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AN EXTRAGRADIENT-TYPE ALGORITHM FOR VARIATIONAL INEQUALITY ON HADAMARD MANIFOLDS

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Abstract. This paper presents an extragradient method for variational inequality associated with a point-to-set vector field in Hadamard manifolds, and a study of its convergence properties. To present our method, the concept of ϵ-enlargement of maximal monotone vector fields is used, and its lower-semicontinuity is established to obtain the method convergence in this new context.

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1. Introduction

In its classical formulation, the variational inequality problem is an inequality involving an operator, which has to be solved for all possible values of a given variable belonging to a convex set in a liner space. It is well known that the variational inequality problem is an abstract model for various problems in classical analysis and its applications, where the convexity and linear structure plays an important role. However, in many practical applications, the natural structure of the data are modeled as constrained optimization problems, where the constraints are non-linear and non-convex; more specifically, they are Riemannian manifolds [1, 4, 8, 8–12, 22, 25, 30, 31, 33, 35, 43, 45]. Due to such applications, interest in the development of optimization tools as well as mathematical programming methods to Riemannian settings has increased significantly. Papers published on this topic include, but are not limited to [2, 3, 7, 17, 18, 23, 26, 32, 46, 47, 49, 53–55].

In this paper, we consider finding a solution to a variational inequality problem on Riemannian context, first introduced and studied by [42], for univalued vector fields on Hadamard manifolds, and for multivalued vector fields on general Riemannian manifolds by [36]. For recent works addressing this subject, see [27, 38, 50, 51]. It is worth noting that constrained optimization problems, and the problem of finding the zero of a multivalued vector field, studied by [2, 7, 21, 28, 37, 53], are particular instances of the variational inequality. The aim of this paper is to present an extragradient-type algorithm for variational inequality associated with a point-to-set vector filed in Hadamard manifolds, and to study its convergence properties.

In our method, we utilize the concept of ϵ-enlargement introduced by [16] in Euclidean spaces, and generalized by [6] to the Riemannian context. It is worth mentioning that the concept of ϵ-enlargement in linear spaces has been successfully employed for a wide range of purposes as presented by [15], and its reference therein. In particular, the ϵ-enlargement concept was used to establish the iteration complexity of the hybrid proximal

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extragradient method, see [40]. The convergence analysis of the extragradient algorithm in linear setting associated to a point-to-point operator cannot be automatically extended to point-to-set operator. As remarked by [34], the reason for this failure is the lack of lower-semicontinuity of the arbitrary maximal monotone operator. [34] showed that the ϵ-enlargement of arbitrary maximal monotone operators are lower-semicontinuity, and then extended the extragradient algorithm to point-to-set operator. In this paper, we show that lower-semicontinuity remains valid for the ϵ-enlargement of arbitrary maximal monotone vector fields, and this allowed us to prove the convergence of the extragradient algorithm for variational inequalities associated to a point-to-set vector filed in Hadamard manifolds. Finally, we state that the proposed method has two important particular instances, namely, the methods (3.1) of [34] and (4.1) of [48].

The remainder of this paper is organized as follows: in Section 1.1, some notations and basic results used in the paper are presented; in Section 2, the concept of enlargement of monotone vector fields is introduced and some properties are obtained; in Section 3, an extragradient algorithm for variational inequalities is presented and its convergence properties are studied; and, finally, the conclusions are presented in Section 5.

1.1. Notation and terminology

In this section, we introduce fundamental properties and notations for Riemannian geometry. These basic facts can be found in any introductory book on Riemannian geometry, such as in [24] and [44].

Let M be a n-dimentional Hadamard manifold. *In this paper, all manifolds M are assumed to be Hadamard finite dimensional.* We denote by T_pM the n-dimentional *tangent space* of M at p, by $TM = \cup_{p\in M} T_pM$ the *tangent bundle* of M, and by $\mathcal{X}(M)$ the space of smooth vector fields on M. The Riemannian metric is denoted by $\langle \, , \, \rangle$, and the corresponding norm by $\| \; \|$. Denote the length of piecewise smooth curves $\gamma : [a, b] \to M$ joining p to q, *i.e.*, such that $\gamma(a) = p$ and $\gamma(b) = q$, by $\ell(\gamma)$ and the Riemannian distance by $\mathrm{d}(p, q)$, which induces the original topology on M, namely, (M, d) is a complete metric space and bounded and closed subsets are compact. For $A \subset M$, the notation $\mathrm{int}(A)$ means the interior of the set A, and if A is a nonempty set, the distance from $p \in M$ to A is given by $\mathrm{d}(p, A) := \inf\{\mathrm{d}(p, q) \; : \; q \in A\}$.

Let ∇ be the Levi-Civita connection associated to $(M, \langle \, , \, \rangle)$. A vector field V along γ is said to be *parallel* if $\nabla_{\gamma'} V = 0$. If γ' itself is parallel, we say that γ is a *geodesic*. Given that geodesic equation $\nabla_{\ \gamma'}\gamma' = 0$ is a second order nonlinear ordinary differential equation, then geodesic $\gamma = \gamma_v(., p)$ is determined by its position p and velocity v at p. It is easy to check that $\|\gamma'\|$ is constant. We say that γ is *normalized* if $\|\gamma'\| = 1$. The restriction of a geodesic to a closed bounded interval is called a *geodesic segment*.

As M is a Hadamard manifolds, the length of the geodesic segment γ joining p to q it is equal to $\mathrm{d}(p, q)$. The parallel transport along γ from p to q is denoted by $P_{pq} : T_pM \to T_qM$. The *exponential map* $\exp_p : T_pM \to M$, defined by $\exp_p v = \gamma_v(1, p)$, is a diffeomorphism and, consequently, M is diffeomorphic to the Euclidean space $\mathbb{R}^n$, $n = dimM$.

Let $q \in M$ and $\exp^{-1}_q : M \to T_pM$ be the inverse of the exponential map. Note that $d(q\,,\,p) = ||\exp^{-1}_p q||$, the map $d^2_q : M \to \mathbb{R}$ defined by $d^2_q(p) = d^2(q, p)$ is C^∞ and $\mathrm{grad}\, d^2_q(p) := -2\exp^{-1}_p q$.

Furthermore, we know that:

$$d^2(p_1, p_3) + d^2(p_3, p_2) - 2\left\langle \exp^{-1}_{p_3} p_1, \exp^{-1}_{p_3} p_2 \right\rangle \leq d^2(p_1, p_2), \qquad p_1, p_2, p_3 \in M. \tag{1.1}$$

$$\left\langle \exp^{-1}_{p_2} p_1,\, \exp^{-1}_{p_2} p_3 \right\rangle + \left\langle \exp^{-1}_{p_3} p_1,\, \exp^{-1}_{p_3} p_2 \right\rangle \geq d^2(p_2, p_3), \qquad p_1, p_2, p_3 \in M. \tag{1.2}$$

A set, $\Omega \subseteq M$ is said to be *convex*, if any geodesic segment with end points in Ω is contained in Ω. That is, if $\gamma : [a, b] \to M$ is a geodesic such that $x = \gamma(a) \in \Omega$ and $y = \gamma(b) \in \Omega$; then $\gamma((1-t)a + tb) \in \Omega$ for all $t \in [0, 1]$. Let $\Omega \subset M$ be a convex set and $p \in M$. Thus, the *projection* $\mathrm{P}_\Omega(p) := \{\tilde{q} \in \Omega \; : \; \mathrm{d}(\tilde{q}, p) \leq \mathrm{d}(q, p), \; q \in \Omega\}$ of p onto Ω satisfies:

$$\left\langle \exp^{-1}_{\mathrm{P}_\Omega(p)} q,\, \exp^{-1}_{\mathrm{P}_\Omega(p)} p \right\rangle \leq 0, \qquad q \in \Omega, \tag{1.3}$$

as presented by Corollary 3.1 from [28]. The metric projection onto a nonempty, closed and convex subset $\Omega \subset M$ is a firmly nonexpansive mapping ([39], Cor. 1). As a consequence, the projection mapping is nonexpansive, *i.e.* there holds

$$\mathrm{d}(\mathrm{P}_\Omega(p), \mathrm{P}_\Omega(q)) \leq \mathrm{d}(p, q) \qquad p, q \in \Omega. \tag{1.4}$$

Lemma 1.1. *Let $\bar{p}, \bar{q} \in M$ and $\{p^k\}, \{q^k\} \subset M$ be such that $\lim_{k\to+\infty} p^k = \bar{p}$ and $\lim_{k\to+\infty} q^k = \bar{q}$. Then, the following assertions hold.*

i) For any $q \in M$, $\lim_{k\to+\infty} \exp^{-1}_{p^k} q = \exp^{-1}_{\bar{p}} q$ and $\lim_{k\to+\infty} \exp^{-1}_{q} p^k = \exp^{-1}_{q} \bar{p}$;
ii) If $v^k \in T_{p^k}M$ and $\lim_{k\to+\infty} v^k = \bar{v}$, then $\bar{v} \in T_{\bar{p}}M$;
iii) For any $u \in T_{\bar{p}}M$, the function $F : M \to TM$ defined by $F(p) = \mathrm{P}_{\bar{p}p}u$ each $p \in M$ is continuous on M;
iv) $\lim_{k\to+\infty} \exp^{-1}_{p^k} q^k = \exp^{-1}_{\bar{p}} \bar{q}$.

Proof. For items (i), (ii) and (iii) see Lemma 2.4 from [37]. For the item (iv), using triangular inequality we obtain $\| \exp^{-1}_{p^k} q^k - \mathrm{P}_{\bar{p}p^k} \exp^{-1}_{\bar{p}} \bar{q} \| \leq \| \exp^{-1}_{p^k} q^k - \exp^{-1}_{p^k} \bar{q} \| + \| \exp^{-1}_{p^k} \bar{q} - \mathrm{P}_{\bar{p}p^k} \exp^{-1}_{\bar{p}} \bar{q} \|$. As M has nonpositive curvature, $\| \exp^{-1}_{p^k} q^k - \exp^{-1}_{p^k} \bar{q} \| \leq \mathrm{d}(q^k, q)$. It follows then that:

$$\| \exp^{-1}_{p^k} q^k - \mathrm{P}_{\bar{p}p^k} \exp^{-1}_{\bar{p}} \bar{q} \| \leq \mathrm{d}(q^k, \bar{q}) + \| \exp^{-1}_{p^k} \bar{q} - \mathrm{P}_{\bar{p}p^k} \exp^{-1}_{\bar{p}} \bar{q} \| .$$

Taking limit as k goes to infinity in the last inequality, and combining items (i) and (iii), we conclude that $\lim_{k\to+\infty} \exp^{-1}_{p^k} q^k = \exp^{-1}_{\bar{p}} \bar{q}$. □

In the following, we present a property of the parallel transport, which will be important to prove our main result:

Lemma 1.2. *Let $\bar{p} \in M$, $\bar{v} \in T_{\bar{p}}M$, $\bar{t} \in [0, 1)$, $\{p^k\} \subset M$, $\{v^k\} \subset T_{p^k}M$ and $\{t_k\} \subset (0, 1)$ be such that $\lim_{k\to+\infty} p^k = \bar{p}$, $\lim_{k\to+\infty} v^k = \bar{v}$ and $\lim_{k\to+\infty} t_k = \bar{t}$. Let $\{q^k\}$ be defined by $q^k := \exp_{p^k} t_k v^k$, for all $k = 0, 1, \ldots$. Then, there holds $\lim_{k\to+\infty} \mathrm{P}_{p^k q^k} v^k = \mathrm{P}_{\bar{p}\bar{q}} \bar{v}$. where $\bar{q} := \exp_{\bar{p}} \bar{t}\bar{v} = \lim_{k\to+\infty} q^k$.*

Proof. As M is a Hadamard manifold and $q^k = \exp_{p^k} t_k v^k$, for all $k = 0, 1, \ldots$, we have

$$-\frac{1}{t_k} \exp^{-1}_{p^k} q^k = v^k, \qquad k = 0, 1, \ldots \tag{1.5}$$

Hence, the definition of a parallel transport along of the geodesic $\gamma_k(t) = \exp_{p^k} t v^k$ implies that:

$$\frac{1}{1 - t_k} \exp^{-1}_{q^k} \exp_{p^k} v^k = -\frac{1}{t_k} \exp^{-1}_{q^k} p^k = P_{p^k q^k} v^k, \qquad k = 0, 1, \ldots$$

Thus, taking the limit as k goes to $+\infty$, and using Lemma 1.1, we conclude that

$$\lim_{k\to+\infty} P_{p^k z^k} v^k = \frac{1}{1 - \bar{t}} \exp^{-1}_{\bar{q}} \exp_{\bar{p}} \bar{v}.$$

As $\bar{q} = \lim_{k\to+\infty} q^k$, taking limit as k goes to $+\infty$ in (1.5), and combining with the last equality, we obtain the desired result. □

Let $\Omega \subset M$ be a convex set, and $p \in \Omega$. Following [37], we define the *normal cone* to Ω at p by:

$$N_\Omega(p) := \left\{ w \in T_pM \ : \ \langle w, \exp^{-1}_p q \rangle \leq 0, \ q \in \Omega \right\}. \tag{1.6}$$

Given a multivalued vector field X, the *domain* of X is the set $\mathrm{dom}X := \{p \in M \ : \ X(p) \neq \varnothing\}$. We also need to define the quantity $m_X(\Omega) := \sup_{q\in\Omega} \{\|u\| \ : \ u \in X(q)\}$. We say that X is *locally bounded* iff, for all $p \in \mathrm{int}(\mathrm{dom}\, X)$, there exist an open set $U \subset M$ such that $p \in U$, and there holds $m_X(U) < +\infty$, and *bounded on bounded sets* . Iff for all bounded set $V \subset M$ such that its closure $\overline{V} \subset \mathrm{int}(\mathrm{dom}\, X)$, it holds that $m_X(V) < +\infty$, see an equivalent definition in [37]. The multivalued vector field X is said to be *upper semicontinuous* at $p \in \mathrm{dom}X$ iff, for any open set $V \subset T_pM$ such that $X(p) \in V$, there exists an open set $U \subset M$ with $p \in U$ such that $P_{qp}X(q) \subset V$, for any $q \in U$. For two multivalued vector fields X, Y on M, the notation $X \subset Y$ implies that $X(p) \subset Y(p)$, for all $p \in M$.

Definition 1.3. A sequence $\{p^k\} \subset (M, d)$ is said to Fejér convergence to a nonempty set $W \subset M$ if, for every $q \in W$, we have $d^2(q, p^{k+1}) \leq d^2(q, p^k)$, for $k = 0, 1, \ldots$ and $q \in W$.

Proposition 1.4. *Let $\{p^k\}$ be a sequence in (M, d). If $\{p^k\}$ is Fejér convergent to nonempty set $W \subset M$, then $\{p^k\}$ is bounded. If furthermore, an accumulation point p of $\{p^k\}$ belongs to W, then $\lim_{k\to\infty} p^k = p$.*

Lemma 1.5. *Let $\{\rho_k\}$ be a sequence of positive real numbers and $\theta_0 > 0$. Define the sequence $\{\theta_k\}$ by $\theta_{k+1} = \mathrm{Minimize}\{\theta_k, \rho_k\}$. The limit $\bar{\theta}$ of $\{\theta_k\}$ is equal to 0 iff 0 is a cluster point of $\{\rho_k\}$.*

Proof. See Lemma 4.9 from [34]. □

A multivalued vector field X is said to be *monotone* iff

$$\left\langle P_{pq}u - v, \, \exp_q^{-1} p \right\rangle \geq 0, \qquad p, \, q \in \mathrm{dom}X, \quad u \in X(p), \ v \in X(q). \tag{1.7}$$

Moreover, a monotone vector field X is said to be *maximal monotone*, iff for each $p \in \mathrm{dom}\, X$ and $u \in T_pM$, there holds:

$$\left\langle P_{pq}u - v, \, \exp_q^{-1} p \right\rangle \geq 0, \qquad q \in \mathrm{dom}X, \quad v \in X(q) \ \Rightarrow \ u \in X(p). \tag{1.8}$$

The concept of monotonicity in the Riemannian context was first introduced in [41], for a single-valued case and in [19] for a multivalued case. Further, the notion of maximal monotonicity of a vector field was introduced by [37], and Theorem 5.1 from [37] showed that the subdifferential of the convex function is maximal monotone. The next result is an extension to the Hadamard manifolds of its counterpart Euclidean and its proof is an immediate consequence of the definition of maximal monotonicity and normal cone, respectively.

Lemma 1.6. *Let $\Omega \subset M$ be a convex set and X be a maximal monotone vector field such that $\mathrm{dom}\, X = M$. Then, $X + N_\Omega$ is a maximal monotone vector field.*

A multivalued vector field X is said to be *upper Kuratowski semicontinuous* at p if, for any open set V satisfying $X(p) \subset V \subset T_pM$, there exists an open neighborhood U of p such that $P_{qp}X(q) \subset V$ for any $q \in U$. When X is upper Kuratowski semicontinuous, for any $p \in \mathrm{dom}X$, we say that X is upper Kuratowski semicontinuous [37]. It has been showed in [37] that maximal monotone vector fields are upper semicontinuous Kuratowski and, if additionally $\mathrm{dom}X = M$, then X is locally bounded M.

2. The enlargement of monotone vector fields

In this section, we recall some properties of the ϵ-enlargement of maximal monotone vector fields and show its lower semicontinuous to obtain the convergence of the extragradient method. We begin by recalling the notion of ϵ-enlargement for multivalued vector fields on Hadamard manifolds introduced in [6].

Definition 2.1. Let X be a multivalued monotone vector field on M and $\epsilon \geq 0$. The ϵ-enlargement $X^\epsilon : M \rightrightarrows TM$ of X is the multivalued vector field defined by:

$$X^\epsilon(p) := \left\{ u \in T_pM \ : \ \left\langle P_{pq}u - v, \, \exp_q^{-1} p \right\rangle \geq -\epsilon, \ q \in \mathrm{dom}X, \ v \in X(q) \right\}, \qquad p \in \mathrm{dom}X. \tag{2.1}$$

Now, we present a specific example to show how large the enlargement can become.

Example 2.2. Let $\epsilon \geq 0$ and $\bar{p} \in M$. Define the closed ball at the origin 0_{T_pM} of T_pM and radius $2\sqrt{2\epsilon}$ by

$$B\left[0_{T_pM},\ 2\sqrt{2\epsilon}\right] := \left\{w \in T_pM\ : \parallel w \parallel \leq 2\sqrt{2\epsilon}\right\}.$$

Denoted by $\partial^\epsilon d^2_{\bar{p}}(\cdot)$, the enlargement of the vector field $\partial d^2_{\bar{p}}(\cdot) = \{\text{grad}\, d^2_{\bar{p}}(\cdot)\}$. In Example 3.1 from [6] we showed that $\partial d^2_{\bar{p}}(p) + B\left[0_{T_pM},\ 2\sqrt{2\epsilon}\right] \subseteq \partial^\epsilon d^2_{\bar{p}}(p)$, for all $p \in M$.

The next proposition is a combination of Propositions 2.6 and 2.2 from [6].

Proposition 2.3. *Let X be a monotone vector field on M and $\epsilon \geq 0$. Then,* $\text{dom}X \subset \text{dom}X^\epsilon$. *In particular, if* $\text{dom}X = M$ *then* $\text{dom}X^\epsilon = \text{dom}X$. *Moreover, if X is maximal monotone then $X^0 = X$ and X^ϵ is bounded on bounded sets.*

Throughout the paper we will also need of the following properties of ϵ-enlargement of maximal monotone vector fields. Its proofs can be found in Proposition 2.3 from [6].

Proposition 2.4. *Let X, X_1 and X_2 be multivalued monotone vector fields on M and $\epsilon, \epsilon_1, \epsilon_2 \geq 0$. Then, there hold:*

i) If $\epsilon_1 \geq \epsilon_2 \geq 0$ then $X^{\epsilon_2} \subset X^{\epsilon_1}$;
ii) $X_1^{\epsilon_1} + X_2^{\epsilon_2} \subset (X_1 + X_2)^{\epsilon_1+\epsilon_2}$;
iii) $X^\epsilon(p)$ is closed and convex for all $p \in M$;
iv) $\alpha X^\epsilon = (\alpha X)^{\alpha\epsilon}$ for all $\alpha \geq 0$;
v) $\alpha X_1^\epsilon + (1-\alpha)X_2^\epsilon \subset (\alpha X_1 + (1-\alpha)X_2)^\epsilon$ for all $\alpha \in [0, 1]$;
vi) If $E \subset \mathbb{R}_+$, then $\bigcap_{\epsilon\in E} X^\epsilon = X^{\bar{\epsilon}}$ with $\bar{\epsilon} = \inf E$.

Moreover, if $\{\epsilon^k\}$ is a sequence of positive numbers and $\{(p^k,\ u^k)\}$ is a sequence in TM such that $\overline{\epsilon} = \lim_{k\to\infty} \epsilon^k$, $\overline{p} = \lim_{k\to\infty} p^k$, $\overline{u} = \lim_{k\to\infty} u^k$ and $u^k \in X^{\epsilon_k}(p^k)$ for all k, then $\overline{u} \in X^{\overline{\epsilon}}(\overline{p})$.

In the next definition we extend the notion of lower semicontinuity of a multivalued operator, which has been introduced in [16], to a vector field.

Definition 2.5. A multivalued vector field $Y : M \rightrightarrows TM$ with $Y(p) \subset T_pM$ is said to be lower semicontinuous at $\bar{p} \in \text{dom}Y$ if, for each sequence $\{p^k\} \subset \text{dom}Y$ such that $\lim_{k\to+\infty} p^k = \bar{p}$ and each $\bar{u} \in Y(\bar{p})$, there exists a sequence $\{w^k\}$ such that $w^k \in Y(p^k)$ and $\lim_{k\to\infty} P_{p^k\bar{p}}w^k = \bar{u}$.

The next result is a generalization of Theorem 4.1 from [34], which will play an important role in the convergence analysis of the extragradient method.

Theorem 2.6. *Let $X : M \rightrightarrows TM$ with $X(p) \subset T_pM$ be a maximal monotone vector field and $\epsilon > 0$. If* $\text{dom}X = M$ *then X^ϵ is lower semicontinuous.*

Proof. As $\text{dom}X = M$, thus Proposition 2.3 implies $\text{dom}X = \text{dom}X^\epsilon$. Take $\{p^k\} \subset M$ such that $\lim_{k\to+\infty} p^k = \bar{p}$, and $\bar{u} \in X^\epsilon(\bar{p})$. First, we prove that the following statements there hold:

(i) For each $0 < \theta < 1$ and $u^k \in X(p^k)$, there exists $k_0 \in \mathbb{N}$ such that $(1-\theta)P_{\bar{p}p^k}\bar{u} + \theta u^k \in X^\epsilon(p^k)$, for all $k > k_0$;
(ii) Take $\nu > 0$. Then, there exist $k_0 \in \mathbb{N}$ and $v_k \in X^\epsilon(p^k)$ such that $\|\bar{u} - P_{p^k\bar{p}}v_k\| \leq \nu$, for all $k > k_0$.

For proving (i), take $q \in M$ and $v \in X(q)$. Then, simple algebraic manipulations yield

$$\left\langle P_{p^kq}[(1-\theta)P_{\bar{p}p^k}\bar{u} + \theta u^k] - v,\ \exp_q^{-1} p^k\right\rangle = (1-\theta)\left\langle P_{\bar{p}q}\bar{u} - v, \exp_q^{-1} p^k\right\rangle + \theta\left\langle P_{p^kq}u^k - v, \exp_q^{-1} p^k\right\rangle.$$

As X is monotone, the second term in the right-hand side of the last inequality is positive. Thus,

$$\left\langle P_{p^k q}[(1-\theta)P_{\bar{p}p^k}\bar{u} + \theta u^k] - v,\ \exp_q^{-1} p^k \right\rangle \geq (1-\theta)\left\langle P_{\bar{p}q}\bar{u} - v, \exp_q^{-1} p^k \right\rangle. \tag{2.2}$$

Considering that $\bar{u} \in X^{\epsilon}(\bar{p})$ and $\lim_{k\to\infty}\left\langle P_{\bar{p}q}\bar{u} - v, \exp_q^{-1} p^k \right\rangle = \left\langle P_{\bar{p}q}\bar{u} - v, \exp_q^{-1} \bar{p} \right\rangle \geq -\epsilon$ then, for all $\delta > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$\left\langle P_{\bar{p}q}\bar{u} - v, \exp_q^{-1} p^k \right\rangle \geq -\epsilon - \delta, \qquad k > k_0. \tag{2.3}$$

Letting $\delta = \theta\epsilon/(1-\theta)$, from (2.2) and (2.3) we have $\langle P_{p^k q}[(1-\theta)P_{\bar{p}p^k}\bar{u} + \theta u^k] - v,\ \exp_q^{-1} p^k\rangle \geq -\epsilon$, for all $k > k_0$, which prove the item (i). For proving the item (ii), take $\eta > 0$ and consider the following constants:

$$\sigma := \sup\left\{\|\ u\ \|:\ u \in X^{\epsilon}(B(\bar{p},\eta))\right\}, \qquad \gamma := \min\{(\epsilon/2\sigma), \eta\}, \qquad 0 < \mu < \min\{1, (\nu/2\sigma)\}.$$

Take any $u^k \in X(p^k)$. Applying item (i) with $\theta = \mu$, we conclude that there exists $k_0 \in \mathbb{N}$ such that $(1-\mu)P_{\bar{p}p^k}\bar{u} + \mu u^k \in X^{\epsilon}(p^k)$, for all $k \geq k_0$. We shall prove that, taking $v_k = (1-\mu)P_{\bar{p}p^k}\bar{u} + \mu u^k$, we have $\|\ \|\bar{u} - P_{p^k\bar{p}}v_k\|\ \| \leq \nu$, for all $k \geq k_0$. First note that, with some manipulation, and taking into account that the parallel transport is an isometry, we have

$$\|\bar{u} - P_{p^k\bar{p}}v_k\| = \mu\ \|\ \bar{u} - P_{p^k\bar{p}}u^k\ \| \leq \mu(\|\ \bar{u}\ \| + \|\ u^k\ \|), \qquad k \geq k_0. \tag{2.4}$$

As $\lim_{k\to+\infty} p^k = \bar{p}$, there exist k_0 such that $p^k \in B(\bar{p},\gamma)$, for all $k \geq k_0$. Thus, with $u^k \in X(p^k) \subset X^{\epsilon}(p^k)$ and $B(\bar{p},\gamma) \subset B(\bar{p},\eta)$, the definition of σ gives $\|\ u^k\ \| \leq \sigma$, for all $k \geq k_0$. Due to $\bar{u} \in X^{\epsilon}(\bar{p})$ we also have $\|\ \bar{u}\ \| \leq \sigma$. Hence, using (2.4) and the definition of μ, we obtain $\|\ \bar{u} - v_k\ \| \leq 2\sigma\mu \leq \nu$, for all $k \geq k_0$, and the proof of item (ii) is proved. Finally, we define the sequence $\{w^k\}$ as $w^k := \text{argmin}\{\|\bar{u} - P_{p^k\bar{p}}u\|\ :\ u \in X^{\epsilon}(p^k)\}$ for each k. As, for each k the set $X^{\epsilon}(p^k)$ is closed and convex, the sequence $\{w^k\}$ is well defined. We claim that $\lim_{k\to\infty} P_{p^k\bar{p}}w^k = \bar{u}$. Otherwise, there exists $\{p^{k_j}\}$, a subsequence of $\{p^k\}$, and some $\nu > 0$ such that $\|\ \bar{u} - P_{p^{k_j}\bar{p}}w^{k_j}\ \| > \nu$ for all j. Definition of $\{w^k\}$ implies that $\|\ \bar{u} - P_{p^{k_j}\bar{p}}u\ \| > \nu$ for all $u \in X^{\epsilon}(p^{k_j})$ and all j. On the other hand, considering that $\lim_{k_j\to+\infty} p^{k_j} = \bar{p}$, $\bar{u} \in X^{\epsilon}(\bar{p})$ and the item (ii) holds, for all $\nu > 0$, we have a contraction. Therefore, the claim is proven and the proof is concluded. □

Remark 2.7. It is well known that the lower semicontinuity of operators is an important property, which is useful to obtain convergence results of iterative process in the Euclidean space. We also remark that, in general, maximal monotone operator are not lower semicontinuous ([34], Sect. 2). The result in Theorem 2.6 establishes this property to vector fields, which will be used in the next section to prove the convergence of the extragradient algorithm on Riemannian manifolds.

3. An extragradient-type algorithm for variational inequality

In this section, we introduce an extragradient-type algorithm for variational inequalities problem in Hadamard manifolds. The variational inequality problem was first introduced in [42], for single-valued vector fields on Hadamard manifolds, and in [36] for multivalued vector fields in general Riemannian manifolds. Let $X : M \rightrightarrows TM$ be a multivalued vector field and $\Omega \subset M$ be a nonempty set. The *variational inequality problem* VIP(X,Ω) consists of finding $p^* \in \Omega$, such that there exists $u^* \in X(p^*)$ satisfying

$$\langle u^*, \exp_{p^*}^{-1} q\rangle \geq 0, \qquad q \in \Omega.$$

Denote by $S^*(X,\Omega)$ *the solution* set of $\mathrm{VIP}(X,\Omega)$. From the definition of a normal cone, we can show that $\mathrm{VIP}(X,\Omega)$ becomes the problem of finding $p^* \in \Omega$, such that

$$0 \in X(p^*) + N_\Omega(p^*). \tag{3.1}$$

Note that, if $\Omega = M$ then $N_\Omega(p) = \{0\}$. Hence, VIP(X,Ω) becomes the problem of finding $p^* \in \Omega$, such that $0 \in X(p^*)$. This particular instance has been studied in several papers, including [29, 37].

Lemma 3.1. *The following statements are equivalent:*

i) p^* *is a solution of* $\mathrm{VIP}(X,\Omega)$*;*
ii) *There exists* $u^* \in X(p^*)$ *such that* $p^* = P_\Omega(\exp_{p^*}(-\alpha u^*))$*, for some* $\alpha > 0$.

Proof. It is an immediate consequence of the inequality (1.3). □

To analyze the extragradient algorithm, we need the following three assumptions:

A1. $\mathrm{dom}\, X = M$ and $\Omega \subset M$ is closed and convex;
A2. X is maximal monotone;
A3. $S^*(X,\Omega) \neq \varnothing$.

We also need the following assumption, which plays an important role in the convergence analyses of our extragradient algorithm in Hadamard manifolds.

A4. For each $y \in M$ and $v \in T_yM$ the following set is convex

$$S := \left\{x \in M : \langle v, \exp_y^{-1} x\rangle \leq 0\right\}. \tag{3.2}$$

Remark 3.2. If the manifold M is the Euclidean space, then S in (3.2) is a closed semi-space. [29] showed that for Hadamard manifolds with constant curvature or two-dimensionality, the set S is convex. However, so far, it is not known if S is or not convex in general Hadamard manifolds. As we shall see, in Lemma 3.3, the hypothesis A2 plays an important role in the proof of Fejér convergence of the sequence generated by our algorithm, which is very important in our analysis. An extensive study about the set S can be found in [56].

Next, we present an *extragradient-type algorithm for finding a solution of VIP*(X,Ω).

Extragradient-type algorithm

Our algorithm requires six exogenous constants:

$$\epsilon > 0, \qquad 0 < \delta_- < \delta_+ < 1, \qquad 0 < \alpha_- < \alpha_+, \qquad 0 < \beta < 1,$$

and two exogenous sequences $\{\alpha_k\}$ and $\{\beta_k\}$ satisfying the following conditions:

$$\alpha_k \in [\alpha^-,\ \alpha^+], \qquad \beta_k \in [\beta,\ 1], \qquad k = 0, 1, \ldots$$

1. Initialization: $p^0 \in \Omega, \qquad \epsilon_0 = \epsilon$.
2. Iterative step: Given p^k and ϵ_k,

(a) *Selection of* u^k : Find

$$u^k \in X^{\epsilon_k}(p^k), \tag{3.3}$$

such that

$$\left\langle w, -\exp_{p^k}^{-1} P_\Omega(\exp_{p^k}(-\alpha_k u^k))\right\rangle \geq \frac{\delta_+}{\alpha_k} d^2\left(p^k, P_\Omega(\exp_{p^k}(-\alpha_k u^k))\right), \quad w \in X^{\epsilon_k}(p^k). \tag{3.4}$$

Define,

$$z^k := P_\Omega\left(\exp_{p^k}(-\alpha_k u^k)\right). \tag{3.5}$$

(b) *Stopping criterion*: If $p^k = z^k$, then stop. Otherwise,
(c) *Selection of λ_k and v^k* : Define $\gamma_k(t) := \exp_{p^k} t \exp_{p^k}^{-1} z^k$ and let

$$i(k) := \min\left\{i \geq 0 \ : \ \exists\, v^{k,i} \in X(y^{k,i}),\ \left\langle v^{k,i}, \gamma_k'(2^{-i}\beta_k)\right\rangle \leq -\frac{\delta_-}{\alpha_k} d^2(p^k, z^k)\right\}, \tag{3.6}$$

where

$$y^{k,i} := \gamma_k\left(2^{-i}\beta_k\right). \tag{3.7}$$

Define

$$y^k := \exp_{p^k} \lambda_k \exp_{p^k}^{-1} z^k, \qquad \lambda_k := 2^{-i(k)}\beta_k, \tag{3.8}$$

$$v^k := v^{k,i(k)}. \tag{3.9}$$

(d) *Definition of p^{k+1} and ϵ_{k+1}*: Define

$$q^k := P_{S_k}(p^k), \qquad S_k := \{p \in M : \langle v^k, \exp_{y^k}^{-1} p\rangle \leq 0\}, \tag{3.10}$$

$$p^{k+1} := P_\Omega(q^k), \qquad \epsilon_{k+1} := \min\left\{\epsilon_k,\ d^2(p^k, z^k)\right\}, \tag{3.11}$$

set $k \leftarrow k+1$ and go to Iterative step.

Remark 3.3. Assuming that X is an univalued monotone vector field and $\epsilon_k = 0$, for all k, the previous algorithm becomes the algorithm presented in [48] (see also [29]). Moreover, in the particular case where $M = \mathbb{R}^n$, it merges into the extragradient algorithm presented in [34]. It is worth noting that if assumption **A4** fails, then step (2) in the algorithm, in general, cannot be performed.

The next result establishes the well-definedness of the above algorithm.

Lemma 3.4. *The following statements hold:*

(i) *$p^k \in \Omega$, for all k;*
(ii) *there exists u^k satisfying (3.3) and (3.4), for each k;*
(iii) *if $p^k \neq z^k$, then $i(k)$ is well defined.*

Proof. The proof of item (i) follows from the initialization step and (3.11). For proving item (ii), we define, for each k, the bifunction $f_k : T_pM \times T_pM \to \mathbb{R}$ by

$$f_k(u, v) = \left\langle \exp_{p^k}^{-1} z^k,\ u - v\right\rangle. \tag{3.12}$$

In view of Proposition 2.3, and item (iii) of Proposition 2.4, we have $X^{\epsilon_k}(p^k)$ compact and convex. Hence, applying Basic Existence Theorem on page 3 from [13] (see also [5], Thm. 3.1) with $C = X^{\epsilon_k}(p^k)$ and $f = f_k$, we conclude that there exists $u^k \in X^{\epsilon_k}(p^k)$ such that $f_k(u^k, v) \geq 0$ for all $v \in X^{\epsilon_k}(p^k)$, and from (3.12) we obtain

$$\langle \exp_{p^k}^{-1} z^k, u^k\rangle \geq \langle \exp_{p^k}^{-1} z^k, v\rangle, \qquad v \in X^{\epsilon_k}(p^k). \tag{3.13}$$

On the other hand, using inequality (1.2) with $p_1 = q$, $p_2 = p^k$ and $p_3 = z^k$ we have

$$\langle \exp_{p^k}^{-1} q,\, \exp_{p^k}^{-1} z^k\rangle + \langle \exp_{z^k}^{-1} q,\, \exp_{z^k}^{-1} p^k\rangle \geq d^2(p^k, z^k),$$

where $q = \exp_{p^k}(-\alpha_k u^k/\delta_+)$. As $z^k = P_\Omega(q)$, it follows from (1.3) and the last inequality that

$$\langle \exp_{p^k}^{-1} q, \exp_{p^k}^{-1} z^k\rangle \geq d^2(p^k, z^k).$$

Thus, considering that $q = \exp_{p^k}(-\alpha_k u^k/\delta_+)$, we have $\langle u^k, \exp_{p^k}^{-1} z^k\rangle \leq -\delta_+ d^2(p^k, z^k)/\alpha_k$ which, combined with (3.13), gives the desired result.

To prove item (iii) we proceed by contradiction. Fix k and assume that, for each i, there holds

$$\left\langle v^{k,i},\ \gamma_k'(2^{-i}\beta_k)\right\rangle > -\frac{\delta_-}{\alpha_k} d^2(p^k, z^k), \qquad v^{k,i} \in X(y^{k,i}). \tag{3.14}$$

First note that, from (3.7), we have $\{y^{k,i}\}$ belonging to the geodesic segment joining p_k to $\gamma_k(\beta_k)$. Thus, $\{y^{k,i}\}$ is a bounded sequence and, consequently, Proposition 3.5 from [37] implies that $\{X(y^{k,i})\}$ is also a bounded. Considering that $v^{k,i} \in X(y^{k,i})$ for each i, without loss of generality, we can assume that $\{v^{k,i}\}$ converges to $\bar{v}$. Letting i goes to infinity in (3.14) and taking into account that $\lim_{i\to\infty} v^{k,i} = \bar{v}$, $\gamma(0) = p^k$ and $\gamma'(0) = \exp_{p^k}^{-1} z^k$, we conclude that

$$\left\langle \bar{v},\ \exp_{p^k}^{-1} z^k \right\rangle \geq -\frac{\delta_-}{\alpha_k} d^2(p^k, z^k). \tag{3.15}$$

On the other hand, as $\lim_{i\to\infty} y^{k,i} = p^k$, $\lim_{i\to\infty} v^{k,i} = \bar{v}$ and $v^{k,i} \in X(y^{k,i})$ for each i, using Proposition 3.5 from [37] we have $\bar{v} \in X(p^k)$. Thus, combining Propositions 2.3 and 2.4 (i) we obtain $\bar{v} \in X^{\epsilon}(p^k)$. Hence, using (3.5) and taking $w = \bar{v}$ the inequality (3.4) becomes

$$\langle \bar{v}, \exp_{p^k}^{-1} z^k)\rangle \leq -\frac{\delta_+}{\alpha_k} d^2(p^k, z^k). \tag{3.16}$$

As $0 < \delta^- < \delta^+$, the inequalities (3.15) and (3.16) imply that $\mathrm{d}(p^k, z^k) = 0$, which is a contradiction with $p^k \neq z^k$. Therefore, $i(k)$ is well defined and the proof of the proposition is done. □

From now on, $\{p^k\}$, $\{q^k\}$, $\{y^k\}$, $\{z^k\}$, $\{v^k\}$, $\{u^k\}$ and $\{\epsilon_k\}$ denote sequences generated by extragradient-type algorithm. To prove the convergence of $\{p^k\}$ to a point of the solution set $S^*(X, \Omega)$, we need some preliminary results.

Lemma 3.5. *The sequence $\{p^k\}$ is Féjer convergent to $S^*(X, \Omega)$ and $\lim_{k\to\infty} \mathrm{d}(q^k, p^k) = 0$.*

Proof. First, let us show that, for all $p^* \in S^*(X, \Omega)$ there holds

$$d^2(p^*, p^{k+1}) \leq d^2(p^*, p^k) - d^2(q^k, p^k), \qquad k = 0, 1, \ldots \tag{3.17}$$

For that, take $u^* \in X(p^*)$ and fix k. Hence, due the monotonicity of X, we conclude that

$$\langle v^k, \exp^{-1}_{y^k} p^* \rangle \leq 0.$$

Thus, in view of (3.10), we obtain $p^* \in S_k$. On the other hand, applying (1.1) with $p_1 = p^*$, $p_2 = p_k$ and $p_3 = q_k$, we have:

$$d^2(p^*, p^k) \geq d^2(p^*, q^k) + d^2(q^k, p^k) - 2\left\langle \exp^{-1}_{q^k} p^*, \exp^{-1}_{q^k} p^k \right\rangle.$$

As $p^* \in S_k$ and $q^k = P_{S_k}(p^k)$, the last inequality implies that

$$d^2(p^*, p^k) \geq d^2(p^*, q^k) + d^2(q^k, p^k).$$

Analogously, applying (1.1) with $p_1 = p^*$, $p_2 = q_k$ and $p_3 = p_{k+1}$, and considering that $p^{k+1} := P_\Omega(q^k)$ and $p^* \in \Omega$, we conclude that:

$$d^2(p^*, q^k) \geq d^2(p^*, p^{k+1}) + d^2(q^k, p^{k+1}).$$

Combining the two last inequalities, we obtain $d^2(p^*, p^k) \geq d^2(q^k, p^k) + d^2(p^*, p^{k+1}) + d^2(q^k, p^{k+1})$, which implies (3.17). In particular, (3.17) implies that $\{p^k\}$ is Féjer convergent to $S^*(X, \Omega)$ and $\{\mathrm{d}(p^*, p^k)\}$ is nonincreasing. Therefore, as $\{\mathrm{d}(p^*, p^k)\}$ is inferiorly limited, we conclude that $\{\mathrm{d}(p^*, p^k)\}$ converges and, taking into account (3.17), the desired result follows. □

Lemma 3.6. *If the sequence $\{p^k\}$ is infinity then $\lim_{k\to\infty} \epsilon_k = 0$. Moreover, all cluster points of $\{p^k\}$ belong to $S^*(X, \Omega)$.*

Proof. Suppose that the sequence $\{p^k\}$ is infinity, *i.e.*, the algorithm does not stop. Thus, by the stopping criterion $\mathrm{d}(p^k, z^k) > 0$, for all k. As (3.11) implies that $\{\epsilon_k\}$ is a nonincreasing monotone sequence and, being nonnegative, it follows that it converges. Set $\bar{\epsilon} := \lim_{k\to+\infty} \epsilon_k$. We are going to prove that $\bar{\epsilon} = 0$. First of all, note that from Lemma 3.5 the sequence $\{p^k\}$ is Fejér convergent to $S^*(X, \Omega)$ and, due to **A3**, we have $S^*(X, \Omega) \neq \varnothing$. Hence, we conclude that $\{p^k\}$ is bounded. On the other hand, considering that $\{p^k\}$ is bounded, Proposition 2.3 implies that $X^{\epsilon_0}(\{p^k\}) = \cup_{k=0}^{\infty} X^{\epsilon_0}(p^k)$ is bounded. As $\epsilon_k \leq \epsilon_0$, the item *i)* of Proposition 2.4 implies that $X^{\epsilon_k} \subset X^{\epsilon_0}$, for all k, and from (3.3) we obtain that $\{u^k\}$ is bounded. Definitions of λ_k and y^k in (3.8) implies that y^k belongs to the geodesic joining p^k to z^k and, using (1.4) and (3.5), we have

$$\mathrm{d}(p^k, y^k) \leq \mathrm{d}(p^k, z^k) = \mathrm{d}\left(P_\Omega(p^k), P_\Omega\left(\exp_{p^k}(-\alpha_k u^k)\right)\right) \leq \mathrm{d}(p^k, \exp_{p^k}(-\alpha_k u^k)) = \|\alpha_k u^k\|,$$

for $k = 0, 1, \ldots$ In view of the boundedness of the sequences $\{p^k\}$, $\{u^k\}$ and $\{\alpha_k\}$, we obtain from the last inequalities that $\{y^k\}$ and $\{z^k\}$ are bounded. Considering that $v^k \in X(y^k)$, for all k, we apply Proposition 2.3 to conclude that $\{v^k\}$ is bounded. Note that (3.6)–(3.9) imply that

$$\left\langle v^k, \gamma_k'(\lambda_k)\right\rangle \leq -\frac{\delta_-}{\alpha_k} d^2(p^k, z^k), \qquad k = 0, 1, \ldots$$

Combining (3.6)–(3.8), it follows that $\gamma'(\lambda_k) = -\lambda_k^{-1} \exp^{-1}_{y^k} p^k$, for $k = 0, 1, \ldots$. Thus, taking into account that $0 < \alpha_k < \alpha_+$, the last inequality becomes

$$\left\langle v^k, \exp^{-1}_{y^k} p^k \right\rangle \geq \frac{\lambda_k \delta_-}{\alpha_+} d^2(p^k, z^k), \qquad k = 0, 1, \ldots. \tag{3.18}$$

As $\{p^k\}$, $\{u^k\}$, $\{v^k\}$, $\{z^k\}$, $\{y^k\}$, $\{\alpha_k\}$, and $\{\lambda_k\}$ are bounded, without loss of generality, we can assume that they have subsequences $\{p^{k_j}\}$, $\{u^{k_j}\}$, $\{v^{k_j}\}$, $\{z^{k_j}\}$, $\{y^{k_j}\}$, $\{\alpha_{k_j}\}$ and $\{\lambda_{k_j}\}$ converging to $\bar{p}$, $\bar{u}$, $\bar{v}$, $\bar{z}$, $\bar{y}$, $\bar{\alpha}$ and $\bar{\lambda}$, respectively. Note that, (3.10) yields

$$q^{k_j} \in S_{k_j} = \left\{p \in M \ : \ \left\langle v^{k_j}, \exp^{-1}_{y^{k_j}} p \right\rangle \leq 0\right\}, \qquad j = 0, 1, \ldots$$

Using Lemma 3.5 we have: $\lim_{j\to\infty} p^{k_j} = \lim_{j\to\infty} q^{k_j} = \bar{p}$. Thereby, the latter inclusion together with item (iv) of Lemma 1.1 and $\lim_{j\to\infty} y^{k_j} = \bar{y}$ imply that

$$\lim_{j\to\infty} \left\langle v^{k_j}, \exp^{-1}_{y^{k_j}} p^{k_j} \right\rangle = \lim_{j\to\infty} \left\langle v^{k_j}, \exp^{-1}_{y^{k_j}} q^{k_j} \right\rangle \leq 0.$$

Thus, combining the last inequality with (3.18), it follows that

$$\lim_{j\to\infty} \lambda_{k_j} d^2(p^{k_j}, z^{k_j}) = 0. \tag{3.19}$$

Considering that $\lim_{j\to\infty} \lambda_{k_j} = \bar{\lambda}$, we have two possibilities: either $\bar{\lambda} > 0$ or $\bar{\lambda} = 0$. First, let us assume that $\bar{\lambda} > 0$. As $\lim_{j\to\infty} p^{k_j} = \bar{p}$ and $\lim_{j\to\infty} z^{k_j} = \bar{z}$, we obtain, from (3.19) that $\mathrm{d}(\bar{p}, \bar{z}) = \lim_{j\to\infty} \mathrm{d}(p^{k_j}, z^{k_j}) = 0$, and, consequently, $\bar{p} = \bar{z}$. Moreover, 0 is a cluster point of $\{d^2(p^k, z^k)\}$. Hence, taking into account that (3.11), we can apply Lemma 1.5 with $\theta_k = \epsilon_k$ and $\rho_k = d^2(p^k, z^k)$ to conclude that $0 = \lim_{k\to+\infty} \epsilon_k = \bar{\epsilon}$. Considering $u^{k_j} \in X^{\epsilon_{k_j}}(p^{k_j})$, combining Propositions 2.4 and 2.3, we conclude that $\bar{u} \in X(\bar{p})$. Therefore, Lemma 3.1 implies that $\bar{p} \in S^*(X, \Omega)$. Now, let us assume that $\bar{\lambda} = 0$. In this case, using Lemma 1.1 and (3.7), we conclude that $\lim_{j\to\infty} y^{k_j, i(k_j)-1} = \bar{p}$. From Proposition 2.3, we can take a sequence $\{\xi^j\}$. such that $\xi^j \in X(y^{k_j, i(k_j)-1})$ with $\lim_{j\to\infty} \xi^j = \bar{\xi}$ and, by using Proposition 3.5 from [37], we conclude that $\bar{\xi} \in X(\bar{p})$. On the other hand, (3.6) implies

$$-\left\langle \xi^j, \gamma'_{k_j}(2^{-i(k_j)+1}\beta_{k_j}) \right\rangle < \frac{\delta^-}{\alpha_{k_j}} d^2(p^{k_j}, z^{k_j}), \quad j = 0, 1, \ldots$$

Considering that $\gamma'_{k_j}(2^{-i(k_j)+1}\beta_{k_j}) = P_{p^{k_j} y^{k_j, i(k_j)-1}} \exp^{-1}_{p^{k_j}} z^{k_j}$ and the parallel transport is an isometry, the last inequality yields

$$-\left\langle \xi^j, P_{p^{k_j} y^{k_j, i(k_j)-1}} \exp^{-1}_{p^{k_j}} z^{k_j} \right\rangle < \frac{\delta^-}{\alpha_{k_j}} d^2(p^{k_j}, z^{k_j}), \quad j = 0, 1, \ldots$$

Taking limits in the above inequality, as j goes to infinity, and using item (iv) of Lemma 1.1 together with Lemma 1.2, we obtain:

$$-\langle \bar{\xi}, \exp^{-1}_{\bar{p}} \bar{z} \rangle \leq \frac{\delta^-}{\bar{\alpha}} d^2(\bar{p}, \bar{z}). \tag{3.20}$$

Assume by contradiction that $\bar{\epsilon} > 0$. First, note that Theorem 2.6 implies that $X^{\bar{\epsilon}}$ is lower semicontinuous. Therefore, due to $\lim_{j\to\infty} p^{k_j} = \bar{p}$ and $\bar{\xi} \in X(\bar{p}) \subset X^{\bar{\epsilon}}(\bar{p})$, there exists a sequence $\{P_{p^{k_j}\bar{p}} w^j\}$ with $w^j \in X^{\bar{\epsilon}}(p^{k_j})$, such that $\lim_{j\to\infty} P_{p^{k_j}\bar{p}} w^j = \bar{\xi}$. Besides, (3.11) implies that $\bar{\epsilon} \leq \epsilon_{k_j}$ and, using item (i) of Proposition 2.4, we conclude that $X^{\bar{\epsilon}}(p^{k_j}) \subset X^{\epsilon_{k_j}}(p^{k_j})$, for all j. Henceforth, $w^j \in X^{\epsilon_{k_j}}(p^{k_j})$, for all j, and from (3.4) we have:

$$\langle w^j, -\exp^{-1}_{p^{k_j}} z^{k_j} \rangle \geq \frac{\delta^+}{\alpha_{k_j}} d^2(p^{k_j}, z^{k_j}), \quad j = 0, 1, \ldots$$

Letting j goes to infinity in the last inequality, and considering Lemma 1.1, we obtain that

$$-\langle \bar{\xi}, \exp_{\bar{p}}^{-1} \bar{z} \rangle \geq \frac{\delta^+}{\bar{\alpha}} d^2(\bar{p}, \bar{z}).$$

As $\bar{\alpha} \geq \alpha^- > 0$ and $0 < \delta^- < \delta^+$, combining the last inequality with (3.20) we conclude that $\bar{p} = \bar{z}$. Again, taking into account that (3.11), we can apply Lemma 1.5 with $\theta_k = \epsilon_k$ and $\rho_k = d^2(p^k, z^k)$ to conclude that $0 = \lim_{k\to+\infty} \epsilon_k = \bar{\epsilon}$, which is a contradiction. Due to $u^{k_j} \in X^{\epsilon_{k_j}}(p^{k_j})$, combining Propositions 2.4 and 2.3, we conclude that $\bar{u} \in X(\bar{p})$. Hence, Lemma 3.1 implies that $\bar{p} \in S^*(X, \Omega)$. Therefore, the proof is concluded. □

Theorem 3.7. *Either the sequence $\{p^k\}$ is finite and ends at iteration k, in which case p^k is ϵ_k-solution of VIP(X, Ω), i.e.,*

$$\sup_{q\in\Omega, v\in X(q)} \left\langle v, \exp_q^{-1} p^k \right\rangle \leq \epsilon_k, \tag{3.21}$$

or it is infinite, in which case it converges to a solution of VIP(X, Ω).

Proof. If the algorithm stops at the iteration k, then we have $p^k = z^k = P_\Omega(\exp_{p^k}(-\alpha_k u^k))$. As $u^k \in X^{\epsilon_k}(p^k)$, Definition 2.1 implies

$$-\left\langle u^k, \exp_{p^k}^{-1} q \right\rangle - \left\langle v, \exp_q^{-1} p^k \right\rangle \geq -\epsilon_k, \qquad q \in \Omega, \quad v \in X(q).$$

Taking into account that $\alpha_k > 0$ and $p^k = z^k$, the last inequality can be written as

$$\frac{1}{\alpha_k} \left\langle \exp_{z^k}^{-1}[\exp_{p^k}(-\alpha_k u^k)], \exp_{z^k}^{-1} q \right\rangle - \left\langle v, \exp_q^{-1} p^k \right\rangle \geq -\epsilon_k, \qquad q \in \Omega, \quad v \in X(q).$$

In view of (1.3), and considering that $z^k = P_\Omega(\exp_{p^k}(-\alpha_k u^k))$, we conclude from the last inequality that

$$\left\langle v, \exp_q^{-1} p^k \right\rangle \leq \epsilon_k, \qquad q \in \Omega, \quad v \in X(q),$$

which implies the desired inequality. Therefore, p^k is an ϵ_k-solution of VIP(X, Ω). Now, if $\{p^k\}$ is infinite, then from Lemma 3.5 the sequence $\{p^k\}$ is Féjer convergent to $S^*(X, \Omega)$. As we are under the assumption **A3**, it follows from Proposition 1.4 that $\{p^k\}$ is bounded. Hence, $\{p^k\}$ has a cluster point $\bar{p}$. Using Lemma 3.6 we obtain that $\bar{p} \in S^*(X, \Omega)$. Therefore, using again Proposition 1.4 we conclude that $\{p^k\}$ converges to $\bar{p} \in S^*(X, \Omega)$, and the theorem is proved. □

4. Example

In this section, numerical examples are provided to illustrate the obtained theoretical results in this new setting. Let M be a Hadamard manifold and $\Omega \subset M$ a convex set. Let us consider the following optimization problem:

$$\min f(p), \qquad \text{such that} \quad p \in \Omega, \tag{4.1}$$

where $f : M \to \mathbb{R}$ is a differentiable convex function. The problem in (4.1) is equivalent to the variational inequality problem VIP$(\operatorname{grad} f, \Omega)$:

$$\text{Find } p^* \in \Omega : \quad \langle \operatorname{grad} f(p^*), \exp_{p^*}^{-1} q \rangle \geq 0, \qquad \forall\, q \in \Omega, \tag{4.2}$$

as in [42]. We need to remember that according to Proposition 3.4 from [20] and Theorem 3.7 from [37], the gradient vector fields of differentiable convex functions are maximal monotone.

Example 4.1. Let $\mathbb{H}^2 := \left\{x = (x_1, x_2) \in \mathbb{R}^2 \ : \ x_2 > 0, \right\}$ be the 2-dimensional hyperbolic plane endowed with the Riemannian metric $g_{ij}(x_1, x_2) := \delta_{ij}/x_2^2$, for $i, j = 1, 2$. It is well known that $\mathbb{H}^2$ is a Hadamard manifolds with curvature $K = -1$, and the geodesics in $\mathbb{H}^2$ are semicircles centered on x_1-axis and vertical lines. For more details see [52], where some examples of differentiable convex functions on $\mathbb{H}^2$ are presented. Let us consider the variational inequality problem $\mathrm{VIP}(\operatorname{grad} f, \Omega)$ on the Hadamard manifold $\mathbb{H}^2$, with

$$f(p) := d^2(p, \bar{p}), \qquad \Omega := \left\{(x, y) \in \mathbb{R}^2 | 2 \le x^2 + y^2 \le 3,\ y > 0\right\},$$

where $\bar{p} = (0, 1)$ and d is the Riemannian metric of $\mathbb{H}^2$. Note that $S^*(\operatorname{grad} f, \Omega) = (0, 2) \ne \emptyset$. It is worth noting that both Ω and f are not Euclidean convex, which implies that the Euclidean Extragradient algorithm [34], can not be applied to solve $\mathrm{VIP}(f', \Omega)$, where f' denotes the Euclidean gradient of f. However, Ω and f are Riemannian convex in $\mathbb{H}^2$ and, in this case, we can apply Riemannian Extragradient-type algorithm to solve $\mathrm{VIP}(\operatorname{grad} f, \Omega)$. To illustrate how Extragradient-type algorithm works, we will generate from the starting point $p^0 = (0, 3)$ the point p^1. For that, we use the notation $X := \partial f$ and set the constants

$$\epsilon_0 = \frac{9}{32} d^2(p^0, \bar{p}), \quad \delta_- = \frac{1}{8}, \quad \delta_+ = \frac{1}{4}, \quad \alpha_- = \frac{1}{2}, \quad \alpha_+ = 1, \quad \alpha_0 = \frac{2}{3}, \quad \beta = \frac{1}{2}, \quad \beta_0 = \frac{1}{2}.$$

The Example 2.2 implies $X^{\epsilon}(p) \supset \partial f(p) + B[0_{T_pM}, 2\sqrt{2\epsilon}]$. As $\partial f(p^0) = -2 \exp^{-1}_{p^0} \bar{p}$ we have $X^{\epsilon_0}(p^0) \supset -2 \exp^{-1}_{p^0} \bar{p} + B[0_{T_{p^0}M}, \frac{3}{2} \mathrm{d}(p^0, \bar{p})]$. Thus, we can take

$$u^0 := -2 \exp^{-1}_{p^0} \bar{p} + \frac{3\alpha_0 - 1}{\alpha_0} \exp^{-1}_{p^0} \bar{p} \subset X^{\epsilon_0}.$$

Moreover, some calculations show that u^0 satisfies (3.4). Hence, as $p^0, \bar{p} \in \Omega$, z^0 satisfying (3.5) is given by

$$z^0 = P_{\Omega}(\exp_{p^0}(1 - \alpha_0) \exp^{-1}_{p^0} \bar{p}) = \exp_{p^0}(1 - \alpha_0) \exp^{-1}_{p^0} \bar{p},$$

and $\epsilon_1 = d^2(p^0, \bar{p})/9$. We proceed by noting that with $i = 0$, $y^{0,0} = \exp_{p^0} 2^{-1} \exp^{-1}_{p^0} z^0$ and $v^{0,0} = -2 \exp^{-1}_{y^{0,0}} \bar{p}$ satisfy (3.6) and then $\lambda_0 = \beta_0 = 1/2$, $y^0 = \exp_{p^0} 2^{-1} \exp^{-1}_{p^0} z^0 = (0, 8/3)$ and $v^0 = -2 \exp^{-1}_{y^0} \bar{p}$. Therefore, S_0 is a semicircle centered on $(0, 0)$ and passing through the point y^0, this implies $q_0 = y^0 = (0, 8/3)$ and then $p^1 = (0, 8/3)$.

5. Conclusions

Theorem 3.7 state that, if the sequence $\{p^k\}$ generated by the extragradient-type algorithm is finite and ends at iteration k, then p^k is ϵ_k-solution of $\mathrm{VIP}(X, \Omega)$. In fact, the concept of approximate solutions of $\mathrm{VIP}(X, \Omega)$ can be related with an important function, namely, the *gap function* $h : \Omega \to \mathbb{R} \cup \{+\infty\}$ defined by

$$h(p) := \sup_{q \in \Omega, v \in X(q)} \left\langle v, \exp^{-1}_q p \right\rangle. \tag{5.1}$$

The relation between the function h and the solutions of $\mathrm{VIP}(X, \Omega)$ is given in the following proposition, which is a Riemannian version of Lemma 4 from [14].

Proposition 5.1. *Let h be the function defined in (5.1). Then, there holds $h^{-1}(0) = S^*(X, \Omega)$.*

Proof. We will see first that a zero of h is a solution of VIP(X,Ω), i.e, $h^{-1}(0) \subset S^*(X,\Omega)$. Let $p \in h^{-1}(0)$. Thus, $h(p) = 0$ and the definition of h in (5.1) implies that

$$\left\langle v, \exp_q^{-1} p \right\rangle \leq 0, \qquad q \in \Omega, \quad v \in X(q).$$

On the other hand, from the definition of the normal cone N_Ω in (1.6), we have

$$\left\langle w, \exp_q^{-1} p \right\rangle \leq 0, \qquad q \in \Omega, \quad w \in N_\Omega(q).$$

Combining two last inequalities, it is easy to conclude that

$$\left\langle 0 - (v + w), \exp_q^{-1} p \right\rangle \geq 0, \qquad q \in \Omega, \quad v \in X(q), \quad w \in N_\Omega(q).$$

Due to Lemma 1.6, the vector field $X + N_\Omega$ is maximal monotone. Then, the maximality property together with the latter inequality yields $0 \in X(p) + N_\Omega(p)$, *i.e.*, $p \in S^*(X,\Omega)$. Now, we are going to show that the solutions of VIP(X,Ω) are zeros of h, i.e, $S^*(X,\Omega) \subset h^{-1}(0)$. Suppose that $p \in S^*(X,\Omega)$. Then, there exists $u \in X(p)$ such that

$$\langle u, \exp_p^{-1} q \rangle \geq 0, \qquad q \in \Omega.$$

Using the last inequality and the monotonicity of the vector field X we obtain

$$\langle v, \exp_q^{-1} p \rangle \leq 0, \qquad q \in \Omega, \quad v \in X(q).$$

Therefore, the definition of h in (5.1) implies that $h(p) \leq 0$ and, considering that $h(p) \geq 0$, we conclude that $h(p) = 0$, which ends the proof. □

Let $\epsilon > 0$. In view of Proposition 5.1, it make sense to define ϵ-solution of VIP(X,Ω) as all point $\bar{p} \in \Omega$ such that

$$h(\bar{p}) = \sup_{q \in \Omega, v \in X(q)} \left\langle v, \exp_q^{-1} \bar{p} \right\rangle \leq \epsilon.$$

We remark that if M is a linear space, then the function gap is convex. However, we do not know if this property is maintained in Hadamard manifolds. It is worth noting that, although [56] showed that the map $q \mapsto \left\langle v, \exp_q^{-1} \bar{p} \right\rangle$ in general is not quasiconvex, this is not sufficient to guarantee that the function gap is not convex.

We end this section with some comments about the implementation of the Extragradient-type algorithm. We remark that the performance of the algorithm depends significantly on how to compute the vector u_k satisfying (3.5) and (3.6), besides the projections P_Ω and P_{S_k}. To compute u_k we need first to deal with the enlargement of $X^{\epsilon_k}(p^k)$. Then, in the same spirit, that the bundle-type methods efficiently approximate the ϵ-subdifferential, we can look for methods approximating X^ϵ, in order to compute u_k. In the same way, we also need to design methods to compute the projections P_Ω and P_{S_k} in this new setting. We felt that these implementation issues are quite challenging, and too technical for the aims of the present paper. Finally, we emphasize that the main contribution of this paper is to extend results related to the extragradient algorithm from Euclidean setting to the Riemannian context, by using the concept of enlargement in order to provide more latitude and more robustness to the method. We expect the results of this paper will be the first step toward practical implementations of the method. We foresee further progress along this path in the near future.

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References

[1] R.L. Adler, J.-P. Dedieu, J.Y. Margulies, M. Martens and M. Shub, Newton's method on Riemannian manifolds and a geometric model for the human spine. *IMA J. Numer. Anal.* **22** (2002) 359–390.

[2] P. Ahmadi and H. Khatibzadeh, On the convergence of inexact proximal point algorithm on Hadamard manifolds. *Taiwan. J. Math.* **18** (2014) 419–433.

[3] M. Bačák, The proximal point algorithm in metric spaces. *Israel J. Math.* **194** (2013) 689–701.

[4] M. Bačák, R. Bergmann, G. Steidl and A. Weinmann, A second order nonsmooth variational model for restoring manifold-valued images. *SIAM J. Sci. Comput.* **38** (2016) A567–A597.

[5] E.E.A. Batista, G.C. Bento and O.P. Ferreira, An existence result for the generalized vector equilibrium problem on Hadamard manifolds. *J. Optim. Theor. Appl.* **167** (2015) 550–557.

[6] E.E.A. Batista, G.d.C. Bento and O.P. Ferreira, Enlargement of monotone vector fields and an inexact proximal point method for variational inequalities in Hadamard manifolds. *J. Optim. Theory Appl.* **170** (2016) 916–931.

[7] G.C. Bento, O.P. Ferreira and P.R. Oliveira, Proximal point method for a special class of nonconvex functions on Hadamard manifolds. *Optimization* **64** (2015) 289–319.

[8] R. Bergmann and A. Weinmann, A second-order TV-type approach for inpainting and denoising higher dimensional combined cyclic and vector space data. *J. Math. Imaging Vis.* **55** (2016) 401–427.

[9] R. Bergmann, J. Persch and G. Steidl, A parallel Douglas-Rachford algorithm for minimizing ROF-like functionals on images with values in symmetric Hadamard manifolds. *SIAM J. Imaging Sci.* **9** (2016) 901–937.

[10] R. Bhattacharya and V. Patrangenaru, Large sample theory of intrinsic and extrinsic sample means on manifolds. I. *Ann. Statist.* **31** (2003) 1–29.

[11] R. Bhattacharya and V. Patrangenaru, Large sample theory of intrinsic and extrinsic sample means on manifolds. II. *Ann. Statist.* **33** (2005) 1225–1259.

[12] A. Bhattacharya and R. Bhattacharya, Statistics on Riemannian manifolds: asymptotic distribution and curvature. *Proc. Am. Math. Soc.* **136** (2008) 2959–2967.

[13] G. Bigi, M. Castellani, M. Pappalardo and M. Passacantando, Existence and solution methods for equilibria. *Eur. J. Oper. Res.* **227** (2013) 1–11.

[14] R.S. Burachik and A.N. Iusem, A generalized proximal point algorithm for the variational inequality problem in a Hilbert space. *SIAM J. Optim.* **8** (1998) 197–216.

[15] R.S. Burachik and A.N. Iusem, Set-valued mappings and enlargements of monotone operators. Vol. 8 of *Springer Optimization and Its Applications*. Springer, New York (2008).

[16] R.S. Burachik, A.N. Iusem and B.F. Svaiter, Enlargement of monotone operators with applications to variational inequalities. *Set-Valued Anal.* **5** (1997) 159–180.

[17] S.-l. Chen and N.-j. Huang, Vector variational inequalities and vector optimization problems on Hadamard manifolds. *Optim. Lett.* **10** (2016) 753–767.

[18] J.X. Cruz Neto, P.S.M. Santos and P.A. Soares, Jr., An extragradient method for equilibrium problems on Hadamard manifolds. *Optim. Lett.* **10** (2016) 1327–1336.

[19] J.X. da Cruz Neto, O.P. Ferreira and L.R. Lucambio Pérez, Monotone point-to-set vector fields. *Balkan J. Geom. Appl.* **5** (2000) 69–79. Dedicated to Professor Constantin Udrişte.

[20] J.X. da Cruz Neto, O.P. Ferreira and L.R. Lucambio Pérez, Contributions to the study of monotone vector fields. *Acta Math. Hung.* **94** (2002) 307–320.

[21] J.X. Da Cruz Neto, O.P. Ferreira, L.R.L. Pérez and S.Z. Németh, Convex- and monotone-transformable mathematical programming problems and a proximal-like point method. *J. Global Optim.* **35** (2006) 53–69.

[22] P. Das, N.R. Chakraborti and P.K. Chaudhuri, Spherical minimax location problem. *Comput. Optim. Appl.* **18** (2001) 311–326.

[23] G. de Carvalho Bento, J.a.X. da Cruz Neto and P.R. Oliveira, A new approach to the proximal point method: convergence on general Riemannian manifolds. *J. Optim. Theory Appl.* **168** (2016) 743–755.

[24] M.P. do Carmo, Riemannian Geometry. *Mathematics: Theory & Applications*. Birkhäuser Boston Inc., Boston, MA (1992). Translated from the second Portuguese edition by Francis Flaherty.

[25] Z. Drezner and G.O. Wesolowsky, Minimax and maximin facility location problems on a sphere. *Naval Res. Logist. Quart.* **30** (1983) 305–312.

[26] R. Espínola and A. Nicolae, Proximal minimization in CAT(κ) spaces. *J. Nonlinear Convex Anal.* **17** (2016) 2329–2338.

[27] C.-j. Fang and S.-l. Chen, A projection algorithm for set-valued variational inequalities on Hadamard manifolds. *Optim. Lett.* **9** (2015) 779–794.

[28] O.P. Ferreira and P.R. Oliveira, Proximal point algorithm on Riemannian manifolds. *Optimization* **51** (2002) 257–270.

[29] O.P. Ferreira, L.R.L. Pérez and S.Z. Németh, Singularities of monotone vector fields and an extragradient-type algorithm. *J. Global Optim.* **31** (2005) 133–151.

[30] P. T. Fletcher, Geodesic regression and the theory of least squares on Riemannian manifolds. *Int. J. Comput. Vis.* **105** (2013) 171–185.

[31] O. Freifeld and M.J. Black, Lie bodies: a manifold representation of 3D human shape, in *Proceedings of ECCV 2012*. Springer, Berlin (2012).

[32] P. Grohs and S. Hosseini, ε-subgradient algorithms for locally Lipschitz functions on Riemannian manifolds. *Adv. Comput. Math.* **42** (2016) 333–360.

[33] S. Hawe, M. Kleinsteuber and K. Diepold, Analysis operator learning and its application to image reconstruction. *IEEE Trans. Image Process.* **22** (2013) 2138–2150.

[34] A.N. Iusem and L.R.L. Pérez, An extragradient-type algorithm for non-smooth variational inequalities. *Optimization* **48** (2000) 309–332.

[35] M. Kleinsteuber and H. Shen, Blind source separation with compressively sensed linear mixtures. *IEEE Signal Process. Lett.* **19** (2012) 107–110.

[36] C. Li and J.-C. Yao, Variational inequalities for set-valued vector fields on Riemannian manifolds: convexity of the solution set and the proximal point algorithm. *SIAM J. Control Optim.* **50** (2012) 2486–2514.

[37] C. Li, G. López and V. Martín-Márquez, Monotone vector fields and the proximal point algorithm on Hadamard manifolds. *J. Lond. Math. Soc.* **79** (2009) 663–683.

[38] S.-L. Li, C. Li, Y.-C. Liou and J.-C. Yao, Existence of solutions for variational inequalities on Riemannian manifolds. *Nonlinear Anal.* **71** (2009) 5695–5706.

[39] C. Li, G. López, V. Martín-Márquez and J.-H. Wang, Resolvents of set-valued monotone vector fields in Hadamard manifolds. *Set-Valued Var. Anal.* **19** (2011) 361–383.

[40] R.D.C. Monteiro and B.F. Svaiter, On the complexity of the hybrid proximal extragradient method for the iterates and the ergodic mean. *SIAM J. Optim.* **20** (2010) 2755–2787.

[41] S.Z. Németh, Monotone vector fields. *Publ. Math. Debrecen* **54** (1999) 437–449.

[42] S.Z. Németh, Variational inequalities on Hadamard manifolds. *Nonlinear Anal.* **52** (2003) 1491–1498.

[43] X. Pennec, Intrinsic statistics on Riemannian manifolds: basic tools for geometric measurements. *J. Math. Imaging Vis.* **25** (2006) 127–154.

[44] T. Sakai, Riemannian Geometry, Vol. 149 of *Translations of Mathematical Monographs.* American Translated from the 1992 Japanese original by the author. Mathematical Society, Providence, RI (1996).

[45] S.T. Smith, Optimization techniques on Riemannian manifolds, in Hamiltonian and Gradient Flows, Algorithms and Control, Vol. 3 of Fields Institute Communications. American Mathematical Society, Providence, RI (1994) 113–136.

[46] J.C.O. Souza and P.R. Oliveira, A proximal point algorithm for DC fuctions on Hadamard manifolds. *J. Global Optim.* **63** (2015) 797–810.

[47] R. Suparatulatorn, P. Cholamjiak and S. Suantai, On solving the minimization problem and the fixed-point problem for nonexpansive mappings in CAT(0) spaces. *Optim. Methods Softw.* **32** (2017) 182–192.

[48] G.-j. Tang and N.-j. Huang, Korpelevich's method for variational inequality problems on Hadamard manifolds. *J. Global Optim.* **54** (2012) 493–509.

[49] G.-j. Tang and N.-j. Huang, An inexact proximal point algorithm for maximal monotone vector fields on Hadamard manifolds. *Oper. Res. Lett.* **41** (2013) 586–591.

[50] G.-j. Tang, L.-w. Zhou and N.-j. Huang, The proximal point algorithm for pseudomonotone variational inequalities on Hadamard manifolds. *Optim. Lett.* **7** (2013) 779–790.

[51] G.-j. Tang, X. Wang and H.-w. Liu, A projection-type method for variational inequalities on Hadamard manifolds and verification of solution existence. *Optimization* **64** (2015) 1081–1096.

[52] C. Udrişte, Convex functions and optimization methods on Riemannian manifolds, Vol. 297 of *Mathematics and its Applications.* Kluwer Academic Publishers Group, Dordrecht (1994).

[53] J. Wang, C. Li, G. Lopez and J.-C. Yao, Convergence analysis of inexact proximal point algorithms on Hadamard manifolds. *J. Global Optim.* **61** (2015) 553–573.

[54] X. Wang, C. Li, J. Wang and J.-C. Yao, Linear convergence of subgradient algorithm for convex feasibility on Riemannian manifolds. *SIAM J. Optim.* **25** (2015) 2334–2358.

[55] J. Wang, C. Li, G. Lopez and J.-C. Yao, Proximal point algorithms on Hadamard manifolds: linear convergence and finite termination. *SIAM J. Optim.* **26** (2016) 2696–2729.

[56] X. Wang, C. Li and J.-C. Yao, On some basic results related to affine functions on Riemannian manifolds. *J. Optim. Theory Appl.* **170** (2016) 783–803.