ESAIM: COCV 26 (2020) 78
https://doi.org/10.1051/cocv/2019068

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

FINITE ELEMENT ANALYSIS OF THE CONSTRAINED DIRICHLET BOUNDARY CONTROL PROBLEM GOVERNED BY THE DIFFUSION PROBLEM*

Thirupathi Gudi** and Ramesh Ch. Sau

Abstract. We study an energy space-based approach for the Dirichlet boundary optimal control problem governed by the Laplace equation with control constraints. The optimality system results in a simplified Signorini type problem for control which is coupled with boundary value problems for state and costate variables. We propose a finite element based numerical method using the linear Lagrange finite element spaces with discrete control constraints at the Lagrange nodes. The analysis is presented in a combination for both the gradient and the L^2 cost functional. *A priori* error estimates of optimal order in the energy norm is derived up to the regularity of the solution for both the cases. Theoretical results are illustrated by some numerical experiments.

Mathematics Subject Classification. 65N30, 65N15, 65N12, 65K10.

Received September 7, 2018. Accepted November 11, 2019.

1. Introduction

The study of the optimal control problems governed by the partial differential equations has been a major research area in the applied mathematics and its allied areas. The optimal control problem consists of finding an optimal control variable that minimizes a cost functional subject to a partial differential equation satisfied by the optimal control and an optimal state. There are many results on the finite element analysis of optimal control problems, see for example the monographs [23, 25, 30] for the theory of optimal control problems and their numerical approximation. In an optimal control problem, the control can act on the system through either a boundary condition or through an interior force. In the latter case, the control is said to be a distributed and in the former case, the control is said to be the boundary control. The choice of the boundary condition leads to several types of boundary controls, *e.g.*, Dirichlet, Neumann or Robin boundary control. There is a lot of work related to the approximation of distributed and Neumann boundary control problems, it is difficult to cite all them here, but we refer to some articles and the references therein. We refer to [6, 17, 19, 22, 27] for the work related to distributed control and to [8, 9, 18] for the work on Neumann boundary control problem. A framework for a priori and *a posteriori* energy norm error analysis for Neumann and distributed control

*The authors acknowledge the support from the UGC center for Advanced Study-II. The first author also thank the support from DST MATRICS Grant.

Keywords and phrases: Diffusion equation, PDE-constrained optimization, control-constraints, finite element method, error bounds.

Department of Mathematics, Indian Institute of Science, Bangalore 560012, India.

** Corresponding author: gudi@iisc.ac.in

problems by discontinuous Galerkin discretization can be found in [11, 15] for biharmonic and Stokes problem respectively.

The study of Dirichlet control problems posed on polygonal domains can be traced back to [7], where a control constrained problem governed by a semilinear elliptic equation posed in a convex polygonal domain is studied. There are various approaches proposed in the literature. One of the approaches is to seek the control from the $L_2(\Gamma)$-space, where Γ is the boundary of the domain Ω. In this approach, the difficulty arises in formulating the weak formulation of the given partial differential equation. The difficulty is addressed by introducing an ultra weak formulation by using the transposition method, see [14, 26]. In the case, if the boundary is of certain smooth, the control is shown to have additional regularity and subsequently the standard weak formulation is recovered. In general, the solution may not be smooth and hence the standard weak formulation can not be recovered. This leads to an irregular solution and hence suboptimal rates of convergence for the corresponding numerical solution. The other approaches based on singular perturbation through Robin condition or using $H^{1/2}(\Gamma)$ or $H^1(\Gamma)$ can be found in [3, 10, 20, 21, 28].

In the recent paper [28], the authors discussed a finite element analysis of Dirichlet boundary control problem where the control is chosen from a subset of $H^{1/2}(\Gamma)$-space. In that paper the continuous and the discrete problems are defined by using the Steklov-Poincaré operator arising from the harmonic extension and then the optimality system is written as an $H^{1/2}(\Gamma)$ -space variational inequality on the boundary. The numerical method in [28] is converted into a system of equations defined in the interior of the domain using harmonic and continuous extension operators. Recently in [12], the authors have proposed another approach for posing the Dirichlet boundary control problem and derived optimal order error estimates. In [28], the Steklov-Poincaré operator was used to define the cost functional with the help of a harmonic extension of the given boundary data. But in [12], the control has been sought in the $H^1(\Omega)$ space and the resulting control is a harmonic function without this being explicitly imposed and this leads to the optimality conditions in a system of PDEs posed over the domain Ω. Based on this formulation, the authors proposed a finite element numerical method and derived its corresponding optimal order error estimates in the energy norm and in the L^2-norm in the absence of point-wise control constraints. In this article, we study the Dirichlet boundary control problem governed by diffusion equation with point-wise control constraints and two cost functionals one is the L^2-cost functional and another one is the gradient cost functional. Here we sought the control from a closed convex subset of $H^1(\Omega)$ with point-wise constraints on the boundary. In [12], the control was taken from the whole of the $H^1(\Omega)$ space and hence the control q satisfies the following Neumann problem:

$$
\begin{aligned}
-\Delta q &= 0 \quad \text{in} \quad \Omega \\
\rho \frac{\partial q}{\partial n} &= \frac{\partial \phi}{\partial n} \quad \text{on} \quad \partial\Omega,
\end{aligned}
$$

where ϕ is the adjoint variable. In the present article, the control is chosen from a subset of $H^1(\Omega)$ and hence we obtain a variational inequality for the optimal control which resembles a simplified Signorini problem (for L^2-cost functional):

$$
\begin{aligned}
-\rho\Delta q = 0 &\quad \text{in} \quad \Omega, \\
q = 0 &\quad \text{on} \quad \Gamma_D, \\
q_a \leq q \leq q_b &\quad \text{on} \quad \Gamma_C,
\end{aligned}
$$

and further the following holds for almost every $x \in \Gamma_C$:

$$
\text{if } q_a < q(x) < q_b \quad \text{then} \quad \left(\rho \frac{\partial q}{\partial n} - \frac{\partial \phi}{\partial n}\right)(x) = 0, \tag{1.1}
$$

$$
\text{if } q_a \leq q(x) < q_b \quad \text{then} \quad \left(\rho \frac{\partial q}{\partial n} - \frac{\partial \phi}{\partial n}\right)(x) \geq 0, \tag{1.2}
$$

$$\text{if } q_a < q(x) \le q_b \quad \text{then} \quad \left(\rho\frac{\partial q}{\partial n} - \frac{\partial \phi}{\partial n}\right)(x) \le 0, \tag{1.3}$$

where ϕ is the adjoint variable and q_a , q_b are scalars and Γ_C, Γ_D are subsets of $\partial\Omega$. Due to this inequality nature, we need to handle in error analysis an integral on Γ_C of the following form:

$$\left[\int_{\Gamma_C} \left(\rho\frac{\partial q}{\partial n} - \frac{\partial \phi}{\partial n}\right)(p_h - q)\mathrm{d}s\right]^{\frac{1}{2}}.$$

If we appeal to the standard error analysis for the above term, we will get the order of convergence $h^{\frac{1}{2}+\frac{\epsilon}{2}}$ $(0 < \epsilon \le \frac{1}{2})$ when $q \in H^{\frac{3}{2}+\epsilon}(\Omega)$. This order of convergence is not optimal since the regularity of $\rho\frac{\partial q}{\partial n} - \frac{\partial \phi}{\partial n}$ is then not utilized properly, see [16] for the optimal error analysis for the Signorini problem. We adopt the ideas in [16] to estimate the term $\rho\frac{\partial q}{\partial n} - \frac{\partial \phi}{\partial n}$ to improve the order of convergence to $h^{\frac{1}{2}+\epsilon}$, which is optimal. The error analysis is developed in an abstract framework so that it can be applied to both the L^2 and the gradient cost functionals. Our analysis makes use of auxiliary systems of PDE to derive optimal order error estimates for the state, the control and the adjoint state variables in the energy norm. Numerical experiments are performed to illustrate the theoretical results.

The rest of the article is organized as follows. In the Section 2, we formulate the Dirichlet boundary control problem with point-wise control constraints. Therein, we prove the well-posedness of the model problem and derive the optimality conditions in a general setting for both the L^2 and gradient cost functional problems. In the Section 3, we define the discrete control problem and deduce the existence and uniqueness of the discrete solution. Further, we derive *a priori* error estimates for both the problems with the L^2 and the gradient cost functionals. Section 4 is devoted to the numerical experiments and finally the Section 5 concludes the article.

2. Control problem for the diffusion equation

In this article, we study the Dirichlet boundary control problem governed by the following diffusion problem:

$$-\Delta u = f \quad \text{in} \quad \Omega, \tag{2.1}$$

$$u = 0 \quad \text{on} \quad \Gamma_D, \tag{2.2}$$

$$u = q \quad \text{on} \quad \Gamma_C, \tag{2.3}$$

where $\Omega \subset \mathbb{R}^2$, be a bounded polygonal domain with boundary $\partial\Omega$ consists of two non-overlapping open subsets Γ_D and Γ_C with $\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_C$.

We make the following assumptions on the data:

- The interior of Γ_D and Γ_C in the induced topology are pairwise disjoint and the one dimensional measure $|\Gamma_C| \neq 0$ so that the model problem does not degenerate to a boundary value problem. Further, for the simplicity we assume that Γ_C is a straight line segment. Such an example of a domain is shown in Figure 1.
- The interior force $f \in L^2(\Omega)$.

In the above model problem, the Dirichlet data function q is the control variable and its admissible space will be defined in the subsequent description. Let $V := H_0^1(\Omega)$, where $H_0^1(\Omega)$ is the closure of C^∞ functions with compact support (in Ω) in $H^1(\Omega)$. Define the admissible control space by

$$Q := \{p \in H^1(\Omega) : \gamma_0(p) = 0 \text{ on } \Gamma_D\},$$

where $\gamma_0 : H^1(\Omega) \to L^2(\Gamma)$ is the trace operator whose range is $H^{1/2}(\Gamma)$. For the setting of the problem, assume that the function q in (2.3) is the trace of some function in Q, which will be denoted again by q. The weak

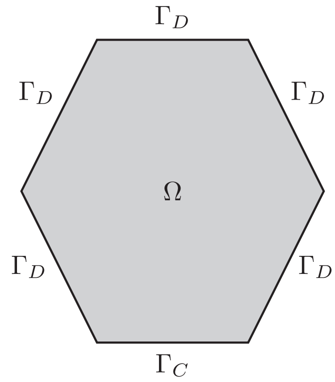

FIGURE 1. An example of domain Ω.

formulation for (2.1)–(2.3) can be written as follows: find $u \in H^1(\Omega)$ such that

$$\begin{aligned} u = w + q, \quad w \in V, \\ a(w,v) = \ell(v) - a(q,v) \quad \forall v \in V, \end{aligned} \tag{2.4}$$

where

$$a(u,v) = (\nabla u, \nabla v) \quad \text{and} \quad \ell(v) = (f,v),$$

with $(\cdot,\cdot)$ denoting the $L^2(\Omega)$ inner-product.

Here and throughout, the $L^2(\Omega)$ norm is denoted by $\|\cdot\|$ and $\|\cdot\|_k (k \geq 1)$ denotes the standard norm on the Sobolev space $H^k(\Omega)$, see for example [13]. The Lax-Milgram lemma [5, 13] implies that for given $q \in Q$, there exists a unique solution w of (2.4). Using this observation, we define the following solution map:

Definition 2.1. For given $q \in Q$, define $S(f,q) := u$, where $u = w + q$ and w solves (2.4).

The associated quadratic cost functional $J : H^1(\Omega) \times Q \to \mathbb{R}$ for the Dirichlet boundary control problem is defined by

$$J(w,p) = \frac{1}{2}\|\nabla^i (w - u_d)\|^2 + \frac{\rho}{2}\|\nabla p\|^2, \quad w \in H^1(\Omega),\ p \in Q. \tag{2.5}$$

where $i = 0, 1$, $\rho > 0$ is the regularizing parameter and $u_d \in H^{2i}(\Omega)$ is a given target function with the notation that $H^0(\Omega) := L^2(\Omega)$.
The control will be sought from the constrained set defined by

$$Q_{ad} := \{p \in Q : q_a \leq \gamma_0(p) \leq q_b \text{ a.e. on } \Gamma_C\},$$

where $q_a, q_b \in \mathbb{R}$ satisfying $q_a < q_b$. Furthermore, whenever Γ_D is non-empty, we assume for consistency that $q_a \leq 0$ and $q_b \geq 0$ so that the admissible set Q_{ad} is nonempty.
Dirichlet control problem: The model control problem consists of finding $(u,q) \in Q \times Q_{ad}$ such that

$$J(u,q) = \min_{(w,p) \in Q \times Q_{ad}} J(w,p), \tag{2.6}$$

subject to the condition that $w = S(f,p)$.

Define the reduced cost functional $j : Q \to \mathbb{R}$ by

$$j(p) = \frac{1}{2}\|\nabla^i (S(f,p) - u_d)\|^2 + \frac{\rho}{2}\|\nabla p\|^2, \quad p \in Q_{ad}, \quad \rho > 0. \tag{2.7}$$

It is easy to check that the derivative of the reduced cost functional j is given by

$$j\,'(q)(p) = \lim_{t\to 0} \frac{j(q+tp) - j(q)}{t} = (\nabla^i (S(f,q) - u_d), \nabla^i S(0,p)) + \rho\, a(q,p).$$

The following proposition establishes the existence and uniqueness of the solution of control problem and the first-order optimality condition:

Proposition 2.2. *There exists a solution $(u,q) \in Q \times Q_{ad}$ for the Dirichlet control problem (2.6). Further, there exists an adjoint state $\phi \in V$ such that for $i = 0,1$, there holds*

$$\begin{aligned} u &= w + q \quad w \in V, \\ a(w,v) &= \ell(v) - a(q,v) \quad \forall v \in V, \qquad (2.8)\\ a(v,\phi) &= (\nabla^i (u - u_d), \nabla^i v) \quad \forall v \in V, \qquad (2.9)\\ \rho\, a(q, p-q) &\geq a(p-q,\phi) - (\nabla^i (u-u_d), \nabla^i (p-q)) \quad \forall p \in Q_{ad}. \qquad (2.10) \end{aligned}$$

Moreover, when $|\Gamma_D| > 0$ (i.e., in this case $\Gamma_C \subsetneq \partial\Omega$), we have the uniqueness of the above solution; and if $\Gamma_C = \partial\Omega$, then we have only the uniqueness of the above solution for the L^2 cost functional problem but solution is unique upto a constant for gradient cost functional problem.

Proof. The proof is split into two cases.
Case I: the set $\mathbf{\Gamma_D}$ has positive measure. Since the set Q_{ad} is a closed and convex subset of Q and the cost functional j is strictly convex, the standard theory of optimal control problems ([30], Thm. 2.14, [29], p. 22) implies that there exists a unique solution $(u,q) \in Q \times Q_{ad}$ to (2.6) for L^2-cost functional and gradient cost functional problem respectively. By the first-order optimality conditions, we find that

$$(\nabla^i (u - u_d), \nabla^i S(0, p-q)) + \rho\, a(q, p-q) \geq 0 \quad \forall p \in Q_{ad}.$$

Define $\phi \in V$ to be the solution of the adjoint problem

$$a(v,\phi) = (\nabla^i (u - u_d), \nabla^i v) \quad \forall v \in V. \tag{2.11}$$

Since $S(0,p-q) - (p-q) \in V$ and $a(S(0,p-q),\phi) = 0$, we note that

$$\begin{aligned} (\nabla^i (u-u_d), \nabla^i S(0,p-q)) &= (\nabla^i (u-u_d), \nabla^i (S(0,p-q) - (p-q))) + (\nabla^i (u-u_d), \nabla^i (p-q)) \\ &= (\nabla^i (u-u_d), \nabla^i (p-q)) - a(p-q,\phi). \end{aligned}$$

Case II: $\mathbf{\Gamma_C} = \partial\mathbf{\Omega}$. For the L^2 cost functional problem the cost functional j is still strictly convex and hence the optimal control problem (2.6) has a unique solution. But for the gradient cost functional problem the cost functional j is only convex, solution is unique upto a constant. This completes the proof. □

Remark 2.3. The solution ϕ of the adjoint problem satisfies

For L^2-cost functional,

$$-\Delta\phi = u - u_d \quad \text{in} \quad \Omega,$$
$$\phi = 0 \quad \text{on} \quad \partial\Omega,$$

For gradient cost functional,

$$-\Delta\phi = -\Delta(u - u_d) \quad \text{in} \quad \Omega,$$
$$\phi = 0 \quad \text{on} \quad \partial\Omega,$$

and the optimal control q for the L^2-cost functional is the weak solutions of the following variational inequality correspond to the simplified Signorini problem:

$$-\rho\Delta q = 0 \quad \text{in} \quad \Omega, \tag{2.12}$$
$$q = 0 \quad \text{on} \quad \Gamma_D, \tag{2.13}$$
$$q_a \leq q \leq q_b \quad \text{on} \quad \Gamma_C, \tag{2.14}$$

further the following holds for almost every $x \in \Gamma_C$:

$$\text{if } q_a < q(x) < q_b \quad \text{then} \quad \left(\rho\frac{\partial q}{\partial n} - \frac{\partial\phi}{\partial n}\right)(x) = 0, \tag{2.15}$$
$$\text{if } q_a \leq q(x) < q_b \quad \text{then} \quad \left(\rho\frac{\partial q}{\partial n} - \frac{\partial\phi}{\partial n}\right)(x) \geq 0, \tag{2.16}$$
$$\text{if } q_a < q(x) \leq q_b \quad \text{then} \quad \left(\rho\frac{\partial q}{\partial n} - \frac{\partial\phi}{\partial n}\right)(x) \leq 0. \tag{2.17}$$

The optimal control q for the gradient cost functional is the weak solutions of the following variational inequality correspond to the simplified Signorini problem:

$$-\rho\Delta q = 0 \quad \text{in} \quad \Omega, \tag{2.18}$$
$$q = 0 \quad \text{on} \quad \Gamma_D, \tag{2.19}$$
$$q_a \leq q \leq q_b \quad \text{on} \quad \Gamma_C, \tag{2.20}$$

and for almost every $x \in \Gamma_C$:

$$\text{if } q_a < q(x) < q_b \quad \text{then} \quad \left(\rho\frac{\partial q}{\partial n} - \frac{\partial\phi}{\partial n} + \frac{\partial(u-u_d)}{\partial n}\right)(x) = 0, \tag{2.21}$$
$$\text{if } q_a \leq q(x) < q_b \quad \text{then} \quad \left(\rho\frac{\partial q}{\partial n} - \frac{\partial\phi}{\partial n} + \frac{\partial(u-u_d)}{\partial n}\right)(x) \geq 0, \tag{2.22}$$
$$\text{if } q_a < q(x) \leq q_b \quad \text{then} \quad \left(\rho\frac{\partial q}{\partial n} - \frac{\partial\phi}{\partial n} + \frac{\partial(u-u_d)}{\partial n}\right)(x) \leq 0. \tag{2.23}$$

We assume that the solution $u \in H^{\frac{3}{2}+\epsilon}(\Omega)$, $\phi \in H^{\frac{3}{2}+\epsilon}(\Omega)$ and $q \in H^{\frac{3}{2}+\epsilon}(\Omega)$ where $0 < \epsilon \leq \frac{1}{2}$ for both the L^2 and the gradient cost functional problem. The definition of fractional order Sobolev space is given below

Definition 2.4. Let $\Omega \subseteq \mathbb{R}^2$ be a domain and $0 < s < 1$. We define the space

$$H^s(\Omega) = \{u \in L^2(\Omega) | \frac{|u(x) - u(y)|}{|x-y|^{s+1}} \in L^2(\Omega \times \Omega)\}.$$

The norm on $H^s(\Omega)$ is defined by

$$\|u\|_{s,\Omega} = \|u\|_{L^2(\Omega)} + \left\|\frac{|u(x) - u(y)|}{|x-y|^{s+1}}\right\|_{L^2(\Omega\times\Omega)}.$$

Remark 2.5. The variational inequality (2.10) is equivalent to the strong Signorini problem (2.18)–(2.23) as soon as $q \in H^{\frac{3}{2}+\epsilon}(\Omega)$, $u \in H^{\frac{3}{2}+\epsilon}(\Omega)$ and $\phi \in H^{\frac{3}{2}+\epsilon}(\Omega)$ where $0 < \epsilon \leq \frac{1}{2}$. This type of regularity assumption is not so abstract, one can achieve this regularity for the model problem (2.6), which we considered in this article for example on a convex polygonal domain. The regularity of the solution of Signorini problem gets impaired due to some reasons, for example regularity of the data and the Signorini condition which generates singularities at contact-noncontact transition points. The first kind of singularity does not depend on the second kind. So, we quote the results on the singularity due to Signorini condition from [2, 4]. Let us first consider the portions of the unilateral boundary (Γ_C) that are straight line segments. Let $\mathbf{m}$ be a contact-noncontact transition point in the interior of a line segment in Γ_C, then $q \in H^{\sigma}(V_{\mathbf{m}})$ for all $\sigma < \frac{5}{2}$ and $V_{\mathbf{m}}$ is an open neighbourhood of $\mathbf{m}$ (see [2], Section 2.3 and [4], Section 2 for details). Now let $\mathbf{m}$ be a vertex on the boundary with an aperture of angle $\omega \in (0, \pi)$ (because of the convexity of the domain Ω). Then three situations arise. First, both the edges connected to the point $\mathbf{m}$ are portions of Γ_C. Now for this case the singularity arises at $\mathbf{m}$ is similar to the previous one (*i.e.*, the contact-noncontact transition case) and hence $q \in H^{\sigma}(V_{\mathbf{m}})$ for all $\sigma < \frac{5}{2}$ and $V_{\mathbf{m}}$ is an open neighbourhood of $\mathbf{m}$ ([2], Section 2.3). Second, one edge connected to $\mathbf{m}$ is in Γ_C and another is in Γ_D and if there is a neighbourhood $V_{\mathbf{m}}$ of $\mathbf{m}$ in Ω such that q vanishes on $V_{\mathbf{m}} \cap \Gamma_C$, then by the regularity theory of Dirichlet boundary value problem on a convex domain, we have $q \in H^2(V_{\mathbf{m}})$ ([2], Section 2.3). Now if q does not vanish on $V_{\mathbf{m}} \cap \Gamma_C$, then $\mathbf{m}$ be a transition point similar to the contact-noncontact type and hence $q \in H^{\sigma}(V_{\mathbf{m}})$ for all $\sigma < \frac{5}{2}$ ([2], Section 2.3). Third, if both the edges connected to $\mathbf{m}$ are in Γ_D, then from the second case we have $q \in H^2(V_{\mathbf{m}})$ ([2], Section 2.3). Now in the rest part of the domain we know that the Signorini solution is sufficiently regular and hence we can conclude that the control q will have $H^{\frac{3}{2}+\epsilon}$ regularity and consequently we have desired regularity ($H^{\frac{3}{2}+\epsilon}$) for the state and adjoint state variables.

Lemma 2.6. *Let the domain Ω be a convex polygon and $\Gamma_C = \partial\Omega$. Then for the L^2 and gradient cost functional problem (i.e., for $i = 0, 1$) the solutions $u \in H^2(\Omega)$, $\phi \in H^2(\Omega)$ and $q \in H^2(\Omega)$.*

Proof. **For the L^2 cost functional problem:** Choosing $p = q + v \in Q_{ad}$ for $v \in V$ in the variational inequality (2.10) and using (2.9) we get $a(q, v) = 0$ for all $v \in V$. Then from the equation (2.8), we get $a(w, v) = \ell(v)$ for all $v \in V$ and have $w \in H^2(\Omega)$ because of the convexity of the domain. Now $u \in H^1(\Omega)$ as $q \in H^1(\Omega)$ and $w \in H^2(\Omega)$. Since $u_d \in L^2(\Omega)$, we have $u - u_d = w + q - u_d \in L^2(\Omega)$ and hence from the equation (2.9) we have $\phi \in H^2(\Omega)$. By using ([28], Thm. 2.2) we can conclude that $q \in H^2(\Omega)$. Hence the optimal state $u \in H^2(\Omega)$ since $u = w + q$.

For the gradient cost functional problem: Similarly, Choosing $p = q + v \in Q_{ad}$ for $v \in V$ in the variational inequality (2.10) and using (2.9) we get $a(q, v) = 0 \quad$ for all $v \in V$. Now from the equation (2.8), we get $a(w, v) = \ell(v)$ for all $v \in V$ and have $w \in H^2(\Omega)$. The right hand side of (2.9) becomes $(\nabla(u - u_d), \nabla v) = (\nabla(w + q - u_d), \nabla v) = (\nabla(w - u_d), \nabla v)$ since $a(q, v) = (\nabla q, \nabla v) = 0$. Hence the adjoint equation takes the form

$$a(v, \phi) = (\nabla(w - u_d), \nabla v) \quad \text{for all } v \in V. \tag{2.24}$$

Using the fact that w, u_d are in $H^2(\Omega)$, we can conclude that $\Delta(w - u_d) \in L^2(\Omega)$. Now, from the equation (2.24) we have $\phi \in H^2(\Omega)$ due to the convexity of the domain. Then using the splitting that $u = w + q$ in the equation (2.10), we obtain

$$(\rho + 1)\, a(q, p - q) \geq a(p - q, \phi) - (\nabla(w - u_d), \nabla(p - q)) \quad \forall p \in Q_{ad},$$

which in the strong form is given by the following:

$$\begin{aligned} -(\rho + 1)\Delta q &= 0 \quad \text{in} \quad \Omega, \\ q &= 0 \quad \text{on} \quad \Gamma_D, \\ q_a \leq q &\leq q_b \quad \text{on} \quad \Gamma_C, \end{aligned}$$

and for almost every $x \in \Gamma_C$:

$$\begin{aligned}
&\text{if } q_a < q(x) < q_b \quad \text{then} \quad \Big((\rho+1)\frac{\partial q}{\partial n} - \frac{\partial \phi}{\partial n} + \frac{\partial (w-u_d)}{\partial n}\Big)(x) = 0,\\
&\text{if } q_a \le q(x) < q_b \quad \text{then} \quad \Big((\rho+1)\frac{\partial q}{\partial n} - \frac{\partial \phi}{\partial n} + \frac{\partial (w-u_d)}{\partial n}\Big)(x) \ge 0,\\
&\text{if } q_a < q(x) \le q_b \quad \text{then} \quad \Big((\rho+1)\frac{\partial q}{\partial n} - \frac{\partial \phi}{\partial n} + \frac{\partial (w-u_d)}{\partial n}\Big)(x) \le 0.
\end{aligned}$$

Since all the data in the above Signorini problem are in $H^2(\Omega)$, we can conclude using ([28], Thm. 2.2) that $q \in H^2(\Omega)$. Hence the optimal state $u \in H^2(\Omega)$ since $u = w + q$. □

3. Discrete control problem

In this section, we define the discrete control problem and derive corresponding *a priori* error estimates. Let $\mathcal{T}_h$ be a regular triangulation of the domain Ω, see [5, 13]. A typical triangle is denoted by T and its diameter by h_T. Set $h = \max_{T\in\mathcal{T}_h} h_T$. Let $\mathcal{V}_h$ denote the set of all the vertices of the triangles in $\mathcal{T}_h$. Let $\mathcal{V}_h^D$ denote the set of vertices on $\overline{\Gamma_D}$ and $\mathcal{V}_h^C$ denote the set of vertices interior to Γ_C.

Define the discrete state space $V_h \subset V$ by

$$V_h := \{v \in V : v|_T \in \mathbb{P}_1(T) \;\; \forall T \in \mathcal{T}_h\},$$

and the discrete control space $Q_h \subset Q$ by

$$Q_h := \{q \in Q : q|_T \in \mathbb{P}_1(T), \;\; \forall T \in \mathcal{T}_h\},$$

where $\mathbb{P}_1(T)$ is the space of polynomials of total degree less than or equal to one restricted to the triangle T. The discrete admissible set of controls is defined by

$$Q_{ad}^h := \{q \in Q_h : q_a \le q(z) \le q_b \text{ for all } z \in \mathcal{V}_h^C\}.$$

It is easy to check that $Q_{ad}^h \subset Q_{ad}$.

Throughout the article, we assume that C denotes a generic positive constant that is independent of the mesh parameter h.

The discrete control problem finds $(w_h, q_h, \phi_h) \in V_h \times Q_{ad}^h \times V_h$ such that

$$u_h = w_h + q_h,$$

$$a(w_h, v_h) = \ell(v_h) - a(q_h, v_h) \quad \forall v_h \in V_h, \tag{3.1}$$

$$a(v_h, \phi_h) = (\nabla^i(u_h - u_d), \nabla^i v_h) \quad \forall v_h \in V_h, \tag{3.2}$$

$$\rho\, a(q_h, p_h - q_h) \ge a(p_h - q_h, \phi_h) - (\nabla^i(u_h - u_d), \nabla^i(p_h - q_h)) \quad \forall p_h \in Q_{ad}^h. \tag{3.3}$$

We note that in the above discrete problem $q_h|_{\Gamma_C}$ is the required discrete control, u_h is the discrete state and ϕ_h is the discrete costate.

Proposition 3.1. *The discrete control problem (3.1)–(3.3) admits a unique solution.*

Proof. The standard theory of optimal control problem [30] can be employed to deduce the existence and uniqueness of the solution to (3.1)–(3.3). □

3.1. *A priori* error analysis

In this section, we derive optimal order *a priori* error estimates in the energy norm. First we note that there hold

$$a(w-w_h,v_h)=-a(q-q_h,v_h)\quad \forall v_h\in V_h, \tag{3.4}$$

and

$$a(v_h,\phi-\phi_h)=(\nabla^i(u-u_h),\nabla^i v_h)\quad \forall v_h\in V_h. \tag{3.5}$$

The error analysis becomes simpler and cleaner, if we use some auxiliary projections. Introduce the projections P_hw and $\bar{P}_h\phi$ by

$$a(P_hw,v_h)=\ell(v_h)-a(q,v_h)\quad \forall v_h\in V_h, \tag{3.6}$$

and

$$a(v_h,\bar{P}_h\phi)=(\nabla^i(u-u_d),\nabla^i v_h)\quad \forall v_h\in V_h. \tag{3.7}$$

It is immediate to see from (3.6)–(2.8) that

$$a(P_hw-w,v_h)=0\quad \forall v_h\in V_h, \tag{3.8}$$

and that from (3.7)–(2.9),

$$a(v_h,\bar{P}_h\phi-\phi)=0\quad \forall v_h\in V_h. \tag{3.9}$$

Using the Galerkin orthogonality defined in (3.9) we have $\left\|\nabla(\phi-\bar{P}_h\phi)\right\|\le C\inf_{v_h\in V_h}\left\|\nabla(\phi-v_h)\right\|.$ Choosing $v_h=\mathcal{I}_h\phi$, we have

$$\left\|\nabla(\phi-\bar{P}_h\phi)\right\|\le Ch^{\frac{1}{2}+\epsilon}\left\|\phi\right\|_{\frac{3}{2}+\epsilon,\Omega}, \tag{3.10}$$

similarly using (3.8) we have the following estimate

$$\left\|\nabla(P_hw-w)\right\|\le Ch^{\frac{1}{2}+\epsilon}\left\|w\right\|_{\frac{3}{2}+\epsilon,\Omega}. \tag{3.11}$$

and by using Aubin-Nitsche lemma [13], (3.11) and the assumed regularity of ϕ we have

$$\left\|P_hw-w\right\|\le Ch^{1+2\epsilon}\left\|w\right\|_{\frac{3}{2}+\epsilon,\Omega.} \tag{3.12}$$

In the above we have taken $0<\epsilon\le\frac{1}{2}$.

The following lemma subsequently facilitate the derivation of error estimates:

Lemma 3.2. *There holds*

$$a(q-q_h,\phi_h-\bar{P}_h\phi)-(\nabla^i(u-u_h),\nabla^i(w-w_h))=(\nabla^i(u-u_h),\nabla^i(P_hw-w)).$$

Proof. The subtraction of (3.1) from (3.6) and the subtraction of (3.2) from (3.7) yields

$$a(P_hw - w_h, v_h) = -a(q - q_h, v_h) \quad \forall v_h \in V_h, \tag{3.13}$$
$$a(v_h, \bar{P}_h\phi - \phi_h) = (\nabla^i(u - u_h), \nabla^i v_h) \quad \forall v_h \in V_h. \tag{3.14}$$

Now substituting $v_h = \bar{P}_h\phi - \phi_h$ in (3.13) and $v_h = P_hw - w_h$ in (3.14) and subtraction of the resulting equations yields

$$a(q - q_h, \bar{P}_h\phi - \phi_h) = -(\nabla^i(u - u_h), \nabla^i(P_hw - w_h))$$

and finally, we have

$$a(q - q_h, \bar{P}_h\phi - \phi_h) + (\nabla^i(u - u_h), \nabla^i(w - w_h)) = -(\nabla^i(u - u_h), \nabla^i(P_hw - w)).$$

This completes the proof. □

We derive below a lemma that gives the first error estimate.

Lemma 3.3. *There holds for $i = 0, 1$ that*

$$\begin{aligned}\|\nabla(q - q_h)\|^2 + \|\nabla^i(u - u_h)\|^2 \le\ & C\big(a(q, p_h - q) - a(p_h - q, \phi) + (\nabla^i(u - u_d), \nabla^i(p_h - q))\big)\\ & + C\left(\|\nabla(q - p_h)\|^2 + \|\nabla(\phi - \bar{P}_h\phi)\|^2 + \|\nabla^i(P_hw - w)\|^2\right)\\ & + C\|\nabla^i(p_h - q)\|^2,\end{aligned}$$

for all $p_h \in Q^h_{ad}$.

Proof. Setting $p = q_h$ in (2.10), we find

$$\rho\, a(q, q_h - q) \ge a(q_h - q, \phi) - (\nabla^i(u - u_d), \nabla^i(q_h - q)). \tag{3.15}$$

Using (3.3), we write

$$\rho\, a(q_h, q - q_h) \ge \rho\, a(q_h, q - p_h) + a(p_h - q_h, \phi_h) - (\nabla^i(u_h - u_d), \nabla^i(p_h - q_h)) \quad \forall p_h \in Q^h_{ad}. \tag{3.16}$$

Adding the equations (3.15) and (3.16), we find for any $p_h \in Q^h_{ad}$ that

$$\begin{aligned}\rho\, a(q - q_h, q_h - q) \ge\ & \rho\, a(q_h, q - p_h) + a(p_h - q_h, \phi_h) + a(q_h - q, \phi) - (\nabla^i(u - u_d), \nabla^i(q_h - q))\\ & - (\nabla^i(u_h - u_d), \nabla^i(p_h - q_h))\\ \ge\ & \rho\, a(q_h, q - p_h) + a(p_h - q, \phi_h) + a(q - q_h, \phi_h - \phi)\\ & + \|\nabla^i(u - u_h)\|^2 - (\nabla^i(u - u_h), \nabla^i(w - w_h)) - (\nabla^i(u_h - u_d), \nabla^i(p_h - q))\\ \ge\ & \rho\, a(q_h, q - p_h) + a(p_h - q, \phi_h) + a(q - q_h, \bar{P}_h\phi - \phi)\\ & + \|\nabla^i(u - u_h)\|^2 - (\nabla^i(u_h - u_d), \nabla^i(p_h - q))\\ & + a(q - q_h, \phi_h - \bar{P}_h\phi) - (\nabla^i(u - u_h), \nabla^i(w - w_h))\\ \ge\ & \rho\, a(q_h, q - p_h) + a(p_h - q, \phi_h) + a(q - q_h, \bar{P}_h\phi - \phi)\\ & + \|\nabla^i(u - u_h)\|^2 - (\nabla^i(u_h - u_d), \nabla^i(p_h - q)) + (\nabla^i(u - u_h), \nabla^i(P_hw - w)).\end{aligned} \tag{3.17}$$

Therefore by rearranging the terms, we obtain

$$\begin{aligned}\rho\, a(q-q_h,q-q_h)+\|\nabla^i(u-u_h)\|^2 &\le \rho\, a(q_h,p_h-q)-a(p_h-q,\phi_h)+(\nabla^i(u_h-u_d),\nabla^i(p_h-q))\\&\quad+a(q_h-q,\bar{P}_h\phi-\phi)-(\nabla^i(u-u_h),\nabla^i(P_hw-w))\\&\le \rho\, a(q,p_h-q)-a(p_h-q,\phi)+(\nabla^i(u-u_d),\nabla^i(p_h-q))\\&\quad+\rho\, a(q_h-q,p_h-q)-a(p_h-q,\phi_h-\phi)+(\nabla^i(u_h-u),\nabla^i(p_h-q))\\&\quad+a(q_h-q,\bar{P}_h\phi-\phi)-(\nabla^i(u-u_h),\nabla^i(P_hw-w)).\end{aligned}\tag{3.18}$$

Using (3.5), we note that

$$\|\nabla(\phi-\phi_h)\|\le C\left(\|\nabla(\phi-\bar{P}_h\phi)\|+\|\nabla^i(u-u_h)\|\right).\tag{3.19}$$

Now using the Cauchy-Schwarz inequality and (3.19), we find

$$\begin{aligned}\|\nabla(q-q_h)\|^2+\|\nabla^i(u-u_h)\|^2 &\le C\big(a(q,p_h-q)-a(p_h-q,\phi)+(\nabla^i(u-u_d),\nabla^i(p_h-q))\big)\\&\quad+C\left(\|\nabla(q-p_h)\|^2+\|\nabla(\phi-\bar{P}_h\phi)\|^2+\|\nabla^i(P_hw-w)\|^2\right)\\&\quad+C\|\nabla^i(p_h-q)\|^2.\end{aligned}\tag{3.20}$$

This completes the proof. □

The right-hand side of the estimate in Lemma 3.3 contains terms corresponding to the interpolation errors except for the term in the first parentheses. We derive two lemmas one each for $i=0$ and $i=1$ estimating those terms. We begin with estimation for $i=0$, *i.e.*, for the L^2-cost functional. Before going to derive the estimates, we will prove the following lemma which will be useful in the analysis.

For a fixed triangle T which shares an edge with Γ_C, define

$$S_{NC}=\{x\in T\cap\Gamma_C:\ q_a<q(x)<q_b\}$$

and

$$S_C=\{x\in T\cap\Gamma_C:\ q(x)=q_a\}\cup\{x\in T\cap\Gamma_C:\ q(x)=q_b\}.$$

Since $q|_{\Gamma_C}$ belongs to $H^{1+\epsilon}(\Gamma_C)$, Sobolev embedding theorem ensures that $q|_{\Gamma_C}$ is a continuous function on Γ_C. The sets S_C and S_{NC} are measurable because they are inverse images of Borel sets by a continuous function. We denote by $|S_C|$, $|S_{NC}|$ their measure in $\mathbb{R}$. Now we are ready to state the lemma.

Lemma 3.4. *Let, h_e be the length of the edge $T\cap\Gamma_C$ and $|S_C|$, $|S_{NC}|$ are defined as above. Assume that $|S_C|>0$, $|S_{NC}|>0$. The following L^2 and L^1 -estimates hold for σ_n and q':*

$$\|\sigma_n\|_{0,T\cap\Gamma_C}\le\frac{1}{|S_{NC}|^{1/2}}\,h_e^{\frac12+\epsilon}\,|\sigma_n|_{\epsilon,T\cap\Gamma_C},\tag{3.21}$$

$$\|\sigma_n\|_{L^1(T\cap\Gamma_C)}\le\frac{|S_C|^{1/2}}{|S_{NC}|^{1/2}}\,h_e^{\frac12+\epsilon}\,|\sigma_n|_{\epsilon,T\cap\Gamma_C},\tag{3.22}$$

$$\|q'\|_{0,T\cap\Gamma_C}\le\frac{1}{|S_C|^{1/2}}\,h_e^{\frac12+\epsilon}\,|q'|_{\epsilon,T\cap\Gamma_C},\tag{3.23}$$

$$\|q'\|_{L^1(T\cap\Gamma_C)}\le\frac{|S_{NC}|^{1/2}}{|S_C|^{1/2}}\,h_e^{\frac12+\epsilon}\,|q'|_{\epsilon,T\cap\Gamma_C},\tag{3.24}$$

where $\sigma_n := \rho\frac{\partial q}{\partial n} - \frac{\partial \phi}{\partial n}$ *and* q' *is the (tangential) distribution derivative of* q *on* $T \cap \Gamma_C$.

Proof. First to find the L^2- estimate of σ_n, consider

$$\begin{aligned}
\|\sigma_n\|^2_{0,T\cap\Gamma_C} &= \int_{T\cap\Gamma_C} (\sigma_n(s))^2 \mathrm{d}s \\
&= \int_{S_C} (\sigma_n(s))^2 \mathrm{d}s \quad (\sigma_n = 0 \;\text{ on }\; S_{NC}) \\
&= \frac{1}{|S_{NC}|} \int_{S_C}\int_{S_{NC}} (\sigma_n(s) - \sigma_n(t))^2 \mathrm{d}t \; \mathrm{d}s \\
&\le \frac{1}{|S_{NC}|} \sup_{S_C \times S_{NC}} |s-t|^{1+2\epsilon} \int_{S_C}\int_{S_{NC}} \frac{(\sigma_n(s) - \sigma_n(t))^2}{|s-t|^{1+2\epsilon}} \mathrm{d}t \; \mathrm{d}s \\
&\le \frac{1}{|S_{NC}|} \, h_e^{1+2\epsilon} \; |\sigma_n|^2_{\epsilon,T\cap\Gamma_C},
\end{aligned}$$

which completes the proof of (3.21). The proof of (3.22) is derived as

$$\begin{aligned}
\int_{T\cap\Gamma_C} |\sigma_n| = \int_{S_C} |\sigma_n|, \\
&\le |S_C|^{1/2} \, \|\sigma_n\|_{0,S_C}, \\
&\le |S_C|^{1/2} \, \|\sigma_n\|_{0,T\cap\Gamma_C}, \\
&\le \frac{|S_C|^{1/2}}{|S_{NC}|^{1/2}} \, h_e^{\frac{1}{2}+\epsilon} \; |\sigma_n|_{\epsilon,T\cap\Gamma_C}.
\end{aligned}$$

Next, we will proof the L^2- estimate of q'. A nontrivial result of Stampacchia, see *e.g.* [24], implies that if $v \in H^1(\omega_0)$ and $v =$ constant a.e. on some $E \subseteq \omega_0$, then $\nabla v = 0$ a.e. on E. In the present study since $v = q$ is continuous on $\omega_0 = \Gamma_C (\subseteq \mathbb{R})$, then the level set can be understood in the classical sense. Since S_C is a level set, we have $q' = 0$ a.e. on S_C. Therefore we have the following:

$$\begin{aligned}
\|q'\|^2_{0,T\cap\Gamma_C} &= \int_{T\cap\Gamma_C} (q'(s))^2 \mathrm{d}s \\
&= \int_{S_{NC}} (q'(s))^2 \mathrm{d}s \quad (q' = 0 \;\text{ on }\; S_C) \\
&= \frac{1}{|S_C|} \int_{S_{NC}}\int_{S_C} (q'(s) - q'(t))^2 \mathrm{d}t \; \mathrm{d}s \\
&\le \frac{1}{|S_C|} \sup_{S_C \times S_{NC}} |s-t|^{1+2\epsilon} \int_{S_{NC}}\int_{S_C} \frac{(q'(s) - q'(t))^2}{|s-t|^{1+2\epsilon}} \mathrm{d}t \; \mathrm{d}s \\
&\le \frac{1}{|S_C|} \, h_e^{1+2\epsilon} \; |q'|^2_{\epsilon,T\cap\Gamma_C}.
\end{aligned}$$

This completes the proof of (3.23). Then the estimate in (3.24) can be obtained by using the Cauchy-Schwarz inequality as it is done in the proof of (3.22). □

Now we will turn to prove the error estimate for $i = 0$, *i.e.*, for the L^2-cost functional. We first prove the following lemma.

Lemma 3.5. **(For L^2-cost functional)** *There holds*

$$|\big(a(q, p_h - q) - a(p_h - q, \phi) + (u - u_d, p_h - q)\big)| \leq Ch^{1+2\epsilon}\left(\|q\|^2_{\frac{3}{2}+\epsilon,\Omega} + \|\phi\|^2_{\frac{3}{2}+\epsilon,\Omega}\right).$$

Proof. Using the integration by parts, we find that

$$a(q, p_h - q) - a(p_h - q, \phi) + (u - u_d, p_h - q) = \int_{\Gamma_C}\left(\rho\frac{\partial q}{\partial n} - \frac{\partial \phi}{\partial n}\right)(p_h - q)\mathrm{d}s. \tag{3.25}$$

Set $\sigma_n := \rho\frac{\partial q}{\partial n} - \frac{\partial \phi}{\partial n}$, and choose $p_h = \mathcal{I}_h q \in Q_h$ where $\mathcal{I}_h$ is the Lagrange interpolation operator. Then the RHS of (3.25) reads as and equals

$$\int_{\Gamma_C}\sigma_n(\mathcal{I}_h q - q)\mathrm{d}s = \sum_{T\in\mathcal{T}_h}\int_{T\cap\Gamma_C}\sigma_n(\mathcal{I}_h q - q)\mathrm{d}s.$$

Therefore it remains to estimate the following:

$$\int_{T\cap\Gamma_C}\sigma_n(\mathcal{I}_h q - q)\mathrm{d}s \quad T\in\mathcal{T}_h. \tag{3.26}$$

Let T be a fixed triangle and h_e be the length of the edge $e = T\cap\Gamma_C$ and obviously $|S_C| + |S_{NC}| = h_e$.

There arises two cases:

(i) either $|S_C|$ or $|S_{NC}|$ equals zero,
(ii) both of $|S_C|$ and $|S_{NC}|$ have positive measure.

For the case (i), it is easy to check that the integral term in (3.26) vanishes. So we need to estimate (3.26) for the case (ii). We will derive two estimates of the same error term (3.26): The first one depending on $|S_{NC}|$ and the second one depending on $|S_C|$.

The estimate of (3.26) depending on S_{NC}: Using the Cauchy-Schwarz inequality, estimate (3.21) in Lemma 3.4, and a standard error estimate of $\mathcal{I}_h$ gives

$$\begin{aligned}\int_{T\cap\Gamma_C}\sigma_n(\mathcal{I}_h q - q)\mathrm{d}s &\leq \|\sigma_n\|_{0,T\cap\Gamma_C}\|\mathcal{I}_h q - q\|_{0,T\cap\Gamma_C}\\ &\leq C\frac{1}{|S_{NC}|^{\frac{1}{2}}}h_e^{\frac{1}{2}+\epsilon}|\sigma_n|_{\epsilon,T\cap\Gamma_C}h^{1+\epsilon}|q'|_{\epsilon,T\cap\Gamma_C}\\ &\leq C\frac{1}{|S_{NC}|^{\frac{1}{2}}}h_e^{\frac{3}{2}+2\epsilon}\big(|\sigma_n|^2_{\epsilon,T\cap\Gamma_C} + |q'|^2_{\epsilon,T\cap\Gamma_C}\big).\end{aligned} \tag{3.27}$$

Estimate of (3.26) depending on S_C: This estimate is obtained in a different way. We now use the standard error estimate on $\mathcal{I}_h$ (see [13]) and bounds (3.21), (3.24) in Lemma 3.4 to find

$$\begin{aligned}\int_{T\cap\Gamma_C}\sigma_n(\mathcal{I}_h q - q)\mathrm{d}s &\leq \|\sigma_n\|_{0,T\cap\Gamma_C}\|\mathcal{I}_h q - q\|_{0,T\cap\Gamma_C}\\ &\leq C\|\sigma_n\|_{0,T\cap\Gamma_C}h_e^{\frac{1}{2}}\|q'\|_{L^1(T\cap\Gamma_C)}\\ &\leq C\frac{1}{|S_C|^{\frac{1}{2}}}h_e^{\frac{3}{2}+2\epsilon}\big(|\sigma_n|^2_{\epsilon,T\cap\Gamma_C} + |q'|^2_{\epsilon,T\cap\Gamma_C}\big).\end{aligned} \tag{3.28}$$

We conclude by noting that either $|S_{NC}|$ or $|S_C|$ is greater than or equal to $h_e/2$ and by choosing the appropriate estimate (3.27) or (3.28), we have

$$\int_{T\cap\Gamma_C} \sigma_n(\mathcal{I}_h q - q)\mathrm{d}s \le Ch_e^{1+2\epsilon}\big(|\sigma_n|^2_{\epsilon,T\cap\Gamma_C} + |q'|^2_{\epsilon,T\cap\Gamma_C}\big).$$

By summation and using the trace theorem, we get

$$\int_{\Gamma_C} \sigma_n(\mathcal{I}_h q - q)\mathrm{d}s \le Ch^{1+2\epsilon}\big(|\sigma_n|^2_{\epsilon,\Gamma_C} + |q'|^2_{\epsilon,\Gamma_C}\big) \le h^{1+2\epsilon}\left(\|q\|^2_{\frac{3}{2}+\epsilon,\Omega} + \|\phi\|^2_{\frac{3}{2}+\epsilon,\Omega}\right).$$

This completes the proof. □

Now we estimate for $i=1$, *i.e.*, for the gradient cost functional.

Lemma 3.6. (For gradient cost functional) *There holds*

$$\begin{aligned}\big(a(q,p_h-q) - a(p_h-q,\phi) + (\nabla(u-u_d),\nabla(p_h-q))\big) &\le Ch^{1+2\epsilon}\left(\|q\|^2_{\frac{3}{2}+\epsilon,\Omega} + \|u_d\|^2_{2,\Omega}\right)\\ &+ Ch^{1+2\epsilon}\left(\|\phi\|^2_{\frac{3}{2}+\epsilon,\Omega} + \|u\|^2_{\frac{3}{2}+\epsilon,\Omega}\right).\end{aligned}$$

Proof. Integration by parts yields

$$\begin{aligned}a(q,p_h-q) - a(p_h-q,\phi) &+ (\nabla(u-u_d),\nabla(p_h-q))\\ &= \int_{\Gamma_C}\left(\rho\frac{\partial q}{\partial n} - \frac{\partial\phi}{\partial n} + \frac{\partial(u-u_d)}{\partial n}\right)(p_h-q)\mathrm{d}s.\end{aligned}$$

Set $\sigma_n := \rho\frac{\partial q}{\partial n} - \frac{\partial\phi}{\partial n} + \frac{\partial(u-u_d)}{\partial n}$. Then, the rest of the proof follows in the same lines as that of the proof of Lemma 3.5. □

In the following theorem, we derive the error estimate for the control:

Theorem 3.7. (Error estimate of control variable for L^2-cost functional)
There holds

$$\|\nabla(q-q_h)\| + \|u-u_h\| \le C\left(h^{\frac{1}{2}+\epsilon}\|q\|_{\frac{3}{2}+\epsilon,\Omega} + h^{\frac{1}{2}+\epsilon}\|\phi\|_{\frac{3}{2}+\epsilon,\Omega} + h^{1+2\epsilon}\|w\|_{\frac{3}{2}+\epsilon,\Omega}\right).$$

Proof. From Lemma 3.3, we have the following:

$$\begin{aligned}\|\nabla(q-q_h)\|^2 + \|u-u_h\|^2 \le\; & C\big(a(q,p_h-q) - a(p_h-q,\phi) + ((u-u_d),(p_h-q))\big)\\ &+ C\left(\|\nabla(q-p_h)\|^2 + \|\nabla(\phi-\bar{P}_h\phi)\|^2 + \|P_hw-w\|^2\right)\\ &+ C\|p_h-q\|^2,\end{aligned} \tag{3.29}$$

for all $p_h \in Q^h_{ad}$.

For the first term in the RHS of (3.29), we have the following estimate from the Lemma 3.5:

$$\big(a(q,p_h-q) - a(p_h-q,\phi) + (u-u_d,p_h-q)\big) \le Ch^{1+2\epsilon}\left(\|q\|^2_{\frac{3}{2}+\epsilon,\Omega} + \|\phi\|^2_{\frac{3}{2}+\epsilon,\Omega}\right).$$

For the second term in the RHS of (3.29), we take $p_h = \mathcal{I}_h q$ and use the standard interpolation estimate which are given by $\|\nabla(q-\mathcal{I}_h q)\| \le Ch^{\frac{1}{2}+\epsilon}\|q\|_{\frac{3}{2}+\epsilon,\Omega}$ and $\|p_h - q\| \le Ch^{\frac{3}{2}+\epsilon}\|q\|_{\frac{3}{2}+\epsilon,\Omega}$. An estimate for the third term in (3.29) has already been done in (3.10) and is given by $\left\|\nabla(\phi-\bar{P}_h\phi)\right\| \leq Ch^{\frac{1}{2}+\epsilon}\left\|\phi\right\|_{\frac{3}{2}+\epsilon,\Omega}$. Furthermore from (3.12), we have $\left\|P_h w - w\right\| \leq Ch^{1+2\epsilon}\left\|w\right\|_{\frac{3}{2}+\epsilon,\Omega}$. Putting all the above estimates together we obtain the proof. □

Theorem 3.8. **(Error estimate of state and adjoint state variable for L^2-cost functional)** *There holds*

$$\|\nabla(u-u_h)\| + \|\nabla(\phi-\phi_h)\| \leq C\left(h^{\frac{1}{2}+\epsilon}\left\|q\right\|_{\frac{3}{2}+\epsilon,\Omega} + h^{\frac{1}{2}+\epsilon}\left\|\phi\right\|_{\frac{3}{2}+\epsilon,\Omega} + h^{\frac{1}{2}+\epsilon}\left\|w\right\|_{\frac{3}{2}+\epsilon,\Omega}\right).$$

Proof. For the energy error in the state, we have by our splitting, $u = w + q$ and $u_h = w_h + q_h$ hence, we have

$$\|\nabla(u-u_h)\| \leq \|\nabla(w-w_h)\| + \|\nabla(q-q_h)\|$$

It remains to estimate the first term of the above equation. By the triangle inequality we have $\|\nabla(w-w_h)\| \leq \|\nabla(w-P_h w)\| + \|\nabla(P_h w - w_h)\|$. Also we know that $a(P_h w - w_h, v_h) = a(q_h - q, v_h) \quad \forall v_h \in V_h$ by subtracting (3.1) from (3.6). Putting $v_h = P_h w - w_h$ in above equation yields $\|\nabla(P_h w - w_h)\| \leq \|\nabla(q-q_h)\|$. Hence we have

$$\|\nabla(u-u_h)\| \leq \|\nabla(w-P_h w)\| + 2\,\|\nabla(q-q_h)\|. \tag{3.30}$$

Now, for the energy error in the adjoint state, by using the triangle inequality, we have

$$\|\nabla(\phi-\phi_h)\| \leq \left\|\nabla(\phi-\bar{P}_h\phi)\right\| + \left\|\nabla(\bar{P}_h\phi-\phi_h)\right\|.$$

We need to estimate the second term of the above equation. We have by subtracting (3.2) from (3.7) that $a(v_h, \bar{P}_h\phi - \phi_h) = (u-u_h, v_h) \quad \forall v_h \in V_h$. By taking $v_h = \bar{P}_h\phi - \phi_h$ in this equation and applying Cauchy-Schwartz inequality we find $\left\|\nabla(\bar{P}_h\phi-\phi_h)\right\| \leq \|u-u_h\|$. Hence we have

$$\|\nabla(\phi-\phi_h)\| \leq \left\|\nabla(\phi-\bar{P}_h\phi)\right\| + \|u-u_h\| \tag{3.31}$$

Adding (3.30) and (3.31) we have

$$\|\nabla(u-u_h)\| + \|\nabla(\phi-\phi_h)\| \leq \|\nabla(w-P_h w)\| + \left\|\nabla(\phi-\bar{P}_h\phi)\right\| + 2\,\|\nabla(q-q_h)\| + \|u-u_h\|.$$

Using the estimate for $\|u-u_h\|$ and $\|\nabla(q-q_h)\|$ from Theorem 3.7 and the projection error estimate in (3.10) and (3.11), we complete the proof. □

We now derive error estimates for the gradient cost functional.

Theorem 3.9. **(Error estimate of state and control variable for gradient cost functional)** *There holds*

$$\|\nabla(q-q_h)\| + \|\nabla(u-u_h)\| \leq Ch^{\frac{1}{2}+\epsilon}\left(\left\|q\right\|_{\frac{3}{2}+\epsilon,\Omega} + \left\|u_d\right\|_{2,\Omega} + \left\|\phi\right\|_{\frac{3}{2}+\epsilon,\Omega} + \left\|w\right\|_{\frac{3}{2}+\epsilon,\Omega}\right).$$

Proof. From Lemma 3.3, we have the following estimate:

$$\begin{aligned}\|\nabla(q-q_h)\|^2 + \|\nabla(u-u_h)\|^2 \leq{}& C\big(a(q,p_h-q) - a(p_h-q,\phi) + (\nabla(u-u_d),\nabla(p_h-q))\big)\\ &+ C\left(\|\nabla(q-p_h)\|^2 + \|\nabla(\phi-\bar{P}_h\phi)\|^2 + \|\nabla(P_h w - w)\|^2\right),\end{aligned} \tag{3.32}$$

for all $p_h \in Q_{ad}^h$.

For the first term in the RHS of (3.32), we have the following estimate from the Lemma 3.6:

$$\big(a(q, p_h - q) - a(p_h - q, \phi) + (\nabla(u - u_d), \nabla(p_h - q))\big) \leq Ch^{1+2\epsilon}\left(\|q\|^2_{\frac{3}{2}+\epsilon,\Omega} + \|u_d\|^2_{2,\Omega}\right) + Ch^{1+2\epsilon}\left(\|\phi\|^2_{\frac{3}{2}+\epsilon,\Omega} + \|u\|^2_{\frac{3}{2}+\epsilon,\Omega}\right).$$

For the second term in the RHS of (3.32), we take $p_h = \mathcal{I}_h q$ and use the standard interpolation estimate $\|\nabla(q - \mathcal{I}_h q)\| \leq Ch^{\frac{1}{2}+\epsilon}\|q\|_{\frac{3}{2}+\epsilon,\Omega}$, and use (3.10) and (3.11) to complete the proof. □

Theorem 3.10. (Error estimate of adjoint state variable for gradient cost functional) *There holds*

$$\|\nabla(\phi - \phi_h)\| \leq Ch^{\frac{1}{2}+\epsilon}\left(\|q\|_{\frac{3}{2}+\epsilon,\Omega} + \|u_d\|_{2,\Omega} + \|\phi\|_{\frac{3}{2}+\epsilon,\Omega} + \|w\|_{\frac{3}{2}+\epsilon,\Omega}\right).$$

Proof. The proof follows the similar lines as that of the proof of Theorem 3.8. □

4. Numerical experiment

In this section, we illustrate the theoretical results by performing some numerical experiments. We conduct two experiments with two model problems. In the first experiment, we test the validity of a priori error estimates derived in Theorems 3.7 and 3.8. In the second experiment, we test the validity of the a priori error estimates derived in Theorems 3.9 and 3.10. For this, we construct the model problems with known solutions. In order to accomplish this, we slightly modify the cost functional J, denoted by $\tilde{J}$, by

$$\tilde{J}(w, p) = \frac{1}{2}\|\nabla^i(w - u_d)\|^2 + \frac{\rho}{2}\|\nabla(p - q_d)\|^2, \quad w \in Q,\ p \in Q_{ad}.$$

where q_d is a given function and $i = 0, 1$. Then the minimization problem reads: Find $(u, q) \in Q \times Q_{ad}$ such that

$$\tilde{J}(u, q) = \min_{(w,p)\in Q\times Q_{ad}} \tilde{J}(w, p)$$

subject to the condition that $(w, p) \in Q \times Q_{ad}$ satisfies $w = S(f, p)$. Then it is easy to check that the optimality conditions take the form

$$\begin{aligned}
u &= u^0 + q, \quad w \in V,\\
a(u^0, v) &= \ell(v) - a(q, v) \quad \forall v \in V,\\
a(v, \phi) &= (\nabla^i(u - u_d), \nabla^i v) \quad \forall v \in V. \quad \forall v \in V,\\
\rho\, a(q, p - q) &\geq a(p - q, \phi) - (\nabla^i(u - u_d), \nabla^i(p - q)) + \rho\, a(q_d, p - q) \quad \forall p \in Q_{ad}.
\end{aligned}$$

Accordingly, the discrete control problem finds $(u_h^0, q_h, \phi_h) \in V_h \times Q_{ad}^h \times V_h$ such that

$$\begin{aligned}
u_h &= u_h^0 + q_h,\\
a(u_h^0, v_h) &= \ell(v_h) - a(q_h, v_h) \quad \forall v_h \in V_h,\\
a(v_h, \phi_h) &= (\nabla^i(u_h - u_d), \nabla^i v_h) \quad \forall v_h \in V_h,\\
\rho\, a(q_h, p_h - q_h) &\geq a(p_h - q_h, \phi_h) - (\nabla^i(u_h - u_d), \nabla^i(p_h - q_h))) + \rho\, a(q_d, p_h - q_h) \quad \forall p_h \in Q_{ad}^h.
\end{aligned}$$

TABLE 1. Errors and orders of convergence (order) for the Example 4.1.

h	$\|\nabla(u-u_h)\|$	Order	$\|\nabla(q-q_h)\|$	Order	$\|\nabla(\phi-\phi_h)\|$	Order
0.250000	0.202248	–	0.206402	–	1.281567	–
0.125000	0.103894	0.961003	0.104789	0.977964	0.668652	0.938581
0.062500	0.052462	0.985768	0.052614	0.993963	0.345636	0.952001
0.031250	0.026311	0.995583	0.026335	0.998458	0.175148	0.980672
0.015625	0.013167	0.998700	0.013171	0.999612	0.087964	0.993595
0.007812	0.006585	0.999627	0.006586	0.999902	0.044041	0.998053

TABLE 2. Errors and orders of convergence (order) for the Example 4.2.

h	$\|\nabla(u-u_h)\|$	Order	$\|\nabla(q-q_h)\|$	Order	$\|\nabla(\phi-\phi_h)\|$	Order
0.353553	0.895935	–	0.900064	–	3.563134	–
0.176776	0.486996	0.879483	0.490611	0.875448	1.769197	1.010051
0.088388	0.250146	0.961139	0.250817	0.967943	0.949273	0.898200
0.044194	0.126011	0.989217	0.126111	0.991931	0.484287	0.970959
0.022097	0.063130	0.997147	0.063144	0.997979	0.243405	0.992500
0.011048	0.031581	0.999261	0.031583	0.999494	0.121862	0.998109

First, we present a numerical example for L^2-cost functional problem.

Example 4.1. (For L^2-cost functional) For this example, the domain Ω is taken to be the unit square $(0,1)\times(0,1)$, $\Gamma_C=(0,1)\times\{0\}$, $\Gamma_D=\partial\Omega\backslash\Gamma_C$ and set $\alpha=1$, $q_a=-0.10$, $q_b=0.25$. We choose the data of the problem as follows. Choose the state to be $u(x,y)=x(1-x)(1-y)\exp(y)$, the adjoint state to be $\phi(x,y)=\sin^2(\pi x)\sin^2(\pi y)$ and the control to be $q(x,y)=x(1-x)(1-y)\exp(y)$. Then compute $f=-\Delta u$ and $u_d=u+\Delta\phi$. The choice of ϕ leads to $q_d=q$. We choose u and q in such a way that $u=q$ on Γ_C and $u=q=0$ on Γ_D and $\frac{\partial q}{\partial n}=0$, $\frac{\partial \phi}{\partial n}=0$, $\frac{\partial q_d}{\partial n}=0$ so, $\sigma_n=\rho\frac{\partial q}{\partial n}-\frac{\partial \phi}{\partial n}-\rho\frac{\partial q_d}{\partial n}=0$ on Γ_C.

We generate a sequence of meshes with mesh size h as shown in Table 1 by uniformly refining successively each triangulation. We have used our in-house MATLAB code for the computations. The computed errors and orders of convergence in H^1-norm are shown in Table 1. The experiment clearly illustrates the expected rates of convergence.

In the next example, we will consider the domain Ω to be L-shaped.

Example 4.2. (For L^2-cost functional) For this example, the domain Ω is taken to be the L shape. The contact boundary $\Gamma_C=(0,1)\times\{0\}\cup\{0\}\times(-1,0)$, $\Gamma_D=\partial\Omega\backslash\Gamma_C$ and set $\alpha=10^{-2}$, $q_a=0$, $q_b=1$. We choose the data of the problem as follows. Choose the state to be $u(x,y)=(1-x^2)^2(1-y^2)^2$ the adjoint state to be $\phi(x,y)=\sin^2(\pi x)\sin^2(\pi y)$ and the control to be $q(x,y)=(1-x^2)^2(1-y^2)^2$. Then compute $f=-\Delta u$ and $u_d=u+\Delta\phi$. The choice of ϕ leads to $q_d=q$. We choose u and q in such a way that $u=q$ on Γ_C and $u=q=0$ on Γ_D and $\frac{\partial \phi}{\partial n}=0$, on Γ_C so, $\sigma_n=\rho\frac{\partial q}{\partial n}-\frac{\partial \phi}{\partial n}-\rho\frac{\partial q_d}{\partial n}=-\frac{\partial \phi}{\partial n}=0$ on Γ_C.

Similarly, as we have done in Example 4.1, we generate a sequence of meshes with mesh size h as shown in Table 2 by uniformly refining successively each triangulation. The computed errors and orders of convergence in H^1-norm are shown in Table 2. The experiment clearly demonstrates the expected rates of convergence.

Now we will present a numerical example for gradient cost functional problem.

Example 4.3. (For gradient cost functional) For this example, the domain Ω is taken to be the unit square $(0,1)\times(0,1)$, $\Gamma_C=(0,1)\times\{0\}$, $\Gamma_D=\partial\Omega\backslash\Gamma_C$ and set $\alpha=10^{-2}$, $q_a=-0.15$, $q_b=1$. We choose the data of the problem as follows. Choose the state to be $u(x,y)=\sin(\pi x)\cos(\frac{\pi}{2}y)$, the adjoint state to be $\phi(x,y)=\sin^2(\pi x)\sin^2(\pi y)$ and the control to be $q(x,y)=\sin(\pi x)\cos(\frac{\pi}{2}y)$. Then compute $f=-\Delta u$ and $\Delta u_d=\Delta u-\Delta\phi$.

Table 3. Errors and orders of convergence (order) for the Example 4.3.

h	$\|\nabla(u-u_h)\|$	Order	$\|\nabla(q-q_h)\|$	Order	$\|\nabla(\phi-\phi_h)\|$	Order
0.250000	0.720803	–	0.736432	–	1.288014	–
0.125000	0.369862	0.962616	0.373585	0.979117	0.682163	0.916959
0.062500	0.186857	0.985048	0.187517	0.994409	0.348756	0.967897
0.0312500	0.093747	0.995087	0.093851	0.998580	0.175697	0.989125
0.015625	0.046921	0.998524	0.046937	0.999643	0.088049	0.996700
0.007812	0.023467	0.999574	0.023470	0.999910	0.044054	0.999043

Table 4. Errors and orders of convergence (order) for the Example 4.4.

h	$\|\nabla(u-u_h)\|$	Order	$\|\nabla(q-q_h)\|$	Order	$\|\nabla(\phi-\phi_h)\|$	Order
0.353553	0.876082	–	0.772173	–	2.900307	–
0.176776	0.447653	0.968684	0.392224	0.977244	1.803048	0.685767
0.088388	0.224819	0.993612	0.196920	0.994068	0.960270	0.908925
0.044194	0.112514	0.998658	0.098562	0.998503	0.487997	0.976566
0.022097	0.056268	0.999715	0.049293	0.999624	0.245000	0.994087
0.011048	0.028135	0.999940	0.024648	0.999906	0.122626	0.998518

The choice of ϕ leads to $q_d = q$. We choose u and q in such a way that $u = q$ on Γ_C and $u = q = 0$ on Γ_D and $\frac{\partial\phi}{\partial n} = 0$, on Γ_C so, $\sigma_n = \rho\frac{\partial q}{\partial n} - \frac{\partial\phi}{\partial n} - \rho\frac{\partial q_d}{\partial n} = -\frac{\partial\phi}{\partial n} = 0$ on Γ_C.

The meshes are taken to be the same as in Example 4.1. The computed errors and orders of convergence in H^1-norm are shown in Table 3. The experiment clearly illustrates the expected rates of convergence.

In the next example, we consider the domain Ω to be L-shaped.

Example 4.4. (For gradient cost functional) For this example, the domain Ω is taken to be the L shape. The contact boundary $\Gamma_C = (0,1)\times\{0\}\cup\{0\}\times(-1,0)$, $\Gamma_D = \partial\Omega\backslash\Gamma_C$ and set $\alpha = 10^{-2}$, $q_a = -0.10$, $q_b = 1$. We choose the data of the problem as follows. Choose the state to be $u(x,y) = \cos(\frac{\pi}{2}x)\cos(\frac{\pi}{2}y)$ the adjoint state to be $\phi(x,y) = \sin^2(\pi x)\sin^2(\pi y)$ and the control to be $q(x,y) = \cos(\frac{\pi}{2}x)\cos(\frac{\pi}{2}y)$. Then compute $f = -\Delta u$ and $\Delta u_d = \Delta u - \Delta\phi$. The choice of ϕ leads to $q_d = q$. We choose u and q in such a way that $u = q$ on Γ_C and $u = q = 0$ on Γ_D and $\frac{\partial\phi}{\partial n} = 0$, on Γ_C so, $\sigma_n = \rho\frac{\partial q}{\partial n} - \frac{\partial\phi}{\partial n} - \rho\frac{\partial q_d}{\partial n} = -\frac{\partial\phi}{\partial n} = 0$ on Γ_C.

The meshes are taken to be the same as in Example 4.2. The computed errors and orders of convergence in H^1-norm are depicted in Table 4. The experiment clearly illustrates the expected theoretical rates of convergence.

Remark 4.5. It is clear that in all the above numerical examples the solutions u, ϕ, q are in $H^2(\Omega)$ *i.e.*, $\epsilon = \frac{1}{2}$, which implies that the theoretically predicted order of convergence is one which is also observed in all the experiments.

5. Conclusions

In this article, we have studied an energy-space based approach for the Dirichlet boundary control problem with control constraints. The optimal control problem is shown to have a unique solution and the optimality system consists of a coupled simplified Signorini type boundary problem for the control variable. A finite element-based numerical method is proposed by considering the linear Lagrange finite element spaces and posing discrete control constraints at the Lagrange nodes. The corresponding *a priori* error estimates are derived which are optimal in the H^1-norm up to the regularity. Numerical experiments have been performed to illustrate the theoretical results.

References

[1] T. Apel, P. Johannes and A. Rösch, Finite element error estimates for Neumann boundary control problems on graded meshes. *Comput. Optim. Appl.* **52** (2012) 3–28.

[2] S. Auliac, Z. Belhachmi, F. Ben Belgacem and F. Hecht, Quadratic finite elements with non matching grids for the unilateral boundary contact. *Math. Model. Numer. Anal.* **47** (2013) 1185–1205.

[3] F.B. Belgacem, H.E. Fekih and H. Metoui, Singular perturbation for the dirichlet boundary control of elliptic problems. *ESAIM: M2AN* **37** (2003) 833–850.

[4] Z. Belhachmi and F. Ben Belgacem, Quadratic finite element approximation of the Signorini problem. *Math. Comp.* **241** (2003) 83–104.

[5] S.C. Brenner and L.R. Scott, The Mathematical Theory of Finite Element Methods (Third Edition). Springer-Verlag, New York (2008).

[6] E. Casas and F. Tröltzsch, Error estimates for linear-quadratic elliptic control problems, in Analysis and Optimization of Differential Systems. Kluwer Academic Publishing, Boston (2003) 89–100.

[7] E. Casas and J.P. Raymond, Error estimates for the numerical approximation of Dirichlet boundary control for semilinear elliptic equations. *SIAM J. Control Optim.* **45** (2006) 1586–1611.

[8] E. Casas and M. Mateos, Error estimates for the numerical approximation of Neumann control problems *Comput. Optim. Appl.* **39** (2008) 265–295.

[9] E. Casas and V. Dhamo, Error estimates for the numerical approximation of Neumann control problems governed by a class of quasilinear elliptic equations. *Comput. Optim. Appl.* **52** (2012) 719–756.

[10] E. Casas, M. Mateos and J.P. Raymond, Penalization of Dirichlet optimal control problems. *ESAIM: COCV* **15** (2009) 782–809.

[11] S. Chowdhury, T. Gudi and A.K. Nandakumaran, A framework for the error analysis of discontinuous finite element methods for elliptic optimal control problems and applications to C^0 IP methods. *Numer. Funct. Anal. Optim.* **36** (2015) 1388–1419.

[12] S. Chowdhury, T. Gudi and A.K. Nandakumaran, On the finite element approximation of the Dirichlet boundary control problem. *Math. Comp.* **86** (2017) 1103–1126.

[13] P.G. Ciarlet, The Finite Element Method for Elliptic Problems. North-Holland, Amsterdam (1978).

[14] K. Deckelnick, A. Günther and M. Hinze, Finite element approximation of Dirichlet boundary control for elliptic PDEs on two- and three-dimensional curved domains. *SIAM J. Numer. Anal.* **48** (2009) 2798–2819.

[15] A.K. Dond, T. Gudi and R.Ch. Sau, An Error analysis of Discontinuous Finite Element Methods for the Optimal Control problems governed by Stokes equation. *Numer. Funct. Anal. Optim.* **40** (2019) 421–460.

[16] G. Drouet and P. Hild, Optimal convergence for discrete variational inequalities modelling Signorini contact in 2D and 3D without additonal assumptions on the unkown contact set. *SIAM J. Numer. Anal.* **53** (2015) 1488–1507.

[17] R. Falk, Approximation of a class of optimal control problems with order of convergence estimates. *J. Math. Anal. Appl.* **44** (1973) 28–47.

[18] T. Geveci, On the approximation of the solution of an optimal control problems governed by an elliptic equation. *ESAIM: M2AN* **13** (1979) 313–328.

[19] A. Günther and M. Hinze, Elliptic control problems with gradient constraints variational discrete versus piecewise constant controls. *Comput. Optim. Appl.* **40** (2011) 549–566.

[20] M.D. Gunzburger, L.S. Hou and T. Swobodny, Analysis and finite element approximation of optimal control problems for the stationary Navier–Stokes equations with Dirichlet controls. *ESAIM: M2AN* **25** (1991) 711–748.

[21] M.D. Gunzburger, L. Hou and T.P. Svobodny, Boundary velocity control of incompressible flow with an application to viscous drag reduction. *SIAM J. Control Optim.* **30** (1992) 167–181.

[22] M. Hinze, A variational discretization concept in control constrained optimization: The linear-quadratic case. *Comput. Optim. Appl.* **30** (2005) 45–61.

[23] M. Hinze, R. Pinnau, M. Ulbrich and S. Ulbrich, Optimization with PDE Constraints. Springer, New York (2009).

[24] H. Le Dret, Equations aux Dérivées Partielles Elliptiques non Linéaires. Springer-Verlag, New York (2013).

[25] W. Liu and N. Yan, Adaptive Finite Element Methods: Optimal Control Governed by PDEs. Beijing Science Press, Beijing (2008).

[26] S. May, R. Rannacher and B. Vexler, Error analysis for a finite element approximation of elliptic Dirichlet boundary control problems. *SIAM J. Control Optim.* **51** (2013) 2585–2611.

[27] C. Meyer and A. Rösch, Superconvergence properties of optimal control problems. *SIAM J. Control Optim.* **43** (2004) 970–985.

[28] G. Of, T.X. Phan and O. Steinbach, An energy space finite element approach for elliptic Dirichlet boundary control problems *Numer. Math.* **129** (2014) 723–748.

[29] J.-P. Raymond, Optimal Control of Partial Differential Equations (lecture notes).

[30] F. Tröltzsch, Optimale Steuerung Partieller Differentialgleichungen, 1st ed. Vieweg. Cambridge University Press (2005).