

## ERRATUM TO: A VARIATIONAL APPROACH TO A STATIONARY FREE BOUNDARY PROBLEM MODELING MEMS

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**Abstract.** An incomplete argument in the proof of Theorem 3.4 from Ph. Laurençot and Ch. Walker [*ESAIM: COCV* **22** (2016) 417–438] is corrected.

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We noticed a gap in the proof of Theorem 3.4 from [2] and the aim of this erratum is to provide a complete argument. Specifically, in Theorem 3.4 from [2], we derive the Euler-Lagrange equation satisfied by a minimizer  $u$  of the functional

$$\mathcal{E}_m(u) := \frac{\beta}{2} \|\partial_x^2 u\|_{L_2(I)}^2 + \frac{1}{2} \left( \tau + \frac{a}{2} \|\partial_x u\|_{L_2(I)}^2 \right) \|\partial_x u\|_{L_2(I)}^2$$

on the set

$$\mathcal{A}_\rho := \{u \in H_D^2(I) : u \text{ is even with } -1 < u \leq 0 \text{ and } \mathcal{E}_e(u) = \rho\},$$

where  $I := (-1, 1)$ ,  $\rho \in (2, \infty)$ ,  $H_D^2(I) := \{u \in H^2(I) : u(\pm 1) = \partial_x u(\pm 1) = 0\}$ , and  $\mathcal{E}_e$  is a non-negative non-linear and nonlocal functional of  $u$ . The computation in [2] of the Euler-Lagrange equation, see equation (3.10) from [2], relies implicitly on the property that minimizers lie in the interior of  $\mathcal{A}_\rho$ , a property which is, however, not known *a priori*. Although knowing that minimizers are strictly greater than  $-1$ , it is actually not known whether minimizers are negative (even though this property can be shown *a posteriori*, which was the main reason to include it in the definition of  $\mathcal{A}_\rho$ ). This issue can be remedied by changing slightly the admissible set  $\mathcal{A}_\rho$  on which the functional  $\mathcal{E}_m$  is minimized. In fact, the non-positivity assumption in  $\mathcal{A}_\rho$  is not needed and our analysis works equally well in the set

$$\mathcal{A}_\rho := \{u \in H_D^2(I) : u \text{ is even with } -1 < u \text{ and } \mathcal{E}_e(u) = \rho\}. \quad (1)$$

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To be more precise, several results in [2] were derived for non-positive functions in

$$\mathcal{K}^s := \{u \in H_D^s(I) : -1 < u \leq 0 \text{ on } I\}, \quad s \geq 1,$$

an assumption which is not required, as it suffices to work in

$$S^s := \{u \in H_D^s(I) : -1 < u \text{ on } I\}, \quad s \geq 1.$$

For  $u \in S^1$ , one shall then rather define the function  $b_u$  in equation (2.1) from [2] as

$$b_u(x, z) := \begin{cases} \frac{1+z}{1+u(x)} & \text{for } (x, z) \in \overline{\Omega(u)}, \\ 1 & \text{for } (x, z) \in \overline{\Omega(M_u)} \setminus \overline{\Omega(u)}, \end{cases}$$

where  $\Omega(M_u) := I \times (-1, M_u + 1)$  with  $M_u := \max\{0, \sup_I u\}$ . Note that  $b_u$  belongs to  $H^1(\Omega(M_u)) \cap C(\overline{\Omega(M_u)})$ , which allows one to redefine  $B_u \in H^{-1}(\Omega(M_u))$  (i.e. the dual space of  $H_D^1(\Omega(M_u))$ ) in equation (2.2) from [2] by

$$\langle B_u, \vartheta \rangle := - \int_{\Omega(M_u)} [\varepsilon^2 \partial_x b_u \partial_x \vartheta + \partial_z b_u \partial_z \vartheta] \, d(x, z), \quad \vartheta \in H_D^1(\Omega(M_u)).$$

Then Lemmas 2.1 and 2.2 from [2] remain true for  $u \in S^1$  (instead of  $u \in \mathcal{K}^1$ ) and Proposition 2.3 from [2] is actually valid for  $u \in S^{2-\alpha}$  (instead of  $u \in \mathcal{K}^{2-\alpha}$ ) when replacing equation (2.5) from [2] by

$$\frac{1+z}{1+M_u} \leq \psi_u(x, z) \leq 1, \quad (x, z) \in \Omega(u).$$

Moreover, Propositions 2.6 and 2.7 from [2] are also true when replacing  $\mathcal{K}^1$  by  $S^1$ . For later use, we note that Proposition 2.6 from [2] implies

$$\mathcal{E}_e(u) \leq \mathcal{E}_e(0) = 2 \quad \text{for } u \in S^1 \quad \text{with } u \geq 0 \text{ in } I. \quad (2)$$

Also Lemma 2.8 from [2] remains true for  $u \in S^1$  (instead of  $u \in \mathcal{K}^1$ ), except that the lower bound on  $\mathcal{E}_e(u)$  has to be replaced by

$$\mathcal{E}_e(u) \geq \int_{-1}^1 \frac{dx}{1+u(x)} \geq \frac{2}{1+M_u}.$$

All other statements of Section 2 from [2] are not affected by these changes.

The minimization of  $\mathcal{E}_m$  in Section 3 from [2] is now performed on the set  $\mathcal{A}_\rho$  defined in (1) for a given  $\rho \in (2, \infty)$ . The statement of Proposition 3.1 from [2] remains true, as it is easily checked that its proof only relies on the continuity of the map  $t \mapsto \mathcal{E}_e(tv)$  for  $v \in \mathcal{A}_\rho$  established in Proposition 2.7 from [2], but not on its monotonicity (which is only true when  $v$  is non-negative). Next, neither Proposition 3.2 from [2], nor Lemma 3.3 from [2] are affected by the change of  $\mathcal{A}_\rho$  to  $\mathcal{A}_\rho$ . Therefore, in the proof of Theorem 3.4 from [2] we can use the same arguments to derive that, if  $u \in \mathcal{A}_\rho$  is an arbitrary minimizer of  $\mathcal{E}_m$  on  $\mathcal{A}_\rho$ , then  $u \in H^4(D) \cap H_D^2(I)$ , and there is a Lagrange multiplier  $\lambda_u \in \mathbb{R}$  such that

$$\beta \partial_x^4 u - (\tau + a \|\partial_x u\|_{L_2(I)}^2) \partial_x^2 u = -\lambda_u g(u), \quad x \in I, \quad (3)$$

where  $g(u) := \partial_u \mathcal{E}_e(u)$  is a non-negative functional of  $u$ , which belongs to  $L_2(I)$ . At this stage, since the non-positivity of  $u$  is not yet guaranteed, we need to employ a slightly different argument than in [2]. Indeed, we first assume for contradiction that  $\lambda_u \leq 0$ . Then  $-\lambda_u g(u)$  is non-negative and it follows of (3) and Theorem 1.1 from [1] that  $u > 0$  in  $I$ . Hence  $\rho = \mathcal{E}_e(u) \leq \mathcal{E}_e(0) = 2$  by (2), contradicting  $\rho \in (2, \infty)$ . Consequently,  $\lambda_u > 0$  and  $-\lambda_u g(u)$  is negative, so that we infer of (3) and Theorem 1.1 from [1] that  $u < 0$  in  $I$ . The remaining arguments in the proof of Theorem 3.4 from [2] are then the same.

Summarizing, the statement of Theorem 3.4 from [2] is correct, once  $\mathcal{A}_\rho$  is replaced by  $\mathcal{A}_\rho$ . Thanks to the above analysis, Theorem 3.4 from [2] may be supplemented with the following result:

**Corollary 1.** *Consider  $\rho \in (2, \infty)$  and let  $u \in \mathcal{A}_\rho$  be an arbitrary minimizer of  $\mathcal{E}_m$  in  $\mathcal{A}_\rho$ . Then  $u < 0$  in  $I$  and  $u \in \mathcal{A}_\rho$ . In addition,*

$$\mathcal{E}_m(u) = \min_{v \in \mathcal{A}_\rho} \mathcal{E}_m(v) = \min_{v \in \mathcal{A}_\rho} \mathcal{E}_m(v) .$$

## REFERENCES

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