ESAIM: COCV 26 (2020) 73
https://doi.org/10.1051/cocv/2019056

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

STABILITY AND REGULARITY TRANSMISSION FOR COUPLED BEAM AND WAVE EQUATIONS THROUGH BOUNDARY WEAK CONNECTIONS*

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Abstract. In this paper, we consider stability for a hyperbolic-hyperbolic coupled system consisting of Euler-Bernoulli beam and wave equations, where the structural damping of the wave equation is taken into account. The coupling is actuated through boundary weak connection in the sense that after differentiation of the total energy for coupled system, only the term of the wave equation appears explicitly. We first show that the spectrum of the closed-loop system consists of three branches: one branch is basically along the real axis and accumulates to a finite point; the second branch is also along the real line; and the third branch distributes along two parabola likewise symmetric with the real axis. The asymptotic expressions of both eigenvalues and eigenfunctions are obtained by means of asymptotic analysis. With an estimation of the resolvent operator, the completeness of the root subspace is proved. The Riesz basis property and exponential stability of the system are then concluded. Finally, we show that the associated C_0-semigroup is of Gevrey class, which shows that not only the stability but also regularity have been transmitted from regular wave subsystem to the whole system through this boundary connections.

Mathematics Subject Classification. 47B06, 74K10, 93D15, 70J25.

Received August 26, 2018. Accepted September 5, 2019.

1. Introduction

System coupling is ubiquitous in systems control. One control plant together with controller represents usually a coupled system. Consider for instance an ODE system with time delay in input:

$$\dot{x}(t) = u(t - \tau), \tag{1.1}$$

where $u(t)$ is the control that has a time delay $\tau > 0$. Make a transform of the following

$$z(x, t) = u(t - x\tau), \quad x \in [0, 1].$$

*This work was carried out with the support of the National Natural Science Foundation of China (No. 61873260), and the Project of Department of Education of Guangdong Province (No. 2017KZDXM087).

Keywords and phrases: Euler-Bernoulli beam equation, wave equation with structural damping, coupled system, Gevrey class.

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Then, $z(x,t)$ satisfies a partial differential equation:

$$\begin{cases} \tau z_t(x,t) + z_x(x,t) = 0, \\ z(0,t) = u(t). \end{cases}$$

System (1.1) is then transformed into a coupled ODE+PDE system:

$$\begin{cases} \dot{x}(t) = z(1,t), \\ \tau z_t(x,t) + z_x(x,t) = 0, \\ z(0,t) = u(t), \end{cases} \tag{1.2}$$

where the ODE part is actuated by PDE part through a boundary connection.

In the past two decades, much effort has been concentrated on control and stability analysis for coupled systems described PDEs. Multiple references have investigated the parabolic-hyperbolic coupled systems like heat-wave system, heat-beam system, heat-Schrödinger system and thermoelastic systems. In [24, 25], stability and controllability for a heat-wave system which is arising from the fluid-structure interaction were analyzed. Stabilization for an interconnected systems of Euler-Bernoulli beam and heat equation with boundary weak connections have been treated in [21, 26] where the heat is the controller to the whole system. The heat controller was also applied to stabilization and the Gevrey regularity property for coupled Schrödinger and heat equations in [22]. The exponential stability and Riesz basis property for coupled heat equation and elastic structure were discussed in [5, 6]. It is seen that the heat equation is mainly motivated for these coupled systems through boundary weak connections. The main reason is that the heat equation has much more regularity which is transmitted without time from boundary to the whole coupled system through boundary connections. From mathematical point of view, all these coupled system are of compact resolvent and hence only the point spectra are available for these systems.

In this paper, we consider a hyperbolic-hyperbolic coupled system consisting of an Euler-Bernoulli beam and a wave equation where the structural damping of the wave equation is taken into account making the wave subsystem part have more regularity likewise the heat equation. The system is described by following partial differential equations:

$$\begin{cases} w_{tt}(x,t) + w_{xxxx}(x,t) = 0, & 0 < x < 1, t > 0, \\ u_{tt}(x,t) = u_{xx}(x,t) + \beta u_{xxt}(x,t), & 0 < x < 1, t > 0, \\ w(1,t) = w_{xx}(1,t) = w(0,t) = 0, & t \geq 0, \\ u(1,t) = 0, & t \geq 0, \\ w_{xx}(0,t) = \alpha u_t(0,t), & t \geq 0, \\ \beta u_{xt}(0,t) + u_x(0,t) = -\alpha w_{xt}(0,t), & t \geq 0, \\ w(x,0) = w_0(x), w_t(x,0) = w_1(x), & 0 \leq x \leq 1, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & 0 \leq x \leq 1, \end{cases} \tag{1.3}$$

where (w_0, w_1, u_0, u_1) is the initial state and $\alpha \neq 0$, $\beta > 0$ are constants. It is seen that the wave subsystem has taken the effect of the structural damping $\beta u_{xxt}(x,t)$ into account, and the connection between the beam and wave is performed only through boundaries. The total energy of system (1.3) is given by

$$E(t) = \frac{1}{2}\int_0^1 [w_t^2(x,t) + w_{xx}^2(x,t) + u_t^2(x,t) + u_x^2(x,t)]\mathrm{d}x.$$

Formally, the derivative of $E(t)$ with respect to time t satisfies

$$\dot{E}(t) = -\int_0^1 \beta u_{xt}^2(x,t)\mathrm{d}x \le 0, \tag{1.4}$$

which shows that $E(t)$ is non-increasing with time. However, the right-hand side of (1.4) has no explicit terms for the part of the beam subsystem. We refer such boundary connections to as boundary weak connections. This gives rise to a serious problem for the stability of system (1.3). In this paper, we adopt the Riesz basis approach to tackle this problem, by which we are not only able to conclude the exponential stability but also the regularity of system (1.3).

A general mathematical model of elastic systems with structural damping was proposed in [2]. The controllability of a wave equation with structural damping was investigated in [16]. Paper [17] studied the rate of decay of solutions to a wave equation with structural damping in the whole spatial space. It is also noted that the structural damping is a special case of the general Kelvin-Voigt damping studied in [10].

We proceed as follows. In next section, Section 2, we transform system (1.3) into an evolution equation in the energy Hilbert space and the well-possedness of the system is then concluded by C_0-semigroup theory, and the main result is stated. In Section 3, we analyze the distribution of spectrum to obtain the asymptotic expansion of eigenvalues and eigenfunctions. It is shown that the eigenvalues of system (1.3) consist of three branches: one branch is basically along the real axis and accumulates to a finite point; the second branch is also along the real line and is represented by the wave equation part; and the third branch, very similar to the one studied in [9, 21], distributes along two parabola likewise symmetric with the real axis, which represents the beam equation part but is strongly affected by the wave part with real parts of spectra approaching infinity. With a careful estimation of the resolvent operator of the system, the completeness of the root subspace of the system is concluded in Section 4. Section 5 is devoted to the Riesz basis generation of the system, by which we can show in Section 6 that the C_0-semigroup associated with system has the Gevrey regularity which lies between differentiable semigroups and analytic semigroups [1, 19, 20]. The exponential stability is concluded by the spectrum-determined growth condition which is a consequence of the Riesz basis property. Some concluding remarks are presented in Section 7.

2. Well-Posedness and main result

We consider system (1.3) in the energy Hilbert space $\mathcal{H} = H_L^2 \times L^2 \times H_L^1 \times L^2$ with $H_L^2 = \{f | f \in H^2(0,1), f(0) = f(1) = 0\}$, $H_L^1 = \{h | h \in H^1(0,1), h(1) = 0\}$ and the norm in $\mathcal{H}$ is induced by the following inner product

$$\langle X_1, X_2\rangle = \int_0^1 [f_1^{''}(x)\overline{f_2^{''}(x)} + g_1(x)\overline{g_2(x)} + h_1^{'}(x)\overline{h_2^{'}(x)} + l_1(x)\overline{l_2(x)}]\mathrm{d}x,$$

for all $X_i = (f_i, g_i, h_i, l_i) \in \mathcal{H}, i = 1, 2$. Define the system operator A by

$$\begin{cases} A(f,g,h,l) = (g, -f^{(4)}, l, (h^{'} + \beta l^{'})^{'}), \forall\, (f,g,h,l) \in D(A), \\ D(A) = \left\{ \begin{array}{c|l} & h^{'} + \beta l^{'} \in H^1(0,1), \\ & f^{''}(1) = 0, \\ (f,g,h,l) \in \mathcal{H}, & g(0) = g(1) = l(1) = 0, \\ A(f,g,h,l) \in \mathcal{H} & f^{''}(0) = \alpha l(0), \\ & \beta l^{'}(0) + h^{'}(0) = -\alpha g^{'}(0) \end{array} \right\}. \end{cases} \tag{2.1}$$

Then system (1.3) can be written as an abstract Cauchy problem in $\mathcal{H}$:

$$\begin{cases} \dot{X}(t) = AX(t), t > 0, \\ X(0) = X_0, \end{cases} \tag{2.2}$$

where $X(t) = (w(\cdot,t), w_t(\cdot,t), u(\cdot,t), u_t(\cdot,t))$ and $X_0 = (\omega_0, \omega_1, u_0, u_1)$.

Theorem 2.1. *Let A be defined by (2.1). Then, A^{-1} exists. Moreover, A generates a C_0-semigroup e^{At} of contractions on $\mathcal{H}$.*

Proof. For any $(\phi, \psi, \omega, \nu) \in \mathcal{H}$, solve

$$A(f, g, h, l) = (\phi, \psi, \omega, \nu)$$

to obtain $g(x) = \phi(x)$ and $l(x) = \omega(x)$. To get $h(x)$, we solve

$$\begin{cases} h^{''}(x) = \nu(x) - \beta\omega^{''}(x), \\ h(1) = 0, h^{'}(0) = -\alpha\phi^{'}(0) - \beta\omega^{'}(0), \end{cases}$$

to obtain

$$\begin{aligned} h(x) &= (\alpha\phi^{'}(0) + \beta\omega^{'}(0))(1-x) \\ &\quad - \left[\int_0^x (1-x)(\nu(\xi) - \beta\omega^{''}(\xi))\mathrm{d}\xi + \int_x^1 (1-\xi)(\nu(\xi) - \beta\omega^{''}(\xi))\mathrm{d}\xi\right]. \end{aligned} \tag{2.3}$$

For $f(x)$, we solve

$$\begin{cases} f^{(4)}(x) = -\psi(x), \\ f(0) = f(1) = f^{''}(1) = 0, f^{''}(0) = \alpha\omega(0), \end{cases}$$

to obtain

$$\begin{cases} f(x) = \displaystyle\int_0^x (x-\xi)p(\xi)\mathrm{d}\xi - x\int_0^1 (1-\xi)p(\xi)\mathrm{d}\xi, \\ p(x) = \displaystyle\int_0^x (\xi - x)\psi(\xi)\mathrm{d}\xi + x\int_0^1 (1-\xi)\psi(\xi)\mathrm{d}\xi + \alpha\omega(0)(1-x). \end{cases} \tag{2.4}$$

By (2.3) and (2.4), $g(x) = \phi(x)$ and $l(x) = \omega(x)$. It is easy to check that

$$h^{'}(x) + \beta l^{'}(x) = \int_0^x \nu(\xi)\mathrm{d}\xi - \alpha\phi^{'}(0) \in H^1(0,1).$$

Thus, we obtain a unique $(f, g, h, l) \in D(A)$. Obviously, A^{-1} is bounded and hence $0 \in \rho(A)$. We next show that A is dissipative in $\mathcal{H}$. Setting $X = (f, g, h, l) \in D(A)$, we have

$$\begin{aligned} \langle AX, X\rangle = \langle (g, -f^{(4)}, l, (h^{'} + \beta l^{'})^{'}), (f, g, h, l)\rangle &= \int_0^1 \left[g^{''}\overline{f^{''}} - f^{(4)}\overline{g} + l^{'}\overline{h^{'}} + (h^{'} + \beta l^{'})^{'}\overline{l}\right]\mathrm{d}x \\ &= \int_0^1 \left[g^{''}\overline{f^{''}} - f^{''}\overline{g^{''}} + l^{'}\overline{h^{'}} - (h^{'} + \beta l^{'})\overline{l^{'}}\right]\mathrm{d}x - f^{(3)}g|_0^1 + f^{''}\overline{g^{'}}|_0^1 + (h^{'} + \beta l^{'})\overline{l}|_0^1 \end{aligned}$$

$$= \int_0^1 \left[g^{''}\overline{f''} - f^{''}\overline{g''} + l^{'}\overline{h'} - h^{'}\overline{l'}\right] \mathrm{d}x + l(0)[\beta\overline{l'(0)} - \overline{h'(0)}] - \overline{l(0)}[\beta l^{'}(0) - h^{'}(0)]$$

$$- \beta \int_0^1 |l^{'}|^2 \mathrm{d}x,$$

and hence

$$\mathrm{Re}\langle AX, X\rangle = -\beta \int_0^1 |l^{'}(x)|^2 \mathrm{d}x \le 0. \tag{2.5}$$

This shows that A is dissipative and therefore, A generates a C_0-semigroup e^{At} of contractions on $\mathcal{H}$ by the Lumer-Philips Theorem ([15], Thm. 4.3, p.14). □

The main result of this paper is stated in succeeding Theorem 2.2 which will be proved at the end of Section 5.

Theorem 2.2. *Let A be defined by (2.1). Then, the spectrum-determined growth condition holds for e^{At}: $\omega(A) = S(A)$ where $\omega(A) = \inf\{\omega|$ there exists an M such that $\|\mathrm{e}^{At}\| \le M e^{\omega t}\}$ is the growth bound of the C_0-semigroup, and $S(A) = \sup\{\mathrm{Re}(\lambda)|\lambda \in \sigma(A)\}$ is the spectral bound of A. Furthermore, the C_0-semigroup e^{At} is exponentially stable:*

$$\|\mathrm{e}^{At}\| \le M\mathrm{e}^{-\mu t},$$

for some $M, \mu > 0$ and $\mu \le 1/\beta$.

3. Spectral analysis

In this section, we consider eigenvalue problem for system operator A. Let $AX = \lambda X$, where $X = (f, g, h, l) \in D(A)$. Then, $g(x) = \lambda f(x), l(x) = \lambda h(x)$, and $f(x)$ and $h(x)$ satisfy:

$$\begin{cases} f^{(4)}(x) + \lambda^2 f(x) = 0, \\ (1+\beta\lambda)h^{''}(x) = \lambda^2 h(x), \\ f(1) = f^{''}(1) = f(0) = h(1) = 0, \\ f^{''}(0) = \alpha\lambda h(0), \\ (1+\beta\lambda)h^{'}(0) = -\alpha\lambda f^{'}(0). \end{cases} \tag{3.1}$$

From the eigenvalue problem (3.1), we can say a few words on the principle of connections of system (1.3). In the first boundary connection $f^{''}(0) = \alpha\lambda h(0)$, the $f''(0)$ is the same order as $\lambda h(0)$; and in the second connection $(1+\beta\lambda)h^{'}(0) = -\alpha\lambda f^{'}(0)$, $h^{'}(0)$ and $f^{'}(0)$ are also of the same order, both with respect to λ. If the wave is only a lower order perturbation of the beam not as strong feedback control as what we have in this paper, the exponential stability for the whole system is not expected. Physically, $w_{xx}(0,t) = \alpha u_t(0,t)$ means that the bending moment of the beam is under the feedback of the velocity of the wave by choosing α so that the both sides are equidimensional. Precisely, the dimension of the bending moment $w_{xx}(0,t)$ is $KN.M$ and the dimension of velocity $u_t(0,t)$ is M/S. To make $w_{xx}(0,t) = \alpha u_t(0,t)$ physically meaningful, we need to take the dimension of α to be $KN.M/(M/S)$.

Lemma 3.1. *Let A be defined by (2.1). Then, $\mathrm{Re}(\lambda) < 0$ for any $\lambda \in \sigma_p(A)$, the point spectrum of A.*

Proof. Since by Theorem 2.1 A is dissipative, it must have $\mathrm{Re}(\lambda) \le 0$ for every $\lambda \in \sigma_p(A)$. We thus only need to show that there is no eigenvalue of A located on the imaginary axis. Letting $0 \ne \lambda = \mathrm{i}\rho^2 \in \sigma_p(A)$ with $\rho \in \mathbb{R}^+$ and $X = (f, g, h, l) \in D(A)$ be the corresponding eigenfunction, it follows from (2.5) that

$$0 = \mathrm{Re}\langle \mathrm{i}\rho^2 X, X\rangle = \mathrm{Re}\langle AX, X\rangle = -\beta \int_0^1 |l^{'}(x)|^2 \mathrm{d}x.$$

Hence, $l^{'}(x) = 0$ and so $h^{'}(x) = 0$. By $h(1) = l(1) = 0$, it has $h(x) = l(x) = 0$, and hence $f^{'}(0) = 0$. This, together with (3.1), shows that $f(x)$ satisfies

$$\begin{cases} f^{(4)}(x) = \rho^4 f(x), \\ f(0) = f(1) = f^{'}(0) = f''(0) = f''(1) = 0. \end{cases}$$

It is easily shown that $f(x) = 0$ and hence $g(x) = 0$. Therefore, there is no eigenvalue of A located on the imaginary axis, proving the lemma. □

Setting $\lambda = \mathrm{i}\rho^2$ in (3.1), we obtain the eigenvalue system of (1.3):

$$\begin{cases} f^{(4)}(x) - \rho^4 f(x) = 0, \\ h^{''}(x) = \dfrac{-\rho^4}{1 + \mathrm{i}\beta\rho^2} h(x), \\ f(0) = f^{''}(1) = f(0) = h(1) = 0, \\ f^{''}(0) = \mathrm{i}\alpha\rho^2 h(0), \\ (1 + \mathrm{i}\beta\rho^2) h^{'}(0) = -\mathrm{i}\alpha\rho^2 f^{'}(0). \end{cases} \tag{3.2}$$

By Lemma 3.1, all eigenvalues are located on the open left complex plane. Let $a = \sqrt{\frac{\lambda^2}{1+\beta\lambda}}$. Then, the general solution of (3.2) can be expressed as

$$f(x) = c_1 \mathrm{e}^{\rho x} + c_2 \mathrm{e}^{-\rho x} + c_3 \mathrm{e}^{\mathrm{i}\rho x} + c_4 \mathrm{e}^{-\mathrm{i}\rho x},\ h(x) = d_1 \mathrm{e}^{ax} + d_2 \mathrm{e}^{-ax}, \tag{3.3}$$

where $c_i, i = 1, 2, 3, 4; d_j, j = 1, 2$ are scalars. By the boundary conditions of (3.2), we obtain that $c_i, i = 1, 2, 3, 4; d_j, j = 1, 2$ are not identical to zero if and only if $\det(B) = 0$, where

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ \mathrm{e}^{\rho} & \mathrm{e}^{-\rho} & \mathrm{e}^{\mathrm{i}\rho} & \mathrm{e}^{-\mathrm{i}\rho} & 0 & 0 \\ \rho^2 \mathrm{e}^{\rho} & \rho^2 \mathrm{e}^{-\rho} & -\rho^2 \mathrm{e}^{\mathrm{i}\rho} & -\rho^2 \mathrm{e}^{-\mathrm{i}\rho} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{e}^{a} & \mathrm{e}^{-a} \\ \rho^2 & \rho^2 & -\rho^2 & -\rho^2 & -\mathrm{i}\alpha\rho^2 & -\mathrm{i}\alpha\rho^2 \\ -\frac{\mathrm{i}a^2\alpha}{\rho} & \frac{\mathrm{i}a^2\alpha}{\rho} & \frac{a^2\alpha}{\rho} & -\frac{a^2\alpha}{\rho} & a & -a \end{pmatrix}. \tag{3.4}$$

Since by Lemma 3.1, all eigenvalues are symmetric to the real axis, we only need to consider those λ which lie in the second quadrant of the complex plane:

$$\lambda := \mathrm{i}\rho^2, \rho \in \mathcal{S} := \left\{ \rho \in \mathbb{C} \middle| 0 \le \arg\rho \le \frac{\pi}{4} \right\}.$$

For any $\rho \in \mathcal{S}$,

$$\mathrm{Re}(-\rho) = -|\rho|\cos(\arg\rho) \le -\frac{\sqrt{2}}{2}|\rho| < 0, \tag{3.5}$$

Denote $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ with

$$\mathcal{S}_1 = \{\rho \in \mathbb{C} | \pi/8 < \arg\rho \le \pi/4\}, \ \ \mathcal{S}_2 = \{\rho \in \mathbb{C} | 0 \le \arg\rho \le \pi/8\}. \tag{3.6}$$

The succeeding Theorem 3.2 gives asymptotic distributions of the eigenvalues in $\mathcal{S}_1$ and $\mathcal{S}_2$.

Theorem 3.2. *Let A be defined by (2.1). Then, the eigenvalues of A have two families:*

$$\sigma_p(A) = \{\lambda_{1n}^+, \lambda_{1n}^-, n \in \mathbb{N}\} \cup \{\lambda_{2n}, \overline{\lambda_{2n}}, n \in \mathbb{N}\}, \tag{3.7}$$

where $\lambda_{1n}^+, \lambda_{1n}^-$ and λ_{2n} have the following asymptotic expansions:

$$\begin{cases} \lambda_{1n}^+ = -\left(n\pi + \dfrac{\theta_1}{2}\right)^2 \beta - \dfrac{1}{\beta} + O\left(\dfrac{1}{n^2}\right), \\ \lambda_{1n}^- = -\dfrac{1}{\beta} - \dfrac{1}{(n\pi + \frac{\theta_1}{2})^2 \beta^3} + O\left(\dfrac{1}{n^3}\right), \\ \lambda_{2n} = \left(n\pi + \dfrac{\theta_2}{2}\right)\ln r + \mathrm{i}\left[\left(n\pi + \dfrac{\theta_2}{2}\right)^2 - \left(\dfrac{\ln r}{2}\right)^2\right] + O\left(\dfrac{1}{n}\right), \end{cases} \tag{3.8}$$

where n are positive integers and

$$\theta_1 = \arctan\frac{2\sqrt{2}\alpha^2\sqrt{\beta}}{2\beta - \alpha^4}, \ \theta_2 = \arctan\frac{\sqrt{2}\alpha^2}{2\sqrt{\beta}}, \ r = \sqrt{\frac{\alpha^4 + 2\beta}{\alpha^4 + 2\sqrt{2}\alpha^2\sqrt{\beta} + 2\beta}} < 1, \ \ln r < 0. \tag{3.9}$$

As a result, for any $\alpha \neq 0$ and $\beta > 0$,

$$\mathrm{Re}(\lambda_{1n}^+), \ \mathrm{Re}(\lambda_{2n}) \to -\infty, \ \mathrm{Re}(\lambda_{1n}^-) \to -\frac{1}{\beta} \ \text{as } n \to \infty. \tag{3.10}$$

Proof. When $\rho \in \mathcal{S}_1$,

$$\mathrm{Re}(\mathrm{i}\rho) = -|\rho|\sin(\arg\rho) \le -|\rho|\sin(\pi/8) < 0. \tag{3.11}$$

Combining with (3.5), it has

$$|\mathrm{e}^{-\rho}| = O(\mathrm{e}^{-|\rho|}), |\mathrm{e}^{\mathrm{i}\rho}| = O(\mathrm{e}^{-|\rho|}). \tag{3.12}$$

Since

$$a = \sqrt{\frac{\lambda^2}{1 + \beta\lambda}} = \sqrt{\frac{-\rho^4}{1 + \mathrm{i}\beta\rho^2}} = \frac{\sqrt{\mathrm{i}}}{\sqrt{\beta}}\rho + O(|\rho|^{-1}) \ \text{as } |\rho| \to \infty$$

and

$$-\mathrm{Re}(\sqrt{\mathrm{i}}\rho) = -|\rho|\cos(\arg\rho + \pi/4) \le 0,$$

it has

$$|\mathrm{e}^{-a}| = |\mathrm{e}^{\frac{\sqrt{i}\rho}{\sqrt{\beta}}}| + O(|\rho|^{-1}) \le 1. \tag{3.13}$$

By multiplying some factors, we make each entry of the matrix $\det(B)$ be bounded as $\rho \to \infty$:

$$\frac{\mathrm{e}^{-\rho}\mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{-a}}{a\rho^4}\det(B) = \begin{vmatrix} \mathrm{e}^{-\rho} & 1 & 1 & \mathrm{e}^{\mathrm{i}\rho} & 0 & 0 \\ 1 & \mathrm{e}^{-\rho} & \mathrm{e}^{\mathrm{i}\rho} & 1 & 0 & 0 \\ 1 & \mathrm{e}^{-\rho} & -\mathrm{e}^{\mathrm{i}\rho} & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \mathrm{e}^{-a} \\ \mathrm{e}^{-\rho} & 1 & -1 & -\mathrm{e}^{\mathrm{i}\rho} & -\mathrm{i}\alpha\mathrm{e}^{-a} & -\mathrm{i}\alpha \\ -\frac{\mathrm{i}a\alpha}{\rho}\mathrm{e}^{-\rho} & \frac{\mathrm{i}a\alpha}{\rho} & \frac{a\alpha}{\rho} & -\frac{a\alpha}{\rho}\mathrm{e}^{\mathrm{i}\rho} & \mathrm{e}^{-a} & -1 \end{vmatrix}. \tag{3.14}$$

By (3.12), (3.13), and the fact $\frac{a\alpha}{\rho} = \frac{\alpha\sqrt{\mathrm{i}}}{\sqrt{\beta}} + O(|\rho|^{-2})$, it is easily seen that

$$\begin{aligned} \frac{\mathrm{e}^{-\rho}\mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{-a}}{a\rho^4}\det(B) &= \begin{vmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \mathrm{e}^{-a} \\ 0 & 1 & -1 & 0 & -\mathrm{i}\alpha\mathrm{e}^{-a} & -\mathrm{i}\alpha \\ 0 & \frac{(-1)^{3/4}\alpha}{\sqrt{\beta}} & \frac{\sqrt[4]{-1}\alpha}{\sqrt{\beta}} & 0 & \mathrm{e}^{-a} & -1 \end{vmatrix} + O(|\rho|^{-2}) \\ &= \left(4 + \mathrm{i}\frac{2\sqrt{2}\alpha^2}{\sqrt{\beta}}\right) + e^{-2a}\left(4 - \mathrm{i}\frac{2\sqrt{2}\alpha^2}{\sqrt{\beta}}\right) + O(|\rho|^{-2}). \end{aligned}$$

From this, we see that $\det(B) = 0$ if and only if

$$\mathrm{e}^{-2a} = \frac{4 + \mathrm{i}\frac{2\sqrt{2}\alpha^2}{\sqrt{\beta}}}{-4 + \mathrm{i}\frac{2\sqrt{2}\alpha^2}{\sqrt{\beta}}} + O(|\rho|^{-2}) = \mathrm{e}^{\mathrm{i}\theta_1} + O(|a|^{-2}), \tag{3.15}$$

where θ_1 is given by (3.9). The roots of $\mathrm{e}^{-2a} = \mathrm{e}^{\mathrm{i}\theta_1}$ are

$$a = -\mathrm{i}\left(n\pi + \frac{\theta_1}{2}\right),\ n = 0, 1, 2, \ldots.$$

By using Rouché's theorem, the roots of (3.15) have the following asymptotic expression

$$a = -\mathrm{i}\left(n\pi + \frac{\theta_1}{2}\right) + O\left(\frac{1}{n^2}\right),\ n > N_1, \tag{3.16}$$

where N_1 is a sufficiently large positive integer. Since $a=\sqrt{\frac{\lambda^2}{1+\beta\lambda}}$ or $\lambda^2-\beta a^2\lambda-a^2=0$, it has

$$\lambda_{1n}^{\pm}=\frac{\beta a^2}{2}\left(1\pm\sqrt{1+\frac{4}{\beta^2a^2}}\right).$$

Using the Taylor expansion, we obtain the expressions of λ_{1n}^{+} and λ_{1n}^{-} given by (3.8). Moreover, by using $\lambda=\mathrm{i}\rho^2$, we obtain the asymptotic expressions of ρ_{1n}^{+} and ρ_{1n}^{-}:

$$\begin{cases}\rho_{1n}^{+}=\sqrt{\mathrm{i}\beta}\left(n\pi+\dfrac{\theta_1}{2}\right)+O\left(\dfrac{1}{n}\right),\\ \rho_{1n}^{-}=\sqrt{\dfrac{\mathrm{i}}{\beta}}\left[1+\dfrac{1}{2\beta^2\left(n\pi+\frac{\theta_1}{2}\right)^2}+O\left(\dfrac{1}{n^3}\right)\right].\end{cases}\tag{3.17}$$

Similarly, when $\rho\in\mathcal{S}_2$, it is easily to verify that there exists a $\gamma>0$ such that

$$\begin{cases}\mathrm{Re}(-a)\le\gamma|\rho|,\\ \mathrm{Re}(\mathrm{i}\rho)=-|\rho|\sin(\arg\rho)\le-|\rho|\sin 0\le 0,\end{cases}\tag{3.18}$$

which, together with (3.5), gives

$$|\mathrm{e}^{-\rho}|=O(\mathrm{e}^{-|\rho|}),\ |\mathrm{e}^{-a}|=O(\mathrm{e}^{-\gamma|\rho|}),\ |\mathrm{e}^{\mathrm{i}\rho}|\le 1.\tag{3.19}$$

By (3.14), (3.19) and the fact $\frac{a\alpha}{\rho}=\frac{\alpha\sqrt{\mathrm{i}}}{\sqrt{\beta}}+O(|\rho|^{-2})$,

$$\begin{aligned}\frac{\mathrm{e}^{-\rho}\mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{-a}}{a\rho^4}\det(B)&=\begin{vmatrix}0&1&1&\mathrm{e}^{\mathrm{i}\rho}&0&0\\1&0&\mathrm{e}^{\mathrm{i}\rho}&1&0&0\\1&0&-\mathrm{e}^{\mathrm{i}\rho}&-1&0&0\\0&0&0&0&1&0\\0&1&-1&-\mathrm{e}^{\mathrm{i}\rho}&0&-\mathrm{i}\alpha\\0&\frac{(-1)^{3/4}\alpha}{\sqrt{\beta}}&\frac{\sqrt[4]{-1}\alpha}{\sqrt{\beta}}&-\frac{\sqrt[4]{-1}\alpha}{\sqrt{\beta}}\mathrm{e}^{\mathrm{i}\rho}&0&-1\end{vmatrix}+O(|\rho|^{-2})\\&=\frac{\mathrm{e}^{2\mathrm{i}\rho}\left(-2\sqrt{2}\alpha^2-4\sqrt{\beta}\right)}{\sqrt{\beta}}+\frac{4\sqrt{\beta}+\mathrm{i}2\sqrt{2}\alpha^2}{\sqrt{\beta}}+O(|\rho|^{-2}).\end{aligned}$$

From this, we can easily derive that $\det(B)=0$ if and only if

$$\mathrm{e}^{2\mathrm{i}\rho}=\frac{2\sqrt{\beta}+\mathrm{i}\sqrt{2}\alpha^2}{\sqrt{2}\alpha^2+2\sqrt{\beta}}+O(|\rho|^{-2})=r\mathrm{e}^{\mathrm{i}\theta_2}+O(|\rho|^{-2}),\tag{3.20}$$

where θ_2 and r are given by (3.9). The roots of $\mathrm{e}^{2\mathrm{i}\rho}=r\mathrm{e}^{\mathrm{i}\theta_2}$ are

$$\rho_{2n}=\frac{1}{2\mathrm{i}}[\ln r+\mathrm{i}(\theta_2+2n\pi)],\ n=0,1,2,\ldots$$

Once again, by Rouché's theorem, the roots of (3.20) have the following asymptotic expression

$$\rho_{2n}=\frac{1}{2\mathrm{i}}\left[\ln r+\mathrm{i}\left(\theta_2+2n\pi\right)\right]+O\left(\frac{1}{n}\right),\ n>N_2,\tag{3.21}$$

where N_2 is a sufficiently large positive integer. By using $\lambda = \mathrm{i}\rho^2$, we eventually get λ_{2n} given by (3.8). □

Remark 3.3. By the asymptotic expression (3.8), we can see the relationship between eigenvalues and α, β. Actually, from θ_1 and θ_2 given by (3.9), we can see that $\alpha = 0$ represents the situation where no connection occurs for wave and beam, and the three branches of the eigenvalues are those of wave and beam systems separately:

$$\begin{cases} \lambda_{1n}^{+} = -(n\pi)^2\beta - \dfrac{1}{\beta} + O\left(\dfrac{1}{n^2}\right), \\ \lambda_{1n}^{-} = -\dfrac{1}{\beta} - \dfrac{1}{(n\pi)^2\beta^3} + O\left(\dfrac{1}{n^3}\right), \\ \lambda_{2n} = \mathrm{i}\,(n\pi)^2 + O\left(\dfrac{1}{n}\right), \end{cases}$$

where the wave eigenvalues are consistent with the eigenvalues of single wave equation obtained in [4, 10]. When $\alpha \neq 0$, these relations can be seen clearly from (3.8) and (3.9), where β plays an important role for the asymptotic behavior of the eigenvalues.

Theorem 3.4. *Let A be defined by (2.1) and $\sigma_p(A) = \{\lambda_{1n}^{+}, \lambda_{1n}^{-}, n \in \mathbb{N}\} \cup \{\lambda_{2n}, \overline{\lambda_{2n}}, n \in \mathbb{N}\}$, be the point spectrum of A. Let $\lambda_{1n}^{+} = \mathrm{i}(\rho_{1n}^{+})^2$, $\lambda_{1n}^{-} = \mathrm{i}(\rho_{1n}^{-})^2$ and $\lambda_{2n} = \mathrm{i}(\rho_{2n})^2$ with ρ_{1n}^{+}, ρ_{1n}^{-} and ρ_{2n} given by (3.17) and (3.21), respectively. Then, there are three families of approximated normalized eigenfunctions of A:*

(1) *One family $\{\Phi_{1n}^{+} = (f_{1n}^{+}, \lambda f_{1n}^{+}, h_{1n}^{+}, \lambda h_{1n}^{+}), n \in \mathbb{N}\}$, where Φ_{1n}^{+} is the eigenfunction of A corresponding to the eigenvalue λ_{1n}^{+}, has the following asymptotic expression:*

$$\begin{pmatrix} (f_{1n}^{+})''(x) \\ \lambda f_{1n}^{+}(x) \\ (h_{1n}^{+})'(x) \\ \lambda h_{1n}^{+}(x) \end{pmatrix} = \begin{pmatrix} \sqrt{2}\mathrm{i}\alpha(\mathrm{e}^{\mathrm{i}\rho_{1n}^{+}x} + \mathrm{e}^{-\rho_{1n}^{+}x}) \\ \sqrt{2}\alpha(\mathrm{e}^{\mathrm{i}\rho_{1n}^{+}x} + \mathrm{e}^{-\rho_{1n}^{+}x}) \\ 0 \\ \mathrm{e}^{-ax}\left(\sqrt{2}\mathrm{i} + \frac{\alpha^2}{\sqrt{\beta}}\right) + \mathrm{e}^{ax}\left(\sqrt{2}\mathrm{i} - \frac{\alpha^2}{\sqrt{\beta}}\right) \end{pmatrix} + O\left(\frac{1}{n}\right), \tag{3.22}$$

where ρ_{1n}^{+} and a are given by (3.17) and (3.16) respectively.

(2) *The second family $\{\Phi_{1n}^{-} = (f_{1n}^{-}, \lambda f_{1n}^{-}, h_{1n}^{-}, \lambda h_{1n}^{-}), n \in \mathbb{N}\}$, where Φ_{1n}^{-} is the eigenfunction of A corresponding to the eigenvalue λ_{1n}^{-}, has the following asymptotic expression:*

$$\begin{pmatrix} (f_{1n}^{-})''(x) \\ \lambda f_{1n}^{-}(x) \\ (h_{1n}^{-})'(x) \\ \lambda h_{1n}^{-}(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \mathrm{e}^{ax} + \mathrm{e}^{-ax} \\ 0 \end{pmatrix} + O\left(\frac{1}{n}\right). \tag{3.23}$$

where a is given by (3.16).

(3) *The third family $\{\Phi_{2n} = (f_{2n}, \lambda f_{2n}, h_{2n}, \lambda h_{2n}), n \in \mathbb{N}\}$, where Φ_{2n} is the eigenfunction of A corresponding to the eigenvalue λ_{2n}, has the following asymptotic expression:*

$$\begin{pmatrix} (f_{2n})''(x) \\ \lambda f_{2n}(x) \\ (h_{2n})'(x) \\ \lambda h_{2n}(x) \end{pmatrix} = \begin{pmatrix} \mathrm{i}(\mathrm{e}^{\mathrm{i}\rho_{2n}(1-x)} - \mathrm{e}^{-\mathrm{i}\rho_{2n}(1-x)} + 2\mathrm{i}\sin\rho_{2n}\mathrm{e}^{-\rho_{2n}x}) \\ \mathrm{e}^{\mathrm{i}\rho_{2n}(1-x)} - \mathrm{e}^{-\mathrm{i}\rho_{2n}(1-x)} - 2\mathrm{i}\sin\rho_{2n}\mathrm{e}^{-\rho_{2n}x} \\ 0 \\ 0 \end{pmatrix} + O\left(\frac{1}{n}\right), \tag{3.24}$$

where ρ_{2n} is given by (3.21).

Proof. Firstly, we look for Φ_{1n}^{+} associated with λ_{1n}^{+}. From the expression of ρ_{1n}^{+} given by (3.17) and a given by (3.16), it has

$$\begin{cases} \mathrm{e}^{\mathrm{i}\rho_{1n}^{+}x} = \mathrm{e}^{(-\frac{\sqrt{2\beta}}{2}+\frac{\sqrt{2\beta}}{2}\mathrm{i})(n\pi+\frac{\theta_1}{2})x+O(\frac{1}{n})},\ \mathrm{e}^{-\rho_{1n}^{+}x} = \mathrm{e}^{(-\frac{\sqrt{2\beta}}{2}-\frac{\sqrt{2\beta}}{2}\mathrm{i})(n\pi+\frac{\theta_1}{2})x+O(\frac{1}{n})}, \\ \mathrm{e}^{ax} = \mathrm{e}^{-\mathrm{i}(n\pi+\frac{\theta_1}{2})x+O(\frac{1}{n^2})},\ \mathrm{e}^{-ax} = \mathrm{e}^{\mathrm{i}(n\pi+\frac{\theta_1}{2})x+O(\frac{1}{n^2})}. \end{cases} \tag{3.25}$$

According to (3.4) and some linear algebra calculations, for ρ_{1n}^{+} given by (3.17), one gets

$$f_1^{+}(x) = \frac{\mathrm{e}^{\mathrm{i}\rho}}{a\rho^6\mathrm{e}^{\rho}} \begin{vmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ \mathrm{e}^{\rho} & \mathrm{e}^{-\rho} & \mathrm{e}^{\mathrm{i}\rho} & \mathrm{e}^{-\mathrm{i}\rho} & 0 & 0 \\ \rho^2\mathrm{e}^{\rho} & \rho^2\mathrm{e}^{-\rho} & -\rho^2\mathrm{e}^{\mathrm{i}\rho} & -\rho^2\mathrm{e}^{-\mathrm{i}\rho} & 0 & 0 \\ \mathrm{e}^{\rho x} & \mathrm{e}^{-\rho x} & \mathrm{e}^{\mathrm{i}\rho x} & \mathrm{e}^{-\mathrm{i}\rho x} & 0 & 0 \\ \rho^2 & \rho^2 & -\rho^2 & -\rho^2 & -\mathrm{i}\alpha\rho^2 & -\mathrm{i}\alpha\rho^2 \\ -\frac{\mathrm{i}a^2\alpha}{\rho} & \frac{\mathrm{i}a^2\alpha}{\rho} & \frac{a^2\alpha}{\rho} & -\frac{a^2\alpha}{\rho} & a & -a \end{vmatrix}.$$

By (3.12), we can write

$$f_1^{+}(x) = \frac{1}{\rho^2} \begin{vmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ \mathrm{e}^{-\rho(1-x)} & \mathrm{e}^{-\rho x} & \mathrm{e}^{\mathrm{i}\rho x} & \mathrm{e}^{\mathrm{i}\rho(1-x)} & 0 & 0 \\ 0 & 1 & -1 & 0 & -\mathrm{i}\alpha & -\mathrm{i}\alpha \\ 0 & \frac{\mathrm{i}a\alpha}{\rho} & \frac{a\alpha}{\rho} & 0 & 1 & -1 \end{vmatrix} + O(\mathrm{e}^{-|\rho|}).$$

By (3.25), it has further that

$$f_1^{+}(x) = \frac{2\mathrm{i}\alpha}{\rho^2} \begin{vmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ \mathrm{e}^{-\rho(1-x)} & \mathrm{e}^{-\rho x} & \mathrm{e}^{\mathrm{i}\rho x} & \mathrm{e}^{\mathrm{i}\rho(1-x)} \end{vmatrix} + O(\mathrm{e}^{-|\rho|}) = -\frac{4\mathrm{i}\alpha}{\rho^2}(\mathrm{e}^{\mathrm{i}\rho x} - \mathrm{e}^{-\rho x}) + O(\mathrm{e}^{-|\rho|}).$$

It then follows that

$$(f_1^{+})''(x) = 4\mathrm{i}\alpha(\mathrm{e}^{\mathrm{i}\rho x} + \mathrm{e}^{-\rho x}) + O(\mathrm{e}^{-|\rho|}),$$

and

$$\lambda f_1^{+}(x) = 4\alpha(\mathrm{e}^{\mathrm{i}\rho x} - \mathrm{e}^{-\rho x}) + O(\mathrm{e}^{-|\rho|}).$$

Similarly, by (3.12) and (3.25),

$$
\begin{aligned}
h_1^+(x) &= \frac{e^{i\rho}}{a\rho^6 e^{\rho}} \begin{vmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ e^{\rho} & e^{-\rho} & e^{i\rho} & e^{-i\rho} & 0 & 0 \\ \rho^2 e^{\rho} & \rho^2 e^{-\rho} & -\rho^2 e^{i\rho} & -\rho^2 e^{-i\rho} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{ax} & e^{-ax} \\ \rho^2 & \rho^2 & -\rho^2 & -\rho^2 & -i\alpha\rho^2 & -i\alpha\rho^2 \\ -\frac{ia^2\alpha}{\rho} & \frac{ia^2\alpha}{\rho} & \frac{a^2\alpha}{\rho} & -\frac{a^2\alpha}{\rho} & a & -a \end{vmatrix} \\
&= \frac{1}{\rho^2} \begin{vmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{ax} & e^{-ax} \\ 0 & 1 & -1 & 0 & -i\alpha & -i\alpha \\ 0 & \frac{ia\alpha}{\rho} & \frac{a\alpha}{\rho} & 0 & 1 & -1 \end{vmatrix} + O(e^{-|\rho|}) \\
&= -\frac{2}{\rho^2} \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & e^{ax} & e^{-ax} \\ 1 & -1 & -i\alpha & -i\alpha \\ \frac{ia\alpha}{\rho} & \frac{a\alpha}{\rho} & 1 & -1 \end{vmatrix} = O\left(\frac{1}{n^2}\right).
\end{aligned}
$$

It then follows that

$$
(h_1^+)'(x) = -\frac{2a}{\rho^2} \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & e^{ax} & e^{-ax} \\ 1 & -1 & -i\alpha & -i\alpha \\ \frac{ia\alpha}{\rho} & \frac{a\alpha}{\rho} & 1 & -1 \end{vmatrix} + O(e^{-|\rho|}) = O\left(\frac{1}{n}\right),
$$

and

$$
\begin{aligned}
\lambda h_1^+(x) &= -2i \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & e^{ax} & e^{-ax} \\ 1 & -1 & -i\alpha & -i\alpha \\ \frac{ia\alpha}{\rho} & \frac{a\alpha}{\rho} & 1 & -1 \end{vmatrix} + O(e^{-|\rho|}) \\
&= e^{-ax}\left[4i + \frac{(2-2i)a\alpha^2}{\rho}\right] + e^{ax}\left[4i - \frac{(2-2i)a\alpha^2}{\rho}\right] + O(e^{-|\rho|}) \\
&= e^{-ax}\left(4i + \frac{2\sqrt{2}\alpha^2}{\sqrt{\beta}}\right) + e^{ax}\left(4i - \frac{2\sqrt{2}\alpha^2}{\sqrt{\beta}}\right) + O(e^{-|\rho|}).
\end{aligned}
$$

By setting

$$
\Phi_{1n}^+ = \begin{pmatrix} f_{1n}^+(x) \\ \lambda f_{1n}^+(x) \\ h_{1n}^+(x) \\ \lambda h_{1n}^+(x) \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} f_1^+(x) \\ \lambda f_1^+(x) \\ h_1^+(x) \\ \lambda h_1^+(x) \end{pmatrix},
$$

we obtain (3.22). Now we look for Φ_{1n}^-. According to the expression of ρ_{1n}^- given by (3.17), we can obtain similarly that

$$f_1^-(x) = \frac{1}{a^3\rho^4}\begin{vmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ e^{\rho} & e^{-\rho} & e^{i\rho} & e^{-i\rho} & 0 & 0 \\ \rho^2 e^{\rho} & \rho^2 e^{-\rho} & -\rho^2 e^{i\rho} & -\rho^2 e^{-i\rho} & 0 & 0 \\ e^{\rho x} & e^{-\rho x} & e^{i\rho x} & e^{-i\rho x} & 0 & 0 \\ \rho^2 & \rho^2 & -\rho^2 & -\rho^2 & -i\alpha\rho^2 & -i\alpha\rho^2 \\ -\frac{ia^2\alpha}{\rho} & \frac{ia^2\alpha}{\rho} & \frac{a^2\alpha}{\rho} & -\frac{a^2\alpha}{\rho} & a & -a \end{vmatrix}$$

$$= \frac{i\alpha}{a^2}\begin{vmatrix} 1 & 1 & 1 & 1 \\ e^{\sqrt{\frac{i}{\beta}}} & e^{-\sqrt{\frac{i}{\beta}}} & e^{i\sqrt{\frac{i}{\beta}}} & e^{-i\sqrt{\frac{i}{\beta}}} \\ e^{\sqrt{\frac{i}{\beta}}} & e^{-\sqrt{\frac{i}{\beta}}} & -e^{i\sqrt{\frac{i}{\beta}}} & -e^{-i\sqrt{\frac{i}{\beta}}} \\ e^{\sqrt{\frac{i}{\beta}}x} & e^{-\sqrt{\frac{i}{\beta}}x} & e^{i\sqrt{\frac{i}{\beta}}x} & e^{-i\sqrt{\frac{i}{\beta}}x} \end{vmatrix} + O\left(\frac{1}{n^4}\right) = O\left(\frac{1}{n^2}\right).$$

It then follows that

$$(f_1^-)''(x) = \frac{i\alpha}{a^2}\begin{vmatrix} 1 & 1 & 1 & 1 \\ e^{\sqrt{\frac{i}{\beta}}} & e^{-\sqrt{\frac{i}{\beta}}} & e^{i\sqrt{\frac{i}{\beta}}} & e^{-i\sqrt{\frac{i}{\beta}}} \\ e^{\sqrt{\frac{i}{\beta}}} & e^{-\sqrt{\frac{i}{\beta}}} & -e^{i\sqrt{\frac{i}{\beta}}} & -e^{-i\sqrt{\frac{i}{\beta}}} \\ \frac{i}{\beta}e^{\sqrt{\frac{i}{\beta}}x} & \frac{i}{\beta}e^{-\sqrt{\frac{i}{\beta}}x} & -\frac{i}{\beta}e^{i\sqrt{\frac{i}{\beta}}x} & -\frac{i}{\beta}e^{-i\sqrt{\frac{i}{\beta}}x} \end{vmatrix} + O\left(\frac{1}{n^4}\right) = O\left(\frac{1}{n^2}\right),$$

and

$$\lambda f_1^-(x) = -\frac{\alpha\rho^2}{a^2}\begin{vmatrix} 1 & 1 & 1 & 1 \\ e^{\sqrt{\frac{i}{\beta}}} & e^{-\sqrt{\frac{i}{\beta}}} & e^{i\sqrt{\frac{i}{\beta}}} & e^{-i\sqrt{\frac{i}{\beta}}} \\ e^{\sqrt{\frac{i}{\beta}}} & e^{-\sqrt{\frac{i}{\beta}}} & -e^{i\sqrt{\frac{i}{\beta}}} & -e^{-i\sqrt{\frac{i}{\beta}}} \\ e^{\sqrt{\frac{i}{\beta}}x} & e^{-\sqrt{\frac{i}{\beta}}x} & e^{i\sqrt{\frac{i}{\beta}}x} & e^{-i\sqrt{\frac{i}{\beta}}x} \end{vmatrix} + O\left(\frac{1}{n^4}\right) = O\left(\frac{1}{n^2}\right).$$

Similarly, we further have

$$h_1^-(x) = \frac{1}{a^3\rho^4}\begin{vmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ e^{\rho} & e^{-\rho} & e^{i\rho} & e^{-i\rho} & 0 & 0 \\ e^{\rho}\rho^2 & e^{-\rho}\rho^2 & -e^{i\rho}\rho^2 & -e^{-i\rho}\rho^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{ax} & e^{-ax} \\ \rho^2 & \rho^2 & -\rho^2 & -\rho^2 & -i\alpha\rho^2 & -i\alpha\rho^2 \\ -\frac{ia^2\alpha}{\rho} & \frac{ia^2\alpha}{\rho} & \frac{a^2\alpha}{\rho} & -\frac{a^2\alpha}{\rho} & a & -a \end{vmatrix}$$

$$= \frac{1}{a}\begin{vmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ e^{\sqrt{\frac{i}{\beta}}} & e^{-\sqrt{\frac{i}{\beta}}} & e^{i\sqrt{\frac{i}{\beta}}} & e^{-i\sqrt{\frac{i}{\beta}}} & 0 & 0 \\ e^{\sqrt{\frac{i}{\beta}}} & e^{-\sqrt{\frac{i}{\beta}}} & -e^{i\sqrt{\frac{i}{\beta}}} & -e^{-i\sqrt{\frac{i}{\beta}}} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{ax} & e^{-ax} \\ 1 & 1 & -1 & -1 & -i\alpha & -i\alpha \\ -\sqrt{i\beta}\alpha & \sqrt{i\beta}\alpha & i\sqrt{i\beta}\alpha & -i\sqrt{i\beta}\alpha & 0 & 0 \end{vmatrix} + O\left(\frac{1}{n}\right)$$

$$= -\frac{\mathrm{i}\alpha C_1}{a}(\mathrm{e}^{ax} - \mathrm{e}^{-ax}) + O\left(\frac{1}{n}\right) = O\left(\frac{1}{n}\right),$$

where $C_1 = \begin{vmatrix} 1 & 1 & 1 & 1 \\ e^{\sqrt{\frac{\mathrm{i}}{\beta}}} & e^{-\sqrt{\frac{\mathrm{i}}{\beta}}} & e^{\mathrm{i}\sqrt{\frac{\mathrm{i}}{\beta}}} & e^{-\mathrm{i}\sqrt{\frac{\mathrm{i}}{\beta}}} \\ e^{\sqrt{\frac{\mathrm{i}}{\beta}}} & e^{-\sqrt{\frac{\mathrm{i}}{\beta}}} & -e^{\mathrm{i}\sqrt{\frac{\mathrm{i}}{\beta}}} & -e^{-\mathrm{i}\sqrt{\frac{\mathrm{i}}{\beta}}} \\ -\alpha\sqrt{\mathrm{i}\beta} & \alpha\sqrt{\mathrm{i}\beta} & \mathrm{i}\alpha\sqrt{\mathrm{i}\beta} & -\mathrm{i}\alpha\sqrt{\mathrm{i}\beta} \end{vmatrix}$. It then follows that

$$(h_1^-)'(x) = -\mathrm{i}\alpha C_1(\mathrm{e}^{ax} + \mathrm{e}^{-ax}) + O\left(\frac{1}{n}\right),$$

and

$$\lambda h_1^-(x) = \frac{\mathrm{i}\alpha C_1}{a\beta}(\mathrm{e}^{ax} - \mathrm{e}^{-ax}) + O\left(\frac{1}{n}\right) = O\left(\frac{1}{n}\right).$$

By setting

$$\Phi_{1n}^- = \begin{pmatrix} f_{1n}^-(x) \\ \lambda f_{1n}^-(x) \\ h_{1n}^-(x) \\ \lambda h_{1n}^-(x) \end{pmatrix} = \frac{1}{-\mathrm{i}\alpha C_1}\begin{pmatrix} f_1^-(x) \\ \lambda f_1^-(x) \\ h_1^-(x) \\ \lambda h_1^-(x) \end{pmatrix},$$

we obtain (3.23). Finally, we look for Φ_{2n}. According to the expression of ρ_{2n} given by (3.21),

$$\begin{cases} \mathrm{e}^{\pm\mathrm{i}\rho_{2n}(1-x)} = \mathrm{e}^{\pm\frac{1}{2}[\mathrm{i}(\theta_2+2n\pi)+\ln r](1-x)+O(\frac{1}{n})}, \\ \mathrm{e}^{-\rho x} = \mathrm{e}^{-\frac{1}{2}[\theta_2+2n\pi-\mathrm{i}\ln r]x+O(\frac{1}{n})}. \end{cases} \tag{3.26}$$

This, together with (3.19), gives analogously that

$$\begin{aligned} f_2(x) &= \frac{1}{\rho^6 e^a e^\rho}\begin{vmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ \mathrm{e}^\rho & \mathrm{e}^{-\rho} & \mathrm{e}^{\mathrm{i}\rho} & \mathrm{e}^{-\mathrm{i}\rho} & 0 & 0 \\ \rho^2\mathrm{e}^\rho & \rho^2\mathrm{e}^{-\rho} & -\rho^2\mathrm{e}^{\mathrm{i}\rho} & -\rho^2\mathrm{e}^{-\mathrm{i}\rho} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{e}^a & \mathrm{e}^{-a} \\ \rho^2 & \rho^2 & -\rho^2 & -\rho^2 & -\mathrm{i}\alpha\rho^2 & -\mathrm{i}\alpha\rho^2 \\ \mathrm{e}^{\rho x} & \mathrm{e}^{-\rho x} & \mathrm{e}^{\mathrm{i}\rho x} & \mathrm{e}^{-\mathrm{i}\rho x} & 0 & 0 \end{vmatrix} \\ &= \frac{1}{\rho^2}\begin{vmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & e^{\mathrm{i}\rho} & e^{-\mathrm{i}\rho} & 0 & 0 \\ 1 & 0 & -e^{\mathrm{i}\rho} & -e^{-\mathrm{i}\rho} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & -\mathrm{i}\alpha \\ e^{-\rho(1-x)} & e^{-\rho x} & e^{\mathrm{i}\rho x} & e^{-\mathrm{i}\rho x} & 0 & 0 \end{vmatrix} + O(\mathrm{e}^{-\gamma|\rho|}) \\ &= -\frac{2\mathrm{i}\alpha}{\rho^2}\begin{vmatrix} 1 & 1 & 1 \\ 0 & \mathrm{e}^{\mathrm{i}\rho} & \mathrm{e}^{-\mathrm{i}\rho} \\ \mathrm{e}^{-\rho x} & \mathrm{e}^{\mathrm{i}\rho x} & \mathrm{e}^{-\mathrm{i}\rho x} \end{vmatrix} + O(\mathrm{e}^{-\gamma|\rho|}). \end{aligned}$$

It then follows that

$$f_2^{''}(x) = -2\mathrm{i}\alpha \begin{vmatrix} 1 & 1 & 1 \\ 0 & \mathrm{e}^{\mathrm{i}\rho} & \mathrm{e}^{-\mathrm{i}\rho} \\ \mathrm{e}^{-\rho x} & -\mathrm{e}^{\mathrm{i}\rho x} & -\mathrm{e}^{-\mathrm{i}\rho x} \end{vmatrix} + O(\mathrm{e}^{-\gamma|\rho|}) = 2\mathrm{i}\alpha(\mathrm{e}^{\mathrm{i}\rho(1-x)} - \mathrm{e}^{-\mathrm{i}\rho(1-x)} + 2\mathrm{i}\sin\rho \mathrm{e}^{-\rho x}) + O(\mathrm{e}^{-\gamma|\rho|}),$$

and

$$\lambda f_2(x) = 2\alpha \begin{vmatrix} 1 & 1 & 1 \\ 0 & \mathrm{e}^{\mathrm{i}\rho} & \mathrm{e}^{-\mathrm{i}\rho} \\ \mathrm{e}^{-\rho x} & \mathrm{e}^{\mathrm{i}\rho x} & \mathrm{e}^{-\mathrm{i}\rho x} \end{vmatrix} + O(\mathrm{e}^{-\gamma|\rho|}) = 2\alpha(\mathrm{e}^{\mathrm{i}\rho(1-x)} - \mathrm{e}^{-\mathrm{i}\rho(1-x)} - 2\mathrm{i}\sin\rho \mathrm{e}^{-\rho x}) + O(\mathrm{e}^{-\gamma|\rho|}).$$

Similarly,

$$\begin{aligned} h_2(x) =& \frac{1}{\rho^6 \mathrm{e}^a \mathrm{e}^\rho} \begin{vmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ \mathrm{e}^\rho & \mathrm{e}^{-\rho} & \mathrm{e}^{\mathrm{i}\rho} & \mathrm{e}^{-\mathrm{i}\rho} & 0 & 0 \\ \mathrm{e}^\rho \rho^2 & \mathrm{e}^{-\rho}\rho^2 & -\mathrm{e}^{\mathrm{i}\rho}\rho^2 & -\mathrm{e}^{-\mathrm{i}\rho}\rho^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^a & \mathrm{e}^{-a} \\ \rho^2 & \rho^2 & -\rho^2 & -\rho^2 & -\mathrm{i}\alpha\rho^2 & -\mathrm{i}\alpha\rho^2 \\ 0 & 0 & 0 & 0 & \mathrm{e}^{ax} & \mathrm{e}^{-ax} \end{vmatrix} \\ =& \frac{1}{\rho^4} \begin{vmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & \mathrm{e}^{\mathrm{i}\rho} & \mathrm{e}^{-\mathrm{i}\rho} & 0 & 0 \\ 1 & 0 & -\mathrm{e}^{\mathrm{i}\rho} & -\mathrm{e}^{-\mathrm{i}\rho} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & -\mathrm{i}\alpha \\ 0 & 0 & 0 & 0 & \mathrm{e}^{-a(1-x)} & \mathrm{e}^{-ax} \end{vmatrix} + O\left(\mathrm{e}^{-\gamma|\rho|}\right) = O\left(\frac{1}{n^5}\right). \end{aligned}$$

It then follows that

$$h_2^{'}(x) = \frac{a}{\rho^4} \begin{vmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & e^{\mathrm{i}\rho} & e^{-\mathrm{i}\rho} & 0 & 0 \\ 1 & 0 & -e^{\mathrm{i}\rho} & -e^{-\mathrm{i}\rho} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & -\mathrm{i}\alpha \\ 0 & 0 & 0 & 0 & e^{-a(1-x)} & -e^{-ax} \end{vmatrix} + O(\mathrm{e}^{-\gamma|\rho|}) = O\left(\frac{1}{n^4}\right),$$

and

$$\lambda h_2(x) = \frac{\mathrm{i}}{\rho^2} \begin{vmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & \mathrm{e}^{\mathrm{i}\rho} & \mathrm{e}^{-\mathrm{i}\rho} & 0 & 0 \\ 1 & 0 & -\mathrm{e}^{\mathrm{i}\rho} & -\mathrm{e}^{-\mathrm{i}\rho} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & -\mathrm{i}\alpha \\ 0 & 0 & 0 & 0 & \mathrm{e}^{-a(1-x)} & \mathrm{e}^{-ax} \end{vmatrix} + O(\mathrm{e}^{-\gamma|\rho|}) = O\left(\frac{1}{n^3}\right).$$

By setting

$$\Phi_{2n} = \begin{pmatrix} f_{2n}(x) \\ \lambda f_{1n}^{-}(x) \\ h_{2n}(x) \\ \lambda h_{2n}(x) \end{pmatrix} = \frac{1}{2\alpha} \begin{pmatrix} f_2(x) \\ \lambda f_2(x) \\ h_2(x) \\ \lambda h_2(x) \end{pmatrix},$$

and substituting (3.21) we can obtain (3.24). This completes the proof of the theorem. □

To end this section, we remark that the same method can be used to produce asymptotic expressions of the eigenpairs of A^*, the adjoint operator of A, which is defined by

$$\begin{cases} A^*(f,g,h,l) = (-g, f^{(4)}, -l, (-h' + \beta l')'), \forall (f,g,h,l) \in D(A^*), \\ D(A^*) = \left\{ \begin{array}{c|l} & -h' + \beta l' \in H^1(0,1) \\ (f,g,h,l) \in \mathcal{H}, & f''(1) = 0 \\ A^*(f,g,h,l) \in \mathcal{H} & g(0) = g(1) = l(1) = 0 \\ & f''(0) = \alpha l(0) \\ & \beta l'(0) - h'(0) = \alpha g'(0) \end{array} \right\}. \end{cases} \tag{3.27}$$

Lemma 3.5. *Let A be defined by (2.1). Then, $\sigma_p(A) = \sigma_p(A^*)$.*

Proof. Let $A^* X = \lambda X$, where $X = (f,g,h,l)$ is the eigenfunction of A^* corresponding to the eigenvalue λ. Then,

$$\begin{cases} f^{(4)}(x) - \rho^4 f(x) = 0, \\ h''(x) = \dfrac{-\rho^4}{1 + \mathrm{i}\beta\rho^2} h(x), \\ f(0) = f''(1) = f(0) = h(1) = 0, \\ f''(0) = -\mathrm{i}\alpha\rho^2 h(0), \\ (1 + \mathrm{i}\beta\rho^2) h'(0) = \mathrm{i}\alpha\rho^2 f'(0). \end{cases} \tag{3.28}$$

With the same process of finding the eigenvalues of A, we know that λ is an eigenvalue of A^* if and only if $\det(\hat{B}) = 0$, where

$$\hat{B} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ \mathrm{e}^{\rho} & \mathrm{e}^{-\rho} & \mathrm{e}^{\mathrm{i}\rho} & \mathrm{e}^{-\mathrm{i}\rho} & 0 & 0 \\ \rho^2 \mathrm{e}^{\rho} & \rho^2 \mathrm{e}^{-\rho} & -\rho^2 \mathrm{e}^{\mathrm{i}\rho} & -\rho^2 \mathrm{e}^{-\mathrm{i}\rho} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{e}^{a} & \mathrm{e}^{-a} \\ \rho^2 & \rho^2 & -\rho^2 & -\rho^2 & \mathrm{i}\alpha\rho^2 & \mathrm{i}\alpha\rho^2 \\ -\frac{\mathrm{i}a^2\alpha}{\rho} & \frac{\mathrm{i}a^2\alpha}{\rho} & \frac{a^2\alpha}{\rho} & -\frac{a^2\alpha}{\rho} & -a & a \end{pmatrix}. \tag{3.29}$$

A direct computation shows that $\det(\hat{B}) = -\det(B)$, that is, A^* has the same eigenvalues with A. □

As in the proof of Theorem 3.4, the eigenfunctions of A^* can be obtained.

Theorem 3.6. *Let A^* be defined by (3.27) and $\sigma_p(A^*) = \sigma_p(A) = \{\lambda_{1n}^+, \lambda_{1n}^-, n \in \mathbb{N}\} \cup \{\lambda_{2n}, \overline{\lambda_{2n}}, n \in \mathbb{N}\}$, where $\lambda_{1n}^+ = \mathrm{i}(\rho_{1n}^+)^2$, $\lambda_{1n}^- = \mathrm{i}(\rho_{1n}^-)^2$ and $\lambda_{2n} = \mathrm{i}(\rho_{2n})^2$ with ρ_{1n}^+, ρ_{1n}^- and ρ_{2n} given by (3.17) and (3.21), respectively. Then, there are three families of approximated formalized eigenfunctions of A^*:*

(1) *One family* $\{\Psi_{1n}^+ = (f_{1n}^+, \lambda f_{1n}^+, h_{1n}^+, \lambda h_{1n}^+), n \in \mathbb{N}\}$*, where* Ψ_{1n}^+ *is the eigenfunction of* A *corresponding to the eigenvalue* λ_{1n}^+*, has the following asymptotic expression:*

$$\begin{pmatrix} (f_{1n}^+)''(x) \\ \lambda f_{1n}^+(x) \\ (h_{1n}^+)'(x) \\ \lambda h_{1n}^+(x) \end{pmatrix} = \begin{pmatrix} -\sqrt{2}\mathrm{i}\alpha(\mathrm{e}^{\mathrm{i}\rho_{1n}^+ x} + \mathrm{e}^{-\rho_{1n}^+ x}) \\ -\sqrt{2}\alpha(\mathrm{e}^{\mathrm{i}\rho_{1n}^+ x} + \mathrm{e}^{-\rho_{1n}^+ x}) \\ 0 \\ \mathrm{e}^{-ax}\left(\sqrt{2}\mathrm{i} + \frac{\alpha^2}{\sqrt{\beta}}\right) + \mathrm{e}^{ax}\left(\sqrt{2}\mathrm{i} - \frac{\alpha^2}{\sqrt{\beta}}\right) \end{pmatrix} + O\left(\frac{1}{n}\right), \tag{3.30}$$

where ρ_{1n}^+ *and* a *are given by (3.17) and (3.16) respectively.*

(2) *The second family* $\{\Psi_{1n}^- = (f_{1n}^-, \lambda f_{1n}^-, h_{1n}^-, \lambda h_{1n}^-), n \in \mathbb{N}\}$*, where* Ψ_{1n}^- *is the eigenfunction of* A^* *corresponding to the eigenvalue* λ_{1n}^-*, has the following asymptotic expression:*

$$\begin{pmatrix} (f_{1n}^-)''(x) \\ \lambda f_{1n}^-(x) \\ (h_{1n}^-)'(x) \\ \lambda h_{1n}^-(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \mathrm{e}^{ax} + \mathrm{e}^{-ax} \\ 0 \end{pmatrix} + O\left(\frac{1}{n}\right). \tag{3.31}$$

where a *is given by (3.16).*

(3) *The third family* $\{\Psi_{2n} = (f_{2n}, \lambda f_{2n}, h_{2n}, \lambda h_{2n}), n \in \mathbb{N}\}$*, where* Ψ_{2n} *is the eigenfunction of* A^* *corresponding to the eigenvalue* λ_{2n}*, has the following asymptotic expression:*

$$\begin{pmatrix} (f_{2n})''(x) \\ \lambda f_{2n}(x) \\ (h_{2n})'(x) \\ \lambda h_{2n}(x) \end{pmatrix} = \begin{pmatrix} \mathrm{i}(\mathrm{e}^{\mathrm{i}\rho_{2n}(1-x)} - \mathrm{e}^{-\mathrm{i}\rho_{2n}(1-x)} + 2\mathrm{i}\sin\rho_{2n}\mathrm{e}^{-\rho_{2n}x}) \\ \mathrm{e}^{\mathrm{i}\rho_{2n}(1-x)} - \mathrm{e}^{-\mathrm{i}\rho_{2n}(1-x)} - 2\mathrm{i}\sin\rho_{2n}\mathrm{e}^{-\rho_{2n}x} \\ 0 \\ 0 \end{pmatrix} + O\left(\frac{1}{n}\right), \tag{3.32}$$

where ρ_{2n} *is given by (3.21).*

4. Completeness of the root subspace

Let A be defined by (2.1) and for $x \in [0,1]$ and $\rho \in \mathbb{C}$, and let

$$\begin{cases} Q_1(x,\xi) = \frac{\mathrm{sign}(x-\xi)}{4\rho^3}[\mathrm{e}^{\rho(x-\xi)} - \mathrm{e}^{-\rho(x-\xi)} + \mathrm{i}\mathrm{e}^{\mathrm{i}\rho(x-\xi)} - \mathrm{i}\mathrm{e}^{-\mathrm{i}\rho(x-\xi)}], \\ Q_2(x,\xi) = \frac{a}{2\rho^4}\mathrm{sign}(x-\xi)[\mathrm{e}^{a(x-\xi)} - \mathrm{e}^{-a(x-\xi)}]. \end{cases} \tag{4.1}$$

For any $\lambda = \mathrm{i}\rho^2 \notin \sigma_p(A)$ with $\lambda \neq 0$ and $(\phi, \psi, \omega, \nu) \in \mathcal{H}$, we consider the invertibility of operator $\lambda - A$. Let

$$F_0(x,\rho) = \int_0^1 Q_1(x,\xi)[\mathrm{i}\rho^2\phi(\xi) + \psi(\xi)]\mathrm{d}x,\ H_0(x,\rho) = \int_0^1 Q_2(x,\xi)[-\beta\omega''(\xi) + \mathrm{i}\rho^2\omega(\xi) + \nu(\xi)]\mathrm{d}x. \tag{4.2}$$

Lemma 4.1. *Assume (4.1) and (4.2). Then, the solution of the equation* $(\lambda - A)(f,g,h,l) = (\phi,\psi,\omega,\nu)$ *is given by*

$$f(x) = \frac{F(x,\rho)}{\det(B)}, g(x) = \lambda f(x) - \phi(x), h(x) = \frac{H(x,\rho)}{\det(B)}, l(x) = \lambda h(x) - \omega(x), \tag{4.3}$$

where the matrix B is defined by (3.4),

$$F(x,\rho)=\begin{vmatrix} e^{x\rho} & e^{-x\rho} & e^{ix\rho} & e^{-ix\rho} & 0 & 0 & F_0(x,\rho)\\ 1 & 1 & 1 & 1 & 0 & 0 & F_1\\ e^{\rho} & e^{-\rho} & e^{i\rho} & e^{-i\rho} & 0 & 0 & F_2\\ \rho^2e^{\rho} & \rho^2e^{-\rho} & -\rho^2e^{i\rho} & -\rho^2e^{-i\rho} & 0 & 0 & F_3\\ 0 & 0 & 0 & 0 & e^{a} & e^{-a} & H_4\\ \rho^2 & \rho^2 & -\rho^2 & -\rho^2 & -i\alpha\rho^2 & -i\alpha\rho^2 & F_5-H_5+\alpha\omega(0)\\ -\frac{ia^2\alpha}{\rho} & \frac{ia^2\alpha}{\rho} & \frac{a^2\alpha}{\rho} & -\frac{a^2\alpha}{\rho} & a & -a & F_6+H_6+\frac{a^2\alpha}{\rho^4}\phi'(0)+\frac{a^2\beta}{\rho^4}\omega'(0) \end{vmatrix}, \tag{4.4}$$

$$H(x,\rho)=\begin{vmatrix} 0 & 0 & 0 & 0 & e^{ax} & e^{-ax} & H_0(x,\rho)\\ 1 & 1 & 1 & 1 & 0 & 0 & F_1\\ e^{\rho} & e^{-\rho} & e^{i\rho} & e^{-i\rho} & 0 & 0 & F_2\\ e^{\rho}\rho^2 & e^{-\rho}\rho^2 & -e^{i\rho}\rho^2 & -e^{-i\rho}\rho^2 & 0 & 0 & F_3\\ 0 & 0 & 0 & 0 & e^{a} & e^{-a} & H_4\\ \rho^2 & \rho^2 & -\rho^2 & -\rho^2 & -i\alpha\rho^2 & -i\alpha\rho^2 & F_5-H_5+\alpha\omega(0)\\ -\frac{ia^2\alpha}{\rho} & \frac{ia^2\alpha}{\rho} & \frac{a^2\alpha}{\rho} & -\frac{a^2\alpha}{\rho} & a & -a & F_6+H_6+\frac{a^2\alpha}{\rho^4}\phi'(0)+\frac{a^2\beta}{\rho^4}\omega'(0) \end{vmatrix}, \tag{4.5}$$

and $F_k, k=1,2,3,5,6,\ H_s, s=4,5,6$ *are constants given by*

$$\begin{cases} F_1=\displaystyle\int_0^1 -\frac{1}{4\rho^3}\left(e^{-\xi\rho}-e^{\xi\rho}+ie^{-i\xi\rho}-ie^{i\xi\rho}\right)\left(i\rho^2\phi(\xi)+\psi(\xi)\right)d\xi,\\ F_2=\displaystyle\int_0^1 \frac{1}{4\rho^3}\left(e^{(1-\xi)\rho}-e^{-(1-\xi)\rho}+ie^{i(1-\xi)\rho}-ie^{-i(1-\xi)\rho}\right)\left(i\rho^2\phi(\xi)+\psi(\xi)\right)d\xi,\\ F_3=\displaystyle\int_0^1 \frac{1}{4\rho}\left(e^{(1-\xi)\rho}-e^{-(1-\xi)\rho}-ie^{i(1-\xi)\rho}+ie^{-i(1-\xi)\rho}\right)\left(i\rho^2\phi(\xi)+\psi(\xi)\right)d\xi,\\ F_5=\displaystyle\int_0^1 -\frac{1}{4\rho}\left(e^{-\xi\rho}-e^{\xi\rho}+ie^{i\xi\rho}-ie^{-i\xi\rho}\right)\left(i\rho^2\phi(\xi)+\psi(\xi)\right)d\xi,\\ F_6=\displaystyle\int_0^1 \frac{ia^2\alpha}{4\rho^4}\left(e^{-\xi\rho}+e^{\xi\rho}-e^{i\xi\rho}-e^{-i\xi\rho}\right)\left(i\rho^2\phi(\xi)+\psi(\xi)\right)d\xi, \end{cases}$$

and

$$\begin{cases} H_4=\displaystyle\int_0^1 \frac{a}{2\rho^4}\left(e^{a(1-\xi)}-e^{-a(1-\xi)}\right)\left(-\beta\omega''(\xi)+i\rho^2\omega(\xi)+\nu(\xi)\right)d\xi,\\ H_5=-i\alpha\rho^2\displaystyle\int_0^1 \frac{a}{2\rho^4}\left(e^{-a\xi}-e^{a\xi}\right)\left(-\beta\omega''(\xi)+i\rho^2\omega(\xi)+\nu(\xi)\right)d\xi,\\ H_6=\displaystyle\int_0^1 -\frac{a^2}{2\rho^4}\left(e^{-a\xi}+e^{a\xi}\right)\left(-\beta\omega''(\xi)+i\rho^2\omega(\xi)+\nu(\xi)\right)d\xi. \end{cases}$$

Proof. For any $(\phi,\psi,\omega,\nu)\in\mathcal{H}$ and $\lambda=i\rho^2\notin\sigma_p(A)$ with $\lambda\neq 0$, solving the equation

$$(\lambda I-A)(f,g,h,l)=(\phi,\psi,\omega,\nu)$$

yields $g=\lambda f-\phi, l=\lambda h-\omega$, with $f(x)$ and $h(x)$ satisfying

$$\begin{cases} f^{(4)}(x)-\rho^4 f(x)=\mathrm{i}\rho^2\phi(x)+\psi(x), \\ h^{''}(x)+\dfrac{\rho^4}{1+\mathrm{i}\beta\rho^2}h(x)=\dfrac{\beta}{1+\mathrm{i}\beta\rho^2}\omega^{''}(x)-\dfrac{\mathrm{i}\rho^2}{1+\mathrm{i}\beta\rho^2}\omega(x)-\dfrac{1}{1+\mathrm{i}\beta\rho^2}\nu(x), \\ f(0)=f(1)=f^{''}(1)=h(1)=0, \\ f^{''}(0)=\alpha\mathrm{i}\rho^2 h(0)-\alpha\omega(0), \\ h^{'}(0)+\dfrac{\alpha\mathrm{i}\rho^2}{1+\mathrm{i}\beta\rho^2}f^{'}(0)=\dfrac{\alpha}{1+\mathrm{i}\beta\rho^2}\omega^{'}(0)+\dfrac{\alpha}{1+\mathrm{i}\beta\rho^2}\phi(0). \end{cases} \tag{4.6}$$

Denoting $a=\sqrt{\frac{-\rho^4}{1+\mathrm{i}\beta\rho^2}}$ as previously, (4.6) has general solution:

$$\begin{cases} f(x)=c_1\mathrm{e}^{\rho x}+c_2\mathrm{e}^{-\rho x}+c_3\mathrm{e}^{\mathrm{i}\rho x}+c_4\mathrm{e}^{-\mathrm{i}\rho x}+F_0(x,\rho), \\ h(x)=d_1\mathrm{e}^{ax}+d_2\mathrm{e}^{-ax}+H_0(x,\rho), \end{cases} \tag{4.7}$$

where $F_0(x,\rho)$ and $H_0(x,\rho)$ are given by (4.2). Substituting $f(x)$ and $h(x)$ into the boundary conditions of (4.6) $c_j, j=1,2,3,4$ and d_1,d_2 satisfy the following algebraic equation:

$$\begin{cases} c_1+c_2+c_3+c_4=-F_1, \\ c_1\mathrm{e}^{\rho}+c_2\mathrm{e}^{-\rho}+c_3\mathrm{e}^{\mathrm{i}\rho}+c_4\mathrm{e}^{-\mathrm{i}\rho}=-F_2, \\ c_1\rho^2\mathrm{e}^{\rho}+c_2\rho^2\mathrm{e}^{-\rho}-c_3\rho^2\mathrm{e}^{\mathrm{i}\rho}-c_4\rho^2\mathrm{e}^{-\mathrm{i}\rho}=-F_3, \\ d_1\mathrm{e}^{a}+d_2\mathrm{e}^{-a}=-H_4, \\ c_1\rho^2+c_2\rho^2-c_3\rho^2-c_4\rho^2-\mathrm{i}\alpha\rho^2 d_1-\mathrm{i}\alpha\rho^2 d_2=-F_5+H_5-\alpha\omega(0), \\ -\dfrac{\mathrm{i}a^2\alpha}{\rho}c_1+\dfrac{\mathrm{i}a^2\alpha}{\rho}c_2+\dfrac{a^2\alpha}{\rho}c_3-\dfrac{a^2\alpha}{\rho}c_4+ad_1-ad_2=-F_6-H_6-\dfrac{a^2\beta}{\rho^4}\omega^{'}(0)-\dfrac{a^2\alpha}{\rho^4}\phi^{'}(0). \end{cases} \tag{4.8}$$

Since $\lambda=\mathrm{i}\rho^2\notin\sigma_p(A)$, namely, $\det(B)\neq 0$, (4.6) admits a unique solution. Moreover, the solution $f(x)$ and $h(x)$ of (4.7) can be written as (4.3). □

Proposition 4.2. *Let A be defined by (2.1). Then, all $\lambda_n=\{\lambda_{1n}^+,\lambda_{1n}^-,\lambda_{2n}\}\in\sigma_p(A)$ represented in (3.8) are algebraically simple with sufficiently large n.*

Proof. We only need to prove the case when $\rho\in S$ since the proof of the symmetrical part is similar. From Lemma 4.1, the order of each $\lambda\in\sigma_p(A)$, as a pole of $(\lambda I-A)^{-1}$, is less than or is equivalent to the multiplicity of λ as a zero of the entire function of ρ $\det(B)=0$ with n large sufficiently. Furthermore, it is easy to see that λ is geometrically simple and from (3.15) and (3.20), we can show that all the zeros of $\det(B)=0$ are simple in S_1 and S_2 respectively with n large sufficiently. Thus, the result follows from the formula: $m_a\leq p\cdot m_g$ [13], where p denotes the order of the pole of the operator $(\lambda I-A)^{-1}$ and m_a, m_g denote the algebraic and geometric multiplicities respectively. □

To estimate the norm of $(\lambda-A)^{-1}$, we recall a lemma in [11] and [18].

Lemma 4.3. *Let*

$$D(\lambda)=1+\sum_i^n Q_i(\lambda)\mathrm{e}^{\alpha_i\lambda},$$

where Q_i are polynomials of λ, α_i are some complex numbers, and n is a positive integer. Then, for all λ outside those circles of radius $\varepsilon > 0$ that centered at the roots of $D(\cdot)$, it has

$$|D(\lambda)| \geq C(\varepsilon) > 0$$

for some constant $C(\varepsilon)$ that depends only on ε.

Theorem 4.4. *Let A be defined by (2.1) and for $\lambda \notin \sigma_p(A) \cup \{-\frac{1}{\beta}\}$. For the operator $(\lambda - A)^{-1}$, there exists $M > 0$ independent of λ such that*

$$\|(\lambda - A)^{-1}\| \leq M(1 + |\lambda|),$$

for all $\lambda = \mathrm{i}\rho^2$ with $\rho \in \mathbb{C}$ lying outside all circles of radius $\varepsilon > 0$ that centered at the zeros of $\det(B)$.

Proof. We first consider those $\lambda = \mathrm{i}\rho^2$ with $\rho \in \mathcal{S}$. Let $\rho \in \mathcal{S}$ with $\rho \neq 0$. For $(\phi, \psi, \omega, \nu) \in \mathcal{H}$, $(f, g, h, l) = (\lambda - A)^{-1}(\phi, \psi, \omega, \nu)$ has the expression given by (4.3). Since in sector $\mathcal{S}$, it holds

$$\mathrm{Re}(-\rho) \leq 0,\ \mathrm{Re}(\sqrt{\mathrm{i}}\rho) \leq 0,\ \mathrm{Re}(-a) \leq 0,$$

and there exits a constant $\hat{M}$ such that $|a/\rho| \leq \hat{M}$ from the expression of a, we need to use the transformation of each entry in the determinant given by (4.4) and (4.5) stable. For (4.4) and (4.5), multiplying the ith column by factors L_i given by

$$\begin{cases} L_1 = \displaystyle\int_0^1 -\frac{1}{4\rho^3}\mathrm{e}^{-\xi\rho}\left(\mathrm{i}\rho^2\phi(\xi) + \psi(\xi)\right)\mathrm{d}\xi, \\ L_2 = \displaystyle\int_0^1 -\frac{1}{4\rho^3}\mathrm{e}^{\xi\rho}\left(\mathrm{i}\rho^2\phi(\xi) + \psi(\xi)\right)\mathrm{d}\xi, \\ L_3 = \displaystyle\int_0^1 \frac{\mathrm{i}}{4\rho^3}\mathrm{e}^{-\mathrm{i}\xi\rho}\left(\mathrm{i}\rho^2\phi(\xi) + \psi(\xi)\right)\mathrm{d}\xi, \\ L_4 = \displaystyle\int_0^1 \frac{\mathrm{i}}{4\rho^3}\mathrm{e}^{\mathrm{i}\xi\rho}\left(\mathrm{i}\rho^2\phi(\xi) + \psi(\xi)\right)\mathrm{d}\xi, \\ L_5 = \displaystyle\int_0^1 -\frac{a}{2\rho^4}\mathrm{e}^{-a\xi}\left(-\beta\omega''(\xi) + \mathrm{i}\rho^2\omega(\xi) + \nu(\xi)\right)\mathrm{d}\xi, \\ L_6 = \displaystyle\int_0^1 -\frac{a}{2\rho^4}\mathrm{e}^{a\xi}\left(-\beta\omega''(\xi) + \mathrm{i}\rho^2\omega(\xi) + \nu(\xi)\right)\mathrm{d}\xi, \end{cases}$$

and adding these columns to the last column of $F(x,\rho)$ and $H(x,\rho)$ respectively, we obtain

$$\frac{\mathrm{e}^{-a}\mathrm{e}^{-\rho}\mathrm{e}^{\mathrm{i}\rho}}{a\rho}F(x,\rho) = \tilde{F}(x,\rho), \quad \frac{\mathrm{e}^{-a}\mathrm{e}^{-\rho}\mathrm{e}^{\mathrm{i}\rho}}{a\rho}H(x,\rho) = \tilde{H}(x,\rho),$$

where for $k=0,1,2$

$$\frac{\partial^k \tilde{F}(x,\rho)}{\rho^k \partial x^k} = \begin{vmatrix} \mathrm{e}^{(x-1)\rho} & (-1)^k \mathrm{e}^{-x\rho} & \mathrm{i}^k \mathrm{e}^{\mathrm{i}x\rho} & (-\mathrm{i})^k \mathrm{e}^{-\mathrm{i}(x-1)\rho} & 0 & 0 & \frac{\partial^k \tilde{F}_0(x,\rho)}{\partial x^k} \\ \mathrm{e}^{-\rho} & 1 & 1 & \mathrm{e}^{\mathrm{i}\rho} & 0 & 0 & \tilde{F}_1 \\ 1 & \mathrm{e}^{-\rho} & \mathrm{e}^{\mathrm{i}\rho} & 1 & 0 & 0 & \tilde{F}_2 \\ 1 & \mathrm{e}^{-\rho} & -\mathrm{e}^{\mathrm{i}\rho} & -1 & 0 & 0 & \tilde{F}_3 \\ 0 & 0 & 0 & 0 & 1 & \mathrm{e}^{-a} & \tilde{H}_4 \\ \mathrm{e}^{-\rho} & 1 & -1 & -\mathrm{e}^{\mathrm{i}\rho} & -\mathrm{i}\alpha \mathrm{e}^{-a} & -\mathrm{i}\alpha & \tilde{F}_5 - \tilde{H}_5 + \alpha\rho\omega(0) \\ -\frac{\mathrm{i}a\alpha}{\rho}\mathrm{e}^{-\rho} & \frac{\mathrm{i}a\alpha}{\rho} & \frac{a\alpha}{\rho} & -\frac{a\alpha}{\rho}\mathrm{e}^{\mathrm{i}\rho} & \mathrm{e}^{-a} & -1 & \tilde{F}_6 + \tilde{H}_6 + \frac{a\alpha}{\rho}\phi'(0) + \frac{a\beta}{\rho}\omega'(0) \end{vmatrix},$$

and for $s=0,1$

$$\frac{\partial^s \tilde{H}(x,\rho)}{\partial x^s} = \begin{vmatrix} 0 & 0 & 0 & 0 & a^s \mathrm{e}^{a(x-1)} & (-a)^s \mathrm{e}^{-ax} & \frac{\partial^s \tilde{H}_0(x,\rho)}{\partial x^s} \\ \mathrm{e}^{-\rho} & 1 & 1 & \mathrm{e}^{\mathrm{i}\rho} & 0 & 0 & \tilde{F}_1 \\ 1 & \mathrm{e}^{-\rho} & \mathrm{e}^{\mathrm{i}\rho} & 1 & 0 & 0 & \tilde{F}_2 \\ 1 & 0 & 0 & -1 & 0 & 0 & \tilde{F}_3 \\ 0 & 0 & 0 & 0 & 1 & \mathrm{e}^{-a} & \tilde{H}_4 \\ \mathrm{e}^{-\rho} & 1 & -1 & -\mathrm{e}^{\mathrm{i}\rho} & -\mathrm{i}\alpha \mathrm{e}^{-a} & -\mathrm{i}\alpha & \tilde{F}_5 - \tilde{H}_5 + \alpha\rho\omega(0) \\ -\frac{\mathrm{i}a\alpha}{\rho}\mathrm{e}^{-\rho} & \frac{\mathrm{i}a\alpha}{\rho} & \frac{a\alpha}{\rho} & -\frac{a\alpha}{\rho}\mathrm{e}^{\mathrm{i}\rho} & \mathrm{e}^{-a} & -1 & \tilde{F}_6 + \tilde{H}_6 + \frac{a\alpha}{\rho}\phi'(0) + \frac{a\beta}{\rho}\omega'(0) \end{vmatrix}.$$

Here

$$\frac{\partial^k \tilde{F}_0(x,\rho)}{\partial x^k} = \frac{1}{2}\int_0^1 \frac{\partial^k P(x,\xi)}{\partial x^k}(\mathrm{i}\rho^2\phi(\xi)+\psi(\xi))\mathrm{d}\xi,$$
$$\frac{\partial^s \tilde{H}_0(x,\rho)}{\partial x^s} = \frac{a}{\rho}\int_0^1 \frac{\partial^s R(x,\xi)}{\partial x^s}(\beta\omega''(\xi)-\mathrm{i}\rho^2\omega(\xi)-\nu(\xi))\mathrm{d}\xi,$$

with

$$P(x,\xi) = \begin{cases} -\mathrm{e}^{-\rho(x-\xi)} + i\mathrm{e}^{\mathrm{i}\rho(x-\xi)}, & x \geq \xi, \\ -\mathrm{e}^{-\rho(\xi-x)} + i\mathrm{e}^{\mathrm{i}\rho(\xi-x)}, & x < \xi, \end{cases}$$

$$R(x,\xi) = \begin{cases} \mathrm{e}^{-a(x-\xi)}, & x \geq \xi, \\ \mathrm{e}^{-a(\xi-x)}, & x < \xi, \end{cases}$$

and

$$\begin{cases} \tilde{F}_1 = -\dfrac{1}{2}\displaystyle\int_0^1 \left(\mathrm{e}^{-\xi\rho} - \mathrm{i}\mathrm{e}^{\mathrm{i}\xi\rho}\right)\left(\mathrm{i}\rho^2\phi(\xi)+\psi(\xi)\right)\mathrm{d}\xi, \\ \tilde{F}_2 = -\dfrac{1}{2}\displaystyle\int_0^1 \left(\mathrm{e}^{-\rho(1-\xi)} - \mathrm{i}\mathrm{e}^{\mathrm{i}\rho(1-\xi)}\right)\left(\mathrm{i}\rho^2\phi(\xi)+\psi(\xi)\right)\mathrm{d}\xi, \\ \tilde{F}_3 = -\dfrac{1}{2}\displaystyle\int_0^1 \left(\mathrm{e}^{-\rho(1-\xi)} + \mathrm{i}\mathrm{e}^{\mathrm{i}(1-\xi)\rho}\right)\left(\mathrm{i}\rho^2\phi(\xi)+\psi(\xi)\right)\mathrm{d}\xi, \\ \tilde{F}_5 = -\dfrac{1}{2}\displaystyle\int_0^1 \left(\mathrm{e}^{-\rho\xi} + \mathrm{i}\mathrm{e}^{\mathrm{i}\rho\xi}\right)\left(\mathrm{i}\rho^2\phi(\xi)+\psi(\xi)\right)\mathrm{d}\xi, \\ \tilde{F}_6 = \dfrac{a\alpha\mathrm{i}}{2\rho}\displaystyle\int_0^1 \left(\mathrm{e}^{-\xi\rho} - \mathrm{e}^{\mathrm{i}\xi\rho}\right)\left(\mathrm{i}\rho^2\phi(\xi)+\psi(\xi)\right)\mathrm{d}\xi, \end{cases}$$

$$\begin{cases} \tilde{H}_4 = \dfrac{a}{\rho}\displaystyle\int_0^1 \mathrm{e}^{-a(1-\xi)} \left(\beta\omega''(\xi) - \mathrm{i}\rho^2\omega(\xi) - \nu(\xi)\right) \mathrm{d}\xi, \\ \tilde{H}_5 = \dfrac{\mathrm{i}\alpha a}{\rho}\displaystyle\int_0^1 \mathrm{e}^{-a\xi} \left(\beta\omega''(\xi) - \mathrm{i}\rho^2\omega(\xi) - \nu(\xi)\right) \mathrm{d}\xi, \\ \tilde{H}_6 = \dfrac{a}{\rho}\displaystyle\int_0^1 \mathrm{e}^{-a\xi} \left(\beta\omega''(\xi) - \mathrm{i}\rho^2\omega(\xi) - \nu(\xi)\right) \mathrm{d}\xi. \end{cases}$$

By Lemma 4.3, there exists $M_1 > 0$ such that

$$|f^{''(x)}| \le \frac{M_1}{|\rho|}\left[\int_0^1 (|\lambda||\phi(\xi)| + |\psi(\xi)| + |\lambda||\omega(\xi)| + |\nu(\xi)|)\,\mathrm{d}\xi\right],$$
$$|g(x)| \le \frac{M_1}{|\rho|}\left[\int_0^1 (|\lambda||\phi(\xi)| + |\psi(\xi)| + |\lambda||\omega(\xi)| + |\nu(\xi)|)\,\mathrm{d}\xi\right] + |\phi(x)|,$$
$$|h^{'}(x)| \le \frac{M_1}{|\rho|^2}\left[\int_0^1 (|\lambda||\phi(\xi)| + |\psi(\xi)| + |\lambda||\omega(\xi)| + |\nu(\xi)|)\,\mathrm{d}\xi\right],$$
$$|l(x)| \le M_1\left[\int_0^1 (|\lambda||\phi(\xi)| + |\psi(\xi)| + |\lambda||\omega(\xi)| + |\nu(\xi)|)\,\mathrm{d}\xi\right] + |\omega(\xi)|,$$

for all $\lambda = \mathrm{i}\rho^2$ with $\rho \in \mathcal{S}$ lying outside all circles of radius $\varepsilon > 0$ that are centered at the zeros of $\det(B)$. Since $|u(x)| \le \|u^{'}\|_{L^2} \le \|u^{''}\|_{L^2}$ for any $x \in [0,1]$ and $u \in H_L^2[0,1]$, it follows that for any $(\phi,\psi,\omega,\nu) \in \mathcal{H}$,

$$\begin{cases} |f^{''(x)}| \le \dfrac{M_1}{|\rho|}\left[|\lambda|\|\phi^{''}\|_{L^2} + \|\psi\|_{L^2} + |\lambda|\|\omega^{'}\|_{L^2} + \|\nu\|_{L^2}\right], \\ |g(x)| \le \dfrac{M_1}{|\rho|}\left[|\lambda|\|\phi^{''}\|_{L^2} + \|\psi\|_{L^2} + |\lambda|\|\omega^{'}\|_{L^2} + \|\nu\|_{L^2}\right] + \|\phi\|_{L^2}, \\ |h^{'}(x)| \le \dfrac{M_1}{|\rho|^2}\left[|\lambda|\|\phi^{''}\|_{L^2} + \|\psi\|_{L^2} + |\lambda|\|\omega^{'}\|_{L^2} + \|\nu\|_{L^2}\right], \\ |l(x)| \le M_1\left[|\lambda|\|\phi^{''}\|_{L^2} + \|\psi\|_{L^2} + |\lambda|\|\omega^{'}\|_{L^2} + \|\nu\|_{L^2}\right] + \|\omega^{'}\|_{L^2}. \end{cases} \tag{4.9}$$

It is seen from (4.9) that we can find constants $M_2 > 0$ and $K > 0$ independent of λ such that

$$\|(f,g,h,l)\| \le M_2(1 + |\lambda|)\|(\phi,\psi,\omega,\nu)\|$$

for all $|\lambda| = |\rho^2| > k > 1$ with $\rho \in S$ lies outside all circles of radius $\varepsilon > 0$ that are centered at the zeros of $\det(B)$. Moreover, for $|\lambda| \le K$, there exists $M > M_2$ such that $\|(f,g,h,l)\| \le M\|(\phi,\psi,\omega,\nu)\|$. Therefore,

$$\|(f,g,h,l)\| \le M(1 + |\lambda|)\|(\phi,\psi,\omega,\nu)\|$$

for all $\lambda = \mathrm{i}\rho^2$ with $\rho \in S$ lying outside all circles of radius $\varepsilon > 0$ that are centered at the zeros of $\det(B)$.

This result can be extended to all other $\rho's$ by the exact same arguments of [14]. This completes the proof of the theorem. □

To get the completeness of the root subspace, we need following result from [12].

Lemma 4.5. *Suppose that $f(z)$ is analytic in the whole complex plane. Denote*

$$M_f(r) = \max_{|z|=r} |f(z)|.$$

If for a nonnegative α, there holds

$$\lim_{r\to\infty} \inf \frac{M_f(r)}{r^\alpha} = 0,$$

then, $f(z)$ is a Polynomial with degree not exceeding α.

Theorem 4.6. *Let A defined by (2.1). Then, both the root subspaces of A and A^* are complete in $\mathcal{H}$, that is,* $\mathrm{Sp}(A) = \mathrm{Sp}(A^*) = \mathcal{H}$ *where* $\mathrm{Sp}(A)$ *denotes the root subspaces of A.*

Proof. Since the completeness for the root subspace of A^* is almost the same, we only show the result for A. By linear operator theory [3], it holds

$$\mathcal{H} = \sigma_\infty(A^*) \oplus \mathrm{Sp}(A),$$

where $\sigma_\infty(A^*)$ consists all of those $Y \in \mathcal{H}$ so that $R(\lambda, A^*)Y$ is analytic with respect to λ in the whole complex plane. We therefore only need to prove $\sigma_\infty(A^*) = \{0\}$.

Suppose for $Y \in \sigma_\infty(A^*)$, $R(\lambda, A^*)Y$ is an analytic function of λ and so is for ρ due to $\lambda = \mathrm{i}\rho^2$. By the maximum modulus principle, $\|R(\lambda, A^*)\| = \|R(\overline{\lambda}, A)\|$, and Theorem 4.4, there exists constant $M > 0$ such that

$$\|R(\lambda, A^*)Y\| \le M(1 + |\lambda|)\|y\|, \forall\, \lambda \in \mathbb{C}.$$

By Lemma 4.5, $R(\lambda, A^*)Y$ is a polynomial of λ with degree less or equal to one, i.e., $R(\lambda, A^*) = Y_0 + \lambda Y_1$ for some $Y_0, Y_1 \in \mathcal{H}$. Thus,

$$Y = (\lambda - A^*)(Y_0 + \lambda Y_1).$$

On account of the closeness of A^*, we conclude that $Y_0, Y_1 \in D(A^*)$. Thus,

$$-A^*Y_0 + \lambda(Y_0 - A^*Y_1) + \lambda^2 Y_1 = Y, \forall\, \lambda \in \mathbb{C}.$$

This leads to $Y = Y_0 = Y_1 = 0$ and the proof is complete. □

5. Riesz basis property and exponential stability

The following Lemma 5.1 comes from [23].

Lemma 5.1. *An approximately normalized sequence $\{\mathrm{e}_i\}_{i=1}^\infty$ and its approximately normalized biorthognal sequence $\{\mathrm{e}_i^*\}_{i=1}^\infty$ are Riesz basis for a Hilbert space H if and only if*

1) *both $\{\mathrm{e}_i\}_{i=1}^\infty$ and $\{\mathrm{e}_i^*\}_{i=1}^\infty$ are complete in H; and*
2) *both $\{\mathrm{e}_i\}_{i=1}^\infty$ and $\{\mathrm{e}_i^*\}_{i=1}^\infty$ are Bessel sequences in H, that is, for any $f \in H$, two sequences $\{\langle f, \mathrm{e}_i\rangle\}_{i=1}^\infty$, $\{\langle f, \mathrm{e}_i^*\rangle\}_{i=1}^\infty$ belong to l^2.*

The succeeding Lemma 5.2 comes from [18].

Lemma 5.2. *Let $\{\mu_n\}$ be a sequence which has asymptotic*

$$\mu_n = u(n + \mathrm{i}v \ln n) + O(1),\ u \ne 0, n = 1, 2, 3, \ldots, \tag{5.1}$$

where v is a real number. If μ_n satisfies $\sup_{n\ge1}\mathrm{Re}(\mu_n)<\infty$, *then, the sequence* $\{e^{\mu_n x}\}_{n=1}^{\infty}$ *is a Bessel sequence in* $L^2(0,1)$.

Lemma 5.3. *Let ρ_{1n}^{+}, a and ρ_{2n} be given by (3.17), (3.16) and (3.21) respectively. Then, the sequences* $\{e^{i\rho_{1n}^{+}x}\}_{n=1}^{\infty}$, $\{e^{-\rho_{1n}^{+}x}\}_{n=1}^{\infty}$, $\{e^{\pm ax}\}_{n=1}^{\infty}$, $\{e^{\pm i\rho_{2n}(1-x)}\}_{n=1}^{\infty}$ *and* $\{e^{-\rho_{2n}x}\}_{n=1}^{\infty}$ *are Bessel sequences in* $L^2(0,1)$.

Proof. By (3.25), if we take $u=i\sqrt{i\beta}\pi$, $v=0$ in $e^{i\rho_{1n}^{+}x}$, $u=-\sqrt{i\beta}\pi$, $v=0$ in $e^{-\rho_{1n}^{+}x}$, $u=-i\pi$, $v=0$ in e^{ax} and $u=i\pi$, $v=0$ in e^{-ax}, respectively, then, by Lemma 5.2, the sequences $\{e^{i\rho_{1n}^{+}x}\}_{n=1}^{\infty}$, $\{e^{-\rho_{1n}^{+}x}\}_{n=1}^{\infty}$ and $\{e^{\pm ax}\}_{n=1}^{\infty}$ are the Bessel sequences in $L^2(0,1)$.

Similarly, by (3.26), if we take $u=i\pi$, $v=0$ in $e^{i\rho_{2n}(1-x)}$, $u=-i\pi$, $v=0$ in $e^{-i\rho_{2n}(1-x)}$ and $u=-\pi$, $v=0$ in $e^{-\rho_{2n}x}$, respectively, then, by Lemma 5.2, it follows that the sequences $\{e^{\pm i\rho_{2n}(1-x)}\}_{n=1}^{\infty}$ and $\{e^{-\rho_{2n}x}\}_{n=1}^{\infty}$ are the Bessel sequences in $L^2(0,1)$. □

Theorem 5.4. *Let A be defined by (2.1). Then, the generalized eigenfunctions of A form a Riesz basis for $\mathcal{H}$.*

Proof. Let $\sigma_p(A)=\{\lambda_{1n}^{+},\lambda_{1n}^{-},\lambda_{2n},\overline{\lambda_{2n}}\}$ be the eigenvalues of A given by Theorem 3.2. By Proposition 4.2, there exists an integer $N>0$ such that all λ_{1n}^{+}, λ_{1n}^{-}, λ_{2n}, $\overline{\lambda_{2n}}$ with $n\le N$, are algebraically simple. For $n<N$, suppose that the algebraic multiplicities of λ_{1n}^{+}, λ_{1n}^{-} and λ_{2n} are m_{1n}^{+}, m_{1n}^{-} and m_{2n} respectively, then, we can find the corresponding generalized eigenfunctions $\{\Phi_{1n,j}^{+}\}_{j=1}^{m_{1n}^{+}}$, $\{\Phi_{1n,j}^{-}\}_{j=1}^{m_{1n}^{-}}$ and $\{\Phi_{2n,j}\}_{j=1}^{m_{2n}}$ by solving

$$\begin{cases}(A-\lambda_{1n}^{\pm})\Phi_{1n,1}^{\pm}=0\\ (A-\lambda_{1n}^{\pm})\Phi_{1n,2}^{\pm}=\Phi_{1n,1}^{\pm}\\ \vdots\\ (A-\lambda_{1n}^{\pm})\Phi_{1n,m_{1n}^{\pm}}^{\pm}=\Phi_{1n,m_{1n}^{\pm}-1}^{\pm}\end{cases}\quad\text{and}\quad\begin{cases}(A-\lambda_{2n})\Phi_{2n,1}=0\\ (A-\lambda_{2n})\Phi_{2n,2}=\Phi_{2n,1}\\ \vdots\\ (A-\lambda_{2n})\Phi_{2n,m_{2n}}=\Phi_{2n,m_{2n}-1}\end{cases}.$$

Then,

$$\Big\{\{\Phi_{1n,j}^{\pm}\}_{j=1}^{m_{1n}^{\pm}}\Big\}_{n<N}\cup\{\Phi_{1n}^{\pm}\}_{n\ge N}\cup\Big\{\{\Phi_{2n,j},\overline{\Phi_{2n,j}}\}_{j=1}^{m_{2n}}\Big\}_{n<N}\cup\{\Phi_{2n},\overline{\Phi_{2n}}\}_{n\ge N}\tag{5.2}$$

are all linearly independent generalized eigenfunctions of A. On the other hand,

$$\Big\{\{\Psi_{1n,j}^{\pm}\}_{j=1}^{m_{1n}^{\pm}}\Big\}_{n<N}\cup\{\Psi_{1n}^{\pm}\}_{n\ge N}\cup\Big\{\{\Psi_{2n,j},\overline{\Psi_{2n,j}}\}_{j=1}^{m_{2n}}\Big\}_{n<N}\cup\{\Psi_{2n},\overline{\Psi_{2n}}\}_{n\ge N}\tag{5.3}$$

are all linearly independent generalized eigenfunctions of A^*. Let

$$\begin{cases}\Phi_{1n,j}^{\pm*}=\dfrac{\Psi_{1n,j}^{\pm}}{\overline{\langle\Phi_{1n,j}^{\pm},\Psi_{1n,j}^{\pm}\rangle}},\ n<N,\ j=1,2,\cdots,m_{1n}^{\pm},\\ \Phi_{1n}^{\pm*}=\dfrac{\Psi_{1n}^{\pm}}{\overline{\langle\Phi_{1n}^{\pm},\Psi_{1n}^{\pm}\rangle}},\ n\ge N,\end{cases}\tag{5.4}$$

and

$$\begin{cases}\Phi_{2n,j}^{*}=\dfrac{\Psi_{2n,j}}{\overline{\langle\Phi_{2n,j},\Psi_{2n,j}\rangle}},\ n<N,\ j=1,2,\cdots,m_{2n},\\ \Phi_{2n}^{\pm*}=\dfrac{\Psi_{2n}}{\overline{\langle\Phi_{2n},\Psi_{2n}\rangle}},\ n\ge N.\end{cases}\tag{5.5}$$

Then,

$$\Big\{\{\Phi^{\pm *}_{1n,j}\}_{j=1}^{m^{\pm}_{1n}}\Big\}_{n<N}\cup\{\Phi^{\pm *}_{1n}\}_{n\geq N}\cup\Big\{\{\Phi^{*}_{2n,j},\overline{\Phi^{*}_{2n,j}}\}_{j=1}^{m_{2n}}\Big\}_{n<N}\cup\{\Phi^{*}_{2n},\overline{\Phi^{*}_{2n}}\}_{n\geq N} \tag{5.6}$$

are all linearly independent generalized eigenfunctions of A^* and they are biorthognal to (5.2). From Theorem 4.6, all the sequences given by (5.2), (5.3) and (5.6) are complete in $\mathcal{H}$.

According to Lemma 5.1, we only need to prove that (5.2) and (5.6) are Bessel sequences in $\mathcal{H}$. Since both $\{\{\Phi^{\pm}_{1n,j}\}_{j=1}^{m^{\pm}_{1n}},\{\Phi_{2n,j}\}_{j=1}^{m_{2n}}\}_{n<N}$ and $\{\{\Phi^{\pm *}_{1n,j}\}_{j=1}^{m^{\pm}_{1n}},\{\Phi^{*}_{2n,j}\}_{j=1}^{m_{2n}}\}_{n<N}$ are finitely many, we only need to show that both the eigenfunctions $\{\Phi^{\pm}_{1n},\Phi_{2n}\}_{n\geq N}$ and $\{\Phi^{\pm *}_{1n},\Phi^{*}_{2n}\}_{n\geq N}$ of A and A^*, respectively, are Bessel sequences in $\mathcal{H}$. Besides, from (5.4) and (5.5), we know that $\{\Phi^{\pm *}_{1n},\Phi^{*}_{2n}\}_{n\geq N}$ is a Bessel sequence if and only if $\{\Psi^{\pm}_{1n},\Psi_{2n}\}_{n\geq N}$ is a Bessel sequence. Therefore, to show the Riesz basis property of the system, it suffices to show that $\{\Phi^{\pm}_{1n},\Phi_{2n}\}_{n\geq N}$ and $\{\Psi^{\pm}_{1n},\Psi_{2n}\}_{n\geq N}$ are Bessel sequences in $\mathcal{H}$.

For all $n\geq N$, from the expressions of $\{\Phi^{\pm}_{1n},\Phi_{2n}\}_{n\geq N}$ and $\{\Psi^{\pm}_{1n},\Psi_{2n}\}_{n\geq N}$ given by Theorem 3.4, Theorem 3.6 and Lemma 5.3, all nonzero components of $\{\Phi^{\pm}_{1n},\Phi_{2n}\}_{n\geq N}$ and $\{\Psi^{\pm}_{1n},\Psi_{2n}\}_{n\geq N}$ are Bessel sequences in $L^2(0,1)$. Hence, $\{\Phi^{\pm}_{1n},\Phi_{2n}\}_{n\geq N}$ and $\{\Psi^{\pm}_{1n},\Psi_{2n}\}_{n\geq N}$ are also Bessel sequences in $\mathcal{H}$. The result then follows from Lemma 5.1. □

Now we are in a position to prove Theorem 2.2.

Proof of Theorem 2.2. The spectrum-determined growth condition follows from Theorem 5.4 and Proposition 4.2. By Lemma 3.1, for every $\lambda\in\sigma_p(A)$, $\mathrm{Re}(\lambda)<0$. According to the distribution of spectra of the operator A given by (3.8)–(3.10), A has one (and only one) continuous spectrum $\lambda=-1/\beta$ which is an accumulation point of the eigenvalues, and hence $\mu\leq 1/\beta$. The exponential stability then follows from the spectrum-determined growth condition, which is similar to [4, 7, 8]. □

6. Gevrey regularity

In what follows, we show that the C_0-semigroup e^{At} generated by A is of a Gevrey class δ with any $\delta>2$, which is the semigroup class between the differentiable Semigroups and the analytical ones.

Definition 6.1. [1, 20] A C_0-semigroup $T(t)$ is of a Gevrey class $\delta>1$ for $t>t_0$ if $T(t)$ is infinitely differentiable for $t>t_0$ and for every compact subset $K\subset(t_0,\infty)$ and each $\theta>0$, there is a constant $C=C(K,\theta)$ such that

$$\|T^{(n)}(t)\|\leq C\theta^n(n!)^{\delta},\ \forall\, t\in K, n=0,1,2,\ldots.$$

In order to get the Gevrey regularity of the system (2.2), we need the following theorem established by Taylor in [20].

Theorem 6.2. *Let* e^{At} *be a* C_0*-semigroup satisfying* $\|\mathrm{e}^{At}\|\leq M\mathrm{e}^{\omega t}$*. Suppose that for some* $\mu\geq\omega$ *and* α *satisfying* $0<\alpha\leq 1$,

$$\lim_{|\tau|\to\infty}\sup|\tau|^{\alpha}\|R(\mu+\mathrm{i}\tau,A)\|=C<\infty,\ \tau\in\mathbb{R}.$$

Then, e^{At} *is of Gevrey class* δ *with* $\delta>1/\alpha$ *for* $t>0$.

Theorem 6.3. *Let A be defined by (2.2). Then, the semigroup e^{At}, generated by A, is of Gevrey class $\delta > 2$ with $t_0 = 0$.*

Proof. From Theorem 2.2, we know that A generates an exponentially stable C_0-semigroup e^{At} in $\mathcal{H}$. By Theorem 6.2, we only need to show

$$\lim_{\tau\to\infty} |\tau| \|R(\mathrm{i}\tau, A)\|^2 = C < \infty,\ \tau \in \mathbb{R}. \tag{6.1}$$

By Theorem 5.4,

$$\left\{\{\Phi_{1n,j}^{\pm}\}_{j=1}^{m_{1n}^{\pm}}\right\}_{n<N} \cup \{\Phi_{1n}^{\pm}\}_{n\geq N} \cup \left\{\{\Phi_{2n,j}, \overline{\Phi_{2n,j}}\}_{j=1}^{m_{2n}}\right\}_{n<N} \cup \{\Phi_{2n}, \overline{\Phi_{2n}}\}_{n\geq N} \tag{6.2}$$

forms a Riesz basis for $\mathcal{H}$. For any $Y \in \mathcal{H}$,

$$\begin{aligned} Y = &\sum_{n=1}^{N-1}\left(\sum_{j=1}^{m_{1n}^+} a_{1n,j}^+\Phi_{1n,j}^+ + \sum_{j=1}^{m_{1n}^-} a_{1n,j}^-\Phi_{1n,j}^-\right) + \sum_{n=N}^{\infty}\left(a_{1n}^+\Phi_{1n}^+ + a_{1n}^-\Phi_{1n}^-\right) \\ &+ \sum_{n=1}^{N-1}\sum_{j=1}^{m_{2n}}\left(a_{2n,j}\Phi_{2n,j} + b_{2n,j}\overline{\Phi_{2n,j}}\right) + \sum_{n=N}^{\infty}\left(a_{2n}\Phi_{2n} + b_{2n}\overline{\Phi_{2n}}\right), \end{aligned} \tag{6.3}$$

and

$$\begin{aligned} \|Y\|^2 \asymp &\sum_{n=1}^{N-1}\left(\sum_{j=1}^{m_{1n}^+} |a_{1n,j}^+|^2 + \sum_{j=1}^{m_{1n}^-} |a_{1n,j}^-|^2\right) + \sum_{n=N}^{\infty}\left(|a_{1n}^+|^2 + |a_{1n}^-|^2\right) \\ &+ \sum_{n=1}^{N-1}\sum_{j=1}^{m_{2n}}\left(|a_{2n,j}|^2 + |b_{2n,j}|^2\right) + \sum_{n=N}^{\infty}\left(|a_{2n}|^2 + |b_{2n}|^2\right), \end{aligned} \tag{6.4}$$

where "$W \asymp Z$" means the equivalence of the norm: There are constants $c_0, c_1 > 0$ independent of W and Z such that $c_0\|Z\| \leq \|W\| \leq c_1\|Z\|$. Let $\tau > 0$. Then, $\mathrm{i}\tau \in \rho(A)$ and hence

$$\begin{aligned} R(\mathrm{i}\tau, A)Y = &\sum_{n=1}^{N-1}\left(\sum_{j=1}^{m_{1n}^+} \frac{a_{1n,j}^+\Phi_{1n,j}^+}{\mathrm{i}\tau - \lambda_{1n}^+} + \sum_{j=1}^{m_{1n}^-} \frac{a_{1n,j}^-\Phi_{1n,j}^-}{\mathrm{i}\tau - \lambda_{1n}^-}\right) + \sum_{n=N}^{\infty}\left(\frac{a_{1n}^+\Phi_{1n}^+}{\mathrm{i}\tau - \lambda_{1n}^+} + \frac{a_{1n}^-\Phi_{1n}^-}{\mathrm{i}\tau - \lambda_{1n}^-}\right) \\ &+ \sum_{n=1}^{N-1}\sum_{j=1}^{m_{2n}}\left(\frac{a_{2n,j}\Phi_{2n,j}}{\mathrm{i}\tau - \lambda_{2n}} + \frac{b_{2n,j}\overline{\Phi_{2n,j}}}{\mathrm{i}\tau - \overline{\lambda_{2n}}}\right) + \sum_{n=N}^{\infty}\left(\frac{a_{2n}\Phi_{2n}}{\mathrm{i}\tau - \lambda_{2n}} + \frac{b_{2n}\overline{\Phi_{2n}}}{\mathrm{i}\tau - \overline{\lambda_{2n}}}\right) \\ &+ \sum_{n=1}^{N-1}\left[O\left(\frac{1}{|\mathrm{i}\tau - \lambda_{1n}^+|^2}\right) + O\left(\frac{1}{|\mathrm{i}\tau - \lambda_{1n}^-|^2}\right)\right. \\ &+ \left.\left(\frac{1}{|\mathrm{i}\tau - \lambda_{2n}|^2}\right) + O\left(\frac{1}{|\mathrm{i}\tau - \overline{\lambda_{2n}}|^2}\right)\right], \end{aligned} \tag{6.5}$$

and

$$\|R(\mathrm{i}\tau,A)Y\|^2 \asymp \sum_{n=1}^{N-1}\left(\sum_{j=1}^{m_{1n}^+}\frac{|a_{1n,j}^+|^2}{|\mathrm{i}\tau-\lambda_{1n}^+|^2}+\sum_{j=1}^{m_{1n}^-}\frac{|a_{1n,j}^-|^2}{|\mathrm{i}\tau-\lambda_{1n}^-|^2}\right)+\sum_{n=N}^{\infty}\left(\frac{|a_{1n}^+|^2}{|\mathrm{i}\tau-\lambda_{1n}^+|^2}+\frac{|a_{1n}^-|^2}{\mathrm{i}\tau-\lambda_{1n}^-}\right)$$
$$+\sum_{n=1}^{N-1}\sum_{j=1}^{m_{2n}}\left(\frac{|a_{2n,j}|^2}{|\mathrm{i}\tau-\lambda_{2n}|^2}+\frac{|b_{2n,j}|^2}{|\mathrm{i}\tau-\overline{\lambda_{2n}}|^2}\right)+\sum_{n=N}^{\infty}\left(\frac{|a_{2n}|^2}{|\mathrm{i}\tau-\lambda_{2n}|^2}+\frac{|b_{2n}|^2}{|\mathrm{i}\tau-\overline{\lambda_{2n}}|^2}\right), \tag{6.6}$$

where $\{\lambda_{1n}^\pm, n\in\mathbb{N}\}$ and $\{\lambda_{2n},\overline{\lambda_{2n}}, n\in\mathbb{N}\}$, given by (3.8), are eigenvalues of A.

Now we estimate $|\mathrm{i}\tau-\lambda_{1n}^\pm|^2$, $|\mathrm{i}\tau-\lambda_{2n}|^2$ and $|\mathrm{i}\tau-\overline{\lambda_{2n}}|^2$. In terms of λ_{1n}^+ and λ_{1n}^- given by (3.8), for n large enough,

$$\begin{aligned}
|\mathrm{i}\tau-\lambda_{1n}^+|^2 &= \left|\mathrm{i}\tau+\left(n\pi+\frac{\theta_1}{2}\right)^2\beta+\frac{1}{\beta}+O\left(\frac{1}{n^2}\right)\right|^2\\
&= \tau^2+\left(n\pi+\frac{\theta_1}{2}\right)^4\beta^2+O(1)\geq M_1\tau^2,\\
|\mathrm{i}\tau-\lambda_{1n}^-|^2 &= \left|\mathrm{i}\tau+\frac{1}{\beta}+\frac{1}{\left(n\pi+\frac{\theta_1}{2}\right)^2\beta^3}+O\left(\frac{1}{n^3}\right)\right|^2=\tau^2+\frac{1}{\beta^2}+O\left(\frac{1}{n^2}\right)\geq M_2\tau^2,
\end{aligned} \tag{6.7}$$

for some $M_1, M_2>0$ independent of τ. In terms of λ_{2n} given by (3.8), for n large enough, there exists $C>0$ such that

$$|\mathrm{Re}(\lambda_{2n})|\geq C|\mathrm{Im}(\lambda_{2n})|^{1/2}.$$

We thus have

$$|\mathrm{i}\tau-\lambda_{2n}|^2\geq|\tau-\mathrm{Im}(\lambda_{2n})|^2+C^2|\mathrm{Im}(\lambda_{2n})|. \tag{6.8}$$

Let $0<\varepsilon<1$ be given. If $\mathrm{Im}(\lambda_{2n})\geq\varepsilon\tau$, then,

$$|\mathrm{i}\tau-\lambda_{2n}|^2\geq C^2|\mathrm{Im}(\lambda_{2n})|\geq C^2\varepsilon\tau. \tag{6.9}$$

If $\mathrm{Im}(\lambda_{2n})<\varepsilon\tau$, then, for $\tau\geq1$,

$$|\mathrm{i}\tau-\lambda_{2n}|^2\geq|\tau-\mathrm{Im}(\lambda_{2n})|^2\geq(1-\varepsilon)^2\tau^2\geq(1-\varepsilon)^2\tau. \tag{6.10}$$

Hence, by (6.9) and (6.10), there exists a constant $M_3>0$ such that

$$|\mathrm{i}\tau-\lambda_{2n}|^2\geq M_3\tau. \tag{6.11}$$

For $\overline{\lambda_{2n}}$ with sufficiently large n, we have

$$|\mathrm{i}\tau-\overline{\lambda_{2n}}|^2=|\tau+\mathrm{Im}(\lambda_{2n})|^2+|\mathrm{Re}(\lambda_{2n})|\geq|\mathrm{i}\tau-\lambda_{2n}|^2\geq M_3\tau. \tag{6.12}$$

By (6.6)–(6.12), there is an $M > 0$ such that

$$\lim_{|\tau|\to+\infty} \sup |\tau| \|R(\mu + \mathrm{i}\tau, A)\|^2 = M. \tag{6.13}$$

On the other hand, with similar way, for $\tau \in \mathbb{R}$ and $\tau < 0$, we also have

$$\lim_{|\tau|\to+\infty} \sup |\tau| \|R(\mu + \mathrm{i}\tau, A)\|^2 = M. \tag{6.14}$$

This proves (6.1), and by Theorem 6.2, the semigroup e^{At} is of Gevrey class $\delta > 2$ with $t_0 = 0$. □

7. Concluding remarks

In this paper, we study stability for a wave-beam coupled system where a damped wave that connects with a beam through boundary weak connections is considered as a controller of the beam system. We develop all properties including mainly the Riesz basis generation, the spectrum-determined growth condition, the exponential stability and the Gevrey regularity for this system, which is parallel to a heat-beam system considered in [21]. It is quite unexpected that a damped wave subsystem which can change shape arbitrarily can stabilize exponentially a beam which cannot change arbitrary the shape. Obviously, the wave controller is much more easily implemented than heat in some engineering beam structures, which is considered as a new discovery from the engineering point of view.

Secondly, in paper [21] and most of other same type of papers, the resolvent of the system operator is always compact. This means that system has only eigenvalues. However, the spectrum of the system operator in this paper has no compact resolvent. It has both eigenvalues and continuous spectrum. This results in the spectrum of the wave part not being symmetric on the real axis. Actually, the wave part has two branches of eigenvalues, where one branch approaches infinity and another finite branch approaches a finite accumulation point. As a result, the eigenfunctions are very different with that in existing literature including [21], in particular for the branch of eigenfunctions corresponding to the eigenvalues that have finite accumulation point. This is another new discovery from the mathematical point of view.

References

[1] B. Belinskiy and I. Lasiecka, Gevrey's and trace regularity of a semigroup associated with beam equation and non-monotone boundary conditions. *J. Math. Anal. Appl.* **332** (2007) 137–154.

[2] G. Chen and D. Russell, A mathematical model for linear elastic systems with structural damping. *Quart. Appl. Math.* **39** (1981/82) 433–454.

[3] N. Dunford and J.T. Schwartz, Linear Operators, Part III. John Wiley, New York (1971).

[4] B.Z. Guo, Riesz basis approach to the stabilization of a flexible beam with a tip mass. *SIAM J. Control Optim.* **39** (2001) 1736–1747.

[5] B.Z. Guo, Further results for a one-dimensional linear thermoelastic equation with dirichlet-dirichlet boundary conditions. *Anziam J.* **43** (2002) 449-462.

[6] B.Z. Guo and S.P. Yung, Asymptotic behavior of the eigenfrequency of a one-dimensional linear thermoelastic system. *J. Math. Anal. Appl.* **213** (1997) 406–421.

[7] B.Z. Guo and K.Y. Chan, Riesz basis generation, eigenvalues distribution, and exponential stability for a Euler-Bernoulli beam with joint feedback control. *Rev. Mat. Complut.* **14** (2001) 205–230.

[8] B.Z. Guo and G.D. Zhang, On spectrum and Riesz basis property for one-dimensional wave equation with Boltzmann damping. *ESAIM: COCV* **18** (2012) 889–913.

[9] B.Z. Guo, J.M. Wang, and S.P. Yung, On the C_0-semigroup generation and exponential stability resulting from a shear force feedback on a rotating beam. *Syst. Control Lett.* **54** (2005) 557–574.

[10] B.Z. Guo, J.M. Wang and G.D. Zhang, Spectral analysis of a wave equation with Kelvin-Voigt damping. *ZAMM Z. Angew. Math. Mech.* **90** (2010) 323–342.

[11] R.E. Langer, On the zeros of exponential sums and integrals. *Bull. Am. Math. Soc.* **37** (1931) 213–239.

[12] B.Ya. Levin, Lectures on Entire Functions. American Mathematical Society, Providence, Rhode Island (1996).

[13] Z.H. Luo, B.Z. Guo and O. Mörgül, Stability and Stabilization of Infinite Dimensional Systems with Applications. Springer-Verlag, London (1999).

[14] M.A. Naimark, Linear Differential Operators, Part I: Elementary Theory of Linear Differential Operators. Frederick Ungar Publishing Co., New York (1967).

[15] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations. Springer-Verlag, New York (1983).

[16] L. Rosier and P. Rouchon, On the controllability of a wave equation with structural damping. *Int. J. Tomogr. Stat.* **5** (2007) 79–84.

[17] Y. Shibata, On the rate of decay of solutions to linear viscoelastic equation. *Math. Methods Appl. Sci.* **23** (2000) 203–226.

[18] A.A. Shkalikov, Boundary-value problems for ordinary differential equations with a parameter in the boundary conditions. *J. Soviet Math.* **33** (1986) 1311–1342.

[19] M.A. Shubov, Generation of gevrey class semigroup by non-selfadjoint Euler-Bernoulli beam model. *Math. Methods Appl. Sci.* **29** (2006) 2181–2199.

[20] S. Taylor, *Gevrey Regularity of Solutions of Evolution Equations and Boundary Controllability, Gevrey Semigroups* (Chapter 5). Ph.D. thesis, School of Mathematics, University of Minnesota (1989).

[21] J.M. Wang and M. Krstic, Stability of an interconnected system of Euler-Bernoulli beam and heat equation with boundary coupling. *ESAIM: COCV* **21** (2015) 1029–1052.

[22] J.M. Wang, B.B. Ren and M. Krstic, Stabilization and Gevrey regularity of a Schrödinger equation in boundary feedback with a heat equation. *IEEE Trans. Automatic Control* **57** (2012) 179–185.

[23] R.M. Young, An Introduction to Nonharmonic Fourier Series. Academic Press, Inc., San Diego, CA, first edition (2001).

[24] X. Zhang and E. Zuazua, Polynomial decay and control of a 1-d hyperbolic-parabolic coupled system. *J. Differ. Equ.* **204** (2004) 380–438.

[25] X. Zhang and E. Zuazua, Asymptotic behavior of a hyperbolic-parabolic coupled system arising in fluid-structure interaction. *Int. Ser. Numer. Math.* **154** (2006) 445–455.

[26] Q. Zhang, J.M. Wang and B.Z. Guo, Stabilization of the Euler-Bernoulli equation via boundary connection with heat equation. *Math. Control Signals Syst.* **26** (2014) 77–118.