^{-1}) + \mathcal{R}_\infty(\Delta B\Delta^{-1})\mathcal{R}_\infty(\Delta K\Delta^{-1}) - e^{-\mu-\varepsilon}|}.$$

By continuity of $t \mapsto \xi_n(t) = \Delta R_n(1,t)$, there exists a $\mathrm{d}t_0 > 0$ independent on p such that:

$$\forall t \in [0, \mathrm{d}t_0],\ |\xi_n|_{\max}(t) < \frac{\mathcal{R}_\infty(\Delta B\Delta^{-1})\sigma_{s,n}}{|\mathcal{R}_\infty(\Delta(H+BK)\Delta^{-1}) + \mathcal{R}_\infty(\Delta B\Delta^{-1})\mathcal{R}_\infty(\Delta K\Delta^{-1}) - e^{-\mu-\varepsilon}|}.$$

Applying Lemma B.8 and noting that condition (B.8) is satisfied for $\bar{t} \leftarrow t \in [0, \mathrm{d}t_0]$, there exists a $\tilde{p}(\mu, H, B, K, \Lambda, \varepsilon) \in \mathbb{N}$ (independent on time):

$$\forall p > \tilde{p},\ \forall t \in [0, \mathrm{d}t_0],\ \frac{\mathrm{d}V_{2p}(R_n(.,t))}{\mathrm{d}t} \le -(\mu\lambda_{\min} - L_{g,\max})V_{2p}(R_n(.,t)).$$

Integrating on $[0, \mathrm{d}t_0]$, it holds

$$\forall p > \tilde{p}, \forall t \in [0, \mathrm{d}t_0],\ V_{2p}(R_n(.,t)) \le e^{-(\mu\lambda_{\min} - L_{g,\max})t}V_{2p}(R_{0,n}).$$

This allows to assert that:

$$\forall p > \tilde{p},\ T_p \ge \mathrm{d}t_0$$

and passing to the limit,

$$T_\infty \ge \mathrm{d}t_0 > 0.$$

(1.2) Proof of the contradiction

As $R_n \in C^0([0,1] \times [0,T])$ for all $T > 0$ and by definition of T_p:

$$\forall p \in \mathbb{N},\ V_{2p}(R_n(.,T_p)) = e^{-(\mu\lambda_{\min} - L_{g,\max})T_p}V_{2p}(R_{0,n}).$$

As $(R_n(T_p))_p$ is a bounded sequence of $H^1([0,1])$ (because R_n is in $C^0([0,T], H^1([0,1]))$ for all $T > 0$), we can use the uniform convergence proven in Lemma B.9 to pass to the limit as p tends towards infinity in last equation:

$$V(R_n(.,T_\infty)) = e^{-(\mu\lambda_{\min} - L_{g,\max})T_\infty}V(R_{0,n}) \le V(R_{0,n}).$$

As a consequence, for all integers i in $[\![1,d]\!]$

$$\xi_{n,i}(T_\infty)e^{-\mu} = \delta_i R_{n,i}(1,T_\infty)e^{-\mu} \le V(R_n(.,T_\infty)) \le V(R_{0,n})$$

and by (B.25), we have:

$$|\xi_n|_{\max}(T_\infty) < \frac{\mathcal{R}_\infty(\Delta B\Delta^{-1})\sigma_{s,n}}{|\mathcal{R}_\infty(\Delta(H+BK)\Delta^{-1}) + \mathcal{R}_\infty(\Delta B\Delta^{-1})\mathcal{R}_\infty(\Delta K\Delta^{-1}) - e^{-\mu-\varepsilon}|}.$$

As $t \mapsto \xi_n(t) = \Delta R_n(1,t)$ is a continuous function (remember that R_n is in $C^0([0,1]\times[0,T])$ for all $T>0$), there exists a $0 < \mathrm{d}t < T_\infty$ independent on p (but dependent on n) such that:

$$\forall t \in [T_\infty - \mathrm{d}t, T_\infty + \mathrm{d}t],\ |\xi_n|_{\max}(t) < \frac{\mathcal{R}_\infty(\Delta B\Delta^{-1})\sigma_{s,n}}{|\mathcal{R}_\infty(\Delta(H+BK)\Delta^{-1}) + \mathcal{R}_\infty(\Delta B\Delta^{-1})\mathcal{R}_\infty(\Delta K\Delta^{-1}) - e^{-\mu-\varepsilon}|}.$$

Applying Lemma B.8 and noting that condition (B.8) is satisfied for $\bar{t} \leftarrow t \in [T_\infty - \mathrm{d}t, T_\infty + \mathrm{d}t]$, there exists $\tilde{p}(\mu, H, B, K, \Lambda, \varepsilon) \in \mathbb{N}$ such that:

$$\forall p > \tilde{p},\ \forall t \in [T_\infty - \mathrm{d}t, T_\infty + \mathrm{d}t],\ \frac{\mathrm{d}V_{2p}(R_n(.,t))}{\mathrm{d}t} \leq -(\mu\lambda_{\min} - L_{g,\max})V_{2p}(R_n(.,t))$$

which is no more than:

$$\forall p > \tilde{p},\ \forall t \in [T_\infty - \mathrm{d}t, T_\infty + \mathrm{d}t],\ \frac{\mathrm{d}[V_{2p}(R_n(.,t))e^{(\mu\lambda_{\min} - L_{g,\max})t}]}{\mathrm{d}t} \leq 0.$$

Integrating last statement on $[T_\infty - \mathrm{d}t, t]$ for $T_\infty - \mathrm{d}t \leq t \leq T_\infty + \mathrm{d}t$:

$$\forall p > \tilde{p},\ \forall t \in [T_\infty - \mathrm{d}t, T_\infty + \mathrm{d}t],\ V_{2p}(R_n(.,t)) \leq e^{-(\mu\lambda_{\min} - L_{g,\max})(t + \mathrm{d}t - T_\infty)}V_{2p}(R_n(., T_\infty - \mathrm{d}t)) \tag{B.26}$$

Moreover, as $T_\infty - \mathrm{d}t < T_\infty$ and $\lim_{p\to\infty} T_p = T_\infty$, there exists a $P(\mathrm{d}t)$ such that for all $p > P(\mathrm{d}t)$, $T_\infty - \mathrm{d}t < T_p$. It implies that:

$$\forall p > P(\mathrm{d}t),\ \forall t \leq T_\infty - \mathrm{d}t,\ V_{2p}(R_n(.,t)) \leq e^{-(\mu\lambda_{\min} - L_{g,\max})t}V_{2p}(R_{0,n}).$$

Using last inequality and (B.26), one has:

$$\forall p > \max(P(\mathrm{d}t), \tilde{p}), \forall t \leq T_\infty + \mathrm{d}t,\ V_{2p}(R_n(.,t)) \leq e^{-(\mu\lambda_{\min} - L_{g,\max})t}V_{2p}(R_{0,n}).$$

As a consequence and by the definition of T_p:

$$\forall p > \max(P(\mathrm{d}t), \tilde{p}),\ T_p \geq T_\infty + \mathrm{d}t$$

passing to the limit as p tends towards infinity, one gets:

$$T_\infty \geq T_\infty + \mathrm{d}t$$

implying that $\mathrm{d}t \leq 0$ which is a contradiction.

(2) Proof of exponential convergence

Hence, $(T_p)_p$ is not a bounded sequence and there exists a non decreasing subsequence $(T_{\phi(p)})_p$ of $(T_p)_p$ such that:

$$\lim_{p\to\infty} T_{\phi(p)} = +\infty.$$

Take an arbitrary time $t \geq 0$, there exists a P such that for $p > P$ then $T_{\phi(p)} \geq t$. It implies that:

$$\forall p > P,\ V_{2\phi(p)}(R_n(.,t)) \leq e^{-(\mu\lambda_{\min} - L_{g,\max})t} V_{2\phi(p)}(R_{0,n})$$

Passing to the limit as p tends towards infinity, we get:

$$V(R_n(.,t)) \leq e^{-(\mu\lambda_{\min} - L_{g,\max})t} V(R_{0,n})$$

and Lemma B.10 is proven. □

B.2 Local $L^\infty([0,1])$ stability for weak solutions

Finally, we prove Theorem 2.9. The integer n is not fixed anymore and we will pass to the limit in the exponential stability of the sequence $(R_n)_n$.

Lemma B.11. *For all $R_0 \in L^\infty([0,1])$, there exists a sequence $(R_{0,n})_n \in C_c^2((0,1))$ converging to R_0 in $L^2([0,1])$ such that*

$$\forall n \in \mathbb{N},\ V(R_{0,n}) \leq V(R_0).$$

Proof. By ([6], Cor. 4.23), for all $R \in L^\infty([0,1])$ there exists a sequence of elements of $C_c^2((0,1))$, $(R_n)_n$, such that:

$$\forall n \in \mathbb{N},\ ||R_n||_{L^\infty([0,1])} \leq ||R||_{L^\infty([0,1])}$$

and

$$R_n \to_{n\to+\infty} R$$

where the convergence holds in $L^2([0,1])$.

Now take R_0 in $L^\infty([0,1])$, $V(R_0) = ||x \mapsto \Delta R_0(x)e^{-\mu x}||_{L^\infty([0,1])}$. Applying the Corollary stated above to $x \mapsto \Delta R_0(x)e^{-\mu x}$, there exists a subsequence $(S_{0,n})_n$ of elements of $C_c^2((0,1))$ such that:

$$\forall n \in \mathbb{N},\ ||S_{0,n}||_{L^\infty([0,1])} \leq V(R_0)$$

and

$$S_{0,n} \to_{n\to+\infty} \left[x \mapsto \Delta R_0(x)e^{-\mu x}\right]$$

in $L^2([0,1])$.

Defining, for all integers n, $R_{0,n} = x \mapsto \Delta^{-1} S_{0,n}(x)e^{\mu x}$, one gets:

$$\forall n \in \mathbb{N},\ V(R_{0,n}) \leq V(R_0)$$

and

$$R_{0,n} \rightarrow_{n\rightarrow+\infty} R_0$$

in $L^2([0,1])$.

This concludes the proof of Lemma B.11.

□

Take the sequence $(R_{0,n})_n \in D(A_n)$ given by preceding Lemma B.11 converging to R_0 in $L^2([0,1])$. We denote also $(R_n)_n = (T_n(.)R_{0,n})_n$.

As for all integers n, $V(R_{0,n}) \leq V(R_0)$, $\sigma_{s,n} \geq \sigma_s$ and (2.6):

$$V(R_{0,n}) < e^{-\mu} \frac{\mathcal{R}_\infty(\Delta B\Delta^{-1})\sigma_{s,n}}{|\mathcal{R}_\infty(\Delta(H+BK)\Delta^{-1}) + \mathcal{R}_\infty(\Delta B\Delta^{-1})\mathcal{R}_\infty(\Delta K\Delta^{-1}) - e^{-\mu}|},$$

we can apply Lemma B.10 for all integers n,

$$\forall n \in \mathbb{N},\ \forall t \geq 0,\ V(R_n(.,t)) \leq e^{-(\mu\lambda_{\min} - L_{g,\max})t} V(R_{0,n}). \tag{B.27}$$

Now let $t \geq 0$ and $p \in \mathbb{N}$. By Fatou's lemma and the fact that $(R_n(.,t))_n$ converges up to a subsequence towards $R(.,t)$ in the almost everywhere sense (because it converges in $L^2([0,1])$ by Thm. A.5):

$$V_{2p}(R(.,t)) \leq \liminf_{n\rightarrow+\infty} V_{2p}(R_n(.,t)).$$

Using the fact that for all $R \in L^\infty([0,1])$, $V_{2p}(R) \leq d^{1/2p}\max_i\{\lambda_i^{-1/2p}\}V(R)$ (Remember (B.14) and the definition of V and V_{2p}), we have:

$$\begin{aligned} V_{2p}(R(.,t)) &\leq d^{1/2p}\max_i\{\lambda_i^{-1/2p}\} \liminf_{n\rightarrow+\infty} V(R_n(.,t)) \\ &\leq d^{1/2p}\max_i\{\lambda_i^{-1/2p}\} e^{-(\mu\lambda_{\min}-L_{g,\max})t} \liminf_{n\rightarrow+\infty} V(R_{0,n}) \\ &\leq d^{1/2p}\max_i\{\lambda_i^{-1/2p}\} e^{-(\mu\lambda_{\min}-L_{g,\max})t} V(R_0) \end{aligned}$$

where we have used the fact that $V(R_{0,n}) \leq V(R_0)$ by construction.

Passing to the limit when p goes to infinity, one gets:

$$V(R(.,t)) \leq e^{-(\mu\lambda_{\min}-L_{g,\max})t} V(R_0).$$

Theorem 2.9 is proven.

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