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ASYMPTOTIC BEHAVIOR OF THE $W^{1/q,q}$-NORM OF MOLLIFIED BV FUNCTIONS AND APPLICATIONS TO SINGULAR PERTURBATION PROBLEMS

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Abstract. Motivated by results of Figalli and Jerison [*J. Funct. Anal.* **266** (2014) 1685–1701] and Hernández [*Pure Appl. Funct. Anal.*, Preprint https://arxiv.org/abs/1709.08262 (2017)], we prove the following formula:

$$\lim_{\varepsilon\to 0^+}\frac{1}{|\ln\varepsilon|}\big\|\eta_\varepsilon * u\big\|^q_{W^{1/q,q}(\Omega)} = C_0\int_{J_u}\Big|u^+(x)-u^-(x)\Big|^q d\mathcal{H}^{N-1}(x),$$

where $\Omega\subset\mathbb{R}^N$ is a regular domain, $u\in BV(\Omega)\cap L^\infty(\Omega)$, $q>1$ and $\eta_\varepsilon(z)=\varepsilon^{-N}\eta(z/\varepsilon)$ is a smooth mollifier. In addition, we apply the above formula to the study of certain singular perturbation problems.

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1. Introduction

Figalli and Jerison found in [12] a relationship between the perimeter of a set and a fractional Sobolev norm of its characteristic function. More precisely, for the mollifying kernel $\eta_\varepsilon(z)=\varepsilon^{-N}\eta(z/\varepsilon)$, where $\eta(z)$ denotes the standard Gaussian in $\mathbb{R}^N$, they showed that there exist constants $C_1>0$ and $C_2>0$ such that for every set $A\subset\mathbb{R}^N$ of finite perimeter $P(A)$ we have

$$C_1P(A)\le\liminf_{\varepsilon\to 0^+}\frac{1}{|\ln\varepsilon|}\big\|\eta_\varepsilon*\chi_A\big\|^2_{H^{1/2}(\mathbb{R}^N)}\le\limsup_{\varepsilon\to 0^+}\frac{1}{|\ln\varepsilon|}\big\|\eta_\varepsilon*\chi_A\big\|^2_{H^{1/2}(\mathbb{R}^N)}\le C_2P(A), \tag{1.1}$$

where χ_A is the characteristic function of A. More recently, Hernández [13] improved this result by showing that there exist a constant $C_0>0$ such that for every $u\in BV(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ we have

$$\lim_{\varepsilon\to 0^+}\frac{1}{|\ln\varepsilon|}\big\|\eta_\varepsilon*u\big\|^2_{H^{1/2}(\mathbb{R}^N)} = C_0\int_{J_u}\Big|u^+(x)-u^-(x)\Big|^2 d\mathcal{H}^{N-1}(x), \tag{1.2}$$

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(See Def. A.2 in the appendix for the definitions of the jump set J_u and the approximate one-side limits u^+, u^- of a BV-function).

In our main result, Theorem 1.1 below, we generalize formula (1.2) in several aspects:

- We allow a general mollifying kernel $\eta \in W^{1,1}(\mathbb{R}^N, \mathbb{R})$ (not only the Gaussian as before),
- We allow a general domain $\Omega \subset \mathbb{R}^N$, of certain regularity, while previous results required $\Omega = \mathbb{R}^N$,
- We treat the $W^{1/q,q}(\Omega)$-norm for any $q > 1$, while previous results were restricted to the case $q = 2$.

Recall that the Gagliardo seminorm $\|u\|_{W^{1/q,q}(\Omega,\mathbb{R}^d)}$ is given by

$$\|u\|_{W^{1/q,q}(\Omega,\mathbb{R}^d)} := \left(\int_\Omega \left(\int_\Omega \frac{\big|u(x) - u(y)\big|^q}{|x-y|^{N+1}} \mathrm{d}y \right) \mathrm{d}x \right)^{\frac{1}{q}}. \tag{1.3}$$

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $u \in BV(\mathbb{R}^N, \mathbb{R}^d) \cap L^\infty(\mathbb{R}^N, \mathbb{R}^d)$ be such that $\|Du\|(\partial\Omega) = 0$. For $\eta \in W^{1,1}(\mathbb{R}^N, \mathbb{R})$, every $x \in \mathbb{R}^N$ and every $\varepsilon > 0$ define*

$$u_\varepsilon(x) := \frac{1}{\varepsilon^N} \int_{\mathbb{R}^N} \eta\Big(\frac{y-x}{\varepsilon}\Big) u(y) \mathrm{d}y = (\eta_\varepsilon * u)(x). \tag{1.4}$$

Then, for any $q > 1$ we have

$$\lim_{\varepsilon \to 0^+} \frac{1}{|\ln \varepsilon|} \|u_\varepsilon\|^q_{W^{1/q,q}(\Omega,\mathbb{R}^d)} = 2 \Big| \int_{\mathbb{R}^N} \eta(z) \mathrm{d}z \Big|^q \left(\int_{\mathbb{R}^{N-1}} \frac{\mathrm{d}v}{\big(\sqrt{1+|v|^2}\big)^{N+1}} \right) \int_{J_u \cap \Omega} \Big| u^+(x) - u^-(x) \Big|^q \mathrm{d}\mathcal{H}^{\mathrm{N}-1}(\mathrm{x}). \tag{1.5}$$

In [18] we showed a result of the same type, but involving a different Sobolev norm, in which the same right hand side as in (1.2) appears. More precisely, we showed that for every radial $\eta \in C_c^\infty(\mathbb{R}^N, \mathbb{R})$ there exists a constant $C = C_\eta > 0$ such that for every $u \in BV(\Omega, \mathbb{R}^d) \cap L^\infty(\Omega, \mathbb{R}^d)$ we have

$$\lim_{\varepsilon \to 0^+} \varepsilon \big\|\eta_\varepsilon * u\big\|^2_{H^1(\Omega)} = C_\eta \int_{J_u} \Big| u^+(x) - u^-(x) \Big|^2 \mathrm{d}\mathcal{H}^{N-1}(x). \tag{1.6}$$

We mention also another related result of us from [19] where the "jump part" of the gradient appears in the limit. In fact, we showed that for any open $\Omega \subset \mathbb{R}^N$ with Lipschitz bounded boundary and every $u \in BV(\Omega, \mathbb{R}^d) \cap L^\infty(\Omega, \mathbb{R}^d)$ and $q > 1$ we have

$$\lim_{\varepsilon \to 0^+} \int\limits_\Omega \int\limits_{B_\varepsilon(x)\cap\Omega} \frac{1}{\varepsilon^N} \frac{\big|u(y) - u(x)\big|^q}{|y-x|} \mathrm{d}y \mathrm{d}x = C_N \int_{J_u} \Big| u^+(x) - u^-(x) \Big|^q \mathrm{d}\mathcal{H}^{N-1}(x), \tag{1.7}$$

for some (explicit) constant $C_N > 0$. The study of the limit in (1.7) is motivated by a special case of the so called "BBM formula" of Bourgain, Brezis and Mironescu [8] (see also Dávila [10]) in which the denominator on the left hand side is $|x-y|^q$, and the limit obtained is then different (see also [4, 20] for the relation of "BBM formula" to the concept of the Γ-limit and see [9, 11] for the properties of the integral energy in the "BBM formula", where u is the characteristic function of a set).

We apply Theorem 1.1 in order to prove an upper bound, in the limit $\varepsilon \to 0^+$, for the following singular perturbation functionals with differential constraints:

(i)

$$E_\varepsilon^{(1)}(v) := \begin{cases} \frac{1}{|\ln\varepsilon|}\|v\|^q_{W^{1/q,q}(\Omega,\mathbb{R}^d)} + \frac{1}{\varepsilon}\int_\Omega W\big(v,x\big)\mathrm{d}x & \text{if} \quad A\cdot\nabla v = 0 \\ +\infty & \text{otherwise,} \end{cases} \tag{1.8}$$

for $v:\Omega\to\mathbb{R}^d$;

(ii)

$$E_\varepsilon^{(2)}(v) := \begin{cases} \frac{1}{|\ln\varepsilon|}\Big(\|v\|^q_{W^{1/q,q}(\mathbb{R}^N,\mathbb{R}^d)} - \|v\|^q_{W^{1/q,q}(\mathbb{R}^N\setminus\overline{\Omega},\mathbb{R}^d)}\Big) + \frac{1}{\varepsilon}\int_\Omega W\big(v,x\big)\mathrm{d}x & \text{if} \quad A\cdot\nabla v = 0 \\ +\infty & \text{otherwise,} \end{cases} \tag{1.9}$$

for $v:\mathbb{R}^N\to\mathbb{R}^d$.

In both cases $A:\mathbb{R}^{d\times N}\to\mathbb{R}^l$ is a linear operator (possibly trivial). The most important particular cases are the following:

(a) $A\equiv 0$ (*i.e.*, without any prescribed differential constraint),
(b) $d=N$, $l=N^2$ and $A\cdot\nabla v\equiv \operatorname{curl} v := \big\{\partial_k v_j - \partial_j v_k\big\}_{1\le k,j\le N}$,
(c) $l=d$ and $A\cdot\nabla v\equiv \operatorname{div} v$.

The Γ-limit of the functional (1.8) in the L^p-topology when $A\equiv 0$, $q=2$, $N=1$ and W is a double-well potential was found by Alberti, Bouchitté and Seppecher [1]. The result was generalized to any dimension $N\ge 1$, for the functional (1.9), by Savin and Valdinoci [21]. The novelty in our second theorem is that it provides an upper bound for energies (1.8) and (1.9) in the case of a general W (not only for the double-well one) and general $q>1$, with or without differential constraints. We hope these upper bounds will allow in the future to find the Γ-limits of these functionals in some special cases (see **(**)** of Rem. A.7 in Sect. A.2 of the appendix).

Theorem 1.2. *Let $\Omega\subset\mathbb{R}^N$ be an open set and let $W:\mathbb{R}^d\times\mathbb{R}^N\to\mathbb{R}$ be a Borel measurable nonnegative function, continuous and continuously differentiable w.r.t. the first argument, such that $W(0,\cdot)\in L^1(\Omega,\mathbb{R})$. Assume further that for every $D>0$ there exists $C:=C_D>0$ such that*

$$\big|\nabla_b W(b,x)\big| \le C_D \qquad \forall x\in\mathbb{R}^N,\ \forall b\in B_D(0). \tag{1.10}$$

Let $u\in BV(\mathbb{R}^N,\mathbb{R}^d)\cap L^\infty(\mathbb{R}^N,\mathbb{R}^d)$ be such that $W\big(u(x),x\big)=0$ a.e. in Ω, $\|Du\|(\partial\Omega)=0$, and $A\cdot Du=0$ in $\mathbb{R}^N$, where $A:\mathbb{R}^{d\times N}\to\mathbb{R}^l$ is a prescribed linear operator (possibly trivial). Then, for any $q>1$ there exists a sequence of functions $\big\{\psi_\varepsilon\big\}_{\varepsilon>0}\subset C^\infty(\mathbb{R}^N,\mathbb{R}^d)\cap W^{1,1}(\mathbb{R}^N,\mathbb{R}^d)\cap W^{1,\infty}(\mathbb{R}^N,\mathbb{R}^d)$ such that $A\cdot D\psi_\varepsilon=0$ in $\mathbb{R}^N$, $\psi_\varepsilon(x)\to u(x)$ strongly in $L^p(\mathbb{R}^N,\mathbb{R}^d)$ for every $p\ge 1$, and

$$\begin{aligned}
&\limsup_{\varepsilon\to 0^+} E_\varepsilon^{(2)}(\psi_\varepsilon) \\
&\qquad := \limsup_{\varepsilon\to 0^+}\left(\frac{1}{|\ln\varepsilon|}\Big(\|\psi_\varepsilon\|^q_{W^{1/q,q}(\mathbb{R}^N,\mathbb{R}^d)} - \|\psi_\varepsilon\|^q_{W^{1/q,q}(\mathbb{R}^N\setminus\overline{\Omega},\mathbb{R}^d)}\Big) + \frac{1}{\varepsilon}\int_\Omega W\Big(\psi_\varepsilon(x),x\Big)\mathrm{d}x\right) \\
&\qquad = \limsup_{\varepsilon\to 0^+} E_\varepsilon^{(1)}(\psi_\varepsilon) := \limsup_{\varepsilon\to 0^+}\left(\frac{1}{|\ln\varepsilon|}\|\psi_\varepsilon\|^q_{W^{1/q,q}(\Omega,\mathbb{R}^d)} + \frac{1}{\varepsilon}\int_\Omega W\Big(\psi_\varepsilon(x),x\Big)\mathrm{d}x\right) \\
&\qquad = \left(\int_{\mathbb{R}^{N-1}}\frac{2}{\big(\sqrt{1+|v|^2}\big)^{N+1}}\mathrm{d}v\right)\int_{J_u\cap\Omega}\big|u^+(y)-u^-(y)\big|^q \mathrm{d}\mathcal{H}^{N-1}(y).
\end{aligned} \tag{1.11}$$

Moreover, in the case $A \equiv 0$ we can choose ψ_ε to satisfy also

$$\int_\Omega \psi_\varepsilon(x)\mathrm{d}x = \int_\Omega u(x)\mathrm{d}x \qquad \forall \varepsilon > 0. \tag{1.12}$$

Note that the functional (1.8) resembles the energy functional in the following singular perturbation problem:

$$\hat{E}_\varepsilon(v) := \begin{cases} \varepsilon^{q-1}\|v\|^q_{W^{1,q}(\Omega,\mathbb{R}^d)} + \frac{1}{\varepsilon}\int_\Omega W\big(v,x\big)\mathrm{d}x & \text{if} \quad A\cdot\nabla v = 0 \\ +\infty \qquad \text{otherwise,} \end{cases} \tag{1.13}$$

that attracted a lot of attention by many authors, starting from Modica and Mortola [15], Modica [14], Sternberg [22] and others, who studied the basic special case of (1.13) with $A \equiv 0$, $q = 2$ and W being a double-well potential. The Γ-limit of (1.13) with $A \equiv 0$, $q = 2$ and a general $W \in C^0$ that does not depend on x, was found by Ambrosio in [2]. As an example with a nontrivial differential constraint we mention the Aviles-Giga functional, that appears in various applications. It is defined for scalar functions ψ by

$$\tilde{E}_\varepsilon(\psi) := \int_\Omega \left\{\varepsilon|\nabla^2\psi|^2 + \frac{1}{\varepsilon}\Big(1-|\nabla\psi|^2\Big)^2\right\}\mathrm{d}x \qquad \text{(see [3, 6, 7])}, \tag{1.14}$$

and the objective is to study the Γ-limit, as $\varepsilon \to 0^+$. This can be seen as a special case of (1.13) if we set $v := \nabla\psi$ and let $A\cdot\nabla v \equiv \operatorname{curl} v$, $q = 2$ and $W(v,x) = (1-|v|^2)^2$.

Unfortunately, the upper bound found in Theorem 1.2 is not sharp in the most general case with a nontrivial prescribed differential constraint. For example, in the particular case of (1.8) with $N = 2$, $A\cdot\nabla v \equiv \operatorname{curl} v$, $q > 3$ and $W(v,x) = (1-|v|^2)^2$, the functional on the right hand side of (1.11) is not lower semicontinuous, hence cannot be the Γ-limit (see [3]). However, we still hope that the result of the above theorem could provide the sharp upper bound in some cases, in particular when $A = 0$. Indeed, the Γ-limit, computed in [1] for the special case of (1.8) with $A \equiv 0$, $q = 2$, $N = 1$ and W being a double well potential, coincides with the upper bound found in Theorem 1.2. Moreover, since the functional in (1.9) is superior to the functional in (1.8), the Γ-limit, found in [21] (see also [17]) for the energy (1.9) in any dimension $N \geq 1$ with $A \equiv 0$, $q = 2$ and W being a double well potential, coincides again with our upper bound.

The paper is organized as follows. In Section 2 we prove our main results. For the convenience of the reader, we recall in the appendix, Section A.1, some known results on BV functions, needed for the proofs, and in Section A.2 we recall some basic facts about Γ-convergence.

2. Proof of the main results

Proposition 2.1. *Let $q > 1$, $\Omega \subset \mathbb{R}^N$ be an open set and $u \in BV(\mathbb{R}^N,\mathbb{R}^d) \cap L^\infty(\mathbb{R}^N,\mathbb{R}^d)$ be such that $\|Du\|(\partial\Omega) = 0$. Let $\eta \in C_c^\infty(\mathbb{R}^N,\mathbb{R})$ and for every $x \in \mathbb{R}^N$ and every $\varepsilon > 0$ define*

$$u_\varepsilon(x) := \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N} \eta\Big(\frac{y-x}{\varepsilon}\Big)u(y)\mathrm{d}y = (\eta_\varepsilon * u)(x). \tag{2.1}$$

Then,

$$\lim_{\varepsilon\to 0^+} \frac{1}{|\ln\varepsilon|}\|u_\varepsilon\|^q_{W^{1/q,q}(\Omega,\mathbb{R}^d)} = 2\left|\int_{\mathbb{R}^N}\eta(z)\mathrm{d}z\right|^q\left(\int_{\mathbb{R}^{N-1}}\frac{1}{\big(\sqrt{1+|v|^2}\big)^{N+1}}\mathrm{d}v\right)\int_{J_u\cap\Omega}\Big|u^+(x)-u^-(x)\Big|^q\mathrm{d}\mathcal{H}^{N-1}(\mathrm{x}). \tag{2.2}$$

Proof. We start with some notations. For every $\boldsymbol{\nu} \in S^{N-1}$ and $x \in \mathbb{R}^N$ set

$$H_+(x,\boldsymbol{\nu}) = \{\xi \in \mathbb{R}^N \,:\, (\xi - x)\cdot\boldsymbol{\nu} > 0\}\,, \tag{2.3}$$
$$H_-(x,\boldsymbol{\nu}) = \{\xi \in \mathbb{R}^N \,:\, (\xi - x)\cdot\boldsymbol{\nu} < 0\} \tag{2.4}$$

and

$$H_0(\boldsymbol{\nu}) = \{\xi \in \mathbb{R}^N \,:\, \xi\cdot\boldsymbol{\nu} = 0\}\,. \tag{2.5}$$

Let $R > 0$ be such that $\operatorname{supp}\eta \subset B_R(0)$. For every $x \in \mathbb{R}^N$ and every $\varepsilon > 0$ we rewrite (2.1) as:

$$u_\varepsilon(x) := \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N} \eta\Big(\frac{y-x}{\varepsilon}\Big)u(y)\mathrm{d}y = \int_{\mathbb{R}^N} \eta(z)u(x+\varepsilon z)\mathrm{d}z = \int_{B_R(0)} \eta(z)u(x+\varepsilon z)\mathrm{d}z. \tag{2.6}$$

By (2.6) we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}u_\varepsilon(x) :&= -\frac{N}{\varepsilon^{N+1}}\int_{\mathbb{R}^N} \eta\Big(\frac{y-x}{\varepsilon}\Big)u(y)\mathrm{d}y - \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\frac{y-x}{\varepsilon^2}\cdot\nabla\eta\Big(\frac{y-x}{\varepsilon}\Big)u(y)\mathrm{d}y \\ &= -\frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\operatorname{div}_y\Big\{\eta\Big(\frac{y-x}{\varepsilon}\Big)\frac{y-x}{\varepsilon}\Big\}u(y)\mathrm{d}y = \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\eta\Big(\frac{y-x}{\varepsilon}\Big)\frac{y-x}{\varepsilon}\cdot\mathrm{d}\big[Du(y)\big].\end{aligned} \tag{2.7}$$

Moreover, by (1.3) we have

$$\begin{aligned}\|u_\varepsilon\|^q_{W^{1/q,q}} = \|u_\varepsilon\|^q_{W^{1/q,q}(\Omega,\mathbb{R}^d)} &= \int_{\mathbb{R}^N}\Bigg(\int_{\mathbb{R}^N}\frac{\big|u_\varepsilon(x)-u_\varepsilon(y)\big|^q}{|x-y|^{N+1}}\chi_\Omega(y)\mathrm{d}y\Bigg)\chi_\Omega(x)\mathrm{d}x \\ &= \int_{\mathbb{R}^N}\Bigg(\int_{\mathbb{R}^N}\frac{\big|u_\varepsilon(x+z)-u_\varepsilon(x)\big|^q}{|z|^{N+1}}\chi_\Omega(x+z)\chi_\Omega(x)\mathrm{d}z\Bigg)\mathrm{d}x,\end{aligned} \tag{2.8}$$

where

$$\chi_\Omega(x) := \begin{cases}1 & \forall x \in \Omega \\ 0 & \forall x \in \mathbb{R}^N\setminus\Omega\end{cases}\,. \tag{2.9}$$

Thus,

$$\frac{1}{-\ln\varepsilon}\|u_\varepsilon\|^q_{W^{1/q,q}} = -\frac{1}{\ln\varepsilon}\int_{\mathbb{R}^N}\Bigg(\int_{\mathbb{R}^N}\frac{\big|u_\varepsilon(x+z)-u_\varepsilon(x)\big|^q}{|z|^{N+1}}\chi_\Omega(x+z)\chi_\Omega(x)\mathrm{d}z\Bigg)\mathrm{d}x. \tag{2.10}$$

Since $-\ln\varepsilon \to +\infty$ as $\varepsilon \to 0^+$, applying L'Hôpital's rule to the expression in (2.10) yields

$$\begin{aligned}&\lim_{\varepsilon\to0^+}\frac{1}{-\ln\varepsilon}\|u_\varepsilon\|^q_{W^{1/q,q}} \\ &= -\lim_{\varepsilon\to0^+}\int\limits_{\mathbb{R}^N}\Bigg(\int\limits_{\mathbb{R}^N}\frac{\varepsilon}{|z|^{N+1}}\bigg(\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big(u_\varepsilon(x+z)-u_\varepsilon(x)\Big)\bigg)\cdot\nabla F_q\big(u_\varepsilon(x+z)-u_\varepsilon(x)\big)\chi_\Omega(x+z)\chi_\Omega(x)\mathrm{d}z\Bigg)\mathrm{d}x,\end{aligned} \tag{2.11}$$

where $F_q \in C^1(\mathbb{R}^d,\mathbb{R})$ is defined by

$$F_q(h) := |h|^q \qquad \forall h \in \mathbb{R}^d. \tag{2.12}$$

Thus, by (2.11), (2.6) and (2.7) we get

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+} \frac{1}{-\ln\varepsilon}\|u_\varepsilon\|^q_{W^{1/q,q}} \\
&\quad= -\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\varepsilon}{|z|^{N+1}}\left\{\frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\left(\eta\Big(\frac{y-(x+z)}{\varepsilon}\Big)\frac{y-(x+z)}{\varepsilon}-\eta\Big(\frac{y-x}{\varepsilon}\Big)\frac{y-x}{\varepsilon}\right)\cdot \mathrm{d}\big[Du(y)\big]\right\} \\
&\qquad\times\nabla F_q\left(\int_{\mathbb{R}^N}\eta(\xi)\Big(u(x+z+\varepsilon\xi)-u(x+\varepsilon\xi)\Big)\mathrm{d}\xi\right)\chi_\Omega(x+z)\chi_\Omega(x)\mathrm{d}z\mathrm{d}x \\
&\quad= -\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\varepsilon}{|z|^{N+1}}\frac{1}{\varepsilon^N}\left(\eta\Big(\frac{y-(x+z)}{\varepsilon}\Big)\frac{y-(x+z)}{\varepsilon}-\eta\Big(\frac{y-x}{\varepsilon}\Big)\frac{y-x}{\varepsilon}\right) \\
&\qquad\times\nabla F_q\left(\int_{\mathbb{R}^N}\eta(\xi)\Big(u(x+z+\varepsilon\xi)-u(x+\varepsilon\xi)\Big)\mathrm{d}\xi\right)\chi_\Omega(x+z)\chi_\Omega(x)\mathrm{d}z\mathrm{d}x\cdot \mathrm{d}\big[Du(y)\big].
\end{aligned} \tag{2.13}$$

Changing variable, $z/\varepsilon \to z$, in the integration on the right hand side of (2.13) gives

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+} \frac{1}{-\ln\varepsilon}\|u_\varepsilon\|^q_{W^{1/q,q}} \\
&\quad= -\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{1}{|z|^{N+1}}\frac{1}{\varepsilon^N}\left(\eta\Big(\frac{y-x}{\varepsilon}-z\Big)\Big(\frac{y-x}{\varepsilon}-z\Big)-\eta\Big(\frac{y-x}{\varepsilon}\Big)\frac{y-x}{\varepsilon}\right) \\
&\qquad\times\nabla F_q\left(\int_{\mathbb{R}^N}\eta(\xi)\Big(u(x+\varepsilon z+\varepsilon\xi)-u(x+\varepsilon\xi)\Big)\mathrm{d}\xi\right)\chi_\Omega(x+\varepsilon z)\chi_\Omega(x)\mathrm{d}z\mathrm{d}x\cdot d\big[Du(y)\big] \\
&\quad= -\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{1}{|z|^{N+1}}\Big(\eta(x-z)(x-z)-\eta(x)x\Big) \\
&\qquad\times\nabla F_q\left(\int_{\mathbb{R}^N}\eta(\xi)\Big(u(y+\varepsilon z+\varepsilon\xi-\varepsilon x)-u(y+\varepsilon\xi-\varepsilon x)\Big)\mathrm{d}\xi\right)\chi_\Omega(y-\varepsilon x+\varepsilon z)\chi_\Omega(y-\varepsilon x)\mathrm{d}z\mathrm{d}x\cdot d\big[Du(y)\big].
\end{aligned} \tag{2.14}$$

Therefore,

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+} \frac{1}{-\ln\varepsilon}\|u_\varepsilon\|^q_{W^{1/q,q}} \\
&\quad= -\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{1}{|z|^{N+1}}\Big(\eta(x-z)(x-z)-\eta(x)x\Big) \\
&\qquad\times\nabla F_q\left(\int_{\mathbb{R}^N}\Big(\eta(\xi-z)-\eta(\xi)\Big)u(y+\varepsilon\xi-\varepsilon x)\mathrm{d}\xi\right)\chi_\Omega(y-\varepsilon x+\varepsilon z)\chi_\Omega(y-\varepsilon x)\mathrm{d}z\mathrm{d}x\cdot d\big[Du(y)\big] \\
&\quad= -\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{1}{|z|^{N+1}}\Big(\eta(x-z)(x-z)-\eta(x)x\Big) \\
&\qquad\times\nabla F_q\left(\int_{\mathbb{R}^N}\Big(\eta(\xi+x-z)-\eta(\xi+x)\Big)u(y+\varepsilon\xi)\mathrm{d}\xi\right)\chi_\Omega(y-\varepsilon x+\varepsilon z)\chi_\Omega(y-\varepsilon x)\mathrm{d}z\mathrm{d}x\cdot d\big[Du(y)\big].
\end{aligned} \tag{2.15}$$

On the other hand, by (A.1) in the appendix, for every $x, z \in \mathbb{R}^N$ and $\mathcal{H}^{N-1}$-a.e. $y \in \mathbb{R}^N$ we have

$$\lim_{\varepsilon\to 0^+} \left\{ \int_{\mathbb{R}^N} \Big(\eta(\xi+x-z)-\eta(\xi+x)\Big)u(y+\varepsilon\xi)\mathrm{d}\xi \right\} \\ = u^+(y)\int_{H_+(0,\boldsymbol{\nu}(y))} \Big(\eta(\xi+x-z)-\eta(\xi+x)\Big)\mathrm{d}\xi + u^-(y)\int_{H_-(0,\boldsymbol{\nu}(y))} \Big(\eta(\xi+x-z)-\eta(\xi+x)\Big)\mathrm{d}\xi, \tag{2.16}$$

with $H_\pm(x,\boldsymbol{\nu})$ as defined in (2.3) and (2.4). Thus, since $\|Du\|(\partial\Omega)=0$, by (2.16) and the Dominated Convergence Theorem we obtain:

$$\begin{aligned} &\lim_{\varepsilon\to 0^+} \frac{1}{-\ln\varepsilon}\|u_\varepsilon\|^q_{W^{1/q,q}} \\ &= -\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{1}{|z|^{N+1}}\Big(\eta(x-z)(x-z)-\eta(x)x\Big)\nabla F_q\Bigg(u^+(y)\int\limits_{H_+(0,\boldsymbol{\nu}(y))}\Big(\eta(\xi+x-z)-\eta(\xi+x)\Big)\mathrm{d}\xi \\ &\qquad + u^-(y)\int\limits_{H_-(0,\boldsymbol{\nu}(y))}\Big(\eta(\xi+x-z)-\eta(\xi+x)\Big)\mathrm{d}\xi\Bigg)\chi^2_\Omega(y)\mathrm{d}z\mathrm{d}x\cdot d\big[Du(y)\big] \\ &= -\int\limits_{\Omega}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{1}{|z|^{N+1}}\Big(\eta(x-z)(x-z)-\eta(x)x\Big)\nabla F_q\Bigg(u^+(y)\int\limits_{H_+(0,\boldsymbol{\nu}(y))}\Big(\eta(\xi+x-z)-\eta(\xi+x)\Big)\mathrm{d}\xi \\ &\qquad + u^-(y)\int\limits_{H_-(0,\boldsymbol{\nu}(y))}\Big(\eta(\xi+x-z)-\eta(\xi+x)\Big)\mathrm{d}\xi\Bigg)\mathrm{d}z\mathrm{d}x\cdot d\big[Du(y)\big]. \end{aligned} \tag{2.17}$$

It follows that

$$\begin{aligned} \lim_{\varepsilon\to 0^+} \frac{1}{-\ln\varepsilon}\|u_\varepsilon\|^q_{W^{1/q,q}} &= -\int_\Omega\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{1}{|z|^{N+1}}\Big(\eta(x-z)(x-z)-\eta(x)x\Big) \\ &\times\nabla F_q\Bigg(\big(u^+(y)-u^-(y)\big)\int_{H_+(0,\boldsymbol{\nu}(y))}\Big(\eta(\xi+x-z)-\eta(\xi+x)\Big)\mathrm{d}\xi \\ &\qquad + u^-(y)\int_{\mathbb{R}^N}\Big(\eta(\xi+x-z)-\eta(\xi+x)\Big)\mathrm{d}\xi\Bigg)\mathrm{d}z\mathrm{d}x\cdot d\big[Du(y)\big] \\ &= -\int_\Omega\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{1}{|z|^{N+1}}\Big(\eta(x-z)(x-z)-\eta(x)x\Big) \\ \times\nabla F_q\Bigg(\big(u^+(y)-u^-(y)\big)&\int_{H_+(0,\boldsymbol{\nu}(y))}\Big(\eta(\xi+x-z)-\eta(\xi+x)\Big)\mathrm{d}\xi\Bigg)\mathrm{d}z\mathrm{d}x\cdot d\big[Du(y)\big], \end{aligned} \tag{2.18}$$

where we used in the last step the fact that $\int_{\mathbb{R}^N}\eta(\xi+x-z)\mathrm{d}\xi = \int_{\mathbb{R}^N}\eta(\xi+x)\mathrm{d}\xi$. Next, by (2.18) and (2.12) we infer that

$$\lim_{\varepsilon\to 0^+} \frac{1}{-\ln\varepsilon}\|u_\varepsilon\|^q_{W^{1/q,q}} = -\int_\Omega\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{1}{|z|^{N+1}}\Big(\eta(x-z)(x-z)-\eta(x)x\Big)$$

$$\times \nabla F_q\Bigg(\Big(u^+(y)-u^-(y)\Big)\Big(\int_{H_+(x-z,\boldsymbol{\nu}(y))}\eta(\xi)\mathrm{d}\xi-\int_{H_+(x,\boldsymbol{\nu}(y))}\eta(\xi)\mathrm{d}\xi\Big)\Bigg)\mathrm{d}z\mathrm{d}x\cdot d\big[Du(y)\big]$$
$$=\int_{J_u\cap\Omega}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{1}{|z|^{N+1}}\Big(\eta\big(x\big)x\cdot\boldsymbol{\nu}(y)-\eta\big(x-z\big)\big(x-z\big)\cdot\boldsymbol{\nu}(y)\Big)$$
$$\times\frac{\mathrm{d}G_q}{\mathrm{d}\rho}\Bigg(\int_{(x-z)\cdot\boldsymbol{\nu}(y)}^{x\cdot\boldsymbol{\nu}(y)}\int_{H_0(\boldsymbol{\nu}(y))}\eta(t\boldsymbol{\nu}(y)+\xi)\mathrm{d}\mathcal{H}^{\mathrm{N}-1}(\xi)\mathrm{d}t\Bigg)\mathrm{d}x\mathrm{d}z\big|\mathrm{u}^+(\mathrm{y})-\mathrm{u}^-(\mathrm{y})\big|^{\mathrm{q}}\mathrm{d}\mathcal{H}^{\mathrm{N}-1}(\mathrm{y}),\quad(2.19)$$

where $G_q(\rho)\in C^1(\mathbb{R},\mathbb{R})$ is defined by

$$G_q(\rho):=|\rho|^q\qquad\forall\rho\in\mathbb{R},\tag{2.20}$$

and $H_0(\boldsymbol{\nu})$ is defined in (2.5). Therefore,

$$\lim_{\varepsilon\to0^+}\frac{1}{-\ln\varepsilon}\|u_\varepsilon\|^q_{W^{1/q,q}}$$
$$=\int_{J_u\cap\Omega}\int_{\mathbb{R}^N}\int_{\mathbb{R}}\int_{H_0(\boldsymbol{\nu}(y))}\frac{1}{|z|^{N+1}}\Bigg(\eta\big(s\boldsymbol{\nu}(y)+\zeta\big)s-\eta\Big(\big(s-z\cdot\boldsymbol{\nu}(y)\big)\boldsymbol{\nu}(y)+\zeta\Big)\big(s-z\cdot\boldsymbol{\nu}(y)\big)\Bigg)$$
$$\times\frac{\mathrm{d}G_q}{\mathrm{d}\rho}\Bigg(\int_{s-z\cdot\boldsymbol{\nu}(y)}^{s}\int_{H_0(\boldsymbol{\nu}(y))}\eta(t\boldsymbol{\nu}(y)+\xi)\mathrm{d}\mathcal{H}^{\mathrm{N}-1}(\xi)\mathrm{dt}\Bigg)\mathrm{d}\mathcal{H}^{\mathrm{N}-1}(\zeta)\mathrm{dsdz}\big|\mathrm{u}^+(\mathrm{y})-\mathrm{u}^-(\mathrm{y})\big|^{\mathrm{q}}\mathrm{d}\mathcal{H}^{\mathrm{N}-1}(\mathrm{y})$$
$$=\int_{J_u\cap\Omega}\Bigg(\int_{\mathbb{R}^{N-1}}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{1}{\big(\sqrt{\tau^2+|w|^2}\big)^{N+1}}$$
$$\times\Bigg(\int_{H_0(\boldsymbol{\nu}(y))}\Big(\eta\big(s\boldsymbol{\nu}(y)+\zeta\big)s-\eta\Big(\big(s-\tau\big)\boldsymbol{\nu}(y)+\zeta\Big)\big(s-\tau\big)\Big)\mathrm{d}\mathcal{H}^{\mathrm{N}-1}(\zeta)\Bigg)$$
$$\times\frac{\mathrm{d}G_q}{\mathrm{d}\rho}\Big(\int_{s-\tau}^{s}\int_{H_0(\boldsymbol{\nu}(y))}\eta(t\boldsymbol{\nu}(y)+\xi)\mathrm{d}\mathcal{H}^{\mathrm{N}-1}(\xi)\mathrm{dt}\Big)\mathrm{d}\tau\mathrm{dsdw}\Bigg)\big|\mathrm{u}^+(\mathrm{y})-\mathrm{u}^-(\mathrm{y})\big|^{\mathrm{q}}\mathrm{d}\mathcal{H}^{\mathrm{N}-1}(\mathrm{y})\,.\quad(2.21)$$

Introducing the notation

$$\Lambda(y,a,b)=\int_a^b\int_{H_0(\boldsymbol{\nu}(y))}\eta(t\boldsymbol{\nu}(y)+\xi)\,\mathrm{d}\mathcal{H}^{\mathrm{N}-1}(\xi)\,\mathrm{dt}\tag{2.22}$$

allows us to rewrite (2.21) as

$$\lim_{\varepsilon\to0^+}\frac{1}{-\ln\varepsilon}\|u_\varepsilon\|^q_{W^{1/q,q}}$$
$$=\int_{J_u\cap\Omega}\Bigg\{\int_{\mathbb{R}^{N-1}}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{1}{\tau^2}\frac{1}{|\tau|^{N-1}}\frac{1}{\big(\sqrt{1+|w/|\tau||^2}\big)^{N+1}}$$
$$\times\Bigg(\int_{H_0(\boldsymbol{\nu}(y))}\Big(\eta\big(s\boldsymbol{\nu}(y)+\zeta\big)s-\eta\Big(\big(s-\tau\big)\boldsymbol{\nu}(y)+\zeta\Big)\big(s-\tau\big)\Big)\mathrm{d}\mathcal{H}^{\mathrm{N}-1}(\zeta)\Bigg)$$
$$\times\frac{\mathrm{d}G_q}{\mathrm{d}\rho}\Big(\Lambda(y,s-\tau,s)\Big)\Bigg\}\big|u^+(y)-u^-(y)\big|^q\mathrm{d}\mathcal{H}^{\mathrm{N}-1}(\mathrm{y}).\quad(2.23)$$

The change of variables $w/|\tau| \to v$ in the right hand side of (2.23) gives

$$\lim_{\varepsilon\to 0^+} \frac{1}{-\ln\varepsilon}\|u_\varepsilon\|^q_{W^{1/q,q}} = D_N \int_{J_u\cap\Omega} \Bigg(\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{\tau^2} \Bigg(\int_{H_0(\boldsymbol{\nu}(y))} \Big(\eta\big(s\boldsymbol{\nu}(y)+\zeta\big)s - \eta\Big(\big(s-\tau\big)\boldsymbol{\nu}(y)+\zeta\Big)\big(s-\tau\big)\Big) \mathrm{d}\mathcal{H}^{\mathrm{N}-1}(\zeta)\Bigg) \times \frac{\mathrm{d}G_q}{\mathrm{d}\rho}\Big(\Lambda(y,s-\tau,s)\Big)\mathrm{d}\tau\mathrm{d}s\Bigg)\big|u^+(y)-u^-(y)\big|^q \mathrm{d}\mathcal{H}^{\mathrm{N}-1}(\mathrm{y}), \tag{2.24}$$

where D_N is the dimensional constant given by

$$D_N := \int_{\mathbb{R}^{N-1}} \frac{1}{\big(\sqrt{1+|v|^2}\big)^{N+1}}\mathrm{d}v. \tag{2.25}$$

Then we rewrite (2.24) as

$$\begin{aligned}\lim_{\varepsilon\to 0^+} \frac{1}{-\ln\varepsilon}\|u_\varepsilon\|^q_{W^{1/q,q}} = \lim_{M\to+\infty}\Bigg(& D_N \int_{J_u\cap\Omega} \Bigg(\int_{\mathbb{R}}\int_{-M}^{M} \frac{1}{\tau^2} \Bigg(\int_{H_0(\boldsymbol{\nu}(y))} s\Big(\eta\big(s\boldsymbol{\nu}(y)+\zeta\big) - \eta\big((s-\tau)\boldsymbol{\nu}(y)+\zeta\big)\Big) \mathrm{d}\mathcal{H}^{\mathrm{N}-1}(\zeta)\Bigg) \\ & \times \frac{\mathrm{d}G_q}{\mathrm{d}\rho}\Big(\Lambda(y,s-\tau,s)\Big)\mathrm{d}\tau\mathrm{d}s\Bigg)\big|u^+(y)-u^-(y)\big|^q \mathrm{d}\mathcal{H}^{\mathrm{N}-1}(\mathrm{y}) \\ & + D_N \int_{J_u\cap\Omega} \Bigg(\int_{\mathbb{R}}\int_{-M}^{M} \frac{1}{\tau} \Bigg(\int_{H_0(\boldsymbol{\nu}(y))} \eta\big((s-\tau)\boldsymbol{\nu}(y)+\zeta\big) \mathrm{d}\mathcal{H}^{\mathrm{N}-1}(\zeta)\Bigg) \\ & \times \frac{\mathrm{d}G_q}{\mathrm{d}\rho}\Big(\Lambda(y,s-\tau,s)\Big)d\tau\mathrm{d}s\Bigg)\big|u^+(y)-u^-(y)\big|^q \mathrm{d}\mathcal{H}^{\mathrm{N}-1}(\mathrm{y})\Bigg).\end{aligned} \tag{2.26}$$

Integration by parts of (2.26) and using (2.20) give

$$\begin{aligned}\lim_{\varepsilon\to 0^+} \frac{1}{-\ln\varepsilon}\|u_\varepsilon\|^q_{W^{1/q,q}} &= -\lim_{M\to+\infty} D_N \int_{J_u\cap\Omega} \big|u^+(y)-u^-(y)\big|^q \Bigg(\int_{\mathbb{R}}\int_{-M}^{M} \frac{1}{\tau^2}\Big|\Lambda(y,s-\tau,s)\Big|^q d\tau\mathrm{d}s\Bigg)\mathrm{d}\mathcal{H}^{\mathrm{N}-1}(\mathrm{y}) \\ &\quad + \lim_{M\to+\infty} D_N \int_{J_u\cap\Omega} \Bigg(\int_{\mathbb{R}}\int_{-M}^{M} \frac{1}{\tau^2}\Big|\Lambda(y,s-\tau,s)\Big|^q d\tau\mathrm{d}s\Bigg)\big|u^+(y)-u^-(y)\big|^q\mathrm{d}\mathcal{H}^{\mathrm{N}-1}(\mathrm{y}) \\ &\quad + \lim_{M\to+\infty} \frac{D_N}{M} \int_{J_u\cap\Omega} \Bigg(\int_{\mathbb{R}}\Big|\Lambda(y,s-M,s)\Big|^q\mathrm{d}s + \int_{\mathbb{R}}\Big|\Lambda(y,s,s+M)\Big|^q\mathrm{d}s\Bigg)\big|u^+(y)-u^-(y)\big|^q\mathrm{d}\mathcal{H}^{\mathrm{N}-1}(\mathrm{y}) \\ &= \lim_{M\to+\infty} \frac{D_N}{M} \int_{J_u\cap\Omega} \Bigg(\int_{\mathbb{R}}\Big|\Lambda(y,s-M,s)\Big|^q\mathrm{d}s + \int_{\mathbb{R}}\Big|\Lambda(y,s,s+M)\Big|^q\mathrm{d}s\Bigg)\big|u^+(y)-u^-(y)\big|^q\mathrm{d}\mathcal{H}^{\mathrm{N}-1}(\mathrm{y}).\end{aligned} \tag{2.27}$$

Therefore, applying L'Hôpital's rule in (2.27), using (2.20), we deduce that

$$\lim_{\varepsilon\to0^+}\frac{1}{-\ln\varepsilon}\|u_\varepsilon\|^q_{W^{1/q,q}} = \lim_{M\to+\infty} D_N\int_{J_u\cap\Omega}\Bigg(\int_{\mathbb{R}}\frac{\mathrm{d}G_q}{\mathrm{d}\rho}\Big(\Lambda(y,s-M,s)\Big)\Big(\int_{H_0(\boldsymbol{\nu}(y))}\eta\big((s-M)\boldsymbol{\nu}(y)+\xi\big)\mathrm{d}\mathcal{H}^{\mathrm{N}-1}(\xi)\Big)\mathrm{ds} + \int_{\mathbb{R}}\frac{\mathrm{d}G_q}{\mathrm{d}\rho}\Big(\Lambda(y,s,s+M)\Big)\Big(\int_{H_0(\boldsymbol{\nu}(y))}\eta\big((s+M)\boldsymbol{\nu}(y)+\xi\big)\mathrm{d}\mathcal{H}^{\mathrm{N}-1}(\xi)\Big)\Bigg)\mathrm{ds} \times\big|u^+(y)-u^-(y)\big|^q\mathrm{d}\mathcal{H}^{\mathrm{N}-1}(\mathrm{y}). \tag{2.28}$$

Changing variables of integration we rewrite (2.28) as

$$\lim_{\varepsilon\to0^+}\frac{1}{-\ln\varepsilon}\|u_\varepsilon\|^q_{W^{1/q,q}} = \lim_{M\to+\infty} D_N\int_{J_u\cap\Omega}\Bigg(\int_{\mathbb{R}}\frac{\mathrm{d}G_q}{\mathrm{d}\rho}\Big(\Lambda(y,s,s+M)\Big)\Big(\int_{H_0(\boldsymbol{\nu}(y))}\eta\big(s\boldsymbol{\nu}(y)+\xi\big)\mathrm{d}\mathcal{H}^{\mathrm{N}-1}(\xi)\Big)\mathrm{ds} + \int_{\mathbb{R}}\frac{\mathrm{d}G_q}{\mathrm{d}\rho}\Big(\Lambda(y,s-M,s)\Big)\Big(\int_{H_0(\boldsymbol{\nu}(y))}\eta\big(s\boldsymbol{\nu}(y)+\xi\big)\mathrm{d}\mathcal{H}^{\mathrm{N}-1}(\xi)\Big)\mathrm{ds}\Bigg) \times\big|u^+(y)-u^-(y)\big|^q\mathrm{d}\mathcal{H}^{\mathrm{N}-1}(\mathrm{y})$$
$$= D_N\int_{J_u\cap\Omega}\Bigg(\int_{\mathbb{R}}\frac{\mathrm{d}G_q}{\mathrm{d}\rho}\Big(\Lambda(y,s,\infty)\Big)\Big(\int_{H_0(\boldsymbol{\nu}(y))}\eta\big(s\boldsymbol{\nu}(y)+\xi\big)\mathrm{d}\mathcal{H}^{\mathrm{N}-1}(\xi)\Big)\mathrm{ds} + \int_{\mathbb{R}}\frac{\mathrm{d}G_q}{\mathrm{d}\rho}\Big(\Lambda(y,-\infty,s)\Big)\Big(\int_{H_0(\boldsymbol{\nu}(y))}\eta\big(s\boldsymbol{\nu}(y)+\xi\big)\mathrm{d}\mathcal{H}^{\mathrm{N}-1}(\xi)\Big)\mathrm{ds}\Bigg) \times\big|u^+(y)-u^-(y)\big|^q\mathrm{d}\mathcal{H}^{\mathrm{N}-1}(\mathrm{y}). \tag{2.29}$$

On the other hand, by (2.22) we deduce

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big(\Lambda(y,-\infty,s)\Big) = \int_{H_0(\boldsymbol{\nu}(y))}\eta(s\boldsymbol{\nu}(y)+\xi)\,\mathrm{d}\mathcal{H}^{N-1}(\xi) = -\frac{\mathrm{d}}{\mathrm{d}s}\Big(\Lambda(y,s,\infty)\Big). \tag{2.30}$$

Thus, inserting (2.30) into (2.29) and using the Chain Rule gives

$$\lim_{\varepsilon\to0^+}\frac{1}{-\ln\varepsilon}\|u_\varepsilon\|^q_{W^{1/q,q}} = D_N\int_{J_u\cap\Omega}\Bigg(-\int_{\mathbb{R}}\frac{\mathrm{d}G_q}{\mathrm{d}\rho}\Big(\Lambda(y,s,\infty)\Big)\frac{\mathrm{d}}{\mathrm{ds}}\Big(\Lambda(y,s,\infty)\Big)\mathrm{d}s + \int_{\mathbb{R}}\frac{\mathrm{d}G_q}{\mathrm{d}\rho}\Big(\Lambda(y,-\infty,s)\Big)\frac{\mathrm{d}}{\mathrm{d}s}\Big(\Lambda(y,-\infty,s)\Big)\mathrm{d}s\Bigg)\big|u^+(y)-u^-(y)\big|^q\mathrm{d}\mathcal{H}^{\mathrm{N}-1}(\mathrm{y})$$
$$= D_N\int_{J_u\cap\Omega}\Bigg(-\int_{\mathbb{R}}\frac{\mathrm{d}}{\mathrm{d}s}\Big(G_q\Big(\Lambda(y,s,\infty)\Big)\Big)\mathrm{d}s + \int_{\mathbb{R}}\frac{\mathrm{d}}{\mathrm{d}s}\Big(G_q\Big(\Lambda(y,-\infty,s)\Big)\Big)\mathrm{d}s\Bigg) \times\big|u^+(y)-u^-(y)\big|^q\mathrm{d}\mathcal{H}^{N-1}(y). \tag{2.31}$$

Finally, applying Newton-Leibniz formula in (2.31) and using (2.20) with (2.22) we obtain that

$$\lim_{\varepsilon\to 0^+}\frac{1}{-\ln\varepsilon}\|u_\varepsilon\|^q_{W^{1/q,q}} = D_N\int_{J_u\cap\Omega}\Bigg(-\Big(G_q\Big(\Lambda(y,\infty,\infty)\Big)-G_q\Big(\Lambda(y,-\infty,\infty)\Big)\Big)$$
$$+\Big(G_q\Big(\Lambda(y,-\infty,\infty)\Big)-G_q\Big(\Lambda(y,-\infty,-\infty)\Big)\Big)\Bigg)\big|u^+(y)-u^-(y)\big|^q d\mathcal{H}^{N-1}(y)$$
$$= 2D_N\int_{J_u\cap\Omega}\bigg|\int_{-\infty}^{\infty}\int_{H_0(\boldsymbol{\nu}(y))}\eta(t\boldsymbol{\nu}(y)+\xi)d\mathcal{H}^{N-1}(\xi)dt\bigg|^q\big|u^+(y)-u^-(y)\big|^q d\mathcal{H}^{N-1}(y)$$
$$= 2D_N\bigg|\int_{\mathbb{R}^N}\eta(z)dz\bigg|^q\int_{J_u\cap\Omega}\big|u^+(y)-u^-(y)\big|^q d\mathcal{H}^{N-1}(y)\,, \tag{2.32}$$

and (2.2) follows. □

Corollary 2.2. *Let $q>1$ and let $\Omega\subset\mathbb{R}^N$ be an open set. Assume $W:\mathbb{R}^d\times\mathbb{R}^N\to\mathbb{R}$ is a Borel measurable function such that, $W(0,\cdot)\in L^1(\Omega,\mathbb{R})$ and for every $D>0$ there exists $C:=C_D>0$ such that*

$$\big|W(b,x)-W(a,x)\big|\le C_D|b-a| \qquad \forall x\in\mathbb{R}^N,\ \forall\, a,b\in B_D(0). \tag{2.33}$$

Let $u\in BV(\mathbb{R}^N,\mathbb{R}^d)\cap L^\infty(\mathbb{R}^N,\mathbb{R}^d)$ be such that $\|Du\|(\partial\Omega)=0$ and $W\big(u(x),x\big)=0$ a.e. in Ω. Let $\eta\in C_c^\infty(\mathbb{R}^N,\mathbb{R})$ be such that $\int_{\mathbb{R}^N}\eta(z)dz=1$ and $\operatorname{supp}\eta\subset B_R(0)$. For every $\rho>0$ set

$$\eta_\rho(z):=\frac{1}{\rho^N}\eta\Big(\frac{z}{\rho}\Big)\quad \forall z\in\mathbb{R}^N. \tag{2.34}$$

Finally, for every $x\in\mathbb{R}^N$ and every $\varepsilon>0$ define

$$u_{\rho,\varepsilon}(x):=\frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\eta_\rho\Big(\frac{y-x}{\varepsilon}\Big)u(y)dy=\int_{\mathbb{R}^N}\eta(z)u(x+\varepsilon\rho z)dz=\int_{B_R(0)}\eta(z)u(x+\varepsilon\rho z)dz. \tag{2.35}$$

Then,

$$\lim_{\rho\to 0^+}\Bigg\{\limsup_{\varepsilon\to 0^+}\Bigg(\frac{1}{-\ln(\varepsilon)}\Big(\|u_{\rho,\varepsilon}\|^q_{W^{1/q,q}(\mathbb{R}^N,\mathbb{R}^d)}-\|u_{\rho,\varepsilon}\|^q_{W^{1/q,q}(\mathbb{R}^N\setminus\overline{\Omega},\mathbb{R}^d)}\Big)+\frac{1}{\varepsilon}\int_\Omega W\Big(u_{\rho,\varepsilon}(x),x\Big)dx\Bigg)\Bigg\}$$
$$=\lim_{\rho\to 0^+}\Bigg\{\limsup_{\varepsilon\to 0^+}\Bigg(\frac{1}{-\ln(\varepsilon)}\|u_{\rho,\varepsilon}\|^q_{W^{1/q,q}(\Omega,\mathbb{R}^d)}+\frac{1}{\varepsilon}\int_\Omega W\Big(u_{\rho,\varepsilon}(x),x\Big)dx\Bigg)\Bigg\}$$
$$=\Bigg(\int_{\mathbb{R}^{N-1}}\frac{2}{\big(\sqrt{1+|v|^2}\big)^{N+1}}dv\Bigg)\int_{J_u\cap\Omega}\big|u^+(y)-u^-(y)\big|^q d\mathcal{H}^{N-1}(y). \tag{2.36}$$

Proof. Since $\int_{\mathbb{R}^N}\eta_\rho(z)dz=1$, applying Proposition 2.1, first for $\mathbb{R}^N$, then for $\mathbb{R}^N\setminus\overline{\Omega}$, and finally for Ω, yields, for every $\rho>0$,

$$\lim_{\varepsilon\to 0^+}\frac{1}{-\ln(\varepsilon)}\Big(\|u_{\rho,\varepsilon}\|^q_{W^{1/q,q}(\mathbb{R}^N,\mathbb{R}^d)}-\|u_{\rho,\varepsilon}\|^q_{W^{1/q,q}(\mathbb{R}^N\setminus\overline{\Omega},\mathbb{R}^d)}\Big)$$

$$= 2D_N \left(\int_{J_u} \left|u^+(y) - u^-(y)\right|^q d\mathcal{H}^{N-1}(y) - \int_{J_u \cap (\mathbb{R}^N \setminus \overline{\Omega})} \left|u^+(y) - u^-(y)\right|^q d\mathcal{H}^{N-1}(y) \right)$$

$$= 2D_N \int_{J_u \cap \Omega} \left|u^+(y) - u^-(y)\right|^q d\mathcal{H}^{N-1}(y) = \lim_{\varepsilon \to 0^+} \left(\frac{1}{-\ln(\varepsilon)} \|u_{\rho,\varepsilon}\|^q_{W^{1/q,q}(\Omega,\mathbb{R}^d)} \right), \quad (2.37)$$

where D_N is the constant defined in (2.25). On the other hand, since $W\big(u(x), x\big) = 0$ a.e. in Ω and $u \in L^\infty(\mathbb{R}^N, \mathbb{R}^d)$, by (2.33) we get that

$$\left| \frac{1}{\varepsilon} \int_\Omega W\Big(u_{\rho,\varepsilon}(x), x\Big) dx \right| = \left| \frac{1}{\varepsilon} \int_\Omega \Big(W\big(u_{\rho,\varepsilon}(x), x\big) - W\big(u(x), x\big) \Big) dx \right| \leq C \int_{\mathbb{R}^N} \frac{1}{\varepsilon} \Big| u_{\rho,\varepsilon}(x) - u(x) \Big| dx$$

$$\leq C \int_{B_R(0)} \big|\eta(z)\big| \left(\int_{\mathbb{R}^N} \frac{1}{\varepsilon} \Big| u(x + \varepsilon\rho z) - u(x) \Big| dx \right) dz$$

$$= C\rho \int_{B_R(0)} |z| \big|\eta(z)\big| \left(\int_{\mathbb{R}^N} \frac{1}{\varepsilon\rho|z|} \Big| u(x + \varepsilon\rho z) - u(x) \Big| dx \right) dz, \quad (2.38)$$

for some constant $C > 0$, independent of ε and ρ. Thus, taking into account the following uniform bound, concluded from Lemma A.5 of the appendix,

$$\int_{\mathbb{R}^N} \frac{1}{\rho\varepsilon|z|} \Big| u(x + \rho\varepsilon z) - u(x) \Big| dx \leq \|Du\|(\mathbb{R}^N) \quad \forall z \in \mathbb{R}^N \setminus \{0\}, \ \ \forall \rho, \varepsilon > 0, \quad (2.39)$$

we obtain that

$$\limsup_{\varepsilon \to 0^+} \left| \frac{1}{\varepsilon} \int_\Omega W\Big(u_{\rho,\varepsilon}(x), x\Big) dx \right| \leq C \|Du\|(\mathbb{R}^N) \rho \int_{B_R(0)} |z| \big|\eta(z)\big| dz = O(\rho). \quad (2.40)$$

By (2.40) and (2.37) we finally derive (2.36). □

Proof of Theorem 1.2. Let η, η_ρ and $u_{\rho,\varepsilon}$ be defined as in Corollary 2.2. Then $u_{\rho,\varepsilon} \in C^\infty(\mathbb{R}^N, \mathbb{R}^d) \cap W^{1,1}(\mathbb{R}^N, \mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^d)$ and by Corollary 2.2 we have

$$\lim_{\rho \to 0^+} \left\{ \limsup_{\varepsilon \to 0^+} \left(\frac{1}{-\ln(\varepsilon)} \Big(\|u_{\rho,\varepsilon}\|^q_{W^{1/q,q}(\mathbb{R}^N,\mathbb{R}^d)} - \|u_{\rho,\varepsilon}\|^q_{W^{1/q,q}(\mathbb{R}^N \setminus \overline{\Omega},\mathbb{R}^d)} \Big) + \frac{1}{\varepsilon} \int_\Omega W\Big(u_{\rho,\varepsilon}(x), x\Big) dx \right) \right\}$$

$$= \lim_{\rho \to 0^+} \left\{ \limsup_{\varepsilon \to 0^+} \left(\frac{1}{-\ln \varepsilon} \|u_{\rho,\varepsilon}\|^q_{W^{1/q,q}(\Omega,\mathbb{R}^d)} + \frac{1}{\varepsilon} \int_\Omega W\Big(u_{\rho,\varepsilon}(x), x\Big) dx \right) \right\}$$

$$= \left(\int_{\mathbb{R}^{N-1}} \frac{2}{\big(\sqrt{1 + |v|^2}\big)^{N+1}} dv \right) \int_{J_u \cap \Omega} \left|u^+(y) - u^-(y)\right|^q d\mathcal{H}^{N-1}(y). \quad (2.41)$$

Clearly, for every $x \in \mathbb{R}^N$ we have $A \cdot \nabla u_{\rho,\varepsilon}(x) = 0$ and $u_{\rho,\varepsilon}(x) \to u(x)$ strongly in $L^p(\mathbb{R}^N, \mathbb{R}^d)$ as $\varepsilon \to 0^+$ for every fixed ρ and p. Therefore, by the above and by (2.41) we can complete the proof of the first assertion of the theorem using a standard diagonal argument.

It remains to show the second assertion of the theorem, namely, that in the case $A \equiv 0$ we can construct ψ_ε satisfying the additional condition (1.12). Let $\varphi \in C_c^\infty(\mathbb{R}^N, \mathbb{R})$ be such that $\int_\Omega \varphi(x) dx = 1$. Define

$$\tilde{u}_{\rho,\varepsilon}(x) := u_{\rho,\varepsilon}(x) - \varphi(x) c_{\varepsilon,\rho}, \quad (2.42)$$

where

$$c_{\varepsilon,\rho} := \int_\Omega u_{\rho,\varepsilon}(y)\mathrm{d}y - \int_\Omega u(y)\mathrm{d}y. \tag{2.43}$$

In particular,

$$\int_\Omega \tilde{u}_{\rho,\varepsilon}(x)\mathrm{d}x = \int_\Omega u(x)\mathrm{d}x, \tag{2.44}$$

and $\lim_{\varepsilon\to 0^+} c_{\varepsilon,\rho} = 0$. On the other hand, since $W\big(u(x),x\big) = 0$ a.e. in Ω, $W(b,x)$ is nonnegative and $W(b,x)$ is differentiable with respect to the b variable, we have

$$\nabla_b W\big(u(x),x\big) = 0 \quad \text{a.e. in } \Omega. \tag{2.45}$$

Thus, since $u \in L^\infty(\mathbb{R}^N,\mathbb{R}^d)$, by (2.42) we get that

$$\begin{aligned}
\left|\frac{1}{\varepsilon}\int_\Omega \Big(W\big(\tilde{u}_{\rho,\varepsilon}(x),x\big) - W\big(u_{\rho,\varepsilon}(x),x\big)\Big)\mathrm{d}x\right| &= \left|\frac{c_{\varepsilon,\rho}}{\varepsilon}\cdot\int_0^1\int_\Omega \nabla_b W\Big(u_{\rho,\varepsilon}(x) - s\varphi(x)c_{\varepsilon,\rho},x\Big)\varphi(x)\mathrm{d}x\mathrm{d}s\right| \\
&\le C\left(\int_{\mathbb{R}^N}\frac{1}{\varepsilon}\Big|u_{\rho,\varepsilon}(x) - u(x)\Big|\mathrm{d}x\right)\left|\int_0^1\int_\Omega \nabla_b W\Big(u_{\rho,\varepsilon}(x) - s\varphi(x)c_{\varepsilon,\rho},x\Big)\varphi(x)\mathrm{d}x\mathrm{d}s\right| \\
&\le C\left(\int_{B_R(0)}\big|\eta(z)\big|\left(\int_{\mathbb{R}^N}\frac{1}{\varepsilon}\Big|u(x+\varepsilon\rho z) - u(x)\Big|\mathrm{d}x\right)\mathrm{d}z\right) \\
&\quad\times\left|\int_0^1\int_\Omega \nabla_b W\Big(u_{\rho,\varepsilon}(x) - s\varphi(x)c_{\varepsilon,\rho},x\Big)\varphi(x)\mathrm{d}x\mathrm{d}s\right| \\
&= C\rho\left(\int_{B_R(0)}|z|\big|\eta(z)\big|\left(\int_{\mathbb{R}^N}\frac{1}{\varepsilon\rho|z|}\Big|u(x+\varepsilon\rho z) - u(x)\Big|\mathrm{d}x\right)\mathrm{d}z\right) \\
&\quad\times\left|\int_0^1\int_\Omega \nabla_b W\Big(u_{\rho,\varepsilon}(x) - s\varphi(x)c_{\varepsilon,\rho},x\Big)\varphi(x)\mathrm{d}x\mathrm{d}s\right|.
\end{aligned} \tag{2.46}$$

On the other hand, taking into account (2.39) and using the Dominated Convergence Theorem and (2.45), we obtain that

$$\begin{aligned}
&\limsup_{\varepsilon\to 0+}\left(\int_{B_R(0)}|z|\big|\eta(z)\big|\left(\int_{\mathbb{R}^N}\frac{1}{\varepsilon\rho|z|}\Big|u(x+\varepsilon\rho z) - u(x)\Big|\mathrm{d}x\right)\mathrm{d}z\right) \\
&\quad\times\left|\int_0^1\int_\Omega \nabla_b W\Big(u_{\rho,\varepsilon}(x) - s\varphi(x)c_{\varepsilon,\rho},x\Big)\varphi(x)\mathrm{d}x\mathrm{d}s\right| \le C_0\Big(\|Du\|(\mathbb{R}^N)\Big)\left(\int_{B_R(0)}|z|\big|\eta(z)\big|\mathrm{d}z\right) \\
&\quad\times\left|\int_0^1\int_\Omega \nabla_b W\Big(\lim_{\varepsilon\to 0+} u_{\rho,\varepsilon}(x) - s\varphi(x)\lim_{\varepsilon\to 0+} c_{\varepsilon,\rho}\,,\,x\Big)\varphi(x)\mathrm{d}x\mathrm{d}s\right| \\
&\quad= C_0\Big(\|Du\|(\mathbb{R}^N)\Big)\left(\int_{B_R(0)}|z|\big|\eta(z)\big|\mathrm{d}z\right)\left|\int_\Omega \nabla_b W\Big(u(x),x\Big)\varphi(x)\mathrm{d}x\right| = 0.
\end{aligned} \tag{2.47}$$

Using (2.47) in (2.46) yields

$$\limsup_{\varepsilon\to 0+}\left|\frac{1}{\varepsilon}\int_\Omega \Big(W\big(\tilde u_{\rho,\varepsilon}(x),x\big)-W\big(u_{\rho,\varepsilon}(x),x\big)\Big)\mathrm{d}x\right|=0. \tag{2.48}$$

Plugging (2.48) into (2.41) we get that

$$\lim_{\rho\to 0^+}\left\{\limsup_{\varepsilon\to 0^+}\left(\frac{1}{-\ln(\varepsilon)}\Big(\|\tilde u_{\rho,\varepsilon}\|^q_{W^{1/q,q}(\mathbb{R}^N,\mathbb{R}^d)}-\|\tilde u_{\rho,\varepsilon}\|^q_{W^{1/q,q}(\mathbb{R}^N\setminus\overline{\Omega},\mathbb{R}^d)}\Big)+\frac{1}{\varepsilon}\int_\Omega W\Big(\tilde u_{\rho,\varepsilon}(x),x\Big)\mathrm{d}x\right)\right\}$$
$$=\lim_{\rho\to 0^+}\left\{\limsup_{\varepsilon\to 0^+}\left(\frac{1}{-\ln\varepsilon}\|\tilde u_{\rho,\varepsilon}\|^q_{W^{1/q,q}(\Omega,\mathbb{R}^d)}+\frac{1}{\varepsilon}\int_\Omega W\Big(\tilde u_{\rho,\varepsilon}(x),x\Big)\mathrm{d}x\right)\right\}$$
$$=\left(\int_{\mathbb{R}^{N-1}}\frac{2}{\big(\sqrt{1+|v|^2}\big)^{N+1}}\mathrm{d}v\right)\int_{J_u\cap\Omega}\big|u^+(y)-u^-(y)\big|^q\mathrm{d}\mathcal{H}^{N-1}(y). \tag{2.49}$$

Moreover, $\tilde u_{\rho,\varepsilon}\to u$ strongly in $L^p(\mathbb{R}^N,\mathbb{R}^d)$ as $\varepsilon\to 0^+$ for every fixed ρ and p. Therefore, by the above and (2.49) we complete again the proof by a standard diagonal argument. □

The next lemma is needed for the proof of Theorem 1.1 (in the general case $\eta\in W^{1,1}$).

Lemma 2.3. *Let $\Omega\subset\mathbb{R}^N$ be an open set and let $u\in BV(\mathbb{R}^N,\mathbb{R}^d)\cap L^\infty(\mathbb{R}^N,\mathbb{R}^d)$. For $\eta\in W^{1,1}(\mathbb{R}^N,\mathbb{R})$, every $x\in\mathbb{R}^N$ and every $\varepsilon>0$ define*

$$u_\varepsilon(x):=\frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\eta\Big(\frac{y-x}{\varepsilon}\Big)u(y)\mathrm{d}y=\int_{\mathbb{R}^N}\eta(z)u(x+\varepsilon z)\mathrm{d}z. \tag{2.50}$$

Then, for every $q>1$ and for every $\varepsilon\in(0,1)$ we have

$$\frac{1}{\omega_{N-1}\big|\ln\varepsilon\big|}\int_\Omega\left(\int_\Omega\frac{\big|u_\varepsilon(x)-u_\varepsilon(y)\big|^q}{|x-y|^{N+1}}\mathrm{d}y\right)\mathrm{d}x\le\frac{2^q\|u\|_{L^1(\mathbb{R}^N,\mathbb{R}^d)}\|u\|^{q-1}_{L^\infty(\mathbb{R}^N,\mathbb{R}^d)}\|\eta\|^q_{L^1(\mathbb{R}^N,\mathbb{R})}}{\big|\ln\varepsilon\big|}$$
$$+\frac{\big(3\|u\|_{L^\infty(\mathbb{R}^N,\mathbb{R}^d)}\|\eta\|_{W^{1,1}(\mathbb{R}^N,\mathbb{R})}\big)^{q-1}\|\eta\|_{L^1(\mathbb{R}^N,\mathbb{R})}\|Du\|(\mathbb{R}^N)}{(q-1)\big|\ln\varepsilon\big|}$$
$$+\big(3\|u\|_{L^\infty(\mathbb{R}^N,\mathbb{R}^d)}\|\eta\|_{W^{1,1}(\mathbb{R}^N,\mathbb{R})}\big)^{q-1}\|\eta\|_{L^1(\mathbb{R}^N,\mathbb{R})}\|Du\|(\mathbb{R}^N), \tag{2.51}$$

where ω_{N-1} denotes the surface area of the unit ball in $\mathbb{R}^N$.

Proof. Assume first that $\eta(z)\in C_c^\infty(\mathbb{R}^N,\mathbb{R})$. Then, by (2.50) we have

$$\varepsilon\nabla u_\varepsilon(x)=-\frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\nabla\eta\Big(\frac{y-x}{\varepsilon}\Big)u(y)\mathrm{d}y=-\int_{\mathbb{R}^N}\nabla\eta(z)u(x+\varepsilon z)\mathrm{d}z\,. \tag{2.52}$$

By (2.50) and (2.52) we get that

$$\|u_\varepsilon\|_{L^\infty(\mathbb{R}^N,\mathbb{R}^d)}+\|\varepsilon\nabla u_\varepsilon\|_{L^\infty(\mathbb{R}^N,\mathbb{R}^d)}\le\|u\|_{L^\infty(\mathbb{R}^N,\mathbb{R}^d)}\|\eta\|_{W^{1,1}(\mathbb{R}^N,\mathbb{R})}\quad\text{and}$$
$$\|u_\varepsilon\|^q_{L^q(\mathbb{R}^N,\mathbb{R}^d)}\le\|u\|_{L^1(\mathbb{R}^N,\mathbb{R}^d)}\|u\|^{q-1}_{L^\infty(\mathbb{R}^N,\mathbb{R}^d)}\|\eta\|^q_{L^1(\mathbb{R}^N,\mathbb{R})}\qquad\forall\varepsilon>0,\ \forall q\in[1,+\infty). \tag{2.53}$$

Next, for every $\varepsilon \in (0,1)$ we have

$$\int_\Omega\bigg(\int_\Omega \frac{\big|u_\varepsilon(x)-u_\varepsilon(y)\big|^q}{|x-y|^{N+1}}\mathrm{d}y\bigg)\mathrm{d}x \le \int_{\mathbb{R}^N}\bigg(\int_{\mathbb{R}^N} \frac{\big|u_\varepsilon(x)-u_\varepsilon(y)\big|^q}{|x-y|^{N+1}}\mathrm{d}y\bigg)\mathrm{d}x$$
$$= \int_{\mathbb{R}^N}\bigg(\int_{\mathbb{R}^N} \frac{\big|u_\varepsilon(x+y)-u_\varepsilon(x)\big|^q}{|y|^{N+1}}\mathrm{d}y\bigg)\mathrm{d}x = \int_{\mathbb{R}^N}\bigg(\int_{B_\varepsilon(0)} \frac{\big|u_\varepsilon(x+y)-u_\varepsilon(x)\big|^q}{|y|^{N+1}}\mathrm{d}y\bigg)\mathrm{d}x$$
$$+ \int_{\mathbb{R}^N}\bigg(\int_{B_1(0)\setminus B_\varepsilon(0)} \frac{\big|u_\varepsilon(x+y)-u_\varepsilon(x)\big|^q}{|y|^{N+1}}\mathrm{d}y\bigg)\mathrm{d}x + \int_{\mathbb{R}^N}\bigg(\int_{\mathbb{R}^N\setminus B_1(0)} \frac{\big|u_\varepsilon(x+y)-u_\varepsilon(x)\big|^q}{|y|^{N+1}}\mathrm{d}y\bigg)\mathrm{d}x$$
$$= \int_{B_\varepsilon(0)}\frac{1}{|y|^{N+1-q}}\bigg(\int_{\mathbb{R}^N} \frac{\big|u_\varepsilon(x+y)-u_\varepsilon(x)\big|^q}{|y|^{q}}\mathrm{d}x\bigg)\mathrm{d}y$$
$$+ \int_{B_1(0)\setminus B_\varepsilon(0)}\frac{1}{|y|^{N}}\bigg(\int_{\mathbb{R}^N} \frac{\big|u_\varepsilon(x+y)-u_\varepsilon(x)\big|^q}{|y|}\mathrm{d}x\bigg)\mathrm{d}y$$
$$+ \int_{\mathbb{R}^N\setminus B_1(0)}\frac{1}{|y|^{N+1}}\bigg(\int_{\mathbb{R}^N} \big|u_\varepsilon(x+y)-u_\varepsilon(x)\big|^q dx\bigg)\mathrm{d}y. \tag{2.54}$$

On the other hand, (2.53) yields

$$\big|u_\varepsilon(x+y)-u_\varepsilon(x)\big| + \frac{\varepsilon\big|u_\varepsilon(x+y)-u_\varepsilon(x)\big|}{|x-y|} \le 3\|u\|_{L^\infty(\mathbb{R}^N,\mathbb{R}^d)}\|\eta\|_{W^{1,1}(\mathbb{R}^N,\mathbb{R})} \qquad \forall\varepsilon>0,\ \forall\, x,y\in\mathbb{R}^N. \tag{2.55}$$

Thus, inserting (2.55) into (2.54) we deduce that

$$\int_\Omega\bigg(\int_\Omega \frac{\big|u_\varepsilon(x)-u_\varepsilon(y)\big|^q}{|x-y|^{N+1}}\mathrm{d}y\bigg)\mathrm{d}x \le 2^q\|u_\varepsilon\|^q_{L^q(\mathbb{R}^N,\mathbb{R}^d)}\int_{\mathbb{R}^N\setminus B_1(0)}\frac{\mathrm{d}y}{|y|^{N+1}}$$
$$+ \frac{\big(3\|u\|_{L^\infty(\mathbb{R}^N,\mathbb{R}^d)}\|\eta\|_{W^{1,1}(\mathbb{R}^N,\mathbb{R})}\big)^{q-1}}{\varepsilon^{q-1}}\int_{B_\varepsilon(0)}\frac{1}{|y|^{N+1-q}}\bigg(\int_{\mathbb{R}^N} \frac{\big|u_\varepsilon(x+y)-u_\varepsilon(x)\big|}{|y|}\mathrm{d}x\bigg)\mathrm{d}y$$
$$+ \big(3\|u\|_{L^\infty(\mathbb{R}^N,\mathbb{R}^d)}\|\eta\|_{W^{1,1}(\mathbb{R}^N,\mathbb{R})}\big)^{q-1}\int_{B_1(0)\setminus B_\varepsilon(0)}\frac{1}{|y|^{N}}\bigg(\int_{\mathbb{R}^N} \frac{\big|u_\varepsilon(x+y)-u_\varepsilon(x)\big|}{|y|}\mathrm{d}x\bigg)\mathrm{d}y. \tag{2.56}$$

Inserting (2.50) into (2.56) and using the second inequality in (2.53) we infer,

$$\int_\Omega\bigg(\int_\Omega \frac{\big|u_\varepsilon(x)-u_\varepsilon(y)\big|^q}{|x-y|^{N+1}}\mathrm{d}y\bigg)\mathrm{d}x \le 2^q\|u\|_{L^1(\mathbb{R}^N,\mathbb{R}^d)}\|u\|^{q-1}_{L^\infty(\mathbb{R}^N,\mathbb{R}^d)}\|\eta\|^q_{L^1(\mathbb{R}^N,\mathbb{R})}\int_{\mathbb{R}^N\setminus B_1(0)}\frac{\mathrm{d}y}{|y|^{N+1}}$$
$$+ \frac{\big(3\|u\|_{L^\infty(\mathbb{R}^N,\mathbb{R}^d)}\|\eta\|_{W^{1,1}(\mathbb{R}^N,\mathbb{R})}\big)^{q-1}}{\varepsilon^{q-1}}$$
$$\times\int_{B_\varepsilon(0)}\frac{1}{|y|^{N+1-q}}\bigg(\int_{\mathbb{R}^N}|\eta(z)|\int_{\mathbb{R}^N} \frac{\big|u_\varepsilon(x+\varepsilon z+y)-u_\varepsilon(x+\varepsilon z)\big|}{|y|}\mathrm{d}x\mathrm{d}z\bigg)\mathrm{d}y$$
$$+ \big(3\|u\|_{L^\infty(\mathbb{R}^N,\mathbb{R}^d)}\|\eta\|_{W^{1,1}(\mathbb{R}^N,\mathbb{R})}\big)^{q-1}$$
$$\times\int_{B_1(0)\setminus B_\varepsilon(0)}\frac{1}{|y|^{N}}\bigg(\int_{\mathbb{R}^N}\big|\eta(z)\big|\int_{\mathbb{R}^N} \frac{\big|u_\varepsilon(x+\varepsilon z+y)-u_\varepsilon(x+\varepsilon z)\big|}{|y|}\mathrm{d}x\mathrm{d}z\bigg)\mathrm{d}y. \tag{2.57}$$

Taking into account the following well known uniform bound from the theory of BV functions:

$$\int_{\mathbb{R}^N} \frac{\big|u(x+\varepsilon z+y)-u(x+\varepsilon z)\big|}{|y|}\mathrm{d}x = \int_{\mathbb{R}^N} \frac{\big|u(x+y)-u(x)\big|}{|y|}\mathrm{d}x \leq \|Du\|(\mathbb{R}^N) \quad \forall y \in \mathbb{R}^N, \tag{2.58}$$

we rewrite (2.57) as

$$\begin{aligned}\int_\Omega \bigg(\int_\Omega \frac{\big|u_\varepsilon(x)-u_\varepsilon(y)\big|^q}{|x-y|^{N+1}}\mathrm{d}y\bigg)\mathrm{d}x \leq 2^q \|u\|_{L^1(\mathbb{R}^N,\mathbb{R}^d)}\|u\|^{q-1}_{L^\infty(\mathbb{R}^N,\mathbb{R}^d)}\|\eta\|^q_{L^1(\mathbb{R}^N,\mathbb{R})}\int_{\mathbb{R}^N\setminus B_1(0)}\frac{\mathrm{d}y}{|y|^{N+1}} \\ + \frac{\big(3\|u\|_{L^\infty(\mathbb{R}^N,\mathbb{R}^d)}\|\eta\|_{W^{1,1}(\mathbb{R}^N,\mathbb{R})}\big)^{q-1}}{\varepsilon^{q-1}}\|\eta\|_{L^1(\mathbb{R}^N,\mathbb{R})}\|Du\|(\mathbb{R}^N)\int_{B_\varepsilon(0)}\frac{\mathrm{d}y}{|y|^{N+1-q}} \\ + \big(3\|u\|_{L^\infty(\mathbb{R}^N,\mathbb{R}^d)}\|\eta\|_{W^{1,1}(\mathbb{R}^N,\mathbb{R})}\big)^{q-1}\|\eta\|_{L^1(\mathbb{R}^N,\mathbb{R})}\|Du\|(\mathbb{R}^N)\int_{B_1(0)\setminus B_\varepsilon(0)}\frac{\mathrm{d}y}{|y|^N}.\end{aligned} \tag{2.59}$$

Computing the integrals on the right hand side of (2.59) yields (2.51) in the case $\eta \in C^\infty_c(\mathbb{R}^N,\mathbb{R})$.

Next consider the general case $\eta \in W^{1,1}(\mathbb{R}^N,\mathbb{R})$. Thanks to the density of $C^\infty_c(\mathbb{R}^N,\mathbb{R})$ in $W^{1,1}(\mathbb{R}^N,\mathbb{R})$, there exists a sequence $\{\eta_n\}_{n=1}^\infty \subset C^\infty_c(\mathbb{R}^N,\mathbb{R})$ such that

$$\lim_{n\to+\infty} \big\|\eta_n-\eta\big\|_{W^{1,1}(\mathbb{R}^N,\mathbb{R})} = 0. \tag{2.60}$$

Thus, if we define

$$u_{n,\varepsilon}(x) := \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\eta_n\Big(\frac{y-x}{\varepsilon}\Big)u(y)\mathrm{d}y = \int_{\mathbb{R}^N}\eta_n(z)u(x+\varepsilon z)\mathrm{d}z, \tag{2.61}$$

then

$$\lim_{n\to+\infty} u_{n,\varepsilon}(x) = u_\varepsilon(x) \quad \forall x\in\mathbb{R}^N,\ \forall\varepsilon>0. \tag{2.62}$$

On the other hand, since we proved (2.51) for the case $\eta_n \in C^\infty_c(\mathbb{R}^N,\mathbb{R})$, for every $q>1$, for every $n=1,2,\ldots$ and for every $\varepsilon\in(0,1)$ we have:

$$\begin{aligned}\frac{1}{\omega_{N-1}\big|\ln\varepsilon\big|}\int_\Omega \bigg(\int_\Omega \frac{\big|u_{n,\varepsilon}(x)-u_{n,\varepsilon}(y)\big|^q}{|x-y|^{N+1}}\mathrm{d}y\bigg)\mathrm{d}x \leq \frac{2^q\|u\|_{L^1(\mathbb{R}^N,\mathbb{R}^d)}\|u\|^{q-1}_{L^\infty(\mathbb{R}^N,\mathbb{R}^d)}\|\eta_n\|^q_{L^1(\mathbb{R}^N,\mathbb{R})}}{\big|\ln\varepsilon\big|} \\ + \frac{\big(3\|u\|_{L^\infty(\mathbb{R}^N,\mathbb{R}^d)}\|\eta_n\|_{W^{1,1}(\mathbb{R}^N,\mathbb{R})}\big)^{q-1}\|\eta_n\|_{L^1(\mathbb{R}^N,\mathbb{R})}\|Du\|(\mathbb{R}^N)}{(q-1)\big|\ln\varepsilon\big|} \\ + \big(3\|u\|_{L^\infty(\mathbb{R}^N,\mathbb{R}^d)}\|\eta_n\|_{W^{1,1}(\mathbb{R}^N,\mathbb{R})}\big)^{q-1}\|\eta_n\|_{L^1(\mathbb{R}^N,\mathbb{R})}\|Du\|(\mathbb{R}^N).\end{aligned} \tag{2.63}$$

Letting n go to infinity in (2.63), using (2.60) in the right hand side and (2.62) together with Fatou's Lemma in the left hand side, we obtain (2.51) in the general case $\eta\in W^{1,1}(\mathbb{R}^N,\mathbb{R})$. □

Proof of Theorem 1.1. In the case $\eta\in C^\infty_c(\mathbb{R}^N,\mathbb{R})$ the result follows by Proposition 2.1. Next consider the general case $\eta\in W^{1,1}(\mathbb{R}^N,\mathbb{R})$. As before, by the density of $C^\infty_c(\mathbb{R}^N,\mathbb{R})$ in $W^{1,1}(\mathbb{R}^N,\mathbb{R})$, there exists a sequence $\{\eta_n\}_{n=1}^\infty \subset C^\infty_c(\mathbb{R}^N,\mathbb{R})$ such that

$$\lim_{n\to+\infty} \big\|\eta_n-\eta\big\|_{W^{1,1}(\mathbb{R}^N,\mathbb{R})} = 0. \tag{2.64}$$

Next, as before, define

$$u_{n,\varepsilon}(x) := \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\eta_n\Big(\frac{y-x}{\varepsilon}\Big)u(y)\mathrm{d}y = \int_{\mathbb{R}^N}\eta_n(z)u(x+\varepsilon z)\mathrm{d}z. \tag{2.65}$$

Defining $u_{n,\varepsilon}$ as in (2.61) we get by Proposition 2.1, for all $n \geq 1$ (see (2.25)),

$$\lim_{\varepsilon\to 0^+}\frac{1}{|\ln\varepsilon|}\|u_{n,\varepsilon}\|^q_{W^{1/q,q}(\Omega,\mathbb{R}^d)} = 2D_N\Big|\int_{\mathbb{R}^N}\eta_n(z)\mathrm{d}z\Big|^q\int_{J_u\cap\Omega}\Big|u^+(x)-u^-(x)\Big|^q\mathrm{d}\mathcal{H}^{\mathrm{N}-1}(\mathrm{x}) := \mathrm{L_n}, \tag{2.66}$$

and then

$$\lim_{n\to\infty}L_n = \bar{L} := 2D_N\Big|\int_{\mathbb{R}^N}\eta(z)\mathrm{d}z\Big|^q\int_{J_u\cap\Omega}\Big|u^+(x)-u^-(x)\Big|^q\mathrm{d}\mathcal{H}^{\mathrm{N}-1}(\mathrm{x}). \tag{2.67}$$

On the other hand, by Lemma 2.3, for all $n \geq 1$ and every $\varepsilon \in (0, 1/e)$ we have

$$\begin{aligned}
&\frac{1}{\omega_{N-1}\big|\ln\varepsilon\big|}\int_\Omega\left(\int_\Omega\frac{1}{|x-y|^{N+1}}\Big|\Big(u_{n,\varepsilon}(x)-u_{n,\varepsilon}(y)\Big)-\Big(u_\varepsilon(x)-u_\varepsilon(y)\Big)\Big|^q\mathrm{d}y\right)\mathrm{d}x\\
&\qquad= \frac{1}{\omega_{N-1}\big|\ln\varepsilon\big|}\int_\Omega\left(\int_\Omega\frac{1}{|x-y|^{N+1}}\Big|\Big(u_{n,\varepsilon}(x)-u_\varepsilon(x)\Big)-\Big(u_{n,\varepsilon}(y)-u_\varepsilon(y)\Big)\Big|^q dy\right)\mathrm{d}x\\
&\qquad\qquad\leq 2^q\|u\|_{L^1(\mathbb{R}^N,\mathbb{R}^d)}\|u\|^{q-1}_{L^\infty(\mathbb{R}^N,\mathbb{R}^d)}\|\eta_n-\eta\|^q_{L^1(\mathbb{R}^N,\mathbb{R})}\\
&\qquad\qquad+\frac{\big(3\|u\|_{L^\infty(\mathbb{R}^N,\mathbb{R}^d)}\|\eta_n-\eta\|_{W^{1,1}(\mathbb{R}^N,\mathbb{R})}\big)^{q-1}\|\eta_n-\eta\|_{L^1(\mathbb{R}^N,\mathbb{R})}\|Du\|(\mathbb{R}^N)}{(q-1)}\\
&\qquad\qquad+\big(3\|u\|_{L^\infty(\mathbb{R}^N,\mathbb{R}^d)}\|\eta_n-\eta\|_{W^{1,1}(\mathbb{R}^N,\mathbb{R})}\big)^{q-1}\|\eta_n-\eta\|_{L^1(\mathbb{R}^N,\mathbb{R})}\|Du\|(\mathbb{R}^N) := H_n.
\end{aligned} \tag{2.68}$$

Thus, by the triangle inequality we get, for every $n \geq 1$ and every $\varepsilon \in (0, 1/e)$,

$$\frac{1}{|\ln\varepsilon|^{1/q}}\Big|\|u_{n,\varepsilon}\|_{W^{1/q,q}}-\|u_\varepsilon\|_{W^{1/q,q}}\Big| \leq \frac{\|u_{n,\varepsilon}-u_\varepsilon\|_{W^{1/q,q}}}{|\ln\varepsilon|^{1/q}} \leq \big(\omega_{N-1}H_n\big)^{1/q}. \tag{2.69}$$

Then, by (2.69) and (2.66), for all $n \geq 1$ we obtain:

$$\begin{aligned}
\limsup_{\varepsilon\to 0^+}\Big|\frac{\|u_\varepsilon\|_{W^{1/q,q}}}{|\ln\varepsilon|^{1/q}}-\bar{L}^{1/q}\Big| &\leq \limsup_{\varepsilon\to 0^+}\frac{1}{\big|\ln\varepsilon\big|^{1/q}}\Big|\|u_{n,\varepsilon}\|_{W^{1/q,q}}-\|u_\varepsilon\|_{W^{1/q,q}}\Big|\\
&+\limsup_{\varepsilon\to 0^+}\Big|\frac{\|u_{n,\varepsilon}\|_{W^{1/q,q}}}{\big|\ln\varepsilon\big|^{1/q}}-L_n^{1/q}\Big|+|L_n^{1/q}-\bar{L}^{1/q}| \leq \big(\omega_{N-1}H_n\big)^{1/q}+0+|L_n^{1/q}-\bar{L}^{1/q}|.
\end{aligned} \tag{2.70}$$

Letting n go to infinity in (2.70), using (2.67), the definition of $\bar{L}$ in (2.67) and the fact that $\lim_{n\to+\infty}H_n = 0$, we finally deduce (1.5). □

Appendix A.

A.1 Some known results about BV-spaces

In what follows we present some known definitions and results on BV-spaces; some of them were used in the previous sections. We rely mainly on the book [5] by Ambrosio, Fusco and Pallara.

Definition A.1. Let Ω be a domain in $\mathbb{R}^N$ and let $f \in L^1(\Omega, \mathbb{R}^m)$. We say that $f \in BV(\Omega, \mathbb{R}^m)$ if the following quantity is finite:

$$\int_\Omega |Df| := \sup\left\{ \int_\Omega f \cdot \operatorname{div}\varphi \, \mathrm{d}x : \ \varphi \in C_c^1(\Omega, \mathbb{R}^{m\times N}), \ |\varphi(x)| \leq 1 \ \forall x \right\}.$$

Definition A.2. Let Ω be a domain in $\mathbb{R}^N$. Consider a function $f \in L^1_{loc}(\Omega, \mathbb{R}^m)$ and a point $x \in \Omega$.
i) We say that x is an *approximate continuity point* of f if there exists $z \in \mathbb{R}^m$ such that

$$\lim_{\rho \to 0^+} \frac{\int_{B_\rho(x)} |f(y) - z| \, \mathrm{d}y}{\rho^N} = 0.$$

In this case we denote z by $\tilde{f}(x)$. The set of approximate continuity points of f is denoted by G_f.
ii) We say that x is an *approximate jump point* of f if there exist $a, b \in \mathbb{R}^m$ and $\boldsymbol{\nu} \in S^{N-1}$ such that $a \neq b$ and

$$\lim_{\rho \to 0^+} \frac{\int_{B_\rho(x)} \big| f(y) - \chi(a, b, \boldsymbol{\nu})(y) \big| \, \mathrm{d}y}{\rho^N} = 0, \tag{A.1}$$

where $\chi(a, b, \boldsymbol{\nu})$ is defined by

$$\chi(a, b, \boldsymbol{\nu})(y) := \begin{cases} b & \text{if } \boldsymbol{\nu} \cdot y < 0, \\ a & \text{if } \boldsymbol{\nu} \cdot y > 0. \end{cases}$$

The triple $(a, b, \boldsymbol{\nu})$, uniquely determined, up to a permutation of (a, b) and a change of sign of $\boldsymbol{\nu}$, is denoted by $(f^+(x), f^-(x), \boldsymbol{\nu}_f(x))$. We shall call $\boldsymbol{\nu}_f(x)$ the *approximate jump vector* and we shall sometimes write simply $\boldsymbol{\nu}(x)$ if the reference to the function f is clear. The set of approximate jump points is denoted by J_f. A choice of $\boldsymbol{\nu}(x)$ for every $x \in J_f$ determines an orientation of J_f. At an approximate continuity point x, we shall use the convention $f^+(x) = f^-(x) = \tilde{f}(x)$.

Theorem A.3 ([5], Thms. 3.69 and 3.78)**.** *Consider an open set $\Omega \subset \mathbb{R}^N$ and $f \in BV(\Omega, \mathbb{R}^m)$. Then:*
i) $\mathcal{H}^{N-1}$-a.e. point in $\Omega \setminus J_f$ is a point of approximate continuity of f.
ii) The set J_f is σ-$\mathcal{H}^{N-1}$-rectifiable Borel set, oriented by $\boldsymbol{\nu}(x)$. I.e., the set J_f is $\mathcal{H}^{N-1}$ σ-finite, there exist countably many C^1 hypersurfaces $\{S_k\}_{k=1}^\infty$ such that $\mathcal{H}^{N-1}\Big(J_f \setminus \bigcup_{k=1}^\infty S_k\Big) = 0$, and for $\mathcal{H}^{N-1}$-a.e. $x \in J_f \cap S_k$, the approximate jump vector $\boldsymbol{\nu}(x)$ is normal to S_k at the point x.
iii) $\big[(f^+ - f^-) \otimes \boldsymbol{\nu}_f\big](x) \in L^1(J_f, \mathrm{d}\mathcal{H}^{\mathrm{N}-1})$.

Theorem A.4 ([5], Thms. 3.92 and 3.78)**.** *Consider an open set $\Omega \subset \mathbb{R}^N$ and $f \in BV(\Omega, \mathbb{R}^m)$. Then, the distributional gradient Df can be decomposed as a sum of two Borel regular finite matrix-valued measures μ_f and $D^j f$ on Ω,*

$$Df = \mu_f + D^j f,$$

where

$$D^j f = (f^+ - f^-) \otimes \boldsymbol{\nu}_f \mathcal{H}^{N-1} \llcorner J_f$$

is called the jump part of Df and

$$\mu_f = (D^a f + D^c f)$$

is a sum of the absolutely continuous and the Cantor parts of Df. The two parts μ_f and $D^j f$ are mutually singular to each other. Moreover, $\mu_f(B) = 0$ for any Borel set $B \subset \Omega$ which is $\mathcal{H}^{N-1}$ σ-finite.

The following simple Lemma is also useful:

Lemma A.5. *For every $u \in BV(\mathbb{R}^N, \mathbb{R}^d)$ we have:*

$$\int_{\mathbb{R}^N} \frac{1}{|y|} \Big| u(x+y) - u(x) \Big| \mathrm{d}x \le \|Du\|(\mathbb{R}^N) \quad \forall y \in \mathbb{R}^N \setminus \{0\}. \tag{A.2}$$

Proof. By Exercise 3.3 and Proposition 3.6 in [5] for every $K \subset\subset \mathbb{R}^N$ we have

$$\int_K \frac{1}{|y|} \Big| u(x+y) - u(x) \Big| \mathrm{d}x \le \|Du\|(\mathbb{R}^N) \quad \forall y \in \mathbb{R}^N \setminus \{0\}. \tag{A.3}$$

Thus taking the supremum of the left hand side of (A.3) over all possible $K \subset\subset \mathbb{R}^N$ we deduce (A.2). □

A.2 The notion of Γ-convergence

The asymptotic behavior, when $\varepsilon \to 0$ of the family $\{I_\varepsilon\}_{\varepsilon>0}$ of the functionals $I_\varepsilon(\phi) : \mathcal{T} \to [0, +\infty]$, where $\mathcal{T}$ is a given metric space, is partially described by the De Giorgi's Γ-limits, defined by:

$$(\Gamma - \liminf I_\varepsilon)(\phi) := \inf \left\{ \liminf_{\varepsilon \to 0^+} I_\varepsilon(\phi_\varepsilon) \ : \ \phi_\varepsilon \to \phi \text{ in } \mathcal{T} \text{ as } \varepsilon \to 0^+ \right\}, \tag{A.4}$$

$$(\Gamma - \limsup I_\varepsilon)(\phi) := \inf \left\{ \limsup_{\varepsilon \to 0^+} I_\varepsilon(\phi_\varepsilon) \ : \ \phi_\varepsilon \to \phi \text{ in } \mathcal{T} \text{ as } \varepsilon \to 0^+ \right\}, \tag{A.5}$$

$$(\Gamma - \lim I_\varepsilon)(\phi) := (\Gamma - \liminf I_\varepsilon)(\phi) = (\Gamma - \limsup I_\varepsilon)(\phi) \quad \text{in the case they are equal.} \tag{A.6}$$

It is useful to know the Γ-limit of I_ε, because it describes the asymptotic behavior as $\varepsilon \downarrow 0$ of minimizers of I_ε, as follows from the following simple well known result:

Proposition A.6 (De-Giorgi)**.** *Assume that ϕ_ε is a minimizer of I_ε for every $\varepsilon > 0$. Then:*

- *If $I_0(\phi) = (\Gamma - \liminf_{\varepsilon \to 0^+} I_\varepsilon)(\phi)$ and $\phi_\varepsilon \to \phi_0$ as $\varepsilon \to 0^+$, then ϕ_0 is a minimizer of I_0.*
- *If $I_0(\phi) = (\Gamma - \lim_{\varepsilon \to 0^+} I_\varepsilon)(\phi)$ (i.e. it is a full Γ-limit of $I_\varepsilon(\phi)$) and for some subsequence $\varepsilon_n \to 0^+$ as $n \to \infty$, we have $\phi_{\varepsilon_n} \to \phi_0$, then ϕ_0 is a minimizer of I_0.*

Remark A.7. Usually, for finding the Γ-limit of $I_\varepsilon(\phi)$, we need to find two bounds:

(*) Firstly, we wish to find a lower bound, i.e. the functional $\underline{I}(\phi)$ such that for every family $\{\phi_\varepsilon\}_{\varepsilon>0}$, satisfying $\phi_\varepsilon \to \phi$ in $\mathcal{T}$ as $\varepsilon \to 0^+$, we have $\liminf_{\varepsilon \to 0^+} I_\varepsilon(\phi_\varepsilon) \ge \underline{I}(\phi)$.

(**) Secondly, we wish to find an upper bound, i.e. the functional $\overline{I}(\phi)$ such that for every $\phi \in \mathcal{T}$ there exists the family $\{\psi_\varepsilon\}_{\varepsilon>0}$, satisfying $\psi_\varepsilon \to \phi$ in $\mathcal{T}$ as $\varepsilon \to 0^+$, and we have $\limsup_{\varepsilon \to 0^+} I_\varepsilon(\psi_\varepsilon) \le \overline{I}(\phi)$.

(***) In the general case we have

$$\underline{I}(\phi) \leq (\Gamma - \liminf_{\varepsilon\to 0^+} I_\varepsilon)(\phi) \leq (\Gamma - \limsup_{\varepsilon\to 0^+} I_\varepsilon)(\phi) \leq \overline{I}(\phi) \qquad \forall \phi \in \mathcal{T}.$$

(****) If we obtain $\underline{I}(\phi) = \overline{I}(\phi) := I(\phi)$, then $I(\phi)$ will be the full Γ-limit of $I_\varepsilon(\phi)$.

The upper and the lower bounds are usually proven separately with the help of completely different technics.

References

[1] G. Alberti, G. Bouchitté and P. Seppecher, Un résultat de perturbations singulières avec la norme $H^{1/2}$. *C. R. Acad. Sci. Paris* **319** (1994) 333–338.

[2] L. Ambrosio, Metric space valued functions of bounded variation. *Ann. Sc. Norm. Super Pisa Cl. Sci.* **17** (1990) 439–478.

[3] L. Ambrosio, C. De Lellis and C. Mantegazza, Line energies for gradient vector fields in the plane. *Calc. Var. Partial Differ. Equ.* **9** (1999) 327–355.

[4] L. Ambrosio, G. De Philippis and L. Martinazzi, Gamma-convergence of nonlocal perimeter functionals. *Manuscripta Math.* **134** (2010) 377–403.

[5] L. Ambrosio, N. Fusco and D. Pallara, Functions of Bounded Variation and Free Discontinuity Problems. *Oxford Mathematical Monographs*. Oxford University Press, New York (2000).

[6] P. Aviles and Y. Giga, A mathematical problem related to the physical theory of liquid crystal configurations. *Proc. Centre Math. Anal. Austral. Nat. Univ.* **12** (1987) 1–16.

[7] P. Aviles and Y. Giga, On lower semicontinuity of a defect energy obtained by a singular limit of the Ginzburg-Landau type energy for gradient fields. *Proc. R. Soc. Edinb. Sect. A* **129** (1999) 1–17.

[8] J. Bourgain, H. Brezis and P. Mironescu, Another Look at Sobolev Spaces. Optimal Control and Partial Differential Equations, edited by J.L. Menaldi, *et al.*, A volume in honour of A. Benssoussan's 60th birthday. IOS Press, Bristol (2001) 439–455.

[9] L. Caffarelli and E. Valdinoci, Regularity properties of nonlocal minimal surfaces via limiting arguments. *Adv. Math.* **248** (2013) 843–871.

[10] J. Dávila, On an open question about functions of bounded variation. *Calc. Var. Partial Differ. Equ.* **15** (2002) 519–527.

[11] S. Dipierro, F. Maggi and E. Valdinoci, Asymptotic expansions of the contact angle in nonlocal capillarity problems. *J. Nonlinear Sci.* **27** (2017) 1531–1550.

[12] A. Figalli and D. Jerison, How to recognize convexity of a set from its marginals. *J. Funct. Anal.* **266** (2014) 1685–1701.

[13] F. Hernández, Properties of a Hilbertian Norm for Perimeter. Preprint https://arxiv.org/abs/1709.08262 (2017).

[14] L. Modica, The gradient theory of phase transitions and the minimal interface criterion. *Arch. Ration. Mech. Anal.* **98** (1987) 123–142.

[15] L. Modica and S. Mortola, Un esempio di Γ^--convergenza. *Boll. Un. Mat. Ital. B* **14** (1977) 285–299.

[16] L. Modica and S. Mortola, Il limite nella Γ-convergenza di una famiglia di funzionali ellittici. *Boll. Un. Mat. Ital. A* **14** (1977) 526–529.

[17] G. Palatucci, O. Savin and E. Valdinoci, Local and global minimizers for a variational energy involving a fractional norm. *Ann. Mat. Pura Appl.* **192** (2013) 673–718.

[18] A. Poliakovsky, Upper bounds for singular perturbation problems involving gradient fields. *J. Eur. Math. Soc.* **9** (2007) 1–43.

[19] A. Poliakovsky, Jump detection in Besov spaces via a new BBM formula. Applications to Aviles-Giga type functionals. *Commun. Contemp. Math.* **20** (2018) 1750096.

[20] A. Ponce, A new approach to Sobolev spaces and connections to Gamma-convergence. *Calc. Var. Partial Differ. Equ.* **19** (2004) 229–255.

[21] O. Savin and E. Valdinoci, Γ-convergence for nonlocal phase transitions. *Ann. Inst. Henri Poincaré Anal. Non Linéaire* **29** (2012) 479–500.

[22] P. Sternberg, The effect of a singular perturbation on nonconvex variational problems. *Arch. Ration. Mech. Anal.* **101** (1988) 209–260.