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STATIONARY KIRCHHOFF EQUATIONS INVOLVING CRITICAL GROWTH AND VANISHING POTENTIAL*

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Abstract. We establish the existence of positive solutions for a class of stationary Kirchhoff-type equations defined in the whole $\mathbb{R}^3$ involving critical growth in the sense of the Sobolev embedding and potentials, which may decay to zero at infinity. We use minimax techniques combined with an appropriate truncated argument and a priori estimate. These results are new even for the local case, which corresponds to nonlinear Schrödinger equations.

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1. Introduction

In the present paper, we prove the existence of positive solutions for stationary Kirchhoff-type equations of the form

$$\begin{cases} -M\left(\|\nabla u\|_2^2\right)\Delta u + V(x)u = f(u) \quad \text{in} \quad \mathbb{R}^N, \\ u > 0 \quad \text{in} \quad \mathbb{R}^N \quad \text{and} \quad u \in D^{1,2}(\mathbb{R}^N). \end{cases} \tag{1.1}$$

Here, $\|\cdot\|_p$ denotes the L_p-norm with respect to the Lebesgue measure. This class of problems has been studied extensively under various assumptions on the function M, the potential V, and the nonlinearity $f = f(s)$. A typical example of M considered in some recent papers is $M(t) = a + bt$. In this case, (1.1) becomes the standard Kirchhoff equation. We note that problem (1.1) with $M \equiv 1$ corresponds to the nonlinear Schrödinger equation

$$\begin{cases} -\Delta u + V(x)u = f(u) \quad \text{in } \mathbb{R}^N, \\ u > 0 \quad \text{in} \quad \mathbb{R}^N \quad \text{and} \quad u \in D^{1,2}(\mathbb{R}^N). \end{cases} \tag{1.2}$$

This class of nonlinear elliptic equations in $\mathbb{R}^N$ has been intensively studied in recent years motivated by a wide variety of problems in mathematics and physics in particular for the search of standing wave solutions by considering different approaches (see [1, 3, 5, 6, 8, 9, 32]).

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Nonlocal elliptic equations as (1.1) were introduced by G. Kirchhoff [23] to describe the transversal oscillations of a stretched string. See also the works of Bernstein [10], Pohozaev [30] and Lions [27] for classical studies on this class of problems. These equations may be considered as the simplest example of quasilinear evolution equations of hyperbolic-type. Considering the intrinsic physical meaning, and also the fact that it can be expanded to more complex equations, Kirchhoff-type equations are a relevant topic for studies. For mathematical and physical background on these problems, we refer the readers to [4, 10, 11].

Problems like (1.1) have been extensively studied by many researchers after the abstract functional analysis framework proposed by Lions [27]. For instance, we refer to [13, 15, 18, 21, 24–26, 29, 31, 33]. We also refer to [19–21], where the authors have discussed about existence of solutions, compactness, uniqueness and stability properties for Kirchhoff-type equations in closed manifolds. There are a few papers in which the existence of solutions is considered for Kirchhoff-type problems in whole $\mathbb{R}^N$ when the potential may decay to zero at infinity. In [16], using a minimization argument and a quantitative deformation lemma, the authors proved the existence of a least energy nodal (or sign-changing) solution for a class of Schrödinger–Kirchhoff problems. Moreover, when the problem presents symmetry, they obtained infinitely many nontrivial solutions.

The main contribution of this article corresponds to the study of the critical case of the Kirchhoff equation of type (1.1) with potentials vanishing at infinity, which is new even for the relevant case of the nonlinear Schrödinger equation (1.2). Thus the present paper can be seen as a natural completion of recent works [1, 3], where it was studied the subcritical case for a certain class of vanishing potentials. We want to mention that V. Benci and G. Cerami in [7] studied standing wave solutions of the critical problem $-\Delta u + a(x)u = u^{(N+2)/(N-2)}$ in $\mathbb{R}^N$ involving vanishing potential requiring also that $a \in L^{N/2}(\mathbb{R}^N)$. They proved that this problem has at least one solution if $\|a\|_{L^{N/2}}$ is sufficiently small. We point out that if $a(x) \approx |x|^{-\theta}$ with $0 < \theta < p-1$ is in the class of potentials satisfying our assumptions, but $a \notin L^{N/2}(\mathbb{R}^N)$ if $\theta \leq 2$, that is, $a(x)$ does not belongs to Benci-Cerami class (see Example 1.2).

As already mentioned, we will focus our study on the stationary Kirchhoff problem involving critical growth defined on the whole 3-dimensional Euclidean space of the form

$$\begin{cases} -M\left(\|\nabla u\|_2^2\right)\Delta u + V_\lambda(x)u = u^5 + \gamma|u|^{p-1}u \quad \text{in } \mathbb{R}^3, \\ u > 0 \quad \text{in} \quad \mathbb{R}^3, \quad u \in D^{1,2}(\mathbb{R}^3), \end{cases} \tag{$\mathcal{P}_{\lambda,\gamma}$}$$

depending on $p \in (3,5)$, the potential $V_\lambda(x) = Z(x) + \lambda V(x)$ and the positive real paramenter λ. We are motivated to study our problem in dimension three because it is where the most interesting physical phenomena occur. We recall that for $N = 3$, we have $2^* - 1 = (N+2)/(N-2) = 5$, where $2^* = 2N/(N-2)$ is the critical Sobolev exponent. This kind of potential $V_\lambda = Z + \lambda V$ appears in some recent works to study a class of nonlinear Schrödinger equations. See for instance [2, 5, 6] and references therein, for the case where the potential is bounded away from zero. In the present paper M is a continuous positive function satisfying some general conditions and the potential $V_\lambda = Z + \lambda V$ may decay to zero at infinity in some direction (Z with compact support, for instance). We mention that because of the first term in the left hand side of ($\mathcal{P}_{\lambda,\gamma}$) this is a nonlocal problem in essence and this leads to some very interesting features.

1.1. Assumptions

To state our main results, let us describe in a more precise way the assumptions on the potential V and the Kirchhoff-type function M.

$$Z(x) \text{ and } V(x) \text{ are continuous and nonnegative functions;} \tag{V_1}$$

$$V(x) \equiv 0 \text{ in some ball } B_{r_1}(x_1) \subset \mathbb{R}^3; \tag{V_2}$$

$$\liminf_{|x|\to\infty} |x|^{p-1}V(x) > 0; \tag{V_3}$$

$$\text{The Kirchhoff function } M : [0,+\infty) \to [0,+\infty) \text{ is continuous;} \tag{M_1}$$

$$M \text{ is increasing on the interval } [0,+\infty) \text{ and } M(0) =: M_o > \frac{1}{p+1}; \tag{M_2}$$

$$\text{The function } M(t)/t \text{ is nonincreasing on the interval } (0,+\infty). \tag{M_3}$$

1.2. Statement of the main results

Our first result for equation ($\mathcal{P}_{\lambda,\gamma}$) is the following.

Theorem 1.1. *Suppose that* (V_1)–(V_3), (M_1)–(M_3) *are satisfied and* $3 < p < 5$*. Then, there exist* $\gamma^* > 0$ *such that for any* $\gamma \geq \gamma^*$ *there exists* $\lambda^* = \lambda^*(\gamma) > 0$ *such that* ($\mathcal{P}_{\lambda,\gamma}$) *possesses a positive solution for all* $\lambda \geq \lambda^*$.

Let us give some examples which illustrates the above result.

Example 1.2. Given $C > 0$, $0 < \theta < p-1$ and $R_o > 0$, we can check that any continuous and nonnegative function $V : \mathbb{R}^3 \to \mathbb{R}$ such that $V(x) = C/|x|^\theta$ for all $|x| \geq R_o$ verifies (V_3).

Example 1.3. Examples of a Kirchhoff function M verifying (M_1)–(M_3) are given below

1. $M(t) = a + bt$ with $a > 0$ and $b \geq 0$.
2. We can also consider $M(t) = a + bt^q \left(\ln(1+t)\right)^\ell$ with $0 \leq q,\ \ell$ and $0 < q + \ell \leq 1$.
3. $M(t) = a + \sum_{j=1}^k b_j t^{p_j}$ with $0 < p_j \leq 1$.

Remark 1.4. One can see that under our assumptions, the natural functional of ($\mathcal{P}_{\lambda,\gamma}$) is not well defined. To face this difficulty we propose a suitable modification on the nonlinearity $f_\gamma(s) := s^5 + \gamma|s|^{p-1}s$ such that the energy functional associated to the modified problem has compactness and allow us to prove the existence of a ground state solution by using the minimax techniques. Next, by choosing a sufficiently large γ, we verify that the solution of the auxiliary problem is indeed a solution to our original problem ($\mathcal{P}_{\lambda,\gamma}$).

Using a similar approach as in Theorem 1.1, with some minor modifications, a more general result for the following Kirchhoff problem can be proved.

$$\begin{cases} -M\left(\|\nabla u\|_2^2\right)\Delta u + W_\lambda(x)u = u^5 + \gamma|u|^{p-1}u & \text{in } \mathbb{R}^3, \\ u > 0 \quad \text{in} \quad \mathbb{R}^3, u \in D^{1,2}(\mathbb{R}^3), \end{cases} \tag{$\mathcal{Q}_{\lambda,\gamma}$}$$

where the family of pontentials W_λ verifies the following hypotheses:

$$\inf_{z\in\mathbb{R}^3} \int_{B_1(z)} W_\lambda(x)\, \mathrm{d}x < 1; \tag{V_4}$$

$$\text{There exists } R_o > 0 \text{ and } C > 0 \text{ such that } \inf_{|x|\geq R_o} W_\lambda(x)|x|^{p-1} > C\lambda. \tag{V_5}$$

Theorem 1.5. *Suppose that* (V_4)–(V_5), (M_1)–(M_3) *are satisfied and* $3 < p < 5$*. Then, there exist* $\gamma^* > 0$ *such that for all* $\gamma \geq \gamma^*$ *there is a* $\lambda^* = \lambda^*(\gamma) > 0$ *such that* ($\mathcal{Q}_{\lambda,\gamma}$) *possesses a positive solution for all* $\lambda \geq \lambda^*$.

Example 1.6. As an example of a class of potentials which satisfies conditions (V_4)–(V_5) is given by $W_\lambda(x) = \lambda^2/(\lambda|x|^\theta + 1)$ where $0 < \theta < p-1$ for $|x-z| \geq r_1$ and W_λ bounded in $|x-z| \leq r_o$ uniformly in $\lambda > 0$. Notice that W_λ does not verifies (V_1)–(V_3).

Remark 1.7. According to some technical difficulties that appear in our argument due to the coefficient of Kirchhoff, we must impose the hypothesis $p > 3$. This assumption implies the following crucial estimate $(p+1)\hat{M}(t) - 2tM(t) \geq M_o(p-3)t$ for all $t > 0$. This fact is important to prove that the Palais-Smale sequences are bounded and converge in the Lebesgue space $L^6(\mathbb{R}^3)$ up to a subsequence. Moreover, we can obtain an upper bound estimate for a mountain pass level associated with an auxiliary functional (see Prop. 2.2).

1.3. Notation

Let us introduce the following notations:

- C, $\tilde{C}$, C_1, C_2,... denote positive constants (possibly different).
- $B_R(x_0)$ denotes the open ball centered at x_0 and radius $R > 0$.
- The norms in $L^p(\mathbb{R}^N)$ and $L^\infty(\mathbb{R}^N)$ will be denoted respectively by $\|\cdot\|_p$ and $\|\cdot\|_\infty$.
- $o_n(1)$ denotes a sequence which converges to 0 as $n \to \infty$.

1.4. Outline

Our paper is organized as follows. In the Section 2, we consider some auxiliary functionals and we obtain estimates for their mountain pass levels. Section 3 is devoted to a study of a L^6-estimate on the solutions of some auxiliary problem and its L^∞-estimate is done in the Section 4. We conclude the proof of Theorem 1.1 in the Section 5.

2. Preliminaries

In this section we show some properties for the Kirchhoff function M and its primitive $\hat{M}$ with $\hat{M}(0) = 0$, that is, $\hat{M}(t) = \int_0^t M(\tau)\,\mathrm{d}\tau$. We also obtain upper bound estimates for mountain pass levels for certain energy functional associated to an auxiliar problem.

Lemma 2.1. *The following properties involving M and its primitive $\hat{M}$ hold:*

(a) $\hat{M}(t) \geq M_o t$, *for all* $t > 0$*;*
(b) *Conditions* (M_2)–(M_3) *imply that the map* $t \mapsto 2\hat{M}(t) - tM(t)$ *is non-decreasing for* $t > 0$*;*
(c) *If* $p > 3$*, conditions* (M_2)–(M_3) *imply that*

$$(p+1)\hat{M}(t) - 2tM(t) \geq M_o(p-3)t, \quad \textit{for all } t > 0.$$

Proof. It is easy to check that (M_1) implies that $\hat{M}(t) \geq M_o t$, for all $t > 0$, thus (a) holds.

Let us define $A(t) = 2\hat{M}(t) - tM(t)$. For $0 \leq s \leq t$, it is not difficult to verify that

$$\frac{1}{2}(A(t) - A(s)) = \int_s^t \left(\frac{M(\tau)}{\tau} - \frac{M(t)}{t}\right)\tau\,\mathrm{d}\tau + \int_0^s \left(\frac{M(s)}{s} - \frac{M(t)}{t}\right)\tau\,\mathrm{d}\tau\,.$$

Hence $A(t)$ is non-decreasing function provided that $M(t)/t$ is non-increasing.

Still about M, condition (M_3) holds if, and only if, $2\hat{M}(t) - tM(t)$ is non-decreasing for $t > 0$. When $p > 3$, we have for all $t > 0$,

$$\begin{aligned}(p+1)\hat{M}(t) - 2tM(t) &= (p-3)\hat{M}(t) + 2[2\hat{M}(t) - tM(t)]\\ &\geq (p-3)\hat{M}(t)\\ &\geq M_o(p-3)t,\end{aligned}$$

which implies that (c) holds. □

We start observing that from (V_1), we can introduce the natural Hilbert space

$$E = \left\{v \in \mathcal{D}^{1,2}(\mathbb{R}^3) : \int_{\mathbb{R}^3} V_\lambda(x)v^2\,\mathrm{d}x < \infty\right\}$$

endowed with the scalar product and norm given, respectively, by

$$\langle u, v\rangle = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + V_\lambda(x)uv)\,\mathrm{d}x, \quad \|u\|^2 = \int_{\mathbb{R}^3} \left(|\nabla u|^2 + V_\lambda(x)u^2\right)\mathrm{d}x.$$

An initial difficulty that appears to attach variational problems like ($\mathcal{P}_{\lambda,\gamma}$) in the case that the potential converges to zero at infinity is that, in general, we do not have the embedding "$E \hookrightarrow L^{p+1}(\mathbb{R}^3)$" for $2 \le p < 5$ and the Euler-Lagrange functional associated to ($\mathcal{P}_{\lambda,\gamma}$) is not well defined in E. For this reason, we will consider an auxiliary problem defined in bounded domains as we will see later.

From (V_2), without loss of generality, we suppose that $V(x) = 0$ for all $x \in B_1(0)$. Now let us consider the energy functional $I_0 : H_o^1(B_1(0)) \to \mathbb{R}$ defined by

$$I_0(u) = \frac{1}{2}\hat{M}(||\nabla u||^2_{L^2(B_1(0))}) + \frac{1}{2}\int_{B_1(0)} Z(x)u^2\,\mathrm{d}x - \frac{\gamma}{p+1}\int_{B_1(0)} |u|^{p+1}\,\mathrm{d}x.$$

It is clear that I_0 is well defined, belongs to class C^1 and does not depend on λ. Moreover, under our assumptions one can verify that I_0 has the mountain pass geometry and thus it is well defined the mountain pass level:

$$d_\gamma = \inf_{v\in H_o^1(B_1(0))} \max_{t>0} I_0(tv).$$

The next proposition is a crucial upper bound estimate on this minimax level.

Proposition 2.2. *There exist constants $C_p > 0$ and $\gamma_o > 0$ such that for all $\gamma \ge \gamma_o$ it holds*

$$0 < d_\gamma \le \frac{C_p}{\gamma^{\frac{2}{p-1}}}.$$

In particular $d_\gamma \to 0$ as $\gamma \to +\infty$.

Proof. Let $v_o \in C_0^\infty(\mathbb{R}^3)$ such that $0 \le v_o \le 1$ and define

$$a := ||\nabla v_o||^2_{L^2(B_1(0))}, \quad b := \int_{B_1(0)} Z(x)v_o^2\,\mathrm{d}x, \quad c := ||v_o||^{p+1}_{L^{p+1}(B_1(0))}.$$

Let us estimate $\max_{t>0} I_0(tv_o)$. It is clear that the function $h(t) := I_0(tv_o)$ has a unique critical point which is a global maximum point. Indeed, $h'(t) = 0$ is equivalent to

$$aM(at^2)t + bt - \gamma ct^p = 0.$$

Thus,

$$\frac{M(at^2)}{at^2} + \frac{b}{a^2t^2} = \frac{\gamma ct^{p-3}}{a^2}.$$

Since the left side in the last equality is decreasing and the right side is increasing, we have only one $t > 0$ such that $h'(t) = 0$.

We also have for $\gamma > 0$ sufficiently large,

$$2h(1) = \hat{M}(a) + b - \frac{2\gamma c}{p+1} < 0.$$

If t_o is the critical point of $h(t)$, it is easy to check that $h(t)$ is increasing in $(0, t_o)$ and it is decreasing in (t_o, ∞). Then, since $h(1) < 0$, we must have $t_o < 1$ such that

$$\max_{t>0} h(t) = h(t_o).$$

From condition (M_1)–(M_2), for any $a > 0$, we have

$$\hat{M}(at) = \int_0^{at} M(s)\,\mathrm{d}s \leq M(a)at, \text{ for all } 0 < t \leq 1,$$

which implies

$$h(t_o) \leq \frac{[M(a)a + b]\, t_o^2}{2} - \frac{\gamma c t_o^{p+1}}{p+1} \leq \max_{t>0} H(t) = H(t_M)$$

where

$$H(t) = \frac{At^2}{2} - \frac{\gamma c t^{p+1}}{p+1} \text{ and } A = M(a)a + b.$$

It is easy to see that

$$\max_{t>0} H(t) = H(t_M) = \frac{A^{\frac{p+1}{p-1}}}{c^{\frac{2}{p-1}}} \left(\frac{1}{2} - \frac{1}{p+1}\right) \frac{1}{\gamma^{\frac{2}{p-1}}} \quad \text{where} \quad t_M = \left(\frac{A}{c}\right)^{\frac{1}{p-1}} \frac{1}{\gamma^{\frac{1}{p-1}}}.$$

Thus, setting

$$C_p := \frac{A^{\frac{p+1}{p-1}}}{c^{\frac{2}{p-1}}} \left(\frac{1}{2} - \frac{1}{p+1}\right),$$

we obtain the following estimate

$$d_\gamma = \inf_{v \in H_o^1(B)} \max_{t>0} I_0(tv) \leq \max_{t>0} I_0(tv_o) \leq H(t_M) = \frac{C_p}{\gamma^{\frac{2}{p-1}}} \tag{2.1}$$

which completes our proof. □

From Proposition 2.2, we can choose $\gamma^* > 0$ such that for all $\gamma \geq \gamma^*$ it holds

$$\frac{2(p+1)}{M_o(p-3)} d_\gamma < \min \left\{ 1, \frac{(M_o S)^{\frac{3}{2}}}{M(1)}, \frac{M_o^{\frac{1}{2}} S^{\frac{3}{2}}}{2\sqrt{2}} \right\}, \tag{2.2}$$

where S is the best constant for the Sobolev embedding $\mathcal{D}^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(R^3)$, that is,

$$S = \inf \left\{ ||\nabla v||_2^2 : v \in \mathcal{D}^{1,2}(\mathbb{R}^3),\ ||v||_6 = 1 \right\}.$$

3. Auxiliary problem

We begin this section by recalling that since we deal with a class of potentials that may decay to zero at infinity, the variational method cannot be applied directly, because the natural Euler-Lagrange functional associated with Problem ($\mathcal{P}_{\lambda,\gamma}$) is not well defined on the space E. To overcome this difficulty, we are going to modify the critical nonlinearity $f_\gamma(s) := s^5 + \gamma|s|^{p-1}s$ as follows: choose $R \geq 1$ and define

$$g(x,s) = \begin{cases} f_\gamma(s), & \text{if} \quad x \in B_R \qquad \text{or} \quad f_\gamma(s) \leq \dfrac{V_\lambda(x)}{p+1}s, \\ \dfrac{V_\lambda(x)}{p+1}s, & \text{if} \quad x \notin B_R \qquad \text{and} \quad f_\gamma(s) > \dfrac{V_\lambda(x)}{p+1}s. \end{cases}$$

Let us consider the auxiliar problem

$$-M(\|\nabla u\|^2)\Delta u + V_\lambda(x)u = g(x,u), \text{ in } \mathbb{R}^3. \tag{$\mathcal{AP}$}$$

It is easy to check that $g(x,s)$ is a Carathéodory function and its primitive

$$G(x,s) = \int_0^s g(x,\tau)\,\mathrm{d}\tau$$

is such that

$$G(x,s) = F_\gamma(s) \quad \text{if} \quad x \in B_R \quad \text{or} \quad f_\gamma(s) \leq \frac{V_\lambda(x)}{p+1}s,$$

where

$$F_\gamma(s) = \int_0^s f_\gamma(\tau)\,\mathrm{d}\tau = \frac{s^6}{6} + \frac{\gamma|s|^{p+1}}{p+1}.$$

Moreover, since $f(s)/s$ is increasing for $s > 0$ and decreasing if $s < 0$, one can see that

$$sg(x,s) \leq s^6 + \gamma|s|^{p+1}, \qquad \text{for all } s \in \mathbb{R}; \tag{g_1}$$

$$sg(x,s) - (p+1)G(x,s) \geq \left[\frac{1}{p+1} - \frac{1}{2}\right] V_\lambda(x)s^2, \qquad \text{for all } s \in \mathbb{R}; \tag{g_2}$$

$$sg(x,s) \leq \frac{V_\lambda(x)}{p+1}s^2, \qquad \text{for all } s \in \mathbb{R} \text{ and } x \in B_R^c; \tag{g_3}$$

Using standard arguments, from condition (g_3), the corresponding energy functional $J : E \to \mathbb{R}$ given by

$$J(u) = \frac{\hat{M}(||\nabla u||_2^2)}{2} + \frac{1}{2}\int_{\mathbb{R}^3} V_\lambda(x)u^2\,\mathrm{d}x - \int_{\mathbb{R}^3} G(x,u)\,\mathrm{d}x,$$

is well defined and of class C^1 with

$$J'(u)v = M(||\nabla u||_2^2)\int_{\mathbb{R}^3} \nabla u \nabla v\,\mathrm{d}x + \int_{\mathbb{R}^3} V_\lambda(x)uv\,\mathrm{d}x - \int_{\mathbb{R}^3} g(x,u)v\,\mathrm{d}x \text{ for all } u, v \in E.$$

From our assumptions, one can see that J fulfills the mountain pass geometry, and then the minimax level

$$c_{\lambda,\gamma} = \inf_{v \in E} \max_{t>0} J(tv)$$

is well defined, and satisfies $0 < c_{\lambda,\gamma} \leq d_\gamma$ due to $J(v) \leq I_0(v)$ for all $v \in H_0^1(B_1(0))$. We can use the Ekeland Variational Principle [14] to produces a Palais-Smale sequence $(u_n) \subset E$ at the minimax level $c_{\lambda,\gamma}$, that is,

$$J(u_n) \to c_{\lambda,\gamma} \text{ and } J'(u_n) \to 0. \tag{3.1}$$

Lemma 3.1. *The sequence* (u_n) *is bounded in* E *and* $||\nabla u_n||_2 \leq 1$, *for large* n.

Proof. Indeed, using (3.1) for n big enough, we have

$$c_{\lambda,\gamma} + 1 + \|u_n\| \geq J(u_n) - (p+1)^{-1} J'(u_n)u_n.$$

Thus, by Lemma 2.1 item (c) and (g_2), we obtain

$$\begin{aligned} d_\gamma + 1 + \|u_n\| &\geq \left[\frac{\hat{M}(||\nabla u_n||_2^2)}{2} - \frac{||\nabla u_n||_2^2 M(||\nabla u_n||_2^2)}{p+1} \right] \\ &+ \left(\frac{1}{2} - \frac{1}{p+1} \right) \int_{\mathbb{R}^3} V_\lambda(x) u_n^2 \, \mathrm{d}x + \int_{\mathbb{R}^3} \left[\frac{1}{p+1} g(x,u_n)u_n - G(x,u_n) \right] \, \mathrm{d}x \\ &\geq \left(\frac{1}{2} - \frac{1}{p+1} \right) \left(1 - \frac{1}{p+1} \right) \int_{\mathbb{R}^3} V_\lambda(x) u_n^2 \, \mathrm{d}x + \frac{M_o(p-3)}{2(p+1)} ||\nabla u_n||_2^2. \end{aligned}$$

This last inequality shows that (u_n) is bounded in E. Besides, using (2.2), for all n large enough, we have

$$||\nabla u_n||_2^2 \leq (d_\gamma + o_n(1)) \frac{2(p+1)}{M_o(p-3)} \leq 1, \tag{3.2}$$

which completes the proof. □

Lemma 3.2. *Up to a subsequence, we have that* (u_n) *converges in* $L^6(\mathbb{R}^3)$.

Proof. We may suppose that $u_n \rightharpoonup u$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^3)$, $|\nabla u_n|^2$ and $|u_n|^6$ converge tightly to μ and ν, where μ and ν are bounded nonnegative measures on $\mathbb{R}^3$. Moreover, $u_n \to u$ in $L^r_{loc}(\mathbb{R}^3)$, for all $2 \leq r < 6$. Then, in view of Lions concentration compactnes principle (see [28], Lem. I.1, p. 158), we have

1. there exists a sequence $(\nu_j)_{j\in\mathbb{N}}$ in $\mathbb{R}_+$, $(x_j)_{j\in\mathbb{N}}$ in $\mathbb{R}^3$ such that

$$\nu = u^6 + \sum_{j=1}^{\infty} \nu_j \delta_{x_j};$$

2. besides, we have

$$\mu \geq |\nabla u|^2 + S \sum_{j=1}^{\infty} \nu_j^{\frac{1}{3}} \delta_{x_j}.$$

Let $\phi \in C_o^\infty(\mathbb{R}^3, [0,1])$ such that $\phi(x) = 1$, if $|x| \leq 1/2$ and $\phi(x) = 0$ if $|x| \geq 1$. For each $\varepsilon \in (0,1)$ let us consider

$$\phi_\varepsilon(x) = \phi\left(\frac{x - x_j}{\varepsilon}\right).$$

Notice that if $2 \leq r < 6$,

$$\lim_n \int_{\mathbb{R}^3} \phi_\varepsilon |u_n|^r \, \mathrm{d}x = \int_{\mathbb{R}^3} \phi_\varepsilon |u|^r \, \mathrm{d}x := B_{\varepsilon,u,r}$$

and for each fixed $u \in E$, we have $\mathrm{supp}(\phi_\varepsilon) \subset B(0,1)$ and $|\phi_\varepsilon |u|^r| \leq |u|^r$. Thus by Lebesgue's dominated convergence theorem,

$$\lim_{\varepsilon\to 0} B_{\varepsilon,u,r} = 0.$$

From (g_1) we get

$$\left|\int_{\mathbb{R}^3} (u_n\phi_\varepsilon) g(x,u_n)\,\mathrm{d}x\right| \leq \gamma \int_{\mathbb{R}^3} \phi_\varepsilon |u_n|^{q+1}\,\mathrm{d}x + \int_{\mathbb{R}^3} \phi_\varepsilon u_n^6\,\mathrm{d}x$$

and consequently

$$\limsup_n \left|\int_{\mathbb{R}^3} (u_n\phi_\varepsilon) g(x,u_n)\,\mathrm{d}x\right| \leq C\left(B_{\varepsilon,u,q+1} + \int_{\mathbb{R}^3} \varphi_\varepsilon\,\mathrm{d}\nu\right).$$

By using a Hölder inequality we obtain

$$\left|\int_{\mathbb{R}^3} u_n \nabla\phi_\varepsilon \nabla u_n\,\mathrm{d}x\right| \leq \varepsilon^{-1}\left(\int_{|x-x_j|\leq 2\varepsilon} u_n^2\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{|x-x_j|\leq 2\varepsilon} |\nabla u_n|^2\,\mathrm{d}x\right)^{\frac{1}{2}}.$$

As (u_n) is bounded in E we have

$$\left|\int_{\mathbb{R}^3} u_n \nabla\phi_\varepsilon \nabla u_n\,\mathrm{d}x\right| \leq C\left(\int_{|x-x_j|\leq 2\varepsilon} u_n^2\,\mathrm{d}x\right)^{\frac{1}{2}}, \text{ for all } n,\ \varepsilon$$

and

$$\limsup_n \left|\int_{\mathbb{R}^3} u_n \nabla\phi_\varepsilon \nabla u_n\,\mathrm{d}x\right| \leq C\left(\int_{|x-x_j|\leq 2\varepsilon} u^2\,\mathrm{d}x\right)^{\frac{1}{2}}, \text{ for all } \varepsilon,$$

which shows that

$$\lim_{\varepsilon\to 0}\left(\limsup_n \left|\int_{\mathbb{R}^3} u_n \nabla\phi_\varepsilon \nabla u_n \mathrm{d}x\right|\right) = 0.$$

Now, we can see that

$$\begin{aligned}
o_n(1) = J'(u_n)(u_n\phi_\varepsilon) &= M(||\nabla u_n||_2^2)\int_{\mathbb{R}^3} \nabla u_n \nabla(u_n\phi_\varepsilon)\,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^3} V(x) u_n (u_n\phi_\varepsilon))\,\mathrm{d}x - \int_{\mathbb{R}^3} g(x,u_n)(u_n\phi_\varepsilon)\,\mathrm{d}x \\
&= M(||\nabla u_n||_2^2)\int_{\mathbb{R}^3} |\nabla u_n|^2\phi_\varepsilon\,\mathrm{d}x + \int_{\mathbb{R}^3} V_\lambda(x) u_n^2 \varphi_\varepsilon\,\mathrm{d}x \\
&\quad + M(||\nabla u_n||_2^2)\int_{\mathbb{R}^3} u_n \nabla\phi_\varepsilon \nabla u_n\,\mathrm{d}x - \int_{\mathbb{R}^3} g(x,u_n)(u_n\phi_\varepsilon))\,\mathrm{d}x,
\end{aligned}$$

or,

$$
\begin{aligned}
&M(||\nabla u_n||_2^2)\int_{\mathbb{R}^3}|\nabla u_n|^2\phi_\varepsilon\ \mathrm{d}x+\int_{\mathbb{R}^3}V_\lambda(x)u_n^2\phi_\varepsilon\ \mathrm{d}x\\
&\quad=-M(||\nabla u_n||_2^2)\int_{\mathbb{R}^3}u_n\nabla\phi_\varepsilon\nabla u_n\ \mathrm{d}x+\int_{\mathbb{R}^3}g(x,u_n)(u_n\phi_\varepsilon))\ \mathrm{d}x+o_n(1).
\end{aligned}
$$

As a consequence of (3.2) and $(M.1)$, $M(||\nabla u_n||_2^2)$ is a bounded sequence between M_o and $M(1)$. We can suppose that $M(||\nabla u_n||_2^2)$ converges to some $m_o\in[M_o,M(1)]$. Passing to the limit as $n\to\infty$ we have:

$$
\left|m_o\int\phi_\varepsilon d\mu+B_{\varepsilon,u,2}-\int\phi_\varepsilon\ \mathrm{d}\nu\right|\le C\left[B_{\varepsilon,u,q+1}+\left(\int_{|x-x_j|\le\varepsilon}u^2\ \mathrm{d}x\right)^{1/2}\right],
$$

for all ε. Passing to the limit as $\varepsilon\to 0$,

$$
m_o\mu(\{x_j\})=\nu(\{x_j\})=\nu_j.
$$

Combining with part (2) of Lions Lemma

$$
\mu(\{x_j\})\ge S\nu_j^{1/3}
$$

we have

$$
\nu_j\ge m_oS\nu_j^{1/3}\ge M_oS\nu_j^{1/3}
$$

and thus, if $\nu_j>0$ we obtain

$$
\nu_j^{2/3}\ge M_oS,
$$

which implies that

$$
m_o\mu(\{x_j\})=\nu_j\ge(M_oS)^{3/2}. \tag{3.3}
$$

We know that,

$$
c_{\lambda,\gamma}+o_n(1)=J(u_n)-\frac{1}{p+1}J'(u_n)u_n,
$$

which together Lemma 2.1 gives

$$
\begin{aligned}
c_{\lambda,\gamma}=&\left[\frac{1}{2}\hat{M}(||\nabla u_n||_2^2)-\frac{1}{p+1}M(||\nabla u_n||_2^2)||\nabla u_n||_2^2\right]\\
&+\left[\frac{1}{2}-\frac{1}{p+1}\right]\int_{\mathbb{R}^3}V_\lambda(x)u_n^2\phi_\varepsilon\ \mathrm{d}x\\
&+\int_{\mathbb{R}^3}\left(\frac{1}{p+1}u_ng(x,u_n)-G(x,u_n)\right)\ \mathrm{d}x+o_n(1)\\
\ge&\frac{M_o(p-3)}{2(p+1)}\int_{\mathbb{R}^3}|\nabla u_n|^2\phi_\varepsilon\ \mathrm{d}x+o_n(1)+o_\varepsilon(1).
\end{aligned}
$$

Passing to the limit as $n \to +\infty$, we obtain

$$c_{\lambda,\gamma} \geq \frac{M_o(p-3)}{2(p+1)} \int_{\mathbb{R}^3} \phi_\varepsilon \, \mathrm{d}\mu + o_\varepsilon(1).$$

Taking to the limit as $\varepsilon \to 0$, we have

$$c_{\lambda,\gamma} \geq \frac{M_o(p-3)}{2(p+1)} \mu(\{x_j\}).$$

We also note that assumption (3.3) implies

$$c_{\lambda,\gamma} \geq \frac{M_o(p-3)}{2m_o(p+1)} (M_o S)^{3/2} \geq \frac{M_o(p-3)}{2M(1)(p+1)} (M_o S)^{3/2}.$$

Now, if $\nu_j > 0$ we can deduce

$$c_{\lambda,\gamma} \geq \frac{M_o(p-3)}{2M(1)(p+1)} (M_o S)^{3/2}, \tag{3.4}$$

which is a contradiction with the inequality $c_{\lambda,\gamma} \leq d_\gamma$ and (2.2). Then $\nu_i = 0$ for all i and, u_n converges to u in $L^6(\mathbb{R}^3)$. □

Lemma 3.3. *The following limits hold for the sequence (u_n):*

$$\lim_n \int_{\mathbb{R}^3} V_\lambda(x) u_n^2 \, \mathrm{d}x = \int_{\mathbb{R}^3} V_\lambda(x) u^2 \, \mathrm{d}x, \tag{3.5}$$

$$\lim_n \int_{\mathbb{R}^3} g(x, u_n) u_n \, \mathrm{d}x = \int_{\mathbb{R}^3} g(x, u) u \, \mathrm{d}x, \tag{3.6}$$

$$\lim_n \int_{\mathbb{R}^3} g(x, u_n) v \, \mathrm{d}x = \int_{\mathbb{R}^3} g(x, u) v \, \mathrm{d}x, \ \forall v \in E \tag{3.7}$$

$$\lim_n \int_{\mathbb{R}^3} G(x, u_n) \, \mathrm{d}x = \int_{\mathbb{R}^3} G(x, u) \, \mathrm{d}x. \tag{3.8}$$

Proof. We start with the following claim:

$$\lim_{r\to\infty} \int_{|x|\geq r} \left[|\nabla u_n|^2 + V_\lambda(x) u_n^2 \right] \mathrm{d}x = 0, \ \text{uniformly in } n. \tag{3.9}$$

In fact, let us consider a cut-off function $\eta \in C_0^\infty(B_r^c, [0,1])$ such that $\eta(x) = 1$ for all $|x| \geq 2r$ and $|\nabla \eta(x)| \leq 2/r$ for all $x \in \mathbb{R}^3$. Since (u_n) is bounded in E, the sequence (ηu_n) is also bounded in E, and then $J'(u_n)(\eta u_n) = o_n(1)$, that is,

$$M(||\nabla u_n||_2^2) \int_{\mathbb{R}^3} \nabla u_n \nabla(\eta u_n) \, \mathrm{d}x + \int_{\mathbb{R}^3} V_\lambda(x) u_n(\eta u_n) \, \mathrm{d}x = \int_{\mathbb{R}^3} g(x, u_n)(\eta u_n) \, \mathrm{d}x + o_n(1).$$

Since $\eta(x) = 0$ for all $|x| \leq r$, using (g_3) we obtain

$$M(||\nabla u_n||_2^2) \int_{|x|\geq r} \eta \left[|\nabla u_n|^2 + V_\lambda(x) u_n^2 \right] \mathrm{d}x \leq$$
$$\frac{1}{p+1} \int_{|x|\geq r} \eta V_\lambda(x) u_n^2 \, \mathrm{d}x - M(||\nabla u_n||_2^2) \int_{|x|\geq r} u_n \nabla u_n \nabla \eta \, \mathrm{d}x + o_n(1),$$

which together with hypothesis (M_2) implies

$$\left(M_o - \frac{1}{p+1}\right)\int_{|x|\geq r}\eta\left[|\nabla u_n|^2 + V_\lambda(x)u_n^2\right] \quad \mathrm{d}x \quad \leq \quad 2\frac{M_1}{r}\int_{r\leq|x|\leq 2r}|u_n||\nabla u_n| \quad \mathrm{d}x \quad + \quad o_n(1), \tag{3.10}$$

where $M_1 = M(1)$. Using Hölder inequality, we can estimate

$$\int_{r\leq|x|\leq 2r}|u_n||\nabla u_n|\ \mathrm{d}x \leq \|\nabla u_n\|_{L^2(\mathbb{R}^3)}\left(\int_{r\leq|x|\leq 2r}|u_n|^2\ \mathrm{d}x\right)^{1/2}$$

Since $u_n \to u$ strongly in $L^2(B_{2r}\setminus B_r)$ and $\|\nabla u_n\|_{L^2(\mathbb{R}^3)} \leq 1$, it follows that

$$\limsup_n \int_{r\leq|x|\leq 2r}|u_n||\nabla u_n|\ \mathrm{d}x \leq \left(\int_{r\leq|x|\leq 2r}|u|^2\ \mathrm{d}x\right)^{1/2} \tag{3.11}$$

On the other hand, Hölder inequality implies

$$\left(\int_{r\leq|x|\leq 2r}|u|^2\ \mathrm{d}x\right)^{1/2} \leq r\left(\int_{r\leq|x|\leq 2r}|u|^6\ \mathrm{d}x\right)^{1/6}\left(\frac{28}{3}\pi\right)^{1/3}$$

which together with (3.11) implies

$$\limsup_n \int_{r\leq|x|\leq 2r}|u_n||\nabla u_n|\ \mathrm{d}x \leq r\left(\frac{28}{3}\pi\right)^{1/3}\left(\int_{r\leq|x|\leq 2r}|u|^6\ \mathrm{d}x\right)^{1/6} \tag{3.12}$$

(3.10) and (3.12) show the claim.

Since $u_n \to u$ strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^3)$, (3.5) follows from (3.9). To prove (3.6)–(3.7), we can use (3.5) together with condition (g_3). □

Using Lemmas 3.2 and 3.3 we can show that u is a weak solution for the problem

$$-m_o\Delta u + V_\lambda(x)u = u^5 + g(x,u), \mathbb{R}^3$$

and

$$m_o||\nabla u||_2^2 + \int_{\mathbb{R}^3} V_\lambda(x)u^2\ \mathrm{d}x = \int_{\mathbb{R}^3} u^6\ \mathrm{d}x + \int_{\mathbb{R}^3} g(x,u)u\ \mathrm{d}x.$$

Now passing to the limit in

$$M(||\nabla u_n||_2^2)||\nabla u_n||_2^2 + \int_{\mathbb{R}^3} V_\lambda(x)u_n^2\ \mathrm{d}x = \int_{\mathbb{R}^3}|u_n|^6\ \mathrm{d}x + \int_{\mathbb{R}^3} g(x,u_n)u_n\ \mathrm{d}x + o_n(1),$$

and recalling that $\lim_n M(||\nabla u_n||_2^2) = m_o$, we conclude that

$$\lim_n ||\nabla u_n||_2^2 = ||\nabla u||_2^2.$$

Then u_n converges to u in E and $J(u) = c_{\lambda,\gamma}$. Therefore u is a ground state solution of auxiliary problem ($\mathcal{AP}$) which depends on R and satisfies

$$||\nabla u||_2^2 \leq d_\gamma \frac{2(p+1)}{M_o(p-3)}, \text{ for all } R > 1,$$

further,

$$||u||_6^2 \leq S^{-1}||\nabla u||_2^2 \leq d_\gamma \frac{2(p+1)}{M_o S(p-3)} \tag{3.13}$$

independent on the choice of $R > 1$. Combining (2.1) and (3.13) we have that

$$||u||_6 \leq C\gamma^{-\frac{1}{p-1}}. \tag{3.14}$$

4. *A priori* estimates in the $L^\infty(\mathbb{R}^3)$ norm

We derive some *a priori* $L^\infty -$ *estimates* for the solutions of Auxiliary Problem ($\mathcal{AP}$). For that we follow some extraordinary ideas due to E. De Giorgi, J. Nash and J. Moser, to obtain regularity results that were discovered in the mid 1950's and early 1960's. For more details see for example [12, 17, 22].

Theorem 4.1. *Let u be a solution of ($\mathcal{AP}$) then*

$$\|u\|_\infty \leq C\gamma^{\frac{7p-19}{6(p-1)}},$$

where C is a positive constant.

Before we prove the above estimate we will need to provide some crucial results. First let us state a version of ([17], Thm. 8.17) which is suitable to our purpose.

Lemma 4.2. *Let $b : \mathbb{R}^N \mapsto \mathbb{R}$ be a nonnegative measurable function and let $h \in L^q_{loc}(\mathbb{R}^N)$ such that*

$$[h]_q = \sup_{z\in\mathbb{R}^N} \left(\int_{B_2(z)} |h|^q \, \mathrm{d}x \right)^{1/q} < \infty,$$

where $3 \leq N < 2q$. Suppose that $v \in E$ is a weak solution of the problem

$$-\Delta v + b(x)v = h(x) \text{ in } \mathbb{R}^N. \tag{4.1}$$

Then we have

$$\sup_{x\in B_1(z)} |v(x)| \leq C\,[h]_q \left(\int_{B_2(z)} |v|^{2^*} \, \mathrm{d}x \right)^{1/2^*} \quad \text{for all } z \in \mathbb{R}^N,$$

where C depends only on q (it does not depend on b or v).

Proposition 4.3. *Let the potential $V_o : \mathbb{R}^3 \mapsto \mathbb{R}$ be a nonnegative measurable function and the nonlinear term $g(x,s)$ be a Caratheodory function such that some $\alpha_o, \beta_o > 0$,*

$$|g(x,s)| \leq \alpha_o|s|^5 + \beta_o|s| \quad \text{for all} \quad (x,s) \in \mathbb{R}^3 \times \mathbb{R}.$$

Suppose that $u \in E$ is a weak solution of the problem

$$-\Delta u + V_o(x)u = g(x,u) \text{ in } \mathbb{R}^3 \tag{4.2}$$

satisfying

$$(\mathcal{C}) \qquad 4\alpha_o ||u||_6^4 \leq S.$$

Then there is Λ *such that*

$$||u||_\infty \leq \Lambda ||u||_6^2,$$

where Λ *does not depend on* V *or* u*, indeed* Λ *depends only on* β_o*. In addition we have* $\Lambda = O(\beta_o^{7/6})$ *as* $\beta_o \to \infty$*.*

Proof. For each $n \in \mathbb{N}$ let us consider the sets

$$A_n = \{x \in \mathbb{R}^3 : u^4 \leq n^2\} \quad \text{and} \quad B_n = \mathbb{R}^3 \setminus A_n.$$

and define the function $v_n \in E$ by

$$v_n = u^5 \quad \text{in } A_n \quad \text{and} \quad v_n = n^2 u \text{ in } B_n.$$

Observe that $v_n \in E$, $v_n u \leq u^6$ in $\mathbb{R}^3$,

$$\nabla v_n = 5u^4 \nabla u \text{ in } A_n \text{ and } \nabla v_n = n^2 \nabla u \text{ in } B_n. \tag{4.3}$$

Then, using v_n as a test function in (4.2),

$$\int_{\mathbb{R}^3} [\nabla u \nabla v_n + V_o(x) u v_n] \, \mathrm{d}x = \int_{\mathbb{R}^3} g(x,u) v_n \, \mathrm{d}x.$$

From (4.3) we have

$$\int_{\mathbb{R}^3} \nabla u \nabla v_n \, \mathrm{d}x = 5 \int_{A_n} u^4 |\nabla u|^2 \, \mathrm{d}x + n^2 \int_{B_n} |\nabla u|^2 \, \mathrm{d}x. \tag{4.4}$$

Now consider

$$\omega_n = u^3 \text{ in } A_n \text{ and } \omega_n = nu \text{ in } B_n.$$

Note that $\omega_n^2 = u v_n \leq |u|^6$, $0 \leq V_o(x)\omega_n^2 = V_o(x) u v_n$ in $\mathbb{R}^3$. Moreover,

$$\nabla \omega_n = 3u^2 \nabla u \text{ in } A_n \quad \text{and} \quad \nabla \omega_n = n \nabla u \text{ in } B_n.$$

Thus,

$$\int_{\mathbb{R}^3} |\nabla \omega_n|^2 \, \mathrm{d}x = 9 \int_{A_n} u^4 |\nabla u|^2 \, \mathrm{d}x + n^2 \int_{B_n} |\nabla u|^2 \, \mathrm{d}x. \tag{4.5}$$

Combining (4.4) and (4.5), we obtain

$$\int_{\mathbb{R}^3} \left[(|\nabla \omega_n|^2 + V_o(x)\omega_n^2 \right] \, \mathrm{d}x - \int_{\mathbb{R}^3} [\nabla u \nabla v_n + V_o(x) u v_n] \, \mathrm{d}x = 4 \int_{A_n} u^4 |\nabla u|^2 \, \mathrm{d}x.$$

From (4.4) we have the inequality

$$5 \int_{A_n} u^4 |\nabla u|^2 \, \mathrm{d}x \leq \int_{\mathbb{R}^3} [\nabla u \nabla v_n + V_o(x) u v_n] \, \mathrm{d}x,$$

and then

$$\int_{\mathbb{R}^3}\left[|\nabla\omega_n|^2+V_o(x)\omega_n^2\right]\,\mathrm{d}x\leq\frac{9}{5}\int_{\mathbb{R}^3}\left[\nabla u\nabla v_n+V_o(x)uv_n\right]\,\mathrm{d}x.$$

Since u a weak solution of (4.2), we have

$$\int_{\mathbb{R}^3}\left[|\nabla\omega_n|^2+V_o(x)\omega_n^2\right]\,\mathrm{d}x\leq\frac{9}{5}\int_{\mathbb{R}^3}g(x,u)v_n\,\mathrm{d}x\leq 2\int_{\mathbb{R}^3}g(x,u)v_n\,\mathrm{d}x. \tag{4.6}$$

Observe that $g(x,u)v_n\leq\alpha_o|u|^5|v_n|+\beta_o|u||v_n|=\alpha_o u^4w_n^2+\beta_o w_n^2$ in $\mathbb{R}^3$. From the Hölder inequality, we get

$$\begin{aligned}\int_{\mathbb{R}^3}\left[|\nabla\omega_n|^2+V_o(x)\omega_n^2\right]\mathrm{d}x&\leq 2\alpha_o\int_{\mathbb{R}^3}|u|^4w_n^2\,\mathrm{d}x+2\beta_o\int_{\mathbb{R}^3}w_n^2\,\mathrm{d}x\\&\leq 2\alpha_o||u||_6^4||w_n||_6^2+2\beta_o\int_{\mathbb{R}^3}\omega_n^2\,\mathrm{d}x.\end{aligned}$$

Combining this last inequality with the Sobolev inequality bellow

$$S||w_n||_6^2\leq\int_{\mathbb{R}^3}|\nabla\omega_n|^2\,\mathrm{d}x\leq\int_{\mathbb{R}^3}\left[|\nabla\omega_n|^2+V_o(x)\omega_n^2\right]\,\mathrm{d}x,$$

under hypothesis $(\mathcal{C})$, we have

$$\left[\int_{A_n}|\omega_n|^6\,\mathrm{d}x\right]^{\frac{2}{6}}\leq\left[\int_{\mathbb{R}^3}|\omega_n|^6\,\mathrm{d}x\right]^{\frac{2}{6}}\leq 4\beta_oS^{-1}\int_{\mathbb{R}^3}\omega_n^2\,\mathrm{d}x,$$

which together with the fact that $|\omega_n|\leq|u|^3$ in $\mathbb{R}^3$ and $|\omega_n|=|v|^3$ in A_n implies

$$\left[\int_{A_n}|u|^{18}\,\mathrm{d}x\right]^{\frac{1}{18}}\leq(4\beta_oS^{-1})^{\frac{1}{6}}\left[\int_{\mathbb{R}^3}|u|^6\,\mathrm{d}x\right]^{\frac{1}{6}}. \tag{4.7}$$

Passing to the liminf in (4.7) and using Fatou's lemma we obtain

$$||u||_{18}\leq(4\beta_oS^{-1})^{\frac{1}{6}}||u||_6. \tag{4.8}$$

Thus $u\in L^{18}(\mathbb{R}^3)\cap L^6(\mathbb{R}^3)$, which implies that $h=\alpha_o|u|^5+\beta_o|u|\in L^{\frac{18}{5}}_{loc}(\mathbb{R}^3)$. Moreover, from (4.8) and condition $(\mathcal{C})$, we obtain

$$\begin{aligned}[h]_{\frac{18}{5}}\leq\alpha_o||u||_{18}^5+C\beta_o||u||_{18}&\leq\alpha_o(4\beta_oS^{-1})^{\frac{5}{6}}||u||_6^5+C\beta_o(4\beta_oS^{-1})^{\frac{1}{6}}||u||_6\\&\leq\left(\alpha_o(4\beta_oS^{-1})^{\frac{5}{6}}||u||_6^4+C\beta_o(4\beta_oS^{-1})^{\frac{1}{6}}\right)||u||_6\\&\leq C\left(\beta_o^{\frac{5}{6}}+\beta_o^{\frac{7}{6}}\right)||u||_6.\end{aligned}$$

From Lemma 4.2 there exists a positive constant Λ which depends only on β_o such that

$$||u||_\infty\leq\Lambda||u||_6^2$$

and the proof is completed. $\square$

4.1. Proof of Theorem 4.1 completed

The choice of d_γ in (2.2) together (3.13) show that a solution $u = u_R$ above satisfies

$$-\Delta u + \frac{V_\lambda(x)}{M(||\nabla u||_2^2)} u = \frac{1}{M(||\nabla u||_2^2)} u^5 + \frac{\gamma}{M(||\nabla u||_2^2)} |u|^{p-1} u$$

and

$$\frac{8}{M(||\nabla u||_2^2)} ||u||_6^4 S^{-1} \leq \frac{8}{M_o} ||u||_6^4 S^{-1} \leq 1.$$

We will use Proposition 4.3 with $\alpha_o = 2M_o^{-1}$. Since

$$|g(x,s)| \leq M_o^{-1}(|s|^5 + \gamma |s|^p) \leq 2M_o^{-1}|s|^5 + M_o^{-1}\gamma^{\frac{4}{5-p}}|s|,$$

from Proposition 4.3 with $\beta_o = M_o^{-1}\gamma^{\frac{4}{5-p}}$ we have:

$$||u||_\infty \leq C\Lambda ||u||_6^2 \leq C\gamma^{\frac{7}{6}}\gamma^{\frac{-2}{p-1}}.$$

Now we have a family of solutions $u = u_R$ of the auxiliary problems $(\mathcal{AP})$ in L^∞ and

$$||u||_\infty \leq C\gamma^{\frac{7p-19}{6(p-1)}}. \tag{4.9}$$

where C is a positive constant.

5. Proof of Theorem 1.1

We need to show that a solution $u \in E$ of the auxiliary problem satisfies

$$f(u) \leq \frac{V_\lambda(x)}{p+1} u \quad \text{in } |x| \geq R. \tag{5.1}$$

Lemma 5.1. *For any ground state solution of Problem* $(\mathcal{AP})$ *it holds*

$$u(x) \leq \frac{R||u||_\infty}{|x|}, \quad \textit{for all} \quad |x| \geq R. \tag{5.2}$$

Proof. We notice that the function

$$v(x) := \frac{R||u||_\infty}{|x|} \quad \text{for} \quad x \in \mathbb{R}^3 \setminus \{0\}$$

is harmonic and $u(x) \leq v(x)$ if $|x| = R$. Let us consider the following test function

$$\omega(x) := \begin{cases} \max\{u - v, 0\}, & \text{if} \quad |x| \geq R; \\ 0, & \text{if} \quad |x| \leq R. \end{cases}$$

It is easy to see that $\omega \geq 0$ and $\omega \in \mathcal{D}^{1,2}(\mathbb{R}^3)$. Since u is ground state solution of Problem ($\mathcal{AP}$) we can see that

$$\begin{aligned}\int_{\mathbb{R}^3} |\nabla \omega|^2 \,\mathrm{d}x &= \int_{\mathbb{R}^3} \nabla(u-v)\nabla\omega \,\mathrm{d}x \\ &= \frac{1}{M(\|u\|_2^2)} \int_{|x|\geq R} (g(x,u)\omega - V_\lambda(x)u\omega)\,\mathrm{d}x\end{aligned}$$

Thus, using (g_3), we obtain

$$\int_{\mathbb{R}^3} |\nabla\omega|^2 \,\mathrm{d}x \leq \left(\frac{1}{p+1} - 1\right)\frac{1}{M(\|u\|_2^2)}\int_{\mathbb{R}^3} V(x)u\omega \,\mathrm{d}x \leq 0,$$

which implies that $\omega \equiv 0$, and consequently we have

$$\omega(x) \leq v(x) = \frac{R||u||_\infty}{|x|} \quad \text{for all} \quad |x| \geq R$$

which is the desired conclusion. □

Lemma 5.2. *There exists $C_o > 0$ such that for any ground state solution of Problem ($\mathcal{AP}$) it holds*

$$\frac{f(u)}{u} \leq C_o\left(\frac{R}{|x|}\right)^{p-1}\gamma^{\frac{7p-13}{6}}, \quad \textit{for all} \quad |x| \geq R. \tag{5.3}$$

Proof. From Lemma 5.1 we have

$$\frac{f(u)}{u} = u^4 + \gamma|u|^{p-1} \leq \frac{R^4||u||_\infty^4}{|x|^4} + \gamma\frac{R^{p-1}||u||_\infty^{p-1}}{|x|^{p-1}}$$

which together with (4.9) gives

$$\begin{aligned}\frac{f(u)}{u} &\leq \frac{R^4C^4\gamma^{\frac{2(7p-19)}{3(p-1)}}}{|x|^4} + \gamma\frac{R^{p-1}C^{p-1}\gamma^{\frac{(p-1)(7p-19)}{6(p-1)}}}{|x|^{p-1}} \\ &\leq \frac{R^4C^4\gamma^{\frac{2(7p-19)}{3(p-1)}}}{|x|^4} + \frac{R^{p-1}C^{p-1}\gamma^{\frac{7p-13}{6}}}{|x|^{p-1}} \\ &\leq \left[\frac{R^4C^4}{|x|^4} + \frac{R^{p-1}C^{p-1}}{|x|^{p-1}}\right]\gamma^{\frac{7p-13}{6}} \\ &= \frac{R^{p-1}}{|x|^{p-1}}\left[C^{p-1} + C^4\frac{R^{5-p}}{|x|^{5-p}}\right]\gamma^{\frac{7p-13}{6}} \\ &\leq C_o\left(\frac{R}{|x|}\right)^{p-1}\gamma^{\frac{7p-13}{6}},\end{aligned}$$

where $C_o = (C^{p-1} + C^4)$ and we have used $|x| \geq R$ and $\gamma \geq 1$. □

5.1. Proof of Theorem 1.1 completed

From condition (V_3) there exists $R_1 > 0$ and $c_1 > 0$ such that

$$|x|^{p-1}V(x) \geq c_1 \quad \text{for all} \quad |x| \geq R_1. \tag{5.4}$$

On the other hand, since $V_\lambda(x) \geq \lambda V(x)$, using (5.3) and taking $R > R_1$ we can see that

$$\frac{f(u)}{u} \leq \frac{V_\lambda(x)}{p+1} \quad \text{for all} \quad |x| \geq R,$$

provided that

$$\lambda \geq \frac{c_o}{c_1}(p+1)\gamma^{\frac{7p-13}{6}}$$

and consequently u solution of auxiliary Problem ($\mathcal{AP}$) is indeed solution of original Problem ($\mathcal{P}_{\lambda,\gamma}$).

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