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ADAPTIVE STABILIZATION BASED ON PASSIVE AND SWAPPING IDENTIFIERS FOR A CLASS OF UNCERTAIN LINEARIZED GINZBURG–LANDAU EQUATIONS*

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Abstract. This paper is devoted to the stabilization for a class of uncertain linearized Ginzburg–Landau equations (GLEs). The distinguishing feature of such system is the presence of serious uncertainties which enlarge the scope of the systems whereas challenge the control problem. Therefore, certain dynamic compensation mechanisms are required to overcome the uncertainties of system. Motivated by the related literature, the original complex-valued GLEs are transformed into a class of real-valued coupled parabolic systems with serious uncertainties and distinctive characteristics. For this, two classes of identifiers respectively based on passive and swapping identifiers are first introduced to design parameter dynamic compensators. Then, by combining infinite-dimensional backstepping method with the dynamic compensators, two adaptive state-feedback controllers are constructed which guarantee all the closed-loop system states are bounded while the original system states converge to zero. A numerical example is provided to validate the effectiveness of the theoretical results.

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1. Introduction and problem formulation

The GLEs are used to model the phenomena of vortex shedding in the wake of bluff body such as circular cylinder. Such phenomena will induce undesirable periodic force acting on the bluff body and give rise to severe damage in certain circumstance. This motivates the investigations for the controls of GLEs. In fact, multiple classes of control problems of GLEs such as chaos control [10–12], fuzzy control [8], robust H_∞ control [16, 17] and stabilization [1–3, 23, 24, 26] have been investigated over the last two decades. However, all the theoretical results proposed in above literature are restricted by the assumption that all the system parameters are exactly known. It is necessary to point out that the actual values of system parameters are rather difficult (or even impossible) to obtain in practice. The uncertainties/unknowns of system parameters are existing ineluctably and

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will make the control problems much more challenging. Therefore, it is meaningful to investigate the controls of uncertain GLEs.

In this paper, we investigate the stabilization of the following uncertain linearized GLEs:

$$\begin{cases} A_t(\breve{x},t) &= a_1 A_{\breve{x}\breve{x}}(\breve{x},t) + a_2 A_{\breve{x}}(\breve{x},t) + a_3 A(\breve{x},t), \\ A(0,t) &= u(t), \\ A(x_d,t) &= 0, \end{cases} \tag{1.1}$$

where $A : [0,\, x_d] \times \mathbf{R}_+ \to \mathbf{C}$ is system state, $u : \mathbf{R}_+ \to \mathbf{C}$ is control input with $\mathbf{R}_+$ and $\mathbf{C}$ denoting the sets of all positive real numbers and complex numbers, respectively; $A_t = \frac{\partial A}{\partial t}$, $A_{\breve{x}} = \frac{\partial A}{\partial \breve{x}}$ and $A_{\breve{x}\breve{x}} = \frac{\partial^2 A}{\partial \breve{x}^2}$; a_1 is a known complex constant with strictly positive real part and a_2, a_3 are unknown complex constants. Due to the presence of unknown parameters, the above system is essentially different from those of [1–3, 23] where a_2 and a_3 are spatial-varying whereas exactly known. Moreover, a_2, a_3 as well as some functions constructed by them (see (1.4) below) don't necessary belong to known or finite intervals and hence allow more serious uncertainties than those of [20, 21].

Specifically, the control objective of the paper is to design adaptive controllers such that all the closed-loop system states are bounded while the original system states converge to zero. Inspired by [1–3, 23], one important idea for the controller design of GLEs is to change the original complex-valued system into a new one represented by real-valued states and controls. Then, the following transformations are used for system (1.1):

$$\rho(x,t) = \mathcal{R}(B(x,t)) = \frac{1}{2}\left(B(x,t) + \overline{B}(x,t)\right), \quad \iota(x,t) = \mathcal{I}(B(x,t)) = \frac{1}{2i}\left(B(x,t) - \overline{B}(x,t)\right),$$

where for a complex number ξ, $\bar{\xi}$ denotes its complex conjugation, $\mathcal{R}(\xi)$ and $\mathcal{I}(\xi)$ respectively denote its real part and imaginary part, and moreover,

$$x = \frac{x_d - \breve{x}}{x_d}, \quad B(x,t) = A(\breve{x},t)e^{\frac{a_2 \breve{x}}{2a_1}}. \tag{1.2}$$

It can be verified that $\rho(x,t)$ and $\iota(x,t)$ satisfy the following equations [2]:

$$\begin{cases} \rho_t(x,t) = a_R \rho_{xx}(x,t) + b_R \rho(x,t) - a_I \iota_{xx}(x,t) - b_I \iota(x,t), \\ \iota_t(x,t) = a_I \rho_{xx}(x,t) + b_I \rho(x,t) + a_R \iota_{xx}(x,t) + b_R \iota(x,t), \\ \rho(0,t) = 0,\ \iota(0,t) = 0, \\ \rho(1,t) = u_R(t),\ \iota(1,t) = u_I(t), \end{cases} \tag{1.3}$$

with

$$\begin{cases} a_R = \frac{\mathcal{R}(a_1)}{x_d^2},\ a_I = \frac{\mathcal{I}(a_1)}{x_d^2}, \\ b_R = \mathcal{R}\left(a_3 - \frac{a_2^2}{4a_1}\right),\ b_I = \mathcal{I}\left(a_3 - \frac{a_2^2}{4a_1}\right), \\ u_R(t) = \mathcal{R}\left(u(t)\right),\ u_I(t) = \mathcal{I}\left(u(t)\right). \end{cases} \tag{1.4}$$

We next turn to discussing system (1.3), to show the relationship between system (1.3) and (1.1), to present the principal methods for control design used in the paper, and then to highlight the main contributions of the paper.

Noting that the transformation from system (1.1) to (1.3) is invertible (see from (1.2)), the stability of real-valued states of system (1.3) (*i.e.*, ρ, ι) will imply that of the complex-valued state of system (1.1) (*i.e.*, A).

Therefore, it is enough to design real-valued control inputs u_R, u_I for system (1.3), and then use the third line of (1.4) to obtain the complex-valued control input u for original system (1.1). Moreover, since a_1 is known and a_2, a_3 are unknown, the new parameters a_R, a_I are known while b_R, b_I being unknown. In this paper, we are going to estimate the new unknown parameters b_R and b_I instead of estimating original parameters a_2 and a_3.

To design stabilizing controller for system (1.3), two adaptive control schemes respectively based on passive and swapping identifiers are proposed in this paper. First, two classes of identifiers are separately constructed, which are sometimes termed observer although their purposes are parameters estimation. Then, by adopting infinite-dimensional bakcstepping transformations, the identifiers are separately changed into different new systems from which two adaptive state-feedback controllers are constructed. It is proved that the proposed controllers guarantee that all the closed-loop system states are bounded while the original system states converge to zero.

To highlight the main contributions of the paper, the following three aspects specify the essential differences between system (1.3) and those of the related literature while analyzing the essential obstacles of the traditional methods for the stabilization of system (1.3):

(i) *The coupling between subsystems makes system* (1.3) *more complex than single parabolic system in* [13, 14, 27, 28]*, and hence makes the control of system* (1.3) *more challenging than those in the literature.* It can be seen from equations (1.3) that both the two PDE subsystem states (*i.e.*, ρ and ι) interact each other. Then system (1.3) is coupled and hence more complex than single parabolic system investigated in [13, 14, 27, 28]. Mainly due to the presence of the coupling in (1.3), the identifiers introduced for parameters estimations and hence the resulting closed-loop system would be coupled. This makes the compensation of unknown parameters and performance analysis of the resulting closed-loop system more difficult than those of [13, 14, 27, 28].

(ii) *System* (1.3) *is described by coupled parabolic equations which are essentially different from the coupled hyperbolic systems in* [4–6] *or the coupled PDE-ODE systems in* [18, 19, 29]. The distinctive characteristics of the systems would give rise to different technique obstacles in control design and performance analysis. For example, new infinite-dimensional backstepping transformations should be searched to change original system into a target system. Moreover, the resulting target system will be coupled parabolic systems rather than coupled hyperbolic or coupled PDE-ODE ones. Then, new framework should be established for stability analysis of the resulting closed-loop system.

(iii) *System* (1.3) *has serious unknowns which make the existing methods in* [2, 3, 7, 20, 21, 23, 25, 30] *ineffective.* In fact, coupled parabolic systems similar as (1.3) have been investigated in [2, 3, 7, 23, 25, 30], but their system parameters are required to be exactly known rather than completely unknown. Then, the control design methods of the literature are incapable for the stabilization of (1.3). Although in [20, 21], all system parameters a_R, a_I, b_R and b_I are unknown, their unknowns are severely restricted since all of them must belong to known finite intervals. In this paper, although a_R, a_I are known, b_R and b_I are unknown and do not necessarily belong to known or finite intervals, and hence allow more serious uncertainties. Moreover, the adaptive method based on projector operator used in [20, 21] depends on the known bounds of unknown parameters, which would be ineffective to the stabilization of system (1.3) since more serious uncertainties exist. To overcome the serious uncertainties, new compensation scheme needs to be developed, under which to construct a stabilizing controller.

Throughout the paper, we assume the existence and uniqueness of the classical solutions for the resulting closed-loop systems since the main purposes of the paper are the designing of adaptive controllers and the proof of stability for the adaptive schemes, and moreover, the parabolic character of the systems ensure their benign behavior (see [13, 14, 27]).

The reminder of the paper is organized as follows. Section 2 presents some mathematical preliminaries for control design where two important functions and their properties are given. Sections 3 and 4 propose the adaptive control design procedures based on passive and swapping identifiers, respectively. Section 5 gives the numerical example to validate the effectiveness of the proposed methods. Section 6 addresses some concluding remarks. This paper ends with an appendix which provides the proofs of some important lemma and propositions, and moreover, collects some useful inequalities and criterions.

Notation. Throughout the paper, the following notations are used. For a function $z(x,t) : [0,1]\times[0,+\infty)\to\mathbf{R}$, let $\|z\| = \sqrt{\int_0^1 z(x,t)^2\mathrm{d}x}$. For a real-valued time-varying function $f(t)$, $f\in\mathcal{L}_p$ is equivalent to $\left(\int_0^\infty |f(t)|^p\mathrm{d}t\right)^{\frac{1}{p}}<\infty$ with $p\geq 1$ and particularly $f\in\mathcal{L}_\infty$ is equivalent to $\sup_{t\geq 0}|f(t)|<\infty$.

2. Mathematical preliminaries for control design

In this section, we present two important functions and their important properties which will be used in the following control design and stability analysis.

The following two functions will be used in the following control design[1]:

$$\hat{k}(x,y)=\sum_{n=0}^{\infty}G_n(x+y,x-y),\qquad \hat{k}^c(x,y)=\sum_{n=0}^{\infty}G_n^c(x+y,x-y), \tag{2.1}$$

where

$$\begin{cases} G_0(\xi,\eta)=-\frac{1}{4}\hat{\beta}\cdot(\xi-\eta),\quad G_0^c(\xi,\eta)=\frac{1}{4}\hat{\beta}^c\cdot(\xi-\eta),\\ G_{n+1}(\xi,\eta)=\frac{1}{4}\hat{\beta}\int_\eta^\xi\int_0^\eta G_n(\tau,s)\mathrm{d}s\mathrm{d}\tau+\frac{1}{4}\hat{\beta}^c\int_\eta^\xi\int_0^\eta G_n^c(\tau,s)\mathrm{d}s\mathrm{d}\tau,\\ G_{n+1}^c(\xi,\eta)=-\frac{1}{4}\hat{\beta}^c\int_\eta^\xi\int_0^\eta G_n(\tau,s)\mathrm{d}s\mathrm{d}\tau+\frac{1}{4}\hat{\beta}\int_\eta^\xi\int_0^\eta G_n^c(\tau,s)\mathrm{d}s\mathrm{d}\tau,\\ \hat{\beta}=\frac{a_R\left(\hat{b}_R+c\right)+a_I\hat{b}_I}{a_R^2+a_I^2},\ \hat{\beta}^c=\frac{a_R\hat{b}_I-a_I\left(\hat{b}_R+c\right)}{a_R^2+a_I^2}, \end{cases} \tag{2.2}$$

c is an arbitrary positive constant, $\hat{b}_R$ and $\hat{b}_I$ are time-varying bounded functions (*i.e.*, the dynamic estimates of unknown constants b_R and b_I, respectively).

The above two functions will be used as kernel functions to construct a pivotal invertible state transformation (see (3.3) below) which plays quite important role in control design. For detail, under the state transformation, original system is changed into a new one (named target system) whose stability in some sense implies that of the original system. By choosing stabilizing controller for the target system, the desirable controller for original system will be obtained by the invertible state transformation.

The specific expressions of kernel function $\hat{k}(x,y)$ and $\hat{k}^c(x,y)$ are obtained from those of [2] by setting $b_R(y)\equiv b_R$, $b_I(y)\equiv b_I$ and $f_R(x)=f_I(x)\equiv 0$ in (28), (29) of the literature. It is necessary to point out that, the kernel functions given by (2.1) are time-varying since they include the dynamic estimates of unknown parameters (*i.e.*, $\hat{b}_R$, $\hat{b}_I$), and hence are essentially different from those of [2] which are time-invariant. The presence of time-variance will make the control design and performance analysis for the control problem under investigation much more difficult.

The following lemma addresses some important properties of $\hat{k}(\cdot)$ and $\hat{k}^c(\cdot)$ which will be used in the following controller design and stability analysis.

Lemma 2.1. *The two infinite series given in (2.1) converge uniformly on* $\Gamma=\{x,y,\hat{b}_R,\hat{b}_I : 0<x<y<1, |\hat{b}_R|\leq\delta_1, |\hat{b}_I|\leq\delta_1\}$ *with* δ_1 *being some positive constant. Moreover,* $\hat{k}(\cdot)$ *and* $\hat{k}^c(\cdot)$ *are continuously*

[1] To reduce the notational burden, we suppress the time dependence and space dependence where it does not lead to a confusion.

differentiable with respect to $\hat{b}_R$, $\hat{b}_I$ *and twice continuously differentiable with respect to* x, y *on* Γ, *and satisfy*

$$\begin{cases} \hat{k}_{xx}(x,y) = \hat{k}_{yy}(x,y) + \hat{\beta}\hat{k}(x,y) + \hat{\beta}^c\hat{k}^c(x,y), \\ \hat{k}^c_{xx}(x,y) = \hat{k}^c_{yy}(x,y) - \hat{\beta}^c\hat{k}(x,y) + \hat{\beta}\hat{k}^c(x,y), \\ \hat{k}(x,x) = -\frac{1}{2}\hat{\beta}x, \quad \hat{k}^c(x,x) = \frac{1}{2}\hat{\beta}^c x, \\ \hat{k}(x,0) = 0, \quad \hat{k}^c(x,0) = 0. \end{cases} \tag{2.3}$$

Proof. See Appendix A in the end of the paper. □

3. Adaptive controller design by passive identifier

In this section, procedure of control design based on passive identifier is given. First, a dynamic system (*i.e.*, passive identifier) is introduced to help the estimation for unknown parameters. Then, the identifier is changed into a new system under some infinite-dimensional backtepping transformation. Finally, an adaptive state-feedback controller is constructed, which guarantees the desirable stability of the resulting closed-loop system.

We first introduce the following passive identifier:

$$\begin{cases} \hat{\rho}_t = a_R\hat{\rho}_{xx} + \hat{b}_R\rho - a_I\hat{\iota}_{xx} - \hat{b}_I\iota + \gamma^2\tilde{\rho}\left(\|\rho\|^2 + \|\iota\|^2\right), \\ \hat{\iota}_t = a_I\hat{\rho}_{xx} + \hat{b}_I\rho + a_R\hat{\iota}_{xx} + \hat{b}_R\iota + \gamma^2\tilde{\iota}\left(\|\rho\|^2 + \|\iota\|^2\right), \\ \hat{\rho}(0) = \hat{\iota}(0) = 0, \\ \hat{\rho}(1) = u_R,\ \hat{\iota}(1) = u_I, \end{cases} \tag{3.1}$$

with $\gamma > 0$ and $\tilde{\rho} = \rho - \hat{\rho}$, $\tilde{\iota} = \iota - \hat{\iota}$ which satisfy the following equations (called error system) by verifying from (1.3) and the above equations:

$$\begin{cases} \tilde{\rho}_t = a_R\tilde{\rho}_{xx} + \tilde{b}_R\rho - a_I\tilde{\iota}_{xx} - \tilde{b}_I\iota - \gamma^2\tilde{\rho}\left(\|\rho\|^2 + \|\iota\|^2\right), \\ \tilde{\iota}_t = a_I\tilde{\rho}_{xx} + \tilde{b}_I\rho + a_R\tilde{\iota}_{xx} + \tilde{b}_R\iota - \gamma^2\tilde{\iota}\left(\|\rho\|^2 + \|\iota\|^2\right), \\ \tilde{\rho}(0) = \tilde{\iota}(0) = 0, \\ \tilde{\rho}(1) = \tilde{\iota}(1) = 0. \end{cases} \tag{3.2}$$

Then, we adopt the following infinite-dimensional backstepping transformation:

$$\begin{cases} \hat{w}^\rho(x) = \hat{\rho}(x) - \int_0^x \big(\hat{k}(x,y)\hat{\rho}(y) + \hat{k}^c(x,y)\hat{\iota}(y)\big)\mathrm{d}y, \\ \hat{w}^\iota(x) = \hat{\iota}(x) - \int_0^x \big(-\hat{k}^c(x,y)\hat{\rho}(y) + \hat{k}(x,y)\hat{\iota}(y)\big)\mathrm{d}y, \end{cases} \tag{3.3}$$

and its inverse transformation

$$\begin{cases} \hat{\rho}(x) = \hat{w}^\rho(x) - \int_0^x \big(\hat{l}(x,y)\hat{w}^\rho(y) + \hat{l}^c(x,y)\hat{w}^\iota(y)\big)\mathrm{d}y, \\ \hat{\iota}(x) = \hat{w}^\iota(x) - \int_0^x \big(-\hat{l}^c(x,y)\hat{w}^\rho(y) + \hat{l}(x,y)\hat{w}^\iota(y)\big)\mathrm{d}y, \end{cases} \tag{3.4}$$

with

$$\hat{l}(x,y)=\sum_{n=0}^{\infty}(-1)^{n+1}G_n(x+y,x-y),\quad \hat{l}^c(x,y)=\sum_{n=0}^{\infty}(-1)^{n+1}G_n^c(x+y,x-y). \tag{3.5}$$

By the above two transformations, the identifier (3.1) is changed into a new system from which it is convenient to design controller and analyze the system performance.

Proposition 3.1. *By transformations* (3.3) *and* (3.4)*, the identifier* (3.1) *is changed into the following system:*

$$\begin{cases} \hat{w}_t^\rho = a_R\hat{w}_{xx}^\rho - a_I\hat{w}_{xx}^\iota - c\hat{w}^\rho + \left(\hat{b}_R + \gamma^2\left(\|\rho\|^2+\|\iota\|^2\right)\right)\tilde{w}^\rho \\ \qquad -\hat{b}_I\tilde{w}^\iota + \int_0^x\left(\hat{w}^\rho(y)H(x,y)+\hat{w}^\iota(y)H^c(x,y)\right)\mathrm{d}y, \\ \hat{w}_t^\iota = a_I\hat{w}_{xx}^\rho + a_R\hat{w}_{xx}^\iota - c\hat{w}^\iota + \left(\hat{b}_R + \gamma^2\left(\|\rho\|^2+\|\iota\|^2\right)\right)\tilde{w}^\iota \\ \qquad +\hat{b}_I\tilde{w}^\rho - \int_0^x\left(\hat{w}^\rho(y)H^c(x,y)-\hat{w}^\iota(y)H(x,y)\right)\mathrm{d}y, \\ \hat{w}^\rho(0)=\hat{w}^\iota(0)=0, \\ \hat{w}^\rho(1)=u_R-\left(\int_0^1\hat{w}^\rho(y)\psi(y)\mathrm{d}y+\int_0^1\hat{w}^\iota(y)\psi^c(y)\mathrm{d}y\right), \\ \hat{w}^\iota(1)=u_I-\left(-\int_0^1\hat{w}^\rho(y)\psi^c(y)\mathrm{d}y+\int_0^1\hat{w}^\iota(y)\psi(y)\mathrm{d}y\right), \end{cases} \tag{3.6}$$

where

$$\begin{cases} \tilde{w}^\rho(x) = \tilde{\rho}(x) - \int_0^x\left(\hat{k}(x,y)\tilde{\rho}(y)+\hat{k}^c(x,y)\tilde{\iota}(y)\right)\mathrm{d}y, \\ \tilde{w}^\iota(x) = \tilde{\iota}(x) - \int_0^x\left(-\hat{k}^c(x,y)\tilde{\rho}(y)+\hat{k}(x,y)\tilde{\iota}(y)\right)\mathrm{d}y, \\ \psi(y) = \hat{k}(1,y) - \int_y^1\left(\hat{k}(1,\xi)\hat{l}(\xi,y)-\hat{k}^c(1,\xi)\hat{l}^c(\xi,y)\right)\mathrm{d}\xi, \\ \psi^c(y) = \hat{k}^c(1,y) - \int_y^1\left(\hat{k}(1,\xi)\hat{l}^c(\xi,y)+\hat{k}^c(1,\xi)\hat{l}(\xi,y)\right)\mathrm{d}\xi, \\ H(x,y) = -\hat{k}_t(x,y) + \int_y^x\left(\hat{k}_t(x,\xi)\hat{l}(\xi,y)-\hat{k}_t^c(x,\xi)\hat{l}^c(\xi,y)\right)\mathrm{d}\xi, \\ H^c(x,y) = -\hat{k}_t^c(x,y) + \int_y^x\left(\hat{k}_t(x,\xi)\hat{l}^c(\xi,y)+\hat{k}_t^c(x,\xi)\hat{l}(\xi,y)\right)\mathrm{d}\xi. \end{cases} \tag{3.7}$$

Proof. See Appendix B in the end of the paper. □

For system (3.6), we choose the following controllers:

$$\begin{cases} u_R = \int_0^1\hat{w}^\rho(y)\psi(y)\mathrm{d}y + \int_0^1\hat{w}^\iota(y)\psi^c(y)\mathrm{d}y, \\ u_I = -\int_0^1\hat{w}^\rho(y)\psi^c(y)\mathrm{d}y + \int_0^1\hat{w}^\iota(y)\psi(y)\mathrm{d}y, \end{cases} \tag{3.8}$$

with $\hat{b}_R$ and $\hat{b}_I$ satisfying

$$\begin{cases} \dot{\hat{b}}_R = \gamma\int_0^1(\rho\tilde{\rho}+\iota\tilde{\iota})\,\mathrm{d}x, \\ \dot{\hat{b}}_I = \gamma\int_0^1(\rho\tilde{\iota}-\iota\tilde{\rho})\,\mathrm{d}x. \end{cases} \tag{3.9}$$

The following two propositions respectively give the performance of parameters estimators and adaptive controller, which will be used in the stability analysis of the closed-loop system.

Proposition 3.2. *The updating law (3.9) guarantees that the following properties hold*

$$\begin{cases} \tilde{b}_R, \tilde{b}_I, \hat{b}_R, \hat{b}_I, \|\tilde{\rho}\|, \|\tilde{\iota}\| \in \mathcal{L}_\infty, \\ \|\tilde{\rho}\|, \|\tilde{\iota}\|, \|\tilde{\rho}_x\|, \|\tilde{\iota}_x\|, \|\rho\|\|\tilde{\rho}\|, \|\iota\|\|\tilde{\iota}\|, \|\iota\|\|\tilde{\rho}\|, \|\rho\|\|\tilde{\iota}\| \in \mathcal{L}_2, \\ \dot{\hat{b}}_R, \dot{\hat{b}}_I \in \mathcal{L}_2. \end{cases} \tag{3.10}$$

Proof. See Appendix C in the end of the paper. □

Proposition 3.3. *The adaptive controller (3.8) and (3.9) guarantees that the following properties hold*

$$\begin{cases} \|\hat{w}^\rho\|, \|\hat{w}^\iota\|, \|\hat{\rho}\|, \|\hat{\iota}\| \in \mathcal{L}_\infty \cap \mathcal{L}_2, \\ \|\tilde{w}^\rho\|, \|\tilde{w}^\iota\|, \|\rho\|, \|\iota\| \in \mathcal{L}_\infty \cap \mathcal{L}_2, \\ \|\hat{\rho}_x\|, \|\hat{\iota}_x\|, \|\rho_x\|, \|\iota_x\| \in \mathcal{L}_\infty. \end{cases} \tag{3.11}$$

Proof. See Appendix D in the end of the paper. □

The following theorem gives the main results of the adaptive control design based on passive identifier in this section.

Theorem 3.4. *The adaptive controller (3.8) and (3.9) guarantees that all the closed-loop system states are bounded and the original system states ρ and ι converge to zero ultimately, i.e.,*

$$\lim_{t\to+\infty} \sup_{x\in[0,\,1]} \left(|\rho(x,t)| + |\iota(x,t)|\right) = 0.$$

Proof. By Agmon's Inequality (see Lem. H.1) and noting that $\|\rho\|$, $\|\iota\|$, $\|\rho_x\|$, $\|\iota_x\| \in \mathcal{L}_\infty$, we obtain that the closed-loop system states ρ, ι are bounded on $[0,\,1]\times[0,\,+\infty)$. Similarly, $\hat{\rho}$, $\hat{\iota}$ are also bounded on $[0,\,1]\times[0,\,+\infty)$. Then, by (3.3) and noting the boundedness of kernel functions, the boundedness of $\hat{w}^\rho$ and $\hat{w}^\iota$ follows.

In the following, we turn to showing the converge of $\rho(x,t)$ and $\iota(x,t)$. By Agmon's Inequality and noting that $\rho(0)=\iota(0)=0$ and $\|\rho_x\|$, $\|\iota_x\| \in \mathcal{L}_\infty$, it suffices to show the converge of $\|\rho\|$ and $\|\iota\|$. For this, by (D.2), we have

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\hat{w}^\rho\|^2 + \|\hat{w}^\iota\|^2\right) \\ &\quad \le \hat{b}_I \int_0^1 \left(\hat{w}^\iota \tilde{w}^\rho - \hat{w}^\rho \tilde{w}^\iota\right)\mathrm{d}x + \left(\hat{b}_R + \gamma^2\|\rho\|^2 + \gamma^2\|\iota\|^2\right)\int_0^1 \left(\hat{w}^\rho \tilde{w}^\rho + \hat{w}^\iota \tilde{w}^\iota\right)\mathrm{d}x \\ &\qquad - \int_0^1 \hat{w}^\rho \int_0^x \big(\hat{w}^\rho(y)H(x,y) + \hat{w}^\iota(y)H^c(x,y)\big)\mathrm{d}y\mathrm{d}x \\ &\qquad - \int_0^1 \hat{w}^\iota \int_0^x \big(\hat{w}^\rho(y)H^c(x,y) - \hat{w}^\iota(y)H(x,y)\big)\mathrm{d}y\mathrm{d}x, \end{aligned}$$

Then, there exists constant M such that $\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\hat{w}^\rho\|^2 + \|\hat{w}^\iota\|^2\right) < M$ since all the terms on the right-hand side of above inequality are bounded. This, together with the proven fact $\left(\|\hat{w}^\rho\|^2 + \|\hat{w}^\iota\|^2\right) \in \mathcal{L}_\infty \cap \mathcal{L}_1$, concludes $\lim_{t\to+\infty}\left(\|\hat{w}^\rho\|^2 + \|\hat{w}^\iota\|^2\right) = 0$ (by Lem. H.4). Then, (3.4) implies that $\lim_{t\to+\infty}\|\hat{\rho}\| = \lim_{t\to+\infty}\|\hat{\iota}\| = 0$. Similarly, there holds $\lim_{t\to+\infty}\|\tilde{\rho}\| = \lim_{t\to+\infty}\|\tilde{\iota}\| = 0$, and hence $\lim_{t\to+\infty}\|\rho\| = \lim_{t\to+\infty}\|\iota\| = 0$. This completes the proof. □

4. Adaptive controller design by swapping identifier

In this section, two classes of filters (*i.e.*, state and input filters) are first introduced whose linear combinations form the estimations of original system (named swapping identifier). Then, by the similar procedure as above section, another adaptive state-feedback controller is constructed which guarantees the desirable stability of the closed-loop system.

First, we employ the following two filters: the state filter

$$\begin{cases} v^\rho_t = a_R v^\rho_{xx} - a_I v^\iota_{xx} + \rho, \\ v^\iota_t = a_I v^\rho_{xx} + a_R v^\iota_{xx} + \iota, \\ v^\rho(0) = v^\iota(0) = 0, \\ v^\rho(1) = v^\iota(1) = 0, \end{cases} \tag{4.1}$$

and the input filter

$$\begin{cases} \eta^\rho_t = a_R \eta^\rho_{xx} - a_I \eta^\iota_{xx}, \\ \eta^\iota_t = a_I \eta^\rho_{xx} + a_R \eta^\iota_{xx}, \\ \eta^\rho(0) = \eta^\iota(0) = 0, \\ \eta^\rho(1) = \rho(1), \\ \eta^\iota(1) = \iota(1). \end{cases} \tag{4.2}$$

Define

$$\begin{cases} \epsilon^\rho \;=\; \rho - b_R v^\rho + b_I v^\iota - \eta^\rho, \\ \epsilon^\iota \;=\; \iota - b_I v^\rho - b_R v^\iota - \eta^\iota. \end{cases} \tag{4.3}$$

It can be verified directly from (4.1) and (4.2) that, ϵ^ρ and ϵ^ι satisfy the following equations:

$$\begin{cases} \epsilon^\rho_t = a_R \epsilon^\rho_{xx} - a_I \epsilon^\iota_{xx}, \\ \epsilon^\iota_t = a_I \epsilon^\rho_{xx} + a_R \epsilon^\iota_{xx}, \\ \epsilon^\rho(0) = \epsilon^\iota(0) = 0, \\ \epsilon^\rho(1) = \epsilon^\iota(1) = 0. \end{cases} \tag{4.4}$$

Then, set

$$\hat\rho = \hat b_R v^\rho - \hat b_I v^\iota + \eta^\rho, \quad \hat\iota = \hat b_I v^\rho + \hat b_R v^\iota + \eta^\iota. \tag{4.5}$$

It can be verified from (4.1) and (4.2) that $\hat\rho$ and $\hat\iota$ satisfy the following equations (called swapping identifier):

$$\begin{cases} \hat\rho_t = a_R \hat\rho_{xx} + \hat b_R \rho - a_I \hat\iota_{xx} - \hat b_I \iota + \dot{\hat b}_R v^\rho - \dot{\hat b}_I v^\iota, \\ \hat\iota_t = a_I \hat\rho_{xx} + \hat b_I \rho + a_R \hat\iota_{xx} + \hat b_R \iota + \dot{\hat b}_I v^\rho + \dot{\hat b}_R v^\iota, \\ \hat\rho(0) = \hat\iota(0) = 0, \\ \hat\rho(1) = u_R,\ \hat\iota(1) = u_I. \end{cases} \tag{4.6}$$

To make the control design convenient, we adopt infinite-dimensional backstepping transformation (3.3) and its inverse one (3.4) with $\hat\rho$, $\hat\iota$ replacing by (4.5), and change the identifier (4.6) into a new system.

Proposition 4.1. *By transformation* (3.3) *and* (3.4) *with* $\hat{\rho}$, $\hat{\iota}$ *being defined by* (4.5)*, the identifier* (4.6) *is changed into the following system:*

$$\begin{cases} \hat{w}^\rho_t = a_R\hat{w}^\rho_{xx} - a_I\hat{w}^\iota_{xx} - c\hat{w}^\rho + \hat{b}_R\tilde{w}^\rho - \hat{b}_I\tilde{w}^\iota + \dot{\hat{b}}_R\breve{v}^\rho - \dot{\hat{b}}_I\breve{v}^\iota \\ \qquad + \int_0^x \big(\hat{w}^\rho(y)H(x,y) + \hat{w}^\iota(y)H^c(x,y)\big)\mathrm{d}y, \\ \hat{w}^\iota_t = a_I\hat{w}^\rho_{xx} + a_R\hat{w}^\iota_{xx} - c\hat{w}^\iota + \hat{b}_R\tilde{w}^\iota + \hat{b}_I\tilde{w}^\rho + \dot{\hat{b}}_I\breve{v}^\rho + \dot{\hat{b}}_R\breve{v}^\iota \\ \qquad - \int_0^x \big(\hat{w}^\rho(y)H^c(x,y) - \hat{w}^\iota(y)H(x,y)\big)\mathrm{d}y, \\ \hat{w}^\rho(0) = \hat{w}^\iota(0) = 0, \\ \hat{w}^\rho(1) = u_R - \left(\int_0^1 \hat{w}^\rho(y)\psi(y)\mathrm{d}y + \int_0^1 \hat{w}^\iota(y)\psi^c(y)\mathrm{d}y\right), \\ \hat{w}^\iota(1) = u_I - \left(-\int_0^1 \hat{w}^\rho(y)\psi^c(y)\mathrm{d}y + \int_0^1 \hat{w}^\iota(y)\psi(y)\mathrm{d}y\right), \end{cases} \tag{4.7}$$

where

$$\begin{cases} \breve{v}^\rho(x) \;=\; v^\rho(x) - \int_0^x \big(\hat{k}(x,y)v^\rho(y) + \hat{k}^c(x,y)v^\iota(y)\big)\mathrm{d}y, \\ \breve{v}^\iota(x) \;=\; v^\iota(x) - \int_0^x \big(-\hat{k}^c(x,y)v^\rho(y) + \hat{k}(x,y)v^\iota(y)\big)\mathrm{d}y. \end{cases} \tag{4.8}$$

Proof. See Appendix E in the end of the paper. □

We choose the following controllers:

$$\begin{cases} u_R \;=\; \int_0^1 \hat{w}^\rho(y)\psi(y)\mathrm{d}y + \int_0^1 \hat{w}^\iota(y)\psi^c(y)\mathrm{d}y, \\ u_I \;=\; -\int_0^1 \hat{w}^\rho(y)\psi^c(y)\mathrm{d}y + \int_0^1 \hat{w}^\iota(y)\psi(y)\mathrm{d}y, \end{cases} \tag{4.9}$$

with $\hat{b}_R$ and $\hat{b}_I$ satisfying

$$\dot{\hat{b}}_R = \frac{\gamma\int_0^1 (\tilde{\rho}v^\rho + \tilde{\iota}v^\iota)\,\mathrm{d}x}{1 + \|v^\rho\|^2 + \|v^\iota\|^2}, \quad \dot{\hat{b}}_I = \frac{\gamma\int_0^1 (\tilde{\iota}v^\rho - \tilde{\rho}v^\iota)\,\mathrm{d}x}{1 + \|v^\rho\|^2 + \|v^\iota\|^2}, \tag{4.10}$$

$\tilde{\rho} = \rho - \hat{\rho}$, $\tilde{\iota} = \iota - \hat{\iota}$, γ being an arbitrary positive constant.

The following two propositions give the important performance of parameters estimators and adaptive controller, which will be used in the stability analysis of the closed-loop system.

Proposition 4.2. *The updating law* (4.10) *guarantees that the following properties hold*

$$\begin{cases} \tilde{b}_R, \tilde{b}_I, \hat{b}_R, \hat{b}_I \in \mathcal{L}_\infty, \quad \dot{\hat{b}}_R, \dot{\hat{b}}_I \in \mathcal{L}_2 \cap \mathcal{L}_\infty, \\ \frac{\|\tilde{\rho}\|}{\sqrt{1+\|v^\rho\|^2+\|v^\iota\|^2}}, \frac{\|\tilde{\iota}\|}{\sqrt{1+\|v^\rho\|^2+\|v^\iota\|^2}} \in \mathcal{L}_2 \cap \mathcal{L}_\infty, \\ \|\epsilon^\rho\|, \|\epsilon^\iota\|, \|\epsilon^\rho_x\|, \|\epsilon^\iota_x\| \in \mathcal{L}_2 \cap \mathcal{L}_\infty. \end{cases} \tag{4.11}$$

Proof. See Appendix F in the end of the paper. □

Proposition 4.3. *The adaptive controller* (4.9) *and* (4.10) *guarantees that the following properties hold*

$$\|\hat{w}^\rho\|, \|\hat{w}^\iota\|, \|v^\rho\|, \|v^\iota\| \in \mathcal{L}_2 \cap \mathcal{L}_\infty, \tag{4.12}$$

$$\|\hat{w}^\rho_x\|, \|\hat{w}^\iota_x\|, \|v^\rho_x\|, \|v^\iota_x\| \in \mathcal{L}_\infty. \tag{4.13}$$

Proof. See Section G of Appendix in the end of the paper. □

The main results of this section are obtained in the following theorem.

Theorem 4.4. *The adaptive controller (4.9) and (4.10) guarantees that all the closed-loop system states are bounded and the original system states $\rho(x,t)$ and $\iota(x,t)$ converge to zero ultimately, i.e.,*

$$\lim_{t\to+\infty}\sup_{x\in[0,\,1]}\left(|\rho(x,t)|+|\iota(x,t)|\right)=0.$$

Proof. By (4.11) and using Agmon's inequality, we known that ϵ^ρ and ϵ^ι are bounded on $[0,\,1]\times[0,\,+\infty)$. Similarly, (4.12) and (4.13) imply that $\hat{w}^\rho$, $\hat{w}^\iota$, v^ρ, v^ι are bounded on $[0,\,1]\times[0,\,+\infty)$. Then, transformation (3.4) implies that $\hat{\rho}$, $\hat{\iota}$, and hence η^ρ, η^ι are bounded on $[0,\,1]\times[0,\,+\infty)$ (see from (4.5)). Thus, we obtain from (4.3) that ρ and ι are bounded on $[0,\,1]\times[0,\,+\infty)$.

By (G.1), we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\hat{w}^\rho\|^2+\|\hat{w}^\iota\|^2\right)\\
&\quad\le \hat{b}_R\int_0^1\left(\hat{w}^\rho\tilde{w}^\rho+\hat{w}^\iota\tilde{w}^\iota\right)\mathrm{d}x+\hat{b}_I\int_0^1\left(\hat{w}^\iota\tilde{w}^\rho-\hat{w}^\rho\tilde{w}^\iota\right)\mathrm{d}x\\
&\qquad+\dot{\hat{b}}_R\int_0^1\left(\hat{w}^\rho\breve{v}^\rho+\hat{w}^\iota\breve{v}^\iota\right)\mathrm{d}x+\dot{\hat{b}}_I\int_0^1\left(\hat{w}^\iota\breve{v}^\rho+\hat{w}^\rho\breve{v}^\iota\right)\mathrm{d}x\\
&\qquad+\int_0^1\hat{w}^\rho\int_0^x\left(\hat{w}^\rho(y)H(x,y)+\hat{w}^\iota(y)H^c(x,y)\right)\mathrm{d}y\mathrm{d}x\\
&\qquad-\int_0^1\hat{w}^\iota\int_0^x\left(\hat{w}^\rho(y)H^c(x,y)-\hat{w}^\iota(y)H(x,y)\right)\mathrm{d}y\mathrm{d}x.
\end{aligned}$$

Noting that all the terms on the right-hand side of the above equality are bounded, there exists constant M such that $\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\hat{w}^\rho\|^2+\|\hat{w}^\iota\|^2\right)\le M$. This, together with the proven fact $\|\hat{w}^\rho\|^2+\|\hat{w}^\iota\|^2\in\mathcal{L}_1$ leads to $\lim_{t\to\infty}\left(\|\hat{w}^\rho\|+\|\hat{w}^\iota\|\right)=0$ (by Lem. H.4). Then, we obtain that $\lim_{t\to+\infty}\sup_{x\in[0,\,1]}\left(|\hat{w}^\rho(x,t)|+|\hat{w}^\iota(x,t)|\right)=0$ by Agmon's inequality and noting that $\|\hat{w}^\rho_x\|$, $\|\hat{w}^\iota_x\|$ are bounded, and $\lim_{t\to+\infty}\sup_{x\in[0,\,1]}\left(|\hat{\rho}(x,t)|+|\hat{\iota}(x,t)|\right)=0$ by (3.4). By the similar derivation as above, we obtain that $\lim_{t\to+\infty}\sup_{x\in[0,\,1]}\left(|v^\rho(x,t)|+|v^\iota(x,t)|\right)=0$ and $\lim_{t\to+\infty}\sup_{x\in[0,\,1]}\left(|\epsilon^\rho(x,t)|+|\epsilon^\iota(x,t)|\right)=0$. Thus, (4.5) implies $\lim_{t\to+\infty}\sup_{x\in[0,\,1]}\left(|\eta^\rho(x,t)|+|\eta^\iota(x,t)|\right)=0$, and hence (4.3) implies that $\lim_{t\to+\infty}\sup_{x\in[0,\,1]}\left(|\rho(x,t)|+|\iota(x,t)|\right)=0$. □

5. Simulation

In this section, we validate the effectiveness of the proposed method for system (1.1) with $x_d=1$, the actual values of system parameters being supposed as $a_1=1+2i$, $a_2=-1+5i$ and $a_3=2-3i$. Then, by (1.4), we obtain that $a_R=1$, $a_I=2$, $b_R=4.2$ and $b_I=-4.9$.

Noting that some controller gain functions (such as, $\hat{k}(\cdot)$, $\hat{k}^c(\cdot)$, $\hat{l}(\cdot)$ and $\hat{l}^c(\cdot)$) are given in the form of infinite series whose compact sums are difficult to obtain, and moreover appropriate truncations of the infinite series are sufficient for the practical implementation, we then replace $\hat{k}(\cdot)$, $\hat{k}^c(\cdot)$, $\hat{l}(\cdot)$ and $\hat{l}^c(\cdot)$ by their appropriations, respectively, *i.e.*,

$$\begin{cases}
\hat{k}(x,y) & \approx -\frac{1}{2}\hat{\beta}y+\frac{\hat{\beta}^{c^2}-\hat{\beta}^2}{16}y(x^2-y^2)+\frac{3\hat{\beta}^{c^2}\hat{\beta}-\hat{\beta}^3}{384}y(x^2-y^2)^2,\\
\hat{k}^c(x,y) & \approx \frac{1}{2}\hat{\beta}^cy+\frac{\hat{\beta}^c\hat{\beta}}{8}y(x^2-y^2)+\frac{3\hat{\beta}^c\hat{\beta}^2-\hat{\beta}^{c^3}}{384}y(x^2-y^2)^2,\\
\hat{l}(x,y) & \approx \frac{1}{2}\hat{\beta}y+\frac{\hat{\beta}^{c^2}-\hat{\beta}^2}{16}y(x^2-y^2)-\frac{3\hat{\beta}^{c^2}\hat{\beta}-\hat{\beta}^3}{384}y(x^2-y^2)^2,\\
\hat{l}^c(x,y) & \approx -\frac{1}{2}\hat{\beta}^cy+\frac{\hat{\beta}^c\hat{\beta}}{8}y(x^2-y^2)-\frac{3\hat{\beta}^c\hat{\beta}^2-\hat{\beta}^{c^3}}{384}y(x^2-y^2)^2,
\end{cases}$$

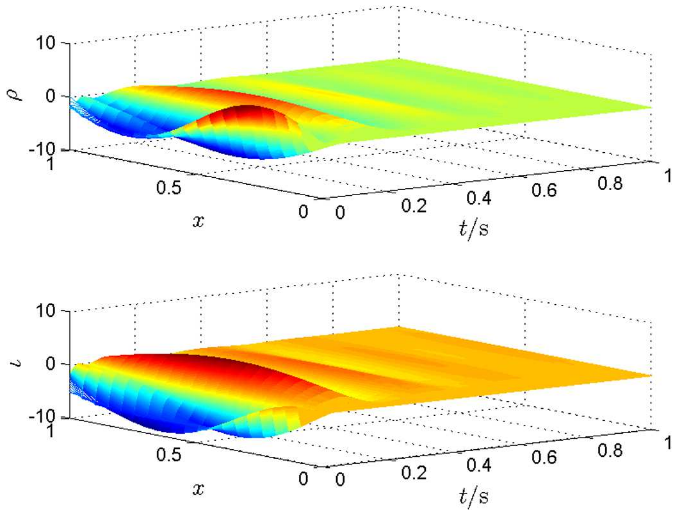

FIGURE 1. Trajectories of closed-loop system states ρ, ι with adaptive controller based on passive identifier.

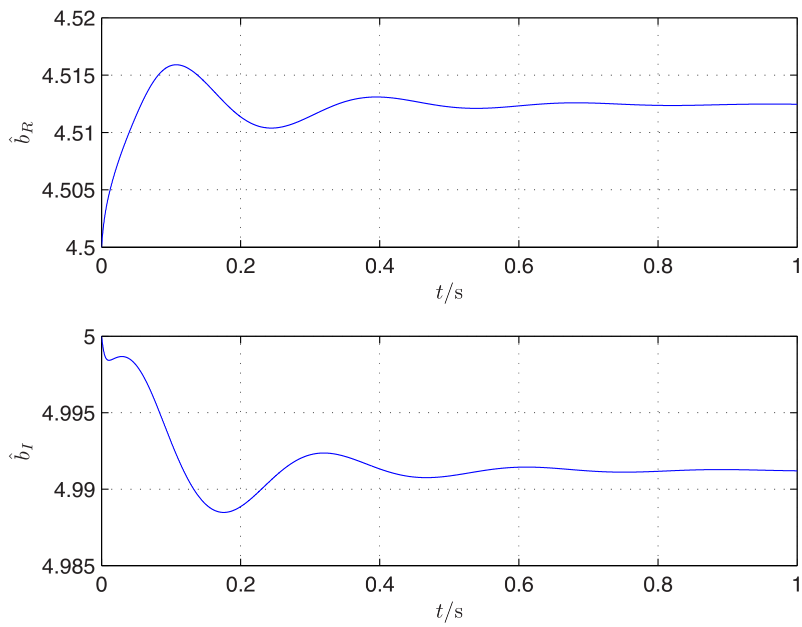

FIGURE 2. Trajectories of parameter estimates $\hat{b}_R$, $\hat{b}_I$ with adaptive controller based on passive identifier.

with $\hat{\beta}$, $\hat{\beta}^c$ being defined in (2.2) and $c = 5$.

Using controller (3.8) and (4.9) with dynamic updating laws (3.9) and (4.10), respectively, and choosing the system initial conditions as $\rho(x, 0) = 5\sin(2\pi x)$, $\iota(x, 0) = 3\cos(2\pi x) + 2\sin(2\pi x) - 3\epsilon^x$, the initial values of dynamic updating laws as $\hat{b}_R(0) = 4.5$, $\hat{b}_I(0) = 5$ and adaptive gains as $\gamma = 0.01$, we implement the simulation by explicit Euler method (see [31]) with 20-step discretization in space. Consequently, four simulation figures are obtained for the closed-loop systems. Specifically, Figures 1 and 3 show that the original closed-loop system

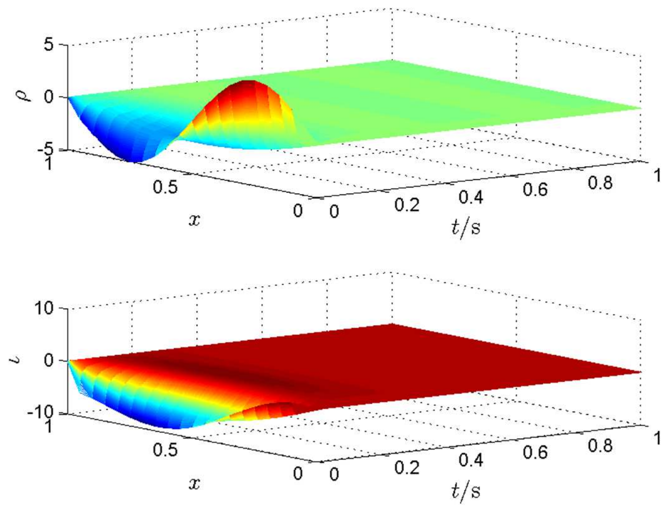

FIGURE 3. Trajectories of closed-loop system states ρ, ι with adaptive controller based on swapping identifier.

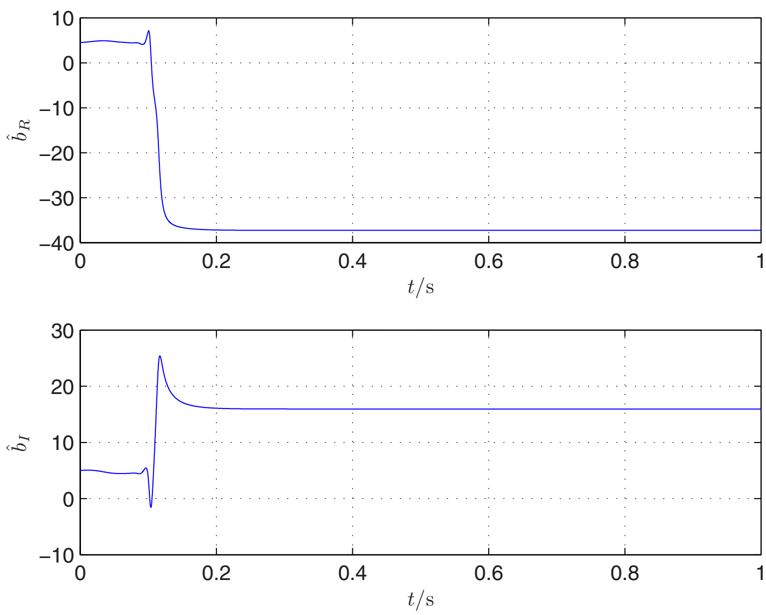

FIGURE 4. Trajectories of parameter estimates $\hat{b}_R$, $\hat{b}_I$ with adaptive controller based on swapping identifier.

states are bounded and converge to zero ultimately, Figures 2 and 4 show that all the parameter estimates are bounded and converge to different constants ultimately.

6. Concluding remarks

In this paper, stabilization of a class of uncertain linearized GLEs has been solved by adaptive control schemes based on passive and swapping identifiers, respectively. Such stabilization problem is equivalent to that of a class of uncertain coupled parabolic systems, which can not be solved by existing methods. By incorporating passive and swapping identifiers into dynamic compensation of unknown parameters, two adaptive state-feedback controllers are constructed which guarantee the desirable stability of the closed-loop systems.

It is worthwhile pointing out that, the adaptive control schemes proposed in this paper can be extended to solve the stabilization problem of more general system. For example, the 1-dimensional coupled parabolic system can be extended to the multidimensional one (*e.g.*, 3-dimensional system defined on a cylinder), unknown constant parameters can be extended to unknown spatial-varying parameters. However, some technique obstacles would appear in the control of system with unknown spatial-varying parameters, since new passive and swapping identifiers should be designed and the dynamic compensators for unknown spatial-varying parameters would be spatio-temporal varying rather than only time-varying.

The future research directions are twofold. First, controllers designed in this paper require all the system states to be available for feedback. Hence, how to design a controller by more less measurements (*e.g.*, only boundary values of GLEs (1.1) or coupled system (1.3)) deserves further investigation. Second, this paper removes the restriction that all the unknown parameters must belong to known finite intervals in the related literature while a_1 of system (1.1) (and hence a_R and a_I of system (1.3)) being known. How to stabilize the system with all the system parameters being unknown and do not belong to known finite intervals will be meaningful and deserves further investigation.

Appendixes

This section collects the proofs of Lemma 1 and Propositions 3.1–4.3, as well as some useful inequalities and criterions which will be frequently used in control design and performance analysis.

Appendix A. Proof of Lemma 2.1

We only show the differentiability of $\hat{k}(\cdot)$ and $\hat{k}^c(\cdot)$ with respect to $\hat{b}_R$, $\hat{b}_I$ since the other claims are direct by Lemma 2 and Theorem 4 in [2].

Firstly, we give the estimations of $G_n(\xi,\eta)$ and $G_n^c(\xi,\eta)$ by induction. Noting that $|\hat{b}_R| \le \delta_1$, $|\hat{b}_I| \le \delta_1$, there is a positive constant c_0 such that $|\hat{\beta}| \le c_0$ and $|\hat{\beta}^c| \le c_0$. Then, the first line of (2.2) gives that $|G_0(\xi,\eta)| \le \frac{c_0}{2}$ and $|G_0^c(\xi,\eta)| \le \frac{c_0}{2}$. Suppose that

$$|G_n(\xi,\eta)| \;\le\; \frac{c_0^{n+1}(\xi+\eta)^n}{2\cdot n!}, |G_n^c(\xi,\eta)| \;\le\; \frac{c_0^{n+1}(\xi+\eta)^n}{2\cdot n!}, \tag{A.1}$$

Then, the second line of (2.2) give that

$$\begin{aligned} |G_{n+1}(\xi,\eta)| &\le \frac{1}{4}\left|\hat{\beta}\right| \int_\eta^\xi \int_0^\eta |G_n(\tau,s)|\,\mathrm{d}s\mathrm{d}\tau + \frac{1}{4}\left|\hat{\beta}^c\right| \int_\eta^\xi \int_0^\eta |G_n^c(\tau,s)|\,\mathrm{d}s\mathrm{d}\tau \\ &\le \frac{c_0^{n+2}}{4\cdot n!}\int_\eta^\xi \int_0^\eta (\tau+s)^n \mathrm{d}s\mathrm{d}\tau. \end{aligned} \tag{A.2}$$

Noting that $\xi = x+y$, $\eta = x-y$, we have $0<\xi-\eta = 2y < 2$, and hence

$$\begin{aligned} \int_\eta^\xi \int_0^\eta (\tau+s)^n \mathrm{d}s\mathrm{d}\tau &= \int_\eta^\xi \frac{(\tau+\eta)^{n+1}-\tau^{n+1}}{n+1}\mathrm{d}\tau \\ &\le \int_\eta^\xi \frac{(\tau+\eta)^{n+1}}{n+1}\mathrm{d}\tau \le \frac{(\xi+\eta)^{n+1}}{n+1}(\xi-\eta) \le \frac{2(\xi+\eta)^{n+1}}{(n+1)}, \end{aligned} \tag{A.3}$$

substituting which into the right-hand side of (A.2) leads to

$$|G_{n+1}(\xi,\eta)| \leq \frac{c_0^{n+2}(\xi+\eta)^{n+1}}{2\cdot(n+1)!}. \tag{A.4}$$

Similarly, the following inequality holds

$$\left|G^c_{n+1}(\xi,\eta)\right| \leq \frac{c_0^{n+2}(\xi+\eta)^{n+1}}{2\cdot(n+1)!},$$

which, together with (A.4), implies that the induce assumption (A.1) holds.

Secondly, we show that $\sum_{n=0}^{\infty}\frac{\partial G_n}{\partial\hat{\beta}}$ and $\sum_{n=0}^{\infty}\frac{\partial G_n}{\partial\hat{\beta}^c}$ converge uniformly on Γ. By the first line of (2.2), we have

$$\left|\frac{\partial G_0}{\partial\hat{\beta}}\right| = \frac{1}{4}(\xi-\eta) \leq \frac{1}{2} \leq c_1, \quad \left|\frac{\partial G^c_0}{\partial\hat{\beta}}\right| = 0 \leq c_1,$$

with $c_1 = 2 + c_0$. Moreover, by the second line of (2.2), we have

$$\frac{\partial G_{n+1}}{\partial\hat{\beta}} = \frac{1}{4}\int_\eta^\xi\int_0^\eta G_n(\tau,s)\mathrm{d}s\mathrm{d}\tau + \frac{1}{4}\hat{\beta}\int_\eta^\xi\int_0^\eta \frac{\partial G_n}{\partial\hat{\beta}}(\tau,s)\mathrm{d}s\mathrm{d}\tau + \frac{1}{4}\hat{\beta}^c\int_\eta^\xi\int_0^\eta \frac{\partial G^c_n}{\partial\hat{\beta}}(\tau,s)\mathrm{d}s\mathrm{d}\tau. \tag{A.5}$$

Suppose that

$$\left|\frac{\partial G_n}{\partial\hat{\beta}}\right| \leq \frac{c_1^{n+1}(\xi+\eta)^n}{n!}, \quad \left|\frac{\partial G^c_n}{\partial\hat{\beta}}\right| \leq \frac{c_1^{n+1}(\xi+\eta)^n}{n!}. \tag{A.6}$$

Then, (A.1), (A.3) and (A.5) give that

$$\begin{aligned}\left|\frac{\partial G_{n+1}}{\partial\hat{\beta}}\right| &\leq \frac{c_0^{n+1}}{8\cdot n!}\int_\eta^\xi\int_0^\eta(\tau+s)^n\mathrm{d}s\mathrm{d}\tau + \frac{c_0c_1^{n+1}}{2\cdot n!}\int_\eta^\xi\int_0^\eta(\tau+s)^n\mathrm{d}s\mathrm{d}\tau \\ &\leq \frac{c_1^{n+1}(\xi+\eta)^{n+1}}{4\cdot(n+1)!} + \frac{c_0c_1^{n+1}(\xi+\eta)^{n+1}}{(n+1)!} \leq \frac{c_1^{n+2}(\xi+\eta)^{n+1}}{(n+1)!}.\end{aligned} \tag{A.7}$$

Similarly, there holds

$$\left|\frac{\partial G^c_{n+1}}{\partial\hat{\beta}}\right| \leq \frac{c_1^{n+2}(\xi+\eta)^{n+1}}{(n+1)!},$$

which,together with (A.4), implies that the induce assumption (A.6) holds.

Noting that $0<\xi+\eta<2$, we obtain from (A.6) that

$$\sum_{n=0}^{+\infty}\frac{\partial G_n}{\partial\hat{\beta}} \leq \sum_{n=0}^{+\infty}\frac{c_1(2c_1)^n}{n!}, \quad \sum_{n=0}^{+\infty}\frac{\partial G^c_n}{\partial\hat{\beta}} \leq \sum_{n=0}^{+\infty}\frac{c_1(2c_1)^n}{n!}.$$

The two infinite series on the right-hand side of above inequalities are convergent. Then, the two ones one the left-hand side of above inequalities are uniformly convergent. Thus, $\hat{k}(\cdot)$ is differentiable with respect to $\hat{\beta}$, and hence is differentiable with respect to $\hat{b}_R$ and $\hat{b}_I$. So does $\hat{k}^c(\cdot)$.

Appendix B. Proof of Proposition 3.1

First, $\hat{w}^\rho(0) = \hat{w}^\iota(0) = 0$ can be directly obtained by setting $x = 0$ in (3.3) while noting $\hat{\rho}(0) = \hat{\iota}(0) = 0$. Moreover, by setting $x = 1$ in (3.3) while noting $\hat{\rho}(1) = u_R$, $\hat{\iota}(1) = u_I$, we have

$$\begin{cases} \hat{w}^\rho(1) &= u_R - \int_0^1 \big(\hat{k}(1,y)\hat{\rho}(y) + \hat{k}^c(1,y)\hat{\iota}(y)\big)\mathrm{d}y, \\ \hat{w}^\iota(1) &= u_I - \int_0^1 \big(-\hat{k}^c(1,y)\hat{\rho}(y) + \hat{k}(1,y)\hat{\iota}(y)\big)\mathrm{d}y. \end{cases}$$

Then, substituting (3.4) into the right-hand side of the above equation and changing the order of integration lead to the last two equations of (3.6).

In the following, we are devoted to the derivation of the first equality of (3.6), and then the second one can be similarly obtained. First, computing the partial derivative with respect to t from the first equation of (3.3) along the solutions of (3.1), we have

$$\begin{aligned}
\hat{w}^\rho_t &= \hat{\rho}_t - \int_0^x \big(\hat{k}(x,y)\hat{\rho}_t(y) + \hat{k}^c(x,y)\hat{\iota}_t(y)\big)\mathrm{d}y - \int_0^x \big(\hat{k}_t(x,y)\hat{\rho}(y) + \hat{k}^c_t(x,y)\hat{\iota}(y)\big)\mathrm{d}y \\
&= a_R\hat{\rho}_{xx} + \hat{b}_R\rho - a_I\hat{\iota}_{xx} - \hat{b}_I\iota + \gamma^2\tilde{\rho}\|\rho\|^2 + \gamma^2\tilde{\rho}\|\iota\|^2 \\
&\quad - \int_0^x \hat{k}(x,y)\left(a_R\hat{\rho}_{yy} + \hat{b}_R\rho - a_I\hat{\iota}_{yy} - \hat{b}_I\iota + \gamma^2\tilde{\rho}\|\rho\|^2 + \gamma^2\tilde{\rho}\|\iota\|^2\right)\mathrm{d}y \\
&\quad - \int_0^x \hat{k}^c(x,y)\left(a_I\hat{\rho}_{yy} + \hat{b}_I\rho + a_R\hat{\iota}_{yy} + \hat{b}_R\iota + \gamma^2\tilde{\iota}\|\iota\|^2 + \gamma^2\tilde{\iota}\|\rho\|^2\right)\mathrm{d}y \\
&\quad - \int_0^x \big(\hat{k}_t(x,y)\hat{\rho}(y) + \hat{k}^c_t(x,y)\hat{\iota}(y)\big)\mathrm{d}y \\
&= a_R\left(\hat{\rho}_{xx} - \int_0^x \left(\hat{k}(x,y)\hat{\rho}_{yy}(y) + \hat{k}^c(x,y)\hat{\iota}_{yy}(y)\right)\mathrm{d}y\right) \\
&\quad - a_I\left(\hat{\iota}_{xx} - \int_0^x \left(-\hat{k}^c(x,y)\hat{\rho}_{yy}(y) + \hat{k}(x,y)\hat{\iota}_{yy}(y)\right)\mathrm{d}y\right) \\
&\quad + \hat{b}_R\left(\rho - \int_0^x \left(\hat{k}(x,y)\rho(y) + \hat{k}^c(x,y)\iota(y)\right)\mathrm{d}y\right) \\
&\quad - \hat{b}_I\left(\iota - \int_0^x \left(-\hat{k}^c(x,y)\rho(y) + \hat{k}(x,y)\iota(y)\right)\mathrm{d}y\right) \\
&\quad + \gamma^2\left(\|\rho\|^2 + \|\iota\|^2\right)\left(\tilde{\rho} - \int_0^x \left(\hat{k}(x,y)\tilde{\rho}(y) + \hat{k}^c(x,y)\tilde{\iota}(y)\right)\mathrm{d}y\right) \\
&\quad - \int_0^x \big(\hat{k}_t(x,y)\hat{\rho}(y) + \hat{k}^c_t(x,y)\hat{\iota}(y)\big)\mathrm{d}y \\
&= a_R\left(\hat{\rho}_{xx} - \int_0^x \left(\hat{k}(x,y)\hat{\rho}_{yy}(y) + \hat{k}^c(x,y)\hat{\iota}_{yy}(y)\right)\mathrm{d}y\right) \\
&\quad - a_I\left(\hat{\iota}_{xx} - \int_0^x \left(-\hat{k}^c(x,y)\hat{\rho}_{yy}(y) + \hat{k}(x,y)\hat{\iota}_{yy}(y)\right)\mathrm{d}y\right) \\
&\quad + \hat{b}_R w^\rho(x) - \hat{b}_I w^\iota(x) + \gamma^2\left(\|\rho\|^2 + \|\iota\|^2\right)\tilde{w}^\rho(x) \\
&\quad - \int_0^x \big(\hat{k}_t(x,y)\hat{\rho}(y) + \hat{k}^c_t(x,y)\hat{\iota}(y)\big)\mathrm{d}y,
\end{aligned} \tag{B.1}$$

with $w^\rho(x) = \rho - \int_0^x \Big(\hat{k}(x,y)\rho(y) + \hat{k}^c(x,y)\iota(y)\Big)\,\mathrm{d}y$ and $w^\iota(x) = \iota - \int_0^x \Big(-\hat{k}^c(x,y)\rho(y) + \hat{k}(x,y)\iota(y)\Big)\,\mathrm{d}y$. The first two terms on the right-hand side of above equality (denoted by ① and ②, respectively) need further treatments. For this, by direct computing from the first equation of (3.3), we obtain

$$\begin{aligned}
\hat{\rho}_{xx} &= \hat{w}^\rho_{xx} + \frac{\partial^2}{\partial x^2}\left(\int_0^x \big(\hat{k}(x,y)\hat{\rho}(y) + \hat{k}^c(x,y)\hat{\iota}(y)\big)\mathrm{d}y\right)\\
&= \hat{w}^\rho_{xx} + \frac{\partial}{\partial x}\left(\hat{k}(x,x)\hat{\rho}(x) + \hat{k}^c(x,x)\hat{\iota}(x) + \int_0^x \big(\hat{k}_x(x,y)\hat{\rho}(y) + \hat{k}^c_x(x,y)\hat{\iota}(y)\big)\mathrm{d}y\right)\\
&= \hat{w}^\rho_{xx} + \frac{\mathrm{d}}{\mathrm{d}x}\hat{k}(x,x)\hat{\rho}(x) + \hat{k}(x,x)\hat{\rho}_x(x) + \frac{\mathrm{d}}{\mathrm{d}x}\hat{k}^c(x,x)\hat{\iota}(x) + \hat{k}^c(x,x)\hat{\iota}_x(x)\\
&\quad + \hat{k}_x(x,x)\hat{\rho}(x) + \hat{k}^c_x(x,x)\hat{\iota}(x) + \int_0^x \big(\hat{k}_{xx}(x,y)\hat{\rho}(y) + \hat{k}^c_{xx}(x,y)\hat{\iota}(y)\big)\mathrm{d}y.
\end{aligned}$$

Moreover, by integration by parts and using (2.3), we have

$$\begin{aligned}
&-\int_0^x \Big(\hat{k}(x,y)\hat{\rho}_{yy}(y) + \hat{k}^c(x,y)\hat{\iota}_{yy}(y)\Big)\,\mathrm{d}y\\
&\qquad = -\hat{k}(x,x)\hat{\rho}_x(x) + \hat{k}_y(x,x)\hat{\rho}(x) - \hat{k}^c(x,x)\hat{\iota}_x(x) + \hat{k}_{c,y}(x,x)\hat{\iota}(x)\\
&\qquad\quad - \int_0^x \Big(\hat{k}_{yy}(x,y)\hat{\rho}(y) + \hat{k}^c_{yy}(x,y)\hat{\iota}(y)\Big)\,\mathrm{d}y.
\end{aligned}$$

Then, by (2.3), we have

$$\begin{aligned}
① &= a_R\hat{w}^\rho_{xx} + a_R\hat{\rho}(x)\left(\frac{\mathrm{d}}{\mathrm{d}x}\hat{k}(x,x) + \hat{k}_x(x,x) + \hat{k}_y(x,x)\right)\\
&\quad + a_R\hat{\iota}(x)\left(\frac{\mathrm{d}}{\mathrm{d}x}\hat{k}^c(x,x) + \hat{k}^c_x(x,x) + k^c_y(x,x)\right)\\
&\quad + a_R\int_0^x \big(\hat{k}_{xx}(x,y) - \hat{k}^c_{xx}(x,y)\big)\hat{\rho}(y)\mathrm{d}y + a_R\int_0^x \big(\hat{k}^c_{xx}(x,y) - \hat{k}^c_{yy}(x,y)\big)\hat{\rho}(y)\mathrm{d}y\\
&= a_R\hat{w}^\rho_{xx} - a_R\hat{\beta}\hat{\rho}(x) + a_R\hat{\beta}^c\hat{\iota}(x) + a_R\int_0^x \big(\hat{\beta}\hat{k}(x,y) + \hat{\beta}^c\hat{k}_c(x,y)\big)\hat{\rho}(y)\mathrm{d}y\\
&\quad + a_R\int_0^x \big(-\hat{\beta}^c\hat{k}(x,y) + \hat{\beta}\hat{k}_c(x,y)\big)\hat{\iota}(y)\mathrm{d}y\\
&= a_R\hat{w}^\rho_{xx} - a_R\hat{\beta}\left(\hat{\rho}(x) - \int_0^x \big(\hat{k}(x,y)\hat{\rho}(y) + \hat{k}_c(x,y)\hat{\iota}(y)\big)\mathrm{d}y\right)\\
&\quad + a_R\hat{\beta}^c\left(\hat{\iota}(x) - \int_0^x \big(-\hat{k}_c(x,y)\hat{\rho}(y) + \hat{k}(x,y)\hat{\iota}(y)\big)\mathrm{d}y\right)\\
&= a_R\hat{w}^\rho_{xx} - a_R\hat{\beta}\hat{w}^\rho + a_R\hat{\beta}^c\hat{w}^\iota.
\end{aligned} \tag{B.2}$$

Similarly, we obtain

$$② = a_I\hat{w}^\iota_{xx} - a_I\hat{\beta}^c\hat{w}^\rho - a_I\hat{\beta}\hat{w}^\iota. \tag{B.3}$$

Substituting the above two equalities into (B.1) while noting that $a_I\hat{\beta}^c - a_R\hat{\beta} = -\hat{b}_R - c$ and $a_I\hat{\beta} + a_R\hat{\beta}^c = \hat{b}_I$, we have

$$\begin{aligned}\hat{w}_t^\rho &= a_R\hat{w}_{xx}^\rho - a_I\hat{w}_{xx}^\iota + \hat{b}_R w^\rho + \left(a_I\hat{\beta}^c - a_R\hat{\beta}\right)\hat{w}^\rho - \hat{b}_I w^\iota + \left(a_I\hat{\beta} + a_R\hat{\beta}^c\right)\hat{w}^\iota \\ &\quad + \left(\gamma^2\|\rho\|^2 + \gamma^2\|\iota\|^2\right)\tilde{w}^\rho(x) - \int_0^x \left(\hat{k}_t(x,y)\hat{\rho}(y) + \hat{k}_t^c(x,y)\hat{\iota}(y)\right)\mathrm{d}y \\ &= a_R\hat{w}_{xx}^\rho - a_I\hat{w}_{xx}^\iota - c\hat{w}^\rho - \hat{b}_I\tilde{w}^\iota + \left(\hat{b}_R + \gamma^2\|\rho\|^2 + \gamma^2\|\iota\|^2\right)\tilde{w}^\rho \\ &\quad - \int_0^x \left(\hat{k}_t(x,y)\hat{\rho}(y) + \hat{k}_t^c(x,y)\hat{\iota}(y)\right)\mathrm{d}y. \end{aligned} \tag{B.4}$$

Then, substituting (3.4) into the last term on the right-hand side of the above equation, we have

$$\begin{aligned}&\int_0^x \left(\hat{k}_t(x,y)\hat{\rho}(y) + \hat{k}_t^c(x,y)\hat{\iota}(y)\right)\mathrm{d}y \\ &\quad = \int_0^x \hat{k}_t(x,y)\left(w^\rho(y) - \int_0^y \left(\hat{l}(y,\xi)w^\rho(\xi) + \hat{l}^c(y,\xi)w^\iota(\xi)\right)\mathrm{d}\xi\right)\mathrm{d}y \\ &\quad\quad + \int_0^x \hat{k}_t^c(x,y)\left(w^\iota(y) - \int_0^y \left(-\hat{l}^c(y,\xi)w^\rho(\xi) + \hat{l}(y,\xi)w^\iota(\xi)\right)\mathrm{d}\xi\right)\mathrm{d}y.\end{aligned}$$

Changing the order of integration, the following equalities hold

$$\begin{cases}\int_0^x \hat{k}_t(x,y)\int_0^y \left(\hat{l}(y,\xi)w^\rho(\xi) + \hat{l}^c(y,\xi)w^\iota(\xi)\right)\mathrm{d}\xi\mathrm{d}y \\ \quad\quad = \int_0^x w^\rho(y)\int_y^x \hat{k}_t(x,\xi)\hat{l}(\xi,y)\mathrm{d}\xi\mathrm{d}y + \int_0^x w^\iota(y)\int_y^x \hat{k}_t(x,\xi)\hat{l}^c(\xi,y)\mathrm{d}\xi\mathrm{d}y, \\ \int_0^x \hat{k}_t^c(x,y)\int_0^y \left(-\hat{l}^c(y,\xi)w^\rho(\xi) + \hat{l}(y,\xi)w^\iota(\xi)\right)\mathrm{d}\xi\mathrm{d}y \\ \quad\quad = -\int_0^x w^\rho(y)\int_y^x \hat{k}_t^c(x,\xi)\hat{l}^c(\xi,y)\mathrm{d}\xi\mathrm{d}y + \int_0^x w^\iota(y)\int_y^x \hat{k}_t^c(x,\xi)\hat{l}(\xi,y)\mathrm{d}\xi\mathrm{d}y,\end{cases}$$

by which and noting the definitions of $H(x,y)$, $H^c(x,y)$, we obtain that

$$\begin{aligned}&\int_0^x \left(\hat{k}_t(x,y)\hat{\rho}(y) + \hat{k}_t^c(x,y)\hat{\iota}(y)\right)\mathrm{d}y \\ &\quad = \int_0^x w^\rho(y)\left(\hat{k}_t(x,y) - \int_y^x \left(\hat{k}_t(x,\xi)\hat{l}(\xi,y) - \hat{k}_t^c(x,\xi)\hat{l}^c(\xi,y)\right)\mathrm{d}\xi\right)\mathrm{d}y \\ &\quad\quad + \int_0^x w^\iota(y)\left(\hat{k}_t^c(x,y) - \int_y^x \left(\hat{k}(x,\xi)\hat{l}^c(\xi,y) + \hat{k}_t^c(x,\xi)\hat{l}(\xi,y)\right)\mathrm{d}\xi\right)\mathrm{d}y \\ &\quad = -\int_0^x \left(w^\rho(y)H(x,y) + w^\iota(y)H^c(x,y)\right)\mathrm{d}y.\end{aligned}$$

Substituting above equality into the last term on the right-hand side of (B.4) leads to the first equation of (3.6).

Appendix C. Proof of Proposition 3.2

Choose the following Lyapunov function:

$$V = \frac{1}{2}\int_0^1 \left(\tilde{\rho}^2 + \tilde{\iota}^2\right)\mathrm{d}x + \frac{\tilde{b}_R^2}{2\gamma} + \frac{\tilde{b}_I^2}{2\gamma}.$$

Computing the time derivative of V along the solutions of error system (3.2) and integrating by parts, we have

$$\begin{aligned}\dot{V} &= \int_0^1 \tilde{\rho}\big(a_R\tilde{\rho}_{xx} + \tilde{b}_R\rho - a_I\tilde{\iota}_{xx} - \tilde{b}_I\iota - \gamma^2\tilde{\rho}\|\rho\|^2 - \gamma^2\tilde{\rho}\|\iota\|^2\big)\mathrm{d}x \\ &\quad + \int_0^1 \tilde{\iota}\big(a_I\tilde{\rho}_{xx} + \tilde{b}_I\rho + a_R\tilde{\iota}_{xx} + \tilde{b}_R\iota - \gamma^2\tilde{\iota}\|\iota\|^2 - \gamma^2\tilde{\iota}\|\rho\|^2\big)\mathrm{d}x + \frac{\tilde{b}_R\dot{\tilde{b}}_R}{\gamma} + \frac{\tilde{b}_I\dot{\tilde{b}}_I}{\gamma} \\ &= -a_R\int_0^1 \left(\tilde{\rho}_x^2 + \tilde{\iota}_x^2\right)\mathrm{d}x + \tilde{b}_R\int_0^1 \left(\tilde{\rho}\rho + \tilde{\iota}\iota\right)\mathrm{d}x + \tilde{b}_I\int_0^1 \left(\tilde{\iota}\rho - \tilde{\rho}\iota\right)\mathrm{d}x \\ &\quad -\gamma^2\|\rho\|^2\|\tilde{\rho}\|^2 - \gamma^2\|\iota\|^2\|\tilde{\rho}\|^2 - \gamma^2\|\iota\|^2\|\tilde{\iota}\|^2 - \gamma^2\|\rho\|^2\|\tilde{\iota}\|^2 - \frac{\tilde{b}_R\dot{\hat{b}}_R}{\gamma} - \frac{\tilde{b}_I\dot{\hat{b}}_I}{\gamma}.\end{aligned}$$

Then, by (3.9), we obtain

$$\dot{V} = -a_R\int_0^1 \left(\tilde{\rho}_x^2 + \tilde{\iota}_x^2\right)\mathrm{d}x - \gamma^2\|\rho\|^2\|\tilde{\rho}\|^2 - \gamma^2\|\iota\|^2\|\tilde{\rho}\|^2 - \gamma^2\|\iota\|^2\|\tilde{\iota}\|^2 - \gamma^2\|\rho\|^2\|\tilde{\iota}\|^2.$$

Integrating both sides of the above equality over $[0,\, t]$ arrives at $V(t) \le V(0)$, and hence gives the first line of (3.10). Moreover, integrating both sides of the above equality over $[0,\, +\infty)$ yields that

$$\begin{aligned}a_R\int_0^{+\infty} \left(\|\tilde{\rho}_x\|^2 + \|\tilde{\iota}_x\|^2\right)\mathrm{d}t + \gamma^2\int_0^{+\infty} \left(\|\rho\|^2\|\tilde{\rho}\|^2 + \|\iota\|^2\|\tilde{\rho}\|^2 + \|\iota\|^2\|\tilde{\iota}\|^2 + \|\rho\|^2\|\tilde{\iota}\|^2\right)\mathrm{d}t \\ = V(0) - V(+\infty) \le V(0),\end{aligned}$$

which shows that $\|\tilde{\rho}_x\|$, $\|\tilde{\iota}_x\|$, $\|\rho\|\|\tilde{\rho}\|$, $\|\iota\|\|\tilde{\iota}\|$, $\|\iota\|\|\tilde{\rho}\|$, $\|\rho\|\|\tilde{\iota}\|$ are square integrable on $[0,\, +\infty)$. Then, by Poincaré's inequality (see Lem. H.2), we obtain that $\|\tilde{\rho}\|$ and $\|\tilde{\iota}\|$ are square integrable on $[0,\, +\infty)$, and hence obtain the second line of (3.10). This, together with (3.9), implies the third line of (3.10).

Appendix D. Proof of Proposition 3.3

We first give the relationships within some closed-loop system signals which will be frequently used in the following proof. Since $\hat{b}_R$ and $\hat{b}_I$ are bounded (see (3.10)) and kernel functions $\hat{k}(\cdot)$, $\hat{k}^c(\cdot)$, $\hat{l}^c(\cdot)$ and $\hat{l}(\cdot)$ are continuous with respect to each variable on bounded intervals, there exists positive constant M_1 such that

$$\begin{cases}\|\tilde{w}^\rho\| \le M_1\left(\|\tilde{\rho}\| + \|\tilde{\iota}\|\right), \quad \|\tilde{w}^\iota\| \le M_1\left(\|\tilde{\rho}\| + \|\tilde{\iota}\|\right), \\ \|\hat{\rho}\| \le M_1\left(\|\hat{w}^\rho\| + \|\hat{w}^\iota\|\right), \quad \|\hat{\iota}\| \le M_1\left(\|\hat{w}^\rho\| + \|\hat{w}^\iota\|\right), \\ \|\rho\| \le \|\tilde{\rho}\| + \|\hat{\rho}\| \le \|\tilde{\rho}\| + M_1\left(\|\hat{w}^\rho\| + \|\hat{w}^\iota\|\right), \\ \|\iota\| \le \|\tilde{\iota}\| + \|\hat{\iota}\| \le \|\tilde{\iota}\| + M_1\left(\|\hat{w}^\rho\| + \|\hat{w}^\iota\|\right),\end{cases} \tag{D.1}$$

which can be derived from (3.4) and the first two equalities of (3.7).

Then, we begin to prove the first two lines of (3.11). The first line of (D.1) implies that $\|\tilde{w}^\rho\|$, $\|\tilde{w}^\iota\| \in \mathcal{L}_\infty \cap \mathcal{L}_2$ since $\|\tilde{\rho}\|$, $\|\tilde{\iota}\| \in \mathcal{L}_\infty \cap \mathcal{L}_2$ (see from (3.10)). Moreover, the last three lines of (D.1) would imply that $\|\hat{\rho}\|$, $\|\hat{\iota}\|$, $\|\rho\|$, $\|\iota\| \in \mathcal{L}_\infty \cap \mathcal{L}_2$ once $\|\hat{w}^\rho\|$, $\|\hat{w}^\iota\| \in \mathcal{L}_\infty \cap \mathcal{L}_2$. For this, we consider

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\hat{w}^\rho\|^2 + \|\hat{w}^\iota\|^2\right) &= \int_0^1 \left(\hat{w}^\rho\hat{w}_t^\rho + \hat{w}^\iota\hat{w}_t^\iota\right)\mathrm{d}x \\ &= \int_0^1 \hat{w}^\rho\Big(a_R\hat{w}_{xx}^\rho - a_I\hat{w}_{xx}^\iota - c\hat{w}^\rho + \big(\hat{b}_R + \gamma^2\|\rho\|^2 + \gamma^2\|\iota\|^2\big)\tilde{w}^\rho - \hat{b}_I\tilde{w}^\iota\end{aligned}$$

$$
\begin{aligned}
&+\int_0^x \left(\hat{w}^\rho(y)H(x,y)+\hat{w}^\iota(y)H^c(x,y)\right)\mathrm{d}y\bigg)\mathrm{d}x\\
&+\int_0^1 \hat{w}^\iota\bigg(a_I\hat{w}^\rho_{xx}+a_R\hat{w}^\iota_{xx}-c\hat{w}^\iota+\left(\hat{b}_R+\gamma^2\|\rho\|^2+\gamma^2\|\iota\|^2\right)\tilde{w}^\iota+\hat{b}_I\tilde{w}^\rho\\
&-\int_0^x \left(\hat{w}^\rho(y)H^c(x,y)-\hat{w}^\iota(y)H(x,y)\right)\mathrm{d}y\bigg)\mathrm{d}x\\
=&-a_R\left(\|\hat{w}^\rho_x\|^2+\|\hat{w}^\iota_x\|^2\right)-c\left(\|\hat{w}^\rho\|^2+\|\hat{w}^\iota\|^2\right)\\
&+\hat{b}_I\int_0^1\left(\hat{w}^\iota\tilde{w}^\rho-\hat{w}^\rho\tilde{w}^\iota\right)\mathrm{d}x+\hat{b}_R\int_0^1\left(\hat{w}^\rho\tilde{w}^\rho+\hat{w}^\iota\tilde{w}^\iota\right)\mathrm{d}x\\
&+\gamma^2\left(\|\rho\|^2+\|\iota\|^2\right)\int_0^1\left(\hat{w}^\rho\tilde{w}^\rho+\hat{w}^\iota\tilde{w}^\iota\right)\mathrm{d}x\\
&+\int_0^1\hat{w}^\rho\int_0^x\left(\hat{w}^\rho(y)H(x,y)+\hat{w}^\iota(y)H^c(x,y)\right)\mathrm{d}y\mathrm{d}x\\
&-\int_0^1\hat{w}^\iota\int_0^x\left(\hat{w}^\rho(y)H^c(x,y)-\hat{w}^\iota(y)H(x,y)\right)\mathrm{d}y\mathrm{d}x. \qquad \text{(D.2)}
\end{aligned}
$$

The last four terms on the right-hand sides of the above equality should be further estimated. First, by Young's inequality and noting that $\|\tilde{w}^\rho\|, \|\tilde{w}^\iota\| \in \mathcal{L}_\infty \cap \mathcal{L}_2$, there exists a positive constant σ_1 such that

$$
\begin{aligned}
\hat{b}_I\int_0^1\left(\hat{w}^\iota\tilde{w}^\rho-\hat{w}^\rho\tilde{w}^\iota\right)\mathrm{d}x &\le \sigma_1\left(\|\hat{w}^\iota\|^2+\|\hat{w}^\rho\|^2\right)+\frac{\bar{b}_I^2}{4\sigma_1}\left(\|\tilde{w}^\iota\|^2+\|\tilde{w}^\rho\|^2\right)\\
&=\sigma_1\left(\|\hat{w}^\iota\|^2+\|\hat{w}^\rho\|^2\right)+l_1, \qquad \text{(D.3)}
\end{aligned}
$$

with $\bar{b}_I$ denoting the bound of $\hat{b}_I$ and l_1 denoting the function which belongs to $\mathcal{L}_1$. Similarly, the following inequality holds

$$
\hat{b}_R\int_0^1\left(\hat{w}^\rho\tilde{w}^\rho+\hat{w}^\iota\tilde{w}^\iota\right)\mathrm{d}x \le \sigma_1\left(\|\hat{w}^\iota\|^2+\|\hat{w}^\rho\|^2\right)+l_1. \qquad \text{(D.4)}
$$

Then, by Young's inequality and (D.1) while noting that $\|\tilde{\rho}\|\|\rho\|, \|\tilde{\iota}\|\|\rho\|, \|\tilde{\rho}\| \in \mathcal{L}_\infty \cap \mathcal{L}_2$ (see (3.10)), we have

$$
\begin{aligned}
&\gamma^2\|\rho\|^2\int_0^1\left(\hat{w}^\rho\tilde{w}^\rho+\hat{w}^\iota\tilde{w}^\iota\right)\mathrm{d}x\\
&\le \gamma^2\|\rho\|^2\|\hat{w}^\rho\|\|\tilde{w}^\rho\|+\gamma^2\|\rho\|^2\|\hat{w}^\iota\|\|\tilde{w}^\iota\|\\
&\le \frac{M_1^2\gamma^4}{\sigma_1}\|\tilde{w}^\rho\|^2\|\rho\|^2\cdot\|\hat{w}^\rho\|^2+\frac{M_1^2\gamma^4}{\sigma_1}\|\tilde{w}^\iota\|^2\|\rho\|^2\cdot\|\hat{w}^\iota\|^2+\frac{\sigma_1}{2M_1^2}\|\rho\|^2\\
&\le \frac{2M_1^4\gamma^4}{\sigma_1}\left(\|\tilde{\rho}\|^2\|\rho\|^2+\|\tilde{\iota}\|^2\|\rho\|^2\right)\left(\|\hat{w}^\rho\|^2+\|\hat{w}^\iota\|^2\right)+2\sigma_1\left(\|\hat{w}^\rho\|^2+\|\hat{w}^\iota\|^2\right)+\frac{\sigma_1}{M_1^2}\|\tilde{\rho}\|^2\\
&\le 2\sigma_1\left(\|\hat{w}^\rho\|^2+\|\hat{w}^\iota\|^2\right)+l_1\left(\|\hat{w}^\rho\|^2+\|\hat{w}^\iota\|^2\right)+l_1. \qquad \text{(D.5)}
\end{aligned}
$$

Similarly, the following inequality holds

$$
\gamma^2\|\iota\|^2\int_0^1\left(\hat{w}^\rho\tilde{w}^\rho+\hat{w}^\iota\tilde{w}^\iota\right)\mathrm{d}x \le 2\sigma_1\left(\|\hat{w}^\rho\|^2+\|\hat{w}^\iota\|^2\right)+l_1\left(\|\hat{w}^\rho\|^2+\|\hat{w}^\iota\|^2\right)+l_1. \qquad \text{(D.6)}
$$

Moreover, by Hölder's inequality, we have

$$\begin{aligned}\int_0^1 \hat{w}^\rho \int_0^x \left(\hat{w}^\rho(y)H(x,y)+\hat{w}^\iota(y)H^c(x,y)\right)\mathrm{d}y\mathrm{d}x \\ &\le \|\hat{w}^\rho\|^2\sqrt{\int_0^1\int_0^x H(x,y)^2\mathrm{d}y\mathrm{d}x}+\|\hat{w}^\rho\|\cdot\|\hat{w}^\iota\|\sqrt{\int_0^1\int_0^x H^c(x,y)^2\mathrm{d}y\mathrm{d}x} \\ &\le \left(\|\hat{w}^\rho\|^2+\|\hat{w}^\iota\|^2\right)\left(\sqrt{\int_0^1\int_0^x H(x,y)^2\mathrm{d}y\mathrm{d}x}+\sqrt{\int_0^1\int_0^x H^c(x,y)^2\mathrm{d}y\mathrm{d}x}\right).\end{aligned}$$

By (2.1), there holds $\hat{k}_t=\frac{\partial\hat{k}}{\partial\hat{b}_R}\dot{\hat{b}}_R+\frac{\partial\hat{k}}{\partial\hat{b}_I}\dot{\hat{b}}_I$, $\hat{k}^c_t=\frac{\partial k^c}{\partial\hat{b}_R}\dot{\hat{b}}_R+\frac{\partial k^c}{\partial\hat{b}_I}\dot{\hat{b}}_I$. Since $\hat{k}(\cdot)$, $\hat{k}^c(\cdot)$ are continuously differentiable with respect to each variable on bounded intervals and $\dot{\hat{b}}_I\in\mathcal{L}_2$, $\dot{\hat{b}}_I\in\mathcal{L}_2$, we obtain that $\int_0^1\int_0^x H(x,y)^2\mathrm{d}y\mathrm{d}x$, $\int_0^1\int_0^x H^c(x,y)^2\mathrm{d}y\mathrm{d}x\in\mathcal{L}_2$. Then, it follows from the above inequality that

$$\int_0^1 \hat{w}^\rho \int_0^x \left(\hat{w}^\rho(y)H(x,y)+\hat{w}^\iota(y)H^c(x,y)\right)\mathrm{d}y\mathrm{d}x \le l_1\left(\|\hat{w}^\rho\|^2+\|\hat{w}^\iota\|^2\right). \tag{D.7}$$

Similarly, there holds

$$-\int_0^1 \hat{w}^\iota \int_0^x \left(\hat{w}^\rho(y)H^c(x,y)-\hat{w}^\iota(y)H(x,y)\right)\mathrm{d}y\mathrm{d}x \le l_1\left(\|\hat{w}^\rho\|^2+\|\hat{w}^\iota\|^2\right). \tag{D.8}$$

Substituting (D.3)–(D.8) into (D.2) leads to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\hat{w}^\rho\|^2+\|\hat{w}^\iota\|^2\right)\le-(c-6\sigma_1)\left(\|\hat{w}^\rho\|^2+\|\hat{w}^\iota\|^2\right)+l_1\left(\|\hat{w}^\rho\|^2+\|\hat{w}^\iota\|^2\right)+l_1. \tag{D.9}$$

Then, by Lemma H.3, we obtain $\|\hat{w}^\rho\|$, $\|\hat{w}^\iota\|\in\mathcal{L}_\infty\cap\mathcal{L}_2$ if $0<\sigma_1<\frac{c}{6}$.

In the following, we turn to showing the last line of (3.11). First, by computing the time derivative and using integration by parts, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\tilde{\rho}_x\|^2+\|\tilde{\iota}_x\|^2\right)&=\int_0^1\tilde{\rho}_x\tilde{\rho}_{xt}\mathrm{d}x+\int_0^1\tilde{\iota}_x\tilde{\iota}_{xt}\mathrm{d}x \\ &=-\int_0^1\tilde{\rho}_{xx}\left(a_R\tilde{\rho}_{xx}+\tilde{b}_R\rho-a_I\tilde{\iota}_{xx}-\tilde{b}_I\iota-\gamma^2\tilde{\rho}\|\rho\|^2-\gamma^2\tilde{\rho}\|\iota\|^2\right)\mathrm{d}x \\ &\quad-\int_0^1\tilde{\iota}_{xx}\left(a_I\tilde{\rho}_{xx}+\tilde{b}_I\rho+a_R\tilde{\iota}_{xx}+\tilde{b}_R\iota-\gamma^2\tilde{\iota}\|\iota\|^2-\gamma^2\tilde{\iota}\|\rho\|^2\right)\mathrm{d}x \\ &=-a_R\int_0^1\left(\tilde{\rho}_{xx}^2+\tilde{\iota}_{xx}^2\right)\mathrm{d}x-\tilde{b}_R\int_0^1\left(\tilde{\rho}_{xx}\rho+\tilde{\iota}_{xx}\iota\right)\mathrm{d}x+\tilde{b}_I\int_0^1\left(\tilde{\rho}_{xx}\iota-\tilde{\iota}_{xx}\rho\right)\mathrm{d}x \\ &\quad+\left(\gamma^2\|\rho\|^2+\gamma^2\|\iota\|^2\right)\int_0^1\left(\tilde{\rho}_{xx}\rho+\tilde{\iota}_{xx}\iota\right)\mathrm{d}x \\ &=-a_R\int_0^1\left(\tilde{\rho}_{xx}^2+\tilde{\iota}_{xx}^2\right)\mathrm{d}x+\int_0^1\tilde{\rho}_{xx}\left(-\tilde{b}_R\rho+\tilde{b}_I\iota+\gamma^2\left(\|\rho\|^2+\|\iota\|^2\right)\rho\right)\mathrm{d}x \\ &\quad-\int_0^1\tilde{\iota}_{xx}\left(\tilde{b}_R\iota+\tilde{b}_I\rho-\gamma^2\left(\|\rho\|^2+\|\iota\|^2\right)\iota\right)\mathrm{d}x.\end{aligned}$$

Then, by Young's inequality and noting that $\tilde{b}_R$, $\tilde{b}_I \in \mathcal{L}_\infty$ and $\|\rho\|$, $\|\iota\| \in \mathcal{L}_\infty \cap \mathcal{L}_2$, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\tilde{\rho}_x\|^2+\|\tilde{\iota}_x\|^2\right) \leq{}& \frac{3}{4a_R}\left(\tilde{b}_R^2\|\rho\|^2+\tilde{b}_I^2\|\iota\|^2+\gamma^2\left(\|\rho\|^2+\|\iota\|^2\right)^2\|\rho\|^2\right)\\ &+\frac{3}{4a_R}\left(\tilde{b}_R^2\|\iota\|^2+\tilde{b}_I^2\|\rho\|^2+\gamma^2\left(\|\rho\|^2+\|\iota\|^2\right)^2\|\iota\|^2\right)\in\mathcal{L}_2. \end{aligned}\tag{D.10}$$

Integrating both sides of the above inequality over $[0, +\infty)$ leads to $\|\tilde{\rho}_x\|$, $\|\tilde{\iota}_x\| \in \mathcal{L}_\infty$.

Similar to the derivation of (D.2), we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\hat{w}^\rho_x\|^2+\|\hat{w}^\iota_x\|^2\right) ={}& \int_0^1\left(\hat{w}^\rho_x\hat{w}_{\rho,xt}+\hat{w}^\iota_x\hat{w}_{\iota,xt}\right)\mathrm{d}x\\ ={}& -\int_0^1\hat{w}^\rho_{xx}\bigg(a_R\hat{w}^\rho_{xx}-a_I\hat{w}^\iota_{xx}-c\hat{w}^\rho+\left(\hat{b}_R+\gamma^2\|\rho\|^2+\gamma^2\|\iota\|^2\right)\tilde{w}^\rho-\hat{b}_I\tilde{w}^\iota\\ &+\int_0^x\left(\hat{w}^\rho(y)H(x,y)+\hat{w}^\iota(y)H^c(x,y)\right)\mathrm{d}y\bigg)\mathrm{d}x\\ &+\int_0^1\hat{w}^\iota_{xx}\bigg(a_I\hat{w}^\rho_{xx}+a_R\hat{w}^\iota_{xx}-c\hat{w}^\iota+\left(\hat{b}_R+\gamma^2\|\rho\|^2+\gamma^2\|\iota\|^2\right)\tilde{w}^\iota+\hat{b}_I\tilde{w}^\rho\\ &-\int_0^x\left(\hat{w}^\rho(y)H^c(x,y)-\hat{w}^\iota(y)H(x,y)\right)\mathrm{d}y\bigg)\mathrm{d}x\\ ={}& -a_R\left(\|\hat{w}^\rho_{xx}\|^2+\|\hat{w}^\iota_{xx}\|^2\right)\mathrm{d}x-c\left(\|\hat{w}^\rho_x\|^2+\|\hat{w}^\iota_x\|^2\right)+\hat{b}_I\int_0^1\left(\hat{w}^\rho_{xx}\tilde{w}^\iota-\hat{w}^\iota_{xx}\tilde{w}^\rho\right)\mathrm{d}x\\ &-\left(\hat{b}_R+\gamma^2\|\rho\|^2+\gamma^2\|\iota\|^2\right)\int_0^1\left(\hat{w}^\rho_{xx}\tilde{w}^\rho+\hat{w}^\iota_{xx}\tilde{w}^\iota\right)\mathrm{d}x\\ &+\int_0^1\hat{w}^\rho_{xx}\int_0^x\left(\hat{w}^\rho(y)H(x,y)+\hat{w}^\iota(y)H^c(x,y)\right)\mathrm{d}y\mathrm{d}x\\ &-\int_0^1\hat{w}^\iota_{xx}\int_0^x\left(\hat{w}^\rho(y)H^c(x,y)-\hat{w}^\iota(y)H(x,y)\right)\mathrm{d}y\mathrm{d}x. \end{aligned}\tag{D.11}$$

By Young's inequality, we obtain that

$$\begin{cases}\hat{b}_I\int_0^1\left(\hat{w}^\rho_{xx}\tilde{w}^\iota-\hat{w}^\iota_{xx}\tilde{w}^\rho\right)\mathrm{d}x\leq\frac{a_R}{4}\left(\|\hat{w}^\rho_{xx}\|^2+\|\hat{w}^\iota_{xx}\|^2\right)+\frac{\bar{b}_I^2}{a_R}\left(\|\tilde{w}^\rho\|^2+\|\tilde{w}^\iota\|^2\right),\\ -\hat{b}_R\int_0^1\left(\hat{w}^\rho_{xx}\tilde{w}^\rho+\hat{w}^\iota_{xx}\tilde{w}^\iota\right)\mathrm{d}x\leq\frac{a_R}{4}\left(\|\hat{w}^\rho_{xx}\|^2+\|\hat{w}^\iota_{xx}\|^2\right)+\frac{\bar{b}_R^2}{a_R}\left(\|\tilde{w}^\rho\|^2+\|\tilde{w}^\iota\|^2\right),\\ \gamma^2\left(\|\rho\|^2+\|\iota\|^2\right)\int_0^1\left(\hat{w}^\rho_{xx}\tilde{w}^\rho+\hat{w}^\iota_{xx}\tilde{w}^\iota\right)\mathrm{d}x\\ \qquad\leq\frac{a_R}{4}\left(\|\hat{w}^\rho_{xx}\|^2+\|\hat{w}^\iota_{xx}\|^2\right)+\frac{\gamma^4}{a_R}\left(\|\rho\|^2+\|\iota\|^2\right)^2\left(\|\tilde{w}^\rho\|^2+\|\tilde{w}^\iota\|^2\right),\\ \int_0^1\hat{w}^\rho_{xx}\int_0^x\left(\hat{w}^\rho(y)H(x,y)+\hat{w}^\iota(y)H^c(x,y)\right)\mathrm{d}y\mathrm{d}x\\ \qquad\leq\frac{a_R}{4}\|\hat{w}^\rho_{xx}\|^2+\frac{2}{a_R}\left(\|\hat{w}^\rho\|^2\int_0^1\int_0^x H(x,y)^2\mathrm{d}y\mathrm{d}x+\|\hat{w}^\iota\|^2\int_0^1\int_0^x H^c(x,y)^2\mathrm{d}y\mathrm{d}x\right),\\ -\int_0^1\hat{w}^\iota_{xx}\int_0^x\left(\hat{w}^\rho(y)H^c(x,y)-\hat{w}^\iota(y)H(x,y)\right)\mathrm{d}y\mathrm{d}x\\ \qquad\leq\frac{a_R}{4}\|\hat{w}^\iota_{xx}\|^2+\frac{2}{a_R}\left(\|\hat{w}^\rho\|^2\int_0^1\int_0^x H(x,y)^2\mathrm{d}y\mathrm{d}x+\|\hat{w}^\iota\|^2\int_0^1\int_0^x H^c(x,y)^2\mathrm{d}y\mathrm{d}x\right).\end{cases}$$

Substituting the above five inequalities into the right hand side of (D.11), we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\hat{w}^\rho_x\|^2+\|\hat{w}^\iota_x\|^2\right)\leq{}&\frac{1}{a_R}\left(\bar{b}_I^2+\bar{b}_R^2+\left(\gamma^2\|\rho\|^2+\gamma^2\|\iota\|^2\right)^2\right)\left(\|\tilde{w}^\rho\|^2+\|\tilde{w}^\iota\|^2\right)\\ &+\frac{4}{a_R}\left(\|\hat{w}^\rho\|^2\int_0^1\int_0^x H(x,y)^2\mathrm{d}y\mathrm{d}x+\|\hat{w}^\iota\|^2\int_0^1\int_0^x H^c(x,y)^2\mathrm{d}y\mathrm{d}x\right)\leq l_1,\end{aligned}$$

which can be obtained by noting that $\|\hat{w}^\rho\|$, $\|\hat{w}^\iota\|$, $\|\tilde{w}^\rho\|$, $\|\tilde{w}^\iota\| \in \mathcal{L}_\infty \cap \mathcal{L}_2$ and $\int_0^1\int_0^x H(x,y)^2 \mathrm{d}y\mathrm{d}x$, $\int_0^1\int_0^x H^c(x,y)^2\mathrm{d}y\mathrm{d}x \in \mathcal{L}_2$. Integrating both side of the above inequality over $[0,\, t]$ arrives at $\|\hat{w}_x^\rho\|$, $\|\hat{w}_x^\iota\| \in \mathcal{L}_\infty$. Then, (3.4) implies that $\|\hat{\rho}_x\|$, $\|\hat{\iota}_x\| \in \mathcal{L}_\infty$, and hence $\|\rho_x\|$, $\|\iota_x\| \in \mathcal{L}_\infty$.

Appendix E. Proof of Proposition 4.1

We just show the derivation of the first equation of (4.7) since the second one can be obtained similarly, and moreover the last three lines of (4.7) can be obtained similarly to those of (3.6).

First, by the similar derivation of (B.4), we have

$$\begin{aligned}
\hat{w}_t^\rho &= \hat{\rho}_t - \int_0^x \big(\hat{k}(x,y)\hat{\rho}_t(y) + \hat{k}^c(x,y)\hat{\iota}_t(y)\big)\mathrm{d}y - \int_0^x \big(\hat{k}_t(x,y)\hat{\rho}(y) + \hat{k}_t^c(x,y)\hat{\iota}(y)\big)\mathrm{d}y \\
&= a_{\scriptscriptstyle R}\hat{\rho}_{xx} + \hat{b}_{\scriptscriptstyle R}\rho - a_{\scriptscriptstyle I}\hat{\iota}_{xx} - \hat{b}_{\scriptscriptstyle I}\iota + \dot{\hat{b}}_R v^\rho - \dot{\hat{b}}_I v^\iota \\
&\quad - \int_0^x \hat{k}(x,y)\left(a_{\scriptscriptstyle R}\hat{\rho}_{yy} + \hat{b}_{\scriptscriptstyle R}\rho - a_{\scriptscriptstyle I}\hat{\iota}_{yy} - \hat{b}_{\scriptscriptstyle I}\iota + \dot{\hat{b}}_R v^\rho - \dot{\hat{b}}_I v^\iota\right)\mathrm{d}y \\
&\quad - \int_0^x \hat{k}^c(x,y)\left(a_{\scriptscriptstyle I}\hat{\rho}_{yy} + \hat{b}_{\scriptscriptstyle I}\rho + a_{\scriptscriptstyle R}\hat{\iota}_{yy} + \hat{b}_{\scriptscriptstyle R}\iota + \dot{\hat{b}}_I v^\rho + \dot{\hat{b}}_R v^\iota\right)\mathrm{d}y \\
&\quad - \int_0^x \big(\hat{k}_t(x,y)\hat{\rho}(y) + \hat{k}_t^c(x,y)\hat{\iota}(y)\big)\mathrm{d}y \\
&= a_{\scriptscriptstyle R}\left(\hat{\rho}_{xx} - \int_0^x \left(\hat{k}(x,y)\hat{\rho}_{yy}(y) + \hat{k}^c(x,y)\hat{\iota}_{yy}(y)\right)\mathrm{d}y\right) \\
&\quad - a_{\scriptscriptstyle I}\left(\hat{\iota}_{xx} - \int_0^x \left(-\hat{k}^c(x,y)\hat{\rho}_{yy}(y) + \hat{k}(x,y)\hat{\iota}_{yy}(y)\right)\mathrm{d}y\right) \\
&\quad + \hat{b}_{\scriptscriptstyle R}\left(\rho - \int_0^x \left(\hat{k}(x,y)\rho(y) + \hat{k}^c(x,y)\iota(y)\right)\mathrm{d}y\right) \\
&\quad - \hat{b}_{\scriptscriptstyle I}\left(\iota - \int_0^x \left(-\hat{k}^c(x,y)\rho(y) + \hat{k}(x,y)\iota(y)\right)\mathrm{d}y\right) \\
&\quad + \dot{\hat{b}}_R\left(v^\rho - \int_0^x \left(\hat{k}(x,y)v^\rho(y) + \hat{k}^c(x,y)v^\iota(y)\right)\mathrm{d}y\right) \\
&\quad - \dot{\hat{b}}_I\left(v^\iota - \int_0^x \left(-\hat{k}^c(x,y)v^\rho(y) + \hat{k}(x,y)v^\iota(y)\right)\mathrm{d}y\right) \\
&\quad - \int_0^x \big(\hat{k}_t(x,y)\hat{\rho}(y) + \hat{k}_t^c(x,y)\hat{\iota}(y)\big)\mathrm{d}y \\
&= a_{\scriptscriptstyle R}\left(\hat{\rho}_{xx} - \int_0^x \left(\hat{k}(x,y)\hat{\rho}_{yy}(y) + \hat{k}^c(x,y)\hat{\iota}_{yy}(y)\right)\mathrm{d}y\right) \\
&\quad - a_{\scriptscriptstyle I}\left(\hat{\iota}_{xx} - \int_0^x \left(-\hat{k}^c(x,y)\hat{\rho}_{yy}(y) + \hat{k}(x,y)\hat{\iota}_{yy}(y)\right)\mathrm{d}y\right) \\
&\quad + \hat{b}_{\scriptscriptstyle R} w_\rho - \hat{b}_{\scriptscriptstyle I} w_\iota + \dot{\hat{b}}_R \breve{v}^\rho - \dot{\hat{b}}_I \breve{v}^\iota - \int_0^x \big(\hat{k}_t(x,y)\hat{\rho}(y) + \hat{k}_t^c(x,y)\hat{\iota}(y)\big)\mathrm{d}y.
\end{aligned}$$

Substituting (B.2) and (B.3) into the first two terms of the above equation and after some simple managements, we arrive at

$$\begin{aligned}
\hat{w}_t^\rho = a_{\scriptscriptstyle R}\hat{w}_{xx}^\rho - a_{\scriptscriptstyle I}\hat{w}_{xx}^\iota - c\hat{w}^\rho - \hat{b}_{\scriptscriptstyle I}\tilde{w}^\iota + \dot{\hat{b}}_R\breve{v}^\rho - \dot{\hat{b}}_I\breve{v}^\iota \\
- \int_0^x \big(\hat{k}_t(x,y)\hat{\rho}(y) + \hat{k}_t^c(x,y)\hat{\iota}(y)\big)\mathrm{d}y.
\end{aligned}$$

Then, substituting (3.3) into the last term on the right-hand side of the above equation and then changing the order of integration, we obtain the first equation of (4.7).

Appendix F. Proof of Proposition 4.2

Choose the following Lyapunov function:

$$V(t) = \frac{1}{2}\left(\|\epsilon^\rho\|^2 + \|\epsilon^\iota\|^2\right) + \frac{\tilde{b}_R^2}{2\sigma_2\gamma} + \frac{\tilde{b}_I^2}{2\sigma_2\gamma},$$

where σ_2 is a positive constant to be determined later. Computing the time derivative of $V(t)$ along the solutions of (4.4) and using (4.10), we have

$$\begin{aligned}
\dot{V}(t) &= \int_0^1 \epsilon^\rho \left(a_R\epsilon^\rho_{xx} - a_I\epsilon^\iota_{xx}\right) \mathrm{d}x + \int_0^1 \epsilon^\iota \left(a_I\epsilon^\rho_{xx} + a_R\epsilon^\iota_{xx}\right) \mathrm{d}x - \frac{1}{\sigma_2\gamma}\left(\tilde{b}_R\dot{\hat{b}}_R + \tilde{b}_I\dot{\hat{b}}_I\right)\\
&= -a_R\left(\|\epsilon^\rho_x\|^2 + \|\epsilon^\iota_x\|^2\right) - \frac{\int_0^1\left(\tilde{b}_R\left(\tilde{\rho}v^\rho + \tilde{\iota}v^\iota\right) + \tilde{b}_I\left(\tilde{\iota}v^\rho - \tilde{\rho}v^\iota\right)\right)\mathrm{d}x}{\sigma_2\left(1 + \|v^\rho\|^2 + \|v^\iota\|^2\right)}\\
&= -a_R\left(\|\epsilon^\rho_x\|^2 + \|\epsilon^\iota_x\|^2\right) - \frac{\int_0^1\left(\tilde{\rho}\left(\tilde{b}_Rv^\rho - \tilde{b}_Iv^\iota\right) + \tilde{\iota}\left(\tilde{b}_Iv^\rho + \tilde{b}_Rv^\iota\right)\right)\mathrm{d}x}{\sigma_2\left(1 + \|v^\rho\|^2 + \|v^\iota\|^2\right)}.
\end{aligned}$$

Noting that $\epsilon^\rho - \tilde{\rho} = \tilde{b}_Iv^\iota - \tilde{b}_Rv^\rho$ and $\epsilon^\iota - \tilde{\iota} = -\tilde{b}_Iv^\rho - \tilde{b}_Rv^\iota$, it follows from the above equality that

$$\begin{aligned}
\dot{V}(t) &= -a_R\left(\|\epsilon^\rho_x\|^2 + \|\epsilon^\iota_x\|^2\right) + \frac{\int_0^1\left(\tilde{\rho}\left(\epsilon^\rho - \tilde{\rho}\right) + \tilde{\iota}\left(\epsilon^\iota - \tilde{\iota}\right)\right)\mathrm{d}x}{\sigma_2\left(1 + \|v^\rho\|^2 + \|v^\iota\|^2\right)}\\
&= -a_R\left(\|\epsilon^\rho_x\|^2 + \|\epsilon^\iota_x\|^2\right) - \frac{\|\tilde{\rho}\|^2 + \|\tilde{\iota}\|^2}{\sigma_2\left(1 + \|v^\rho\|^2 + \|v^\iota\|^2\right)} + \frac{\int_0^1\left(\tilde{\rho}\epsilon^\rho + \tilde{\iota}\epsilon^\iota\right)\mathrm{d}x}{\sigma_2\left(1 + \|v^\rho\|^2 + \|v^\iota\|^2\right)}\\
&\le -a_R\left(\|\epsilon^\rho_x\|^2 + \|\epsilon^\iota_x\|^2\right) - \frac{\|\tilde{\rho}\|^2 + \|\tilde{\iota}\|^2}{\sigma_2\left(1 + \|v^\rho\|^2 + \|v^\iota\|^2\right)} + \frac{\|\tilde{\rho}\|\|\epsilon^\rho\| + \|\tilde{\iota}\|\|\epsilon^\iota\|}{\sigma_2\left(1 + \|v^\rho\|^2 + \|v^\iota\|^2\right)}.
\end{aligned} \tag{F.1}$$

Moreover, by Poincare's inequality, Agmon's inequality and then Young's inequality, we have

$$\begin{aligned}
\frac{\|\tilde{\rho}\|\|\epsilon^\rho\| + \|\tilde{\iota}\|\|\epsilon^\iota\|}{\sigma_2\left(1 + \|v^\rho\|^2 + \|v^\iota\|^2\right)} &\le \frac{2\|\tilde{\rho}\|\|\epsilon^\rho_x\| + 2\|\tilde{\iota}\|\|\epsilon^\iota_x\|}{\sigma_2\sqrt{1 + \|v^\rho\|^2 + \|v^\iota\|^2}}\\
&\le \frac{a_R}{2}\left(\|\epsilon^\rho_x\|^2 + \|\epsilon^\iota_x\|^2\right) + \frac{2\left(\|\tilde{\rho}\|^2 + \|\tilde{\iota}\|^2\right)}{\sigma_2^2a_R\left(1 + \|v^\rho\|^2 + \|v^\iota\|^2\right)}.
\end{aligned}$$

Substituting the above inequality into (F.1), we have

$$\dot{V}(t) \le -\frac{a_R}{2}\left(\|\epsilon^\rho_x\|^2 + \|\epsilon^\iota_x\|^2\right) - \frac{\|\tilde{\rho}\|^2 + \|\tilde{\iota}\|^2}{\sigma_2\left(1 + \|v^\rho\|^2 + \|v^\iota\|^2\right)}\left(1 - \frac{2}{\sigma_2a_R}\right).$$

Choosing $\sigma_2 > \frac{2}{a_R}$ and integrating the above inequality over $[0, t]$ and $[0, +\infty)$, respectively, we obtain that $\|\epsilon^\rho\|$, $\|\epsilon^\iota\|$, $\tilde{b}_R$, $\tilde{b}_I \in \mathcal{L}_\infty$, and $\|\epsilon^\rho_x\|$, $\|\epsilon^\iota_x\|$, $\frac{\|\tilde{\rho}\|}{\sqrt{1+\|v^\rho\|^2+\|v^\iota\|^2}}$, $\frac{\|\tilde{\iota}\|}{\sqrt{1+\|v^\rho\|^2+\|v^\iota\|^2}} \in \mathcal{L}_2$.

On one hand, by Poincare's inequality and noting that $\|\epsilon_x^\rho\|$, $\|\epsilon_x^\iota\| \in \mathcal{L}_2$, we have $\|\epsilon^\rho\|$, $\|\epsilon^\iota\| \in \mathcal{L}_2$. On the other hand, replacing $\tilde{\rho}$ by $\epsilon^\rho - \tilde{b}_I v^\iota + \tilde{b}_R v^\rho$ and noting the boundedness of $\|\epsilon^\rho\|$, $\tilde{b}_R$ and $\tilde{b}_I$, we have

$$\frac{\|\tilde{\rho}\|}{\sqrt{1+\|v^\rho\|^2+\|v^\iota\|^2}} \leq \frac{\|\epsilon^\rho\| + |\tilde{b}_R|\cdot\|v^\rho\| + |\tilde{b}_I|\cdot\|v^\iota\|}{\sqrt{1+\|v^\rho\|^2+\|v^\iota\|^2}} \leq \|\epsilon^\rho\| + |\tilde{b}_R| + |\tilde{b}_I| \in \mathcal{L}_\infty.$$

Similarly, there holds $\frac{\|\tilde{\iota}\|}{\sqrt{1+\|v^\rho\|^2+\|v^\iota\|^2}} \in \mathcal{L}_\infty$. Then, from (4.10), we arrive at

$$\left|\dot{\hat{b}}_R\right| \leq \frac{\gamma\|\tilde{\rho}\|\|v^\rho\| + \gamma\|\tilde{\iota}\|\|v^\iota\|}{1+\|v^\rho\|^2+\|v^\iota\|^2} \leq \frac{\gamma\|\tilde{\rho}\|}{\sqrt{1+\|v^\rho\|^2+\|v^\iota\|^2}} + \frac{\gamma\|\tilde{\iota}\|}{\sqrt{1+\|v^\rho\|^2+\|v^\iota\|^2}} \in \mathcal{L}_2 \cap \mathcal{L}_\infty,$$

and $\left|\dot{\hat{b}}_I\right| \in \mathcal{L}_2 \cap \mathcal{L}_\infty$. Then, the first two lines of (4.11) hold.

To derive the last line of (4.11), we consider

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\epsilon_x^\rho\|^2 + \|\epsilon_x^\iota\|^2\right) = -a_R\left(\|\epsilon_{xx}^\rho\|^2 + \|\epsilon_{xx}^\iota\|^2\right) \leq 0.$$

By integrating both sides of the above inequality over $[0, t]$, we obtain that $\|\epsilon_x^\rho\|$, $\|\epsilon_x^\iota\| \in \mathcal{L}_\infty$, and hence $\|\epsilon_x^\rho\|$, $\|\epsilon_x^\iota\| \in \mathcal{L}_2 \cap \mathcal{L}_\infty$. Then, we obtain that $\|\epsilon^\rho\|$, $\|\epsilon^\iota\| \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ by Poincare's inequality.

Appendix G. Proof of Proposition 4.3

Part I: Proof of (4.12):

Consider the following computation

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\hat{w}^\rho\|^2 + \|\hat{w}^\iota\|^2\right)\\
&\quad= \int_0^1 \hat{w}^\rho\bigg(a_R\hat{w}_{xx}^\rho - a_I\hat{w}_{xx}^\iota - c\hat{w}^\rho + \hat{b}_R\tilde{w}^\rho - \hat{b}_I\tilde{w}^\iota\\
&\qquad+\dot{\hat{b}}_R\breve{v}^\rho - \dot{\hat{b}}_I\breve{v}^\iota + \int_0^x \left(\hat{w}^\rho(y)H(x,y) + \hat{w}^\iota(y)H^c(x,y)\right)\mathrm{d}y\bigg)\mathrm{d}x\\
&\qquad+ \int_0^1 \hat{w}^\iota\bigg(a_I\hat{w}_{xx}^\rho + a_R\hat{w}_{xx}^\iota - c\hat{w}^\iota + \hat{b}_R\tilde{w}^\iota + \hat{b}_I\tilde{w}^\rho\\
&\qquad+\dot{\hat{b}}_I\breve{v}^\rho + \dot{\hat{b}}_R\breve{v}^\iota - \int_0^x \left(\hat{w}^\rho(y)H^c(x,y) - \hat{w}^\iota(y)H(x,y)\right)\mathrm{d}y\bigg)\mathrm{d}x\\
&\quad= -a_R\left(\|\hat{w}_x^\rho\|^2 + \|\hat{w}_x^\iota\|^2\right) - c\left(\|\hat{w}^\rho\|^2 + \|\hat{w}^\iota\|^2\right)\\
&\qquad+\hat{b}_R\int_0^1 \left(\hat{w}^\rho\tilde{w}^\rho + \hat{w}^\iota\tilde{w}^\iota\right)\mathrm{d}x + \hat{b}_I\int_0^1 \left(\hat{w}^\iota\tilde{w}^\rho - \hat{w}^\rho\tilde{w}^\iota\right)\mathrm{d}x\\
&\qquad+\dot{\hat{b}}_R\int_0^1 \left(\hat{w}^\rho\breve{v}^\rho + \hat{w}^\iota\breve{v}^\iota\right)\mathrm{d}x + \dot{\hat{b}}_I\int_0^1 \left(\hat{w}^\iota\breve{v}^\rho + \hat{w}^\rho\breve{v}^\iota\right)\mathrm{d}x\\
&\qquad+\int_0^1 \hat{w}^\rho\int_0^x \left(\hat{w}^\rho(y)H(x,y) + \hat{w}^\iota(y)H^c(x,y)\right)\mathrm{d}y\mathrm{d}x\\
&\qquad-\int_0^1 \hat{w}^\iota\int_0^x \left(\hat{w}^\rho(y)H^c(x,y) - \hat{w}^\iota(y)H(x,y)\right)\mathrm{d}y\mathrm{d}x.
\end{aligned} \tag{G.1}$$

From (4.8), we obtain that $\|\breve{v}^\rho\| \le M_1\left(\|v^\rho\|+\|v^\iota\|\right)$ and $\|\breve{v}^\iota\| \le M_1\left(\|v^\rho\|+\|v^\iota\|\right)$, by which, (D.1) and using Young's inequality, we have

$$\begin{cases} \hat{b}_R\int_0^1\left(\hat{w}^\rho\tilde{w}^\rho+\hat{w}^\iota\tilde{w}^\iota\right)\mathrm{d}x \le 2\sigma_3\left(\|\hat{w}^\rho\|^2+\|\hat{w}^\iota\|^2\right)+\frac{\bar{b}_R^2M_1^2}{2\sigma_3}\left(\|\tilde{\rho}\|^2+\|\tilde{\iota}\|^2\right),\\ \hat{b}_I\int_0^1\left(\hat{w}^\iota\tilde{w}^\rho-\hat{w}^\rho\tilde{w}^\iota\right)\mathrm{d}x \le 2\sigma_3\left(\|\hat{w}^\rho\|^2+\|\hat{w}^\iota\|^2\right)+\frac{\bar{b}_I^2M_1^2}{2\sigma_3}\left(\|\tilde{\rho}\|^2+\|\tilde{\iota}\|^2\right),\\ \dot{\hat{b}}_R\int_0^1\left(\hat{w}^\iota\breve{v}^\rho+\hat{w}^\rho\breve{v}^\iota\right)\mathrm{d}x \le \sigma_3\left(\|\hat{w}^\rho\|^2+\|\hat{w}^\iota\|^2\right)+\frac{M_1^2|\dot{\hat{b}}_R|^2}{2\sigma_3}\left(\|v^\rho\|^2+\|v^\iota\|^2\right),\\ \dot{\hat{b}}_I\int_0^1\left(\hat{w}^\iota\breve{v}^\rho+\hat{w}^\rho\breve{v}^\iota\right)\mathrm{d}x \le \sigma_3\left(\|\hat{w}^\rho\|^2+\|\hat{w}^\iota\|^2\right)+\frac{M_1^2|\dot{\hat{b}}_I|^2}{2\sigma_3}\left(\|v^\rho\|^2+\|v^\iota\|^2\right). \end{cases}$$

Substituting this and (D.7), (D.8) into (G.1) while noting that $|\dot{\hat{b}}_R|$, $|\dot{\hat{b}}_I| \in \mathcal{L}_2$, leads to

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\hat{w}^\rho\|^2+\|\hat{w}^\iota\|^2\right) \le{}& -a_R\left(\|\hat{w}_x^\rho\|^2+\|\hat{w}_x^\iota\|^2\right)-(c-6\sigma_3)\left(\|\hat{w}^\rho\|^2+\|\hat{w}^\iota\|^2\right)\\ &+\frac{\left(\bar{b}_R^2+\bar{b}_I^2\right)M_2^2}{2\sigma_3}\left(\|\tilde{\rho}\|^2+\|\tilde{\iota}\|^2\right)+l_1\left(\|v^\rho\|^2+\|v^\iota\|^2\right)+l_1\left(\|\hat{w}^\rho\|^2+\|\hat{w}^\iota\|^2\right). \end{aligned} \tag{G.2}$$

Moreover, $\frac{\|\tilde{\rho}\|}{\sqrt{1+\|v^\rho\|^2+\|v^\iota\|^2}}$, $\frac{\|\tilde{\iota}\|}{\sqrt{1+\|v^\rho\|^2+\|v^\iota\|^2}} \in \mathcal{L}_2$ (see (4.11)) implies that

$$\|\tilde{\rho}\|^2 \le l_1+l_1\left(\|v^\rho\|^2+\|v^\iota\|^2\right),\quad \|\tilde{\iota}\|^2 \le l_1+l_1\left(\|v^\rho\|^2+\|v^\iota\|^2\right). \tag{G.3}$$

Then, (G.2) changes into

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\hat{w}^\rho\|^2+\|\hat{w}^\iota\|^2\right) \le{}& -a_R\left(\|\hat{w}_x^\rho\|^2+\|\hat{w}_x^\iota\|^2\right)-(c-6\sigma_3)\left(\|\hat{w}^\rho\|^2+\|\hat{w}^\iota\|^2\right)\\ &+l_1\left(\|v^\rho\|^2+\|v^\iota\|^2\right)+l_1\left(\|\hat{w}^\rho\|^2+\|\hat{w}^\iota\|^2\right)+l_1. \end{aligned} \tag{G.4}$$

Moreover, from (4.1), we have

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|v^\rho\|^2+\|v^\iota\|^2\right) &= \int_0^1 v^\rho\left(a_Rv_{xx}^\rho-a_Iv_{xx}^\iota+\rho\right)\mathrm{d}x+\int_0^1 v^\rho\left(a_Iv_{xx}^\rho+a_Rv_{xx}^\iota+\iota\right)\mathrm{d}x\\ &= -a_R\left(\|v_x^\rho\|^2+\|v_x^\iota\|^2\right)+\int_0^1 v^\rho\rho\mathrm{d}x+\int_0^1 v^\iota\iota\mathrm{d}x. \end{aligned} \tag{G.5}$$

By (D.1) and Young's inequality, we have

$$\begin{aligned} \int_0^1 v^\rho\rho\mathrm{d}x &\le \|v^\rho\|\|\rho\| \le \|v^\rho\|\|\tilde{\rho}\|+M_1\|v^\rho\|\left(\|\hat{w}^\rho\|+\|\hat{w}^\iota\|\right)\\ &\le \frac{a_R}{8}\|v^\rho\|^2+\frac{4}{a_R}\|\tilde{\rho}\|^2+\frac{8M_1^2}{a_R}\left(\|\hat{w}^\rho\|^2+\|\hat{w}^\iota\|^2\right). \end{aligned}$$

Then, by Poincaré's inequality and (G.3), it follows that

$$\int_0^1 v^\rho\rho\mathrm{d}x \le \frac{a_R}{2}\|v_x^\rho\|^2+l_1+l_1\left(\|v^\rho\|^2+\|v^\iota\|^2\right)+\frac{8M_1^2}{a_R}\left(\|\hat{w}^\rho\|^2+\|\hat{w}^\iota\|^2\right).$$

Similarly, the following inequality holds

$$\int_0^1 v^\iota \iota \mathrm{d}x \leq \frac{a_R}{2}\|v_x^\iota\|^2 + l_1 + l_1\left(\|v^\rho\|^2 + \|v^\iota\|^2\right) + \frac{8M_1^2}{a_R}\left(\|\hat{w}^\rho\|^2 + \|\hat{w}^\iota\|^2\right).$$

Substituting both of the above two inequalities into (G.5), we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|v^\rho\|^2 + \|v^\iota\|^2\right) \leq & -\frac{a_R}{2}\left(\|v_x^\rho\|^2 + \|v_x^\iota\|^2\right) \\ & + l_1 + l_1\left(\|v^\rho\|^2 + \|v^\iota\|^2\right) + \frac{16M_1^2}{a_R}\left(\|\hat{w}^\rho\|^2 + \|\hat{w}^\iota\|^2\right).\end{aligned} \tag{G.6}$$

Choose the following Lyapunov function:

$$V(t) = \frac{\delta}{2}\left(\|\hat{w}^\rho\|^2 + \|\hat{w}^\iota\|^2\right) + \frac{1}{2}\left(\|v^\rho\|^2 + \|v^\iota\|^2\right).$$

Computing the time derivative of $V(t)$ and using (G.4), (G.6), we have

$$\begin{aligned}\dot{V}(t) \leq & -\left(\delta\left(c - 6\sigma_3\right) - \frac{16M_1^2}{a_R}\right)\left(\|\hat{w}^\rho\|^2 + \|\hat{w}^\iota\|^2\right) + l_1\left(\|v^\rho\|^2 + \|v^\iota\|^2\right) \\ & + l_1\left(\|\hat{w}^\rho\|^2 + \|\hat{w}^\iota\|^2\right) + l_1 - \frac{a_R}{2}\left(\|v_x^\rho\|^2 + \|v_x^\iota\|^2\right) \\ \leq & -\left(\delta\left(c - 6\sigma_3\right) - \frac{16M_1^2}{a_R}\right)\left(\|\hat{w}^\rho\|^2 + \|\hat{w}^\iota\|^2\right) + l_1\left(\|v^\rho\|^2 + \|v^\iota\|^2\right) \\ & + l_1\left(\|\hat{w}^\rho\|^2 + \|\hat{w}^\iota\|^2\right) + l_1 - \frac{a_R}{8}\left(\|v^\rho\|^2 + \|v^\iota\|^2\right).\end{aligned}$$

Choosing $0 < \sigma_3 < \frac{c}{6}$, $\delta > \frac{16M_1^2}{a_R(c-6\sigma_3)}$ and letting $q = \min\left\{2\left(c - 6\sigma_3\right) - \frac{16M_1^2}{\delta a_R},\ \frac{a_R}{4}\right\}$, we have

$$\dot{V}(t) \leq -qV(t) + l_1 V(t) + l_1.$$

Then, by Lemma H.3, we have $V \in \mathcal{L}_1 \cap \mathcal{L}_\infty$, and hence $\|\hat{w}^\rho\|, \|\hat{w}^\iota\|, \|v^\rho\|, \|v^\iota\| \in \mathcal{L}_2 \cap \mathcal{L}_\infty$.

Part II: Proof of (4.13):

By integration by parts, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|v_x^\rho\|^2 + \|v_x^\iota\|^2\right) \\ &\quad = -\int_0^1 v_{xx}^\rho\left(a_R v_{xx}^\rho - a_I v_{xx}^\iota + \rho\right)\mathrm{d}x - \int_0^1 v_{xx}^\iota\left(a_I v_{xx}^\rho + a_R v_{xx}^\iota + \iota\right)\mathrm{d}x \\ &\quad = -a_R\left(\|v_{xx}^\rho\|^2 + \|v_{xx}^\iota\|^2\right)\mathrm{d}x - \int_0^1 v_{xx}^\rho \rho \mathrm{d}x - \int_0^1 v_{xx}^\iota \iota \mathrm{d}x.\end{aligned} \tag{G.7}$$

The last two terms need to be further estimated. For this, by (D.1), (G.3) and noting that $\|\hat{w}^\rho\|, \|\hat{w}^\iota\| \in \mathcal{L}_2$, we obatin

$$\begin{aligned}-\int_0^1 v_{xx}^\rho \rho \mathrm{d}x & \leq a_R\|v_{xx}^\rho\|^2 + \frac{1}{4a_R}\|\rho\|^2 \\ & \leq a_R\|v_{xx}^\rho\|^2 + \frac{1}{2a_R}\|\tilde{\rho}\|^2 + \frac{M_1^2}{a_R}\left(\|\hat{w}^\rho\|^2 + \|\hat{w}^\iota\|^2\right) \\ & \leq a_R\|v_{xx}^\rho\|^2\mathrm{d}x + l_1 + l_1\left(\|v^\rho\|^2 + \|v^\iota\|^2\right),\end{aligned}$$

and

$$-\int_0^1 v^\iota_{xx}\iota \mathrm{d}x \leq a_R\|v^\iota_{xx}\|^2 + l_1 + l_1\left(\|v^\rho\|^2+\|v^\iota\|^2\right).$$

Substituting the above two inequalities into (G.7) leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|v^\rho_x\|^2+\|v^\iota_x\|^2\right) \leq l_1 + l_1\left(\|v^\rho\|^2+\|v^\iota\|^2\right).$$

Then, integrating both sides of the above inequality over $[0,\,t]$ and noting $\|v^\rho\|$, $\|v^\iota\| \in \mathcal{L}_2$ directly arrive at $\|v^\rho_x\|$, $\|v^\iota_x\| \in \mathcal{L}_\infty$.

Moreover, similar to the derivation of (G.1), we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\hat{w}^\rho_x\|^2+\|\hat{w}^\iota_x\|^2\right)\\
&\quad= -\int_0^1 \hat{w}^\rho_{xx}\Big(a_R\hat{w}^\rho_{xx} - a_I\hat{w}^\iota_{xx} - c\hat{w}^\rho + \hat{b}_R\tilde{w}^\rho - \hat{b}_I\tilde{w}^\iota\\
&\qquad+\dot{\hat{b}}_R\breve{v}^\rho - \dot{\hat{b}}_I\breve{v}^\iota + \int_0^x\left(\hat{w}^\rho(y)H(x,y)+\hat{w}^\iota(y)H^c(x,y)\right)\mathrm{d}y\Big)\mathrm{d}x\\
&\qquad-\int_0^1 \hat{w}^\iota_{xx}\Big(a_I\hat{w}^\rho_{xx} + a_R\hat{w}^\iota_{xx} - c\hat{w}^\iota + \hat{b}_R\tilde{w}^\iota + \hat{b}_I\tilde{w}^\rho\\
&\qquad+\dot{\hat{b}}_I\breve{v}^\rho + \dot{\hat{b}}_R\breve{v}^\iota - \int_0^x\left(\hat{w}^\rho(y)H^c(x,y)-\hat{w}^\iota(y)H(x,y)\right)\mathrm{d}y\Big)\mathrm{d}x\\
&\quad= -a_R\left(\|\hat{w}^\rho_{xx}\|^2+\|\hat{w}^\iota_{xx}\|^2\right) - c\left(\|\hat{w}^\rho\|^2+\|\hat{w}^\iota\|^2\right)\\
&\qquad-\int_0^1 \hat{w}^\rho_{xx}\Big(\hat{b}_R\tilde{w}^\rho - \hat{b}_I\tilde{w}^\iota + \dot{\hat{b}}_R\breve{v}^\rho - \dot{\hat{b}}_I\breve{v}^\iota + \int_0^x\left(\hat{w}^\rho(y)H^c(x,y)-\hat{w}^\iota(y)H(x,y)\right)\mathrm{d}y\Big)\mathrm{d}x\\
&\qquad-\int_0^1 \hat{w}^\iota_{xx}\Big(\hat{b}_R\tilde{w}^\iota + \hat{b}_I\tilde{w}^\rho + \dot{\hat{b}}_I\breve{v}^\rho + \dot{\hat{b}}_R\breve{v}^\iota - \int_0^x\left(\hat{w}^\rho(y)H^c(x,y)-\hat{w}^\iota(y)H(x,y)\right)\mathrm{d}y\Big)\mathrm{d}x.
\end{aligned}\tag{G.8}$$

Denote the third and fourth terms on the right-hand side of the above equality by ① and ②, respectively, we obtain their estimates as follows:

$$\begin{aligned}
① &\leq a_R\|\hat{w}^\rho_{xx}\|^2 + \frac{5\bar{b}_R^2}{4}\|\tilde{w}^\rho\|^2 + \frac{5\bar{b}_I^2}{4}\|\tilde{w}^\iota\|^2 + \frac{5\dot{\hat{b}}_R^2}{4}\|\breve{v}^\rho\|^2\\
&\quad+\frac{5\dot{\hat{b}}_I^2}{4}\|\breve{v}^\iota\|^2 + \frac{5}{2}\|\hat{w}^\rho\|^2\int_0^1\int_0^x H(x,y)^2\mathrm{d}y\mathrm{d}x + \frac{5}{2}\|\hat{w}^\iota\|^2\int_0^1\int_0^x H^c(x,y)^2\mathrm{d}y\mathrm{d}x\\
&\leq a_R\|\hat{w}^\rho_{xx}\|^2 + \frac{5M_1\left(\bar{b}_R^2+\bar{b}_I^2\right)}{4}\left(\|\tilde{\rho}\|^2+\|\tilde{\iota}\|^2\right) + \frac{5M_1^2\left(\dot{\hat{b}}_R^2+\dot{\hat{b}}_I^2\right)}{2}\left(\|v^\rho\|^2+\|v^\iota\|^2\right)\\
&\quad+\frac{5}{2}\left(\|\hat{w}^\rho\|^2+\|\hat{w}^\iota\|^2\right)\left(\int_0^1\int_0^x H(x,y)^2\mathrm{d}y + \int_0^1\int_0^x H^c(x,y)^2\mathrm{d}y\mathrm{d}x\right)\\
&\leq a_R\|\hat{w}^\rho_{xx}\|^2 + l_1\left(\|v^\rho\|^2+\|v^\iota\|^2\right) + l_1\\
&\leq a_R\|\hat{w}^\rho_{xx}\|^2 + l_1,
\end{aligned}\tag{G.9}$$

where (D.1), (G.3) and (4.12) have been used. Similarly, we obtain

$$② \leq a_R\|\hat{w}^\iota_{xx}\|^2 + l_1.$$

Substituting this and (G.9) into (G.1) arrives at

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\hat{w}_x^{\varrho}\|^2 + \|\hat{w}_x^{\iota}\|^2\right) \leq l_1.$$

Then, by integrating both sides of the above inequality over $[0, t]$, we yield that $\|\hat{w}_x^{\varrho}\|, \|\hat{w}_x^{\iota}\| \in \mathcal{L}_\infty$.

Appendix H. Some useful inequalities and criterions

The following four lemmas respectively give four important inequalities (see Lems. H.1 and H.2 later) and two useful criterions (see Lems. H.3 and H.4 later) which will be used frequently in the stability analysis of the closed-loop systems.

Lemma H.1. [9] (*Agmon's Inequality*) *For any* $w \in \mathbf{C}^1[0, D]$, *there hold:*

$$\begin{cases} w(x)^2 \leq w(0)^2 + 2\|w\| \cdot \|w_x\|, \\ w(x)^2 \leq w(D)^2 + 2\|w\| \cdot \|w_x\|. \end{cases}$$

Lemma H.2. [9] (*Poincaré's Inequality*) *For any* $w \in \mathbf{C}^1[0, D]$, *there hold*

$$\begin{cases} \|w\|^2 \leq 2Dw(0)^2 + 4D^2\|w_x\|^2, \\ \|w\|^2 \leq 2Dw(D)^2 + 4D^2\|w_x\|^2. \end{cases}$$

Lemma H.3. [15] *Let σ be a positive constant, v be a real-valued function defined on R_+ and l_1, l_2 be real-valued positive functions defined on R_+. If $l_1, l_2 \in \mathcal{L}_1$ and satisfy*

$$\dot{v} \leq -\sigma v + l_1 v + l_2,\ v(0) \geq 0,$$

then, $v \in \mathcal{L}_\infty \cap \mathcal{L}_1$.

Lemma H.4. [22] *Suppose the function $f(t)$ defined on $[0, +\infty)$ satisfies:*
(1) *$f(t) \geq 0$ for all $t \in [0, +\infty)$,*
(2) *$f(t)$ is differentiable on $[0, +\infty)$ and there exists a constant M such that $\frac{\mathrm{d}f(t)}{\mathrm{d}t} \leq M$, $\forall t \geq 0$,*
(3) *$\int_0^{+\infty} f(t)\mathrm{d}t < \infty$.*
Then, there holds

$$\lim_{t\to+\infty} f(t) = 0.$$

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