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ON THE PARTIAL CONTROLLABILITY OF SDES AND THE EXACT CONTROLLABILITY OF FBSDES*

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Abstract. A notion of partial controllability (also can be called directional controllability or output controllability) is proposed for linear controlled (forward) stochastic differential equations (SDEs), which characterizes the ability of the state to reach some given random hyperplane. It generalizes the classical notion of exact controllability. For time-invariant system, checkable rank conditions ensuring SDEs' partial controllability are provided. With some special setting, the partial controllability for SDEs is proved to be equivalent to the exact controllability for linear controlled forward-backward stochastic differential equations (FBSDEs). Moreover, we obtain some equivalent conclusions to partial controllability for SDEs or exact controllability for FBSDEs, including the validity of observability inequalities for the adjoint equations, the solvability of some optimal control problems, the solvability of norm optimal control problems, and the non-singularity of a random version of Gramian matrix.

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1. Introduction

In this paper, we consider the following controlled linear stochastic system:

$$\mathrm{d}X(t) = \Big[A_0(t)X(t) + B_0(t)v(t)\Big]\mathrm{d}t + \sum_{j=1}^{d}\Big[A_j(t)X(t) + B_j(t)v(t)\Big]\mathrm{d}W_j(t), \quad t\in[0,T], \tag{1.1}$$

where $W(\cdot)\equiv(W_1(\cdot),W_2(\cdot),\ldots,W_d(\cdot))^\top$ is a d-dimensional Brownian motion and $A_j(\cdot)$, $B_j(\cdot)$ $(j=0,1,\ldots,d)$ are given matrix-valued stochastic processes (which will be defined precisely in the next section). In the above, $X(\cdot)$ is the *state process* valued in $\mathbb{R}^n$ and $v(\cdot)$ is the *control process* valued in $\mathbb{R}^k$.

System (1.1) is called *exactly controllable* on $[0,T]$ if for any initial state $X_0\in\mathbb{R}^n$ and any terminal state $\xi\in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$ (the definition is given in the next section), there exists a control process $v(\cdot)$ such that

$$X(T;X_0,v(\cdot)) = \xi. \tag{1.2}$$

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The notion of exact controllability characterizes the ability of the system to attain accurately any given terminal point from an arbitrary initial point. *Controllability* is one of the most important concepts in the deterministic/stochastic control theory, and there exist extensive works on this topic. Among them, for a comprehensive survey of the controllability theory on deterministic systems, one can refer to [11] for ordinary differential equation (ODE, for short) systems and [5] for partial differential equation (PDE, for short) systems. For stochastic systems, there mainly exist two different definitions of controllability. One is driving the state to the target exactly (*exact controllability*, see [1, 12, 13, 15, 18, 19, 24], and so on) or to a neighborhood of the target under an appropriate norm (*approximate controllability*, see [4, 6, 10, 16]), which is similar with that in deterministic systems. The other one is driving the state to the target with a positive probability (*stochastic controllability*, see [2, 7, 20, 26], and the reference therein).

According to the notion of exact controllability, one can classify all the linear SDE systems (1.1) into two classes: one is exactly controllable and the other is not. However, the exact controllability is dichotomic, and some weaker concepts should be introduced to get some detailed classification. Inspired by this point, in this paper, we introduce a notion called H-partial controllability. In detail, let $H \in L^\infty_{\mathcal{F}_T}(\Omega;\mathbb{R}^{l\times n})$ be a given $\mathcal{F}_T$-measurable matrix-valued bounded random variable. System (1.1) is said to be *H-partially controllable* on $[0,T]$ if for any $X_0 \in \mathbb{R}^n$ and any $\xi \in L^2_{\mathcal{F}_T}(\Omega;\mathcal{R}(H))$, where $\mathcal{R}(H)$ denote the range of H, there exists a control process $v(\cdot)$ such that

$$HX(T;X_0,v(\cdot)) = \xi. \tag{1.3}$$

If $\xi = 0$, system (1.1) is called *H-partially null-controllable* on $[0,T]$. Clearly, by selecting H to be the identity matrix I, one can find that the I-partial controllability coincides with the classical notion of exact controllability. Therefore, the H-partial controllability can be regarded as a natural generalization of exact controllability. Moreover, the notion of H-partial null-controllability characterizes the ability of system (1.1) to reach a random subspace of $\mathbb{R}^n$ (*i.e.*, $\mathcal{N}(H)$, the kernel of H) from an arbitrary initial state. Similarly, the notion of H-partial controllability characterizes the ability to reach any given random hyperplane which parallel $\mathcal{N}(H)$. In other words, instead of exact controllability, the H-partial controllability provides a more detailed classification for all linear SDE systems (1.1). Partial controllability is also proposed in [3], which generalizes the notion of *stochastic controllability*, and is different from our definition.

Inspired by the duality relation between the notions of controllability and *observability* (which is equivalent to an inequality named *observability inequality*), and the powerful effectiveness of observability in studying controllability for ODE, PDE and SDE systems, we introduce a new observability inequality for the adjoint equation, the validity of which is proved to be equivalent to the H-partial controllability of system (1.1). This provides an approach to study the H-partial controllability by establishing an inequality for backward stochastic differential equation (BSDE, for short) which, at least conceptually, is a simpler problem. Moreover, we introduce a family of optimal control problems for the adjoint equation (see Problem (O) in the next section), and the solvability of these optimal control problems is also proved to be equivalent to the H-partial controllability of system (1.1). In other words, we provide a second alternative method to study the H-partial controllability through optimal control theory. Furthermore, we introduce another family of norm optimal control problems for the original system (1.1) (see Problem (N) in the next section), and we prove the equivalence between the H-partial controllability and the solvability of this family of norm optimal control problems.

The norm optimal control problem for ODE and PDE systems has been widely investigated (see, for example, [21, 22]), while there exist a few works on the SDE systems (see [9, 23, 24]). Notice that in [9, 24] the norm optimal control problems were studied under the assumption that the system is exactly controllable (see (1.2)). However, in the present paper, we study the problem under the constraint that the system is H-partially controllable (see (1.3)).

Denote $X(\cdot)^\top = (x(\cdot)^\top, y(\cdot)^\top)$, where $x(\cdot)$ and $y(\cdot)$ take values in $\mathbb{R}^{n'}$ and $\mathbb{R}^{m'}$ respectively, with $n'+m'=n$. Denote $v(\cdot)^\top = (u(\cdot)^\top, z_1(\cdot)^\top, \dots, z_d(\cdot)^\top)$, where $u(\cdot)$ and $z_j(\cdot)$ $(j=1,2,\dots,d)$ take values in $\mathbb{R}^{k'}$ and $\mathbb{R}^{m'}$ respectively, with $k = k' + m'd$. Moreover, let $H = (-M, I_{m'})$ and the system (1.1) satisfy a special setting (see Setting (FB) in Section 4 for detail) or it can be transformed into this special setting (see Appendix for detail).

Then the H-partial controllability and H-partial null-controllability of system (1.1) are equivalent, and both of them are also equivalent to the exact controllability of the following system:

$$\begin{cases} \mathrm{d}x(t) = \Big[C_0(t)x(t) + D_0(t)u(t)\Big]\mathrm{d}t + \displaystyle\sum_{j=1}^{d}\Big[C_j(t)x(t) + D_j(t)u(t)\Big]\mathrm{d}W_j(t), \quad t \in [0,T], \\ \mathrm{d}y(t) = \Big[F(t)x(t) + G(t)u(t) + R(t)y(t) + \displaystyle\sum_{j=1}^{d} S_j(t)z_j(t)\Big]\mathrm{d}t + \displaystyle\sum_{j=1}^{d} z_j(t)\mathrm{d}W_j(t), \quad t \in [0,T], \\ y(T) = Mx(T), \end{cases} \tag{1.4}$$

where $C_0(\cdot)$, $D_0(\cdot)$, $C_j(\cdot)$, $D_j(\cdot)$, $F(\cdot)$, $G(\cdot)$, $R(\cdot)$, $S_j(\cdot)$ $(j = 1, 2, \ldots d)$ are given matrix-valued stochastic processes. With any given initial value x_0 of $x(\cdot)$ and $u(\cdot)$, (1.4) becomes a forward-backward stochastic differential equation (FBSDE, for short), which admits a unique solution $(x(\cdot), y(\cdot), z(\cdot))$ where we denote $z(\cdot) = (z_1(\cdot), z_2(\cdot), \ldots, z_d(\cdot))$. In the viewpoint of FBSDEs, $(x(\cdot), y(\cdot), z(\cdot))$ is the *state process* and $u(\cdot)$ is the *control process*. Similarly, system (1.4) is called *exactly controllable* on $[0,T]$ if for any $x_0 \in \mathbb{R}^{n'}$ and any $y_0 \in \mathbb{R}^{m'}$, there exists a control process $u(\cdot)$ such that

$$y(0; x_0, u(\cdot)) = y_0. \tag{1.5}$$

The equivalence between the H-partial controllability of SDE system (1.1) and the exact controllability of FBSDE system (1.4) provides a nice motivation to study the controllability theory for FBSDE systems. For (1.4) with any given $x(0) = x_0$ and $u(\cdot)$, we can firstly solve $x(\cdot)$ from the forward equation, and then solve $(y(\cdot), z(\cdot))$ from the backward equation. Since FBSDE systems are well-defined dynamic systems, besides the relation with H-partial controllability of SDE system (1.1), it is also natural and appealing to study the exact controllability of FBSDE system (1.4) in its own right. For more details on the theory of BSDEs and FBSDEs, one can refer to [8, 14] and the reference therein.

It is worth noting that, although they are equivalent, there are two different viewpoints between the H-partial controllability of (1.1) and the exact controllability of (1.4). For the former one, both $u(\cdot)$ and $z(\cdot)$ are regarded as control processes (noticing that $v(\cdot)^\top = (u(\cdot)^\top, z_1(\cdot)^\top, \ldots, z_d(\cdot)^\top)$), and for the latter one, only $u(\cdot)$ is regarded as a control process, while $z(\cdot)$ is regarded as a state process (noticing that $(x(\cdot), y(\cdot), z(\cdot))$ is the solution to the FBSDE). There are some subtle differences in these two viewpoints. Keeping the last viewpoint in mind, we develop some different and deeper conclusions. In detail, we obtain the equivalence among the exact controllability of the FBSDE system (1.4), the validity of a new observability inequality for the adjoint equation (which is also a FBSDE system), the existence of minimizers to a family of functions (see Problem (O$'$) in Sect. 4), the solvability of a new family of norm optimal control problems (see Problem (N$'$) in Sect. 4), and the non-singularity of a random version of Gramian matrix. We point out the last equivalent condition is not obtained for the H-partial controllability of system (1.1). With the help of Gramian matrix and some other techniques, for the one dimensional case with deterministic time-invariant coefficients, we give a clear characterization for the exact controllability of (1.4) (see Prop. 4.7 in Sect. 4.1).

The rest of this paper is organized as follows. In Section 2, we introduce some notations, and then study the H-partial controllability for SDE system (1.1). In Section 3, we provide checkable necessary condition and equivalent Kalman type rank condition ensuring time-invariant system's partial controllability. In Section 4 we devote ourselves to investigating the exact controllability for FBSDE system (1.4).

2. Notations and partial controllability of SDEs

Recall that $\mathbb{R}^n$ is the n-dimensional Euclidean space with Euclidean scalar product $\langle\cdot,\ \cdot\rangle$ and the induced Euclidean norm $|\cdot|$. Let $\mathbb{R}^{m\times n}$ be the collection of $(m \times n)$ matrices which is an $(m \times n)$-dimensional Euclidean space obviously. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space on which a d-dimensional standard Brownian motion $W(\cdot) \equiv (W_1(\cdot), W_2(\cdot), \ldots, W_d(\cdot))^\top$ is defined, where the superscript $\top$ denotes the transpose of a vector or a matrix, such that $\mathbb{F} \equiv \{\mathcal{F}_t\}_{t\geq 0}$ is its natural filtration augmented by all the $\mathbb{P}$-null sets. Let $T > 0$ be a fixed time horizon. Now we introduce some spaces.

- $L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$, the set of $\mathcal{F}_T$-measurable random variables $\xi : \Omega \to \mathbb{R}^n$ such that $\|\xi\|_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)} \equiv \{\mathbb{E}[|\xi|^2]\}^{\frac{1}{2}} < \infty$;
- $L^\infty_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$, the subset of $L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$ where each element ξ is essentially bounded;
- $L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$, the set of $\mathbb{F}$-progressively measurable stochastic processes $x : \Omega \times [0,T] \to \mathbb{R}^n$ such that $\|x(\cdot)\|_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)} \equiv \{\mathbb{E}\int_0^T |x(t)|^2\mathrm{d}t\}^{\frac{1}{2}} < \infty$;
- $L^\infty_{\mathbb{F}}(0,T;\mathbb{R}^n)$, the subset of $L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$ where each element $x(\cdot)$ is essentially bounded;
- $S^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$, the subset of $L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$, in which each element $x(\cdot)$ has continuous paths and satisfies $\|x(\cdot)\|_{S^2_{\mathbb{F}}(0,T;\mathbb{R}^n)} = \{\mathbb{E}[\sup_{t\in[0,T]} |x(t)|^2]\}^{\frac{1}{2}} < \infty$.

Let us begin with the following controlled linear stochastic system (which is (1.1), for convenience, we rewrite it here):

$$\mathrm{d}X(t) = \Big[A_0(t)X(t) + B_0(t)v(t)\Big]\mathrm{d}t + \sum_{j=1}^{d}\Big[A_j(t)X(t) + B_j(t)v(t)\Big]\mathrm{d}W_j(t), \quad t \in [0,T], \tag{2.1}$$

where $A_j(\cdot) \in L^\infty_{\mathbb{F}}(0,T;\mathbb{R}^{n\times n})$ and $B_j(\cdot) \in L^\infty_{\mathbb{F}}(0,T;\mathbb{R}^{n\times k})$ for $j = 0,1,2,\dots,d$. Obviously, with any given initial value $X(0) = X_0 \in \mathbb{R}^n$ and any given $v(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)$, (2.1) becomes a (forward) SDE. By the classical result, SDE (2.1) admits a unique solution $X(\cdot) \equiv X(\cdot;X_0,v(\cdot)) \in S^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$.

For the system (2.1), we will propose a concept which can be regarded as a generalization of the traditional notion of exact controllability (see [18] for example). For this aim, we introduce a matrix-valued random variable $H \in L^\infty_{\mathcal{F}_T}(\Omega;\mathbb{R}^{l\times n})$. Define

$$\mathcal{X} = \Big\{\xi \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^l) \;\Big|\; \xi(\omega) \in \mathcal{R}(H(\omega)),\ \mathbb{P}\text{-a.s. } \omega \in \Omega\Big\},$$

where $\mathcal{R}(H(\omega))$ denotes the range of $H(\omega)$. It is easy to know $\mathcal{X}$ is a Hilbert space. Particularly, when the rank of H equals to l $\mathbb{P}$-a.s., we have $\mathcal{X} = L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^l)$.

Definition 2.1. Let a matrix $H \in L^\infty_{\mathcal{F}_T}(\Omega,\mathbb{R}^{l\times n})$ be given.
(i). System (2.1) is called H-partially controllable on the time interval $[0,T]$, if for any $(X_0,\xi) \in \mathbb{R}^n \times \mathcal{X}$, there exists a $v(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)$ such that the solution $X(\cdot;X_0,v(\cdot))$ to the SDE (2.1) with the initial condition $X(0) = X_0$ satisfies $HX(T) = \xi$ $\mathbb{P}$-a.s.
(ii). System (2.1) is called H-partially null-controllable on the time interval $[0,T]$, if for any $X_0 \in \mathbb{R}^n$, there exists a $v(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)$ such that the solution $X(\cdot;X_0,v(\cdot))$ to the SDE (2.1) with the initial condition $X(0) = X_0$ satisfies $HX(T) = 0$ $\mathbb{P}$-a.s.

Remark 2.2. 1. Clearly, when $H = I$ which is the $(n \times n)$ identity matrix, the corresponding I-partial controllability coincides with the traditional notion of *exact controllability* [18, 23].
2. The notion of H-partial null-controllability characterizes the ability of the terminal state to reach a random subspace of $\mathbb{R}^n$ (*i.e.*, $\mathcal{N}(H)$, the kernel of H) from an arbitrary initial state. Moreover, the notion of H-partial controllability characterizes the ability to reach all given random hyperplanes which parallel $\mathcal{N}(H)$.

As we know that for linear ODE systems and SDE systems [24], the exact controllability of the original systems is equivalent to the observability of the adjoint equations. We would like to see how such a result will look like for our generalized H-partial controllability. For this aim, we introduce the adjoint equation of (2.1) as follows:

$$\begin{cases} \mathrm{d}Y(t) = -\Big[A_0(t)^\top Y(t) + \sum_{j=1}^{d} A_j(t)^\top Z_j(t)\Big]\mathrm{d}t + \sum_{j=1}^{d} Z_j(t)\mathrm{d}W_j(t), \quad t \in [0,T], \\ Y(T) = H^\top \eta. \end{cases} \tag{2.2}$$

By the classical result on BSDEs, for any $\eta \in \mathcal{X}$, (2.2) admits a unique solution $(Y(\cdot), Z(\cdot)) \in S^2_{\mathbb{F}}(0,T;\mathbb{R}^n) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^{n\times d})$ where we denote $Z(\cdot) = (Z_1(\cdot), Z_2(\cdot), \ldots, Z_d(\cdot))$ for simplicity. By applying Itô's formula to $\langle X(\cdot),\ Y(\cdot)\rangle$ on the interval $[0,T]$, we have the following duality:

$$\mathbb{E}\int_0^T \Big\langle v(t),\ B_0(t)^\top Y(t) + \sum_{j=1}^{d} B_j(t)^\top Z_j(t)\Big\rangle \mathrm{d}t + \langle X_0,\ Y(0)\rangle - \mathbb{E}\langle HX(T),\ \eta\rangle = 0. \tag{2.3}$$

For any $(X_0, v(\cdot)) \in \mathbb{R}^n \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)$, let $X(\cdot) \equiv X(\cdot; X_0, v(\cdot))$ be the unique solution to (2.1). Due to the linearity of (2.1), we have

$$X(\cdot; X_0, v(\cdot)) = X(\cdot; X_0, 0) + X(\cdot; 0, v(\cdot)).$$

Define

$$\begin{cases} \mathbb{L}_0 X_0 = X(T; X_0, 0), \quad \forall\ X_0 \in \mathbb{R}^n, \\ \mathbb{L}v(\cdot) = X(T; 0, v(\cdot)), \quad \forall\ v(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^k). \end{cases} \tag{2.4}$$

By the linearity and stability result of (2.1) (see [8]), both $\mathbb{L}_0 : \mathbb{R}^n \to L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$ and $\mathbb{L} : L^2_{\mathbb{F}}(0,T;\mathbb{R}^k) \to L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$ are bounded linear operators. Denote $\mathbb{L}_0^* : L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n) \to \mathbb{R}^n$ and $\mathbb{L}^* : L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n) \to L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)$, the adjoint operators of $\mathbb{L}_0$ and $\mathbb{L}$, respectively. From the duality relation (2.3), we have

$$\langle X_0,\ Y(0)\rangle = \mathbb{E}\langle HX(T; X_0, 0),\ \eta\rangle = \mathbb{E}\langle H\mathbb{L}_0 X_0,\ \eta\rangle = \langle X_0, \mathbb{L}_0^*(H^\top \eta)\rangle, \tag{2.5}$$

and

$$\begin{aligned} &\mathbb{E}\int_0^T \Big\langle v(t),\ B_0(t)^\top Y(t) + \sum_{j=1}^{d} B_j(t)^\top Z_j(t)\Big\rangle \mathrm{d}t \\ =\ & \mathbb{E}\langle HX(T; 0, v(\cdot)),\ \eta\rangle = \mathbb{E}\langle \mathbb{L}v(\cdot),\ H^\top \eta\rangle = \langle v(\cdot),\ \mathbb{L}^*(H^\top \eta)\rangle_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)}. \end{aligned} \tag{2.6}$$

Hence

$$\begin{cases} (\mathbb{L}_0^* \circ H^\top)\eta = \mathbb{L}_0^*(H^\top \eta) = Y(0), \quad \eta \in \mathcal{X}, \\ (\mathbb{L}^* \circ H^\top)\eta = \mathbb{L}^*(H^\top \eta) = B_0(\cdot)^\top Y(\cdot) + \sum_{j=1}^{d} B_j(\cdot)^\top Z_j(\cdot), \quad \eta \in \mathcal{X}, \end{cases} \tag{2.7}$$

where $(Y(\cdot), Z(\cdot))$ is the unique solution to BSDE (2.2) with the terminal value $Y(T) = H^\top \eta$.

Definition 2.3. (i) The composition operator $\mathbb{L}^* \circ H^\top$ given by (2.7) is called an observer of system (2.2) on the time interval $[0,T]$.
(ii). System (2.2) is said to be exactly observable on the time interval $[0,T]$ if from the observation $(\mathbb{L}^* \circ H^\top)\eta$, the random variable $\eta \in \mathcal{X}$ can be uniquely determined, *i.e.*, the composition operator $\mathbb{L}^* \circ H^\top : \mathcal{X} \to L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)$ is injective.

Remark 2.4. In the viewpoint of Banach inverse operator theorem, system (2.2) is exactly observable if and only if, there exists a constant $\delta > 0$ such that

$$\|(\mathbb{L}^* \circ H^\top)\eta\|_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)} \geq \delta \|\eta\|_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^l)}, \quad \forall\, \eta \in \mathcal{X}, \tag{2.8}$$

where $\mathbb{L}^* \circ H^\top$ is defined by (2.7). Equivalently,

$$\|B_0(\cdot)^\top Y(\cdot) + \sum_{j=1}^{d} B_j(\cdot)^\top Z_j(\cdot)\|_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)} \geq \delta \|\eta\|_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^l)}, \quad \forall\, \eta \in \mathcal{X}, \tag{2.9}$$

where $(Y(\cdot), Z(\cdot))$ is the unique solution to (2.2). The above inequality is called the observability inequality for equation (2.2).

Now, we introduce a family of functionals which is associated with the adjoint equation (2.2). In detail, for any $(X_0, \xi) \in \mathbb{R}^n \times \mathcal{X}$, we define $J(\cdot; X_0, \xi) : \mathcal{X} \to \mathbb{R}$ as follows:

$$\begin{aligned} J(\eta; X_0, \xi) &= \frac{1}{2}\big\|(\mathbb{L}^* \circ H^\top)\eta\big\|^2_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)} + \big\langle X_0,\ (\mathbb{L}^*_0 \circ H^\top)\eta\big\rangle - \mathbb{E}\langle \xi,\ \eta\rangle \\ &\equiv \frac{1}{2}\mathbb{E}\int_0^T \Big|B_0(t)^\top Y(t) + \sum_{j=1}^{d} B_j(t)^\top Z_j(t)\Big|^2 \mathrm{d}t + \langle X_0,\ Y(0)\rangle - \mathbb{E}\langle \xi,\ \eta\rangle, \quad \eta \in \mathcal{X}, \end{aligned} \tag{2.10}$$

where $\mathbb{L}^* \circ H^\top$ and $\mathbb{L}^*_0 \circ H^\top$ are given by (2.7), and $(Y(\cdot), Z(\cdot))$ is the unique solution to BSDE (2.2). We pose an optimal control problem as follows.

Problem (O). For any $(X_0, \xi) \in \mathbb{R}^n \times \mathcal{X}$, find a $\tilde{\eta} \in \mathcal{X}$ such that

$$J(\tilde{\eta}; X_0, \xi) = \inf_{\eta \in \mathcal{X}} J(\eta; X_0, \xi).$$

We also introduce a family of norm optimal control problems which is associated with the original system (2.1). For any $(X_0, \xi) \in \mathbb{R}^n \times \mathcal{X}$, we define an admissible control set

$$\mathcal{V}(X_0, \xi) \equiv \Big\{ v(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^k) \ \Big|\ HX(T; X_0, v(\cdot)) = \xi \Big\}.$$

Problem (N). For any $(X_0, \xi) \in \mathbb{R}^n \times \mathcal{X}$, find a $\tilde{v}(\cdot) \in \mathcal{V}(X_0, \xi)$ such that

$$\|\tilde{v}(\cdot)\|_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)} = \inf_{v(\cdot) \in \mathcal{V}(X_0,\xi)} \|v(\cdot)\|_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)}.$$

We are in the position to give the main result of this section.

Theorem 2.5. *For any given $H \in L^\infty_{\mathcal{F}_T}(\Omega; \mathbb{R}^{l\times n})$, the following statements are equivalent:*
(i). *System (2.1) is H-partially controllable;*
(ii). *The observability inequality (2.8) for the adjoint equation (2.2) holds true;*
(iii). *Problem (O) admits a unique solution $\tilde{\eta} \in \mathcal{X}$;*
(iv). *Problem (N) admits a unique solution $\tilde{v}(\cdot) \in \mathcal{V}(X_0, \xi)$.*
Moreover, the unique norm optimal control $\tilde{v}(\cdot)$ to Problem (N) is given by

$$\tilde{v}(\cdot) = (\mathbb{L}^* \circ H^\top)\tilde{\eta} \equiv B_0(\cdot)\tilde{Y}(\cdot) + \sum_{j=1}^{d} B_j(\cdot)^\top \tilde{Z}_j(\cdot), \tag{2.11}$$

where $\tilde{\eta} \in \mathcal{X}$ is the unique optimal solution to Problem (O), $\mathbb{L}^* \circ H^\top$ is given by (2.7), and $(\tilde{Y}(\cdot), \tilde{Z}(\cdot))$ is the solution to BSDE (2.2) with the terminal value $\tilde{Y}(T) = H^\top \tilde{\eta}$. Furthermore, the minimal norm to Problem (N) *is given by*

$$\|\tilde{v}(\cdot)\|_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)} = \sqrt{\mathbb{E}\langle \xi,\ \tilde{\eta}\rangle - \langle X_0,\ (\mathbb{L}_0^* \circ H^\top)\tilde{\eta}\rangle} \equiv \sqrt{\mathbb{E}\langle \xi,\ \tilde{\eta}\rangle - \langle X_0,\ \tilde{Y}(0)\rangle}. \tag{2.12}$$

The minimal value of functional $J(\cdot; X_0, \xi)$ is given by

$$J(\tilde{\eta}; X_0, \xi) = -\frac{1}{2}\|\tilde{v}(\cdot)\|^2_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)} \equiv -\frac{1}{2}\big(\mathbb{E}\langle \xi,\ \tilde{\eta}\rangle - \langle X_0,\ \tilde{Y}(0)\rangle\big). \tag{2.13}$$

In [24], the exact controllability for SDEs (*i.e.*, the special case: $H = I$) was studied. The above Theorem 2.5 can be regarded as a generalization of some corresponding results in [24]. Moreover, in what follows, we shall give a different proof to the one appearing in [24]. Since the proof is long, we split it into several parts.

Lemma 2.6. *If the system (2.1) is H-partially controllable, then the observability inequality (2.8) for the system (2.2) holds true.*

Proof. Setting $X(0) = 0$ in (2.1), we recall the bounded linear operator $H \circ \mathbb{L} : L^2_{\mathbb{F}}(0, T; \mathbb{R}^k) \to \mathcal{X}$ where $\mathbb{L}$ is defined by (2.4). Due to the H-partial controllability, $H \circ \mathbb{L}$ is also an onto map. By the classical Open Mapping Theorem, we get that $H \circ \mathbb{L}$ is an open mapping. Then for the unit open ball $B_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)}(0, 1)$, there exists a constant $\delta > 0$ such that $B_{\mathcal{X}}(0, 2\delta) \subset (H \circ \mathbb{L})(B_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)}(0, 1))$.

For any $\eta \in \mathcal{X}$, if $\eta = 0$, then the observability inequality (2.8) holds true obviously. Next we assume $\eta \neq 0$. Let

$$\eta' = \frac{\delta}{\|\eta\|_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^l)}}\eta \in B_{\mathcal{X}}(0, 2\delta).$$

Then there exists a $v'(\cdot) \in B_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)}(0, 1)$ such that $(H \circ \mathbb{L})v'(\cdot) = \eta'$. Let

$$v(\cdot) = \frac{\|\eta\|_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^l)}}{\delta}v'(\cdot).$$

By the linearity of $H \circ \mathbb{L}$, we have $(H \circ \mathbb{L})v(\cdot) = \eta$. Then, with the help of Hölder's inequality and the definition of $v(\cdot)$, we derive

$$\begin{aligned}
\|\eta\|^2_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^l)} &= \langle (H \circ \mathbb{L})v(\cdot),\ \eta\rangle_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^l)} = \langle v(\cdot),\ (\mathbb{L}^* \circ H^\top)\eta\rangle_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)} \\
&\leq \|v(\cdot)\|_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)}\|(\mathbb{L}^* \circ H^\top)\eta\|_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)} \\
&\leq \frac{\|\eta\|_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^l)}}{\delta}\|(\mathbb{L}^* \circ H^\top)\eta\|_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)},
\end{aligned}$$

where $\mathbb{L}^*$ is the adjoint operator of $\mathbb{L}$. Subsequently, we obtain that the observability inequality (2.8) holds true. □

Lemma 2.7. *For any $(X_0, \xi) \in \mathbb{R}^n \times \mathcal{X}$, the functional $J(\cdot; X_0, \xi)$ defined by (2.10) is weak lower semi-continuous.*

Proof. Let $\{\eta^i\}_{i=1}^\infty$ be a sequence converging to η weakly in $\mathcal{X}$. Let $(Y^i(\cdot), Z^i(\cdot))$ $(i = 1, 2, \dots)$ and $(Y(\cdot), Z(\cdot))$ denote the solutions to the system (2.2) with terminal values $H^\top\eta^i$ and $H^\top\eta$ respectively. For any $X_0 \in \mathbb{R}^n$, and any $v(\cdot) \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^k)$, let $X(\cdot)$ be the solution to SDE (2.1). From the duality relation (2.3), we have

$$\mathbb{E}\langle HX(T),\ \eta^i\rangle = \langle X_0,\ (\mathbb{L}_0^* \circ H^\top)\eta^i\rangle + \langle v(\cdot),\ (\mathbb{L}^* \circ H^\top)\eta^i\rangle_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)},$$

and

$$\mathbb{E}\langle HX(T),\ \eta\rangle = \langle X_0,\ (\mathbb{L}_0^* \circ H^\top)\eta\rangle + \langle v(\cdot),\ (\mathbb{L}^* \circ H^\top)\eta\rangle_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)},$$

respectively. Due to the arbitrariness of $X_0,\ v(\cdot)$ and the fact $\lim\limits_{i\to\infty} \mathbb{E}\langle HX(T),\ \eta^i\rangle = \mathbb{E}\langle HX(T),\ \eta\rangle$, we have

$$\begin{aligned}&(\mathbb{L}^* \circ H^\top)\eta^i \to (\mathbb{L}^* \circ H^\top)\eta \text{ weakly in } L^2_{\mathbb{F}}(0,T;\mathbb{R}^k),\\ &(\mathbb{L}_0^* \circ H^\top)\eta^i \to (\mathbb{L}_0^* \circ H^\top)\eta \text{ in } \mathbb{R}^n.\end{aligned}$$

By virtue of the weak lower semi-continuity of the norms to Banach spaces, we obtained the weak lower semi-continuity of $J(\cdot; X_0, \xi)$. □

We shall use the standard notion of coercivity. For the convenience of readers, we would like to present it in this paper.

Definition 2.8. A functional $L(\cdot)$ is called to be coercive in $\mathcal{X}$ if $\lim_{i\to\infty} L(\eta^i) = \infty$, where $\{\eta^i\}_{i=1}^\infty \subset \mathcal{X}$ is any given sequence satisfying $\lim_{i\to\infty} \|\eta^i\|_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^l)} = \infty$.

Lemma 2.9. *If the observability inequality (2.8) for the system (2.2) holds true, then $J(\cdot; X_0, \xi)$ defined by (2.10) is coercive and strictly convex in $\mathcal{X}$. Moreover, Problem (O) admits a unique optimal control.*

Proof. Firstly, we prove the coercivity of $J(\cdot; X_0, \xi)$. Let $\{\eta^i\}_{i=1}^\infty \subset \mathcal{X}$ be an arbitrary sequence such that $\lim_{i\to\infty} \|\eta^i\|_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^l)} = \infty$. By the observability inequality (2.8),

$$J(\eta^i; X_0, \xi) \geq \frac{\delta^2}{2}\|\eta^i\|^2_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^l)} - |X_0|\|\mathbb{L}_0^* \circ H^\top\|\|\eta^i\|_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^l)} - \|\xi\|_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^l)}\|\eta^i\|_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^l)},$$

which yields the coercivity of $J(\cdot; X_0, \xi)$.

Secondly, we prove the existence of infimum of $J(\cdot; X_0, \xi)$. For any $M > 0$, and any $\eta \in \mathcal{X}$ satisfying $\|\eta\|_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^l)} \leq M$, since $\mathbb{L}^* \circ H^\top$ and $\mathbb{L}_0^* \circ H^\top$ are bounded linear operators, we have

$$|J(\eta; X_0, \xi)| \leq \frac{1}{2}\|\mathbb{L}^* \circ H^\top\|^2\|\eta\|^2_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^l)} + |X_0|\|\mathbb{L}_0^* \circ H^\top\|\|\eta\|_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^l)} + \|\xi\|_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^l)}\|\eta\|_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^l)},$$

which implies $J(\cdot; X_0, \xi)$ is bounded in the open ball $B_{\mathcal{X}}(0, M)$. Especially, there exists a $K_1 > 0$ such that

$$|J(\eta; X_0, \xi)| \leq K_1, \qquad \forall\ \eta \in B_{\mathcal{X}}(0, 1).$$

Now we consider the values of $J(\cdot; X_0, \xi)$ outside $B_{\mathcal{X}}(0, 1)$. We assert that there exists another positive number K_2 such that

$$|J(\eta; X_0, \xi)| \geq -K_2, \qquad \forall\ \eta\ \notin\ B_{\mathcal{X}}(0, 1).$$

If not, then there exists a sequence $\{\eta^i\}_{i=1}^\infty$ such that $\lim_{i\to\infty} J(\eta^i; X_0, \xi) = -\infty$. Obviously, there exists a subsequence $\{\eta^{i_j}\}_{j=1}^\infty$ such that $\lim_{j\to\infty} \|\eta^{i_j}\|_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^l)} = \infty$, which is a contradiction with the coercivity. We obtain the existence of infimum.

Thirdly, we prove the existence of minimum. Denote $\kappa = \inf_{\eta\in\mathcal{X}} J(\eta; X_0, \xi)$. Then there exists a sequence $\{\eta^i\}_{i=1}^\infty$ such that

$$\kappa \leq J(\eta^i; X_0, \xi) < \kappa + \frac{1}{i}.$$

By the coercivity, there exists a constant $M > 0$ such that $\|\eta^i\|_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^l)} \le M$ for any $i \in \mathbb{N}$. Then there exists a subsequence $\{\eta^{i_j}\}_{j=1}^\infty$ converging to some $\tilde\eta$ weakly in $L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^l)$ as $j \to \infty$. Since the functional $J(\cdot;X_0,\xi)$ is weak lower semi-continuous (see Lem. 2.7), then

$$\kappa \le J(\tilde\eta;X_0,\xi) \le \varliminf_{j\to\infty} J(\eta^{i_j};X_0,\xi) \le \lim_{j\to\infty}\Big(\kappa + \frac{1}{i_j}\Big) = \kappa.$$

We obtain that $\tilde\eta$ is a minimizer of the functional $J(\cdot;X_0,\xi)$.

Due to the fact that the uniqueness of the minimizer is a consequence of the strict convexity of $J(\cdot;X_0,\xi)$, hence the remaining thing is to examine the strict convexity. From the definition (2.10) of $J(\cdot;X_0,\xi)$ and the linearity of $\langle X_0,\ (\mathbb{L}_0^*\circ H^\top)\eta\rangle$ and $\mathbb{E}\langle\xi,\ \eta\rangle$, the requirement is reduced to prove the strict convexity of $\|(\mathbb{L}^*\circ H^\top)\eta\|^2_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)}$. For any $\eta^{(1)},\eta^{(2)} \in \mathcal{X}$ satisfying $\|\eta^{(1)}-\eta^{(2)}\|_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^l)} \ne 0$, by the observability inequality (2.8), $\|(\mathbb{L}^*\circ H^\top)\eta^{(1)} - (\mathbb{L}^*\circ H^\top)\eta^{(2)}\|_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)} \ne 0$. For any $\lambda \in (0,1)$, we have

$$\begin{aligned}
&\|(\mathbb{L}^*\circ H^\top)(\lambda\eta^{(1)}+(1-\lambda)\eta^{(2)})\|^2_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)}\\
&= \|\lambda(\mathbb{L}^*\circ H^\top)\eta^{(1)}+(1-\lambda)(\mathbb{L}^*\circ H^\top)\eta^{(2)}\|^2_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)}\\
&\le \Big[\lambda\|(\mathbb{L}^*\circ H^\top)\eta^{(1)}\|_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)} + (1-\lambda)\|(\mathbb{L}^*\circ H^\top)\eta^{(2)}\|_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)}\Big]^2\\
&\le \lambda\|(\mathbb{L}^*\circ H^\top)\eta^{(1)}\|^2_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)} + (1-\lambda)\|(\mathbb{L}^*\circ H^\top)\eta^{(2)}\|^2_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)}.
\end{aligned}\tag{2.14}$$

The above (2.14) implies the convexity of $\|(\mathbb{L}^*\circ H^\top)\eta\|^2_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)}$. Without loss of generality, we assume that $\|(\mathbb{L}^*\circ H^\top)\eta^{(2)}\|_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)} \ne 0$. In the above (2.14), both the last two equalities hold true if and only if

$$\begin{aligned}
&\text{(a).}\quad \lambda(\mathbb{L}^*\circ H^\top)\eta^{(1)} = c(1-\lambda)(\mathbb{L}^*\circ H^\top)\eta^{(2)}, \quad \text{for some constant } c \ge 0;\\
&\text{(b).}\quad \|(\mathbb{L}^*\circ H^\top)\eta^{(1)}\|_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)} = \|(\mathbb{L}^*\circ H^\top)\eta^{(2)}\|_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)},
\end{aligned}$$

which is equivalent to $(\mathbb{L}^*\circ H^\top)\eta^{(1)} = (\mathbb{L}^*\circ H^\top)\eta^{(2)}$. This contradiction implies that the last two equations in (2.14) cannot hold true at the same time. Then we get

$$\begin{aligned}
&\|(\mathbb{L}^*\circ H^\top)(\lambda\eta^{(1)}+(1-\lambda)\eta^{(2)})\|^2_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)}\\
&< \lambda\|(\mathbb{L}^*\circ H^\top)\eta^{(1)}\|^2_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)} + (1-\lambda)\|(\mathbb{L}^*\circ H^\top)\eta^{(2)}\|^2_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)}.
\end{aligned}$$

We obtain the strict convexity of $J(\cdot;X_0,\xi)$, and finish the proof. □

Moreover, for Problem (O), the variational analysis leads to the following necessary condition.

Lemma 2.10 (Euler-Lagrange equation). *Let $\tilde\eta \in \mathcal{X}$ be an optimal control of Problem (O), and $(\tilde Y(\cdot),\tilde Z(\cdot))$ denote the solution to BSDE (2.2) with terminal value $H^\top\tilde\eta$. For any $\eta\in\mathcal{X}$, we also use $(Y(\cdot),Z(\cdot))$ to denote the solution to BSDE (2.2) with terminal value $H^\top\eta$. Then*

$$\begin{aligned}
0 &= \langle(\mathbb{L}^*\circ H^\top)\tilde\eta,\ (\mathbb{L}^*\circ H^\top)\eta\rangle_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)} + \langle X_0,\ (\mathbb{L}_0^*\circ H^\top)\eta\rangle - \mathbb{E}\langle\xi,\ \eta\rangle\\
&\equiv \mathbb{E}\int_0^T\Big\langle B_0(t)^\top\tilde Y(t)+\sum_{j=1}^d B_j(t)^\top\tilde Z_j(t),\ B_0(t)^\top Y(t)+\sum_{j=1}^d B_j(t)^\top Z_j(t)\Big\rangle \mathrm{d}t\\
&\quad + \langle X_0,\ Y(0)\rangle - \mathbb{E}\langle\xi,\ \eta\rangle.
\end{aligned}\tag{2.15}$$

Proof. Let $\tilde{\eta} \in \mathcal{X}$ be an optimal control of Problem (O). For any $\eta \in \mathcal{X}$, any $-1 < \varepsilon < 1$, we take the variational control $\eta^\varepsilon = \tilde{\eta} + \varepsilon\eta$. Since $\tilde{\eta}$ is optimal, we have

$$\begin{aligned} 0 = & \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\Big[J(\eta^\varepsilon; X_0, \xi) - J(\tilde{\eta}; X_0, \xi)\Big] \\ = & \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\Big\{\frac{1}{2}\|(\mathbb{L}^* \circ H^\top)\tilde{\eta} + \varepsilon(\mathbb{L}^* \circ H^\top)\eta\|^2_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)} - \frac{1}{2}\|(\mathbb{L}^* \circ H^\top)\tilde{\eta}\|^2_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)}\Big\} \\ & + \langle X_0,\ (\mathbb{L}_0^* \circ H^\top)\eta\rangle - \mathbb{E}\langle \xi,\ \eta\rangle \\ = & \frac{1}{2}\frac{d}{d\varepsilon}\Big\{\|(\mathbb{L}^* \circ H^\top)\tilde{\eta} + \varepsilon(\mathbb{L}^* \circ H^\top)\eta\|^2_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)}\Big\}\Big|_{\varepsilon=0} + \langle X_0,\ (\mathbb{L}_0^* \circ H^\top)\eta\rangle - \mathbb{E}\langle \xi,\ \eta\rangle \\ = & \langle (\mathbb{L}^* \circ H^\top)\tilde{\eta},\ (\mathbb{L}^* \circ H^\top)\eta\rangle_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)} + \langle X_0,\ (\mathbb{L}_0^* \circ H^\top)\eta\rangle - \mathbb{E}\langle \xi,\ \eta\rangle. \end{aligned}$$

That completes the proof. □

With the above preparations, now we can prove Theorem 2.5.

Proof of Theorem 2.5. By Lemma 2.6, the statement (i) implies the statement (ii). By Lemma 2.9, the statement (ii) implies the statement (iii).

(iii) ⇒ (i). For any $(X_0, \xi) \in \mathbb{R}^n \times \mathcal{X}$, we denote the unique optimal control of Problem (O) by $\tilde{\eta}$. By Lemma 2.10, the corresponding Euler-Lagrange equation is read as follows (see (2.15)):

$$\langle \tilde{v}(\cdot),\ (\mathbb{L}^* \circ H^\top)\eta\rangle_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)} + \langle X_0,\ (\mathbb{L}_0^* \circ H^\top)\eta\rangle - \mathbb{E}\langle \xi,\ \eta\rangle = 0, \qquad \forall\ \eta \in \mathcal{X}, \tag{2.16}$$

where $\tilde{v}(\cdot)$ is defined by (2.11). On the other hand, the duality between $X(\cdot)$ and $Y(\cdot)$ is read as (see (2.3)):

$$\begin{aligned} \langle v(\cdot),\ (\mathbb{L}^* \circ H^\top)\eta\rangle_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)} + \langle X_0,\ (\mathbb{L}_0^* \circ H^\top)\eta\rangle - \mathbb{E}\langle HX(T; X_0, v(\cdot)),\ \eta\rangle = 0, \\ \forall\ \eta \in \mathcal{X}, \quad \forall v(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^k). \end{aligned} \tag{2.17}$$

Then, letting $v(\cdot) = \tilde{v}(\cdot)$ in (2.17), and comparing with (2.16), we have

$$\mathbb{E}\langle HX(T; X_0, \tilde{v}(\cdot)) - \xi,\ \eta\rangle = 0, \qquad \forall\ \eta \in \mathcal{X}.$$

That is $HX(T; X_0, \tilde{v}(\cdot)) = \xi$ P-a.s. Hence the system (2.1) is H-partially controllable.

(iii) ⇒ (iv). From the proof of the previous part, $\tilde{v}(\cdot)$ defined by (2.11) is an admissible control of Problem (N), *i.e.*, $\tilde{v}(\cdot) \in \mathcal{V}(X_0, \xi)$. By letting $\eta = \tilde{\eta}$ in both (2.16) and (2.17), we obtain

$$\|\tilde{v}(\cdot)\|^2_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)} = \mathbb{E}\langle \xi,\ \tilde{\eta}\rangle - \langle X_0,\ (\mathbb{L}_0^* \circ H^\top)\tilde{\eta}\rangle = \langle v(\cdot),\ \tilde{v}(\cdot)\rangle_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)}, \qquad \forall v(\cdot) \in \mathcal{V}(X_0, \xi).$$

By Hölder's inequality, we have

$$\|\tilde{v}(\cdot)\|_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)} \le \|v(\cdot)\|_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)}, \qquad \forall v(\cdot) \in \mathcal{V}(X_0, \xi).$$

We get the optimality of $\tilde{v}(\cdot)$ defined by (2.11). The uniqueness of the optimal control to Problem (N) follows immediately from the classical parallelogram rule of the norm $\|\cdot\|_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)}$.

Let us come back to the Euler-Lagrange equation. Letting $\eta = \tilde{\eta}$ and from the definition (2.11) of $\tilde{v}(\cdot)$, (2.15) is reduced to

$$\|\tilde{v}(\cdot)\|^2_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)} = \mathbb{E}\langle \xi,\ \tilde{\eta}\rangle - \langle X_0,\ (\mathbb{L}_0^* \circ H^\top)\tilde{\eta}\rangle,$$

which implies (2.12). Then we calculate

$$J(\tilde{\eta}; X_0, \xi) = \frac{1}{2}\|\tilde{v}(\cdot)\|^2_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)} + \langle X_0,\ (\mathbb{L}_0^* \circ H^\top)\tilde{\eta}\rangle - \mathbb{E}\langle \xi,\ \tilde{\eta}\rangle = -\frac{1}{2}\|\tilde{v}(\cdot)\|^2_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)},$$

which is (2.13).

(iv) ⇒ (i). Since for any $(X_0,\xi)\in\mathbb{R}^n\times\mathcal{X}$, Problem (N) admits an optimal control, then the corresponding admissible control set $\mathcal{V}(X_0,\xi)$ is nonempty. Therefore, system (2.1) is H-partially controllable. The proof is completed. □

3. Partial controllability of time-invariant SDEs

In this section, we assume that all the matrices in system (2.1) and H are deterministic and time-invariant, and the dimension of the Brownian motion $d=1$. Under aforementioned assumptions, we tend to investigate a necessary condition and a Kalman type rank criterion ensuring system (2.1)'s H-partial controllability.

Let $H\in\mathbb{R}^{l\times n}$ be given. If $\mathrm{Rank}(H)=\bar{l}<l$, then there exists an invertible matrix $P\in\mathbb{R}^{l\times l}$ such that $PH=\begin{pmatrix}\bar{H}\\ O\end{pmatrix}$ satisfying $\bar{H}\in\mathbb{R}^{\bar{l}\times n}$ and $\mathrm{Rank}(\bar{H})=\bar{l}$. The following proposition shows that, in this case, we can substitute $\bar{H}$ for H to investigate the corresponding $\bar{H}$-partial controllability.

Proposition 3.1. *System (2.1) is H-partially controllable if and only if system (2.1) is $\bar{H}$-partially controllable.*

Proof. (⇒). For any $X_0\in\mathbb{R}^n$ and any $\bar{\xi}\in\bar{\mathcal{X}}\equiv\{\bar{\xi}\in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^{\bar{l}})\mid\bar{\xi}(\omega)\in\mathcal{R}(\bar{H}),\mathbb{P}-\text{a.s. }\omega\in\Omega\}=L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^{\bar{l}})$, we want to find a control $v(\cdot)$ such that $\bar{H}X(T;X_0,v(\cdot))=\bar{\xi}$. Indeed, since $\bar{\xi}\in\bar{\mathcal{X}}$, for $\mathbb{P}$-a.s. $\omega\in\Omega$, there exists a $\beta(\omega)$ satisfying $\bar{H}\beta(\omega)=\bar{\xi}(\omega)$. Therefore,

$$\begin{pmatrix}\bar{H}\beta(\omega)\\ O\beta(\omega)\end{pmatrix}=\begin{pmatrix}\bar{\xi}(\omega)\\ O\end{pmatrix},\ \text{a.s.}\,\omega\in\Omega,$$

which implies that $PH\beta=\begin{pmatrix}\bar{\xi}\\ O\end{pmatrix}$. Denote $\xi\equiv P^{-1}\begin{pmatrix}\bar{\xi}\\ O\end{pmatrix}=H\beta$. Clearly $\xi\in\mathcal{X}$. By the H-partial controllability of system (2.1), there exists a $v(\cdot)$ such that $HX(T;X_0,v(\cdot))=\xi=P^{-1}\begin{pmatrix}\bar{\xi}\\ O\end{pmatrix}$, which is equivalent to $\begin{pmatrix}\bar{H}\\ O\end{pmatrix}X(T;X_0,v(\cdot))=\begin{pmatrix}\bar{\xi}\\ O\end{pmatrix}$. This proves system (2.1)'s $\bar{H}$-partially controllability.

(⇐). For any $X_0\in\mathbb{R}^n$ and any $\xi\in\mathcal{X}$, we tend to find a control process $v(\cdot)$ such that $HX(T;X_0,v(\cdot))=\xi$. Indeed, since $\xi\in\mathcal{X}$ for $\mathbb{P}$-a.s. $\omega\in\Omega$, then there exists a $\beta(\omega)$ satisfying $H\beta(\omega)=\xi(\omega)$. Then $PH\beta(\omega)=P\xi(\omega)$, which is $\begin{pmatrix}\bar{H}\beta(\omega)\\ O\end{pmatrix}=\begin{pmatrix}(P\xi(\omega))_1\\ (P\xi(\omega))_2\end{pmatrix}$. Hence, one gets $(P\xi)_1\in\bar{\mathcal{X}}$ and $(P\xi)_2=O$. Denote $\bar{\xi}=(P\xi)_1$. By the $\bar{H}$-partial controllability of system (2.1), we know that there exists a $v(\cdot)\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$ satisfying $\bar{H}X(T;X_0,v(\cdot))=\bar{\xi}$. Therefore,

$$\begin{pmatrix}\bar{H}X(T;X_0,v(\cdot))\\ OX(T;X_0,v(\cdot))\end{pmatrix}=\begin{pmatrix}(P\xi)_1\\ (P\xi)_2\end{pmatrix},$$

which implies $HX(T;X_0,v(\cdot))=\xi$. We complete the proof. □

Applying Proposition 3.1, from now on, we suppose that $\mathrm{Rank}(H)=l$. The following result states a necessary condition ensuring (2.1)'s partial controllability.

Theorem 3.2. *Suppose that* $\mathrm{Rank}(H)=l$. *If system (2.1) is H-partially controllable, then*

$$\mathrm{Rank}(HD)=l.\tag{3.1}$$

Proof. We borrow the idea from ([18], Prop. 2.1). If $\mathrm{Rank}(HD) < l$, then there exists a nonzero vector $\gamma \in \mathbb{R}^l$, satisfying $\gamma^\top HD = 0$. Take $\xi = HH^\top\gamma \int_0^T \eta(t)\mathrm{d}W(t) \in \mathcal{X}$. Here $\eta(\cdot)$ is defined as follows:

$$\eta(t) = \begin{cases} 1, & t \in \left[\left(1 - \frac{1}{2^{2i}}\right)T, \left(1 - \frac{1}{2^{2i+1}}\right)T\right), \ i = 0, 1, 2, \dots \\ -1, & \text{others.} \end{cases}$$

For such $\eta(\cdot)$, we know that there exists a positive constant β, such that for any $c \in \mathbb{R}$, $t \in [0, T]$,

$$\int_t^T |\eta(s) - c|^2 \mathrm{d}s \geq 4\beta(T - t). \tag{3.2}$$

Since (2.1) is H-partially controllable, for any $X_0 \in \mathbb{R}^n$, there exists a control $v(\cdot) \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^m)$, such that

$$HH^\top\gamma \int_0^T \eta(t)\mathrm{d}W(t) = HX_0 + \int_0^T H\left(AX(t) + Bv(t)\right)\mathrm{d}t + \int_0^T H\left(CX(t) + Dv(t)\right)\mathrm{d}W(t).$$

And then

$$\begin{aligned} &\gamma^\top HH^\top\gamma \int_0^T \eta(t)\mathrm{d}W(t) \\ =&\gamma^\top HX_0 + \int_0^T \gamma^\top H\left(AX(t) + Bv(t)\right)\mathrm{d}t + \int_0^T \gamma^\top H\left(CX(t) + Dv(t)\right)\mathrm{d}W(t) \\ =&\gamma^\top HX_0 + \int_0^T \gamma^\top H\left(AX(t) + Bv(t)\right)\mathrm{d}t + \int_0^T \gamma^\top HCX(t)\mathrm{d}W(t). \end{aligned}$$

Taking conditional expectation with respect to $\mathcal{F}_t$, one has

$$\begin{aligned} &\gamma^\top HH^\top\gamma \int_0^t \left(\eta(s) - \frac{1}{\gamma^\top HH^\top\gamma}\gamma^\top HCX(s)\right)\mathrm{d}W(s) \\ &= \gamma^\top HX_0 + \int_0^t \gamma^\top H\left(AX(s) + Bv(s)\right)\mathrm{d}s + \mathbb{E}\left(\int_t^T \gamma^\top H\left(AX(s) + Bv(s)\right)\mathrm{d}s\Big|\mathcal{F}_t\right). \end{aligned}$$

Here we use the fact $HH^\top$ is positive, which is deduced by $\mathrm{Rank}(H) = l$. Above two equalities yield

$$\begin{aligned} &\gamma^\top HH^\top\gamma \int_t^T \left(\eta(s) - \frac{1}{\gamma^\top HH^\top\gamma}\gamma^\top HCX(s)\right)\mathrm{d}W(s) \\ &= \int_t^T \gamma^\top H\left(AX(s) + Bv(s)\right)\mathrm{d}s - \mathbb{E}\left(\int_t^T \gamma^\top H\left(AX(s) + Bv(s)\right)\mathrm{d}s\Big|\mathcal{F}_t\right). \end{aligned}$$

Squaring and taking expectation, one gets

$$\begin{aligned} &\left|\gamma^\top HH^\top\gamma\right|^2 \mathbb{E}\int_t^T \left|\eta(s) - \frac{1}{\gamma^\top HH^\top\gamma}\gamma^\top HCX(s)\right|^2 \mathrm{d}s \\ \leq&\mathbb{E}\left|\int_t^T \gamma^\top H\left(AX(s) + Bv(s)\right)\mathrm{d}s\right|^2 \\ \leq&(T - t)\mathbb{E}\int_t^T \left|\gamma^\top H\left(AX(s) + Bv(s)\right)\right|^2 \mathrm{d}s. \end{aligned} \tag{3.3}$$

On the other side, since $X(\cdot) \in L^2_{\mathbb{F}}(\Omega; C([0,T];\mathbb{R}^n))$, $\lim_{s\to T} \mathbb{E}|X(T) - X(s)|^2 = 0$. Then there exists $T_0 \in [0,T)$, such that

$$\mathbb{E}\left|\frac{1}{\gamma^\top HH^\top\gamma}\left(\gamma^\top HCX(T) - \gamma^\top HCX(s)\right)\right|^2 < \beta,\ \forall s \in [T_0, T). \tag{3.4}$$

And then by (3.2) and (3.4), for any $t \in [T_0, T)$, one has

$$\begin{aligned}
&\mathbb{E}\int_t^T \left|\eta(s) - \frac{1}{\gamma^\top HH^\top\gamma}\gamma^\top HCX(s)\right|^2 \mathrm{d}s\\
\geq&\frac{1}{2}\mathbb{E}\int_t^T \left|\eta(s) - \frac{1}{\gamma^\top HH^\top\gamma}\gamma^\top HCX(T)\right|^2 \mathrm{d}s\\
&-\mathbb{E}\int_t^T \left|\frac{1}{\gamma^\top HH^\top\gamma}\left(\gamma^\top HCX(T) - \gamma^\top HCX(s)\right)\right|^2 \mathrm{d}s\\
\geq&2\beta(T-t) - \beta(T-t)\\
\geq&\beta(T-t).
\end{aligned} \tag{3.5}$$

(3.3) and (3.5) yield

$$\mathbb{E}\int_t^T \left|\gamma^\top H\left(AX(s) + Bv(s)\right)\right|^2 \mathrm{d}s \geq \beta,\ \ \forall t \in [T_0, T),$$

which is in contradiction with $X(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$ and $v(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$. That completes the proof. □

Now we tend to search a sufficient condition guaranteeing (2.1)'s partial controllability. By Theorem 3.2, $\mathrm{Rank}(HD) = l$, then there exists an invertible matrix K, such that $HDK = \big(I_l, O\big)$. Setting

$$K^{-1}v(\cdot) = \begin{pmatrix} v_2(\cdot)\\ v_1(\cdot)\end{pmatrix},\ HBK = \left(A_2, B_1\right), \tag{3.6}$$

one can get

$$\begin{aligned}
HX(t) =&HX_0 + \int_0^t (HAX(s) + HBKK^{-1}v(s))\mathrm{d}s + \int_0^t (HCX(s) + HDKK^{-1}v(s))\mathrm{d}W(s)\\
=&HX_0 + \int_0^t \left(HAX(s) + \left(A_2, B_1\right)\begin{pmatrix} v_2(s)\\ v_1(s)\end{pmatrix}\right)\mathrm{d}s\\
&+\int_0^t \left(HCX(s) + \left(I_l, O\right)\begin{pmatrix} v_2(s)\\ v_1(s)\end{pmatrix}\right)\mathrm{d}W(s)\\
=&HX_0 + \int_0^t \left((HA - A_2HC)X(s) + A_2(HCX(s) + v_2(s)) + B_1v_1(s)\right)\mathrm{d}s\\
&+\int_0^t \left(HCX(s) + v_2(s)\right)\mathrm{d}W(s).
\end{aligned} \tag{3.7}$$

Taking $u(\cdot) = HCX(\cdot) + v_2(\cdot)$, by (3.7), one has

$$HX(t) = HX_0 + \int_0^t \left((HA - A_2HC)X(s) + A_2u(s) + B_1v_1(s)\right)\mathrm{d}s + \int_0^t u(s)\mathrm{d}W(s). \tag{3.8}$$

The following result provides a verifiable algebra rank criterion guaranteeing (2.1)'s partial controllability.

Theorem 3.3. *Suppose that there exist $\bar{A}$, $\bar{C} \in \mathbb{R}^{p\times p}$, satisfying*

$$HA = \bar{A}H,\ HC = \bar{C}H. \tag{3.9}$$

Then, system (2.1) is H-partially controllable if and only if

$$\mathrm{Rank}(B_1, A_1B_1, A_2B_1, A_1^2B_1, A_1A_2B_1, A_2^2B_1, A_2A_1B_1, \dots) = l, \tag{3.10}$$

where $A_1 = \bar{A} - A_2\bar{C}$.

Proof. Setting $\bar{X}(\cdot) = HX(\cdot)$, by (3.8) and (3.9), one obtains that (2.1)'s H-partial controllability is equivalent to the solvability of the following stochastic system:

$$\begin{cases} \mathrm{d}\bar{X}(t) = (A_1\bar{X}(t) + A_2u(t) + B_1v_1(t))\mathrm{d}t + u(t)\mathrm{d}W(t),\ t \in [0,T] \\ \bar{X}(0) = HX_0,\ \bar{X}(T) = \xi, \end{cases} \tag{3.11}$$

for any $X_0 \in \mathbb{R}^n$, $\xi \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^l)$. Hence, system (2.1)'s H-partial controllability is equivalent to system (3.11)'s controllability, by Theorem 3.2 in [18], which is equivalent to (3.10). That completes the proof. □

We can adopt the idea appeared in the above result to explore other sufficient conditions. The following example gives another condition, under which, (3.10) can guarantee (2.1)'s partial controllability.

Example 3.4. If

$$\mathrm{Rank}(H) = \mathrm{Rank}\begin{pmatrix} H \\ A \\ C \end{pmatrix} = l, \tag{3.12}$$

then, there exists a invertible matrix $K_0 \in \mathbb{R}^{n\times n}$, such that

$$\begin{pmatrix} H \\ A \\ C \end{pmatrix} K_0 = \begin{pmatrix} I_l & O \\ A_0 & O \\ C_0 & O \end{pmatrix}. \tag{3.13}$$

Therefore, setting $K_0^{-1}X(\cdot) = \begin{pmatrix} \bar{X}_1(\cdot) \\ \bar{X}_2(\cdot) \end{pmatrix}$

$$\begin{aligned} HX(\cdot) &= (HK_0)K_0^{-1}X(\cdot) = \bar{X}_1(\cdot), \\ AX(\cdot) &= (AK_0)K_0^{-1}X(\cdot) = A_0\bar{X}_1(\cdot), \\ CX(\cdot) &= (CK_0)K_0^{-1}X(\cdot) = C_0\bar{X}_1(\cdot), \end{aligned} \tag{3.14}$$

and applying (3.8), we have

$$\begin{cases} \mathrm{d}\bar{X}_1(t) = ((HA_0 - A_2HC_0)\bar{X}_1(t) + A_2u(t) + B_1v_1(t))\mathrm{d}t + u(t)\mathrm{d}W(t),\ t \in [0,T] \\ \bar{X}_1(0) = HX_0,\ \bar{X}_1(T) = \xi, \end{cases} \tag{3.15}$$

for any $X_0 \in \mathbb{R}^n$, $\xi \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^l)$. Applying Theorem 3.3, we know that if

$$\mathrm{Rank}(B_1, A_1B_1, A_2B_1, A_1^2B_1, A_1A_2B_1, A_2^2B_1, A_2A_1B_1, \dots) = l, \tag{3.16}$$

then system (2.1) is H-partially controllable, where A_0, C_0 is defined in (3.13), and $A_1 = HA_0 - A_2HC_0$.

4. Exact controllability of FBSDEs

In Section 2, for any given $H \in L^\infty_{\mathcal{F}_T}(\Omega;\mathbb{R}^l)$, we studied the H-partial controllability for SDE (2.1), and obtained some equivalent conditions. In this section, we continue to study an important special case. For this special case, we will obtain some further conclusions. In detail, we introduce the following

Setting (FB). Let $X(\cdot)^\top = (x(\cdot)^\top, y(\cdot)^\top)$, where $x(\cdot)$ and $y(\cdot)$ take values in $\mathbb{R}^{n'}$ and $\mathbb{R}^{m'}$ respectively, with $n'+m'=n$. Let $v(\cdot)^\top = (u(\cdot)^\top, z_1(\cdot)^\top, \dots, z_d(\cdot)^\top)$, where $u(\cdot)$ and $z_j(\cdot)$ $(j=1,2,\dots,d)$ take values in $\mathbb{R}^{k'}$ and $\mathbb{R}^{m'}$ respectively, with $k = k' + m'd$. Let

$$A_0(\cdot) = \begin{pmatrix} C_0(\cdot) & O \\ F(\cdot) & R(\cdot) \end{pmatrix}, \qquad B_0(\cdot) = \begin{pmatrix} D_0(\cdot) & O & \cdots & O \\ G(\cdot) & S_1(\cdot) & \cdots & S_d(\cdot) \end{pmatrix},$$
$$A_j(\cdot) = \begin{pmatrix} C_j(\cdot) & O \\ O & O \end{pmatrix}, \qquad B_j(\cdot) = \begin{pmatrix} D_j(\cdot) & O & \cdots & O & \cdots & O \\ O & O & \cdots & I_{m'} & \cdots & O \end{pmatrix}, \quad j = 1,2,\dots,d,$$

where $C_0(\cdot), C_j(\cdot) \in L^\infty_{\mathbb{F}}(0,T;\mathbb{R}^{n'\times n'})$, $D_0(\cdot), D_j(\cdot) \in L^\infty_{\mathbb{F}}(0,T;\mathbb{R}^{n'\times k'})$, $F(\cdot) \in L^\infty_{\mathbb{F}}(0,T;\mathbb{R}^{m'\times n'})$, $G(\cdot) \in L^\infty_{\mathbb{F}}(0,T;\mathbb{R}^{m'\times k'})$, and $R(\cdot), S_j(\cdot) \in L^\infty_{\mathbb{F}}(0,T;\mathbb{R}^{m'\times m'})$ for $j=1,2,\dots,d$. Let $H = (-M, I_{m'})$ where $M \in L^\infty_{\mathcal{F}_T}(\Omega;\mathbb{R}^{m'\times n'})$.

When the general situation does not satisfy Setting (FB), sometimes we can do some transformation to change it into Setting (FB). In Appendix, we provide an example for this aim.

Proposition 4.1. *With Setting (FB), system (2.1) is H-partially controllable on $[0,T]$ if and only if system (2.1) is H-partially null-controllable on $[0,T]$.*

Proof. The necessity is trivial. Now we consider the sufficiency. For any $X_0 = (x_0^\top, y_0^\top)^\top \in \mathbb{R}^{n'+m'}$ and $\xi \in \mathcal{X} = L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^{m'})$, we need prove the following system on $[0,T]$:

$$\begin{cases} \mathrm{d}x(t) = \Big[C_0(t)x(t) + D_0(t)u(t)\Big]\mathrm{d}t + \displaystyle\sum_{j=1}^d \Big[C_j(t)x(t) + D_j(t)u(t)\Big]\mathrm{d}W_j(t), \\ \mathrm{d}y(t) = \Big[F(t)x(t) + G(t)u(t) + R(t)y(t) + \displaystyle\sum_{j=1}^d S_j(t)z_j(t)\Big]\mathrm{d}t + \displaystyle\sum_{j=1}^d z_j(t)\mathrm{d}W_j(t), \\ x(0) = x_0, \quad y(0) = y_0, \quad y(T) = Mx(T) + \xi \end{cases} \tag{4.1}$$

has a solution $(x(\cdot), y(\cdot), u(\cdot), z(\cdot)) \in S^2_{\mathbb{F}}(0,T;\mathbb{R}^{n'}) \times S^2_{\mathbb{F}}(0,T;\mathbb{R}^{m'}) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^{k'}) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^{m'\times d})$ where we denote $z(\cdot) = (z_1(\cdot), z_2(\cdot), \dots, z_d(\cdot))$ for simplicity.

Firstly, by the classical results on SDEs and BSDEs, the following forward-backward stochastic differential equation on $[0,T]$:

$$\begin{cases} \mathrm{d}x^0(t) = C_0(t)x^0(t)\mathrm{d}t + \displaystyle\sum_{j=1}^d C_j(t)x^0(t)\mathrm{d}W_j(t), \\ \mathrm{d}y^0(t) = \Big[F(t)x^0(t) + R(t)y^0(t) + \displaystyle\sum_{j=1}^d S_j(t)z_j^0(t)\Big]\mathrm{d}t + \displaystyle\sum_{j=1}^d z_j^0(t)\mathrm{d}W_j(t), \\ x^0(0) = 0, \quad y^0(T) = Mx^0(T) + \xi \end{cases} \tag{4.2}$$

admits a unique solution $(x^0(\cdot), y^0(\cdot), z^0(\cdot)) \in S^2_{\mathbb{F}}(0,T;\mathbb{R}^{n'}) \times S^2_{\mathbb{F}}(0,T;\mathbb{R}^{m'}) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^{m'\times d})$. Secondly, the H-partial null-controllability leads to the solvability of the following system

$$\begin{cases} \mathrm{d}x^1(t) = \Big[C_0(t)x^1(t) + D_0(t)u^1(t)\Big]\mathrm{d}t + \sum_{j=1}^{d}\Big[C_j(t)x^1(t) + D_j(t)u^1(t)\Big]\mathrm{d}W_j(t), \\ \mathrm{d}y^1(t) = \Big[F(t)x^1(t) + G(t)u^1(t) + R(t)y^1(t) + \sum_{j=1}^{d} S_j(t)z^1_j(t)\Big]\mathrm{d}t + \sum_{j=1}^{d} z^1_j(t)\mathrm{d}W_j(t), \\ x^1(0) = x_0, \quad y^1(0) = y_0 - y^0(0), \quad y^1(T) = Mx^1(T). \end{cases} \tag{4.3}$$

We denote the solution by $(x^1(\cdot), y^1(\cdot), u^1(\cdot), z^1(\cdot)) \in S^2_{\mathbb{F}}(0,T;\mathbb{R}^{n'}) \times S^2_{\mathbb{F}}(0,T;\mathbb{R}^{m'}) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^{k'}) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^{m'\times d})$. By the linearity, $(x^0(\cdot) + x^1(\cdot), y^0(\cdot) + y^1(\cdot), u^1(\cdot), z^0(\cdot) + z^1(\cdot))$ solves (4.1). We get the sufficiency. □

Under Setting (FB), the H-partial null-controllability can be reformulated by the notion of exact controllability for some forward-backward system. Precisely, let us introduce the following controlled linear forward-backward stochastic system:

$$\begin{cases} \mathrm{d}x(t) = \Big[C_0(t)x(t) + D_0(t)u(t)\Big]\mathrm{d}t + \sum_{j=1}^{d}\Big[C_j(t)x(t) + D_j(t)u(t)\Big]\mathrm{d}W_j(t), \quad t \in [0,T], \\ \mathrm{d}y(t) = \Big[F(t)x(t) + G(t)u(t) + R(t)y(t) + \sum_{j=1}^{d} S_j(t)z_j(t)\Big]\mathrm{d}t + \sum_{j=1}^{d} z_j(t)\mathrm{d}W_j(t), \quad t \in [0,T], \\ y(T) = Mx(T). \end{cases} \tag{4.4}$$

Clearly, with any given initial value $x(0) = x_0 \in \mathbb{R}^{n'}$, and any given $u(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^{k'})$, (4.4) becomes a decoupled forward-backward stochastic differential equation. By the classical results on SDEs and BSDEs, FBSDE (4.4) admits a unique solution $(x(\cdot), y(\cdot), z(\cdot)) \in S^2_{\mathbb{F}}(0,T;\mathbb{R}^{n'}) \times S^2_{\mathbb{F}}(0,T;\mathbb{R}^{m'}) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^{m'\times d})$ where we denote $z(\cdot) = (z_1(\cdot), z_2(\cdot), \ldots, z_d(\cdot))$ for simplicity.

We now give a precise definition of exact controllability for system (4.4).

Definition 4.2. System (4.4) is said to be exactly controllable on the time interval $[0,T]$, if for any $(x_0, y_0) \in \mathbb{R}^{n'} \times \mathbb{R}^{m'}$, there exists a $u(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^{k'})$ such that the solution $(x(\cdot), y(\cdot), z(\cdot))$ to FBSDE (4.4) with $x(0) = x_0$ and $u(\cdot)$ satisfies $y(0) = y_0$.

Remark 4.3. By Proposition 4.1, it is clear that under Setting (FB), the H-partial controllability of system (2.1) is equivalent to the exact controllability of system (4.4). However, we take two different viewpoints for these two notions. Precisely, in the viewpoint of the H-partial controllability of system (2.1), both $u(\cdot)$ and $z(\cdot)$ are regarded as control processes (noticing that $v(\cdot)^\top = (u(\cdot)^\top, z_1(\cdot)^\top, \ldots, z_d(\cdot)^\top)$). While, when we consider the exact controllability of system (4.4), $u(\cdot)$ is regarded as a control process, and $z(\cdot)$ is regarded as a state process.

Now keeping the new viewpoint stated in Remark 4.3, we reconsider the issues studied in Section 2, and try to find some different conclusions. We introduce the adjoint equation of system (4.4) as follows:

$$\begin{cases} \mathrm{d}p(t) = -R(t)^\top p(t)\mathrm{d}t - \sum_{j=1}^{d} S_j(t)^\top p(t)\mathrm{d}W_j(t), \quad t \in [0,T], \\ \mathrm{d}q(t) = -\Big[F(t)^\top p(t) + C_0(t)^\top q(t) + \sum_{j=1}^{d} C_j(t)^\top r_j(t)\Big]\mathrm{d}t + \sum_{j=1}^{d} r_j(t)\mathrm{d}W_j(t), \quad t \in [0,T], \\ p(0) = p_0, \qquad q(T) = -M^\top p(T). \end{cases} \tag{4.5}$$

For any $p(0) = p_0 \in \mathbb{R}^{m'}$, the classical results on SDEs and BSDEs work again to ensure that FBSDE (4.5) admits a unique solution $(p(\cdot), q(\cdot), r(\cdot)) \in S^2_{\mathbb{F}}(0,T;\mathbb{R}^{m'}) \times S^2_{\mathbb{F}}(0,T;\mathbb{R}^{n'}) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^{n' \times d})$, where we denote $r(\cdot) = (r_1(\cdot), r_2(\cdot), \dots, r_d(\cdot))$ similarly. By applying Itô's formula to $\langle x(\cdot),\ q(\cdot)\rangle + \langle y(\cdot),\ p(\cdot)\rangle$ on the interval $[0,T]$, we have the following duality:

$$\mathbb{E}\int_0^T \Big\langle u(t),\ G(t)^\top p(t) + D_0(t)^\top q(t) + \sum_{j=1}^{d} D_j(t)^\top r_j(t)\Big\rangle \mathrm{d}t + \langle x_0,\ q(0)\rangle + \langle y(0),\ p_0\rangle = 0. \tag{4.6}$$

We define two bounded linear operators $\mathbb{K}_0 : \mathbb{R}^{n'} \to \mathbb{R}^{m'}$ and $\mathbb{K} : L^2_{\mathbb{F}}(0,T;\mathbb{R}^{k'}) \to \mathbb{R}^{m'}$ as follows

$$\begin{cases} \mathbb{K}_0 x_0 = y(0; x_0, 0), \quad \forall\ x_0 \in \mathbb{R}^{n'}, \\ \mathbb{K}u(\cdot) = y(0; 0, u(\cdot)), \quad \forall\ u(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^{k'}). \end{cases} \tag{4.7}$$

Denote $\mathbb{K}_0^* : \mathbb{R}^{m'} \to \mathbb{R}^{n'}$ and $\mathbb{K}^* : \mathbb{R}^{m'} \to L^2_{\mathbb{F}}(0,T;\mathbb{R}^{k'})$ be the adjoint operators of $\mathbb{K}_0$ and $\mathbb{K}$, respectively. From the duality relation (4.6), we have

$$\begin{cases} \mathbb{K}_0^* p_0 \equiv -q(0), \quad \forall\ p_0 \in \mathbb{R}^{m'}, \\ \mathbb{K}^* p_0 \equiv -\Big[G(\cdot)^\top p(\cdot) + D_0(\cdot)^\top q(\cdot) + \sum_{j=1}^{d} D_j(\cdot)^\top r_j(\cdot)\Big], \quad \forall\ p_0 \in \mathbb{R}^{m'}, \end{cases} \tag{4.8}$$

where $(p(\cdot), q(\cdot), r(\cdot))$ is the unique solution to FBSDE (4.5) with the initial value $p(0) = p_0$.

Definition 4.4. (i). The operator $\mathbb{K}^*$ defined by (4.8) is called an observer of system (4.5) on the time interval $[0,T]$.

(ii). System (4.5) is said to be exactly observable on the time interval $[0,T]$ if from the observation $\mathbb{K}^* p_0$, the initial value $p_0 \in \mathbb{R}^{m'}$ of $p(\cdot)$ can be uniquely determined, *i.e.*, the operator $\mathbb{K}^* : \mathbb{R}^{m'} \to L^2_{\mathbb{F}}(0,T;\mathbb{R}^{k'})$ is injective.

Remark 4.5. In the viewpoint of Banach inverse operator theorem, system (4.5) is exactly observable if and only if, there exists a constant $\delta > 0$ such that

$$\|\mathbb{K}^* p_0\|_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^{k'})} \geq \delta |p_0|, \quad \forall\ p_0 \in \mathbb{R}^{m'}, \tag{4.9}$$

where $\mathbb{K}^*$ is defined by (4.8). Equivalently,

$$\Big\| G(\cdot)^\top p(\cdot) + D_0(\cdot)^\top q(\cdot) + \sum_{j=1}^{d} D_j(\cdot)^\top r_j(\cdot) \Big\|_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^{k})} \geq \delta |p_0|, \quad \forall\ p_0 \in \mathbb{R}^{m'}, \tag{4.10}$$

where $(p(\cdot), q(\cdot), r(\cdot))$ is the unique solution to (4.5). The above inequality is called the observability inequality for system (4.5).

The main purpose of this section is to get the equivalence among the exact controllability of the original system (4.4), the validity of the observability inequality (4.9) for the adjoint equation (4.5), the non-singularity of a random version of Gramian matrix, the existence of minimizers to a family of functions, and the solvability of a family of norm optimal control problems.

Firstly, we introduce a matrix-valued decoupled FBSDE as follows:

$$\begin{cases} \mathrm{d}\Phi(t) = -R(t)^\top \Phi(t)\mathrm{d}t - \sum_{j=1}^{d} S_j(t)^\top \Phi(t)\mathrm{d}W_j(t), \quad t\in[0,T], \\ \mathrm{d}\Psi(t) = -\Big[F(t)^\top\Phi(t) + C_0(t)^\top \Psi(t) + \sum_{j=1}^{d} C_j(t)^\top \Gamma_j(t)\Big]\mathrm{d}t + \sum_{j=1}^{d}\Gamma_j(t)\mathrm{d}W_j(t), \quad t\in[0,T], \\ \Phi(0)=I, \quad \Psi(T) = -M^\top \Phi(T), \end{cases} \tag{4.11}$$

which admits a unique solution. Moreover, we defined a matrix which can be regarded as a random version of Gramian matrix:

$$\begin{aligned} \Lambda = \mathbb{E}\int_0^T &\Big[G(t)^\top\Phi(t) + D_0(t)^\top\Psi(t) + \sum_{j=1}^{d} D_j(t)^\top\Gamma_j(t)\Big]^\top \\ &\times \Big[G(t)^\top\Phi(t) + D_0(t)^\top\Psi(t) + \sum_{j=1}^{d} D_j(t)^\top\Gamma_j(t)\Big]\mathrm{d}t. \end{aligned} \tag{4.12}$$

Obviously, Λ is positive semi-definite. Comparing with the adjoint equation (4.5), by the linearity, we have

$$p(\cdot) = \Phi(\cdot)p_0, \quad q(\cdot)=\Psi(\cdot)p_0, \quad r_j(\cdot) = \Gamma_j(\cdot)p_0, \quad j=1,2,\ldots,d. \tag{4.13}$$

With the above relation, (4.8) reads

$$\begin{cases} \mathbb{K}_0^* p_0 \equiv -\Psi(0)p_0, \quad \forall\ p_0\in\mathbb{R}^{m'}, \\ \mathbb{K}^* p_0 \equiv -\Big[G(\cdot)^\top\Phi(\cdot) + D_0(\cdot)^\top\Psi(\cdot) + \sum_{j=1}^{d} D_j(\cdot)^\top\Gamma_j(\cdot)\Big]p_0, \quad \forall\ p_0\in\mathbb{R}^{m'}. \end{cases} \tag{4.14}$$

Secondly, for any $(x_0,y_0)\in\mathbb{R}^{n'}\times\mathbb{R}^{m'}$, we introduce a function $f(\cdot;x_0,y_0):\mathbb{R}^{m'}\to\mathbb{R}$ by

$$\begin{aligned} f(p_0;x_0,y_0) &= \frac{1}{2}\|\mathbb{K}^*p_0\|^2_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^{k'})} - \langle x_0,\ \mathbb{K}_0^*p_0\rangle + \langle y_0,\ p_0\rangle \\ &\equiv \frac{1}{2}\mathbb{E}\int_0^T \Big| G(t)^\top p(t) + D_0(t)^\top q(t) + \sum_{j=1}^{d} D_j(t)^\top r_j(t)\Big|^2 \mathrm{d}t + \langle x_0,\ q(0)\rangle + \langle y_0,\ p_0\rangle \\ &\equiv \frac{1}{2}\langle \Lambda p_0,\ p_0\rangle + \langle \Psi(0)^\top x_0 + y_0,\ p_0\rangle, \qquad p_0\in\mathbb{R}^{m'}, \end{aligned} \tag{4.15}$$

where $\mathbb{K}^*$ and $\mathbb{K}_0^*$ are given by (4.8), $(p(\cdot),q(\cdot),r(\cdot))$ solves FBSDE (4.5) with the initial value $p(0)=p_0$, and Λ defined by (4.12). Similarly, we pose the following

Problem (O′). For any $(x_0,y_0)\in\mathbb{R}^{n'}\times\mathbb{R}^{m'}$, find a $\bar p_0\in\mathbb{R}^{m'}$ such that

$$f(\bar p_0;x_0,y_0) = \inf_{p_0\in\mathbb{R}^{m'}} f(p_0;x_0,y_0).$$

Similar with that in Section 2, we also pose a family of norm optimal control problems. For any $(x_0, y_0) \in \mathbb{R}^{n'} \times \mathbb{R}^{m'}$, we introduce an admissible control set

$$\mathcal{U}(x_0, y_0) \equiv \Big\{ u(\cdot) \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^{k'}) \ \Big|\ y(0; x_0, u(\cdot)) = y_0 \Big\},$$

and pose the following

Problem (N′). For any $(x_0, y_0) \in \mathbb{R}^{n'} \times \mathbb{R}^{m'}$, find a $\bar{u}(\cdot) \in \mathcal{U}(x_0, y_0)$ such that

$$\|\bar{u}(\cdot)\|_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^{k'})} = \inf_{u(\cdot)\in\mathcal{U}(x_0,y_0)} \|u(\cdot)\|_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^{k'})}.$$

As a counterpart of Theorem 2.5, we present the following result. We notice that, besides the first four equivalent statements corresponding to the ones appearing in Theorem 2.5, we also have the fifth equivalent statement which is on the Gramian matrix Λ.

Theorem 4.6. *The following statements are equivalent:*
(i′). *The original system (4.4) is exactly controllable;*
(ii′). *The observability inequality (4.9) for the adjoint equation (4.5) holds true;*
(iii′). *Problem (O′) admits a unique solution $\bar{p}_0 \in \mathbb{R}^{m'}$;*
(iv′). *Problem (N′) admits a unique solution $\bar{u}(\cdot) \in \mathcal{U}(x_0, y_0)$;*
(v′). *The matrix Λ defined by (4.12) is non-singular.*
Moreover, in this case, the optimal solution of Problem (O′) is given by

$$\bar{p}_0 = -\Lambda^{-1}(\Psi(0)^\top x_0 + y_0), \tag{4.16}$$

and the minimum of function $f(\cdot; x_0, y_0)$ is given by

$$f(\bar{p}_0; x_0, y_0) = -\frac{1}{2}\Big\langle \Lambda^{-1}\big(\Psi(0)^\top x_0 + y_0\big),\ \big(\Psi(0)^\top x_0 + y_0\big)\Big\rangle, \tag{4.17}$$

where $(\Phi(\cdot), \Psi(\cdot), \Gamma(\cdot))$ is the solution to (4.11). Furthermore, the unique norm optimal control to Problem (N′) is given by

$$\bar{u}(\cdot) = -\mathbb{K}^*\bar{p}_0 \equiv \Big[G(\cdot)^\top\Phi(\cdot) + D_0(\cdot)^\top\Psi(\cdot) + \sum_{j=1}^d D_j(\cdot)^\top\Gamma_j(\cdot)\Big]\bar{p}_0, \tag{4.18}$$

where $\bar{p}_0$ is given by (4.16), and the minimal norm is

$$\|\bar{u}(\cdot)\|_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^{k'})} = \sqrt{\Big\langle \Lambda^{-1}\big(\Psi(0)^\top x_0 + y_0\big),\ \big(\Psi(0)^\top x_0 + y_0\big)\Big\rangle}. \tag{4.19}$$

Proof. (i′) $\Rightarrow$ (ii′). The proof is similar with the one of Lemma 2.6. Here we would like to omit it.

(ii′) $\Rightarrow$ (v′). We notice that

$$\|\mathbb{K}^*p_0\|_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^{k'})} = \mathbb{E}\int_0^T \Big|\Big(G(t)^\top\Phi(t) + D_0(t)^\top\Psi(t) + \sum_{j=1}^d D_j(t)^\top\Gamma_j(t)\Big)p_0\Big|^2 \mathrm{d}t = \langle \Lambda p_0,\ p_0\rangle. \tag{4.20}$$

By the observability inequality (4.9), there exists a constant $\delta > 0$ such that

$$\langle \Lambda p_0,\ p_0\rangle \geq \delta |p_0|^2, \quad \forall\ p_0 \in \mathbb{R}^{m'},$$

which means Λ is a positive definite matrix. Consequently, it is non-singular.

(v′) ⇒ (iii′). Since Λ is non-singular and positive semi-definite, then it is positive definite. From the third representation (see (4.15)) of function $f(\cdot;x_0,y_0)$, we calculate

$$\begin{aligned} f(p_0;x_0,y_0) &= \frac{1}{2}\langle \Lambda p_0,\ p_0\rangle + \langle \Psi(0)^\top x_0 + y_0,\ p_0\rangle \\ &= \frac{1}{2}\Big\langle \Lambda^{-1}\Big[\Lambda p_0 + \big(\Psi(0)^\top x_0 + y_0\big)\Big],\ \Big[\Lambda p_0 + \big(\Psi(0)^\top x_0 + y_0\big)\Big]\Big\rangle \\ &\quad - \frac{1}{2}\Big\langle \Lambda^{-1}\big(\Psi(0)^\top x_0 + y_0\big),\ \big(\Psi(0)^\top x_0 + y_0\big)\Big\rangle. \end{aligned}$$

Therefore, Problem (O′) is uniquely solvable. Moreover, its unique optimal solution is given by (4.16), and the minimum of function $f(\cdot;x_0,y_0)$ is given by (4.17).

(iii′) ⇒ (i′). Now, for any given $(x_0,y_0)\in\mathbb{R}^{n'}\times\mathbb{R}^{m'}$, we look for an admissible control in $\mathcal{U}(x_0,y_0)$. For any $u(\cdot)\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^{k'})$, let $(x(\cdot;x_0,u(\cdot)),y(\cdot;x_0,u(\cdot)),z(\cdot;x_0,u(\cdot)))$ be the solution to (4.4). By means of the solution $(\Phi(\cdot),\Psi(\cdot),\Gamma(\cdot))$ to (4.11), the duality relation (4.6) can be rewritten as

$$\begin{aligned} &\mathbb{E}\int_0^T \Big\langle u(t),\ \Big[G(t)^\top\Phi(t) + D_0(t)^\top\Psi(t) + \sum_{j=1}^d D_j(t)^\top\Gamma_j(t)\Big]p_0\Big\rangle \mathrm{d}t \\ &\quad + \big\langle \Psi(0)^\top x_0 + y(0;x_0,u(\cdot)),\ p_0\big\rangle = 0, \quad \forall\, u(\cdot)\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^{k'}),\ \forall\, p_0\in\mathbb{R}^{m'}. \end{aligned} \tag{4.21}$$

Let $\bar p_0$ be the unique minimizer of Problem (O′), and $\bar u(\cdot)$ be given by (4.18). Selecting $u(\cdot)=\bar u(\cdot)$ in (4.21) and noticing the definition of Λ (see (4.12)), we have

$$\big\langle \Lambda\bar p_0 + \big(\Psi(0)^\top x_0 + y(0;x_0,\bar u(\cdot))\big),\ p_0\big\rangle = 0, \quad \forall\, p_0\in\mathbb{R}^{m'}.$$

Due to the arbitrariness of p_0, we have

$$\Lambda\bar p_0 + \big(\Psi(0)^\top x_0 + y(0;x_0,\bar u(\cdot))\big) = 0.$$

Noticing that $\bar p_0$ satisfies (4.16), *i.e.*, $\Lambda\bar p_0 + (\Psi(0)^\top x_0 + y_0) = 0$, we obtain

$$y(0;x_0,\bar u(\cdot)) = y_0.$$

We proved the exact controllability of system (4.4).

(iii′) ⇒ (iv′). From the proof of the previous part, $\bar u(\cdot)$ defined by (4.18) is an admissible control to Problem (N′), *i.e.*, $\bar u(\cdot)\in\mathcal{U}(x_0,y_0)$. By letting $p_0=\bar p_0$, for any $u(\cdot)\in\mathcal{U}(x_0,y_0)$, from (4.21), we have

$$\begin{aligned} &\mathbb{E}\int_0^T \Big\langle u(t),\ \Big[G(t)^\top\Phi(t) + D_0(t)^\top\Psi(t) + \sum_{j=1}^d D_j(t)^\top\Gamma_j(t)\Big]\bar p_0\Big\rangle \mathrm{d}t \\ &= -\big\langle \Psi(0)^\top x_0 + y_0,\ \bar p_0\big\rangle, \quad \forall\, u(\cdot)\in\mathcal{U}(x_0,y_0), \end{aligned}$$

i.e.,

$$\|\bar u(\cdot)\|^2_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^{k'})} = -\big\langle \Psi(0)^\top x_0 + y_0,\ \bar p_0\big\rangle = \langle u(\cdot),\ \bar u(\cdot)\rangle_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^{k'})}, \quad \forall\, u(\cdot)\in\mathcal{U}(x_0,y_0). \tag{4.22}$$

By Hölder's inequality, we have

$$\|\bar u(\cdot)\|_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^{k'})} \le \|u(\cdot)\|_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^{k'})}, \quad \forall\, u(\cdot)\in\mathcal{U}(x_0,y_0).$$

We get the optimality of $\bar u(\cdot)$ defined by (4.18). Moreover, (4.22) and (4.16) lead to the value (4.19) of minimal norm.

The uniqueness of the optimal control to Problem (N$'$) follows immediately from the classical parallelogram rule of the norm $\|\cdot\|_{L^2_{\mathbb{F}}(0,T;\mathbb{R}^{k'})}$.

(iv$'$) $\Rightarrow$ (i$'$). The proof is the same as the corresponding part of the proof of Theorem 2.5. We omit it. □

4.1. A special case

In order to describe in depth the results of Theorem 4.6, we would like to reduce it to a simple case, *i.e.* $n'=m'=d=1$ (noticing that in general $k'\neq 1$) and all the coefficients are deterministic and time-invariant. In this case, we can solve FBSDE (4.11) explicitly.

Firstly, by the classical theory of SDEs, it is well known that the process $\Phi(\cdot)$ can be given by

$$\Phi(t)=\exp\Big\{\Big(-R-\frac{1}{2}S_1^2\Big)t-S_1W(t)\Big\},\quad t\in[0,T]. \tag{4.23}$$

Secondly, in order to solve the process $\Psi(\cdot)$, we introduce a family of SDEs parameterized by $t\in[0,T]$:

$$\begin{cases} \mathrm{d}\mathbb{X}(t,s)=C_0\mathbb{X}(t,s)\mathrm{d}s+C_1\mathbb{X}(t,s)\mathrm{d}W(s), & s\in[t,T],\\ \mathbb{X}(t,t)=1. \end{cases}$$

Applying Itô's formula to $\mathbb{X}(t,\cdot)\Psi(\cdot)$ on the interval $[t,T]$, we have

$$\Psi(t)=-M\mathbb{E}_t\Big[\mathbb{X}(t,T)\Phi(T)\Big]+F\int_t^T\mathbb{E}_t\Big[\mathbb{X}(t,s)\Phi(s)\Big]\mathrm{d}s, \tag{4.24}$$

where $\mathbb{E}_t[\cdot]=\mathbb{E}[\cdot|\mathcal{F}_t]$. Applying Itô's formula to $\mathbb{X}(t,\cdot)\Phi(\cdot)$, we have

$$\begin{cases} \mathrm{d}\mathbb{X}(t,s)\Phi(s)=(C_0-R-C_1S_1)\mathbb{X}(t,s)\Phi(s)\mathrm{d}s+(C_1-S_1)\mathbb{X}(t,s)\Phi(s)\mathrm{d}W(s), & s\in[t,T],\\ \mathbb{X}(t,t)\Phi(t)=\Phi(t). \end{cases}$$

Then,

$$\begin{cases} \dfrac{d}{\mathrm{d}s}\mathbb{E}_t\Big[\mathbb{X}(t,s)\Phi(s)\Big]=(C_0-R-C_1S_1)\mathbb{E}_t\Big[\mathbb{X}(t,s)\Phi(s)\Big], & s\in[t,T],\\ \mathbb{E}_t\Big[\mathbb{X}(t,t)\Phi(t)\Big]=\Phi(t). \end{cases}$$

Explicitly,

$$\mathbb{E}_t\Big[\mathbb{X}(t,s)\Phi(s)\Big]=\Phi(t)\exp\Big\{(C_0-R-C_1S_1)(s-t)\Big\},\quad s\in[t,T].$$

Substituting the above equation into (4.24), we obtain

$$\Psi(t)=\Phi(t)\rho(t),\quad t\in[0,T], \tag{4.25}$$

where $\Phi(\cdot)$ is given by (4.23) and we denote

$$\rho(t)=\begin{cases} \Big[\dfrac{F}{C_0-R-C_1S_1}-M\Big]\exp\Big\{(C_0-R-C_1S_1)(T-t)\Big\}-\dfrac{F}{C_0-R-C_1S_1}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\text{when } C_0-R-C_1S_1\neq 0,\\ -M+F(T-t),\qquad \text{when } C_0-R-C_1S_1=0. \end{cases} \tag{4.26}$$

Thirdly, we introduce some technique of Malliavin analysis to solve the process $\Gamma(\cdot)$ explicitly. By the result of [8, Proposition 5.3] (see also [17, 25]), the solution $(\Phi(\cdot),\Psi(\cdot),\Gamma(\cdot))$ to FBSDE (4.11) is Malliavin differentiable and $D_\cdot\Psi(\cdot)$ provides a version of process $\Gamma(\cdot)$. Precisely, for any $\theta\in[0,T]$,

$$\Gamma(\theta)=D_\theta\Psi(\theta), \tag{4.27}$$

where the Malliavin derivative processes $(D_\theta\Phi(\cdot),D_\theta\Psi(\cdot),D_\theta\Gamma(\cdot))$ satisfies the following FBSDE:

$$\begin{cases} \mathrm{d}D_\theta\Phi(t)=-RD_\theta\Phi(t)\mathrm{d}t-S_1D_\theta\Phi(t)\mathrm{d}W(t), \quad t\in[\theta,T],\\ \mathrm{d}D_\theta\Psi(t)=-\Big[FD_\theta\Phi(t)+C_0D_\theta\Psi(t)+C_1D_\theta\Gamma(t)\Big]\mathrm{d}t+D_\theta\Gamma(t)\mathrm{d}W(t), \quad t\in[\theta,T],\\ D_\theta\Phi(\theta)=-S_1\Phi(\theta), \quad D_\theta\Psi(T)=-MD_\theta\Phi(T). \end{cases} \tag{4.28}$$

Comparing with (4.11), we notice that the above FBSDE (4.28) just has a different initial pair $(\theta,-S_1\Phi(\theta))$. Therefore, by the same derivation, we have

$$D_\theta\Psi(t)=D_\theta\Phi(t)\rho(t), \quad t\in[\theta,T].$$

By virtue of (4.27) and the initial condition $D_\theta\Phi(\theta)=-S_1\Phi(\theta)$ (see (4.28)), we obtain

$$\Gamma(t)=-S_1\Phi(t)\rho(t), \quad t\in[0,T], \tag{4.29}$$

where $\Phi(\cdot)$ is given by (4.23) and $\rho(\cdot)$ is given by (4.26).

Proposition 4.7. *Let $n'=m'=d=1$ and all the coefficients are deterministic and time-invariant. System (4.4) is exactly controllable if and only if*

$$\begin{cases} G^\top-\big(D_0^\top-D_1^\top S_1\big)M\neq 0, \quad \text{when } C_0-R-C_1S_1\neq 0 \text{ and } \dfrac{F}{C_0-R-C_1S_1}=M,\\ \big|G^\top\big|^2+\big|D_0^\top-D_1^\top S_1\big|^2\neq 0, \quad \text{others.} \end{cases} \tag{4.30}$$

Moreover, in this case, the optimal solution to Problem (O′) is given by

$$\bar p_0=-\Lambda^{-1}\big(\rho(0)x_0+y_0\big), \tag{4.31}$$

and the minimum of function $f(\cdot;x_0,y_0)$ is given by

$$f(\bar p_0;x_0,y_0)=-\frac{1}{2}\Lambda^{-1}\big|\rho(0)x_0+y_0\big|^2, \tag{4.32}$$

where

$$\Lambda=\int_0^T\big|G^\top+\big(D_0^\top-D_1^\top S_1\big)\rho(t)\big|^2\exp\{(-2R+|S_1|^2)t\}\mathrm{d}t, \tag{4.33}$$

and $\rho(\cdot)$ is defined by (4.26). Furthermore, the unique norm optimal control to Problem (N′) is given by

$$\bar u(t)=-\Lambda^{-1}(\rho(0)x_0+y_0)\big[G^\top+\big(D_0^\top-D_1^\top S_1\big)\rho(t)\big]\Phi(t), \quad t\in[0,T], \tag{4.34}$$

where $\Phi(\cdot)$ is given by (4.23), and the minimal norm is

$$\|\bar u(\cdot)\|_{L^2_{\mathbb F}(0,T;\mathbb R^{k'})}=\Lambda^{-\frac{1}{2}}|\rho(0)x_0+y_0|. \tag{4.35}$$

Proof. By virtue of the explicit representation of $(\Phi(\cdot),\Psi(\cdot),\Gamma(\cdot))$ (see (4.23), (4.25) and (4.29)), we provide an explicit representation of Λ (which is defined by (4.12)) as follows:

$$\begin{aligned}\Lambda &= \int_0^T \big|G^\top + \big(D_0^\top - D_1^\top S_1\big)\rho(t)\big|^2 \mathbb{E}\big[|\Phi(t)|^2\big]\mathrm{d}t\\ &= \int_0^T \big|G^\top + \big(D_0^\top - D_1^\top S_1\big)\rho(t)\big|^2 \exp\{(-2R+|S_1|^2)t\}\mathrm{d}t,\end{aligned}$$

which is (4.33). By Theorem 4.6, system (4.4) is not exactly controllable if and only if $\Lambda=0$ which is also equivalent to

$$G^\top + \big(D_0^\top - D_1^\top S_1\big)\rho(t) = 0, \quad \forall\, t\in[0,T]. \tag{4.36}$$

When $C_0 - R - C_1S_1 \neq 0$ and $F/(C_0-R-C_1S_1)=M$, we have

$$\rho(t) \equiv -\frac{F}{C_0-R-C_1S_1} = -M.$$

Then (4.36) reads

$$G^\top - \big(D_0^\top - D_1^\top S_1\big)M = 0.$$

When $C_0-R-C_1S_1=0$ or $F/(C_0-R-C_1S_1)\neq M$, function $\rho(\cdot)$ is not a constant. Then (4.36) reads

$$\big|G^\top\big|^2 + \big|D_0^\top - D_1^\top S_1\big|^2 = 0.$$

We have proved (4.30). The rest conclusions can be proved by a straightforward calculation. □

In the following example, we would calculate the norm optimal control of Problem (N′) and simulate the state $(x(\cdot),y(\cdot))$.

Example 4.8. Suppose that the initial state $x_0=0$, $y_0=1$, and in (4.4), $d=T=1$, $R=2$, and the other coefficients are all equal to 1. Then (4.4) turns to

$$\begin{cases}\mathrm{d}x(t) = \Big[x(t)+u(t)\Big]\mathrm{d}t + \Big[x(t)+u(t)\Big]\mathrm{d}W(t), \quad t\in[0,1],\\ \mathrm{d}y(t) = \Big[x(t)+u(t)+2y(t)+z(t)\Big]\mathrm{d}t + z(t)\mathrm{d}W(t), \quad t\in[0,1],\\ y(1)=x(1).\end{cases} \tag{4.37}$$

By Proposition 4.7, we can easily check that (4.37) is exactly controllable. We now calculate the norm optimal control $\bar u(\cdot)$. By (4.23) and (4.26), we have

$$\Phi(t) = \exp\Big\{-\frac{5}{2}t - W(t)\Big\}, \quad \rho(t) = \frac{1}{2} - \frac{3}{2}\exp\{2(t-1)\}.$$

Hence, the Gramian matrix turns out

$$\Lambda = \int_0^1 \exp\{-3t\}\mathrm{d}t = \frac{1-\exp\{-3\}}{3}.$$

Therefore, we can get the norm optimal control to Problem (N′):

$$\bar u(t) = -\Lambda^{-1}(\rho(0)x_0+y_0)\Phi(t) = -\frac{3}{1-\exp\{-3\}}\exp\Big\{-\frac{5}{2}t - W(t)\Big\}.$$

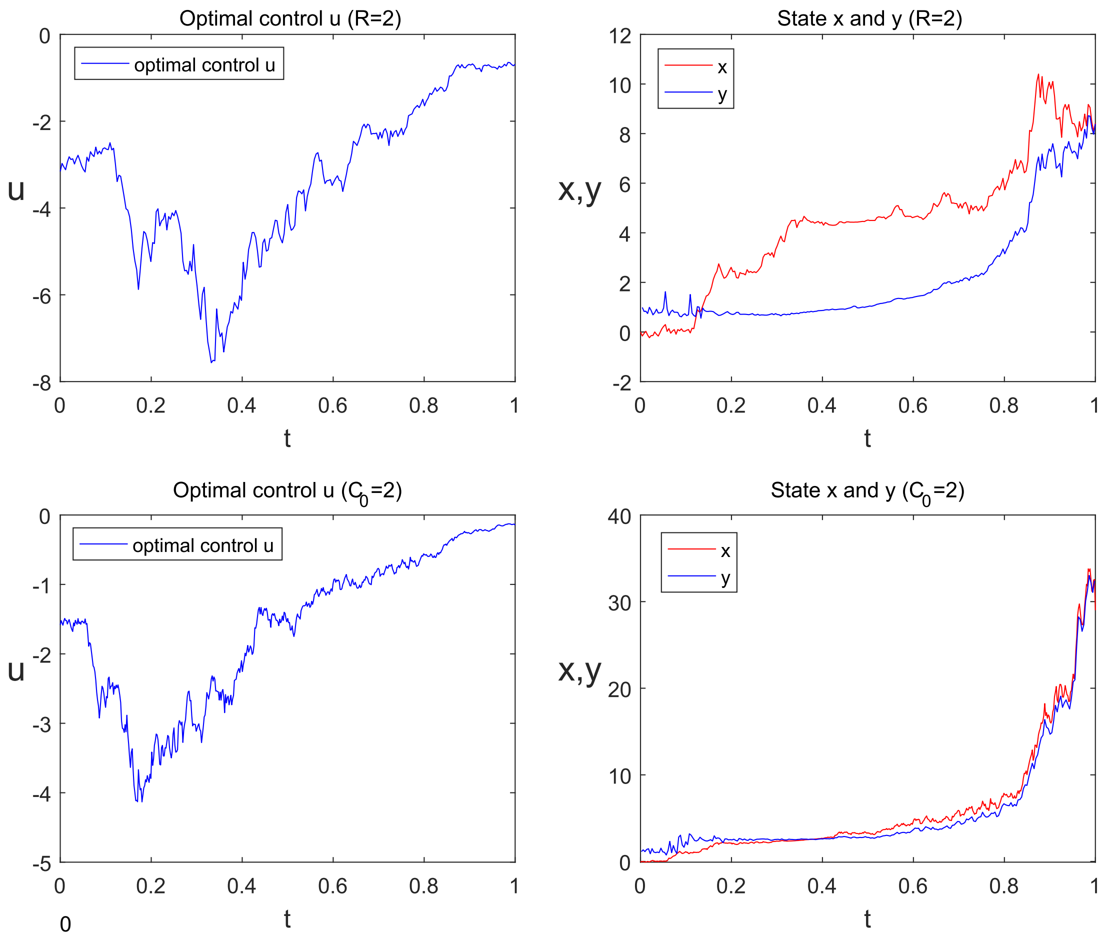

FIGURE 1. The norm optimal control u and the related state x and y are simulated in the first two figures in the case $R = 2$. The norm optimal control and the corresponding state in the case $C_0 = 2$ are simulated in the last two figures.

Similarly, when we suppose that $x_0 = 0$, $y_0 = 1$, and in (4.4), $d = T = 1$, $C_0 = 2$ and other coefficients are all 1, we obtain the corresponding norm optimal control to Problem (N′):

$$\bar{u}'(t) = -\frac{1}{1 - \exp\{-1\}} \exp\Big\{ -\frac{3}{2}t - W(t)\Big\}.$$

To illustrate intuitively, in the following figures, we simulate the norm optimal control u and the related state (x, y) in the above two cases (Case 1: $R = 2$; Case 2: $C_0 = 2$). In the first two figures, we simulate the optimal control and the related state, respectively, in the case $R = 2$. Particularly, in the second figure, firstly, we simulate the state x (solution to forward SDE with $u(\cdot) = \bar{u}(\cdot)$ and $x(0) = 0$), and we get the terminal value $y(1)$ $(= x(1))$ of the BSDE in (4.37), then we could simulate the state y. In the last two figures, we simulate the same issues in the case $C_0 = 2$. By the figures, we see that the initial state $y(0)$ is approximately equal to 1 (in Case 1, $|y(0) - 1| = 0.0094$, and in Case 2, $|y(0) - 1| = 0.1415$).

Appendix A. Transform the general case to FBSDE case satisfying Setting (FB)

In this appendix, we consider the problem how to transform the general situation on the H-partial controllability of system (2.1) to satisfy Setting (FB). Our analysis will be under some assumptions.

(H1). The dimension of Brownian motion $d = 1$. All the coefficients $A_0(\cdot)$, $A_1(\cdot)$, $B_0(\cdot)$, $B_1(\cdot)$ and H are deterministic and time-invariant.

(H2). $\mathrm{Rank}(H) = l$ and the last l column vectors of H are linearly independent. Moreover, $l < n$.

For convenience, we denote $m' = l$, $n' = n - m'$. Then we can rewrite $H = (H_1, H_2)$, where $H_1 \in \mathbb{R}^{m'\times n'}$ and $H_2 \in \mathbb{R}^{m'\times m'}$. Since H_2 is invertible, it is easy to check that, the H-partial controllability of system (2.1) is equivalent to the $(H_2^{-1}H)$-partial controllability. Denote

$$M = -H_2^{-1}H_1.$$

Clearly, we have $H_2^{-1}H = (-M, I_{m'})$ which coincides with the corresponding part of Setting (FB).

We also denote $X(\cdot)^\top = (x(\cdot)^\top, y(\cdot)^\top)$ where $x(\cdot)$ and $y(\cdot)$ take values in $\mathbb{R}^{n'}$ and $\mathbb{R}^{m'}$ respectively. Denote

$$A_0 = \begin{pmatrix} A_0^{11} & A_0^{12} \\ A_0^{21} & A_0^{22} \end{pmatrix}, \quad A_1 = \begin{pmatrix} A_1^{11} & A_1^{12} \\ A_1^{21} & A_1^{22} \end{pmatrix}, \quad B_0 = \begin{pmatrix} B_0^{1} \\ B_0^{2} \end{pmatrix}, \quad B_1 = \begin{pmatrix} B_1^{1} \\ B_1^{2} \end{pmatrix}.$$

With the above notations, we can write system (2.1) as follows (here and after, the argument t is suppressed):

$$\begin{cases} \mathrm{d}x = \Big[A_0^{11}x + A_0^{12}y + B_0^1 v\Big]\mathrm{d}t + \Big[A_1^{11}x + A_1^{12}y + B_1^1 v\Big]\mathrm{d}W, \\ \mathrm{d}y = \Big[A_0^{21}x + A_0^{22}y + B_0^2 v\Big]\mathrm{d}t + \Big[A_1^{21}x + A_1^{22}y + B_1^2 v\Big]\mathrm{d}W. \end{cases}$$

Now, we introduce the following hypothesis:

(H3). $\mathrm{Rank}(B_1^2) = m'$. Moreover, $m' < k$.

Under the assumption (H3), we know that there exists an invertible matrix $N \in \mathbb{R}^{k\times k}$ such that $B_1^2 N = (I_{m'}, O)$. Setting

$$N^{-1}v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^{m'} \times \mathbb{R}^{k-m'}, \quad B_0^1 N = (B_0^{11}, B_0^{12}), \quad B_0^2 N = (B_0^{21}, B_0^{22}), \quad B_1^1 N = (B_1^{11}, B_1^{12}),$$

we get

$$\begin{cases} \mathrm{d}x = \Big[A_0^{11}x + A_0^{12}y + B_0^{11}v_1 + B_0^{12}v_2\Big]\mathrm{d}t + \Big[A_1^{11}x + A_1^{12}y + B_1^{11}v_1 + B_1^{12}v_2\Big]\mathrm{d}W, \\ \mathrm{d}y = \Big[A_0^{21}x + A_0^{22}y + B_0^{21}v_1 + B_0^{22}v_2\Big]\mathrm{d}t + \Big[A_1^{21}x + A_1^{22}y + v_1\Big]\mathrm{d}W. \end{cases}$$

Denote $z = A_1^{21}x + A_1^{22}y + v_1$. By the above equation, one has

$$\begin{cases} \mathrm{d}x = \Big[(A_0^{11} - B_0^{11}A_1^{21})x + (A_0^{12} - B_0^{11}A_1^{22})y + B_0^{11}z + B_0^{12}v_2\Big]\mathrm{d}t \\ \qquad\quad + \Big[(A_1^{11} - B_1^{11}A_1^{21})x + (A_1^{12} - B_1^{11}A_1^{22})y + B_1^{11}z + B_1^{12}v_2\Big]\mathrm{d}W, \\ \mathrm{d}y = \Big[(A_0^{21} - B_0^{21}A_1^{21})x + (A_0^{22} - B_0^{21}A_1^{22})y + B_0^{21}z + B_0^{22}v_2\Big]\mathrm{d}t + z\mathrm{d}W. \end{cases}$$

For further processing, we introduce

(H4). $k > 2n' + m'$.

We denote

$$v_2=\begin{pmatrix} v_{21}\\ u\end{pmatrix}\in\mathbb{R}^{2n'}\times\mathbb{R}^{k-m'-2n'},\quad \begin{pmatrix} B_0^{12}\\ B_1^{12}\end{pmatrix}=\begin{pmatrix} B_0^{121} & B_0^{122}\\ B_1^{121} & B_1^{122}\end{pmatrix},\quad B_0^{22}=(B_0^{221},B_0^{222}),$$

where matrices $B_0^{121},B_1^{121}\in\mathbb{R}^{n'\times(2n')}$, $B_0^{122},B_1^{122}\in\mathbb{R}^{n'\times(k-m'-2n')}$, $B_0^{221}\in\mathbb{R}^{m'\times(2n')}$, and $B_0^{222}\in\mathbb{R}^{m'\times(k-m'-2n')}$.

(H5). The matrix $\begin{pmatrix} B_0^{121}\\ B_1^{121}\end{pmatrix}$ is invertible.

With the help of (H5), we choose

$$v_{21}=-\begin{pmatrix} B_0^{121}\\ B_1^{121}\end{pmatrix}^{-1}\begin{pmatrix} A_0^{12}-B_0^{11}A_1^{22} & B_0^{11}\\ A_1^{12}-B_1^{11}A_1^{22} & B_1^{11}\end{pmatrix}\begin{pmatrix} y\\ z\end{pmatrix}.$$

Then,

$$\begin{cases} \mathrm{d}x=\Big[(A_0^{11}-B_0^{11}A_1^{21})x+B_0^{122}u\Big]\mathrm{d}t+\Big[(A_1^{11}-B_1^{11}A_1^{21})x+B_1^{122}u\Big]\mathrm{d}W,\\ \mathrm{d}y=\Bigg\{(A_0^{21}-B_0^{21}A_1^{21})x+B_0^{222}u\\ \qquad+\left[A_0^{22}-B_0^{21}A_1^{22}-B_0^{221}\begin{pmatrix} B_0^{121}\\ B_1^{121}\end{pmatrix}^{-1}\begin{pmatrix} A_0^{12}-B_0^{11}A_1^{22}\\ A_1^{12}-B_1^{11}A_1^{22}\end{pmatrix}\right]y\\ \qquad+\left[B_0^{21}-B_0^{221}\begin{pmatrix} B_0^{121}\\ B_1^{121}\end{pmatrix}^{-1}\begin{pmatrix} B_0^{11}\\ B_1^{11}\end{pmatrix}\right]z\Bigg\}\mathrm{d}t+z dW. \end{cases}$$

Denote

$$\begin{aligned} &C_0=A_0^{11}-B_0^{11}A_1^{21},\qquad D_0=B_0^{122},\qquad C_1=A_1^{11}-B_1^{11}A_1^{21},\qquad D_1=B_1^{122},\\ &F=A_0^{21}-B_0^{21}A_1^{21},\qquad G=B_0^{222},\\ &R=A_0^{22}-B_0^{21}A_1^{22}-B_0^{221}\begin{pmatrix} B_0^{121}\\ B_1^{121}\end{pmatrix}^{-1}\begin{pmatrix} A_0^{12}-B_0^{11}A_1^{22}\\ A_1^{12}-B_1^{11}A_1^{22}\end{pmatrix},\\ &S_1=B_0^{21}-B_0^{221}\begin{pmatrix} B_0^{121}\\ B_1^{121}\end{pmatrix}^{-1}\begin{pmatrix} B_0^{11}\\ B_1^{11}\end{pmatrix}. \end{aligned}$$

Therefore, by the above analysis, system (2.1) can be written as the form in Setting (FB):

$$\begin{aligned} \mathrm{d}\begin{pmatrix} x\\ y\end{pmatrix}=&\left[\begin{pmatrix} C_0 & O\\ F & R\end{pmatrix}\begin{pmatrix} x\\ y\end{pmatrix}+\begin{pmatrix} D_0 & O\\ G & S_1\end{pmatrix}\begin{pmatrix} u\\ z\end{pmatrix}\right]\mathrm{d}t\\ &+\left[\begin{pmatrix} C_1 & O\\ O & O\end{pmatrix}\begin{pmatrix} x\\ y\end{pmatrix}+\begin{pmatrix} D_1 & O\\ O & I_{m'}\end{pmatrix}\begin{pmatrix} u\\ z\end{pmatrix}\right]\mathrm{d}W. \end{aligned}\tag{A.1}$$

To sum up, we give the following result.

Theorem A.1. *Under the assumptions (H1)–(H5), system* (2.1) *is H-partially controllable on the time interval $[0,T]$ if and only if system* (A.1) *is $(H_2^{-1}H)$-partially controllable on the time interval $[0,T]$.*

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