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ANALYSIS OF THE CONTROLLABILITY FROM THE EXTERIOR OF STRONG DAMPING NONLOCAL WAVE EQUATIONS*

Mahamadi Warma[1,**] and Sebastián Zamorano[2]

Abstract. We make a complete analysis of the controllability properties from the exterior of the (possible) strong damping wave equation associated with the fractional Laplace operator subject to the non-homogeneous Dirichlet type exterior condition. In the first part, we show that if $0 < s < 1$, $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded Lipschitz domain and the parameter $\delta > 0$, then there is no control function g such that the following system

$$\begin{cases} u_{tt} + (-\Delta)^s u + \delta(-\Delta)^s u_t = 0 & \text{in } \Omega \times (0,T),\\ u = g\chi_{\mathcal{O}\times(0,T)} & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T),\\ u(\cdot,0) = u_0, u_t(\cdot,0) = u_1 & \text{in } \Omega, \end{cases}$$

is exact or null controllable at time $T > 0$. In the second part, we prove that for every $\delta \geq 0$ and $0 < s < 1$, the system is indeed approximately controllable for any $T > 0$ and $g \in \mathcal{D}(\mathcal{O} \times (0,T))$, where $\mathcal{O} \subset \mathbb{R}^N \setminus \Omega$ is any non-empty open set.

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded open set with a Lipschitz continuous boundary $\partial\Omega$. The aim of the present paper is to study completely the controllability properties from the exterior of the (possible) strong damping nonlocal wave equation associated with the fractional Laplace operator. More precisely, we consider

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the system

$$\begin{cases} u_{tt} + (-\Delta)^s u + \delta(-\Delta)^s u_t = 0 & \text{in } \ \Omega \times (0,T), \\ u = g\chi_{\mathcal{O}\times(0,T)} & \text{in } \ (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ u(\cdot,0) = u_0, \ \ u_t(\cdot,0) = u_1 & \text{in } \ \Omega, \end{cases} \tag{1.1}$$

where $u = u(x,t)$ is the state to be controlled, $g = g(x,t)$ is the control function which is localized on a subset $\mathcal{O}$ of $\mathbb{R}^N \setminus \Omega$, $\delta \geq 0$, $T > 0$, $0 < s < 1$ are real numbers and $(-\Delta)^s$ denotes the fractional Laplace operator, which is given formally by the following singular integral:

$$(-\Delta)^s u(x) = C_{N,s}\text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, \mathrm{d}y, \quad x \in \mathbb{R}^N,$$

where $C_{N,s}$ is a normalization constant that depends on N and s only. We refer to Section 3 for the precise definition.

We mention that it is nowadays known that the system (1.1) is not well-posed if g is prescribed at the boundary $\partial\Omega$. This follows from the fact that the stationary problem, that is, the following Dirichlet problem

$$(-\Delta)^s u = 0 \quad \text{in} \ \ \Omega, \ \ u = g \ \ \text{on} \ \ \partial\Omega,$$

is not well-posed. The well-posed Dirichlet problem for $(-\Delta)^s$ is given by (see Prop. 3.2)

$$(-\Delta)^s u = 0 \quad \text{in} \ \ \Omega, \ \ u = g \ \ \text{in} \ \ \mathbb{R}^N \setminus \Omega.$$

That is, the function g must be prescribed in $\mathbb{R}^N \setminus \Omega$. For these reasons, it has been shown in [46] for the first time, that the classical notion of boundary controllability for local partial differential equations (PDEs) does not make sense for fractional PDEs involving $(-\Delta)^s$, and therefore it should be replaced by a control that is localized outside the domain where the PDE is satisfied. This is a consequence of the nonlocality of $(-\Delta)^s$. We refer to [46] and the references therein for more details on these topics.

We shall show that for every $g \in L^2((0,T); W^{s,2}(\mathbb{R}^N \setminus \Omega))$, if u_0 and u_1 belong to a suitable Banach space, then the system (1.1) has a unique solution (u, u_t) satisfying the regularity $u \in C([0,T]; W^{s,2}(\mathbb{R}^N)) \cap C^1([0,T]; L^2(\Omega))$. In that case, the set of reachable states is given by

$$\mathcal{R}((u_0,u_1),T) = \Big\{ (u(\cdot,T), u_t(\cdot,T)) : \ g \in L^2((0,T); W^{s,2}(\mathbb{R}^N \setminus \Omega)) \Big\}.$$

Let $W^{-s,2}(\overline{\Omega})$ denote the dual of the energy space $W_0^{s,2}(\overline{\Omega})$ (see Sect. 3).

We shall consider the following three notions of controllability.

- The system is said to be null controllable at $T > 0$, if

$$(0,0) \in \mathcal{R}((u_0,u_1),T).$$

  In other words, there is a control function g such that the unique solution (u,u_t) satisfies $u(\cdot,T) = u_t(\cdot,T) = 0$ almost everywhere in Ω.

– The system is said to be exact controllable at $T > 0$, if

$$\mathcal{R}((u_0, u_1), T) = L^2(\Omega) \times W^{-s,2}(\overline{\Omega}).$$

– The system is said to be approximately controllable at $T > 0$, if

$$\mathcal{R}((u_0, u_1), T) \text{ is dense in } L^2(\Omega) \times W^{-s,2}(\overline{\Omega}),$$

or equivalently, for every $(\tilde{u}_0, \tilde{u}_1) \in L^2(\Omega) \times W^{-s,2}(\overline{\Omega})$ and $\varepsilon > 0$, there is a control g such that the corresponding unique solution (u, u_t) of (1.1) with $u_0 = u_1 = 0$ satisfies

$$\|u(\cdot, T) - \tilde{u}_0\|_{L^2(\Omega)} + \|u_t(\cdot, T) - \tilde{u}_1\|_{W^{-s,2}(\overline{\Omega})} \le \varepsilon. \tag{1.2}$$

In the present article we have obtained the following specific results.

(i) Our first main result says that if $\delta > 0$, then the system is not exact or null controllable at any time $T > 0$.
(ii) We also obtain that the adjoint system associated with (1.1) satisfies the unique continuous property for evolution equations.
(iii) The third main result states that the system (1.1) is approximately controllable, for every $\delta \ge 0$, $0 < s < 1$, $T > 0$, and for every $g \in \mathcal{D}(\mathcal{O} \times (0, T))$, where $\mathcal{O} \subset \mathbb{R}^N \setminus \Omega$ is any non-empty open set. Since the system is not exact or null controllable (if $\delta > 0$), it is the best possible result that can be obtained regarding the controllability of such systems.

The null/exact controllabilty from the interior of the pure (without damping) wave equation (with strong zero Dirichlet exterior condition) associated with the bi-fractional Laplace operator has been investigated in [6] by using a Pohozaev identity for the fractional Laplacian established in [37]. More precisely, the author in [6] has considered the following problem:

$$\begin{cases} u_{tt} + (-\Delta)^{2s} u = f\chi_{\omega \times (0,T)} & \text{in } \Omega \times (0, T), \\ u = (-\Delta)^s u = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T), \\ u(\cdot, 0) = u_0, \ \ u_t(\cdot, 0) = u_1 & \text{in } \Omega, \end{cases} \tag{1.3}$$

where $\frac{1}{2} \le s < 1$, u is the state to be controlled and f is the control function localized in a certain neighborhood $\omega \subset \Omega$ of the boundary $\partial\Omega$. He has shown that the system (1.3) is exact/null controllable at any time $T > 0$, if $\frac{1}{2} < s < 1$ and at any time $T > T_0$ if $s = \frac{1}{2}$, where T_0 is a certain positive constant. We notice that, since the system (1.3) is reversible in time (which is not the case for (1.1) if $\delta > 0$), then in this case, the null and exact controllabilties are the same notions.

Always in the case $\delta = 0$, most recently, we have studied in [28] the controllability of the space-time fractional wave equation, that is, the case where in (1.1), we have replaced u_{tt} by the Caputo time fractional derivative $\mathbb{D}_t^\alpha$ $(1 < \alpha < 2)$. We have obtained a positive result about the approximate controllability from the exterior. The corresponding problem for interior control has been studied in [26]. The case of the fractional diffusion equation from the exterior, that is, when $0 < \alpha \le 1$, has been completely investigated in [46]. We mention that due to the results in [29], fractional in time evolution equations can never be null/exact controllable. The null controllabilty from the interior of the heat equation associated with the fractional Laplace operator (with zero Dirichlet exterior condition) has been recently studied in one space-dimension in [7] by using the asymptotic gap of the eigenvalues of the realization in $L^2(\Omega)$ of $(-\Delta)^s$ with the zero Dirichlet exterior condition. The null controllability from the exterior for the one-dimensional fractional heat equation has been very recently studied in [47] by using the above mentioned asymptotic gap on the eigenvalues. The case $N \ge 2$ is still an open problem.

In the present paper, using some ideas that we have recently developed in [28, 46], we shall study the controllability of the nonlocal wave or/and the strong damping nonlocal wave equations with the control function localized in the exterior of the domain Ω where the evolution equation is satisfied. To the best of our knowledge, it is the third work (after our work [46] for the case $0 < \alpha \leq 1$ and [28] for the case $1 < \alpha < 2$) that addresses the controllability of nonlocal equations from the exterior of the domain involved, and it is the first work that studies the controllability from the exterior of wave and/or strong damping nonlocal wave equations involving the fractional Laplace operator.

We also notice that from our results, when taking the limit as $s \uparrow 1^-$, we can recover the known results on the topics regarding the controllability from the boundary of the local wave or the strong damping local wave equations studied in [38, 51] and their references. That is, the case where the control function is localized on a subset ω of $\partial\Omega$.

In many real life applications a source or a control is placed outside (disjoint from) the observation domain Ω where the PDE is satisfied. Some examples of control problems where this situation may arise (since the topic is new, our findings have been not yet validated by concrete examples) are: (i) Acoustic testing, when the loudspeakers are placed far from the aerospace structures [27]; (ii) Magnetotellurics (MT), which is a technique to infer earth's subsurface electrical conductivity from surface measurements [41, 48]; (iii) Magnetic drug targeting (MDT), where drugs with ferromagnetic particles in suspension are injected into the body and the external magnetic field is then used to steer the drug to relevant areas, for example, solid tumors [2, 3, 30]; (iv) Electroencephalography (EEG) is used to record electrical activities in brain [34, 49], in case one accounts for the neurons disjoint from the brain, one will obtain an external source problem. This is different from the traditional approaches where the source/control is placed either inside the domain Ω or on the boundary $\partial\Omega$ of Ω. We refer to our very recent paper [1] for some preliminary numerical analysis results.

Anomalous diffusion and wave equations are of great interest in physics. In [31] it has been shown that the fractional wave equation governs the propagation of mechanical diffusion waves in viscoelastic media. Fractional order operators have also recently emerged as a modeling alternative in various branches of science. A number of stochastic models for explaining anomalous diffusion have been introduced in the literature; among them we quote the fractional Brownian motion; the continuous time random walk; the Lévy flights; the Schneider grey Brownian motion; and more generally, random walk models based on evolution equations of single and distributed fractional order in space (see *e.g.* [16, 23, 32, 39, 50]). In general, a fractional diffusion operator corresponds to a diverging jump length variance in the random walk. We refer to [14, 19, 20, 42] and the references therein for a complete analysis, the derivation and the applications of the fractional Laplace operator.

The rest of the paper is structured as follows. In Section 2 we state the main results of the article. The first one (Thm. 2.1) says that if $\delta > 0$, then the system (1.1) is not exact/null controllable at time $T > 0$. The second main result (Thm. 2.4) shows that the adjoint system associated with (1.1) satisfies the unique continuation property for evolution equations and the third main result (Thm. 2.5) states that for every $\delta \geq 0$, the system (1.1) is approximately controllable at any time $T > 0$. In Section 3 we introduce the function spaces needed to study our problem and we give some known results that are used in the proof of our main results. This is followed in Section 4 by the proof of the existence, uniqueness, regularity and the representation of solutions of (1.1) and its associated dual system in terms of series. Finally in Section 5, we give the proof of the main results stated in Section 2.

2. Main results

In this section we state the main results of the article. Throughout the remainder of the paper, without any mention, $\delta \geq 0$ and $0 < s < 1$ are real numbers and $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ denotes a bounded open set with a Lipschitz continuous boundary. Given a measurable set $E \subset \mathbb{R}^N$, we shall denote by $(\cdot,\cdot)_{L^2(E)}$ the scalar product in $L^2(E)$. We refer to Section 3 for a rigorous definition of the function spaces and operators involved. Let $W_0^{s,2}(\overline{\Omega})$ denote the energy space and we denote by $W^{-s,2}(\overline{\Omega})$ its dual. We shall let $\langle\cdot,\cdot\rangle_{-\frac{1}{2},\frac{1}{2}}$ be their duality pairing.

Our first main result is the following theorem.

Theorem 2.1. *Let $\delta > 0$. Then the system (1.1) is not exact or null controllable at time $T > 0$.*

Next, we introduce our notion of solution. Let $(u_0, u_1) \in L^2(\Omega) \times W^{-s,2}(\overline{\Omega})$ and consider the following two systems:

$$\begin{cases} v_{tt} + (-\Delta)^s v + \delta(-\Delta)^s v_t = 0 & \text{in } \Omega \times (0,T), \\ v = g & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ v(\cdot,0) = 0, \;\; v_t(\cdot,0) = 0 & \text{in } \Omega, \end{cases} \tag{2.1}$$

and

$$\begin{cases} w_{tt} + (-\Delta)^s w + \delta(-\Delta)^s w_t = 0 & \text{in } \Omega \times (0,T), \\ w = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ w(\cdot,0) = u_0, \;\; w_t(\cdot,0) = u_1 & \text{in } \Omega. \end{cases} \tag{2.2}$$

Then it is clear that $u = v + w$ solves the system (1.1).

Definition 2.2. Let g be a given function. A function (v, v_t) is said to be a weak solution of (2.1), if the following properties hold.

– Regularity:

$$\begin{cases} v \in C([0,T]; L^2(\Omega)) \cap C^1([0,T]; W^{-s,2}(\overline{\Omega})), \\ v_{tt} \in C((0,T); W^{-s,2}(\overline{\Omega})). \end{cases} \tag{2.3}$$

– Variational identity: For every $w \in W_0^{s,2}(\overline{\Omega})$ and a.e. $t \in (0,T)$,

$$\langle v_{tt}, w\rangle_{-\frac{1}{2},\frac{1}{2}} + \langle(-\Delta)^s(v + \delta v_t), w\rangle_{-\frac{1}{2},\frac{1}{2}} = 0.$$

– Initial and exterior conditions:

$$v(\cdot,0) = 0, \; v_t(\cdot,0) = 0 \;\; \text{in } \; \Omega \;\; \text{and} \;\; v = g \text{ in } \; (\mathbb{R}^N \setminus \Omega) \times (0,T). \tag{2.4}$$

It follows from Definition 2.2, that for a weak solution (v, v_t) of the system (2.1), we have that the functions $v(\cdot,T) \in L^2(\Omega)$ and $v_t(\cdot,T)) \in W^{-s,2}(\overline{\Omega})$.

Now, let us consider the following backward homogeneous system

$$\begin{cases} \psi_{tt} + (-\Delta)^s \psi - \delta(-\Delta)^s \psi_t = 0 & \text{in } \Omega \times (0,T), \\ \psi = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ \psi(\cdot,T) = \psi_0, \quad \psi_t(\cdot,T) = -\psi_1 & \text{in } \Omega, \end{cases} \tag{2.5}$$

which is the dual system associated with (2.2). In fact, multiplying the first equation in (2.2) with a function $\psi = \psi(t,x)$ satisfying $\psi(t,\cdot) \in D((-\Delta)_D^s)$ (the domain of the self-adjoint operator $(-\Delta)_D^s$ defined in (3.4) below), integrating by parts over $(0,T)$, and then over Ω by using the integration by parts formula (3.11), or by noticing that $((-\Delta)_D^s \phi_1, \phi_2)_{L^2(\Omega)} = (\phi_1, (-\Delta)_D^s \phi_2)_{L^2(\Omega)}$ for every $\phi_1, \, \phi_2 \in D((-\Delta)_D^s)$, we get the system (2.5). By abuse of terminology, one usually says that (2.5) is the adjoint system associated with (1.1).

Our notion of weak solution to (2.5) is as follows.

Definition 2.3. Let $(\psi_0, \psi_1) \in W_0^{s,2}(\overline{\Omega}) \times L^2(\Omega)$. A function (ψ, ψ_t) is said to be a weak solution of (2.5), if for a.e. $t \in (0,T)$, the following properties hold.

– Regularity and final data:

$$
\begin{cases}\psi \in C([0,T]; W_0^{s,2}(\overline{\Omega})) \cap C^1([0,T]; L^2(\Omega)),\\ \psi_{tt} \in C((0,T); W^{-s,2}(\overline{\Omega})),\end{cases} \tag{2.6}
$$

and $\psi(\cdot,T)=\psi_0$, $\psi_t(\cdot,T)=\psi_1$ in Ω.

– Variational identity: For every $w\in W_0^{s,2}(\overline{\Omega})$ and a.e. $t\in(0,T)$,

$$
\langle \psi_{tt}, w\rangle_{-\frac{1}{2},\frac{1}{2}} + \langle (-\Delta)^s(\psi-\delta\psi_t), w\rangle_{-\frac{1}{2},\frac{1}{2}} = 0.
$$

The next theorem, which is our second main result, says that the adjoint system (2.5) satisfies the unique continuation property for evolution equations.

Theorem 2.4. *Let $(\psi_0,\psi_1)\in W_0^{s,2}(\overline{\Omega})\times L^2(\Omega)$ and (ψ,ψ_t) the unique weak solution of (2.5). Let $\mathcal{O}\subset\mathbb{R}^N\setminus\Omega$ be an arbitrary non-empty open set. If $\mathcal{N}_s\psi=0$ in $\mathcal{O}\times(0,T)$, then $\psi=0$ in $\Omega\times(0,T)$. Here, $\mathcal{N}_s\psi$ is the nonlocal normal derivative of ψ defined in (3.9) below.*

The last main result of the article concerns the approximate controllability of (1.1). For this, we notice that the study of the approximate controllability of (1.1) can be reduced to the case $u_0=u_1=0$ (see *e.g.* [26, 28, 38, 45, 46, 51]).

Theorem 2.5. *The system (1.1) is approximately controllable for any $T>0$ and any control function $g\in\mathcal{D}(\mathcal{O}\times(0,T))$, where $\mathcal{O}\subset\mathbb{R}^N\setminus\Omega$ is an arbitrary non-empty open set. That is,*

$$
\begin{aligned}
\overline{\mathcal{R}((0,0),T)}^{L^2(\Omega)\times W^{-s,2}(\overline{\Omega})} &= \overline{\{(u(\cdot,T),u_t(\cdot,T)):\ g\in\mathcal{D}(\mathcal{O}\times(0,T))\}}^{L^2(\Omega)\times W^{-s,2}(\overline{\Omega})}\\
&= L^2(\Omega)\times W^{-s,2}(\overline{\Omega}),
\end{aligned}
$$

where (u,u_t) is the unique weak solution of (1.1) with $u_0=u_1=0$.

3. Preliminaries

In this section we give some notations and recall some known results as they are needed in the proof of our main results. We start with fractional order Sobolev spaces. Given $0<s<1$, we let

$$
W^{s,2}(\Omega) := \left\{ u\in L^2(\Omega):\ \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y<\infty\right\},
$$

and we endow it with the norm

$$
\|u\|_{W^{s,2}(\Omega)} := \left(\int_\Omega |u(x)|^2\,\mathrm{d}x + \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}}.
$$

We set

$$
W_0^{s,2}(\overline{\Omega}) := \left\{ u\in W^{s,2}(\mathbb{R}^N):\ u=0 \text{ in } \mathbb{R}^N\setminus\Omega\right\}.
$$

For more information on fractional order Sobolev spaces, we refer to [14, 43].

Next, we give a rigorous definition of the fractional Laplace operator. Let

$$\mathcal{L}_s^1(\mathbb{R}^N) := \left\{ u : \mathbb{R}^N \to \mathbb{R} \ \text{ measurable}, \ \int_{\mathbb{R}^N} \frac{|u(x)|}{(1+|x|)^{N+2s}}\, \mathrm{d}x < \infty \right\}.$$

For $u \in \mathcal{L}_s^1(\mathbb{R}^N)$ and $\varepsilon > 0$ we set

$$(-\Delta)_\varepsilon^s u(x) := C_{N,s} \int_{\{y\in\mathbb{R}^N:\, |x-y|>\varepsilon\}} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, \mathrm{d}y, \quad x \in \mathbb{R}^N,$$

where $C_{N,s}$ is a normalization constant given by

$$C_{N,s} := \frac{s2^{2s}\Gamma\left(\frac{2s+N}{2}\right)}{\pi^{\frac{N}{2}}\Gamma(1-s)}, \tag{3.1}$$

with Γ being the usual Euler-Gamma function. The fractional Laplacian $(-\Delta)^s$ is defined by the following singular integral:

$$(-\Delta)^s u(x) := C_{N,s}\, \mathrm{P.V.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, \mathrm{d}y = \lim_{\varepsilon\downarrow 0} (-\Delta)_\varepsilon^s u(x), \quad x \in \mathbb{R}^N, \tag{3.2}$$

provided that the limit exists. Note that $\mathcal{L}_s^1(\mathbb{R}^N)$ is the right space for which $v := (-\Delta)_\varepsilon^s u$ exists for every $\varepsilon > 0$, v being also continuous at the continuity points of u. For more details on the fractional Laplace operator we refer to [11, 13, 14, 18, 20, 21, 43, 44] and their references.

Next, we consider the following Dirichlet problem:

$$\begin{cases} (-\Delta)^s u = 0 & \text{in } \ \Omega, \\ u = g & \text{in } \ \mathbb{R}^N \setminus \Omega. \end{cases} \tag{3.3}$$

Definition 3.1. Let $g \in W^{s,2}(\mathbb{R}^N \setminus \Omega)$ and $G \in W^{s,2}(\mathbb{R}^N)$ be such that $G|_{\mathbb{R}^N\setminus\Omega} = g$. A function $u \in W^{s,2}(\mathbb{R}^N)$ is said to be a weak solution of (3.3) if $u - G \in W_0^{s,2}(\overline{\Omega})$ and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, \mathrm{d}x\mathrm{d}y = 0, \quad \forall\, v \in W_0^{s,2}(\overline{\Omega}).$$

The following existence result is taken from [24] (see also [22]).

Proposition 3.2. *For any $g \in W^{s,2}(\mathbb{R}^N \setminus \Omega)$, there is a unique $u \in W^{s,2}(\mathbb{R}^N)$ satisfying (3.3) in the sense of Definition 3.1. In addition, there is a constant $C > 0$ such that*

$$\|u\|_{W^{s,2}(\mathbb{R}^N)} \le C\|g\|_{W^{s,2}(\mathbb{R}^N\setminus\Omega)}.$$

Next, we consider the closed, symmetric, continuous and non-negative bilinear form

$$\mathcal{F}(u,v) := \frac{C_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, \mathrm{d}x\mathrm{d}y, \quad u, v \in W_0^{s,2}(\overline{\Omega}).$$

Let $(-\Delta)_D^s$ be the self-adjoint operator in $L^2(\Omega)$ associated with $\mathcal{F}$ in the sense that

$$\begin{cases} D((-\Delta)_D^s) := \Big\{u \in W_0^{s,2}(\overline{\Omega}),\ \exists\, f \in L^2(\Omega),\ \mathcal{F}(u,v) = (f,v)_{L^2(\Omega)}\ \forall\, v \in W_0^{s,2}(\overline{\Omega})\Big\}, \\ (-\Delta)_D^s u := f. \end{cases}$$

More precisely, we have that

$$D((-\Delta)_D^s) = \Big\{u \in W_0^{s,2}(\overline{\Omega}),\ (-\Delta)^s u \in L^2(\Omega)\Big\}, \quad (-\Delta)_D^s u = (-\Delta)^s u \ \text{ in } \ \Omega. \tag{3.4}$$

Then $(-\Delta)_D^s$ is the realization of $(-\Delta)^s$ in $L^2(\Omega)$ with the condition $u = 0$ in $\mathbb{R}^N \setminus \Omega$. It is well-known (see *e.g.* [19, 40, 46]) that $(-\Delta)_D^s$ has a compact resolvent and its eigenvalues form a non-decreasing sequence of real numbers

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots \ \text{ satisfying } \ \lim_{n\to\infty} \lambda_n = \infty. \tag{3.5}$$

In addition, the eigenvalues are of finite multiplicity. Let $(\varphi_n)_{n\in\mathbb{N}}$ be the orthonormal basis of eigenfunctions associated with $(\lambda_n)_{n\in\mathbb{N}}$. Then $\varphi_n \in D((-\Delta)_D^s)$ for every $n \in \mathbb{N}$, $(\varphi_n)_{n\in\mathbb{N}}$ is total in $L^2(\Omega)$ and satisfies

$$\begin{cases} (-\Delta)^s \varphi_n = \lambda_n \varphi_n & \text{in } \ \Omega, \\ \varphi_n = 0 & \text{in } \ \mathbb{R}^N \setminus \Omega. \end{cases} \tag{3.6}$$

With this setting, we have that for $u \in W_0^{s,2}(\overline{\Omega})$,

$$\|u\|_{W_0^{s,2}(\overline{\Omega})} := \left(\sum_{n=1}^{\infty} \left|\lambda_n^{\frac{1}{2}} (u,\varphi_n)_{L^2(\Omega)}\right|^2\right)^{\frac{1}{2}}, \tag{3.7}$$

defines an equivalent norm on $W_0^{s,2}(\overline{\Omega})$. If $u \in D((-\Delta)_D^s)$, then

$$\|u\|^2_{D((-\Delta)_D^s)} = \|(-\Delta)_D^s u\|^2_{L^2(\Omega)} = \sum_{n=1}^{\infty} \left|\lambda_n (u,\varphi_n)_{L^2(\Omega)}\right|^2.$$

In addition, for $u \in W^{-s,2}(\overline{\Omega})$, we have that

$$\|u\|^2_{W^{-s,2}(\overline{\Omega})} = \sum_{n=1}^{\infty} \left|\lambda_n^{-\frac{1}{2}} (u,\varphi_n)_{L^2(\Omega)}\right|^2. \tag{3.8}$$

In that case, using the Gelfand triple (see *e.g.* [5]), we get the following continuous embeddings: $W_0^{s,2}(\overline{\Omega}) \hookrightarrow L^2(\Omega) \hookrightarrow W^{-s,2}(\overline{\Omega})$.

Next, for $u \in W^{s,2}(\mathbb{R}^N)$ we introduce the *nonlocal normal derivative* $\mathcal{N}_s$ defined by

$$\mathcal{N}_s u(x) := C_{N,s} \int_\Omega \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, \mathrm{d}y, \quad x \in \mathbb{R}^N \setminus \overline{\Omega}, \tag{3.9}$$

where $C_{N,s}$ is the constant given in (3.1). By ([22], Lem. 3.2), for every $u \in W^{s,2}(\mathbb{R}^N)$, we have that $\mathcal{N}_s u \in L^2(\mathbb{R}^N \setminus \Omega)$.

The following unique continuation property, which shall play an important role in the proof of our main results has been recently obtained in ([46], Thm. 3.10).

Lemma 3.3. *Let $\lambda>0$ be a real number and $\mathcal{O}\subset\mathbb{R}^N\setminus\overline{\Omega}$ an arbitrary non-empty open set. If $\varphi\in D((-\Delta)_D^s)$ satisfies*

$$(-\Delta)_D^s\varphi=\lambda\varphi \;\text{ in }\; \Omega \;\text{ and }\; \mathcal{N}_s\varphi=0 \;\text{ in }\; \mathcal{O},$$

then $\varphi=0$ in $\mathbb{R}^N$.

Remark 3.4. The following important identity has been recently proved in ([46], Rem. 3.11). Let $g\in W^{s,2}(\mathbb{R}^N\setminus\Omega)$ and $U_g\in W^{s,2}(\mathbb{R}^N)$ the associated unique weak solution of the Dirichlet problem (3.3). Then

$$\int_{\mathbb{R}^N\setminus\Omega} g\mathcal{N}_s\varphi_n\,\mathrm{d}x=-\lambda_n\int_\Omega\varphi_n U_g\,\mathrm{d}x. \tag{3.10}$$

For more details on the Dirichlet problem associated with the fractional Laplace operator, we refer the interested reader to [9–11, 13, 24, 35, 36, 43, 46] and their references.

The following integration by parts formula is contained in ([15], Lem. 3.3) for smooth functions. The version given here can be obtained by using a simple density argument (see *e.g.* [46]).

Proposition 3.5. *Let $u\in W^{s,2}(\mathbb{R}^N)$ be such that $(-\Delta)^s u\in L^2(\Omega)$. Then for every $v\in W^{s,2}(\mathbb{R}^N)$, we have*

$$\begin{aligned}\frac{C_{N,s}}{2}&\int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y\\&=\int_\Omega v(-\Delta)^s u\,\mathrm{d}x+\int_{\mathbb{R}^N\setminus\Omega}v\mathcal{N}_s u\,\mathrm{d}x.\end{aligned}\tag{3.11}$$

Remark 3.6. We mention the following facts regarding the identity (3.11).

(a) Firstly, we notice that

$$\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2=(\Omega\times\Omega)\cup(\Omega\times(\mathbb{R}^N\setminus\Omega))\cup((\mathbb{R}^N\setminus\Omega)\times\Omega).$$

(b) Secondly, if $u=0$ in $\mathbb{R}^N\setminus\Omega$ or $v=0$ in $\mathbb{R}^N\setminus\Omega$, then

$$\int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y,$$

so that the identity (3.11) becomes

$$\frac{C_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y=\int_\Omega v(-\Delta)^s u\,\mathrm{d}x+\int_{\mathbb{R}^N\setminus\Omega}v\mathcal{N}_s u\,\mathrm{d}x. \tag{3.12}$$

We conclude this section with the following convergence result.

Lemma 3.7. *Let $u\in W_0^{1,2}(\Omega)\hookrightarrow W_0^{s,2}(\overline{\Omega})$ be such that $(-\Delta)^s u,\ \Delta u\in L^2(\Omega)$. Then the following assertions hold.*

(a) For every $v\in W_0^{1,2}(\Omega)$,

$$\lim_{s\uparrow 1^-}\int_\Omega v(-\Delta)^s u\,\mathrm{d}x=-\int_\Omega v\Delta u\,\mathrm{d}x. \tag{3.13}$$

(b) For every $v \in W^{1,2}(\mathbb{R}^N)$,

$$\lim_{s\uparrow 1^-}\int_{\mathbb{R}^N\setminus\Omega} v\mathcal{N}_s u\,\mathrm{d}x = \int_{\partial\Omega} v\partial_\nu u\,\mathrm{d}\sigma, \tag{3.14}$$

where $\partial_\nu u$ is the normal derivative of u in direction of the outer normal vector $\vec{\nu}$.

We refer to ([12], Prop. 2.2) for Part (a) and to ([15], Prop. 5.1) for Part (b).

4. Series representation of solutions

In this section we give a representation in terms of series of weak solutions to the system (2.1) and the dual system (2.5). Evolution equations with non-homogeneous boundary or exterior conditions are in general not so easy to solve since one cannot apply directly semigroup methods due the fact that the associated operator is in general not a generator of a semigroup. For this reason, we shall give more details in the proofs. The representation of solutions in terms of series shall play an important role in the proof of our main results.

We recall that $(\varphi_n)_{n\in\mathbb{N}}$ denotes the orthonormal basis of eigenfunctions of the operator $(-\Delta)_D^s$ associated with the eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$.

Let $\delta \ge 0$ and set

$$\mathbf{D}_{\mathbf{n}}^{\delta} := \delta^2\lambda_n^2 - 4\lambda_n. \tag{4.1}$$

We have the following two situations.

– If $\delta > 0$, since $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots$ and $\lim_{n\to\infty}\lambda_n = +\infty$, it follows that there is a number $N_0 \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}$ such that $\delta^2\lambda_n < 4$ for all $n \le N_0$. In that case we shall use the following notations.

  (a) If $\mathbf{D}_{\mathbf{n}}^{\delta} \ge 0$, that is, if $\delta^2\lambda_n - 4 \ge 0$, then we let

$$\lambda_n^{\pm} := \frac{-\delta\lambda_n \pm \sqrt{\mathbf{D}_{\mathbf{n}}^{\delta}}}{2}. \tag{4.2}$$

  (b) If $\mathbf{D}_{\mathbf{n}}^{\delta} < 0$, that is, if $\delta^2\lambda_n - 4 < 0$, then we let

$$\widetilde{\lambda}_n^{\pm} := \frac{-\delta\lambda_n \pm i\sqrt{-\mathbf{D}_{\mathbf{n}}^{\delta}}}{2},\quad \alpha_n := \mathrm{Re}(\widetilde{\lambda}_n^{+}) = \frac{-\delta\lambda_n}{2} \text{ and } \beta_n = \mathrm{Im}(\widetilde{\lambda}_n^{+}) = \frac{\sqrt{-\mathbf{D}_{\mathbf{n}}^{\delta}}}{2}. \tag{4.3}$$

– If $\delta = 0$, then $\mathbf{D}_{\mathbf{n}}^{\mathbf{0}} := -4\lambda_n < 0$ for all $n \in \mathbb{N}$. In that case we let

$$\widetilde{\lambda}_n^{\pm} := \pm i\sqrt{\lambda_n},\ \ \alpha_n = 0 \ \text{ and } \ \beta_n = \sqrt{\lambda_n}. \tag{4.4}$$

Remark 4.1. An immediate and important consequence is the following. If $\mathbf{D}_{\mathbf{n}}^{\delta} \ge 0$, then we have that $\lambda_n^{\pm} < 0$ for all $n > N_0$, and in addition

$$\lambda_n^{+} \to -\frac{1}{\delta},\quad \lambda_n^{-} \to -\infty,\quad \text{as } \ n \to \infty. \tag{4.5}$$

This fact will be used in the proof of our main results.

4.1. Series solutions of the system (2.1)

Recall that we have shown in Section 2 that a solution (u, u_t) of (1.1) can be decomposed as $u = v + w$ where (v, v_t) solves (2.1) and (w, w_t) is a solution of (2.2). Let $\delta \geq 0$ and consider the system (2.2). That is,

$$\begin{cases} w_{tt} + (-\Delta)^s w + \delta(-\Delta)^s w_t = 0 & \text{in } \Omega \times (0,T), \\ w = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ w(\cdot,0) = u_0, \quad w_t(\cdot,0) = u_1 & \text{in } \Omega. \end{cases} \tag{4.6}$$

Let

$$W = \begin{pmatrix} w \\ w_t \end{pmatrix} \quad \text{and} \quad W_0 = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}.$$

Then (4.6) can be rewritten as the following first order Cauchy problem:

$$\begin{cases} W_t + \mathcal{A}_\delta W = 0 & \text{in } \Omega \times (0,T), \\ W(\cdot,0) = W_0 & \text{in } \Omega, \end{cases} \tag{4.7}$$

where the operator matrix $\mathcal{A}_\delta$ with domain $D(\mathcal{A}_\delta) := D((-\Delta)^s_D) \times D((-\Delta)^s_D)$ is given by

$$\mathcal{A}_\delta := \begin{pmatrix} 0 & -I \\ (-\Delta)^s_D & \delta(-\Delta)^s_D \end{pmatrix}. \tag{4.8}$$

Let $\mathcal{H} := W_0^{s,2}(\overline{\Omega}) \times L^2(\Omega)$ be the Hilbert space equipped with the scalar product

$$\begin{aligned} \langle (v_1,v_2),(w_1,w_2)\rangle_{\mathcal{H}} := &\int_\Omega v_1 w_1 \, \mathrm{d}x + \int_\Omega v_2 w_2 \, \mathrm{d}x \\ &+ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(v_1(x)-v_1(y))(w_1(x)-w_1(y))}{|x-y|^{N+2s}} \, \mathrm{d}x\mathrm{d}y. \end{aligned}$$

The following result is classical in an abstract form. We include the proof for the sake of completeness.

Lemma 4.2. *The operator $-\mathcal{A}_\delta$ generates a strongly continuous semigroup on $\mathcal{H}$.*

Proof. We prove the lemma in several steps. We shall apply the Lumer-Phillips theorem (see *e.g.* [4], Thm. 3.4.5) to the operator $\mathcal{B}_\delta = -\mathcal{A}_\delta - I$.

Step 1: We claim that $\mathcal{B}_\delta$ is a closed operator. Indeed, assume that $U_n \in D(\mathcal{B}_\delta)$ satisfies $U_n \to U$ in $\mathcal{H}$ and $\mathcal{B}_\delta U_n \to V$ in $\mathcal{H}$, as $n \to \infty$, where

$$U_n := \begin{pmatrix} u_1^n \\ u_2^n \end{pmatrix}, \quad U := \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \text{and} \quad V := \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

This means that $u_1^n \to u_1$ in $W_0^{s,2}(\overline{\Omega})$, $u_2^n \to u_2$ in $L^2(\Omega)$ and

$$\begin{cases} -u_1^n + u_2^n \to v_1 & \text{in } W_0^{s,2}(\overline{\Omega}), \\ -u_2^n - (-\Delta)^s_D(u_1^n + \delta u_2^n) \to v_2 & \text{in } L^2(\Omega), \end{cases}$$

as $n \to \infty$. Thus $u_1^n + u_2^n \to u_1 + u_2$ and $(-\Delta)_D^s(u_1^n + \delta u_2^n) \to -(u_2 + v_2)$, in $L^2(\Omega)$, as $n \to \infty$. Recall that $(-\Delta)_D^s$ is a closed operator in $L^2(\Omega)$. So, $u_1 + \delta u_2 \in D((-\Delta)_D^s)$ and $(-\Delta)_D^s(u_1 + \delta u_2) = -(u_2 + v_2)$. We have shown that $U \in D(\mathcal{B}_\delta) = D(\mathcal{A}_\delta)$ and $\mathcal{B}_\delta(U) = V$. Hence, the operator $\mathcal{B}_\delta$ is closed.

Step 2: We show that $\mathcal{B}_\delta$ is a dissipative operator, that is,

$$\lambda\|U\|_{\mathcal{H}} \le \|\mathcal{B}_\delta U - \lambda U\|_{\mathcal{H}} \text{ for any } \lambda > 0 \text{ and } U \in D(\mathcal{B}_\delta). \tag{4.9}$$

Indeed, let $\lambda > 0$ and $U \in D(\mathcal{B}_\delta)$. Then

$$\begin{aligned}\langle \mathcal{B}_\delta U, U\rangle_{\mathcal{H}} =& -(u_1,u_1)_{L^2(\Omega)} + (u_2,u_1)_{L^2(\Omega)}\\ &- \Big(((-\Delta)_D^s)^{1/2}u_1, ((-\Delta)_D^s)^{1/2}u_1\Big)_{L^2(\Omega)} + \Big(((-\Delta)_D^s)^{1/2}u_2, ((-\Delta)_D^s)^{1/2}u_1\Big)_{L^2(\Omega)}\\ &- \Big((-\Delta)_D^s u_1, u_2\Big)_{L^2(\Omega)} - \delta\Big((-\Delta)_D^s u_2, u_2\Big)_{L^2(\Omega)} - (u_2,u_2)_{L^2(\Omega)}.\end{aligned}$$

Since

$$\Big(((-\Delta)_D^s)^{1/2}u_2, ((-\Delta)_D^s)^{1/2}u_1\Big)_{L^2(\Omega)} = \Big(u_2, (-\Delta)_D^s u_1\Big)_{L^2(\Omega)} = \Big((-\Delta)_D^s u_1, u_2\Big)_{L^2(\Omega)},$$

it follows from the preceding identity that

$$\begin{aligned}\langle \mathcal{B}_\delta U, U\rangle_{\mathcal{H}} =& -(u_1,u_1)_{L^2(\Omega)} + (u_2,u_1)_{L^2(\Omega)}\\ &- \Big(((-\Delta)_D^s)^{1/2}u_1, ((-\Delta)_D^s)^{1/2}u_1\Big)_{L^2(\Omega)} + \Big((-\Delta)_D^s u_1, u_2\Big)_{L^2(\Omega)}\\ &- \Big((-\Delta)_D^s u_1, u_2\Big)_{L^2(\Omega)} - \delta\Big((-\Delta)_D^s u_2, u_2\Big)_{L^2(\Omega)} - (u_2,u_2)_{L^2(\Omega)}\\ =& -(u_1,u_1)_{L^2(\Omega)} + (u_2,u_1)_{L^2(\Omega)} - \Big(((-\Delta)_D^s)^{1/2}u_1, ((-\Delta)_D^s)^{1/2}u_1\Big)_{L^2(\Omega)}\\ &- \delta\Big(((-\Delta)_D^s)^{1/2}u_2, ((-\Delta)_D^s)^{1/2}u_2\Big)_{L^2(\Omega)} - (u_2,u_2)_{L^2(\Omega)}.\end{aligned} \tag{4.10}$$

Since $|(u_1,u_2)_{L^2(\Omega)}| \le \frac{1}{2}\Big(\|u_1\|^2_{L^2(\Omega)} + \|u_2\|^2_{L^2(\Omega)}\Big)$, it follows from (4.10) that

$$\langle \mathcal{B}_\delta U, U\rangle_{\mathcal{H}} \le -\left(\frac{1}{2}\|u_1\|^2_{L^2(\Omega)} + \frac{1}{2}\|u_2\|^2_{L^2(\Omega)} + \|((-\Delta)_D^s)^{1/2}u_1\|^2_{L^2(\Omega)} + \|((-\Delta)_D^s)^{1/2}u_2\|^2_{L^2(\Omega)}\right) \le 0.$$

Thus

$$\langle \mathcal{B}_\delta U - \lambda U, U\rangle_{\mathcal{H}} = \langle \mathcal{B}_\delta U, U\rangle_{\mathcal{H}} - \lambda\|U\|^2_{\mathcal{H}} \le -\lambda\|U\|^2_{\mathcal{H}}, \;\; U \in D(\mathcal{B}_\delta).$$

This implies that $\lambda\|U\|^2_{\mathcal{H}} \le |\langle \mathcal{B}_\delta U - \lambda U, U\rangle_{\mathcal{H}}| \le \|\mathcal{B}_\delta U - \lambda U\|_{\mathcal{H}}\|U\|_{\mathcal{H}}$ and we have shown (4.9).

Step 3: Since $D((-\Delta)_D^s)$ is dense in $L^2(\Omega)$ and in $W_0^{s,2}(\overline{\Omega})$, it follows that $D(\mathcal{B}_\delta)$ is dense in $\mathcal{H}$.

Step 4: We show that $\mathcal{R}(I-\mathcal{B}_\delta)=\mathcal{H}$. We have to solve the equation $(I-\mathcal{B}_\delta)U=V$ for every $V\in\mathcal{H}$. That is, the system

$$\begin{cases} 2u_1 - u_2 & = v_1,\\ 2u_2 + (-\Delta)_D^s(u_1+\delta u_2) & = v_2. \end{cases}$$

Since $(-\Delta)_D^s$ is a non-negative self-adjoint operator in $L^2(\Omega)$, a simple calculation gives

$$\begin{aligned} u_1 &= \Big((-\Delta)_D^s + 2\delta I\Big)\Big(4I+(1+2\delta)(-\Delta)_D^s\Big)^{-1}v_1 + \Big(4I+(1+2\delta)(-\Delta)_D^s\Big)^{-1}v_2\\ u_2 &= \Big(4(\delta-1)I+(1-2\delta)(-\Delta)_D^s\Big)\Big(4I+3(-\Delta)_D^s\Big)^{-1}v_1 + 2\Big(4I+(1+2\delta)(-\Delta)_D^s\Big)^{-1}v_2, \end{aligned}$$

and we have proved that $\mathcal{R}(I-\mathcal{B}_\delta)=\mathcal{H}$.

Finally, it follows from the Lumer-Phillips theorem, that the operator $\mathcal{B}_\delta$ generates a strongly continuous semigroup $(e^{t\mathcal{B}_\delta})_{t\geq 0}$ on $\mathcal{H}$ and thus, the operator $-\mathcal{A}_\delta$ generates the strongly continuous semigroup $(e^{-t\mathcal{A}_\delta})_{t\geq 0} = (e^t e^{t\mathcal{B}_\delta})_{t\geq 0}$ on $\mathcal{H}$. The proof is finished. □

Remark 4.3. As a consequence of Lemma 4.2, it follows from semigroup theory (see *e.g.* [4], Chap. 3) that, for every $(u_0,u_1)\in W_0^{s,2}(\overline{\Omega})\times L^2(\Omega)$, the first order Cauchy problem (4.7) has a unique strong solution $W\in C([0,T];\mathcal{H})$. Hence, the system (4.6), has a unique (weak) solution (w,w_t) satisfying

$$w\in C([0,T];W_0^{s,2}(\overline{\Omega}))\cap C^1([0,T];L^2(\Omega)). \tag{4.11}$$

Next we give the representation of solutions in terms of series.

Proposition 4.4. *Let $(u_0,u_1)\in W_0^{s,2}(\overline{\Omega})\times L^2(\Omega)$. Then the solution (w,w_t) of (4.6) is given by*

$$w(x,t)=\sum_{n=1}^{\infty}\Big(A_n(t)(u_0,\varphi_n)_{L^2(\Omega)}+B_n(t)(u_1,\varphi_n)_{L^2(\Omega)}\Big)\varphi_n(x), \tag{4.12}$$

where

$$A_n(t):=\begin{cases}\Big(\cos(\beta_n t)-\dfrac{\alpha_n}{\beta_n}\sin(\beta_n t)\Big)e^{\alpha_n t} & \text{if } n\leq N_0,\\ \dfrac{\lambda_n^- e^{\lambda_n^+ t}-\lambda_n^+ e^{\lambda_n^- t}}{\lambda_n^- - \lambda_n^+} & \text{if } n> N_0,\end{cases} \tag{4.13}$$

and

$$B_n(t):=\begin{cases}\dfrac{\sin(\beta_n t)}{\beta_n}e^{\alpha_n t} & \text{if } n\leq N_0,\\ \dfrac{e^{\lambda_n^- t}-e^{\lambda_n^+ t}}{\lambda_n^- - \lambda_n^+} & \text{if } n> N_0.\end{cases} \tag{4.14}$$

Here, $\lambda_n^\pm$, α_n and β_n are the real numbers given in (4.2), (4.3) and (4.4).

Proof. Using the spectral theorem for self-adjoint operators, we can proceed as follows. We look for a solution (w, w_t) of (4.6) in the form

$$w(x,t) = \sum_{n=1}^{\infty} (w(\cdot,t), \varphi_n)_{L^2(\Omega)}\, \varphi_n(x). \tag{4.15}$$

For the sake of simplicity we let $w_n(t) = (w(\cdot,t), \varphi_n)_{L^2(\Omega)}$. Replacing (4.15) in the first equation of (4.6), then multiplying both sides with φ_k and integrating over Ω, we get that $w_n(t)$ satisfies the following ordinary differential equation:

$$w_n''(t) + \lambda_n w_n(t) + \delta\lambda_n w_n'(t) = 0. \tag{4.16}$$

Solving (4.16), calculating, letting $u_{0,n} = (u_0, \varphi_n)_{L^2(\Omega)}$ and $u_{1,n} = (u_1, \varphi_n)_{L^2(\Omega)}$, we get

$$w(x,t) = \sum_{n=1}^{N_0} \Big(a_n^1 \cos(\beta_n t) + a_n^2 \sin(\beta_n t)\Big) e^{\alpha_n t} \varphi_n(x) + \sum_{n=N_0+1}^{\infty} \Big(a_n^3 e^{\lambda_n^+ t} + a_n^4 e^{\lambda_n^- t}\Big)\varphi_n(x), \tag{4.17}$$

where

$$a_n^1 = u_{0,n}, \quad a_n^2 = \frac{u_{1,n} - \alpha_n u_{0,n}}{\beta_n} \;\text{ for }\; n \le N_0,$$

and

$$a_n^3 = \frac{u_{0,n}\lambda_n^- - u_{1,n}}{\lambda_n^- - \lambda_n^+}, \quad a_n^4 = \frac{u_{1,n} - u_{0,n}\lambda_n^+}{\lambda_n^- - \lambda_n^+} \;\text{ for }\; n > N_0.$$

Therefore, we obtain the following expression of w:

$$\begin{aligned} w(x,t) = &\sum_{n=1}^{N_0} \left[u_{0,n}\left(\cos(\beta_n t) - \frac{\alpha}{\beta_n}\sin(\beta_n t)\right) + u_{1,n}\frac{\sin(\beta_n t)}{\beta}\right] e^{\alpha_n t}\varphi_n(x) \\ &+ \sum_{n=N_0+1}^{\infty} \left[u_{0,n}\left(\frac{\lambda_n^- e^{\lambda_n^+ t} - \lambda_n^+ e^{\lambda_n^- t}}{\lambda_n^- - \lambda_b^+}\right) + u_{1,n}\left(\frac{e^{\lambda_n^- t} - e^{\lambda_n^+ t}}{\lambda_n^- - \lambda_n^+}\right)\right]\varphi_n(x). \end{aligned} \tag{4.18}$$

Let $A_n(t)$ and $B_n(t)$ be given by (4.13) and (4.14), respectively. Then (4.12) follows from (4.18).

A simple calculation gives

$$w(x,0) = \sum_{n=1}^{\infty} \Big(A_n(0)u_{0,n} + B_n(0)u_{1,n}\Big)\varphi_n(x) = \sum_{n=1}^{\infty} u_{0,n}\varphi_n(x) = u_0(x),$$

and

$$w_t(x,0) = \sum_{n=1}^{\infty} \Big(A_n'(0)u_{0,n} + B_n'(0)u_{1,n}\Big)\varphi_n(x) = \sum_{n=1}^{\infty} u_{1,n}\varphi_n(x) = u_1(x).$$

It is straightforward to verify that w given in (4.12) has the regularity (4.11). Since we are not interested with solutions of (4.6), we will not go into details. The proof is finished. $\square$

Next, we consider the non-homogeneous system (2.1), that is,

$$\begin{cases} v_{tt} + (-\Delta)^s v + \delta(-\Delta)^s v_t = 0 & \text{in } \Omega \times (0,T), \\ v = g & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ v(\cdot,0) = 0, v_t(\cdot,0) = 0 & \text{in } \Omega. \end{cases} \tag{4.19}$$

We have the following result.

Theorem 4.5. *For every $g \in \mathcal{D}((\mathbb{R}^N \setminus \Omega) \times (0,T))$, the system (4.19) has a unique weak (classical solution) (v, v_t) satisfying $v \in C^\infty([0,T]; W^{s,2}(\mathbb{R}^N))$ and is given by*

$$v(x,t) = \sum_{n=1}^{\infty} \left(\int_0^t \Big(g(\cdot,\tau), \mathcal{N}_s \varphi_n \Big)_{L^2(\mathbb{R}^N \setminus \Omega)} \frac{1}{\lambda_n} B_n''(t-\tau) \mathrm{d}\tau \right) \varphi_n(x). \tag{4.20}$$

Moreover, there is a constant $C > 0$ such that for all $t \in [0,T]$ and $m \in \mathbb{N}_0$,

$$\|\partial_t^m v(\cdot,t)\|_{W^{s,2}(\mathbb{R}^N)} \le C \left(\|\partial_t^{m+2} g\|_{L^\infty((0,T);W^{s,2}(\mathbb{R}^N \setminus \Omega))} + \|\partial_t^m g(\cdot,t)\|_{W^{s,2}(\mathbb{R}^N \setminus \Omega)} \right). \tag{4.21}$$

Proof. We proof the theorem in several steps.

Step 1: Consider the following elliptic Dirichlet exterior problem:

$$\begin{cases} (-\Delta)^s \phi = 0 & \Omega, \\ \phi = g & \mathbb{R}^N \setminus \Omega. \end{cases} \tag{4.22}$$

We have shown in Proposition 3.2 that for every $g \in W^{s,2}(\mathbb{R}^N \setminus \Omega)$, there exists a unique function $\phi \in W^{s,2}(\mathbb{R}^N)$ solution of (4.22), and there is a constant $C > 0$ such that

$$\|\phi\|_{W^{s,2}(\mathbb{R}^N)} \le C \|g\|_{W^{s,2}(\mathbb{R}^N \setminus \Omega)}. \tag{4.23}$$

Since g depends on (x,t), then ϕ also depends on (x,t). If in (4.22) one replaces g by $\partial_t^m g$, $m \in \mathbb{N}$, then the associated unique solution is given by $\partial_t^m \phi$ for every $m \in \mathbb{N}_0$. From this, we can deduce that $\phi \in C^\infty([0,T]; W^{s,2}(\mathbb{R}^N))$.

Now let (v, v_t) be a solution of (4.19) and set $w := v - \phi$. Then a simple calculation gives

$$\begin{aligned} w_{tt} + (-\Delta)^s w + \delta(-\Delta)^s w_t &= v_{tt} - \phi_{tt} + (-\Delta)^s v - (-\Delta)^s \phi + \delta(-\Delta)^s v_t - \delta(-\Delta)^s \phi_t \\ &= v_{tt} + (-\Delta)^s v + \delta(-\Delta)^s v_t - \phi_{tt} = -\phi_{tt} \quad \text{in } \Omega \times (0,T). \end{aligned}$$

In addition

$$w = v - \phi = g - g = 0 \ \text{ in } \ (\mathbb{R}^N \setminus \Omega) \times (0,T),$$

and

$$\begin{cases} w(\cdot,0) = v(\cdot,0) - \phi(\cdot,0) = -\phi(\cdot,0) & \text{in } \Omega, \\ w_t(\cdot,0) = v_t(\cdot,0) - \phi_t(\cdot,0) = -\phi_t(\cdot,0) & \text{in } \Omega. \end{cases}$$

Since $g \in \mathcal{D}((\mathbb{R}^N \setminus \Omega) \times (0,T))$, we have that $\phi(\cdot,0) = \partial_t \phi(\cdot,0) = 0$ in Ω. We have shown that a solution (v, v_t) of (4.19) can be decomposed as $v = \phi + w$, where (w, w_t) solves the system

$$\begin{cases} w_{tt} + (-\Delta)^s w + \delta(-\Delta)^s w_t = -\phi_{tt} & \text{in } \Omega \times (0,T), \\ w = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ w(\cdot,0) = 0, \ \partial_t w(\cdot,0) = 0 & \text{in } \Omega. \end{cases} \tag{4.24}$$

We notice that $\phi_{tt} \in C^\infty([0,T]; W^{s,2}(\mathbb{R}^N))$.

Step 2: We observe that letting

$$W = \begin{pmatrix} w \\ w_t \end{pmatrix} \quad \text{and} \quad \Phi_{tt} = \begin{pmatrix} 0 \\ -\phi_{tt} \end{pmatrix},$$

then the system (4.24) can be rewritten as the following first order Cauchy problem

$$\begin{cases} W_t + \mathcal{A}_\delta W = \Phi_{tt} & \text{in } \Omega \times (0,T), \\ W(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{in } \Omega, \end{cases} \tag{4.25}$$

where $\mathcal{A}_\delta$ is the matrix operator defined in (4.8). Proceeding as the proof of Proposition 4.4 and using semigroup theory, we get that (4.25) has a unique classical solution W and hence, (4.24) has a unique weak (classical) solution (w, w_t) such that $w \in C^\infty([0,T]; W^{s,2}(\mathbb{R}^N))$ and is given by

$$w(x,t) = -\sum_{n=1}^{\infty} \left(\int_0^t (\phi_{\tau\tau}(\cdot,\tau), \varphi_n)_{L^2(\Omega)} B_n(t-\tau) \mathrm{d}\tau \right) \varphi_n(x), \tag{4.26}$$

where we recall that B_n is given in (4.14). Integrating (4.26) by parts we get that

$$\begin{aligned} w(x,t) = &-\sum_{n=1}^{\infty} \left(\int_0^t (\phi(\cdot,\tau), \varphi_n)_{L^2(\Omega)} B_n''(t-\tau) \mathrm{d}\tau \right) \varphi_n(x) \\ &-\sum_{n=1}^{\infty} \left((\phi_\tau(\cdot,\tau), \varphi_n)_{L^2(\Omega)} B_n(t-\tau) \Big|_{\tau=0}^{\tau=t} \right) \varphi_n(x) \\ &-\sum_{n=1}^{\infty} \left((\phi(\cdot,\tau), \varphi_n)_{L^2(\Omega)} B_n'(t-\tau) \Big|_{\tau=0}^{\tau=t} \right) \varphi_n(x). \end{aligned}$$

We observe that $B_n(0) = 0$ and $B_n'(0) = 1$ for all $n \in \mathbb{N}$. Since $\phi(\cdot,0) = \phi_t(\cdot,0) = 0$, we get

$$w(x,t) = -\phi(x,t) - \sum_{n=1}^{\infty} \left(\int_0^t (\phi(\cdot,\tau), \varphi_n)_{L^2(\Omega)} B_n''(t-\tau) \mathrm{d}\tau \right) \varphi_n(x). \tag{4.27}$$

Using the fact that φ_n satisfies (3.6) and the integration by parts formula (3.11)–(3.12), we get

$$\begin{aligned}\Big(\phi(\cdot,\tau),\lambda_n\varphi_n\Big)_{L^2(\Omega)} &= \Big(\phi(\cdot,\tau),(-\Delta)^s\varphi_n\Big)_{L^2(\Omega)}\\ &= \Big((-\Delta)^s\phi(\cdot,\tau),\varphi_n\Big)_{L^2(\Omega)} - \int_{\mathbb{R}^N\setminus\Omega}\left(\phi\mathcal{N}_s\varphi_n - \varphi_n\mathcal{N}_s\phi\right)\,\mathrm{d}x\\ &= -\int_{\mathbb{R}^N\setminus\Omega} g\mathcal{N}_s\varphi_n \mathrm{d}x. \end{aligned} \tag{4.28}$$

From (4.27) and (4.28) we can deduce that

$$w(x,t) = -\phi(x,t) + \sum_{n=1}^{\infty}\left(\int_0^t \Big(g(\cdot,\tau),\mathcal{N}_s\varphi_n\Big)_{L^2(\mathbb{R}^N\setminus\Omega)}\frac{1}{\lambda_n}B_n''(t-\tau)\mathrm{d}\tau\right)\varphi_n(x).$$

We have shown (4.20). Since $\phi, w\in C^\infty([0,T];W^{s,2}(\mathbb{R}^N))$, then $v\in C^\infty([0,T];W^{s,2}(\mathbb{R}^N))$.

Step 3: Using (4.26) and calculating, we get that for every $t\in[0,T]$,

$$\begin{aligned}\|(-\Delta)_D^s w(\cdot,t)\|^2_{L^2(\Omega)} &= \left\|\sum_{n=1}^{\infty}\lambda_n\left(\int_0^t\left(\phi_{\tau\tau}(\cdot,\tau),\varphi_n\right)_{L^2(\Omega)}B_n(t-\tau)\mathrm{d}\tau\right)\varphi_n\right\|^2_{L^2(\Omega)}\\ &\le \int_0^t\left\|\sum_{n=1}^{\infty}\lambda_n\left(\phi_{\tau\tau}(\cdot,\tau),\varphi_n\right)_{L^2(\Omega)}B_n(t-\tau)\,\mathrm{d}\tau\varphi_n\right\|^2_{L^2(\Omega)}\\ &\le \int_0^t\sum_{n=1}^{\infty}\Big|\left(\phi_{\tau\tau}(\cdot,\tau),\varphi_n\right)_{L^2(\Omega)}\Big|^2\Big|\lambda_nB_n(t-\tau)\Big|^2\,\mathrm{d}\tau.\end{aligned}\tag{4.29}$$

We claim that there is a constant $C>0$ (independent of n) such that

$$|\lambda_nB_n(t)|\le C,\;\;\forall\, n\in\mathbb{N}\;\text{ and }\; t\in[0,T]. \tag{4.30}$$

We have the following situations.

– If $\delta=0$, then $\mathbf{D_n^0}<0$ so that $\alpha_n=0$, $\beta_n=\sqrt{\lambda_n}$, $\lambda_n^{\pm}=0$ for every $n\in\mathbb{N}$. Thus

$$|\lambda_nB_n(t)|^2 = \frac{\lambda_n^2\sin^2(\sqrt{\lambda_n}t)}{\lambda_n^2}\le 1,\;\;\forall\, n\in\mathbb{N}.$$

– If $\delta>0$ and $N_0=0$, then since $\lambda_n^{\pm}<0$, we have that

$$|\lambda_nB_n(t)|^2 = \lambda_n^2\frac{e^{2\lambda_n^-t}-2e^{(\lambda_n^-+\lambda_n^+)t}+e^{2\lambda_n^+t}}{(\lambda_n^--\lambda_n^+)^2}\le\frac{4\lambda_n^2}{\delta^2\lambda_n^2-4\lambda_n}\le\frac{4\lambda_1}{\delta\lambda_1-4},\;\;\forall\, n\in\mathbb{N}.$$

– If $\delta>0$ and $1\le N_0<\infty$, then since $\alpha_n<0$, we have that

$$|\lambda_nB_n(t)|^2 = \frac{\lambda_n^2\sin^2(\beta_nt)}{\beta_n^2}e^{2\alpha_nt}\le\frac{\lambda_n^2}{\beta_n^2}=\frac{4\lambda_n}{4-\delta^2\lambda_n}\le\frac{4\lambda_{N_0}}{4-\delta^2\lambda_{N_0}},\;\;\forall\, 1\le n\le N_0,$$

and since $\lambda_n^\pm < 0$ for all $n > N_0$, we have that

$$|\lambda_n B_n(t)|^2 = \lambda_n^2 \frac{e^{2\lambda_n^- t} - 2e^{(\lambda_n^- + \lambda_n^+)t} + e^{2\lambda_n^+ t}}{(\lambda_n^- - \lambda_n^+)^2} \leq \frac{4\lambda_n^2}{\delta^2\lambda_n^2 - 4\lambda_n} \leq \frac{4\lambda_1}{\delta\lambda_1 - 4}, \ \ \forall\, n > N_0.$$

The proof of the claim (4.30) is complete.

It follows from (4.29), (4.30) and (4.23) that for every $t \in [0, T]$,

$$\begin{aligned} \|(-\Delta)_D^s w(\cdot,t)\|_{L^2(\Omega)}^2 &\leq C\int_0^t \|\phi_{\tau\tau}(\cdot,\tau)\|_{L^2(\Omega)}^2\,\mathrm{d}\tau \leq C\int_0^t \|\phi_{\tau\tau}(\cdot,\tau)\|_{W^{s,2}(\mathbb{R}^N)}^2\,\mathrm{d}\tau \\ &\leq C\int_0^t \|g_{\tau\tau}(\cdot,\tau)\|_{W^{s,2}(\mathbb{R}^N\setminus\Omega)}^2\,\mathrm{d}\tau \leq CT\|g_{tt}\|_{L^\infty((0,T);W^{s,2}(\mathbb{R}^N\setminus\Omega))}^2. \end{aligned} \tag{4.31}$$

Using (4.31), we get that for every $t \in [0, T]$,

$$\begin{aligned} \|v(\cdot,t)\|_{W^{s,2}(\mathbb{R}^N)} &\leq C\left(\|(-\Delta)_D^s w(\cdot,t)\|_{L^2(\Omega)} + \|\phi(\cdot,t)\|_{W^{s,2}(\mathbb{R}^N)}\right) \\ &\leq C\left(\|g_{tt}\|_{L^\infty((0,T);W^{s,2}(\mathbb{R}^N\setminus\Omega))} + \|g(\cdot,t)\|_{W^{s,2}(\mathbb{R}^N\setminus\Omega)}\right). \end{aligned}$$

We have shown (4.21) for $m = 0$. Proceeding by induction on m we get (4.21) for $m \in \mathbb{N}_0$. The proof is finished. ☐

Next, we give the following result of existence and representation of solutions.

Corollary 4.6. *For every $(u_0, u_1) \in W_0^{s,2}(\overline{\Omega}) \times L^2(\Omega)$ and $g \in \mathcal{D}((\mathbb{R}^N \setminus \Omega) \times (0, T))$, the system (1.1) has a unique weak solution (u, u_t) given by*

$$\begin{aligned} u(x,t) =& \sum_{n=1}^\infty \Big(A_n(t)u_{0,n} + B_n(t)u_{1,n}\Big)\varphi_n(x) \\ &+ \sum_{n=1}^\infty \left(\int_0^t \Big(g(\cdot,\tau), \mathcal{N}_s\varphi_n\Big)_{L^2(\mathbb{R}^N\setminus\Omega)} \frac{1}{\lambda_n} B_n''(t-\tau)\mathrm{d}\tau\right)\varphi_n(x). \end{aligned}$$

Proof. Since a weak solution (u, u_t) of (1.1) can de decomposed into $u = v + w$ where (v, v_t) solves the system (2.1) and (w, w_t) is the unique weak solution of (2.2), the result follows from Proposition 4.4 and Theorem 4.5. The proof is finished. ☐

We conclude this subsection with the following remark.

Remark 4.7. We mention the following facts regarding the total energy of our system.

(a) If $g = 0$, then the system (1.1) can be rewritten as an abstract Cauchy problem (see (4.6) and (4.7)) involving the self-adjoint operator $(-\Delta)_D^s$ defined in (3.4). In that case, if $\delta > 0$, then the total energy of the associated system is given by

$$E_u(t) := \frac{1}{2}\left(\|u_t(\cdot,t)\|_{L^2(\Omega)}^2 + \|(u(\cdot,t)\|_{W_0^{s,2}(\overline{\Omega})}^2\right).$$

Using the abstract result in ([25], Thm. 1.1), we can deduce that, if $u_0 \in W_0^{s,2}(\overline{\Omega})$ and $u_1 \in L^2(\Omega)$, then there are two positive constants C and η such that

$$\begin{aligned} E_u(t) \leq & C \left(\|e^{-\eta t(-\Delta)_D^s} u_0\|^2_{L^2(\Omega)} + \|e^{-\eta t(-\Delta)_D^s} u_1\|^2_{L^2(\Omega)} \right) \\ & + Ce^{-\eta t} \left(\|u_0\|^2_{W_0^{s,2}(\overline{\Omega})} + \|u_1\|^2_{L^2(\Omega)} \right), \qquad t \geq 0, \end{aligned} \tag{4.32}$$

where $(e^{-\tau(-\Delta)_D^s})_{\tau \geq 0}$ is the semigroup on $L^2(\Omega)$ generated by $-(-\Delta)_D^s$. Since the first eigenvalue $\lambda_1 = \lambda_1(s)$ of $(-\Delta)_D^s$ is strictly positive (see (3.5)), then using semigroup theory, we have that there is a constant $C > 0$ such that

$$\|e^{-t(-\Delta)_D^s} f\|_{L^2(\Omega)} \leq Ce^{-\lambda_1 t} \|f\|_{L^2(\Omega)}, \quad t \geq 0, \quad f \in L^2(\Omega).$$

This estimate together with (4.32) imply the following exponential decay of the energy:

$$E_u(t) \leq C \Big(e^{-2\lambda_1 \eta t} + e^{-2\eta t} \Big) \Big(\|u_0\|^2_{W_0^{s,2}(\overline{\Omega})} + \|u_1\|^2_{L^2(\Omega)} \Big), \qquad t \geq 0. \tag{4.33}$$

(b) If g is non-zero, then a semigroup method cannot be used and we do not know if we have an exponential decay as in (4.33) of the associated energy.

4.2. Series solutions of the dual system

Now we consider the dual system (2.5). That is, the backward system

$$\begin{cases} \psi_{tt} + (-\Delta)^s \psi - \delta(-\Delta)^s \psi_t = 0 & \text{in } \Omega \times (0,T), \\ \psi = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ \psi(\cdot, T) = \psi_0, \quad -\psi_t(\cdot, T) = \psi_1 & \text{in } \Omega, \end{cases} \tag{4.34}$$

Let $\psi_{0,n} := (\psi_0, \varphi_n)_{L^2(\Omega)}$ and $\psi_{1,n} := (\psi_1, \varphi_n)_{L^2(\Omega)}$. We have the following existence result.

Theorem 4.8. *For every $(\psi_0, \psi_1) \in W_0^{s,2}(\overline{\Omega}) \times L^2(\Omega)$, the dual system (4.34) has a unique weak solution (ψ, ψ_t) given by*

$$\psi(x,t) = \sum_{n=1}^{\infty} \Big(\psi_{0,n} C_n(T-t) + \psi_{1,n} D_n(T-t) \Big) \varphi_n(x), \tag{4.35}$$

where $C_n(t) = A_n(t)$, $D_n(t) = -B_n(t)$ and we recall that $A_n(t)$ and $B_n(t)$ are given in (4.13) and (4.14), respectively. In addition the following assertions hold.

(a) There is a constant $C > 0$ such that for all $t \in [0,T]$,

$$\|\psi(\cdot,t)\|^2_{W_0^{s,2}(\overline{\Omega})} + \|\psi_t(\cdot,t)\|^2_{L^2(\Omega)} \leq C \left(\|\psi_0\|^2_{W_0^{s,2}(\overline{\Omega})} + \|\psi_1\|^2_{L^2(\Omega)} \right), \tag{4.36}$$

and

$$\|\psi_{tt}(\cdot,t)\|^2_{W^{-s,2}(\overline{\Omega})} \leq \left(\|\psi_0\|^2_{W_0^{s,2}(\overline{\Omega})} + \|\psi_1\|^2_{L^2(\Omega)} \right). \tag{4.37}$$

(b) We have that $\psi \in C([0,T]; D((-\Delta)_D^s)) \cap L^\infty((0,T); L^2(\Omega))$.

(c) The mapping

$$[0,T) \ni t \mapsto \mathcal{N}_s\psi(\cdot,t) \in L^2(\mathbb{R}^N \setminus \Omega),$$

can be analytically extended to the half-plane $\Sigma_T := \{z \in \mathbb{C} : \operatorname{Re}(z) < T\}$.

The proof of the theorem uses heavily the following result.

Lemma 4.9. *There is a constant $C > 0$ (independent of n) such that for every $t \in [0,T]$,*

$$\max\left\{|C_n(t)|^2, \left|\lambda_n^{\frac{1}{2}} C_n(t)\right|^2, |C_n'(t)|^2\right\} \le C, \tag{4.38}$$

and

$$\max\left\{\left|\lambda_n^{\frac{1}{2}} D_n(t)\right|^2, |\lambda_n D_n(t)|^2, |D_n'(t)|^2, \left|\lambda_n^{\frac{1}{2}} D_n'(t)\right|^2\right\} \le C. \tag{4.39}$$

Proof. Firstly, we notice that it suffices to prove (4.38) and (4.39) for $n > N_0$. Secondly, we recall that $\lambda_n^{\pm} < 0$ for every $n \in \mathbb{N}$. Thirdly, it is easy to show that there is a constant $C > 0$ such that $\left|\lambda_n^{\pm} e^{\lambda_n^{\pm} t}\right| \le C$ for every $n > N_0$. From this estimate, we can deduce (4.38) and (4.39) by using some easy computations as in the proof of (4.30). ☐

Proof of Theorem 4.8. Let

$$\psi_0 = \sum_{n=1}^{\infty} \psi_{0,n}\varphi_n, \quad \psi_1 = \sum_{n=1}^{\infty} \psi_{1,n}\varphi_n.$$

We proof the theorem in several steps. Here we include more details.

Step 1: Proceeding in the same way as the proof of Proposition 4.4, we easily get that

$$\psi(x,t) = \sum_{n=1}^{\infty} \Big[C_n(T-t)\psi_{0,n} + D_n(T-t)\psi_{1,n}\Big]\varphi_n(x), \tag{4.40}$$

where we recall that $C_n(t) = A_n(t)$ and $D_n(t) = -B_n(t)$. In addition, a simple calculation gives $\psi(x,T) = \psi_0(x)$ and $\psi_t(x,T) = -\psi_1(x)$ for a.e. $x \in \Omega$.

Let us show that ψ satisfies the regularity and variational identity requirements. Let $1 \le n \le m$ and set

$$\psi_m(x,t) := \sum_{n=1}^{m} \Big[C_n(T-t)\psi_{0,n} + D_n(T-t)\psi_{1,n}\Big]\varphi_n(x).$$

For every $m, \tilde{m} \in \mathbb{N}$ with $m > \tilde{m}$ and $t \in [0,T]$, we have that

$$\begin{aligned}\|\psi_m(x,t) - \psi_{\tilde{m}}(x,t)\|^2_{W_0^{s,2}(\overline{\Omega})} &= \sum_{n=\tilde{m}+1}^{m} \left|\lambda_n^{\frac{1}{2}} C_n(T-t)\psi_{0,n} + \lambda_n^{\frac{1}{2}} D_n(T-t)\psi_{1,n}\right|^2 \\ &\le 2\sum_{n=\tilde{m}+1}^{m} \left|\lambda_n^{\frac{1}{2}} C_n(T-t)\psi_{0,n}\right|^2 + 2\sum_{n=\tilde{m}+1}^{m} \left|\lambda_n^{\frac{1}{2}} D_n(T-t)\psi_{1,n}\right|^2.\end{aligned} \tag{4.41}$$

Using (4.38) and (4.39) we get from (4.41) that for every $m, \tilde{m} \in \mathbb{N}$ with $m > \tilde{m}$ and $t \in [0,T]$,

$$\|\psi_m(x,t) - \psi_{\tilde{m}}(x,t)\|^2_{W_0^{s,2}(\overline{\Omega})} \leq C\left(\sum_{n=\tilde{m}+1}^{m}\left|\lambda_n^{\frac{1}{2}}\psi_{0,n}\right|^2 + \sum_{n=\tilde{m}+1}^{m}\left|\psi_{1,n}\right|^2\right) \longrightarrow 0 \text{ as } m, \tilde{m} \to \infty.$$

We have shown that the series

$$\sum_{n=1}^{\infty}\Big[C_n(T-t)\psi_{0,n} + D_n(T-t)\psi_{1,n}\Big]\varphi_n \longrightarrow v(\cdot,t) \text{ in } W_0^{s,2}(\overline{\Omega}),$$

and that the convergence is uniform in $t \in [0,T]$. Hence, $\psi \in C([0,T]; W_0^{s,2}(\overline{\Omega}))$. Using (4.38) and (4.39) again we get that there is a constant $C > 0$ such that for every $t \in [0,T]$,

$$\|\psi(\cdot,t)\|_{W_0^{s,2}(\overline{\Omega})} \leq C\Big(\|\psi_0\|_{W_0^{s,2}(\overline{\Omega})} + \|\psi_1\|_{L^2(\Omega)}\Big). \tag{4.42}$$

Step 2: Next, we claim that $\psi_t \in C([0,T]; L^2(\Omega))$. Calculating, we get that

$$(\psi_m)_t(x,t) = -\sum_{n=1}^{m}\Big[C_n'(T-t)\psi_{0,n} + D_n'(T-t)\psi_{1,n}\Big]\varphi_n(x).$$

Proceeding as above, we can easily deduce that the series

$$\sum_{n=1}^{\infty}\Big[C_n'(T-t)\psi_{0,n} + D_n'(T-t)\psi_{1,n}\Big]\varphi_n \longrightarrow \psi_t(\cdot,t) \text{ in } L^2(\Omega),$$

and the convergence is uniform in $t \in [0,T]$. In addition using (4.38) and (4.39), we get that there is a constant $C > 0$ such that for every $t \in [0,T]$,

$$\|\psi_t(\cdot,t)\|^2_{L^2(\Omega)} \leq C\Big(\|\psi_0\|^2_{W_0^{s,2}(\overline{\Omega})} + \|\psi_1\|^2_{L^2(\Omega)}\Big). \tag{4.43}$$

The estimate (4.36) follows from (4.42) and (4.43).

Step 3: We show that $\psi_{tt} \in C([0,T); W^{-s,2}(\overline{\Omega}))$. Using (3.8), (4.38) and (4.39), we get that for every $t \in [0,T]$,

$$\begin{aligned}
\|(-\Delta)_D^s\psi(\cdot,t)\|^2_{W^{-s,2}(\overline{\Omega})} &\leq 2\sum_{n=1}^{\infty}\left(\left|\lambda_n^{-\frac{1}{2}}\lambda_n C_n(T-s)\psi_{0,n}\right|^2 + \left|\lambda_n^{-\frac{1}{2}}\lambda_n D_n(T-s)\psi_{1,n}\right|^2\right)\\
&\leq 2\sum_{n=1}^{\infty}\left(\left|\lambda_n^{\frac{1}{2}}C_n(T-s)\psi_{0,n}\right|^2 + \left|\lambda_n^{\frac{1}{2}}D_n(T-s)\psi_{1,n}\right|^2\right)\\
&\leq C\Big(\|\psi_0\|^2_{W_0^{s,2}(\overline{\Omega})} + \|\psi_1\|^2_{L^2(\Omega)}\Big).
\end{aligned} \tag{4.44}$$

Using (3.8), (4.38) and (4.39) again we get that there is a constant $C > 0$ such that for every $t \in [0, T]$,

$$\begin{aligned}\|(-\Delta)_D^s \psi_t(\cdot,t)\|^2_{W^{-s,2}(\overline{\Omega})} &\le 2\sum_{n=1}^{\infty}\left(\left|\lambda_n^{-\frac{1}{2}}\lambda_n C_n'(T-s)\psi_{0,n}\right|^2 + \left|\lambda_n^{-\frac{1}{2}}\lambda_n D_n'(T-s)\psi_{1,n}\right|^2\right)\\ &\le 2\sum_{n=1}^{\infty}\left(\left|\lambda_n^{\frac{1}{2}} C_n'(T-s)\psi_{0,n}\right|^2 + \left|\lambda_n^{\frac{1}{2}} D_n'(T-s)\psi_{1,n}\right|^2\right)\\ &\le C\Big(\|\psi_0\|^2_{W_0^{s,2}(\overline{\Omega})} + \|\psi_1\|^2_{L^2(\Omega)}\Big). \end{aligned} \tag{4.45}$$

Since $\psi_{tt}(\cdot,t) = -(-\Delta)_D^s\psi(\cdot,t) + \delta(-\Delta)_D^s\psi_t(\cdot,t)$, it follows from (4.44) and (4.45) that

$$\|\psi_{tt}(\cdot,t)\|^2_{W^{-s,2}(\overline{\Omega})} \le C\Big(\|\psi_0\|^2_{W_0^{s,2}(\overline{\Omega})} + \|\psi_1\|^2_{L^2(\Omega)}\Big),$$

and we have also shown (4.37). We can also easily deduce that $\psi_{tt} \in C([0,T]; W^{-s,2}(\overline{\Omega}))$.

Step 4: We claim that $\psi \in C([0,T]; D((-\Delta)_D^s)) \cap L^\infty((0,T); L^2(\Omega))$. It follows from the estimate (4.36) that $\psi \in L^\infty((0,T); L^2(\Omega))$. Proceeding as above we get that

$$\begin{aligned}\|\psi(\cdot,t)\|^2_{D((-\Delta)_D^s)} &= \|(-\Delta)_D^s\psi(\cdot,t)\|^2_{L^2(\Omega)}\\ &\le 2\sum_{n=1}^{\infty}\left(\left|\lambda_n^{\frac{1}{2}} C_n(T-t)\lambda_n^{\frac{1}{2}}\psi_{0,n}\right|^2 + \left|\lambda_n D_n(T-t)\psi_{1,n}\right|^2\right). \end{aligned} \tag{4.46}$$

It follows from (4.46), (4.38) and (4.39) that

$$\|\psi(\cdot,t)\|^2_{D((-\Delta)_D^s)} \le C\Big(\|\psi_0\|^2_{W_0^{s,2}(\overline{\Omega})} + \|\psi_1\|^2_{L^2(\Omega)}\Big),$$

and we can also deduce that $\psi \in C([0,T]; D((-\Delta)_D^s))$. We have shown the claim.

Step 5: It is easy to see that the mapping $[0,T) \ni t \to \psi(\cdot,t) \in L^2(\mathbb{R}^N\setminus\Omega)$ can be analytically extended to Σ_T. We also recall that for every $t \in [0,T)$ fixed, we have that $\psi(\cdot,t) \in D((-\Delta)_D^s) \subset W^{s,2}(\mathbb{R}^N)$. Therefore, $\mathcal{N}_s v(\cdot,t)$ exists and belongs to $L^2(\mathbb{R}^N\setminus\Omega)$.

We claim that

$$\mathcal{N}_s\psi(x,t) = \sum_{n=1}^{\infty}\Big(C_n(T-t)\psi_{0,n} + D_n(T-t)\psi_{1,n}\Big)\mathcal{N}_s\varphi_n(x), \tag{4.47}$$

and the series is convergent in $L^2(\mathbb{R}^N\setminus\Omega)$ for every $t \in [0,T)$. Indeed, let $\xi > 0$ be fixed but arbitrary and let $t \in [0, T-\xi]$. It is sufficient to prove that

$$\left\|\sum_{n=N_0+1}^{\infty}\Big[C_n(T-t)\psi_{0,n} + D_n(T-t)\psi_{1,n}\Big]\mathcal{N}_s\varphi_n\right\|_{L^2(\mathbb{R}^N\setminus\Omega)} \longrightarrow 0 \text{ as } N_0 \to \infty.$$

Since $\mathcal{N}_s : W^{s,2}(\mathbb{R}^N) \to L^2(\mathbb{R}^N \setminus \Omega)$ is bounded, then using (4.38) and (4.39), we get that there is a constant $C > 0$ such that

$$\begin{aligned}\Big\|\sum_{n=N_0+1}^{\infty}\Big(C_n(T-t)\psi_{0,n}+D_n(T-t)\psi_{1,n}\Big)\mathcal{N}_s\varphi_n\Big\|^2_{L^2(\mathbb{R}^N\setminus\Omega)}\\ \le C\Big\|\sum_{n=N_0+1}^{\infty}\Big(C_n(T-t)\psi_{0,n}+D_n(T-t)\psi_{1,n}\Big)\varphi_n\Big\|^2_{W_0^{s,2}(\overline{\Omega})}\\ \le C\left(\sum_{n=N_0+1}^{\infty}|\psi_{0,n}|^2+\sum_{n=N_0+1}^{\infty}|\psi_{1,n}|^2\right)\longrightarrow 0 \text{ as } N_0\to\infty.\end{aligned}$$

Thus, $\mathcal{N}_s$ is given by (4.47) and the series is convergent in $L^2(\mathbb{R}^N \setminus \Omega)$ uniformly on any compact subset of $[0, T)$.

Besides, we obtain the following continuous dependence on the data. Let $m \in \mathbb{N}$ be such that $m > N_0$ and consider

$$\psi_m(x,t)=\sum_{k=1}^{m}\Big(C_n(T-t)\psi_{0,n}+D_n(T-t)\psi_{1,n}\Big)\mathcal{N}_s\varphi_n(x). \tag{4.48}$$

Using the fact that the operator $\mathcal{N}_s : W_0^{s,2}(\overline{\Omega}) \longrightarrow L^2(\mathbb{R}^N \setminus \Omega)$ is bounded, the continuous embedding $W_0^{s,2}(\overline{\Omega}) \hookrightarrow L^2(\Omega)$, (4.38) and (4.39), we get that there is a constant $C > 0$ such that for every $t \in [0, T]$,

$$\begin{aligned}&\Big\|\sum_{k=1}^{m}C_k(T-t)\psi_{0,k}\mathcal{N}_s\varphi_k\Big\|^2_{L^2(\mathbb{R}^N\setminus\Omega)}\\ &\le 2\Big\|\sum_{k=1}^{N_0}C_k(T-t)\psi_{0,k}\mathcal{N}_s\varphi_k\Big\|^2_{L^2(\mathbb{R}^N\setminus\Omega)}+2\Big\|\sum_{k=N_0+1}^{m}C_k(T-t)\psi_{0,k}\mathcal{N}_s\varphi_k\Big\|^2_{L^2(\mathbb{R}^N\setminus\Omega)}\\ &\le C\Big\|\sum_{k=1}^{N_0}C_k(T-t)\psi_{0,k}\varphi_k\Big\|^2_{W_0^{s,2}(\overline{\Omega})}+C\Big\|\sum_{k=N_0+1}^{m}C_k(T-t)\psi_{0,k}\varphi_k\Big\|^2_{W_0^{s,2}(\overline{\Omega})}\\ &\le C\left(\sum_{k=1}^{N_0}|C_k(T-t)\lambda_k^{\frac12}\psi_{0,k}|^2+\sum_{k=N_0+1}^{m}|C_k(T-t)\lambda_k^{\frac12}\psi_{0,k}|^2\right)\le C\|\psi_0\|^2_{W^{s,2}(\overline{\Omega})}.\end{aligned} \tag{4.49}$$

Analogously, we obtain that there is a constant $C > 0$ such that for every $t \in [0, T]$,

$$\Big\|\sum_{k=1}^{m}D_k(T-t)\psi_{1,k}\mathcal{N}_s\varphi_k\Big\|^2_{L^2(\mathbb{R}^N\setminus\Omega)}\le C\|\psi_1\|^2_{L^2(\Omega)}. \tag{4.50}$$

It follows from (4.49) and (4.50) that

$$\|\mathcal{N}_s\psi(\cdot,t)\|^2_{L^2(\mathbb{R}^N\setminus\Omega)}\le C\Big(\|\psi_0\|^2_{W^{s,2}(\overline{\Omega})}+\|\psi_1\|^2_{L^2(\Omega)}\Big). \tag{4.51}$$

Next, since the functions $C_n(z)$ and $D_n(z)$ are entire functions, it follows that the function

$$\sum_{n=1}^{m}\Big[C_n(T-z)\psi_{0,n}+D_n(T-z)\psi_{1,n}\Big]\mathcal{N}_s\varphi_n$$

is analytic in Σ_T.

Let $\tau>0$ be fixed but otherwise arbitrary. Let $z\in\mathbb{C}$ satisfy $\mathrm{Re}(z)\le T-\tau$. Then proceeding as above by using (4.38) and (4.39), we get that

$$\begin{aligned}&\left\|\sum_{n=m+1}^{\infty}\psi_{0,n}C_n(T-z)\mathcal{N}_s\varphi_n\right\|^2_{L^2(\mathbb{R}^N\setminus\Omega)}+\left\|\sum_{n=m+1}^{\infty}\psi_{1,n}D_n(T-z)\mathcal{N}_s\varphi_n\right\|^2_{L^2(\mathbb{R}^N\setminus\Omega)}\\ \le& C\sum_{n=m+1}^{\infty}\left|\lambda_n^{\frac12}\psi_{0,n}\right|^2+C\sum_{n=m+1}^{\infty}|\psi_{1,n}|^2\longrightarrow 0\ \text{ as }\ m\to\infty.\end{aligned}$$

We have shown that

$$\mathcal{N}_s\psi(\cdot,z)=\sum_{n=1}^{\infty}\psi_{0,n}C_n(T-z)\mathcal{N}_s\varphi_n+\sum_{n=1}^{\infty}\psi_{1,n}D_n(T-z)\mathcal{N}_s\varphi_n, \tag{4.52}$$

and the series is uniformly convergent in any compact subset of Σ_T. Thus, $\mathcal{N}_s\psi$ given in (4.52) is also analytic in Σ_T. The proof is finished. □

5. Proof of the main results

In this section we prove the main results stated in Section 2.

5.1. The lack of exact or null controllability

We start with the proof of the lack of null/exact controllability of the system (1.1). For this purpose, we will use the following concept of controllability.

Definition 5.1. The system (1.1) is said to be *spectrally controllable* if any finite linear combination of eigenvectors, that is,

$$u_0=\sum_{n=1}^{M}u_{0,n}\varphi_n,\quad u_1=\sum_{n=1}^{M}u_{1,n}\varphi_n,\ \ M\ge 1\ \text{ arbitrary},$$

can be steered to zero by a control function g.

Remark 5.2. We mention that to prove the lack of null/exact controllability of the system (1.1), we shall prove that the system is not spectrally controllable. Using some classical results on control theory (see *e.g.* [17, 33, 38]), this implies that the system is not null or exact controllable. One can also see it directly from the proof as we shall mention below.

Let (u,u_t) and (ψ,ψ_t) be the weak solutions of (1.1) and (2.5), respectively. Multiplying the first equation in (1.1) by ψ, then integrating by parts over $(0,T)$ and over Ω and using the integration by parts formulas (3.11) and (3.12), we get

$$\int_\Omega \Big(-u_t\psi + u\psi_t - \delta u(-\Delta)^s\psi\Big)\Big|_{t=0}^{t=T}\mathrm{d}x = \int_0^T \int_{\mathbb{R}^N\setminus\Omega} \Big(g(x,t)+\delta g_t(x,t)\Big)\mathcal{N}_s\psi(x,t)\mathrm{d}x\mathrm{d}t. \tag{5.1}$$

Using the identity (5.1) and a density argument to pass to the limit, we obtain the following criteria of null and exact controllabilities.

Lemma 5.3. *The following assertions hold.*

(a) The system (1.1) is null controllable if and only if for each initial condition $(u_0,u_1)\in W_0^{s,2}(\overline{\Omega})\times L^2(\Omega)$, there exists a control function g such that the solution (ψ,ψ_t) of the dual system (2.5) satisfies

$$\begin{aligned}&(u_1,\psi(\cdot,0))_{L^2(\Omega)} - \langle u_0,\psi_t(\cdot,0)\rangle_{\frac12,-\frac12} + \langle u_0,\delta(-\Delta)^s\psi(\cdot,0)\rangle_{\frac12,-\frac12}\\ &\quad= \int_0^T\int_{\mathbb{R}^N\setminus\Omega}\Big(g(x,t)+\delta g_t(x,t)\Big)\mathcal{N}_s\psi(x,t)\mathrm{d}x\mathrm{d}t,\end{aligned} \tag{5.2}$$

for each $(\psi_0,\psi_1)\in L^2(\Omega)\times W^{-s,2}(\overline{\Omega})$.

(b) The system (1.1) is exact controllable at time $T>0$, if and only if there exists a control function g such that the solution (ψ,ψ_t) of (2.5) satisfies

$$\begin{aligned}&-(u_t(\cdot,T),\psi_0)_{L^2(\Omega)} + \langle u(\cdot,T),\psi_1\rangle_{\frac12,-\frac12} - \langle u(\cdot,T),\delta(-\Delta)^s\psi_0\rangle_{\frac12,-\frac12}\\ &\quad= \int_0^T\int_{\mathbb{R}^N\setminus\Omega}\Big(g(x,t)+\delta g_t(x,t)\Big)\mathcal{N}_s\psi(x,t)\mathrm{d}x\mathrm{d}t,\end{aligned} \tag{5.3}$$

for each $(\psi_0,\psi_1)\in L^2(\Omega)\times W^{-s,2}(\overline{\Omega})$.

Now, we are able to give the proof of the first main result of this work.

Proof of Theorem 2.1. Firstly, since $\varphi_n\in W_0^{s,2}(\overline{\Omega})\hookrightarrow L^2(\Omega)\hookrightarrow W^{-s,2}(\overline{\Omega})$, it suffices to prove that the system is not spectrally controllable.

Secondly, using Definition 5.1, we show that no non-trivial finite linear combination of eigenvectors can be driven to zero in finite time. To do this, we write the solution of (2.5) in a better way. With a simple calculation, it is easy to see that

$$\begin{aligned}\psi(x,t) =& \sum_{n=1}^{N_0}\left(\widetilde{a}_n e^{\widetilde{\lambda}_n^+(T-t)} + \widetilde{b}_n e^{\widetilde{\lambda}_n^-(T-t)}\right)\varphi_n(x)\\ &+ \sum_{n=N_0+1}^{\infty}\left(a_n e^{\lambda_n^+(T-t)} + b_n e^{\lambda_n^-(T-t)}\right)\varphi_n(x),\end{aligned} \tag{5.4}$$

where

$$\begin{cases}\widetilde{a}_n = \dfrac12\left((1-\dfrac{\alpha_n}{\beta_n}i)\psi_{0,n} - \dfrac{i}{\beta_n}\psi_{1,n}\right),\\ \widetilde{b}_n = \dfrac12\left((1+\dfrac{\alpha_n}{\beta_n}i)\psi_{0,n} + \dfrac{i}{\beta_n}\psi_{1,n}\right),\end{cases} \tag{5.5}$$

and

$$\begin{cases} a_n = \dfrac{\lambda_n^-}{\lambda_n^- - \lambda_n^+}\psi_{0,n} + \dfrac{1}{\lambda_n^- - \lambda_n^+}\psi_{1,n}, \\ b_n = \dfrac{-\lambda_n^+}{\lambda_n^- - \lambda_n^+}\psi_{0,n} - \dfrac{1}{\lambda_n^- - \lambda_n^+}\psi_{1,n}. \end{cases} \tag{5.6}$$

Now, we write the initial data in Fourier series

$$u_0 = \sum_{n=1}^{\infty} u_{0,n}\varphi_n, \quad u_1 = \sum_{n=1}^{\infty} u_{1,n}\varphi_n, \tag{5.7}$$

and suppose that there exists $M \in \mathbb{N}$ such that

$$u_{0,n} = u_{1,n} = 0, \quad \forall\, n \geq M. \tag{5.8}$$

Assume that (1.1) is spectrally controllable. Then, there exists a control function g such that the solution (u, u_t) of (1.1), with u_0, u_1 given by (5.7)–(5.8), satisfies $u(\cdot, T) = u_t(\cdot, T) = 0$ in Ω. From Lemma 5.3 we have

$$\begin{aligned} &(u_1, \psi(\cdot,0))_{L^2(\Omega)} - \langle u_0, \psi_t(\cdot,0)\rangle_{\frac{1}{2},-\frac{1}{2}} + \langle u_0, \delta(-\Delta)^s\psi(\cdot,0)\rangle_{\frac{1}{2},-\frac{1}{2}} \\ &= \int_0^T \int_{\mathbb{R}^N\setminus\Omega} \Big(g(x,t) + \delta g_t(x,t)\Big)\mathcal{N}_s\psi(x,t)\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{5.9}$$

for any solution (ψ, ψ_t) of the dual system (2.5).

We divide the proof in the following two cases.

Case 1. $M > N_0$.

We consider the following trajectories:

$$\psi(x,t) = e^{\widetilde{\lambda}_n^+(T-t)}\varphi_n(x), \quad \psi(x,t) = e^{\widetilde{\lambda}_n^-(T-t)}\varphi_n(x) \text{ for } n \leq N_0 \tag{5.10}$$

$$\psi(x,t) = e^{\lambda_n^+(T-t)}\varphi_n(x), \quad \psi(x,t) = e^{\lambda_n^-(T-t)}\varphi_n(x) \text{ for } N_0 < n < M. \tag{5.11}$$

Replacing (5.10) in (5.9) we obtain, for any $n \in (0, N_0]$, the following system:

$$u_{1,n} + u_{0,n}\widetilde{\lambda}_n^+ + \delta u_{0,n}\lambda_n = \int_0^T \int_{\mathbb{R}^N\setminus\Omega} (g(x,t) + \delta g_t(x,t))e^{-\widetilde{\lambda}_n^+ t}\mathcal{N}_s\varphi_n(x)\mathrm{d}x\mathrm{d}t, \tag{5.12}$$

$$u_{1,n} + u_{0,n}\widetilde{\lambda}_n^- + \delta u_{0,n}\lambda_n = \int_0^T \int_{\mathbb{R}^N\setminus\Omega} (g(x,t) + \delta g_t(x,t))e^{-\widetilde{\lambda}_n^- t}\mathcal{N}_s\varphi_n(x)\mathrm{d}x\mathrm{d}t, \tag{5.13}$$

and replacing (5.11) in (5.9), it follows that for any $N_0 < n < M$,

$$u_{1,n} + u_{0,n}\lambda_n^+ + \delta u_{0,n}\lambda_n = \int_0^T \int_{\mathbb{R}^N\setminus\Omega} (g(x,t) + \delta g_t(x,t))e^{-\lambda_n^+ t}\mathcal{N}_s\varphi_n(x)\mathrm{d}x\mathrm{d}t, \tag{5.14}$$

$$u_{1,n} + u_{0,n}\lambda_n^- + \delta u_{0,n}\lambda_n = \int_0^T \int_{\mathbb{R}^N\setminus\Omega} (g(x,t) + \delta g_t(x,t))e^{-\lambda_n^- t}\mathcal{N}_s\varphi_n(x)\mathrm{d}x\mathrm{d}t. \tag{5.15}$$

Next, define the complex-valued function of complex variable,

$$F(z) = \int_0^T \left(\int_{\mathbb{R}^N \setminus \Omega} (g(x,t) + \delta g_t(x,t)) \mathcal{N}_s \varphi_n(x) \mathrm{d}x \right) e^{izt} \mathrm{d}t. \tag{5.16}$$

According to Paley–Wiener theorem, F is an entire function. Due to (5.8), from (5.14) and (5.15) we obtain that F satisfies $F(i\lambda_n^+) = F(i\lambda_n^-) = 0$, for all $n > M$. Besides, we know that $\lambda_n^+ \to -\delta$ as $n \to \infty$ (see Rem. 4.1). Then, F is zero in a set with finite accumulation point. This implies that $F \equiv 0$. In particular, $F(i\widetilde{\lambda}_n^+) = F(i\widetilde{\lambda}_n^-) = 0$. From (5.12) to (5.15), it is easy to see that $u_{0,n} = u_{1,n} = 0$ for $n \leq N_0$ and $u_{0,n} = u_{1,n} = 0$ for $N_0 < n < M$. Thus the trivial state is the only one which can be steered to zero.

Case 2. $M = N_0$ or $M < N_0$.
In these cases the identities (5.14) and (5.15) do not appear. Proceeding as above we get the desired result.

We have shown that the system (1.1) is not spectrally controllable. That is, there exist initial conditions (u_0, u_1) (nontrivial) such that for any exterior control function g, the associated solution (u, u_t) of the system (1.1) is not identically zero at time T. Namely, the system (1.1) is not null controllable. On the other hand, since the exact controllability implies the null controllability and the latter one fails, we finally obtain that the system (1.1) is not exact controllable. The proof is finished. □

Remark 5.4. We can observe that in the case $\delta = 0$, the previous conclusion is not valid. This is due to the fact that, when $\delta = 0$, the previous computation gives the following system for $n \leq M$:

$$u_{1,n} + u_{0,n}\widetilde{\lambda}_n^+ = \int_0^T \int_{\mathbb{R}^N \setminus \Omega} g(x,t) e^{-\widetilde{\lambda}_n^+ t} \mathcal{N}_s \varphi_n(x) \mathrm{d}x \mathrm{d}t,$$
$$u_{1,n} + u_{0,n}\widetilde{\lambda}_n^- = \int_0^T \int_{\mathbb{R}^N \setminus \Omega} g(x,t) e^{-\widetilde{\lambda}_n^- t} \mathcal{N}_s \varphi_n(x) \mathrm{d}x \mathrm{d}t.$$

Since $\widetilde{\lambda}_n^\pm \to \pm i\infty$, as $n \to \infty$, it follows that the set $\{\widetilde{\lambda}_n^\pm\}_{n\in\mathbb{N}}$, on which the function F defined in (5.16) is zero, does not have a finite accumulation point. Thus we cannot conclude that $u_{0,n} = u_{1,n} = 0$ for all $n \leq M$. This shows that the analysis of the null/exact controllability of the pure wave equation (without damping) with the fractional Laplacian must be done by using other techniques as in the classical case $s = 1$. Finally we mention that in the case $\delta = 0$, except the bi-fractional system (1.3), it is still unknown if the fractional wave equation $(0 < s < 1)$ is null or exact controllable from the interior (that is, when the control function is localized in a subset ω of Ω) or from the exterior (that is, as the system (1.1) with $\delta = 0$).

We conclude this subsection with the following observation.

Remark 5.5. We mention the following situations.

(a) Firstly, we notice that if s is close to 1, using the results obtained in [8], we can deduce that the eigenfuctions $\varphi_n \in W_0^{1,2}(\Omega)$, for every $n \in \mathbb{N}$, and the net $\{\varphi_n\} = \{\varphi_{n,s}\}_{0<s<1}$ converges as $s \uparrow 1^-$ to the eigenfunctions of the Laplace operator with the zero Dirichlet boundary condition. This implies that if $(\psi_0, \psi_1) \in W_0^{1,2}(\Omega) \times L^2(\Omega) \hookrightarrow W_0^{s,2}(\overline{\Omega}) \times L^2(\Omega)$ and s is close to 1, then the solution (ψ, ψ_t) of the dual system has the following regularity: $\psi \in C([0,T]; W_0^{1,2}(\Omega)) \cap C^1([0,T]; L^2(\Omega)$ and $\psi_{tt} \in C([0,T); (W_0^{1,2}(\Omega))^\star))$. Therefore, if the control function g has enough regularity as in Lemma 3.7 and $(u_0, u_1) \in W_0^{1,2}(\Omega) \times L^2(\Omega)$, then in Lemma 5.3, using (3.13) and (3.14), and taking the limit of (5.2) and (5.3) as $s \uparrow 1^-$, we can deduce that

$$\begin{aligned}&(u_1,\psi(\cdot,0))_{L^2(\Omega)}-\langle u_0,\psi_t(\cdot,0)\rangle_{1,-1}-\langle u_0,\delta\Delta\psi(\cdot,0)\rangle_{1,-1}\\&=\int_0^T\int_{\partial\Omega}\Big(g(x,t)+\delta g_t(x,t)\Big)\frac{\partial\psi(x,t)}{\partial\nu}\,\mathrm{d}\sigma\mathrm{d}t,\end{aligned}$$

for every $(\psi_0,\psi_1)\in L^2(\Omega)\times(W_0^{1,2}(\Omega))^\star$, and

$$\begin{aligned}&-(u_t(\cdot,T),\psi_0)_{L^2(\Omega)}+\langle u(\cdot,T),\psi_1\rangle_{1,-1}-\langle u(\cdot,T),\delta\Delta\psi_0\rangle_{1,-1}\\&=\int_0^T\int_{\partial\Omega}\Big(g(x,t)+\delta g_t(x,t)\Big)\frac{\partial\psi(x,t)}{\partial\nu}\,\mathrm{d}\sigma\mathrm{d}t,\end{aligned}$$

respectively. These are the notions of null and exact controllabilities, respectively, of the following (possible) strong damping local wave equation:

$$\begin{cases}u_{tt}-\Delta u-\delta\Delta u_t=0 & \text{in }\ \Omega\times(0,T),\\ u=g\chi_{\omega\times(0,T)} & \text{on }\ \partial\Omega\times(0,T);\\ u(\cdot,0)=u_0,\ u_t(\cdot,0)=u_1 & \text{in }\ \Omega,\end{cases}\tag{5.17}$$

studied by several authors (see *e.g.* [38, 51] and the references therein). Here, $\langle\cdot,\cdot\rangle_{1,-1}$ denotes the duality pairing between $W_0^{1,2}(\Omega)$ and $(W_0^{1,2}(\Omega))^\star$.

(b) In the above sense, the results obtained in the present paper for the fractional case $0<s<1$ are consistent with the ones obtained for the case of the Laplace operator in one space dimension $N=1$ in [38]. For this reason, following the techniques we developed in the present article, we anticipate that the approximate controllability or the lack of exact/null controllability of the system (5.17) proved in [38] for one space dimension, that is, $N=1$, is also valid for any dimension $N\geq 1$. These anticipated results will be written rigorously in a forthcoming paper.

5.2. The unique continuation property

Here we show that the dual system satisfies the unique continuation property.

Proof of Theorem 2.4. Let $\mathcal{O}\subset\mathbb{R}^N\setminus\Omega$ be an arbitrary non-empty open set. Suppose that $\mathcal{N}_s\psi=0$ in $\mathcal{O}\times(0,T)$. Then,

$$\mathcal{N}_s\psi(x,t)=\sum_{n=1}^{\infty}\Big(C_n(T-t)\psi_{0,n}+D_n(T-t)\psi_{1,n}\Big)\mathcal{N}_s\varphi_n(x)=0,\quad\forall\ (x,t)\in\mathcal{O}\times(0,T).\tag{5.18}$$

Since $\mathcal{N}_s\psi$ can be analytically extended to Σ_T (by Thm. 4.8(c)), it follows from (5.18) that

$$\mathcal{N}_s\psi(x,t)=\sum_{n=1}^{\infty}\Big(C_n(T-t)\psi_{0,n}+D_n(T-t)\psi_{1,n}\Big)\mathcal{N}_s\varphi_n(x)=0,\quad(x,t)\in\mathcal{O}\times(-\infty,T).\tag{5.19}$$

Let $\{\lambda_k\}_{k\in\mathbb{N}}$ be the set of all eigenvalues of the operator $(-\Delta)_D^s$ and let $\{\varphi_{k_j}\}_{1\le j\le m_k}$ be an orthonormal basis for $\ker(\lambda_k-(-\Delta)_D^s)$. Then, (5.19) can be rewritten as

$$\begin{aligned}\mathcal{N}_s\psi(x,t) =&\sum_{k=1}^{\infty}\left(\sum_{j=1}^{m_k}\psi_{0,k_j}\mathcal{N}_s\varphi_{k_j}(x)\right)C_k(T-t)\\ &+\sum_{k=1}^{\infty}\left(\sum_{j=1}^{m_k}\psi_{1,k_j}\mathcal{N}_s\varphi_{k_j}(x)\right)D_k(T-t)=0,\quad \forall\,(x,t)\in\mathcal{O}\times(-\infty,T).\end{aligned}\tag{5.20}$$

Let $z\in\mathbb{C}$ with $\mathrm{Re}(z)=\eta>0$ and let $m\in\mathbb{N}$. Since φ_{k_j}, $1\le j\le m_k$, are orthonormal, then using the fact that the operator $\mathcal{N}_s: D((-\Delta)_D^s)\subset W^{s,2}(\mathbb{R}^N)\to L^2(\mathbb{R}^N\setminus\Omega)$ is bounded, the continuous dependence on the data of $\mathcal{N}_s$ (see (4.51)), and letting

$$\begin{aligned}\psi_m(\cdot,t):=&\sum_{k=1}^{m}\left(\sum_{j=1}^{m_k}\psi_{0,k_j}\mathcal{N}_s\varphi_{k_j}(x)\right)e^{z(t-T)}C_k(T-t)\\ &+\sum_{k=1}^{m}\left(\sum_{j=1}^{m_k}\psi_{1,k_j}\mathcal{N}_s\varphi_{k_j}(x)\right)e^{z(t-T)}D_k(T-t),\end{aligned}$$

we obtain that there is a constant $C>0$ such that for every $t\in[0,T]$,

$$\|\psi_m(\cdot,t)\|_{L^2(\mathbb{R}^N\setminus\Omega)}\le Ce^{\eta(t-T)}\Big(\|\psi_0\|_{W^{s,2}(\overline{\Omega})}+\|\psi_1\|_{L^2(\Omega)}\Big).\tag{5.21}$$

The right hand side of (5.21) is integrable over $t\in(-\infty,T)$ and

$$\int_{-\infty}^{T}e^{\eta(t-T)}\Big(\|\psi_0\|_{W^{s,2}(\overline{\Omega})}+\|\psi_1\|_{L^2(\Omega)}\Big)\,\mathrm{d}t=\frac{1}{\eta}\left(\|\psi_0\|_{W^{s,2}(\overline{\Omega})}+\|\psi_1\|_{L^2(\Omega)}\right).$$

By the Lebesgue dominated convergence theorem, we can deduce that

$$\begin{aligned}&\int_{-\infty}^{T}e^{z(t-T)}\left[\sum_{k=1}^{\infty}\left(\sum_{j=1}^{m_k}\psi_{0,k_j}\mathcal{N}_s\varphi_{k_j}(x)\right)C_k(T-t)+\sum_{k=1}^{\infty}\left(\sum_{j=1}^{m_k}\psi_{1,k_j}\mathcal{N}_s\varphi_{k_j}(x)\right)D_k(T-t)\right]\mathrm{d}t\\ &=\sum_{k=1}^{\infty}\sum_{j=1}^{m_k}\Big(E_k(z)\psi_{0,k_j}+F_k(z)\psi_{1,k_j}\Big)\mathcal{N}_s\varphi_{k_j},\quad x\in\mathbb{R}^N\setminus\Omega,\ \mathrm{Re}(z)>0,\end{aligned}$$

where

$$E_k(z)=\begin{cases}\dfrac{1-\frac{\alpha_k+\beta_k}{2\beta_k}}{z-(\alpha_k+\beta_k)}+\dfrac{\frac{\alpha_k+\beta_k}{2\beta_k}}{z-(\alpha_k-\beta_k)} & \text{if } k\le N_0,\\ \dfrac{1}{\lambda_k^--\lambda_k^+}\left(\dfrac{\lambda_k^-}{z-\lambda_k^+}-\dfrac{\lambda_k^+}{z-\lambda_k^-}\right) & \text{if } k>N_0,\end{cases}$$

and

$$F_k(z) = \begin{cases} \dfrac{-1}{2\beta_k(z-(\alpha_k+\beta_k))} + \dfrac{1}{2\beta_k(z-(\alpha_k-\beta_k))} & \text{if } k \le N_0, \\ \dfrac{1}{\lambda_k^- - \lambda_k^+}\left(\dfrac{1}{z-\lambda_k^+} - \dfrac{1}{z-\lambda_k^-}\right) & \text{if } k > N_0. \end{cases}$$

From (5.20) we get that

$$\sum_{k=1}^{\infty}\sum_{j=1}^{m_k}\Big(E_k(z)\psi_{0,k_j} + F_k(z)\psi_{1,k_j}\Big)\mathcal{N}_s\varphi_{k_j}(x) = 0, \quad x \in \mathcal{O},\ \mathrm{Re}(z) > 0. \tag{5.22}$$

Using the analytic continuation in z, we get that (5.22) holds for every $z \in \mathbb{C} \setminus \{\lambda_k^+, \lambda_k^-\}_{k>N_0}$ and also for every $z \in \mathbb{C} \setminus \{\alpha_k+\beta_k, \alpha_k-\beta_k\}_{k\le N_0}$.

For $k \le N_0$, we take a small circle about $\alpha_k+\beta_k$, but not including $\{\alpha_l+\beta_l\}_{l\neq k}$. Then, integrating over that circle we get

$$\sum_{j=1}^{m_k}\left[\left(1-\frac{\alpha_{k_j}+\beta_{k_j}}{2\beta_{k_j}}\right)\psi_{0,k_j} - \frac{1}{2\beta_{k_j}}\psi_{1,k_j}\right]\mathcal{N}_s\varphi_{k_j} = 0, \quad x \in \mathcal{O}. \tag{5.23}$$

Now, integrating over a small circle about $\alpha_k-\beta_k$ and not including $\{\alpha_l-\beta_l\}_{l\neq k}$, we obtain

$$\sum_{j=1}^{m_k}\left(\frac{\alpha_{k_j}+\beta_{k_j}}{2\beta_{k_j}}\psi_{0,k_j} + \frac{1}{2\beta_{k_j}}\psi_{1,k_j}\right)\mathcal{N}_s\varphi_{k_j} = 0, \quad x \in \mathcal{O}. \tag{5.24}$$

Let

$$\begin{aligned} \psi_k^1 &:= \sum_{j=1}^{m_k}\left[\left(1-\frac{\alpha_{k_j}+\beta_{k_j}}{2\beta_{k_j}}\right)\psi_{0,k_j} - \frac{1}{2\beta_{k_j}}\psi_{1,k_j}\right]\varphi_{k_j}, \\ \psi_k^2 &:= \sum_{j=1}^{m_k}\left(\frac{\alpha_{k_j}+\beta_{k_j}}{2\beta_{k_j}}\psi_{0,k_j} + \frac{1}{2\beta_{k_j}}\psi_{1,k_j}\right)\varphi_{k_j}. \end{aligned}$$

It follows from (5.23) and (5.24) that $\mathcal{N}_s\psi_k^1 = \mathcal{N}_s\psi_k^2 = 0$ in $\mathcal{O}$. We have shown that

$$(-\Delta)^s\psi_k^l = \lambda_k\psi_k^l \quad \text{in } \Omega \quad \text{and} \quad \mathcal{N}_s\psi_k^l = 0 \quad \text{in } \mathcal{O},\ l = 1,2.$$

It follows from Lemma 3.3 that $\psi_k^l = 0$ $(l = 1,2)$ for every k. Using the fact that $\{\varphi_{k_j}\}_{1\le j\le m_k}$ is linearly independent in $L^2(\Omega)$, we get that

$$\begin{aligned} \left(\left(1-\frac{\alpha_{k_j}+\beta_{k_j}}{2\beta_{k_j}}\right)\psi_{0,k_j} - \frac{1}{2\beta_{k_j}}\psi_{1,k_j}, \varphi_{k_j}\right) &= 0, \quad 1 \le j \le m_k, \\ \left(\frac{\alpha_{k_j}+\beta_{k_j}}{2\beta_{k_j}}\psi_{0,k_j} + \frac{1}{2\beta_{k_j}}\psi_{1,k_j}, \varphi_{k_j}\right) &= 0, \quad 1 \le j \le m_k. \end{aligned}$$

Therefore, we can deduce that

$$\psi_{0,k} = \psi_{1,k} = 0, \;\; k \leq N_0. \tag{5.25}$$

On the other hand, since the real number λ_k^+ and λ_k^- (see (4.5)) satisfy

$$\lambda_k^+ \sim -\frac{1}{\delta} \quad \text{ and } \quad \lambda_k^- \sim -\lambda_k \quad \text{ as } \quad k \to \infty,$$

then for $k > N_0$, we can take a suitable small circle about λ_k^+ and not including $\{\lambda_l^+\}_{l\neq k}$ and also not including $\{\lambda_k^-\}_{k>N_0}$, and integrating (5.22) over that circle, we get that

$$\sum_{j=1}^{m_k} \left(\frac{\lambda_{k_j}^-}{\lambda_{k_j}^- - \lambda_{k_j}^+} \psi_{0,k_j} + \frac{1}{\lambda_{k_j}^- - \lambda_{k_j}^+} \psi_{1,k_j} \right) \mathcal{N}_s \varphi_{k_j} = 0, \quad x \in \mathcal{O}. \tag{5.26}$$

Let us consider

$$\psi_k^1 := \sum_{j=1}^{m_k} \left(\frac{\lambda_{k_j}^-}{\lambda_{k_j}^- - \lambda_{k_j}^+} \psi_{0,k_j} + \frac{1}{\lambda_{k_j}^- - \lambda_{k_j}^+} \psi_{1,k_j} \right) \varphi_{k_j}.$$

It follows from (5.26) that $\mathcal{N}_s \psi_k^1 = 0$ in $\mathcal{O}$. Thus, ψ_k^1 solves the elliptic problem

$$(-\Delta)^s \psi_k^1 = \lambda_k \psi_k^1 \text{ in } \Omega \text{ and } \mathcal{N}_s \psi_k^1 = 0 \text{ in } \mathcal{O}.$$

From Lemma 3.3 we get that $\psi_k^1 = 0$ in Ω for every k. Since $\{\varphi_{k_j}\}_{1\leq j\leq m_k}$ is linearly independent in $L^2(\Omega)$, we deduce that

$$\left(\frac{\lambda_{k_j}^-}{\lambda_{k_j}^- - \lambda_{k_j}^+} \psi_{0,k_j} + \frac{1}{\lambda_{k_j}^- - \lambda_{k_j}^+} \psi_{1,k_j}, \varphi_{k_j} \right) = 0, \quad \forall \, 1 \leq j \leq m_k.$$

Thus,

$$\frac{\lambda_k^-}{\lambda_k^- - \lambda_k^+} \psi_{0,k} + \frac{1}{\lambda_k^- - \lambda_k^+} \psi_{1,k} = 0. \tag{5.27}$$

Similarly, taking a circle about λ_k^- and not including $\{\lambda_l^-\}_{l\neq k}$ and also not including $\{\lambda_k^+\}_{k>N_0}$, we obtain that

$$\sum_{j=1}^{m_k} \left(\frac{-\lambda_{k_j}^+}{\lambda_{k_j}^- - \lambda_{k_j}^+} \psi_{0,k_j} - \frac{1}{\lambda_{k_j}^- - \lambda_{k_j}^+} \psi_{1,k_j} \right) \mathcal{N}_s \varphi_{k_j} = 0, \quad x \in \mathcal{O}.$$

Let

$$\psi_k^2 := \sum_{j=1}^{m_k} \left(\frac{-\lambda_{k_j}^+}{\lambda_{k_j}^- - \lambda_{k_j}^+} \psi_{0,k_j} - \frac{1}{\lambda_{k_j}^- - \lambda_{k_j}^+} \psi_{1,k_j} \right) \varphi_{k_j}.$$

It follows from (5.26) that $\mathcal{N}_s\psi_k^2 = 0$ in $\mathcal{O}$. Hence, ψ_k^2 solves the elliptic problem

$$(-\Delta)^s\psi_k^2 = \lambda_k\psi_k^2 \text{ in } \Omega \text{ and } \mathcal{N}_s\psi_k^2 = 0 \text{ in } \mathcal{O}.$$

From Lemma 3.3, we get that $\psi_k^2 = 0$ in Ω for every k. Therefore,

$$\frac{-\lambda_k^+}{\lambda_k^- - \lambda_k^+}\psi_{0,k} - \frac{1}{\lambda_k^- - \lambda_k^+}\psi_{1,k} = 0. \tag{5.28}$$

Finally, from (5.27) and (5.28) we get that

$$\psi_{0,k} = \psi_{1,k} = 0,\ \ k > N_0. \tag{5.29}$$

From (5.25) and (5.29), we finally obtain that $\psi_0 = \psi_1 = 0$. Since the solution (ψ, ψ_t) of the adjoint system is unique, we can conclude that $\psi = 0$ in $\Omega \times (0, T)$. The proof is finished. ☐

5.3. The approximate controllability

We obtain the result as a direct consequence of the unique continuation property for the adjoint system (Thm. 2.4).

Proof of Theorem 2.5. Let $g \in \mathcal{D}(\mathcal{O} \times (0, T))$, (u, u_t) the unique weak solution of (1.1) with $u_0 = u_1 = 0$ and let (ψ, ψ_t) be the unique weak solution of (2.5) with $(\psi_0, \psi_1) \in W_0^{s,2}(\overline{\Omega}) \times L^2(\Omega)$. Firstly, it follows from Theorem 4.5 that $u \in C^\infty([0, T]; W^{s,2}(\mathbb{R}^N))$. Thus $u(\cdot, T) \in L^2(\Omega)$ and $u_t(\cdot, T) \in W^{-s,2}(\overline{\Omega})$. Secondly, it follows from Theorem 4.8 that $\psi \in L^\infty((0, T); L^2(\Omega))$. Therefore, using the identity (5.1) we can deduce that

$$\begin{aligned} &- \langle u_t(\cdot, T), \psi_0\rangle_{-\frac{1}{2},\frac{1}{2}} + (u(\cdot, T), \psi_1)_{L^2(\Omega)} - \langle u(\cdot, T), \delta(-\Delta)^s\psi_0\rangle_{\frac{1}{2},-\frac{1}{2}} \\ &= \int_0^T \int_{\mathbb{R}^N\setminus\Omega} \Big(g(x, t) + \delta g_t(x, t)\Big)\mathcal{N}_s\psi(x, t)\ \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{5.30}$$

If $(\psi_0, \psi_1) \in D((-\Delta)_D^s) \times L^2(\Omega) \hookrightarrow W_0^{s,2}(\overline{\Omega}) \times L^2(\Omega)$, then (5.30) becomes

$$\begin{aligned} &- \langle u_t(\cdot, T), \psi_0\rangle_{-\frac{1}{2},\frac{1}{2}} + \Big(u(\cdot, T), \psi_1 - \delta(-\Delta)^s\psi_0\Big)_{L^2(\Omega)} \\ &= \int_0^T \int_{\mathbb{R}^N\setminus\Omega} \Big(g(x, t) + \delta g_t(x, t)\Big)\mathcal{N}_s\psi(x, t)\ \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{5.31}$$

Since $D((-\Delta)_D^s) \times L^2(\Omega)$ is dense in $W_0^{s,2}(\overline{\Omega}) \times L^2(\Omega)$, to prove that the set $\Big\{\Big(u(\cdot, T), u_t(\cdot, T)\Big) :\ g \in \mathcal{D}(\mathcal{O} \times (0, T))\Big\}$ is dense in $L^2(\Omega) \times W^{-s,2}(\overline{\Omega})$, it suffices to show that if the pair of functions $(\psi_0, \psi_1) \in D((-\Delta)_D^s) \times L^2(\Omega)$ is such that

$$-\langle u_t(\cdot, T), \psi_0\rangle_{-\frac{1}{2},\frac{1}{2}} + \Big(u(\cdot, T), \psi_1 - \delta(-\Delta)^s\psi_0\Big)_{L^2(\Omega)} = 0, \tag{5.32}$$

for any $g \in \mathcal{D}(\mathcal{O} \times (0, T))$, then $\psi_0 = \psi_1 = 0$.

Indeed, let $(\psi_0, \psi_1) \in D((-\Delta)_D^s) \times L^2(\Omega)$ satisfy (5.32). It follows from (5.31) and (5.32) that

$$\int_0^T \int_{\mathbb{R}^N\setminus\Omega} \Big(g(x, t) + \delta g_t(x, t)\Big)\mathcal{N}_s\psi(x, t)\ \mathrm{d}x\mathrm{d}t = 0,$$

for any $g \in \mathcal{D}(\mathcal{O} \times (0,T))$. By the fundamental lemma of the calculus of variations, we have that

$$\mathcal{N}_s \psi = 0 \quad \text{in} \;\; \mathcal{O} \times (0,T).$$

It follows from Theorem 2.4 that $\psi = 0$ in $\mathcal{O} \times (0,T)$. Since the solution (ψ, ψ_t) of (2.5) is unique, we can conclude that $\psi_0 = \psi_1 = 0$ in Ω. The proof is finished. □

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