

MULTIPLICATIVE CONTROLLABILITY FOR NONLINEAR DEGENERATE PARABOLIC EQUATIONS BETWEEN SIGN-CHANGING STATES*

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Abstract. In this paper we study the global approximate multiplicative controllability for nonlinear degenerate parabolic Cauchy problems. In particular, we consider a one-dimensional semilinear degenerate reaction-diffusion equation in divergence form governed via the coefficient of the reaction term (bilinear or multiplicative control). The above one-dimensional equation is degenerate since the diffusion coefficient is positive on the interior of the spatial domain and vanishes at the boundary points. Furthermore, two different kinds of degenerate diffusion coefficient are distinguished and studied in this paper: the weakly degenerate case, that is, if the reciprocal of the diffusion coefficient is summable, and the strongly degenerate case, that is, if that reciprocal isn't summable. In our main result we show that the above systems can be steered from an initial continuous state that admits a finite number of points of sign change to a target state with the same number of changes of sign in the same order. Our method uses a recent technique introduced for uniformly parabolic equations employing the shifting of the points of sign change by making use of a finite sequence of initial-value pure diffusion problems. Our interest in degenerate reaction-diffusion equations is motivated by the study of some energy balance models in climatology (see, *e.g.*, the Budyko-Sellers model) and some models in population genetics (see, *e.g.*, the Fleming-Viot model).

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1. INTRODUCTION

This paper is concerned with the study of the global approximate controllability of one-dimensional semilinear degenerate reaction-diffusion equations governed in the bounded domain $(-1, 1)$ by means of the *bilinear controls*

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$\alpha(x, t)$, of the form

$$\left\{ \begin{array}{l} u_t - (a(x)u_x)_x = \alpha(x, t)u + f(x, t, u) \quad \text{in } Q_T := (-1, 1) \times (0, T) \\ \left\{ \begin{array}{l} \beta_0 u(-1, t) + \beta_1 a(-1)u_x(-1, t) = 0 \quad t \in (0, T) \\ \gamma_0 u(1, t) + \gamma_1 a(1)u_x(1, t) = 0 \quad t \in (0, T) \\ a(x)u_x(x, t)|_{x=\pm 1} = 0 \quad t \in (0, T) \\ u(x, 0) = u_0(x) \quad x \in (-1, 1) . \end{array} \right. \quad \begin{array}{l} \text{(for } WDP) \\ \text{(for } SDP) \end{array} \end{array} \right. \quad (1.1)$$

In the semilinear Cauchy problem (1.1) the diffusion coefficient $a \in C([-1, 1]) \cap C^1(-1, 1)$, positive on $(-1, 1)$, vanishes at the boundary points of $[-1, 1]$, leading to a degenerate parabolic equation. Furthermore, two kinds of degenerate diffusion coefficient can be distinguished and are studied in this paper. (1.1) is a weakly degenerate problem (*WDP*) (see [10, 30]) if the diffusion coefficient is such that $\frac{1}{a} \in L^1(-1, 1)$, while the problem (1.1) is called a strongly degenerate problem (*SDP*) (see [9, 29]) if $\frac{1}{a} \notin L^1(-1, 1)$.

In this paper, we assume that the reaction coefficient, that is the bilinear control $\alpha(x, t)$, is bounded on Q_T , and the initial datum $u_0(x)$ is continuous on the open interval $(-1, 1)$, since u_0 belongs to the weighted Sobolev space $H_a^1(-1, 1)$, defined as either

$$\left\{ \begin{array}{l} \{u \in L^2(-1, 1) \mid u \text{ is absolutely continuous in } [-1, 1] \text{ and } \sqrt{a}u_x \in L^2(-1, 1)\} \text{ for (WDP),} \\ \text{or} \\ \{u \in L^2(-1, 1) \mid u \text{ is locally absol. continuous in } (-1, 1) \text{ and } \sqrt{a}u_x \in L^2(-1, 1)\} \text{ for (SDP).} \end{array} \right. \quad (1.2)$$

See [1] for the main functional properties of this kind of weighted Sobolev spaces, in particular we note that the space $H_a^1(-1, 1)$ is embedded in $L^\infty(-1, 1)$ only in the weakly degenerate case (see also [9, 10, 29, 30]). After introducing in Section 1.4 the problem formulation of (1.1), in Section 1.5 we recall the main properties of this weighted Sobolev spaces and the well-posedness of (1.1) ⁽¹⁾. In the following we introduce the main motivations for studying degenerate parabolic problems with the above structure.

1.1. Reaction-diffusion equations and their applications

It is well-known that reaction-diffusion equations can be linked to various applied models such as chemical reactions, nuclear chain reactions, social sciences and biomedical models. More generally, reaction-diffusion equations or systems describe how the concentration of one or more substances changes under the influence of some processes such as local reactions, where substances are transformed into each other, and diffusion which causes substances to spread out in space. See, *e.g.*, some recent papers by Enrique Zuazua and coauthors for the control theory of reaction-diffusion equations: [40, 42, 44] (see also [41]).

In the contest of degenerate reaction-diffusion equations there are several interesting models. In particular, we recall some models in population genetics, see, *e.g.*, the Fleming-Viot model (for a comprehensive literature of these applications see Epstein's and Mazzeo's book [24]), and some models arising in mathematical finance, see, *e.g.*, the Black-Scholes equation in the theory of option pricing. Our interest in degenerate parabolic problem (1.1) is also motivated by the study of an energy balance model in climatology: the Budyko-Sellers model. This model was introduced, independently, by Budyko (see [7]) and Sellers (see [43]). The Budyko-Sellers model studies the role played by continental and oceanic areas of ice on the evolution of the climate. A complete

¹ We recall that it is well-known (see, *e.g.*, [1]) that in the (*WDP*) case all functions in the domain of the corresponding differential operator possess a trace on the boundary, in spite of the fact that the operator degenerates at such points. Thus, in the (*WDP*) case we can consider in (1.1) the general Robin type boundary conditions. On the other hand, in the (*SDP*) one is forced to restrict to the Neumann type boundary conditions.

treatment of the mathematical formulation of the Budyko-Sellers model has been obtained by Diaz in [21] (see also [9, 16]).

1.2. Structure of the paper

In Section 1.3 we show the mathematical motivations, the state of the art and the contents of this paper. Then, in Section 1.4 we give the problem formulation and in Section 1.5 we recall the well-posedness of (1.1) (obtained in [30] for (WDP) , and in [29] for (SDP)). Thus, in Section 2 we introduce the main result for system (1.1), that is, Theorem 2.4, together with some of its consequences. In Sections 2.1–2.4, we explain the iterative structure of the proof of the main result, and, after introducing two necessary technical tools, Theorem 2.11 and Theorem 2.12, we proceed with the proof of Theorem 2.4. Section 3 deals with the proof of Theorem 2.11: a controllability result for pure diffusion problems. Section 4 is devoted to the proof of Theorem 2.12: a smoothing result intended to attain suitable intermediate data while preserving the points of sign change.

1.3. Mathematical motivations, state of the art and contents

The interest in degenerate parabolic equations is motivated by several mathematical models (see Epstein's and Mazzeo's book [24]) and is dated back by many decades, in particular significant contributions are due to Fichera's and Oleinik's recherches (see *e.g.*, respectively, [28, 38]). In control theory only in the last fifteen years several contributions about degenerate PDEs appeared, in particular we recall some papers due to Cannarsa and collaborators, see, *e.g.*, [1, 2, 14, 18] (principally, we call to mind the pioneering and fundamental paper [15], obtained in collaboration with Martinez and Vancostenoble). In the above papers, in [19, 36], and also in many works about controllability for non-degenerate equations (see, *e.g.*, [3, 6, 25, 27]), boundary and interior locally distributed controls are usually employed, these controls are additive terms in the equation and have localized support. Additive control problems for the Budyko-Sellers model have been studied by J.I. Diaz in [22] (see also [21]).

However, such controls are unfit to study several interesting applied problems such as chemical reactions controlled by catalysts, and also smart materials, which are able to change their principal parameters under certain conditions. In the present work, the control action takes the form of a bilinear control, that is, a control given by the multiplicative coefficient α in (1.1). General references in the area of multiplicative controllability are the seminal work [4] by Ball, Marsden, and Slemrod, some important results about bilinear control of the Schrödinger equation obtained by Beauchard, Coron, Gagnon, Laurent and Morancey in [5, 20], and in the references therein, and some results obtained by Khapalov for parabolic and hyperbolic equations, see [33], and the references therein. See also some results for reaction-diffusion equations (both degenerate and uniformly parabolic) obtained by Cannarsa and Floridia in [9, 10], by Floridia in [29], and by Cannarsa, Floridia and Khapalov in [12]. Moreover, we mention the recent papers [23, 39] about multiplicative controllability of heat and wave equation, respectively. New perspectives in the area of the multiplicative controllability are suggested by Enrique Zuazua and coauthors in [44], in that paper the bilinear control for reaction-diffusion equations has a big applied importance, it represents the so-called Allee threshold.

Additive vs multiplicative controllability. Historically, the concept of controllability emerged in the second half of the twentieth century in the context of linear ordinary differential equations and was motivated by several engineering, economics and Life sciences applications. Then, it was extended to various linear partial differential equations governed by *additive* locally distributed (*i.e.*, supported on a bounded subdomain of the space domain), lumped (acting at a point), and boundary controls (see, *e.g.* Fattorini and Russell in [26], and many papers by J.L. Lions and collaborators). Methodologically, these studies are typically based on the *linear* duality pairing technique between the control-to-state mapping at hand and its dual observation map (see in [3] the Hilbert Uniqueness Method, HUM, introduced by J.L. Lions in 1988), using in some cases Carleman estimates tool (see [1, 11, 13]). When this mapping is nonlinear, as it happens in the case of the multiplicative controllability, the aforementioned approach does not apply and the above-stated concept of controllability becomes, in general, unachievable.

In the last years, in spite of the mentioned difficulties, many researchers started to study multiplicative controllability, since additive controls (see also [1, 14]–[17]) are unable to treat application problems that require inputs with high energy levels or they are not available due to the physical nature of the process at hand. Thus, an approach based on multiplicative controls, where the coefficient α in (1.1) is used to change the main physical characteristics of the system at hand, seems realistic.

State of the art for uniformly parabolic equations. To motivate the multiplicative controllability results obtained in this paper for degenerate equation, we start to present the state of the art for uniformly parabolic equations. Let us introduce the following semilinear Dirichlet boundary value problem, studied in [12],

$$\begin{cases} u_t = u_{xx} + v(x, t)u + f(u) & \text{in } (0, 1) \times (0, T), \quad T > 0, \\ u(0, t) = u(1, t) = 0, & t \in (0, T), \\ u|_{t=0} = u_0, \end{cases} \quad (1.3)$$

where $u_0 \in H_0^1(0, 1)$ ⁽²⁾, $v \in L^\infty(Q_T)$ is a bilinear control, the nonlinear term $f: \mathbb{R} \rightarrow \mathbb{R}$ is assumed to be a Lipschitz function, differentiable at $u = 0$, and satisfying $f(0) = 0$.

There are some important obstructions to the multiplicative controllability ⁽³⁾ of (1.3). We note that system (1.3) cannot be steered anywhere from the origin. Moreover, if $u_0(x) \geq 0$ in $(0, 1)$, then the strong maximum principle ⁽⁴⁾ demands that the respective solution to (1.3) remains nonnegative at any moment of time, regardless of the choice of the bilinear control v . This means that system (1.3) cannot be steered from any such u_0 to any target state which is negative on a nonzero measure set in the space domain. Owing to the previous obstruction to the multiplicative controllability two kinds of controllability are worth studying: nonnegative controllability and controllability between sign-changing states.

First, in [33] Khapalov studied global nonnegative approximate controllability of the one-dimensional nondegenerate semilinear convection-diffusion-reaction equation governed in a bounded domain via bilinear controls. Finally, in [12] Cannarsa, Floridia and Khapalov established an approximate controllability property for the semilinear system (1.3) in suitable classes of functions that change sign, not arbitrarily but respecting the structure imposed by the strong maximum principle, like in the seminal paper by Matano [37].

State of the art for degenerate parabolic equations: nonnegative controllability. With regard to the degenerate reaction-diffusion equations, similar results about global nonnegative approximate multiplicative controllability were obtained in [9, 10, 29] ⁽⁵⁾.

At first, Cannarsa and Floridia considered the linear degenerate problem associated to (1.1) (*i.e.* when $f \equiv 0$) in the two distinct kinds of set-up. Namely, in [10] the *weakly degenerate* linear problem (*WDP*) (that is, when $\frac{1}{\alpha} \in L^1(-1, 1)$) was investigated, and in [9] the *strongly degenerate* linear problem (*SDP*) (that is, when $\frac{1}{\alpha} \notin L^1(-1, 1)$) was studied. Then, in [29] Floridia focused on the semilinear strongly degenerate case. So, in the three above intermediate steps, studied in the papers [9, 10, 29], the authors obtained global nonnegative approximate controllability of (1.1), *in large time*, via bilinear piecewise static controls with initial state $u_0 \in L^2(-1, 1)$. That is, it has been showed that the above system can be steered *in large time*, in the space of square-summable functions, from any nonzero, nonnegative initial state into any neighborhood of any desirable nonnegative target-state by bilinear piecewise static controls.

Contents of the paper: controllability of (1.1) between sign-changing states. In this paper, we study the multiplicative controllability of the semilinear degenerate reaction-diffusion system (1.1) when both the initial and target states admit a finite number of points of sign change, in particular we extend to the degenerate settings

² $H_0^1(-1, 1) = \{u \in L^2(-1, 1) | u_x \in L^2(-1, 1) \text{ and } u(\pm 1) = 0\}$.

³ Let us recall that, in general terms, an evolution system is called globally approximately controllable in a given space H at time $T > 0$, if any initial state in H can be steered into any neighborhood of any desirable target state at time T , by selecting a suitable control.

⁴ In Remark 2.1 of [12], it was observed that the strong maximum principle for linear parabolic PDEs can be extended to the semilinear parabolic system (1.3).

⁵ In [9, 10, 29], first, the authors proved that, also in the degenerate case, if $u_0 \geq 0$ then the respective solution remains nonnegative at any moment of time, regardless of the choice of the bilinear control.

the results obtained in [12] for uniformly parabolic equations. There are some substantial differences with respect to the work [12]. The main technical difficulty to overcome with respect to the uniformly parabolic case, is the fact that functions in $H_a^1(-1, 1)$ need not be necessarily bounded when the operator is strongly degenerate. The (WDP) case is somewhat similar to the uniformly parabolic case, however in our control problem we have to cope with further difficulties given by the general *Robin* boundary conditions. In spite of the above difficulties we are able to prove for the degenerate problem (1.1) that given an initial datum $u_0 \in H_a^1(-1, 1)$ with a finite number of changes of sign, any target state $u^* \in H_a^1(-1, 1)$, with as many changes of sign *in the same order* (in the sense of Def. 2.2) as the given u_0 , can be approximately reached in the $L^2(-1, 1)$ -norm at some time $T > 0$, choosing suitable reaction coefficients (see Thm. 2.4). We adapt to the degenerate system (1.1) a technique introduced in [12], for uniformly parabolic equations, employing the shifting of the points of sign change by making use of a finite sequence of initial-value pure diffusion problems. In particular, we proceed by splitting the time interval $[0, T]$ into $2N$ time intervals ($N \in \mathbb{N}$ will be determined after the crucial Prop. 3.9)

$$[0, T] = [0, S_1] \cup [S_1, T_1] \cup \cdots \cup [T_{N-1}, S_N] \cup [S_N, T_N] \cup [T_N, T],$$

on which two alternative actions are applied. On the even intervals $[S_k, T_k]$ we choose suitable initial data, $w_k \in H_a^1(-1, 1) \cap C^{2+\beta}([a_0^*, b_0^*])$ (with suitable $[a_0^*, b_0^*] \subset (-1, 1)$), in pure diffusion problems ($\alpha \equiv 0$) to move the points of sign change to their desired location, whereas on the odd intervals $[T_{k-1}, S_k]$ we use piecewise static multiplicative controls α_k to attain such w_k 's as intermediate final conditions. More precisely, on $\bigcup_{k=1}^N [S_k, T_k]$ we make use of the boundary problems

$$\left\{ \begin{array}{l} w_t = (a(x)w_x)_x + f(x, t, w), \quad \text{in } (-1, 1) \times \bigcup_{k=1}^N [S_k, T_k] \\ \left\{ \begin{array}{l} \beta_0 w(-1, t) + \beta_1 a(-1)w_x(-1, t) = 0 \\ \gamma_0 w(1, t) + \gamma_1 a(1)w_x(1, t) = 0 \end{array} \right. \quad t \in \bigcup_{k=1}^N (S_k, T_k) \quad (\text{for WDP}) \\ a(x)w_x(x, t)|_{x=\pm 1} = 0 \quad t \in \bigcup_{k=1}^N (S_k, T_k) \quad (\text{for SDP}) \\ w|_{t=S_k} = w_k(x), \quad x \in (-1, 1), \quad k = 1, \dots, N. \end{array} \right.$$

where the w_k 's are viewed as control parameters to be chosen to generate suitable curves of sign change, which have to be continued along all the N time intervals $[S_k, T_k]$ until each point has reached the desired final position. In order to fill the gaps between two successive $[S_k, T_k]$'s, on $[T_{k-1}, S_k]$ we construct α_k that steers the solution of

$$\left\{ \begin{array}{l} u_t = (a(x)u_x)_x + \alpha_k(x, t)u + f(x, t, u) \quad \text{in } (-1, 1) \times [T_{k-1}, S_k], \\ \left\{ \begin{array}{l} \beta_0 u(-1, t) + \beta_1 a(-1)u_x(-1, t) = 0 \\ \gamma_0 u(1, t) + \gamma_1 a(1)u_x(1, t) = 0 \end{array} \right. \quad t \in (T_{k-1}, S_k) \quad (\text{for WDP}) \\ a(x)u_x(x, t)|_{x=\pm 1} = 0 \quad t \in (T_{k-1}, S_k) \quad (\text{for SDP}) \\ u|_{t=T_{k-1}} = u_{k-1} + r_{k-1} \in H_a^1(-1, 1), \end{array} \right.$$

from $u_{k-1} + r_{k-1}$ to w_k , where u_{k-1} and w_k have the same points of sign change, and $\|r_{k-1}\|_{L^2(-1,1)}$ is small. The above result is contained in Theorem 2.12.

The fact that such an iterative process can be completed within a finite number of steps ($2N$, for suitable $N \in \mathbb{N}$) is an important point of the proof. Such a point follows from precise estimates which is, essentially, the consequence of the following facts:

- (a) the sum of the distances of each branch of the null set of the resulting solution of (1.3) from its target points of sign change decreases at a linear-in-time rate for curves which are still far away from their corresponding target points;
- (b) the error caused by the possible displacement of points already near their targets is negligible.

Some open problems. In the future, we intend to investigate the multiplicative controllability for both degenerate and uniformly parabolic equations in higher space dimensions on domains with specific geometries (see, *e.g.*, Sect. 6 in [12]). Moreover, we would like to extend our approach to study the approximate controllability of the general formulation of the Budyko-Sellers differential problem on a compact surface without boundary. Finally, we would like to extend our approach to other nonlinear systems of parabolic type, such as the systems of fluid dynamics (see, *e.g.*, KCPF), and the porous medium equation.

1.4. Problem formulation

In this paper, we consider the degenerate problem (1.1) under the following assumptions:

(A.1) $u_0 \in H_a^1(-1, 1)$; ⁽⁶⁾

(A.2) $\alpha \in L^\infty(Q_T)$;

(A.3) $f : Q_T \times \mathbb{R} \rightarrow \mathbb{R}$ is such that

- $(x, t, u) \mapsto f(x, t, u)$ is a Carathéodory function on $Q_T \times \mathbb{R}$, ⁽⁷⁾
- $u \mapsto f(x, t, u)$ is differentiable at $u = 0$,
- $t \mapsto f(x, t, u)$ is locally absolutely continuous for a.e. $x \in (-1, 1)$, for every $u \in \mathbb{R}$,
- there exist constants $\gamma_* \geq 0, \vartheta \geq 1$ and $\nu \geq 0$ such that, for a.e. $(x, t) \in Q_T, \forall u, v \in \mathbb{R}$, we have

$$|f(x, t, u)| \leq \gamma_* |u|^\vartheta, \quad (1.4)$$

$$-\nu(1 + |u|^{\vartheta-1} + |v|^{\vartheta-1})(u - v)^2 \leq (f(x, t, u) - f(x, t, v))(u - v) \leq \nu(u - v)^2, \quad (1.5)$$

$$f_t(x, t, u) u \geq -\nu u^2; \quad (8)$$

(A.4) $a \in C([-1, 1]) \cap C^1(-1, 1)$ is such that

$$a(x) > 0, \quad \forall x \in (-1, 1), \quad a(-1) = a(1) = 0,$$

moreover, we have the following two alternative assumptions:

(A.4_{WD}) if $\frac{1}{a} \in L^1(-1, 1)$, in (1.1) let us consider the Robin boundary conditions, where

⁶ The definition of the weighted Sobolev space $H_a^1(-1, 1)$ is given in (1.2).

⁷ We say that $f : Q_T \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function on $Q_T \times \mathbb{R}$ if the following properties hold:

- * $(x, t) \mapsto f(x, t, u)$ is measurable, for every $u \in \mathbb{R}$;
- * $u \mapsto f(x, t, u)$ is a continuous function, for a.e. $(x, t) \in Q_T$.

⁸ This assumption is necessary and used only for the well-posedness, obtained in the paper [30] for the (WDP) case, and in [29] for the (SDP) case.

★ $\beta_0, \beta_1, \gamma_0, \gamma_1 \in \mathbb{R}$, $\beta_0^2 + \beta_1^2 > 0$, $\gamma_0^2 + \gamma_1^2 > 0$, satisfy the sign condition

$$\beta_0\beta_1 \leq 0 \quad \text{and} \quad \gamma_0\gamma_1 \geq 0;$$

(A.4_{SD}) if $\frac{1}{a} \notin L^1(-1, 1)$ ⁽⁹⁾ and the function $\xi_a(x) := \int_0^x \frac{1}{a(s)} ds \in L^{2\vartheta-1}(-1, 1)$, in (1.1) let us consider the weighted Neumann boundary conditions.

Remark 1.1. The principal part of the operator in (1.1) coincides with that of the Budyko-Sellers climate model for $a(x) = 1 - x^2$. In this case $\frac{1}{1-x^2} \notin L^1(-1, 1)$, but $\xi_a(x) = \frac{1}{2} \log\left(\frac{1+x}{1-x}\right) \in L^p(-1, 1)$, for every $p \geq 1$, so this is an example of strongly degenerate equation, while an example of weakly degenerate coefficient is $a(x) = \sqrt{1-x^2}$.

Remark 1.2. We note that the inequalities (1.5) imply that

$$|f(x, t, u) - f(x, t, v)| \leq \nu(1 + |u|^{\vartheta-1} + |v|^{\vartheta-1})|u - v|, \quad \text{for a.e. } (x, t) \in Q_T, \forall u, v \in \mathbb{R}.$$

Moreover, under the further assumption $u \mapsto f(x, t, u)$ is locally absolutely continuous respect to u , the inequalities (1.5) in hypothesis (A.3) is equivalent to the following conditions on the function f

$$-\nu(1 + |u|^{\vartheta-1}) \leq f_u(x, t, u) \leq \nu, \quad \text{for a.e. } (x, t) \in Q_T, \forall u \in \mathbb{R},$$

thus

$$|f_u(x, t, u)| \leq \nu(1 + |u|^{\vartheta-1}), \quad \text{for a.e. } (x, t) \in Q_T, \forall u \in \mathbb{R}.$$

In order to clarify the previous Remark 1.2, we note that, since, for a.e. $(x, t) \in Q_T$, $f(x, t, u)$ is locally absolutely continuous respect to u , we have

$$(f(x, t, u) - f(x, t, v))(u - v) = (u - v) \int_v^u f_\xi(x, t, \xi) d\xi \leq (u - v) \int_v^u \nu d\xi \leq \nu(u - v)^2,$$

$$|f(x, t, u) - f(x, t, v)| \leq \int_{\min\{u, v\}}^{\max\{u, v\}} |f_\xi(x, t, \xi)| d\xi \leq \nu \int_{\min\{u, v\}}^{\max\{u, v\}} (1 + |\xi|^{\vartheta-1}) d\xi \leq \nu(1 + |u|^{\vartheta-1} + |v|^{\vartheta-1})|u - v|,$$

for a.e. $(x, t) \in Q_T$, for every $u, v \in \mathbb{R}$.

Example 1.3. An example of function f that satisfies the assumptions (A.3) is the following

$$f(x, t, u) = c(x, t) \min\{|u|^{\vartheta-1}, 1\}u - |u|^{\vartheta-1}u,$$

where c is a Lipschitz continuous function.

Remark 1.4. We note that system (1.1) cannot be steered anywhere from the origin. Moreover, in [29] it was proved that if $u_0(x) \geq 0$ in $(-1, 1)$ the respective solution to (1.1) remains nonnegative at any moment of time, regardless of the choice of $\alpha(x, t)$. This means that system (1.1) cannot be steered from any such u_0 to any target state which is negative on a nonzero measure set in the space domain.

1.5. Well-posedness

The well-posedness of the (SDP) problem (1.1) under the assumptions (A.1) – (A.4_{SD}) is obtained in [29], while the well-posedness of the (WDP) problem (1.1) under the assumptions (A.1) – (A.4_{WD}) is obtained in

⁹ We note that if $a \in C^1([-1, 1])$ follows $\frac{1}{a} \notin L^1(-1, 1)$.

with the following norm

$$\|u\|_{\mathcal{H}(Q_T)}^2 = \sup_{[0,T]} (\|u\|^2 + \|\sqrt{a}u_x\|^2) + \int_0^T (\|u_t\|^2 + \|(au_x)_x\|^2) dt. \quad (10)$$

Here we give the definition of “*strict solutions* to (1.1)” (introduced in [29] for (SDP) and in [30] for (WDP)), that is the notion of solution with initial state belongs to $H_a^1(-1, 1)$.

Definition 1.6. If $u_0 \in H_a^1(-1, 1)$, u is a *strict solution* to (1.1), if $u \in \mathcal{H}(Q_T)$ and

$$\left\{ \begin{array}{ll} u_t - (a(x)u_x)_x = \alpha(x, t)u + f(x, t, u) & \text{a.e. in } Q_T := (-1, 1) \times (0, T) \\ \left\{ \begin{array}{l} \beta_0 u(-1, t) + \beta_1 a(-1)u_x(-1, t) = 0 \\ \gamma_0 u(1, t) + \gamma_1 a(1)u_x(1, t) = 0 \end{array} \right. & \text{a.e. } t \in (0, T) \quad (\text{for WDP}) \\ \left\{ \begin{array}{l} a(x)u_x(x, t)|_{x=\pm 1} = 0 \\ u(x, 0) = u_0(x) \end{array} \right. & \text{a.e. } t \in (0, T) \quad (\text{for SDP}) \\ & x \in (-1, 1). \end{array} \right. \quad (11)$$

We proved the following result in [29] (see Appendix B) for (SDP) and in [30] for (WDP).

Proposition 1.7. For all $u_0 \in H_a^1(-1, 1)$ there exists a unique *strict solution* $u \in \mathcal{H}(Q_T)$ to (1.1).

Remark 1.8. In [29] (for the (SDP)) and in [30] (for the (WDP)), for initial data in $L^2(-1, 1)$ the notion of “*strong solutions*” was defined by approximation⁽¹²⁾. In this paper, we consider states continuous on the open interval $(-1, 1)$, then we use initial states in $H_a^1(-1, 1)$, so consequently we only consider the notion of “*strict solution*”.

2. MAIN RESULTS

Our main goal is to show that, given a initial datum $u_0 \in H_a^1(-1, 1)$ with a finite number of changes of sign, any target state $u^* \in H_a^1(-1, 1)$, with as many changes of sign *in the same order* (see Def. 2.2) as the given u_0 , can be approximately reached in the $L^2(-1, 1)$ -norm at some time $T > 0$, choosing suitable reaction coefficients. Thanks to this result, in Corollary 2.5 we easily show, by approximation argument, that the system (1.1) can be also steered toward the target states such that the amount of points of sign change is no more than the one of the given initial data. Now, we give some definitions to clarify and simplify the notation.

Definition 2.1. We say that $\bar{u} \in H_a^1(-1, 1)$ has n *points of sign change*, if there exist n points \bar{x}_l , $l = 1, \dots, n$, with

$$-1 < \bar{x}_1 < \dots < \bar{x}_n < 1$$

such that

$$\star \bar{u}(x) = 0, \quad x \in (-1, 1) \quad \iff \quad x = \bar{x}_l, \quad l = 1, \dots, n;$$

¹⁰ It's well known that this norm is equivalent to the Hilbert norm

$$\|u\|_{\mathcal{H}(Q_T)}^2 = \int_0^T (\|u\|^2 + \|\sqrt{a}u_x\|^2 + \|u_t\|^2 + \|(au_x)_x\|^2) dt.$$

¹¹ Since $u \in \mathcal{H}(Q_T) \subseteq L^2(0, T; D(A))$ we have $u(\cdot, t) \in D(A)$ for a.e. $t \in (0, T)$, so we deduce the associated boundary condition.

¹² The notions of “*strict/strong solutions*” are classical in PDEs theory, see, for instance, [6], pp. 62–64.

★ for $l = 1, \dots, n$,

$$\bar{u}(x)\bar{u}(y) < 0, \quad \forall x \in (\bar{x}_{l-1}, \bar{x}_l), \forall y \in (\bar{x}_l, \bar{x}_{l+1}),$$

where let us set $\bar{x}_0 := -1$ and $\bar{x}_{n+1} := 1$.

Definition 2.2. We say that $u_0, u^* \in H_a^1(-1, 1)$ have the n points of sign change in the same order, if denoting by x_l^0, x_l^* , $l = 1, \dots, n$, the zeros of u_0 and u^* , respectively, we have

$$u_0(x)u^*(y) > 0, \quad \forall x \in (x_{l-1}^0, x_l^0), \forall y \in (x_{l-1}^*, x_l^*), \text{ for } l = 1, \dots, n+1,$$

where let us set $x_0^0 = x_0^* = -1$ and $x_{n+1}^0 = x_{n+1}^* = 1$.

Definition 2.3. We say that a function $\alpha \in L^\infty(Q_T)$ is *piecewise static*, if there exist $m \in \mathbb{N}$, $c_k(x) \in L^\infty(-1, 1)$ and $t_k \in [0, T]$, $t_{k-1} < t_k$, $k = 1, \dots, m$ with $t_0 = 0$ and $t_m = T$, such that

$$\alpha(x, t) = c_1(x)\mathbb{1}_{[t_0, t_1]}(t) + \sum_{k=2}^m c_k(x)\mathbb{1}_{(t_{k-1}, t_k]}(t),$$

where $\mathbb{1}_{[t_0, t_1]}$ and $\mathbb{1}_{(t_{k-1}, t_k]}$ are the indicator function of $[t_0, t_1]$ and $(t_{k-1}, t_k]$, respectively.

Theorem 2.4. Let $u_0 \in H_a^1(-1, 1)$. Assume that u_0 has a finite number of points of sign change. Consider any $u^* \in H_a^1(-1, 1)$ which has exactly the same number of points of sign change in the same order as u_0 . Then, for any $\eta > 0$, there exists $T = T(\eta, u_0, u^*) > 0$ and a piecewise static multiplicative control $\alpha = \alpha(\eta, u_0, u^*) \in L^\infty(Q_T)$ such that the respective solution u to (1.1) satisfies

$$\|u(\cdot, T) - u^*\|_{L^2(-1, 1)} \leq \eta.$$

In Figure 1 we explain the statement of Theorem 2.4.

Further results

In the following, we derive two results that generalize Theorem 2.4.

Corollary 2.5. Let $u_0, u^* \in H_a^1(-1, 1)$. Assume that u_0 and u^* have finitely many points of sign change and the amount of points of sign change of u^* is less than the one of u_0 . Then, for any $\eta > 0$ there exist $T = T(\eta, u_0, u^*) > 0$ and a piecewise static multiplicative control $\alpha = \alpha(\eta, u_0, u^*) \in L^\infty(Q_T)$ such that the solution u to (1.1) satisfies

$$\|u(\cdot, T) - u^*\|_{L^2(-1, 1)} \leq \eta.$$

Proof of Corollary 2.5. Corollary 2.5 easily follows from Theorem 2.4. Indeed, all the target states described in Corollary 2.5 can be approximated in $L^2(-1, 1)$ by those in Theorem 2.4. \square

In the following Remark 2.6 we clarify the statement of Corollary 2.5.

Remark 2.6. We note that by Corollary 2.5 we can steer the system (1.1) from the initial state u_0 toward those states whose points of change of sign are organized in any order. We explain the statement of Corollary 2.5 by the following example. Let us denote by x_l^0 , $l = 1, \dots, n$ the points of sign change of u_0 . Let us consider an interval $(-1, x_1^0)$ of positive values of u_0 followed by an interval (x_1^0, x_2^0) of negative values of $u_0(x)$, which in turn is followed by an interval (x_2^0, x_3^0) of positive values of $u_0(x)$ and so forth. Then, the merging of the respective two points of sign change x_1^0 and x_2^0 will result in one single interval $(-1, x_3^0)$ of positive otherwise negative values.

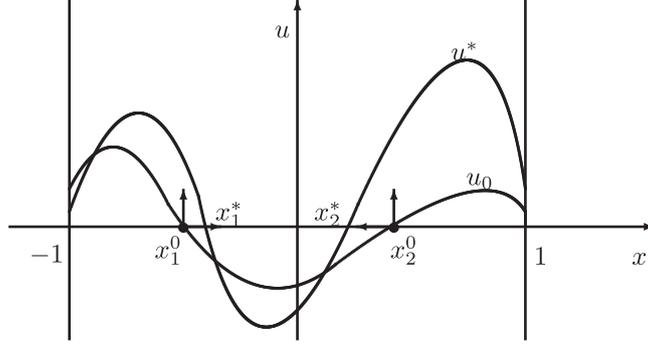


FIGURE 1. Control of two points of sign change.

In Figure 2 we describe one of the situations discussed in Remark 2.6 in the particular case $-1 = x_0^0 < x_1^0 < x_2^0 < x_3^0 = 1$.

The following approximate controllability property can be deduced from Corollary 2.5.

Corollary 2.7. *Let u_0 and u^* be given in $L^2(-1, 1)$. Then, for any $\eta > 0$ there exists $u_0^\eta \in H_a^1(-1, 1)$ such that $\|u_0^\eta - u_0\|_{L^2(-1, 1)} < \eta$, and there exist $T = T(\eta, u_0, u^*) > 0$ and a piecewise static multiplicative control $\alpha = \alpha(\eta, u_0, u^*) \in L^\infty(Q_T)$ such that the solution u to*

$$\begin{cases} u_t - (a(x)u_x)_x = \alpha(x, t)u + f(x, t, u) & \text{in } Q_T := (-1, 1) \times (0, T) \\ B.C. \\ u(0, x) = u_0^\eta \in H_a^1(-1, 1) & x \in (-1, 1) . \end{cases}$$

satisfies

$$\|u(\cdot, T) - u^*\|_{L^2(-1, 1)} \leq \eta.$$

Proof of Corollary 2.7. The proof of Corollary 2.7 is similar to one of Corollary 2.2 of [12]. \square

2.1. Control strategy for the proof of the main result (Thm. 2.4)

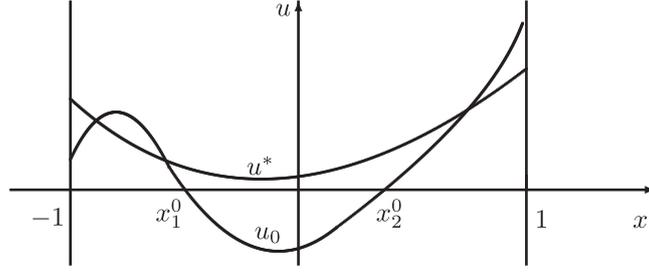
Let us consider the initial state $u_0 \in H_a^1(-1, 1)$ and the target state $u^* \in H_a^1(-1, 1)$. Both data have n points of sign change. Set $x_0^0 := -1$, $x_{n+1}^0 := 1$, and consider the set of points of sign change of u_0 , $X^0 = (x_1^0, \dots, x_n^0)$ where $-1 = x_0^0 < x_l^0 < x_{l+1}^0 \leq x_{n+1}^0 = 1$ for all $l = 1, \dots, n$. Similarly, let $x_0^* := -1$, $x_{n+1}^* := 1$, and consider the set of target points $X^* = (x_1^*, \dots, x_n^*)$, where $-1 = x_0^* < x_l^* < x_{l+1}^* \leq x_{n+1}^* = 1$ for all $l = 1, \dots, n$.

Some notations

Let us introduce some notations.

Notation for space intervals

Set $\rho_0^* = \min_{l=0, \dots, n} \{x_{l+1}^* - x_l^*, x_{l+1}^0 - x_l^0\}$, we define $a_0^* := -1 + \frac{\rho_0^*}{2}$ and $b_0^* := 1 - \frac{\rho_0^*}{2}$, then $(a_0^*, b_0^*) \subset (-1, 1)$.

FIGURE 2. u_0, u^* : merging of the points of change of sign.

Notation for time intervals

Given $N \in \mathbb{N}$, for every $(\tau_1, \dots, \tau_N) = (\tau_k)_1^N \in \mathbb{R}_+^N$, $(\sigma_1, \dots, \sigma_N) = (\sigma_k)_1^N \in \mathbb{R}_+^N$,⁽¹³⁾ we define

$$T_0 := 0, \quad S_k := T_{k-1} + \sigma_k, \quad T_k := S_k + \tau_k, \quad k = 1, \dots, N. \quad (14) \quad (2.1)$$

Noting that $0 = T_0 < S_k < T_k \leq T_N$, $k = 1, \dots, N$, we consider the following partition of $[0, T_N]$ in $2N$ intervals:

$$[0, T_N] = [0, S_1] \cup [S_1, T_1] \cup \dots \cup [T_{N-1}, S_N] \cup [S_N, T_N] = \bigcup_{k=1}^N (\mathcal{O}_k \cup \mathcal{E}_k), \quad (2.2)$$

where, for every $k = 1, \dots, N$, we have set $\mathcal{O}_k := [T_{k-1}, S_k]$ (k^{th} odd interval) and $\mathcal{E}_k := [S_k, T_k]$ (k^{th} even interval).

Notation for parabolic domains

For every $k = 1, \dots, N$, let us set $Q_{\mathcal{E}_k} := (-1, 1) \times [S_k, T_k]$ and $Q_{\mathcal{E}_k}^* := (a_0^*, b_0^*) \times [S_k, T_k] \subset Q_{\mathcal{E}_k}$.

Let $Q_{\mathcal{E}} := (-1, 1) \times \bigcup_{k=1}^N [S_k, T_k]$ and $Q_{\mathcal{E}}^* := (a_0^*, b_0^*) \times \bigcup_{k=1}^N [S_k, T_k] \subset Q_{\mathcal{E}}$.

Outline and main ideas for the proof of Theorem 2.4

The proof of Theorem 2.4 uses the partition introduced in (2.1)–(2.2) and two alternative control actions: on the even interval $\mathcal{E}_k = [S_k, T_k]$ we choose suitable initial data, w_k , in pure diffusion problems ($v \equiv 0$) as control parameters to move the points of sign change to their desired location (see Sect. 2.2), whereas on the odd interval $\mathcal{O}_k = [T_{k-1}, S_k]$ we give a smoothing result to preserve the reached points of sign change and attain such w_k 's as intermediate final conditions, using piecewise static multiplicative controls α_k , $\alpha_k \neq 0$ (see Sect. 2.3). The complete proof of Theorem 2.4 is achieved in Section 2.4. In Figure 3 we outline the iterative control strategy used to prove Theorem 2.4, for simplicity, in the case of two points of sign change and Dirichlet boundary conditions, that is, in the (WDP) case, with $\beta_1 = \gamma_1 = 0$.

¹³ $\mathbb{R}_+^N = \{(a_1, \dots, a_N) \mid a_k \in \mathbb{R}, a_k > 0, k = 1, \dots, N\}$.

¹⁴ We note that $T_0 = 0$, $S_1 = \sigma_1$, $T_1 = \sigma_1 + \tau_1$, and $S_k = \sum_{h=1}^{k-1} (\sigma_h + \tau_h) + \sigma_k$, $T_k = \sum_{h=1}^k (\sigma_h + \tau_h)$, $\forall k = 2, \dots, N$.

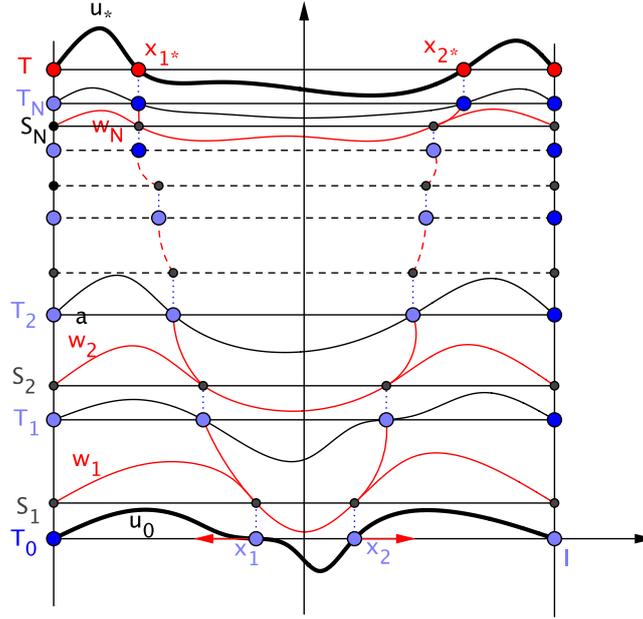


FIGURE 3. Iterative control strategy.

2.2. Controllability for initial-value pure diffusion problems

Let us fix a number $\beta \in (0, 1)$ to be used in whole the paper. Let $N \in \mathbb{N}$. For any fixed $(\sigma_1, \dots, \sigma_N) \in \mathbb{R}_+^N$, let us consider a generic $(\tau_1, \dots, \tau_N) \in \mathbb{R}_+^N$ and, recalling (2.1)–(2.2), let us introduce the following initial value pure diffusion problems on disjoint time intervals

$$\left\{ \begin{array}{l} w_t = (a(x)w_x)_x + f(x, t, w), \quad \text{in } Q_\varepsilon = (-1, 1) \times \bigcup_{k=1}^N [S_k, T_k] \\ \left\{ \begin{array}{l} \beta_0 w(-1, t) + \beta_1 a(-1)w_x(-1, t) = 0 \\ \gamma_0 w(1, t) + \gamma_1 a(1)w_x(1, t) = 0 \end{array} \right. \quad t \in \bigcup_{k=1}^N (S_k, T_k) \quad (\text{for } WDP) \\ a(x)w_x(x, t)|_{x=\pm 1} = 0 \quad t \in \bigcup_{k=1}^N (S_k, T_k) \quad (\text{for } SDP) \\ w|_{t=S_k} = w_k(x), \quad x \in (-1, 1), \quad k = 1, \dots, N. \end{array} \right. \quad (2.3)$$

Let us suppose that the assumptions (A.3) and (A.4) hold. Moreover, we will consider the initial data w_k and times τ_k ($\tau_k = T_k - S_k$), $k = 1, \dots, N$, as control parameters, where the w_k 's belong to $H_a^1(-1, 1) \cap C^{2+\beta}([a_0^*, b_0^*])$ (15), with $a_0^* = -1 + \frac{\rho_0^*}{2}$ and $b_0^* = 1 - \frac{\rho_0^*}{2}$ ($\rho_0^* = \min_{l=0, \dots, n} \{x_{l+1}^* - x_l^*, x_{l+1}^0 - x_l^0\}$).

¹⁵ We recall the following spaces of Hölder continuous functions (see also [31]):

$$C^\beta([a_0^*, b_0^*]) := \left\{ w \in C([a_0^*, b_0^*]) : \sup_{x, y \in [a_0^*, b_0^*]} \frac{|w(x) - w(y)|}{|x - y|^\beta} < +\infty \right\},$$

$$C^{2+\beta}([a_0^*, b_0^*]) := \left\{ w \in C^2([a_0^*, b_0^*]) : w'' \in C^\beta([a_0^*, b_0^*]) \right\}.$$

Definition 2.8. We call solution of (2.3) the function defined in $(-1, 1) \times \bigcup_{k=1}^N [S_k, T_k]$ as

$$w(x, t) = W_k(x, t), \quad \forall (x, t) \in (-1, 1) \times [S_k, T_k], \quad k = 1, \dots, N,$$

where W_k is the unique strict solution on $(-1, 1) \times [S_k, T_k]$ of the k^{th} problem in (2.3).

Remark 2.9. We observe that a solution of (2.3) is a collection of solutions of a finite number of problems which are set on disjoint time intervals. Therefore, it is independent of the choice of $(\sigma_k)_1^N$. We prefer to give the following Definition 2.10 for a fixed $(\sigma_k)_1^N$, just for technical purposes that will be clear in the sequel (see Theorem 2.11).

Definition 2.10. Let $u_0 \in H_a^1(-1, 1)$ be a function with the n points of sign change x_l^0 , $l = 1, \dots, n$. For every fixed $N \in \mathbb{N}$ and $(\sigma_k)_1^N \in \mathbb{R}_+^N$, we call a finite ‘‘family of Times and Initial Data’’ of (2.3) associated with u_0 , a set of the form $\{(\tau_k)_1^N, (w_k)_1^N\}$ such that

- ★ $(\tau_k)_1^N \in \mathbb{R}_+^N$;
- ★ for all $k = 1, \dots, N$, $w_k \in H_a^1(-1, 1) \cap C^{2+\beta}([a_0^*, b_0^*])$ satisfies the following:
 1. w_1 and u_0 have the same points of sign change, in the same order as the points of sign change of u_0 ;
 2. for $k = 2, \dots, N$, $w_k(\cdot)$ and $w(\cdot, T_{k-1})$ have the same points as the points of sign change, in the same order of sign change of u_0 , where w is the solution of (2.3) on $(-1, 1) \times \bigcup_{h=1}^{k-1} [S_h, T_h]$.

All Section 3 of this paper is devoted to the proof of the following Theorem 2.11.

Theorem 2.11. Let $u_0 \in H_a^1(-1, 1)$ have n points of sign change at $x_l^0 \in (-1, 1)$, $l = 1, \dots, n$ with

$$-1 := x_0^0 < x_l^0 < x_{l+1}^0 \leq x_{n+1}^0 := 1, \quad l = 1, \dots, n.$$

Let $x_l^* \in (-1, 1)$, $l = 1, \dots, n$, be such that $-1 := x_0^* < x_l^* < x_{l+1}^* \leq x_{n+1}^* := 1$. Then, for every $\varepsilon > 0$ there exist $N_\varepsilon \in \mathbb{N}$ and a finite family of times and initial data $\{(\tau_k)_1^{N_\varepsilon}, (w_k)_1^{N_\varepsilon}\}$ such that, for any $(\sigma_k)_1^{N_\varepsilon} \in \mathbb{R}_+^{N_\varepsilon}$, the solution w^ε of problem (2.3) satisfies

$$w^\varepsilon(x, T_{N_\varepsilon}) = 0 \quad \iff \quad x = x_l^\varepsilon, \quad l = 1, \dots, n,$$

for some points $x_l^\varepsilon \in (-1, 1)$, with $-1 := x_0^\varepsilon < x_l^\varepsilon < x_{l+1}^\varepsilon \leq x_{n+1}^\varepsilon := 1$ for $l = 1, \dots, n$, such that

$$\sum_{l=1}^n |x_l^* - x_l^\varepsilon| < \varepsilon.$$

Moreover, $w^\varepsilon(\cdot, T_{N_\varepsilon})$ has the same order of sign change as u_0 .

In Section 2.4 we will use Theorem 2.11 to prove Theorem 2.4.

2.3. A control result to preserve the reached points of sign change and to obtain suitable smooth intermediate data

In this section we introduce a smoothing result to preserve the reached points of sign change and attain smooth intermediate final conditions w_k 's.

Let $N \in \mathbb{N}$. For any fixed $(\tau_1, \dots, \tau_N) \in \mathbb{R}_+^N$, let us consider a generic $(\sigma_1, \dots, \sigma_N) \in \mathbb{R}_+^N$ and, for $k = 1, \dots, N$, recalling (2.1)–(2.2), given $u_{k-1}, r_{k-1} \in H_a^1(-1, 1)$, $\alpha_k \in L^\infty((-1, 1) \times [T_{k-1}, S_k])$, let us introduce the following problem

$$\left\{ \begin{array}{l} u_t = (a(x)u_x)_x + \alpha_k(x, t)u + f(x, t, u) \quad \text{in } Q_{\mathcal{O}_k} = (-1, 1) \times [T_{k-1}, T_{k-1} + \sigma_k], \\ \left\{ \begin{array}{l} \beta_0 u(-1, t) + \beta_1 a(-1)u_x(-1, t) = 0 \quad t \in (T_{k-1}, T_{k-1} + \sigma_k) \\ \gamma_0 u(1, t) + \gamma_1 a(1)u_x(1, t) = 0 \quad t \in (T_{k-1}, T_{k-1} + \sigma_k) \\ a(x)u_x(x, t)|_{x=\pm 1} = 0 \quad t \in (T_{k-1}, T_{k-1} + \sigma_k) \end{array} \right. \quad \text{(for WDP)} \\ u|_{t=T_{k-1}} = u_{k-1} + r_{k-1} \in H_a^1(-1, 1), \quad \text{(for SDP)} \end{array} \right. \quad (2.4)$$

where we recall that $S_k = T_{k-1} + \sigma_k$, and we suppose that the assumptions (A.3) and (A.4) hold. All Section 4 of this paper is devoted to the proof of the following Theorem 2.12.

Theorem 2.12. *Let $u_{k-1}, r_{k-1}, w_k \in H_a^1(-1, 1)$. Let u_{k-1} and w_k have the same n points of sign change, in the same order. Then, for every $\eta > 0$ there exist $\sigma_k > 0$, $C_k \geq 1$, and a piecewise static control $\alpha_k \in L^\infty((-1, 1) \times (T_{k-1}, S_k))$ (depending only on η, u_{k-1} , and w_k) such that*

$$\|U_k(\cdot, S_k) - w_k(\cdot)\|_{L^2(-1, 1)} \leq \eta + C_k \|r_{k-1}\|_{L^2(-1, 1)},$$

where U_k is the solution of (2.4) on $(-1, 1) \times [T_{k-1}, S_k]$.

In Section 2.4 we will use Theorem 2.12 to prove Theorem 2.4.

Remark 2.13. We note that Theorem 2.12, in the particular case $r_{k-1} = 0$, gives the approximate controllability in the subspace of states with the same changes of sign, in the same order of sign change.

2.4. Proof of Theorem 2.4

As soon as we prove Theorem 2.11 and Theorem 2.12, combining these two results the proof of Theorem 2.4 is easily obtained, using an idea introduced in Section 3.3 of [12], through an intermediate result, Lemma 2.14 (this lemma is similar to Lemma 3.1 of [12]). In this section we will avoid some repetitions, so we put only a sketch of the iterative idea and we invite the reader to see Section 3.3 of [12], that contains every technical detail.

Lemma 2.14. *Let $u_0 \in H_a^1(-1, 1)$ be a function with n points of sign change, let $\{(\tau_k)_1^N, (w_k)_1^N\}$ be a finite family of times and initial data of (2.3) associated with u_0 , and let*

$w : (-1, 1) \times \bigcup_{k=1}^N [S_k, T_k] \rightarrow \mathbb{R}$ be the solution of (2.3). Then for every $\delta > 0$ there exists $\sigma_\delta = (\sigma_k)_1^N \in \mathbb{R}_+^N$ and $\alpha_\delta \in L^\infty((-1, 1) \times (0, T_N))$ such that, denoting by $u_\delta : (0, 1) \times [0, T_N] \rightarrow \mathbb{R}$ the solution of (1.1) with bilinear control α_δ , we have

$$\|u_\delta(\cdot, T_k) - w(\cdot, T_k)\|_{L^2(-1, 1)} \leq \delta, \quad \forall k = 1, \dots, N. \quad (2.5)$$

Sketch of the proof of Lemma 2.14. Fix $\{(\tau_k)_1^N, (w_k)_1^N\}$ and $\delta > 0$. Let us consider the partition of $[0, T_N]$ in $2N$ intervals introduced in (2.1)–(2.2). In particular, we will show that the bilinear control α_δ has the following

expression

$$\alpha_\delta(x, t) = \begin{cases} \alpha_k^\delta(x, t) & \text{in } Q_{\mathcal{O}_k} = (-1, 1) \times [T_{k-1}, S_k], \quad k = 1, \dots, N, \\ 0 & \text{in } Q_{\mathcal{E}_k} = (-1, 1) \times [S_k, T_k], \quad k = 1, \dots, N. \end{cases}$$

In the following, for every $k = 1, \dots, N$, we will consider the following problem on $Q_{\mathcal{E}_k}$

$$\begin{cases} u_t = (a(x)u_x)_x + f(x, t, u), & \text{in } Q_{\mathcal{E}_k} = (-1, 1) \times [S_k, T_k], \\ B.C. \\ u|_{t=S_k} = w_k + p_k, \end{cases}$$

where $w_k \in H_a^1(-1, 1) \cap C^{2+\beta}([a_0^*, b_0^*])$, $p_k \in H_a^1(-1, 1)$ are given functions, and we will represent its solution as the sum of two functions $w(x, t)$ and $h(x, t)$, which solve the following problems in $Q_{\mathcal{E}_k}$

$$\begin{cases} w_t = (a(x)w_x)_x + f(x, t, w), \\ B.C. \\ w|_{t=S_k} = w_k, \end{cases} \quad \begin{cases} h_t = (a(x)h_x)_x + (f(x, t, w+h) - f(x, t, w)), \\ B.C. \\ h|_{t=S_k} = p_k. \end{cases} \quad (2.6)$$

Multiplying by h the equation of the second problem of (2.6) and integrating by parts over $Q_{\mathcal{E}_k}$, since $[a(x)h_x(x, t)h(x, t)]_{-1}^1 \leq 0$, by (1.5) we obtain

$$\begin{aligned} \int_{-1}^1 h^2(x, T_k) dx &\leq \int_{-1}^1 p_k^2(x) dx + 2 \int_{S_k}^{T_k} \int_{-1}^1 (f(x, t, w+h) - f(x, t, w)) h \, dx dt \\ &\leq \int_{-1}^1 p_k^2(x) dx + 2\nu \int_{S_k}^{T_k} \int_{-1}^1 h^2 dx dt, \quad t \in (S_k, T_k), \end{aligned}$$

thus applying Grönwall's inequality we deduce

$$\|h(\cdot, T_k)\|_{L^2(-1, 1)} \leq e^{\nu \tilde{T}} \|p_k\|_{L^2(-1, 1)}, \quad \text{with } \tilde{T} := \sum_{k=1}^N \tau_k, \quad k = 1, \dots, N. \quad (2.7)$$

Using the energy estimate (2.7) we can conclude this proof proceeding exactly with the same technical and iterative proof of Lemma 3.1 of [12]. \square

3. PROOF OF THEOREM 2.11

In this section we refer to the notation introduced in Section 2.2. The plan of this section is as follows.

In Section 3.1, we start by a regularity result for the problem (2.3), contained in Proposition 3.1. By Lemma 3.2 we give the existence of suitable *initial data* w_k 's to be used in the proof of Theorem 2.11. By Lemma 3.3 we construct the n *curves of sign change* associated with the n initial points of sign change.

In Section 3.2, we construct a suitable particular family of *times and initial data*, that allows to move the n initial points of sign change towards the n target points of sign change. In this section, we also introduce the definitions of *gap and target distance functional*.

In Section 3.3, after obtaining Proposition 3.9, we show how to steer the points of sign change of the solution arbitrarily close to the target points.

3.1. Preliminary results

Let us prove the following Proposition 3.1.

Proposition 3.1. *Let $[a_0^*, b_0^*] \subset (-1, 1)$, let $k = 1, \dots, N$. If $w_k \in H_a^1(-1, 1) \cap C^{2+\beta}([a_0^*, b_0^*])$, then the k^{th} initial-value problem in (2.3) has a unique strict solution $W_k(x, t)$ on $\overline{Q_{\mathcal{E}_k}}$ and*

$$W_k \in \mathcal{H}(Q_{\mathcal{E}_k}) \cap C^{2+\beta, 1+\beta/2}(\overline{Q_{\mathcal{E}_k}^*}). \quad (16)$$

Proof. We note that our assumptions permit to apply a well known *interior regularity* result, contained in Section 5 and 6 of Chapter V in [35] (in particular see Theorem 5.4, pp. 448–449, and Theorem 6.1, pp. 452–453). Indeed, on the domain $Q_{\mathcal{E}_k}^* = (a_0^*, b_0^*) \times [S_k, T_k] \subset \subset Q_{\mathcal{E}_k}$ the equation in (2.3) is uniformly parabolic, thus the unique strict solution of the k^{th} problem in (2.3), $W_k \in \mathcal{H}(Q_{\mathcal{E}_k})$, is bounded on $Q_{\mathcal{E}_k}^*$, therefore there exists a positive constant M_k , such that

$$|W_k(x, t)| \leq M_k, \quad \text{for a.e. } (x, t) \in Q_{\mathcal{E}_k}^*.$$

Let us set

$$f_{M_k}(x, t, w) := \begin{cases} f(x, t, w), & \text{if } |w| \leq M_k, \\ f(x, t, M_k), & \text{if } w > M_k, \\ f(x, t, -M_k), & \text{if } w < -M_k. \end{cases}$$

Thus, by the inequalities (1.5) and Remark 1.2 we deduce that, for a.e. $(x, t) \in Q_T, \forall w_1, w_2 \in \mathbb{R}$,

$$|f_{M_k}(x, t, w_1) - f_{M_k}(x, t, w_2)| \leq \nu(1 + |M_k|^{\vartheta-1} + |M_k|^{\vartheta-1})|w_1 - w_2| = L(\vartheta, M_k)|w_1 - w_2|,$$

where $L(\vartheta, M_k) := \nu(1 + 2|M_k|^{\vartheta-1})$, and ν is the constant of (1.5). Then, $w \mapsto f_{M_k}(x, t, w)$ is a Lipschitz continuous function on $Q_{\mathcal{E}_k}^*$, and we can apply the aforementioned interior regularity result of [35] to the problem

$$\begin{cases} w_t = (a(x)w_x)_x + f_{M_k}(x, t, w), & \text{in } Q_{\mathcal{E}_k}^* = (a_0^*, b_0^*) \times [S_k, T_k], \\ w|_{t=S_k} = w_k(x), \end{cases}$$

thus, the unique solution W_k belongs to $C^{2+\beta, 1+\beta/2}(\overline{Q_{\mathcal{E}_k}^*})$. \square

By a simple exercise we can obtain the following.

Lemma 3.2 (Existence of suitable initial data w_k 's). *Let $x_l \in (-1, 1)$, $l = 1, \dots, n$, be such that $-1 := x_0 < x_1 < x_{l+1} \leq x_{n+1} := 1$, $l = 1, \dots, n$. Let $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$, $(\omega_1, \dots, \omega_n) \in \mathbb{R}^n$ be such that $\lambda_l \in$*

¹⁶ Let $Q_{\mathcal{E}_k}^* = (a_0^*, b_0^*) \times [S_k, T_k]$. We define

$$\mathcal{H}(Q_{\mathcal{E}_k}) := L^2(S_k, T_k; H_a^2(-1, 1)) \cap H^1(S_k, T_k; L^2(-1, 1)) \cap C([S_k, T_k]; H_a^1(-1, 1)),$$

$$C^{\beta, \frac{\beta}{2}}(\overline{Q_{\mathcal{E}_k}^*}) := \left\{ u \in C(\overline{Q_{\mathcal{E}_k}^*}) : \sup_{x, y \in [a_0^*, b_0^*]} \frac{|u(x, t) - u(y, t)|}{|x - y|^\beta} + \sup_{t, s \in [S_k, T_k]} \frac{|u(x, t) - u(x, s)|}{|t - s|^{\frac{\beta}{2}}} < +\infty \right\},$$

$$C^{2, 1}(\overline{Q_{\mathcal{E}_k}^*}) := \left\{ u : \overline{Q_{\mathcal{E}_k}^*} \rightarrow \mathbb{R} : \exists u_{xx}, u_x, u_t \in C(\overline{Q_{\mathcal{E}_k}^*}) \right\},$$

$$C^{2+\beta, 1+\frac{\beta}{2}}(\overline{Q_{\mathcal{E}_k}^*}) := \left\{ u \in C^{2, 1}(\overline{Q_{\mathcal{E}_k}^*}) : u_{xx}, u_x, u_t \in C^{\beta, \frac{\beta}{2}}(\overline{Q_{\mathcal{E}_k}^*}) \right\}.$$

$\{-1, 1\}$, $\omega_l \in \{-1, 0, 1\}$, $l = 1, \dots, n$, and $\lambda_l \lambda_{l+1} < 0$, $l = 1, \dots, n-1$. Let $\tilde{\rho} = \min_{l=0, \dots, n} \{x_{l+1} - x_l\}$, $\tilde{a} := -1 + \frac{\tilde{\rho}}{2}$ and $\tilde{b} := -1 + \frac{\tilde{\rho}}{2}$. Then, there exists $w \in H_a^1(-1, 1) \cap C^\infty([\tilde{a}, \tilde{b}])$ such that

- ★ $w(x) = 0 \iff x = x_l$, $l = 1, \dots, n$;
- ★ $w'(x_l) = \lambda_l$, $w''(x_l) = \omega_l$, $l = 1, \dots, n$;
- ★ $\|w\|_{C^m([\tilde{a}, \tilde{b}])} \leq C(m, \tilde{\rho})$, $\forall m \in \mathbb{N}$.

Lemma 3.3 (Construction of the curves of sign change). Let $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ be such that $\lambda_l \in \{-1, 1\}$, $l = 1, \dots, n$, and $\lambda_l \lambda_{l+1} < 0$, $l = 1, \dots, n-1$. Let $\tilde{\rho} > 0$ and let $x_l \in (-1, 1)$, $l = 1, \dots, n$, be such that $-1 := x_0 < x_l < x_{l+1} \leq x_{n+1} := 1$, $l = 1, \dots, n$, and $\min_{l=0, \dots, n} \{x_{l+1} - x_l\} = \tilde{\rho}$. Let $a_0^* := -1 + \frac{\tilde{\rho}}{2}$ and $b_0^* := 1 - \frac{\tilde{\rho}}{2}$.

Let $w_k \in H_a^1(-1, 1) \cap C^{2+\beta}([a_0^*, b_0^*])$ be such that

- ★ $w_k(x) = 0 \iff x = x_l$, $l = 1, \dots, n$;
- ★ $w_k'(x_l) = \lambda_l$, $l = 1, \dots, n$;
- ★ $\|w_k\|_{C^{2+\beta}([a_0^*, b_0^*])} \leq c$, for some positive constant $c = c(\tilde{\rho})$.

Let $T > 0$ and let w be the solution of

$$\begin{cases} w_t = (a(x)w_x)_x + f(x, t, w) & \text{in } Q_T = (-1, 1) \times (0, T) \\ B.C. & \\ w(x, 0) = w_k(x) & x \in (-1, 1). \end{cases} \quad (17) \quad (3.1)$$

Then, for every $\rho \in (0, \tilde{\rho}]$ there exist $\tilde{\tau} = \tilde{\tau}(\rho) > 0$ and $M = M(\rho) > 0$ such that, for each $l = 1, \dots, n$, there exists a unique solution $\xi_l : [0, \tilde{\tau}] \rightarrow \mathbb{R}$ of the initial-value problem

$$\begin{cases} \dot{\xi}_l(t) = -\frac{a(\xi_l(t))w_{xx}(\xi_l(t), t)}{w_x(\xi_l(t), t)} - a'(\xi_l(t)), & t \in [0, \tilde{\tau}], \\ \xi_l(0) = x_l, \end{cases}$$

that satisfies

- $w(\xi_l(t), t) = 0$, $\forall t \in [0, \tilde{\tau}]$;
- $\xi_l \in C^{1+\frac{\beta}{2}}([0, \tilde{\tau}])$ and $\|\xi_l\|_{C^{1+\frac{\beta}{2}}([0, \tilde{\tau}])} \leq M$;
- $\|\xi_l(\cdot) - x_l\|_{C([0, \tilde{\tau}])} < \frac{\rho}{2}$.

Definition 3.4. We call the functions $\xi_l : [0, \tilde{\tau}] \rightarrow \mathbb{R}$, $l = 1, \dots, n$, given by Lemma 3.3, Curves of Sign Change associated with the set of initial points of sign change $X = (x_1, \dots, x_n)$.

Remark 3.5. As consequence of Lemma 3.3, since $\|\xi_l(\cdot) - x_l\|_{C([0, \tilde{\tau}])} < \frac{\rho}{2}$ for each $l = 1, \dots, n$, we have

$$a_0^* < \xi_l(t) < \xi_{l+1}(t) < b_0^*, \quad \forall t \in [0, \tilde{\tau}], \quad \forall l = 1, \dots, n-1.$$

Therefore, two adjacent curves of sign change don't intersect, so they remain separated.

Proof of Lemma 3.3. Let us fix $\rho \in (0, \tilde{\rho}]$. Due to Proposition 3.1, the solution w of (3.1) is such that

$$w \in C^{2+\beta, 1+\frac{\beta}{2}}(\overline{(a_0^*, b_0^*) \times (0, T)}) \quad \text{and} \quad \|w\|_{C^{2+\beta, 1+\frac{\beta}{2}}(\overline{(a_0^*, b_0^*) \times (0, T)})} \leq K,$$

¹⁷ For the existence, uniqueness and regularity of problem (3.1) see Proposition 3.1.

for some positive constant $K = K(\|w_k\|_{C^{2+\beta}([a_0^*, b_0^*])})$ (see (6.8)-(6.12) on pp. 451-452 in [35]). Thus, since $\|w_k\|_{C^{2+\beta}([a_0^*, b_0^*])} \leq c(\tilde{\rho})$, we have that

$$\|w\|_{C^{2+\beta, 1+\frac{\beta}{2}}(\overline{Q_T})} \leq K(\|w_k\|_{C^{2+\beta}([a_0^*, b_0^*])}) \leq C, \quad (3.2)$$

for some positive constant $C = C(\tilde{\rho})$.

Existence and regularity of curves of sign change. For any fixed $l = 1, \dots, n$, since $w_x(x_l, 0) = \lambda_l \neq 0$ and $w_x(x, t)$ is a continuous function in $(x_l, 0) \in (a_0^*, b_0^*) \times (0, T)$, there exist $\delta_l \in (0, \min\{\frac{1}{2C}, \rho\})$ ⁽¹⁸⁾ and $T_l > 0$ such that $w_x(x, t) \neq 0, \forall (x, t) \in [x_l - \delta_l, x_l + \delta_l] \times [0, T_l]$.

For every $l = 1, \dots, n$, we consider the Cauchy problems

$$\begin{cases} \dot{\xi}_l(t) = -\frac{w_t(\xi_l(t), t)}{w_x(\xi_l(t), t)}, & t > 0, \\ \xi_l(0) = x_l. \end{cases} \quad (3.3)$$

Let $\delta := \min_{l=1, \dots, n} \delta_l$, we note that $G(x, t) := -\frac{w_t(x, t)}{w_x(x, t)}$ is continuous on $[x_l - \delta, x_l + \delta] \times [0, T_l]$. Therefore, for every $l = 1, \dots, n$, the problem (3.3) has a solution ξ_l of class C^1 on some interval $[0, \tau_l]$, with $0 < \tau_l \leq T_l$. Let $\tau := \min_{l=1, \dots, n} \tau_l$, since $G \in C^{\frac{\beta}{2}}([x_l - \delta, x_l + \delta] \times [0, T_l])$, we conclude that $\xi_l \in C^{1+\frac{\beta}{2}}([0, \tau])$ and there exists $M = M(\rho) > 0$ such that $\|\xi_l\|_{C^{1+\frac{\beta}{2}}([0, \tau])} \leq M$.

Moreover, since by (3.3) we deduce that

$$\frac{d}{dt} w(\xi_l(t), t) = w_t(\xi_l(t), t) + w_x(\xi_l(t), t) \dot{\xi}_l(t) = 0, \forall t \in [0, \tau], \quad \text{and } w(\xi_l(0), 0) = w_k(x_l) = 0,$$

we obtain that

$$w(\xi_l(t), t) = 0, \quad \forall t \in [0, \tau].$$

Furthermore, since by (1.4) $f(w(\xi_l(t)), t) = 0, \forall t \in [0, \tau]$, we also have that, for every $t \in [0, \tau]$,

$$\dot{\xi}_l(t) = -\frac{w_t(\xi_l(t), t)}{w_x(\xi_l(t), t)} = -\frac{(a(\xi_l(t))w_x(\xi_l(t), t))_x}{w_x(\xi_l(t), t)} = -\left[a'(\xi_l(t)) + \frac{a(\xi_l(t))w_{xx}(\xi_l(t), t)}{w_x(\xi_l(t), t)} \right].$$

Uniform estimates for the curves of sign change. For any fixed $l = 1, \dots, n$, we consider the uniform time $\tilde{\tau} := \min\left\{\left(\frac{1}{2C} - \delta\right)^{\frac{2}{\beta}}, \frac{\delta^2}{3}, \tau\right\}$, $\tilde{\tau} = \tilde{\tau}(\rho)$ (independent of l) since $\delta < \min\{\frac{1}{2C}, \rho\}$. Recall that the function $t \mapsto w_x(x, t)$ belongs to $C^{\frac{\beta}{2}}([0, \tilde{\tau}])$ and the function $x \mapsto w_x(x, t)$ belongs to $C^{1+\beta}([0, \tilde{\tau}])$. Thus, for every $(x, t) \in (x_l - \delta, x_l + \delta) \times (0, \tilde{\tau})$, by (3.2) we have

$$\begin{aligned} |w_x(x, t) - \lambda_l| &\leq |w_x(x, t) - w_x(x, 0)| + |w_x(x, 0) - w_x(x_l, 0)| \\ &\leq \|w\|_{C^{2+\beta, 1+\frac{\beta}{2}}(\overline{Q_T})} (t^{\frac{\beta}{2}} + |x - x_l|) \leq C(\tilde{\tau}^{\frac{\beta}{2}} + \delta). \end{aligned} \quad (3.4)$$

Since $\delta < \frac{1}{2C}$ and $\tilde{\tau} \leq \left(\frac{1}{2C} - \delta\right)^{\frac{2}{\beta}}$, we deduce $C(\tilde{\tau}^{\frac{\beta}{2}} + \delta) \leq \frac{1}{2}$, so by (3.4) we obtain

$$\left| |w_x(x, t)| - |\lambda_l| \right| \leq |w_x(x, t) - \lambda_l| \leq \frac{1}{2}.$$

¹⁸ C is the constant present in (3.2).

Therefore, for every $l = 1, \dots, n$, having in mind that $|\lambda_l| = 1$, we have

$$|w_x(x, t)| \geq |\lambda_l| - \frac{1}{2} = \frac{1}{2}, \quad \forall (x, t) \in (x_l - \delta, x_l + \delta) \times (0, \tilde{\tau}). \quad (3.5)$$

Then, by (3.2) and (3.5), keeping in mind that $\tilde{\tau} \leq \delta^2/3$ and $\delta < \min\{\frac{1}{2C}, \rho\}$, for every $t \in [0, \tilde{\tau}]$, we deduce

$$|\xi_l(t) - x_l| = \left| \int_0^t \dot{\xi}_l(s) ds \right| \leq \int_0^{\tilde{\tau}} \frac{|w_{xx}(\xi_l(s), s)|}{|w_x(\xi_l(s), s)|} ds \leq 2\tilde{\tau} \|w\|_{C^{2+\beta, 1+\frac{\beta}{2}}(\bar{Q}_T)} \leq 2\tilde{\tau} C < \frac{\tilde{\tau}}{\delta} < \frac{\rho}{3}. \quad (3.6)$$

Uniqueness “a posteriori” of the curves of sign change. We observe that, although one cannot claim uniqueness for the Cauchy problem (3.3), *a posteriori* the ξ_l 's turn out to be uniquely determined. Indeed, let $\bar{a} \in (-1, a_0^*]$ and $\bar{b} \in [b_0^*, 1)$, setting $\xi_0(t) \equiv \bar{a}$, $\xi_{n+1}(t) \equiv \bar{b}$, $\forall t \in [0, \tilde{\tau}]$, since by (3.6) and Remark 3.5 one can apply the strong maximum principle for uniformly parabolic equations on the domains $\{(x, t) | x \in [\xi_l(t), \xi_{l+1}(t)], t \in [0, \tilde{\tau}]\}$, for every $l = 0, \dots, n$. The fact that the initial datum $w_k(x)$ doesn't change sign on (x_l, x_{l+1}) implies that (thanks to the fact that \bar{a} and \bar{b} are arbitrary), for every $t^* \in [0, \tilde{\tau}]$,

$$w(x, t^*) = 0 \iff x = \xi_l(t^*), \quad l = 0, \dots, n+1,$$

completing the proof of Lemma 3.3. □

3.2. Construction of order processing steering sets

In the following we define *Order Processing Steering Times and Initial Data* that permit to move the points of sign change towards the desired targets. In this section we use the notation introduced in Section 2.1 and in Section 2.2.

Given the initial state $u_0 \in H_a^1(-1, 1)$, let us consider the n points of sign change of u_0 , $X^0 = (x_1^0, \dots, x_n^0)$, where $-1 := x_0^0 < x_l^0 < x_{l+1}^0 \leq x_{n+1}^0 := 1$, for $l = 1, \dots, n$. Let us define, for every $l = 1, \dots, n$,

$$\lambda(x_l^0) = \begin{cases} 1, & \text{if } u_0(x) > 0 \text{ on } (x_l^0, x_{l+1}^0), \\ -1, & \text{if } u_0(x) < 0 \text{ on } (x_l^0, x_{l+1}^0). \end{cases} \quad (3.7)$$

Since x_l^0 , $l = 1, \dots, n$, are points of sign change, we note that $\lambda(x_{l+1}^0) = -\lambda(x_l^0)$, $l = 1, \dots, n-1$.

Let us set $x_0^* := -1$ and $x_{n+1}^* := 1$, and let us consider the set of n target points $X^* = (x_1^*, \dots, x_n^*)$, where $-1 = x_0^* < x_l^* < x_{l+1}^* \leq x_{n+1}^* = 1$, $l = 1, \dots, n$.

Let $\rho_0^* = \min_{l=0, \dots, n} \{x_{l+1}^0 - x_l^0, x_{l+1}^* - x_l^*\}$, then let us set $a_0^* := -1 + \frac{\rho_0^*}{2}$ and $b_0^* := 1 - \frac{\rho_0^*}{2}$.

Order processing steering times and initial data

Let $\beta \in (0, 1)$ be the number that was fixed at the beginning of Section 2.2. Let $N \in \mathbb{N}$ and let us fix $(\sigma_k)_1^N \in \mathbb{R}_+^N$. Now, we will construct a *finite a family of Times and Initial Data* for (2.3) associated with u_0 , $\{(\tau_k)_1^N, (w_k)_1^N\}$ (see Def. 2.10), in order to move the points of sign change towards the desired targets. We denote by $\mathcal{W}^*(u_0)$ the subclass of such special families, which we call *Order Processing Steering Times and Initial Data* associated with u_0 and X^* .

In the following, we will define the times S_k, T_k , $k = 1, \dots, N$, in the same way of (2.1).

Construction of $\{\tau_1, w_1\}$.

By Lemma 3.2, there exists $w_1 \in H_a^1(-1, 1) \cap C^{2+\beta}([a_0^*, b_0^*])$, with $\|w_1\|_{C^{2+\beta}([a_0^*, b_0^*])} \leq c_1$, for some positive constant $c_1 = c(\rho_0^*)$, such that

$$\begin{aligned} \star w_1(x) = 0 &\iff x = x_l^0, \quad l = 1, \dots, n; \\ \star w_1'(x_l^0) = \lambda(x_l^0)a(x_l^0), \quad w_1''(x_l^0) = -\lambda(x_l^0) [\mu_1(x_l^* - x_l^0) + a'(x_l^0)], \quad l = 1, \dots, n, \end{aligned}$$

$$\text{where } \mu_1(x_l^* - x_l^0) = \text{sgn}(x_l^* - x_l^0) = \begin{cases} 1, & \text{if } x_l^0 < x_l^*, \\ 0, & \text{if } x_l^0 = x_l^*, \\ -1, & \text{if } x_l^0 > x_l^*. \end{cases}$$

Let w be the solution to

$$\begin{cases} w_t = (a(x)w_x)_x + f(x, t, w) & (x, t) \in (-1, 1) \times (S_1, +\infty) \\ B.C. \\ w(x, S_1) = w_k(x), \end{cases}$$

where $S_1 = \sigma_1$. By Lemma 3.3, for every $\rho \in (0, \rho_0^*)$ there exist $\tilde{\tau}_1 = \tilde{\tau}_1(\rho) > 0$, $M_1 = M_1(\rho) > 0$ and n curves of sign change (associated to the points of sign change $X^0 = (x_1^0, \dots, x_n^0)$) $\xi_l^1 \in C^{1+\frac{\beta}{2}}([S_1, \tilde{T}_1])$, $l = 1, \dots, n$, with $\tilde{T}_1 = S_1 + \tilde{\tau}_1$, such that $\|\xi_l^1\|_{C^{1+\frac{\beta}{2}}([S_1, \tilde{T}_1])} \leq M_1$, $w(\xi_l^1(t), t) = 0 \forall t \in [S_1, \tilde{T}_1]$, and

$$\begin{cases} \dot{\xi}_l^1(t) = - \left[a'(\xi_l^1(t)) + \frac{a(\xi_l^1(t))w_{xx}(\xi_l^1(t), t)}{w_x(\xi_l^1(t), t)} \right], \quad t \in [S_1, \tilde{T}_1], \\ \xi_l^1(S_1) = x_l^0. \end{cases} \quad (3.8)$$

Let us set $\xi_0^1(t) \equiv a_0^*$ and $\xi_{n+1}^1(t) \equiv b_0^*$ on $[S_1, \tilde{T}_1]$, by Remark 3.5, for every $l = 1, \dots, n-1$, we have

$$a_0^* = \xi_0^1(t) < \xi_l^1(t) < \xi_{l+1}^1(t) < \xi_{n+1}^1(t) = b_0^*, \quad \forall t \in [S_1, \tilde{T}_1]. \quad (3.9)$$

Let us introduce the *Inactive Set* ⁽¹⁹⁾

$$L_{IS}^0 := \{l \mid l \in \{1, \dots, n\}, x_l^0 = x_l^*\}$$

and let us consider the set of the *stopping times*

$$\Theta_1 := \{s \in (0, \tilde{\tau}_1] \mid \xi_l^1(S_1 + s) = x_l^*, \text{ for some } l \in \{1, \dots, n\} \setminus L_{IS}^0\}.$$

Let us set

$$\tau_1 = \begin{cases} \tilde{\tau}_1 & \text{if } \Theta_1 = \emptyset, \\ \min \Theta_1, & \text{otherwise,} \end{cases} \quad (3.10)$$

by (2.1) we have $T_1 = S_1 + \tau_1$.

Construction of $\{\tau_k, w_k\}$, $k = 2, \dots, N$.

By the previous step we have obtained the vector $X^{k-1} = (x_1^{k-1}, \dots, x_n^{k-1})$, where $x_l^{k-1} := \xi_l^{k-1}(T_{k-1})$ for $l = 1, \dots, n$, and ξ_l^{k-1} , defined on $[S_{k-1}, T_{k-1}]$, are the n curves of sign change associated with the initial state w_{k-1} and to the set of points of sign change $X^{k-2} = (x_1^{k-2}, \dots, x_n^{k-2})$. Let us set $x_0^{k-1} := a_0^*$,

¹⁹ The *Inactive Set* L_{IS}^0 is the set of the indexes such that the corresponding points of sign change don't need to be moved.

$x_{n+1}^{k-1} =: b_0^*$, and $\rho_{k-1}^* = \min_{l=0, \dots, n} \{x_{l+1}^{k-1} - x_l^{k-1}, x_{l+1}^* - x_l^*\}$. Let us introduce the *Inactive Set*

$$L_{IS}^{k-1} := \{l, l \in \{1, \dots, n\} | \exists h_l \in \{1, \dots, k-1\} : x_l^{h_l} = x_l^*\},$$

which consists of the indexes of the points of sign change that have already reached the corresponding target points $X^* = (x_1^*, \dots, x_n^*)$ at some previous time ⁽²⁰⁾, so these points don't need to be moved. Then, let us set

$$\mu_k(x_l^* - x_l^0) = \begin{cases} 0 & \text{if } l \in L_{IS}^{k-1}, \\ \text{sgn}(x_l^* - x_l^0) & \text{if } l \notin L_{IS}^{k-1}. \end{cases} \quad (21)$$

By Lemma 3.2 we can choose $w_k \in H_a^1(-1, 1) \cap C^{2+\beta}([a_0^*, b_0^*])$, with $\|w_k\|_{C^{2+\beta}([a_0^*, b_0^*])} \leq c_k$, for some positive constant $c_k = c(\rho_{k-1}^*)$, such that

$$\star w_k(x) = 0 \iff x = x_l^{k-1}, \quad l = 1, \dots, n;$$

$$\star w_k'(x_l^{k-1}) = \lambda(x_l^0) a(x_l^{k-1}), w_k''(x_l^{k-1}) = -\lambda(x_l^0) [\mu_k(x_l^* - x_l^0) + a'(x_l^{k-1})], \quad l = 1, \dots, n.$$

Let w be the solution to

$$\begin{cases} w_t = (a(x)w_x)_x + f(x, t, w) & (x, t) \in (-1, 1) \times (S_k, +\infty) \\ B.C. \\ w(x, S_k) = w_k(x). \end{cases}$$

where $S_k = \sum_{h=1}^{k-1} (\sigma_h + \tau_h) + \sigma_k$. By Lemma 3.3, for every $\rho \in (0, \rho_{k-1}^*]$ there exist $\tilde{\tau}_k = \tilde{\tau}_k(\rho) > 0$, $M_k = M_k(\rho) > 0$ and n curves of sign change (associated to the points of sign change $X^{k-1} = (x_1^{k-1}, \dots, x_n^{k-1})$), $\xi_l^k \in C^{1+\frac{\beta}{2}}([S_k, T_k])$, $l = 1, \dots, n$, with $\tilde{T}_k = S_k + \tilde{\tau}_k$, such that $\|\xi_l^k\|_{C^{1+\frac{\beta}{2}}([S_k, \tilde{T}_k])} \leq M_k$, $w(\xi_l^k(t), t) = 0 \forall t \in [S_k, \tilde{T}_k]$, and

$$\begin{cases} \dot{\xi}_l^k(t) = - \left[a'(\xi_l^k(t)) + \frac{a(\xi_l^k(t))w_{xx}(\xi_l^k(t), t)}{w_x(\xi_l^k(t), t)} \right], \\ \xi_l^k(S_k) = x_l^{k-1}. \end{cases} \quad (3.11)$$

Let us set $\xi_0^k(t) \equiv a_0^*$ and $\xi_{n+1}^k(t) \equiv b_0^*$ on $[S_k, \tilde{T}_k]$, by Remark 3.5, for every $l = 1, \dots, n-1$, we have that

$$a_0^* = \xi_0^k(t) < \xi_l^k(t) < \xi_{l+1}^k(t) < \xi_{n+1}^k(t) = b_0^*, \quad \forall t \in [S_k, \tilde{T}_k]. \quad (3.12)$$

Let us consider the set of the *stopping times*

$$\Theta_k := \{s \in (0, \tilde{\tau}_k] | \xi_l^k(S_k + s) = x_l^*, \text{ for some } l \in \{1, \dots, n\} \setminus L_{IS}^{k-1}\},$$

and let us set

$$\tau_k = \begin{cases} \tilde{\tau}_k & \text{if } \Theta_k = \emptyset, \\ \min \Theta_k, & \text{otherwise,} \end{cases} \quad (3.13)$$

²⁰ We note that $L_{IS}^{k-1} \subseteq \{1, \dots, n\}$ is an increasing family of sets.

²¹ We remark that the definition of μ_k , $k = 2, \dots, N$, is consistent with the one of μ_1 .

by (2.1) we have $T_k = S_k + \tau_k$.

Remark 3.6. We note that $\tau_k < \tilde{\tau}_k$ for at most n values of $k \in \{1, \dots, N\}$.

An important remark. Let us give an important remark about $\mathcal{W}^*(u_0)$, that is, the previous subclass of special families, which we have called *Order Processing Steering Times and Initial Data* associated with u_0 and X^* , and we will show that a generic $\{(\tau_k)_1^N, (w_k)_1^N\} \in \mathcal{W}^*(u_0)$ moves the points of sign change towards the desired targets.

Remark 3.7. We note that, for each index $l \notin L_{IS}^{k-1}$ ($k = 1, \dots, N$), by (3.8) and (3.11) and the choice of the initial data w_k we deduce that

$$\dot{\xi}_l^k(S_k) = \text{sgn}(x_l^* - x_l^0), \quad \xi_l^k(S_k) = x_l^{k-1}.$$

If $x_l^0 < x_l^*$, we have that $\dot{\xi}_l^k(S_k) = 1 > 0$, thus the initial conditions w_k permit to move the points of sign change x_l^{k-1} to the right towards x_l^* . Similarly, if $x_l^* < x_l^0$, the initial condition w_k permits to move the points of sign change to the left.

Curves of sign change, gap and target distance functional

Given $W^N = \{(\tau_k)_1^N, (w_k)_1^N\} \in \mathcal{W}^*(u_0)$, we introduce the n curves of sign change associated with W^N as the functions $\xi_l^W : \bigcup_{k=1}^N [S_k, T_k] \rightarrow \mathbb{R}$, $l = 1, \dots, n$, such that

$$\xi_l^W(t) = \xi_l^k(t), \quad S_k \leq t \leq T_k, \quad k = 1, \dots, N,$$

where the curves ξ_l^k are previously been constructed. We also set $\xi_0^W(t) \equiv 0$ and $\xi_{n+1}^W(t) \equiv 1$. Moreover, by (3.9) and (3.12), for $l = 1, \dots, n-1$, we deduce that

$$a_0^* = \xi_0^W(t) < \xi_l^W(t) < \xi_{l+1}^W(t) < \xi_{n+1}^W(t) = b_0^*, \quad \text{for all } t \in \bigcup_{k=1}^N [S_k, T_k].$$

Definition 3.8. For all $W^N = \{(\tau_k)_1^N, (w_k)_1^N\} \in \mathcal{W}^*(u_0)$ we define the **gap functional** by

$$\rho(W^N) = \min_{l=0, \dots, n} \min_{t \in \bigcup_{k=1}^N [S_k, T_k]} \{\xi_{l+1}^W(t) - \xi_l^W(t)\}$$

and the **target distance functional** by

$$J^*(W^N) = \sum_{l=1}^n |\xi_l^W(T_N) - x_l^*|.$$

3.3. Proof of Theorem 2.11 completed

Let us consider the initial state $u_0 \in H_a^1(-1, 1)$, and consider the set of points of sign change of u_0 , $X^0 = (x_1^0, \dots, x_n^0)$ where $0 = x_0^0 < x_l^0 < x_{l+1}^0 \leq x_{n+1}^0 = 1$ for all $l = 1, \dots, n$. Similarly, let consider the set of target points $X^* = (x_1^*, \dots, x_n^*)$, where $-1 = x_0^* < x_l^* < x_{l+1}^* \leq x_{n+1}^* = 1$ for all $l = 1, \dots, n$. Set

$$\rho_0^* = \min_{l=0, \dots, n} \{x_{l+1}^* - x_l^*, x_{l+1}^0 - x_l^0\}$$

and let $\tau_0^* = \tau(\frac{\rho_0^*}{2}) > 0$ and $M_0^* = M(\frac{\rho_0^*}{2}) > 0$ be the positive time and constant of Lemma 3.3, associated with $\rho = \frac{\rho_0^*}{2}$. The following proposition is crucial to obtain the proof of Theorem 2.11.

Proposition 3.9. *There exists $\varepsilon_0^* \in (0, 1)$ such that for all $\varepsilon \in (0, \varepsilon_0^*]$ and $N \in \mathbb{N}$, $N > n$ there exists $W^N = \{(\tau_k)_1^N, (w_k)_1^N\} \in \mathcal{W}^*(u_0)$ such that $\rho(W^N) \geq \frac{\rho_0^*}{2}$ and*

$$J^*(W^N) \leq \sum_{l=1}^n |x_l^0 - x_l^*| + c_1(\varepsilon) \sum_{k=1}^N \frac{1}{k^{1+\frac{\beta}{2}}} - c_2(\varepsilon) \sum_{k=n+1}^N \frac{1}{k}, \quad (3.14)$$

where $c_1(\varepsilon) = \frac{\varepsilon \rho_0^* n}{4 s_\beta}$, $c_2(\varepsilon) = (\frac{\varepsilon \rho_0^*}{4 M_0^* s_\beta})^{\frac{2}{2+\beta}}$, and $s_\beta = \sum_{k=1}^{\infty} \frac{1}{k^{1+\frac{\beta}{2}}}$.

Moreover, for such W^N , for $k = 1, \dots, N$, we have that

$$\tau_k \leq \tilde{\tau}_k := \left(\frac{\varepsilon \rho_0^*}{4 M_0^* s_\beta} \right)^{\frac{2}{2+\beta}} \frac{1}{k}, \quad (22) \quad (3.15)$$

and if $l \in L_{IS}^{k-1}$ the following inequality holds

$$|\xi_l^h(t) - x_l^*| \leq \varepsilon \frac{\rho_0^*}{4}, \quad \forall t \in [S_h, T_h], \quad \forall h = k, \dots, N. \quad (3.16)$$

Remark 3.10. We note that for each inactive index $l \in L_{IS}^{k-1}$ ($k = 1, \dots, N$) we have chosen the initial data such that $\xi_l^h(S_h) = 0$, $h = k, \dots, N$. So, by the inequality (3.16) of Lemma 3.9, the corresponding points of sign change remain forever near the target points that they have already reached.

We omit the proof of Proposition 3.9, because using Remark 3.7 and Remark 3.10 we can repeat a proof similar to that of Proposition 4.1 of [12].

We give the following definition.

Definition 3.11. A set of times and initial data $W^N \in \mathcal{W}^*(u_0)$ is said to be *separating* if $\rho(W^N) \geq \frac{\rho_0^*}{2}$. We set $\mathcal{W}_S^*(u_0) := \{W^N \in \mathcal{W}^*(u_0) : \rho(W^N) \geq \frac{\rho_0^*}{2}\}$.

Finally, we can prove Theorem 2.11.

Proof of Theorem 2.11. We will prove that

$$\forall \varepsilon > 0 \exists N_\varepsilon \in \mathbb{N} \exists W^{N_\varepsilon} \in \mathcal{W}_S^*(u_0) \text{ such that } J^*(W^{N_\varepsilon}) < \varepsilon \quad \text{and} \quad L_{IS}^{N_\varepsilon} = \{1, \dots, n\},$$

which implies the conclusion of Theorem 2.11. Arguing by contradiction, suppose

$$\exists \varepsilon > 0 \exists j \in \{1, \dots, n\} : \forall N \in \mathbb{N}, \forall W^N \in \mathcal{W}_S^*(u_0) \text{ we have } |\xi_j^N(T_N) - x_j^*| > \varepsilon.$$

Moreover, we can assume $\varepsilon \leq \varepsilon_0^*$, where $\varepsilon_0^* \in (0, 1)$ is given by Proposition 3.9. For every $N > n$, by Proposition 3.9, there exists $W^N = \{(\tau_k)_1^N, (w_k)_1^N\} \in \mathcal{W}_S^*(u_0)$ such that $\tau_k \leq \tilde{\tau}_k = \left(\frac{\varepsilon \rho_0^*}{4 M_0^* s_\beta} \right)^{\frac{2}{2+\beta}} \frac{1}{k}$, $k = 1, \dots, N$

²² $\tilde{\tau}_k$ and τ_k are defined in (3.10) and (3.13), see also Remark 3.6.

and, by (3.14), we obtain

$$\varepsilon < |\xi_j^N(T_N) - x_j^*| \leq J^*(W^N) \leq \sum_{l=1}^n |x_l^0 - x_l^*| + c_1 \sum_{k=1}^N \frac{1}{k^{1+\frac{\beta}{2}}} - c_2 \sum_{k=n+1}^N \frac{1}{k}.$$

Since $\sum_{k=1}^{\infty} \frac{1}{k} = +\infty$, the previous inequality gives a contradiction. \square

4. PROOF OF THEOREM 2.12

This section is devoted to the proof of Theorem 2.12, obtained in Section 4.2 after proving Lemma 4.7. Let us start with Section 4.1, where we recall some preliminary results obtained in [29] (for (SDP)) and in [30] (for (WDP)). In this section we use the notation introduced in Section 1.5.

4.1. Some spectral properties and some estimates for the semilinear degenerate problem (1.1)

Let us observe that the semilinear problem (1.1) can be recast as

$$\begin{cases} u'(t) = A u(t) + \phi(u), & t > 0 \\ u(0) = u_0 \in H_a^1(-1, 1), \end{cases}$$

where the operator $(A, D(A))$ is defined in (1.6) and, for every $u \in \mathcal{H}(Q_T)$,

$$\phi(u)(x, t) := f(x, t, u(x, t)) \quad \forall (x, t) \in Q_T.$$

Let us consider the operator $(A_0, D(A_0))$ defined as

$$\begin{cases} D(A_0) = D(A) \\ A_0 u = (au_x)_x, \quad \forall u \in D(A_0), \end{cases} \quad (4.1)$$

for this operator the following Proposition 4.1 and Proposition 4.2 is obtained in [9] for (SDP) ⁽²³⁾ and in [10] for (WDP).

Proposition 4.1. *$(A_0, D(A_0))$ is a closed, self-adjoint, dissipative operator with dense domain in $L^2(-1, 1)$. Therefore, A_0 is the infinitesimal generator of a strongly continuous semigroup of bounded linear operator on $L^2(-1, 1)$.*

Proposition 4.1 permits to obtain the following.

Proposition 4.2. *There exists an increasing sequence $\{\lambda_p\}_{p \in \mathbb{N}}$, with $\lambda_p \rightarrow +\infty$ as $p \rightarrow \infty$, such that the eigenvalues of the operator $(A_0, D(A_0))$ are given by $\{-\lambda_p\}_{p \in \mathbb{N}}$, and the corresponding eigenfunctions $\{\omega_p\}_{p \in \mathbb{N}}$ form a complete orthonormal system in $L^2(-1, 1)$.*

Remark 4.3. In the case $a(x) = 1 - x^2$, that is in the case of the Budyko-Sellers model, the orthonormal eigenfunctions of the operator $(A_0, D(A_0))$ are reduced to Legendre's polynomials $Q_p(x)$, and the eigenvalues

²³ In the (SDP) case, in [9], we showed that for this result it is sufficient that the diffusion coefficient $a(\cdot)$ satisfies the assumption (A.4_{SD}) with $\xi_a(x) = \int_0^x \frac{ds}{a(s)} \in L^1(-1, 1)$ instead of $\xi_a \in L^{2\theta-1}(-1, 1)$.

are $\mu_p = (p-1)p$, $p \in \mathbb{N}$. $Q_p(x)$ is equal to $\sqrt{\frac{2}{2p-1}}L_p(x)$, where $L_p(x)$ is assigned by *Rodrigues's formula*: $L_p(x) = \frac{1}{2^{p-1}(p-1)!} \frac{d}{dx^{p-1}} (x^2-1)^{p-1}$, $p \geq 1$.

Some estimates. By the next Lemma 4.4 and Lemma 4.5 (obtained in [29, 30]) we can deduce the Proposition 4.6.

Lemma 4.4. *Let $T > 0$ and $\vartheta \geq 1$. Let $a \in C([-1, 1]) \cap C^1(-1, 1)$ such that the assumption (A.4) ((A.4_{SD}) or (A.4_{WD})) holds, then $\mathcal{H}(Q_T) \subset L^{2\vartheta}(Q_T)$ and*

$$\|u\|_{L^{2\vartheta}(Q_T)} \leq cT^{\frac{1}{2\vartheta}} \|u\|_{\mathcal{H}(Q_T)},$$

where c is a positive constant.

Lemma 4.5. *Let $T > 0$, $u_0 \in H_a^1(-1, 1)$ and let $\alpha \in L^\infty(-1, 1)$. The strict solution $u \in \mathcal{H}(Q_T)$ of system (1.1), under the assumptions (A.3) – (A.4), satisfies the following estimate*

$$\|u\|_{\mathcal{H}(Q_T)} \leq c e^{kT} \|u_0\|_{1,a},$$

where $c = c(\|u_0\|_{1,a})$ and k are positive constants.

So, we can deduce the following.

Proposition 4.6. *Let $T > 0$, $u_0 \in H_a^1(-1, 1)$ and let $\alpha \in L^\infty(-1, 1)$. Let $u \in \mathcal{H}(Q_T)$ the strict solution $u \in \mathcal{H}(Q_T)$ of system (1.1), under the assumptions (A.3) – (A.4). Then, the function $(t, x) \mapsto f(t, x, u(t, x))$ belongs to $L^2(Q_T)$ and the following estimate holds*

$$\int_{Q_T} |f(x, t, u(x, t))|^2 dx dt \leq C e^{2k\vartheta T} T \|u_0\|_{1,a}^{2\vartheta},$$

where $C = C(\|u_0\|_{1,a})$ and k are positive constants.

Proof. Using Lemma 4.4 and Lemma 4.5 we obtain

$$\int_0^T \int_{-1}^1 f^2(x, t, w) dx dt \leq \gamma_*^2 \int_0^T \int_{-1}^1 |w|^{2\vartheta} dx dt \leq cT \|w\|_{\mathcal{H}(Q_T)}^{2\vartheta} \leq C e^{2k\vartheta T} T \|u_0\|_{1,a}^{2\vartheta},$$

where $c = c(\|u_0\|_{1,a})$, $C = C(\|u_0\|_{1,a})$ and k are positive constants. □

4.2. Proof

In this section we reformulate the problem (2.4), using a lighter notation than one introduced in Section 2.3 in the statement of Theorem 2.12, in the following way

$$\begin{cases} u_t = (a(x)u_x)_x + \alpha(x, t)u + f(x, t, u) & \text{in } Q_T = (-1, 1) \times (0, T), \\ B.C. & t \in (0, T), \\ u|_{t=0} = u_{in} + r_{in}, \end{cases} \quad (4.2)$$

where $(0, T)$ is a generic time interval, $u_{in}, r_{in} \in H_a^1(-1, 1)$, and u_{in} has n points of sign change at $x_l \in (-1, 1)$, $l = 1, \dots, n$, with $-1 := x_0 < x_l < x_{l+1} \leq x_{n+1} := 1$. Moreover, we will denote the target state by $\bar{u} \in H_a^1(-1, 1)$ instead of w_k .

Throughout this section, we represent the solution $u(x, t)$ of (4.2) as the sum of two functions $w(x, t)$ and $h(x, t)$, which solve the following problems in Q_T :

$$\begin{cases} w_t = (a(x)w_x)_x + \alpha w + f(x, t, w) \\ B.C. \\ w|_{t=0} = u_{in}, \end{cases} \quad \begin{cases} h_t = (a(x)h_x)_x + \alpha h + (f(x, t, w+h) - f(x, t, w)) \\ B.C. \\ h|_{t=0} = r_{in}. \end{cases} \quad (4.3)$$

Let us start with the following Lemma 4.7.

Lemma 4.7. *Let $\bar{u} \in H_a^1(-1, 1)$ have the same points of sign change as u_{in} in the same order of sign change. Let us suppose that*

$$\exists \delta^* > 0 : \delta^* \leq \frac{\bar{u}(x)}{u_{in}(x)} < 1, \quad \forall x \in (-1, 1) \setminus \bigcup_{l=1}^n \{x_l\}. \quad (4.4)$$

Then, for every $\eta > 0$ there exist a small time $T = T(\eta, u_{in}, \bar{u}) > 0$ and a static bilinear control $\tilde{\alpha}(x)$, $\tilde{\alpha} = \tilde{\alpha}(\eta, u_{in}, \bar{u}) \in C^2([-1, 1])$ such that

$$\|u(\cdot, T) - \bar{u}(\cdot)\|_{L^2(-1, 1)} \leq \eta + \sqrt{2}\|r_{in}\|_{L^2(-1, 1)}, \quad (4.5)$$

where u is the corresponding solution of (4.2) on Q_T .

Proof. Let us represent the solution u of (4.2) as the sum of two functions $w(x, t)$ and $h(x, t)$, which solve the two problems introduced in (4.3), respectively. For this proof it need to obtain some preliminary estimates that will deduced in the Step 1.

Step 1: *Evaluation of $\|w\|_{C([0, T]; L^2(-1, 1))}$ and $\|h(\cdot, T)\|_{L^2(-1, 1)}$.* For every $\alpha \in L^\infty(Q_T)$, with $\alpha(x, t) \leq 0 \quad \forall (x, t) \in Q_T$, multiplying by w the equation in the first problem of (4.3) and integrating by parts, using (1.5), since $[a(x)w_x(x, t)w(x, t)]_{-1}^1 \leq 0$ we obtain

$$\begin{aligned} \frac{1}{2} \int_0^t \int_{-1}^1 (w^2)_t dx ds &= \int_0^t \int_{-1}^1 (a(x)w_x)_x w dx ds + \int_0^t \int_{-1}^1 \alpha w^2 dx ds + \int_0^t \int_{-1}^1 f(x, s, w) w dx ds \\ &\leq - \int_0^t \int_{-1}^1 a(x)w_x^2 dx ds + \nu \int_0^T \int_{-1}^1 w^2 dx dt \leq \nu \int_0^T \int_{-1}^1 w^2 dx dt, \end{aligned}$$

where ν is the constant of (1.5). Then, for $T \in (0, \frac{1}{4\nu})$ we deduce

$$\begin{aligned} \int_{-1}^1 w^2(x, t) dx &\leq \int_{-1}^1 u_{in}^2(x) dx + 2\nu \int_0^T \int_{-1}^1 w^2 dx dt \\ &\leq \int_{-1}^1 u_{in}^2(x) dx + 2\nu T \|w\|_{C([0, T], L^2(-1, 1))}^2 \leq \|u_{in}\|_{L^2(-1, 1)}^2 + \frac{1}{2} \|w\|_{C([0, T], L^2(-1, 1))}^2, \quad t \in (0, T), \end{aligned}$$

so,

$$\|w\|_{C([0, T]; L^2(-1, 1))} \leq \sqrt{2} \|u_{in}\|_{L^2(-1, 1)}. \quad (4.6)$$

Proceeding as for the estimate (4.6) we evaluate $\|h(\cdot, T)\|_{L^2(-1, 1)}$. Namely, multiplying by h the equation of the second problem of (4.3) and integrating by parts over Q_T , using (1.5), for $T \in (0, \frac{1}{4\nu})$ and for every

$t \in (0, T)$, since $[a(x)h_x(x, t)h(x, t)]_{-1}^1 \leq 0$ we obtain

$$\begin{aligned} \int_{-1}^1 h^2(x, t) dx &\leq \int_{-1}^1 r_{in}^2(x) dx + 2 \int_0^T \int_{-1}^1 (f(x, t, w+h) - f(x, t, w)) h dx dt \\ &\leq \int_{-1}^1 r_{in}^2(x) dx + 2\nu \int_0^T \int_{-1}^1 h^2 dx dt \leq \|r_{in}\|_{L^2(-1,1)}^2 + 2\nu T \|h\|_{C([0,T];L^2(-1,1))}^2 \\ &\leq \|r_{in}\|_{L^2(-1,1)}^2 + \frac{1}{2} \|h\|_{C([0,T];L^2(-1,1))}^2. \end{aligned}$$

Hence,

$$\|h(\cdot, T)\|_{L^2(-1,1)} \leq \|h\|_{C([0,T];L^2(-1,1))} \leq \sqrt{2} \|r_{in}\|_{L^2(-1,1)}. \quad (4.7)$$

Step 2: *Choice of the bilinear control α .* Let us consider the following function defined on $[-1, 1]$

$$\alpha_0(x) = \begin{cases} \log\left(\frac{\bar{u}(x)}{u_{in}(x)}\right), & \text{for } x \neq -1, 1, x_l \ (l = 1 \dots, n) \\ 0, & \text{for } x = -1, 1, x_l \ (l = 1 \dots, n). \end{cases}$$

Using the assumption (4.4), we deduce that $\alpha_0 \in L^\infty(-1, 1)$ and $\alpha_0(x) \leq 0$, for every $x \in [-1, 1]$. Now, we select the following bilinear control

$$\alpha(x, t) := \frac{1}{T} \alpha_0(x).$$

For every fixed $x \in (-1, 1)$, by the classical technique for solving first order ODEs, applied to the equation $w_t(x, t) = \frac{\alpha_0(x)}{T} w(x, t) + ((a(x)w_x(x, t))_x + f(x, t, w))$ $t \in (0, T)$, we compute at time T the solution w to the first problem in (4.3), so the following representation formula holds for every $x \in (-1, 1)$

$$w(x, T) = e^{\alpha_0(x)} u_{in}(x) + \int_0^T e^{\alpha_0(x) \frac{(T-\tau)}{T}} ((a(x)w_x)_x(x, \tau) + f(x, \tau, w(x, \tau))) d\tau. \quad (4.8)$$

Let us show that $w(\cdot, T) \rightarrow \bar{u}$ in $L^2(-1, 1)$, as $T \rightarrow 0^+$. In advance, since $\alpha_0(x) \leq 0$ let us note that by the above formula we deduce

$$\begin{aligned} \|w(\cdot, T) - \bar{u}(\cdot)\|_{L^2(-1,1)}^2 &= \int_{-1}^1 \left(\int_0^T e^{\alpha_0(x) \frac{(T-\tau)}{T}} ((a(x)w_x)_x(x, \tau) + f(x, \tau, w(x, \tau))) d\tau \right)^2 dx \\ &\leq T \| (a(\cdot)w_x)_x + f(\cdot, \cdot, w) \|_{L^2(Q_T)}^2. \end{aligned} \quad (4.9)$$

Let us prove that the right-hand side of (4.9) tends to zero as $T \rightarrow 0^+$.

Step 3: *Evaluation of $\|(a(\cdot)w_x)_x\|_{L^2(Q_T)}^2$.* Let us suppose, without loss of generality, that α_0 satisfies the further properties:

$$\alpha_0 \in C^2([-1, 1]) \text{ with } \lim_{x \rightarrow \pm 1} \frac{\alpha_0'(x)}{a(x)} = 0 \quad \text{and} \quad \lim_{x \rightarrow \pm 1} \alpha_0'(x) a'(x) = 0. \quad (4.10)$$

Moreover, let us consider the (WDP) problem with $\beta_0 \gamma_0 \neq 0$ in the assumption (A.4_{WDP}). These assumptions will be removed in Step 4.

Multiplying by $(a(x)w_x)_x$ the equation in the first problem in (4.3), with $\alpha(x, t) = \frac{\alpha_0(x)}{T} \leq 0$, integrating over Q_T , we have

$$\begin{aligned} \|(a(\cdot)w_x)_x\|_{L^2(Q_T)}^2 &\leq \int_0^T \int_{-1}^1 w_t (a(x)w_x)_x \, dx dt - \frac{1}{T} \int_0^T \int_{-1}^1 \alpha_0 w (a(x)w_x)_x \, dx dt \\ &\quad + \frac{1}{2} \int_0^T \int_{-1}^1 f^2(x, t, w) \, dx dt + \frac{1}{2} \int_0^T \int_{-1}^1 |(a(x)w_x)_x|^2 \, dx dt. \end{aligned} \quad (4.11)$$

Let us estimate the first two terms of the right-hand side of (4.11). Integrating by parts and using the sign condition $\beta_0\beta_1 \leq 0$ and $\gamma_0\gamma_1 \geq 0$ we deduce

$$\begin{aligned} \int_0^T \int_{-1}^1 w_t (a(x)w_x)_x \, dx dt &= \int_0^T [w_t (a(x)w_x)]_{-1}^1 \, dt - \frac{1}{2} \int_0^T \int_{-1}^1 a(x) (w_x^2)_t \, dx dt \\ &\leq \frac{1}{2} \frac{\gamma_1}{\gamma_0} a^2(1) u_{inx}^2(1) - \frac{1}{2} \frac{\beta_1}{\beta_0} a^2(-1) u_{inx}^2(-1) + \frac{1}{2} \int_{-1}^1 a(x) u_{inx}^2 \, dx dt, \end{aligned} \quad (4.12)$$

moreover, by (4.10) and (4.6) we obtain

$$\begin{aligned} \int_0^T \int_{-1}^1 \alpha_0(x) w (a(x)w_x)_x \, dx dt &= - \int_0^T \int_{-1}^1 \alpha_0(x) a(x) w_x^2 \, dx dt - \frac{1}{2} \int_0^T \int_{-1}^1 \alpha'_0(x) a(x) (w^2)_x \, dx dt \\ &\geq -\frac{1}{2} \int_0^T \int_{-1}^1 (\alpha'_0(x) a(x) w^2)_x \, dx dt + \frac{1}{2} \int_0^T \int_{-1}^1 (\alpha''_0(x) a(x) + \alpha'_0(x) a'(x)) w^2 \, dx dt \\ &\geq -\frac{1}{2} \sup_{x \in [-1, 1]} |\alpha''_0(x) a(x) + \alpha'_0(x) a'(x)| \int_0^T \int_{-1}^1 w^2 \, dx dt \\ &\geq -T \sup_{x \in [-1, 1]} |\alpha''_0(x) a(x) + \alpha'_0(x) a'(x)| \|u_{in}\|_{L^2(-1, 1)}^2. \end{aligned} \quad (4.13)$$

From (4.9), applying (4.11)–(4.13) and Proposition 4.6, we deduce

$$\begin{aligned} \|w(x, T) - \bar{u}(x)\|_{L^2(-1, 1)}^2 &\leq 2T \left(\|(a(\cdot)w_x)_x\|_{L^2(Q_T)}^2 + \|f(\cdot, \cdot, w)\|_{L^2(Q_T)}^2 \right) \\ &\leq 2T \left(\frac{\gamma_1}{\gamma_0} a^2(1) u_{inx}^2(1) - \frac{\beta_1}{\beta_0} a^2(-1) u_{inx}^2(-1) + |u_{in}|_{1, a}^2 \right) \\ &\quad + 4T \left(\sup_{x \in [-1, 1]} |\alpha''_0(x) a(x) + \alpha'_0(x) a'(x)| \|u_{in}\|_{L^2(-1, 1)}^2 + C e^{2k\vartheta T} \|u_{in}\|_{1, a}^{2\vartheta} \right), \end{aligned} \quad (4.14)$$

where $C = C(\|u_{in}\|_{1, a})$ and k are the positive constants of Proposition 4.6.

Step 4: *Convergence of $w(\cdot, T)$ to $\bar{u}(\cdot)$.* The previous estimates of Step.3 hold also in the (WDP) case with $\beta_0\gamma_0 = 0$ and in the (SDP) case. In effect, the simple weighted Neumann boundary condition permits some simplifications in the previous estimates (see (4.12)), and in particular in the last estimate (4.14).

In order to remove the assumption (4.10) we observe that we can approximate in $L^2(-1, 1)$ the reaction coefficient $\alpha_0 \in L^\infty(-1, 1)$, introduced in Step.2, by a sequence of uniformly bounded functions $\{\alpha_{0j}\}_{j \in \mathbb{N}} \subset C^2([-1, 1])$ such that

$$\alpha_{0j}(x) \leq 0 \quad \forall x \in [-1, 1], \quad \alpha_{0j}(\pm 1) = 0, \quad \lim_{x \rightarrow \pm 1} \frac{\alpha'_{0j}(x)}{a(x)} = 0 \quad \text{and} \quad \lim_{x \rightarrow \pm 1} \alpha'_{0j}(x) a'(x) = 0.$$

We remark that, for every $j \in \mathbb{N}$, the representation formula (4.8) still holds for the corresponding solutions w_j :

$$w_j(x, T) = e^{\alpha_{0j}(x)} u_{in}(x) + \int_0^T e^{\alpha_{0j}(x) \frac{(T-\tau)}{T}} ((a(x)w_{jx})_x(x, \tau) + f(x, \tau, w_j(x, \tau))) d\tau.$$

Let us fix $\eta > 0$, then, making use of the following limit relation

$$\bar{u}_j(x) := e^{\alpha_{0j}(x)} u_{in}(x) \longrightarrow e^{\alpha_0(x)} u_{in}(x) = \bar{u}(x) \text{ in } L^2(-1, 1) \text{ as } j \rightarrow \infty,$$

there exists $j^* \in \mathbb{N}$ and a positive constant $K = K(\alpha_{0j^*}, \alpha'_{0j^*}, \alpha''_{0j^*}, u_{in})$, given by (4.14) (written with α_{0j^*} instead of α_0), such that we deduce that

$$\|w_{j^*}(x, T) - \bar{u}(x)\|_{L^2(-1, 1)} \leq \|w_{j^*}(x, T) - \bar{u}_{j^*}(x)\|_{L^2(-1, 1)} + \|\bar{u}_{j^*}(x) - \bar{u}(x)\|_{L^2(-1, 1)} \leq KT + \frac{\eta}{2} \leq \eta, \quad (4.15)$$

for $T \leq \frac{\eta}{2K}$. Keeping in mind that $u_{j^*} = w_{j^*} + h_{j^*}$, by combining (4.15) and (4.7) we obtain that there exist a small time $T = T(\eta, u_{in}, \bar{u}) < \min\{\frac{\eta}{2K}, \frac{1}{4\nu}\}$ and a static bilinear control $\tilde{\alpha} = \tilde{\alpha}(\eta, u_{in}, \bar{u}) \in C^2([-1, 1])$ ($\tilde{\alpha}(x, t) = \frac{\alpha_{j^*}(x)}{T}$, $\forall(x, t) \in Q_T$) such that we obtain the conclusion. \square

Now, using the notation in (4.2), let us give the proof of Theorem 2.12.

Proof of Theorem 2.12. Let us fix $\eta > 0$. For any $\rho \in [0, \frac{\rho_0}{2})$, with $\rho_0 := \min_{l=0, \dots, n} \{x_{l+1} - x_l\}$, let us consider

the set $\Omega_\rho := \bigcup_{l=0}^n (x_l + \rho, x_{l+1} - \rho)$. Since $\bar{u}, u_{in} \in H_a^1(-1, 1)$, there exist $\bar{u}_\eta, u_{in}^\eta \in C^1([-1, 1])$ such that

- * $\bar{u}_\eta(x) = 0 \iff x = x_l, l = 0, \dots, n+1$ and $\|\bar{u}_\eta - \bar{u}\|_{L^2(-1, 1)} < \frac{\eta}{4}$;
- * $u_{in}^\eta(x) = 0 \iff x = x_l, l = 0, \dots, n+1, |(u_{in}^\eta)'(x_l)| = 1, l = 0, \dots, n+1$, and there exists $\bar{\rho} \in (0, \frac{\rho_0}{2})$ such that

$$\|u_{in}^\eta - u_{in}\|_{L^2(-1, 1)} < \frac{\sqrt{2}}{16Me^\nu} \eta, \quad (4.16)$$

where ν is the nonnegative constant of (1.5),

$$M := \max_{x \in \Omega_{\bar{\rho}}} \left\{ \frac{\bar{u}(x)}{u_{in}(x)} \right\} + 1 > \sup_{x \in \Omega_0} \left\{ \frac{\bar{u}_\eta(x)}{u_{in}^\eta(x)} \right\}, \quad (24) \text{ and } \|Mu_{in}^\eta\|_{L^2((-1, 1) \setminus \Omega_{\bar{\rho}})} < \frac{\eta}{4}. \quad (4.17)$$

Step 1: *Steering the system from $u_{in} + r_{in}$ to Mu_{in}^η .* In this step, we represent the solution $u(x, t)$ of (4.2) as the sum of two functions $w(x, t)$ and $h(x, t)$, which solve the problems in (4.3) in $(-1, 1) \times (0, t_1)$, $t_1 > 0$, with

²⁴ We note that the auxiliary function $\Phi : [-1, 1] \longrightarrow \mathbb{R}$, defined in the following way

$$\Phi(x) = \begin{cases} \frac{\bar{u}_\eta(x)}{u_{in}^\eta(x)}, & \text{if } x \in (-1, 1) \setminus \bigcup_{l=1}^n \{x_l\} \\ |\bar{u}'_\eta(x)|, & \text{if } x = x_l, l = 0, \dots, n+1, \end{cases}$$

is continuous on $[-1, 1]$, since we have $\lim_{x \rightarrow x_l} \frac{\bar{u}_\eta(x)}{u_{in}^\eta(x)} = |\bar{u}'_\eta(x_l)|$, for every $l = 0, \dots, n+1$.

the modified initial states:

$$w|_{t=0} = u_{in}^\eta \quad \text{and} \quad h|_{t=0} = r_{in} + (u_{in} - u_{in}^\eta).$$

Let us choose

$$\alpha(x, t) = \alpha_1 := \frac{\log M}{t_1}, \quad (x, t) \in (-1, 1) \times (0, t_1),$$

for some $t_1 > 0$. Applying the constant bilinear control $\alpha(x, t) = \alpha_1 > 0, \forall x \in (-1, 1)$, on the interval $(0, t_1)$, the solution of the first problem in (4.3) is given by

$$\begin{aligned} w(x, t_1) &= \sum_{p=1}^{\infty} \left[\int_0^{t_1} e^{(\alpha_1 - \lambda_p^2)(t_1 - t)} \left(\int_{-1}^1 f(r, t, w(r, t)) \omega_p(r) dr \right) dt \right] \omega_p(x) \\ &\quad + M \sum_{p=1}^{\infty} e^{-\lambda_p^2 t_1} \left(\int_{-1}^1 u_{in}^\eta(r) \omega_p(r) dr \right) \omega_p(x), \end{aligned} \quad (4.18)$$

where $\{-\lambda_p\}_{p \in \mathbb{N}}$ are the eigenvalues of the operator $(A_0, D(A_0))$, and $\{\omega_p\}_{p \in \mathbb{N}}$ are the corresponding eigenfunctions, that form a complete orthonormal system in $L^2(-1, 1)$ (see also Proposition 4.2). By the strong continuity of the semigroup, see Proposition 4.1, we have that

$$\sum_{p=1}^{\infty} e^{-\lambda_p^2 t_1} \left(\int_0^1 u_{in}^\eta(r) \omega_p(x) dr \right) \omega_p(x) \longrightarrow u_{in}^\eta(\cdot) \quad \text{in } L^2(-1, 1) \quad \text{as } t_1 \rightarrow 0. \quad (4.19)$$

Moreover, using Proposition 4.6 and Hölder's inequality we deduce

$$\begin{aligned} &\left\| \sum_{p=1}^{\infty} \left[\int_0^{t_1} e^{(\alpha_1 - \lambda_p^2)(t_1 - t)} \left(\int_{-1}^1 f(r, t, w(r, t)) \omega_p(r) dr \right) dt \right] \omega_p(x) \right\|_{L^2(-1, 1)}^2 \\ &= \sum_{p=1}^{\infty} \left| \int_0^{t_1} e^{(\alpha_1 - \lambda_p^2)(t_1 - t)} \left(\int_{-1}^1 f(r, t, w(r, t)) \omega_p(r) dr \right) dt \right|^2 \\ &\leq \sum_{p=1}^{\infty} \left(\int_0^{t_1} e^{2(\alpha_1 - \lambda_p^2)(t_1 - t)} dt \right) \int_0^{t_1} \left| \int_{-1}^1 f(r, t, w(r, t)) \omega_p(r) dr \right|^2 dt \\ &\leq \sum_{p=1}^{\infty} e^{2\alpha_1 t_1} t_1 \int_0^{t_1} \left| \int_{-1}^1 f(r, t, w(r, t)) \omega_p(r) dr \right|^2 dt = M^2 t_1 \int_0^{t_1} \sum_{p=1}^{\infty} \left| \int_{-1}^1 f(r, t, w(r, t)) \omega_p(r) dr \right|^2 dt \\ &= M^2 t_1 \int_0^{t_1} \int_{-1}^1 f^2(r, t, w(r, t)) dr dt \leq C M^2 e^{2k\vartheta t_1} t_1 \|u_{in}^\eta\|_{1, a}^{2\vartheta}. \end{aligned} \quad (4.20)$$

Making use of (4.18)–(4.20), we have, as $t_1 \rightarrow 0$,

$$w(\cdot, t_1) \longrightarrow M u_{in}^\eta(\cdot) \quad \text{in } L^2(-1, 1). \quad (4.21)$$

Now, we evaluate $\|h(\cdot, t_1)\|_{L^2(-1,1)}$. Let $t_1 > 0$, multiplying by h both members of the equation in the second problem of (4.3) and integrating by parts, since $[a(x)h_x(x, t)h(x, t)]_{-1}^1 \leq 0$ we obtain

$$\begin{aligned} \int_{-1}^1 h^2(x, t) dx &\leq \int_{-1}^1 (r_{in} + (u_{in} - u_{in}^\eta))^2 dx + 2\alpha_1 \int_0^{t_1} \int_{-1}^1 h^2 dx dt \\ &\quad + 2 \int_0^{t_1} \int_{-1}^1 (f(x, t, w + h) - f(x, t, w)) h dx dt \\ &\leq \int_{-1}^1 (r_{in} + (u_{in} - u_{in}^\eta))^2 dx + 2(\alpha_1 + \nu) \int_0^{t_1} \int_{-1}^1 h^2 dx dt, \quad t \in (0, t_1). \end{aligned}$$

Thus, by Gronwall's inequality we have

$$\|h(\cdot, t)\|_{L^2(-1,1)}^2 \leq e^{2(\alpha_1 + \nu)t_1} \|r_{in} + (u_{in} - u_{in}^\eta)\|_{L^2(-1,1)}^2, \quad t \in (0, t_1),$$

then, for every $t_1 \in (0, 1)$,

$$\|h(\cdot, t_1)\|_{L^2(-1,1)} \leq \|h\|_{C([0, t_1]; L^2(-1,1))} \leq Me^\nu (\|r_{in}\|_{L^2(-1,1)} + \|u_{in} - u_{in}^\eta\|_{L^2(-1,1)}). \quad (4.22)$$

Since $u = w + h$, by (4.21), (4.22) and (4.16), there exists a small time $t_1 \in (0, 1)$ such that

$$\|u(\cdot, t_1) - Mu_{in}^\eta(\cdot)\|_{L^2(-1,1)} \leq \frac{\sqrt{2}}{8} \eta + Me^\nu \|r_{in}\|_{L^2(-1,1)}. \quad (4.23)$$

Step 2: *Steering the system from $Mu_{in}^\eta + \bar{r}_{in}$ to \bar{u} .* In this step, the solution $u(x, t)$ of (4.2) is still represented as the sum of $w(x, t)$ and $h(x, t)$, which solve the problems in (4.3) in $(-1, 1) \times (t_1, T)$, with the modified initial states Mu_{in}^η instead of u_{in} and $\bar{r}_{in}(\cdot) := u(\cdot, t_1) - Mu_{in}^\eta(\cdot)$ instead of $r_{in}(\cdot)$. By (4.16) it follows that

$$\exists \delta^* > 0 : \quad \delta^* \leq \frac{\bar{u}_\eta(x)}{Mu_{in}^\eta(x)} < 1, \quad \forall x \in \Omega_{\bar{p}}. \quad (4.24)$$

Owing to (4.24), assumption (4.4) of Lemma 4.7 is satisfied on $\Omega_{\bar{p}}$ with initial state Mu_{in}^η and target state \bar{u}_η . Then, adapting the proof of Lemma 4.7 we can conclude that there exists $T > t_1$ (such that $T - t_1 > 0$ is small) and a bilinear control $\bar{\alpha}(x, t) = \frac{\bar{\alpha}(x)}{T - t_1}$, $\forall (x, t) \in (-1, 1) \times (t_1, T)$, where $\bar{\alpha} \in C^2([-1, 1])$ is very close in $L^2(-1, 1)$ -norm to the function

$$\alpha_\eta(x) = \begin{cases} \log\left(\frac{\bar{u}_\eta(x)}{Mu_{in}^\eta(x)}\right) & \text{if } x \in \Omega_{\bar{p}}, \\ 0 & \text{elsewhere in } [-1, 1], \end{cases} \quad \text{such that}$$

$$\|u(\cdot, T) - \bar{u}_{\bar{p}}(\cdot)\|_{L^2(-1,1)} \leq \frac{\eta}{4} + \sqrt{2} \|u(\cdot, t_1) - Mu_{in}^\eta(\cdot)\|_{L^2(-1,1)}, \quad (4.25)$$

$$\text{where } \bar{u}_{\bar{p}}(x) = \begin{cases} \bar{u}_\eta(x) & \text{if } x \in \Omega_{\bar{p}}, \\ Mu_{in}^\eta(x) & \text{elsewhere in } [-1, 1]. \end{cases}$$

Since $\|\bar{u}_\eta - \bar{u}\|_{L^2(-1,1)} < \frac{\eta}{4}$, from (4.25), (4.23) and (4.17) we obtain

$$\begin{aligned} \|u(\cdot, T) - \bar{u}(\cdot)\|_{L^2(-1,1)} &\leq \|u(\cdot, T) - \bar{u}_{\bar{p}}(\cdot)\|_{L^2(-1,1)} + \|\bar{u}_{\bar{p}} - \bar{u}_\eta\|_{L^2(-1,1)} + \|\bar{u}_\eta - \bar{u}\|_{L^2(-1,1)} \\ &\leq \frac{\eta}{2} + \sqrt{2} \|u(\cdot, t_1) - Mu_{in}^\eta(\cdot)\|_{L^2(-1,1)} + \|Mu_{in}^\eta\|_{L^2((-1,1) \setminus \Omega_{\bar{p}})} \\ &\leq \eta + \sqrt{2} Me^\nu \|r_{in}\|_{L^2(-1,1)}, \end{aligned}$$

from which the conclusion follows. \square

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