ESAIM: COCV 26 (2020) 43
https://doi.org/10.1051/cocv/2019027

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

A NON-HOMOGENEOUS BOUNDARY VALUE PROBLEM FOR THE KURAMOTO-SIVASHINSKY EQUATION POSED IN A FINITE INTERVAL*

Jing Li[1], Bing-Yu Zhang[2] and Zhixiong Zhang[3,**]

Abstract. This paper studies the initial boundary value problem (IBVP) for the dispersive Kuramoto-Sivashinsky equation posed in a finite interval $(0, L)$ with non-homogeneous boundary conditions. It is shown that the IBVP is globally well-posed in the space $H^s(0, L)$ for any $s > -2$ with the initial data in $H^s(0, L)$ and the boundary value data belonging to some appropriate spaces. In addition, the IBVP is demonstrated to be ill-posed in the space $H^s(0, L)$ for any $s < -2$ in the sense that the corresponding solution map fails to be in C^2.

Mathematics Subject Classification. 35A01, 35C15, 35D30, 35K20, 35K55.

Received March 3, 2018. Accepted April 23, 2019.

1. Introduction

In the late 1970s and early 1980s, Kuramoto [13, 14] and Sivashinsky [22, 23] derived the Kuramoto–Sivashinsky equation independently as a model for phase turbulence in reaction-diffusion systems, or as a model for plane flame propagation, describing the combined influence of diffusion and thermal conduction of the gas on stability of a plane flame front. Since then, many works addressing the Kuramoto–Sivashinsky equation have appeared in the fields of both physics and mathematics.

From the point of view of mathematics, the well-posedness problem of the Kuramoto–Sivashinsky equation has been studied extensively. Most of these existing literatures deal with the Cauchy problem, the initial boundary value problem (IBVP) with periodic boundary conditions, or the IBVP with homogeneous boundary conditions. We list a few references in these respects. For the Cauchy problem (*i.e.*, pure IVP), Pilod [21] studied the dispersive Kuramoto–Velarde equation with initial datum in $H^s(\mathbb{R})$ $(s > -1)$. For the IBVP with periodic boundary conditions, Nicolaenko and Scheurer [18] used the standard energy methods. In [24], Tadmor obtained the existence and stability by another approach. For the IBVP with homogeneous boundary values in a non-cylindrical domain, Cousin and Larkin [8] used the Faedo-Galerkin method to obtain the existence and uniqueness of global weak, strong and smooth solutions. For the non-homogeneous IBVP, there

*This work was partially supported by the National Natural Science Foundation of China (No. 11301425, No. 11571244 and No. 11231007) and the PCSIRT of the Ministry of Education of China under grant IRT_16R53.

Keywords and phrases: Kuramoto-Sivashinsky equation, initial boundary value problem, well-posedness.

[1] School of Economics and Mathematics, Southwestern University of Finance and Economics, Chengdu 611130, China.
[2] Department of Mathematical Sciences, University of Cincinnati, Cincinnati, OH 45221, USA.
[3] School of Mathematics, Sichuan University, Chengdu 610064, China.

** Corresponding author: zxzhang@amss.ac.cn

exist relatively few articles. Cerpa and Mercado [6] obtained the well-posedness results with small initial datum $u(x,0) \in H^{-2}(0,1)$ and non-homogeneous boundary conditions.

Besides the aforementioned papers on the well-posedness, there is a rich literature concerning other topics for the Kuramoto-Sivashinsky equation. To list a few of them, see [7, 9, 11, 12, 19] for the long time behavior, see [1, 5, 6] for the control problem.

This paper considers the well-posedness of the non-homogeneous IBVP for the following dispersive Kuramoto–Sivashinsky equation (sometimes, we call it Korteweg-de Vries-Kuramoto-Sivashinsky equation) defined on a finite interval $(0,L)$:

$$\begin{cases} u_t + u_{xxxx} + \delta u_{xxx} + u_{xx} + uu_x = 0, & (x,t) \in (0,L) \times (0,T), \\ u(x,0) = \phi(x), & x \in (0,L), \\ u(0,t) = h_1(t),\ u(L,t) = h_2(t),\ u_x(0,t) = h_3(t),\ u_x(L,t) = h_4(t), & t \in (0,T), \end{cases} \tag{1.1}$$

where $L > 0$, $T > 0$ and $\delta \in \mathbb{R}$. Assuming that the initial datum $\phi \in H^s(0,L)$ and the boundary data

$$\vec{h}(t) := (h_1(t),\ h_2(t),\ h_3(t),\ h_4(t)) \in \mathcal{H}^s(0,T),$$

where

$$\mathcal{H}^s(0,T) := H^{\frac{s}{4}+\frac{3}{8}}(0,T) \times H^{\frac{s}{4}+\frac{3}{8}}(0,T) \times H^{\frac{s}{4}+\frac{1}{8}}(0,T) \times H^{\frac{s}{4}+\frac{1}{8}}(0,T),$$

we seek the lowest Sobolev index s such that the solution of (1.1) exists globally in $C([0,T];H^s(0,L))$.

Recall that the Cauchy problem

$$\partial_t w + \delta\partial_x^3 w + \mu(\partial_x^4 w + \partial_x^2 w) = \alpha(\partial_x w)^2, \quad w(x,0) = w_0(x), \quad \mu > 0$$

was shown in [21] to be well-posed in $H^s(\mathbb{R})$ when $s > -1$, and is ill-posed when $s < -1$ in the sense that the corresponding solution map fails to be C^2. Taking $\mu = 1$, $\alpha = -\frac{1}{2}$, and setting $u = w_x$, it leads to the global well-posedness in $H^s(\mathbb{R})$ $(s > -2)$ for the following dispersive Kuramto–Sivashinsky equation:

$$\begin{cases} u_t + u_{xxxx} + \delta u_{xxx} + u_{xx} + uu_x = 0, & (x,t) \in \mathbb{R} \times (0,T), \\ u(x,0) = \phi(x), & x \in \mathbb{R}. \end{cases} \tag{1.2}$$

In addition, the IBVP of the following dispersive Kuramoto–Sivashinsky equation posed in a half line $\mathbb{R}^+$

$$\begin{cases} u_t + u_{xxxx} + \delta u_{xxx} + u_{xx} + uu_x = 0, & (x,t) \in \mathbb{R}^+ \times (0,T), \\ u(x,0) = \phi(x), & x \in \mathbb{R}^+, \\ u(0,t) = h_1(t), \quad u_x(0,t) = h_2(t), & t \in (0,T), \end{cases} \tag{1.3}$$

was shown by us to be globally well-posed in the space $H^s(\mathbb{R}^+)$ for any $s > -2$ with $\phi \in H^s(\mathbb{R}^+)$, and $(h_1(t),\ h_2(t))$ belongs to some subspace of $H^{\frac{s}{4}+\frac{3}{8}}(0,T) \times H^{\frac{s}{4}+\frac{1}{8}}(0,T)$ (see [16], Thm. 1.1).

Now we turn to consider the IBVP (1.1). Before we present the global well-posedness result for IBVP (1.1) in our main theorems, we give the following definition of s-compatibility conditions.

Definition 1.1. Let $T>0$ and $s>\frac{1}{2}$ be given. $(\phi,\vec{h})\in H^s(0,L)\times\mathcal{H}^s(0,T)$ is said to be s-compatible if

(i)

$$(\phi_k(0),\phi_k(L),\phi_k'(0),\phi_k'(L))=\vec{h}^{(k)}(0):=(h_1^{(k)}(0),\ h_2^{(k)}(0),\ h_3^{(k)}(0),\ h_4^{(k)}(0)),\quad k=0,1,\dots,[\tfrac{s}{4}]-1$$

when $s-4[\frac{s}{4}]\le\frac{1}{2}$;

(ii)

$$\begin{aligned}&(\phi_k(0),\phi_k(L),\phi_k'(0),\phi_k'(L))=\vec{h}^{(k)}(0), && k=0,1,\dots,[\tfrac{s}{4}]-1;\\ &(\phi_k(0),\phi_k(L))=(h_1^{(k)}(0),h_2^{(k)}(0)), && k=[\tfrac{s}{4}]\end{aligned}$$

when $\frac{1}{2}<s-4[\frac{s}{4}]\le\frac{3}{2}$;

(iii)

$$(\phi_k(0),\phi_k(L),\phi_k'(0),\phi_k'(L))=\vec{h}^{(k)}(0),\qquad k=0,1,\dots,[\tfrac{s}{4}]$$

when $s-4[\frac{s}{4}]>\frac{3}{2}$.

Here,

$$\begin{cases}\phi_0(x)=\phi(x),\\ \phi_k(x)=-\phi_{k-1}''''(x)-\delta\phi_{k-1}'''(x)-\phi_{k-1}''(x)-\displaystyle\sum_{j=0}^{k-1}C_{k-1}^j\phi_j(x)\phi_{k-j-1}'(x),\quad k=1,2,\dots\end{cases}$$

where $C_{k-1}^j=\frac{(k-1)!}{j!(k-j-1)!}$ and $[\cdot]$ is the floor function.

Our first main result in this paper is stated as follows.

Theorem 1.2. *Let $L>0$, $T>0$, $\delta\in\mathbb{R}$ and $(\phi,\vec{h})\in H^s(0,L)\times\mathcal{H}^s(0,T)$ is s-compatible.*

(i) If $s\ge 0$, then IBVP (1.1) admits a unique solution $u\in C([0,T];H^s(0,L))$.
(ii) If $-2<s<0$, and in addition $t^{\frac{|s|}{4}+\varepsilon}\vec{h}\in\mathcal{H}^0(0,T)$ for some $0<\varepsilon\ll 1$, then IBVP (1.1) admits a unique solution $u\in C([0,T];H^s(0,L))$.

Moreover, in both (i) and (ii), the corresponding solution map from the space of initial and boundary data to the solution space is continuous.[1]

Our next main result is about the ill-posedness of the following Kuramoto–Sivashinsky equation defined on a finite interval $(0,L)$ with homogeneous boundary conditions:

$$\begin{cases}u_t+u_{xxxx}+u_{xx}+uu_x=0, & (x,t)\in(0,L)\times(0,T),\\ u(x,0)=\phi(x), & x\in(0,L),\\ u(0,t)=0,\ u(L,t)=0,\ u_{xx}(0,t)=0,\ u_{xx}(L,t)=0, & t\in(0,T).\end{cases}\tag{1.4}$$

[1]The solution map is in fact real analytic.

Theorem 1.3. *Let $L > 0$, $T > 0$ and $s < -2$. Then the flow map of the homogeneous IBVP (1.4):*

$$\Xi : H^s(0, L) \to C([0, T]; H^s(0, L)), \quad \phi \mapsto u(t)$$

is not C^2 at the origin.

Remark 1.4. Theorem 1.3 is some kind of "weak" ill-posedness result. It implies that one cannot use the contraction mapping argument to obtain the well-posedness of IBVP (1.4) when $s < -2$. To our best knowledge, how to construct a counterexample directly to show the ill-posedness when $s < -2$ is still open.

This paper mainly focuses on lower regularity solutions of IBVP (1.1), especially when the Sobolev space index s is negative. For smoother solutions when $s \geq 0$, the global well-posedness result can be obtained by the same approach developed in Section 4 of [16]. Hence, we omit the proof for the first part of Theorem 1.2.

Our strategy to obtain the well-posedness in this paper is similar to that in [16]. Both papers have used the contraction mapping principle to yield the local well-posedness, based on the analysis of the associated linear problems. In the analysis of the boundary integral operator $W_{bdr}(t)$, both papers have borrowed the idea developed in [2–4] for the Korteweg-de Vries equation. However, in this paper, we choose a different kind of working space from the working space $X^{\varepsilon}_{s,T}$ defined in (3.3) of [16]. As a consequence, there exist many technical differences both in establishing smoothing properties and nonlinear estimates between the two papers.

On the other hand, our paper also has many similarities to [21]. For instance, our working space $\mathcal{X}_{s,T}$ defined at the beginning of Section 3 is very much like the space X^s_T defined in (12) of [21]. Our smoothing properties are expressed in a style like Proposition 1 in [21]. Our estimate for the nonlinear term uu_x in Section 3 is nearly along the same way as Proposition 2 in [21]. In some sense, the approach developed in [21] for the Cauchy problem has been adapted to treat the non-homogeneous IBVP in this paper.

The advantage of our new approach used in this paper lies in two aspects. One is to make our proof more concise and shorter compared with the one in [16]. The other is to enable us to follow the existing proof method for the Cauchy problem when one continues to study the IBVP.

When $-2 < s \leq 0$, both approaches adopted in this paper and [16] can yield the well-posedness of IBVP (1.1) and IBVP (1.3), respectively. Unfortunately, these approaches can not effectively tackle with the limiting case $s = -2$. We believe that the well-posedness with big initial data for $s = -2$ may be obtained by an alternative method. We do not discuss the ill-posedness of (1.1) when $s < -2$ in this paper for technical reasons. Instead, we study the ill-posedness of (1.4) when $s < -2$ and show that the corresponding solution map fails to be C^2. This means that it is impossible to use the contraction mapping principle to prove the well-posedness of IBVP (1.4).

The rest of this paper is organized as follows. Section 2 is devoted to study the associated linear problems to obtain various smoothing properties for their solutions. In Section 3, we deal with the nonlinear estimates. Thanks to the fixed point theory, the local well-posedness is established. In Section 4, a priori estimate is derived when the Sobolev index $s = 0$. And we obtain the global well-posedness for the IBVP (1.1). In Section 5, we obtain the ill-posedness of IBVP (1.4) when $s < -2$.

2. Smoothing properties for the associated linear problems

In this section, we discuss the associated linear problem, which is fundamental for the analysis of the nonlinear problem given in Sections 3 and 4.

2.1. Statements of the results

Recall $\vec{h}(t) := (h_1(t),\ h_2(t),\ h_3(t),\ h_4(t))$ and

$$\mathcal{H}^s(0, T) := H^{\frac{s}{4}+\frac{3}{8}}(0, T) \times H^{\frac{s}{4}+\frac{3}{8}}(0, T) \times H^{\frac{s}{4}+\frac{1}{8}}(0, T) \times H^{\frac{s}{4}+\frac{1}{8}}(0, T)$$

for any $s \in \mathbb{R}$.

It is worth mentioning that the following two propositions still hold for an arbitrary Sobolev index s. However, we only give the proof for $-2 \le s \le 0$.

Proposition 2.1. *Let $\delta \in \mathbb{R}$, $L > 0$, $T > 0$, $-2 \le s \le 0$ and $\phi \in H^s(0,L)$. Then IBVP*

$$\begin{cases} v_t + v_{xxxx} + \delta v_{xxx} = 0, & (x,t) \in (0,L) \times (0,T), \\ v(x,0) = \phi(x), & x \in (0,L), \\ v(0,t) = 0,\ v(L,t) = 0,\ v_x(0,t) = 0,\ v_x(L,t) = 0, \ t \in (0,T) \end{cases} \tag{2.1}$$

admits a unique solution $v \in C([0,T]; H^s(0,L))$. Moreover, there exists a constant $C > 0$ such that for any $\phi \in H^s(0,L)$ and $0 < \epsilon \ll 1$, it holds

$$\|v\|_{C([0,T];H^s(0,L))} \le C\|\phi\|_{H^s(0,L)}, \qquad \|t^{\frac{|s|}{4}+\epsilon} v\|_{C([0,T];L^2(0,L))} \le C\|\phi\|_{H^s(0,L)}. \tag{2.2}$$

Proposition 2.2. *Let $\delta \in \mathbb{R}$, $L > 0$, $T > 0$, $-2 \le s \le 0$ and $\vec{h} \in \mathcal{H}^s(0,T)$. Then IBVP*

$$\begin{cases} v_t + v_{xxxx} + \delta v_{xxx} = 0, & (x,t) \in (0,L) \times (0,T), \\ v(x,0) = 0, & x \in (0,L), \\ v(0,t) = h_1(t),\ v(L,t) = h_2(t),\ v_x(0,t) = h_3(t),\ v_x(L,t) = h_4(t), & t \in (0,T) \end{cases} \tag{2.3}$$

admits a unique solution $v \in C([0,T]; H^s(0,L))$. Moreover, there exists a constant $C > 0$ such that for any $\vec{h} \in \mathcal{H}^s(0,T)$, it holds

$$\|v\|_{C([0,T];H^s(0,L))} \le C\|\vec{h}\|_{\mathcal{H}^s(0,T)}. \tag{2.4}$$

Furthermore, if $t^{\frac{|s|}{4}+\varepsilon}\vec{h} \in \mathcal{H}^0(0,T)$ for some $0 < \varepsilon \ll 1$, then it holds

$$\|t^{\frac{|s|}{4}+\varepsilon} v\|_{C([0,T];L^2(0,L))} \le C\left(\|\vec{h}\|_{\mathcal{H}^s(0,T)} + \|t^{\frac{|s|}{4}+\varepsilon}\vec{h}\|_{\mathcal{H}^0(0,T)}\right). \tag{2.5}$$

2.2. Proof of Proposition 2.1

Step 1. Define a linear operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset L^2(0,L) \to L^2(0,L)$ as below

$$\mathcal{A}w := -w_{xxxx}, \qquad \mathcal{D}(\mathcal{A}) = H^4(0,L) \cap H_0^2(0,L).$$

By semigroup theory (*cf*. Thm. 2.7 in Sect. 7.2 of [20]), the operator $\mathcal{A}$ generates an analytic semigroup in $L^2(0,L)$. With the perturbation of a lower order operator, $\mathcal{A}_1 w := \mathcal{A}w - \delta w_{xxx}$ is still the infinitesimal generator of an analytic semigroup $W_c(t)$ in $L^2(0,L)$. By restriction, $\mathcal{A}_1$ can generate a C_0-semigroup in $V_\alpha = D({\mathcal{A}_1}^\alpha)$ for $0 < \alpha \le 1$ (for the definition of fractional powers of closed operators, see Sect. 2.6 of [20]). Then by duality, $\mathcal{A}_1$ generates a C_0-semigroup in $H^s(0,L)$ for any $-2 \le s < 0$. We still denote this semigroup by $W_c(t)$ for simplicity.

Hence, for any $-2 \le s \le 0$, IBVP (2.1) admits a unique solution v in $C([0,T]; H^s(0,L))$ satisfying $\|v\|_{C([0,T];H^s(0,L))} \le C\|\phi\|_{H^s(0,L)}$ for some constant C depending only on s. Thus the first estimate in (2.2) is true.

Step 2. We prove that the second estimate in (2.2) holds for any $-2 \le s \le 0$.

When $-2<s\le 0$, multiplying the first equation in (2.1) by $\tau^{\frac{|s|}{2}+2\epsilon}v$, integrating by parts and noting that $v(0,t)=0,\ v(L,t)=0,\ v_x(0,t)=0,\ v_x(L,t)=0$, we arrive at

$$\frac{1}{2}\int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau}\left\|\tau^{\frac{|s|}{4}+\epsilon}v\right\|^2_{L^2(0,L)}\mathrm{d}\tau-\left(\frac{|s|}{4}+\epsilon\right)\int_0^t \tau^{\frac{|s|}{2}-1+2\epsilon}\|v\|^2_{L^2(0,L)}\mathrm{d}\tau+\int_0^t\|\tau^{\frac{|s|}{4}+\epsilon}v_{xx}\|^2_{L^2(0,L)}\mathrm{d}\tau=0.$$

This yields that for any $t\in[0,T]$,

$$\frac{1}{2}\|t^{\frac{|s|}{4}+\epsilon}v\|^2_{L^2(0,L)}+\int_0^t\|\tau^{\frac{|s|}{4}+\epsilon}v_{xx}\|^2_{L^2(0,L)}\mathrm{d}\tau=(\frac{|s|}{4}+\epsilon)\int_0^t\tau^{\frac{|s|}{2}-1+2\epsilon}\|v\|^2_{L^2(0,L)}\mathrm{d}\tau$$

which, using the Hölder inequality, leads to

$$\begin{aligned}&\|t^{\frac{|s|}{4}+\epsilon}v\|^2_{C([0,T];L^2(0,L))}+\|t^{\frac{|s|}{4}+\epsilon}v\|^2_{L^2(0,T;H^2(0,L))}\\ &\le C\left(\int_0^T\tau^{\frac{\frac{|s|}{2}-1+2\epsilon}{1-\frac{|s|}{2}}}\mathrm{d}\tau\right)^{1-\frac{|s|}{2}}\left(\int_0^T\|v\|^{\frac{4}{|s|}}_{L^2(0,L)}\mathrm{d}\tau\right)^{\frac{|s|}{2}},\\ &\le C\|v\|^2_{L^{\frac{4}{|s|}}(0,T;L^2(0,L))}\end{aligned}\tag{2.6}$$

here, $\displaystyle\int_0^T\tau^{\frac{\frac{|s|}{2}-1+2\epsilon}{1-\frac{|s|}{2}}}\mathrm{d}\tau<+\infty$ for any $-2<s\le 0$ and $0<\epsilon\ll 1$, and C is a common constant that may change its value from line to line.

When $s=-2$, multiplying the first equation in (2.1) by τv, we get that for any $t\in[0,T]$,

$$\frac{1}{2}\int_0^t\frac{\mathrm{d}}{\mathrm{d}\tau}\|\tau^{\frac{1}{2}}v\|^2_{L^2(0,L)}\mathrm{d}\tau-\frac{1}{2}\int_0^t\|v\|^2_{L^2(0,L)}\mathrm{d}\tau+\int_0^t\|\tau^{\frac{1}{2}}v_{xx}\|^2_{L^2(0,L)}\mathrm{d}\tau=0,$$

which gives

$$\|t^{\frac{1}{2}}v\|^2_{C([0,T];L^2(0,L))}+\|t^{\frac{1}{2}}v\|^2_{L^2(0,T;H^2(0,L))}\le C\|v\|^2_{L^2(0,T;L^2(0,L))}\mathrm{d}\tau.\tag{2.7}$$

Combining (2.6) and (2.7) together yields that for any $-2\le s\le 0$ and $0<\epsilon\ll 1$, it holds

$$\|t^{\frac{|s|}{4}+\epsilon}v\|^2_{C([0,T];L^2(0,L))}+\|t^{\frac{|s|}{4}+\epsilon}v\|^2_{L^2(0,T;H^2(0,L))}\le C\|v\|^2_{L^{\frac{4}{|s|}}(0,T;L^2(0,L))}.\tag{2.8}$$

From the results in Section 15 of Chapter 4 in [15], it follows that the solution of IBVP (2.1) satisfies $v\in H^{\frac{s}{4}+\frac{1}{2}}(0,T;L^2(0,L))$ and

$$\|v\|_{H^{\frac{s}{4}+\frac{1}{2}}(0,T;L^2(0,L))}\le C\|\phi\|_{H^s(0,L)}.\tag{2.9}$$

Therefore, by (2.8), (2.9) and the embedding $H^{\frac{s}{4}+\frac{1}{2}}(0,T)\hookrightarrow L^{\frac{4}{|s|}}(0,T)$,

$$\|t^{\frac{|s|}{4}+\epsilon}v\|_{C([0,T];L^2(0,L))}+\|t^{\frac{|s|}{4}+\epsilon}v\|_{L^2(0,T;H^2(0,L))}\le C\|\phi\|_{H^s(0,L)}.\tag{2.10}$$

Hence, (2.10) implies that the second estimate in (2.2) holds. □

2.3. Proof of Proposition 2.2

In this proof, we borrow some ideas from [2–4], which were used to study non-homogenous boundary value problems of the Korteweg-de Vries equation. By harmonic analysis tools, the solution of IBVP (2.3) can be represented explicitly. First, we recall the following lemma, which is Lemma 2.5 in [3] with a minor modification.

Lemma 2.3. *Let $\gamma(\rho)$ be a continuous complex-valued function defined on $(0,+\infty)$ satisfying the following two conditions:*

(i) There exist $\delta>0$ and $b>0$ such that $\sup\limits_{0<\rho<\delta}\dfrac{|Re\gamma(\rho)|}{\rho}\geq b$;

(ii) There exists a complex number $\alpha+i\beta$ such that $\lim\limits_{\rho\to+\infty}\dfrac{\gamma(\rho)}{\rho}=\alpha+i\beta$.

Then there exists a constant $C>0$ such that for all $f\in L^2(0,+\infty)$,

$$\left\|\int_0^{+\infty}e^{\gamma(\rho)x}f(\rho)\mathrm{d}\rho\right\|_{L^2(0,L)}\leq C\left(\|e^{LRe\gamma(\cdot)}f(\cdot)\|_{L^2(\mathbb{R}^+)}+\|f(\cdot)\|_{L^2(\mathbb{R}^+)}\right).$$

We now return to the proof of Proposition 2.2. The proof is divided into three steps.

Step 1. Apply the Laplace transform with respect to t in equation (2.3) to yield that for any τ with $\mathrm{Re}\tau>0$,

$$\begin{cases}\tau\hat{v}+\hat{v}_{xxxx}+\delta\hat{v}_{xxx}=0, & (x,\tau)\in(0,L)\times\mathbb{C},\\ \hat{v}(0,\tau)=\hat{h}_1(\tau),\ \hat{v}(L,\tau)=\hat{h}_2(\tau),\ \hat{v}_x(0,\tau)=\hat{h}_3(\tau),\ \hat{v}_x(L,\tau)=\hat{h}_4(\tau), & \tau\in\mathbb{C},\end{cases}\tag{2.11}$$

where

$$\hat{v}(x,\tau)=\int_0^{+\infty}e^{-\tau t}v(x,t)\mathrm{d}t,\qquad \hat{h}_j(\tau)=\int_0^{+\infty}e^{-\tau t}h_j(t)\mathrm{d}t,\quad j=1,2,3,4.$$

The solution of IBVP (2.11) can be expressed as below

$$\hat{v}(x,\tau)=\sum_{j=1}^4 c_j(\tau)e^{\lambda_j(\tau)x},\tag{2.12}$$

where $\lambda_j(\tau)$ $(j=1,2,3,4)$ are the solutions of the characteristic equation

$$\lambda^4+\delta\lambda^3+\tau=0.\tag{2.13}$$

And $c_j(\tau)$ $(j=1,2,3,4)$ are the solutions of

$$\begin{cases}c_1+c_2+c_3+c_4=\hat{h}_1(\tau),\\ c_1e^{L\lambda_1(\tau)}+c_2e^{L\lambda_2(\tau)}+c_3e^{L\lambda_3(\tau)}+c_4e^{L\lambda_4(\tau)}=\hat{h}_2(\tau),\\ c_1\lambda_1(\tau)+c_2\lambda_2(\tau)+c_3\lambda_3(\tau)+c_4\lambda_4(\tau)=\hat{h}_3(\tau)\\ c_1\lambda_1(\tau)e^{L\lambda_1(\tau)}+c_2\lambda_2(\tau)e^{L\lambda_2(\tau)}+c_3\lambda_3(\tau)e^{L\lambda_3(\tau)}+c_4\lambda_4(\tau)e^{L\lambda_4(\tau)}=\hat{h}_4(\tau).\end{cases}\tag{2.14}$$

From (2.12)–(2.14) and by Cramer's rule, one finds that for any fixed $r > 0$, the solution of (2.3) takes the form of

$$v(x,t) = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{\tau t}\hat{v}(x,\tau)\mathrm{d}\tau = \sum_{j=1}^{4}\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{\tau t}\frac{\Delta_j(\tau)}{\Delta(\tau)}e^{\lambda_j(\tau)x}\mathrm{d}\tau, \tag{2.15}$$

where $\Delta(\tau)$ denotes the determinant of the coefficient matrix of (2.14), and $\Delta_j(\tau)$ denotes the determinants of the matrices which replace the jth-column of $\Delta(\tau)$ by the column vector $(\hat{h}_1(\tau), \hat{h}_2(\tau), \hat{h}_3(\tau), \hat{h}_4(\tau))^T$.

Let $v_m(x,t)$ solve IBVP (2.3) with $h_j(t) = 0$ when $j \neq m$. Therefore, (2.15) shows that

$$v(x,t) = \sum_{m=1}^{4} v_m(x,t) = \sum_{m=1}^{4}\sum_{j=1}^{4}\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{\tau t}\frac{\Delta_{j,m}(\tau)}{\Delta(\tau)}e^{\lambda_j(\tau)x}\hat{h}_m(\tau)\mathrm{d}\tau, \tag{2.16}$$

where $\Delta_{j,m}(\tau)$ is obtained from $\Delta_j(\tau)$ by letting $\hat{h}_m(\tau) = 1$ and $\hat{h}_l(\tau) = 0$ for $l \neq m$. Since the right-hand side of (2.16) is continuous with respect to r and the left-hand side does not depend on r, we can take $r = 0$ in (2.16). Consequently,

$$\begin{aligned} v(x,t) &= \sum_{m=1}^{4}\sum_{j=1}^{4}\frac{1}{2\pi i}\int_{0}^{+i\infty} e^{\tau t}\frac{\Delta_{j,m}(\tau)}{\Delta(\tau)}e^{\lambda_j(\tau)x}\hat{h}_m(\tau)\mathrm{d}\tau \\ &\quad + \sum_{m=1}^{4}\sum_{j=1}^{4}\frac{1}{2\pi i}\int_{-i\infty}^{0} e^{\tau t}\frac{\Delta_{j,m}(\tau)}{\Delta(\tau)}e^{\lambda_j(\tau)x}\hat{h}_m(\tau)\mathrm{d}\tau \\ &:= v^+(x,t) + v^-(x,t). \end{aligned} \tag{2.17}$$

To obtain the estimate for $v^+(x,t)$, we set $\tau = i8\rho^4$ with $0 \leq \rho < +\infty$ in (2.13). Then the solutions of characteristic equation (2.13) also satisfy

$$\left(\frac{\lambda}{\rho}\right)^4 + \delta\frac{1}{\rho}\left(\frac{\lambda}{\rho}\right)^3 + 8i = 0$$

when $\rho > 0$. For any given $\delta \in \mathbb{R}$, as $\rho \to +\infty$, it has $\dfrac{\lambda}{\rho} = \dfrac{\lambda^0}{\rho} + o(1)$, where $\dfrac{\lambda^0}{\rho}$ satisfies

$$\left(\frac{\lambda^0}{\rho}\right)^4 + 8i = 0.$$

A direct computation shows that those $\dfrac{\lambda^0}{\rho}$ satisfying the above equation are just

$$-\sqrt{\sqrt{2}+1} \pm i\sqrt{\sqrt{2}-1}, \qquad \sqrt{\sqrt{2}+1} \pm i\sqrt{\sqrt{2}-1}.$$

Thus, the four solutions of characteristic equation (2.13) can be written as below

$$\lambda_1^+(\rho) = -\rho\sqrt{\sqrt{2}+1} + i\rho\sqrt{\sqrt{2}-1} + o(\rho), \qquad \lambda_2^+(\rho) = \rho\sqrt{\sqrt{2}+1} - i\rho\sqrt{\sqrt{2}-1} + o(\rho),$$
$$\lambda_3^+(\rho) = -\rho\sqrt{\sqrt{2}-1} - i\rho\sqrt{\sqrt{2}+1} + o(\rho), \qquad \lambda_4^+(\rho) = \rho\sqrt{\sqrt{2}-1} + i\rho\sqrt{\sqrt{2}+1} + o(\rho).$$

Hence, from (2.17), one gets that

$$v^+(x,t) = \sum_{m=1}^{4}\sum_{j=1}^{4} \frac{16}{\pi}\int_0^{+\infty} e^{i8\rho^4 t} e^{\lambda_j^+(\rho)x} \frac{\Delta_{j,m}^+(\rho)}{\Delta^+(\rho)} \rho^3 \hat{h}_m^+(\rho)\mathrm{d}\rho, \tag{2.18}$$

where $\Delta_{j,m}^+(\rho) := \Delta_{j,m}(i8\rho^4)$, $\Delta^+(\rho) := \Delta(i8\rho^4)$, $\hat{h}_m^+(\rho) := \hat{h}_m(i8\rho^4)$.

Similarly, to estimate $v^-(x,t)$, we put $\tau = \overline{i8\rho^4} = -i8\rho^4$ with $0 \le \rho < +\infty$ in characteristic equation (2.13). Then the corresponding solutions of (2.13) are

$$\lambda_j^-(\rho) = \overline{\lambda_j^+(\rho)}, \quad j = 1,2,3,4,$$

where $\overline{\lambda_j^+(\rho)}$ is the conjugation of $\lambda_j^+(\rho)$. Hence, (2.17) yields that

$$v^-(x,t) = \sum_{m=1}^{4}\sum_{j=1}^{4} \frac{16}{\pi}\int_0^{+\infty} e^{-i8\rho^4 t} e^{\lambda_j^-(\rho)x} \frac{\Delta_{j,m}^-(\rho)}{\Delta^-(\rho)} \rho^3 \hat{h}_m^-(\rho)\mathrm{d}\rho, \tag{2.19}$$

where $\Delta_{j,m}^-(\rho) := \Delta_{j,m}(-i8\rho^4)$, $\Delta^-(\rho) := \Delta(-i8\rho^4)$, $\hat{h}_m^-(\rho) := \hat{h}_m(-i8\rho^4)$. Comparing (2.18) and (2.19), we deduce that

$$v^-(x,t) = \overline{v^+(x,t)},$$

which gives

$$v(x,t) = v^+(x,t) + \overline{v^+(x,t)}. \tag{2.20}$$

Now we calculate the large-ρ asymptotics of the ratios $\frac{\Delta_{j,m}^+(\rho)}{\Delta^+(\rho)}$ ($m = 1,2,3,4;\ j = 1,2,3,4$). It is easy to check that

$$\begin{cases} \Delta_{1,1}^+(\rho) = \lambda_4^+(\lambda_2^+ - \lambda_3^+)e^{L(\lambda_2^+ + \lambda_3^+)} + \lambda_2^+(\lambda_3^+ - \lambda_4^+)e^{L(\lambda_3^+ + \lambda_4^+)} + \lambda_3^+(\lambda_4^+ - \lambda_2^+)e^{L(\lambda_4^+ + \lambda_2^+)}, \\ \Delta_{2,1}^+(\rho) = \lambda_4^+(\lambda_3^+ - \lambda_1^+)e^{L(\lambda_1^+ + \lambda_3^+)} + \lambda_1^+(\lambda_4^+ - \lambda_3^+)e^{L(\lambda_3^+ + \lambda_4^+)} + \lambda_3^+(\lambda_1^+ - \lambda_4^+)e^{L(\lambda_4^+ + \lambda_1^+)}, \\ \Delta_{3,1}^+(\rho) = \lambda_4^+(\lambda_1^+ - \lambda_2^+)e^{L(\lambda_1^+ + \lambda_2^+)} + \lambda_1^+(\lambda_2^+ - \lambda_4^+)e^{L(\lambda_2^+ + \lambda_4^+)} + \lambda_2^+(\lambda_4^+ - \lambda_1^+)e^{L(\lambda_4^+ + \lambda_1^+)}, \\ \Delta_{4,1}^+(\rho) = \lambda_3^+(\lambda_2^+ - \lambda_1^+)e^{L(\lambda_1^+ + \lambda_2^+)} + \lambda_1^+(\lambda_3^+ - \lambda_2^+)e^{L(\lambda_2^+ + \lambda_3^+)} + \lambda_2^+(\lambda_1^+ - \lambda_3^+)e^{L(\lambda_3^+ + \lambda_1^+)}, \end{cases}$$

and

$$\begin{aligned}\Delta^+(\rho) \quad &= (\lambda_1^+ - \lambda_2^+)(\lambda_4^+ - \lambda_3^+)\left(e^{L(\lambda_1^+ + \lambda_2^+)} + e^{L(\lambda_3^+ + \lambda_4^+)}\right)\\ &+ (\lambda_1^+ - \lambda_3^+)(\lambda_2^+ - \lambda_4^+)\left(e^{L(\lambda_1^+ + \lambda_3^+)} + e^{L(\lambda_2^+ + \lambda_4^+)}\right)\\ &+ (\lambda_1^+ - \lambda_4^+)(\lambda_3^+ - \lambda_2^+)\left(e^{L(\lambda_1^+ + \lambda_4^+)} + e^{L(\lambda_2^+ + \lambda_3^+)}\right).\end{aligned}$$

Therefore, as $\rho \to +\infty$, it holds

$$\begin{cases} \dfrac{\Delta_{1,1}^+(\rho)}{\Delta^+(\rho)} \sim 1,\ \dfrac{\Delta_{2,1}^+(\rho)}{\Delta^+(\rho)} \sim e^{-L\rho\left(\sqrt{\sqrt{2}+1}+\sqrt{\sqrt{2}-1}\right)}, \\ \dfrac{\Delta_{3,1}^+(\rho)}{\Delta^+(\rho)} \sim 1,\ \dfrac{\Delta_{4,1}^+(\rho)}{\Delta^+(\rho)} \sim e^{-2L\rho\sqrt{\sqrt{2}-1}}. \end{cases} \tag{2.21}$$

Similarly, one can show that

$$\begin{cases} \dfrac{\Delta_{1,2}^+(\rho)}{\Delta^+(\rho)} \sim e^{-L\rho\sqrt{\sqrt{2}-1}},\ \dfrac{\Delta_{2,2}^+(\rho)}{\Delta^+(\rho)} \sim e^{-L\rho\sqrt{\sqrt{2}+1}}, \\ \dfrac{\Delta_{3,2}^+(\rho)}{\Delta^+(\rho)} \sim e^{-L\rho\sqrt{\sqrt{2}-1}},\ \dfrac{\Delta_{4,2}^+(\rho)}{\Delta^+(\rho)} \sim e^{-L\rho\sqrt{\sqrt{2}-1}}. \end{cases} \tag{2.22}$$

$$\begin{cases} \dfrac{\Delta_{1,3}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-1},\ \dfrac{\Delta_{2,3}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-1}e^{-L\rho\left(\sqrt{\sqrt{2}+1}+\sqrt{\sqrt{2}-1}\right)}, \\ \dfrac{\Delta_{3,3}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-1},\ \dfrac{\Delta_{4,3}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-1}e^{-2L\rho\sqrt{\sqrt{2}-1}}. \end{cases} \tag{2.23}$$

$$\begin{cases} \dfrac{\Delta_{1,4}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-1}e^{-L\rho\sqrt{\sqrt{2}-1}},\ \dfrac{\Delta_{2,4}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-1}e^{-L\rho\sqrt{\sqrt{2}+1}}, \\ \dfrac{\Delta_{3,4}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-1}e^{-L\rho\sqrt{\sqrt{2}-1}},\ \dfrac{\Delta_{4,4}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-1}e^{-L\rho\sqrt{\sqrt{2}-1}}. \end{cases} \tag{2.24}$$

Let $v(x,t) = W_{bdr}(t)\vec{h}$ be the solution map of equation (2.3), then $W_{bdr}(t)\vec{h}$ has been represented explicitly by (2.18) and (2.20). In Step 2, we will show that $W_{bdr}(t)$ is actually a continuous map form $\mathcal{H}^s(0,T)$ to $C([0,T];H^s(0,L))$.

Step 2. After a continuous extension (for instance, a zero extension), $\vec{h}(t)$ can be defined on $\mathbb{R}^+$. For simplicity, we still denote the extension by $\vec{h}(t)$, and denote the corresponding solution of the pure BVP by v. Therefore, to prove (2.4), it suffices to show that for $-2 \le s \le 0$,

$$\sup_{t\in\mathbb{R}^+} \|v(\cdot,t)\|_{H^s(0,L)} \le C\|\vec{h}\|_{\mathcal{H}^s(\mathbb{R}^+)}. \tag{2.25}$$

First, we prove that estimate (2.25) holds when $s=0,-1,-2$. This technical requirement for s appears in (2.26), where the $H^s(0,L)$-norm estimate can be transfered to the $L^2(0,L)$-norm one. Then, for non-integer negative values of s, estimate (2.25) can be obtained by standard interpolation theory.

From (2.18), we see that for any $t\in\mathbb{R}^+$,

$$\begin{aligned}\|v^+(\cdot,t)\|^2_{H^s(0,L)} &\le C\sum_{m=1}^4\sum_{j=1}^4\left\|\int_0^{+\infty} e^{i8\rho^4t}e^{\lambda_j^+(\rho)x}\frac{\Delta^+_{j,m}(\rho)}{\Delta^+(\rho)}\rho^3\hat{h}^+_m(\rho)\mathrm{d}\rho\right\|^2_{H^s(0,L)}\\ &\le C\sum_{m=1}^4\sum_{j=1}^4\left\|\int_0^{+\infty}\left(\lambda_j^+(\rho)\right)^s e^{\lambda_j^+(\rho)x}e^{i8\rho^4t}\frac{\Delta^+_{j,m}(\rho)}{\Delta^+(\rho)}\rho^3\hat{h}^+_m(\rho)\mathrm{d}\rho\right\|^2_{L^2(0,L)}.\end{aligned}\tag{2.26}$$

By (2.26) and Lemma 2.3, one has that for any $t\in\mathbb{R}^+$,

$$\|v^+(\cdot,t)\|^2_{H^s(0,L)}\le C\sum_{m=1}^4\sum_{j=1}^4\int_0^{+\infty}\left(e^{L\mathrm{Re}\lambda_j^+(\rho)}+1\right)^2\left|\lambda_j^+(\rho)\right|^{2s}\left|\frac{\Delta^+_{j,m}(\rho)}{\Delta^+(\rho)}\right|^2\rho^6\left|\hat{h}^+_m(\rho)\right|^2\mathrm{d}\rho.$$

Therefore, from (2.21)–(2.24), the above inequality yields

$$\begin{aligned}&\sup_{t\in\mathbb{R}^+}\|v^+(\cdot,t)\|^2_{H^s(0,L)}\\ &\le C\sum_{m=1}^4\sum_{j=1}^4\int_0^{+\infty}\rho^{2s+6}\left(e^{L\mathrm{Re}\lambda_j^+(\rho)}+1\right)^2\left|\frac{\Delta^+_{j,m}(\rho)}{\Delta^+(\rho)}\right|^2\left|\hat{h}^+_m(\rho)\right|^2\mathrm{d}\rho\\ &\le C\int_0^{+\infty}\rho^{2s+6}\left|\hat{h}^+_1(\rho)\right|^2\mathrm{d}\rho+C\int_0^{+\infty}\rho^{2s+6}\left|\hat{h}^+_2(\rho)\right|^2\mathrm{d}\rho\\ &\quad+C\int_0^{+\infty}\rho^{2s+4}\left|\hat{h}^+_3(\rho)\right|^2\mathrm{d}\rho+C\int_0^{+\infty}\rho^{2s+4}\left|\hat{h}^+_4(\rho)\right|^2\mathrm{d}\rho.\end{aligned}\tag{2.27}$$

Setting $\mu=8\rho^4$ in (2.27), and recalling that $\hat{h}^+_m(\rho):=\hat{h}_m(i8\rho^4)$ and $\hat{h}_m(\tau)=\int_0^{+\infty}e^{-\tau t}h_m(t)\mathrm{d}t$, we conclude that

$$\begin{aligned}\sup_{t\in\mathbb{R}^+}\|v^+(\cdot,t)\|^2_{H^s(0,L)}&\le C\int_0^{+\infty}(1+\mu^2)^{\frac{s}{4}+\frac{3}{8}}\left|\int_0^{+\infty}e^{-i\mu\tau}h_1(\tau)\mathrm{d}\tau\right|^2\mathrm{d}\mu\\ &\quad+C\int_0^{+\infty}(1+\mu^2)^{\frac{s}{4}+\frac{3}{8}}\left|\int_0^{+\infty}e^{-i\mu\tau}h_2(\tau)\mathrm{d}\tau\right|^2\mathrm{d}\mu\\ &\quad+C\int_0^{+\infty}(1+\mu^2)^{\frac{s}{4}+\frac{1}{8}}\left|\int_0^{+\infty}e^{-i\mu\tau}h_3(\tau)\mathrm{d}\tau\right|^2\mathrm{d}\mu\\ &\quad+C\int_0^{+\infty}(1+\mu^2)^{\frac{s}{4}+\frac{1}{8}}\left|\int_0^{+\infty}e^{-i\mu\tau}h_4(\tau)\mathrm{d}\tau\right|^2\mathrm{d}\mu\\ &\le C\|\vec{h}\|^2_{\mathcal{H}^s(\mathbb{R}^+)}.\end{aligned}\tag{2.28}$$

Identity (2.20) and estimate (2.28) yield that (2.25) holds for $s=0,-1,-2$.

Step 3. We prove that estimate (2.5) is true.

It follows from (2.3) that $q := t^{\frac{|s|}{4}+\varepsilon}v$ satisfies

$$\begin{cases} q_t + q_{xxxx} + \delta q_{xxx} = (\frac{|s|}{4}+\varepsilon)t^{\frac{|s|}{4}+\varepsilon-1}v, & (x,t)\in(0,L)\times(0,T),\\ q(x,0)=0, & x\in(0,L),\\ q(0,t)=t^{\frac{|s|}{4}+\varepsilon}h_1(t), \quad q(L,t)=t^{\frac{|s|}{4}+\varepsilon}h_2(t), & t\in(0,T),\\ q_x(0,t)=t^{\frac{|s|}{4}+\varepsilon}h_3(t), \quad q_x(L,t)=t^{\frac{|s|}{4}+\varepsilon}h_4(t), & t\in(0,T). \end{cases} \tag{2.29}$$

Let θ and ϑ be solutions of

$$\begin{cases} \theta_t + \theta_{xxxx} + \delta\theta_{xxx} = (\frac{|s|}{4}+\varepsilon)t^{\frac{|s|}{4}+\varepsilon-1}v, & (x,t)\in(0,L)\times(0,T),\\ \theta(x,0)=0, & x\in(0,L),\\ \theta(0,t)=0,\ \theta(L,t)=0,\ \theta_x(0,t)=0,\ \theta_x(L,t)=0, & t\in(0,T) \end{cases} \tag{2.30}$$

and

$$\begin{cases} \vartheta_t + \vartheta_{xxxx} + \delta\vartheta_{xxx} = 0, & (x,t)\in(0,L)\times(0,T),\\ \vartheta(x,0)=0, & x\in(0,L),\\ \vartheta(0,t)=t^{\frac{|s|}{4}+\varepsilon}h_1(t), \quad \vartheta(L,t)=t^{\frac{|s|}{4}+\varepsilon}h_2(t), & t\in(0,T),\\ \vartheta_x(0,t)=t^{\frac{|s|}{4}+\varepsilon}h_3(t), \quad \vartheta_x(L,t)=t^{\frac{|s|}{4}+\varepsilon}h_4(t), & t\in(0,T), \end{cases} \tag{2.31}$$

respectively. It follows from (2.29)–(2.31) that

$$t^{\frac{|s|}{4}+\varepsilon}v = q = \theta + \vartheta. \tag{2.32}$$

By the Minkowski's inequality, the solution of IBVP (2.30) satisfies that for any $t\in[0,T]$,

$$\begin{aligned} \|\theta(\cdot,t)\|_{L^2(0,L)} &= \left\|\int_0^t W_c(t-\tau)\left((\frac{|s|}{4}+\varepsilon)\tau^{\frac{|s|}{4}+\varepsilon-1}v(\tau)\right)\mathrm{d}\tau\right\|_{L^2(0,L)}\\ &\le C\int_0^t \left\|W_c(t-\tau)(\tau^{\frac{|s|}{4}+\varepsilon-1}v(\tau))\right\|_{L^2(0,L)}\mathrm{d}\tau\\ &\le C\int_0^t (t-\tau)^{-\frac{|s|}{4}-\frac{\varepsilon}{2}}\left\|(t-\tau)^{\frac{|s|}{4}+\frac{\varepsilon}{2}}W_c(t-\tau)(\tau^{\frac{|s|}{4}+\varepsilon-1}v(\tau))\right\|_{L^2(0,L)}\mathrm{d}\tau. \end{aligned} \tag{2.33}$$

From Proposition 2.1 (take $\epsilon=\frac{\varepsilon}{2}$ in (2.2)), then (2.33) implies that for any $t\in[0,T]$,

$$\begin{aligned} \|\theta(\cdot,t)\|_{L^2(0,L)} &\le C\int_0^t (t-\tau)^{-\frac{|s|}{4}-\frac{\varepsilon}{2}}\|\tau^{\frac{|s|}{4}+\varepsilon-1}v(\tau)\|_{H^s(0,L)}\mathrm{d}\tau\\ &\le C\sup_{\tau\in[0,T]}\|v(\tau)\|_{H^s(0,L)}\int_0^t (t-\tau)^{-\frac{|s|}{4}-\frac{\varepsilon}{2}}\tau^{\frac{|s|}{4}+\varepsilon-1}\mathrm{d}\tau\\ &\le C\|v\|_{C([0,T];H^s(0,L))}. \end{aligned}$$

Recalling estimate (2.4) just obtained in Step 2, we know that the above estimate implies

$$\|\theta\|_{C([0,T];L^2(0,L))} \leq C\|\vec{h}\|_{\mathcal{H}^s(0,T)}. \tag{2.34}$$

Again by estimate (2.4) and noting $t^{\frac{|s|}{4}+\varepsilon}\vec{h} \in \mathcal{H}^0(0,T)$, we conclude that the solution ϑ of IBVP (2.31) satisfies

$$\|\vartheta\|_{C([0,T];L^2(0,L))} \leq C\|t^{\frac{|s|}{4}+\varepsilon}\vec{h}\|_{\mathcal{H}^0(0,T)}. \tag{2.35}$$

Therefore, (2.32), (2.34) and (2.35) deduce the desired estimate (2.5). □

3. Local well-posedness

The main purpose of this section is to discuss the local well-posedness of the dispersive Kuramoto-Sivashinsky equation (1.1). By the results of the associated linear equation presented in Section 2, if we further get a suitable estimate for $u_{xx} + uu_x$ (see Lem. 3.1), then using Banach contraction principle we can seek a fixed point solution.

The following unified notation $\mathcal{X}_{s,T}$ will be used in the sequel:

$$\|w\|_{\mathcal{X}_{s,T}} = \begin{cases} \|w\|_{C([0,T];L^2(0,L))}, & s = 0, \\ \|w\|_{C([0,T];H^s(0,L))} + \|t^{\frac{|s|}{4}+\varepsilon}w\|_{C([0,T];L^2(0,L))}, & -2 < s < 0. \end{cases}$$

Lemma 3.1. *Let $L > 0$, $0 < T \leq 1$. For any given $-2 < s \leq 0$, choose $0 < \varepsilon < \min\{\frac{2+s}{8}, \frac{1}{80}\}$. Then there exists a constant $C > 0$ such that for any u, $v \in \mathcal{X}_{s,T}$, it holds*

$$\left\|\int_0^t W_c(t-\tau)\left(u_{xx} + uv_x\right)(\tau)\mathrm{d}\tau\right\|_{\mathcal{X}_{s,T}} \leq CT^{\alpha(s,\varepsilon)}\left(\|u\|_{\mathcal{X}_{s,T}} + \|u\|_{\mathcal{X}_{s,T}}\|v\|_{\mathcal{X}_{s,T}}\right),$$

where $\alpha(s,\varepsilon) = \min\{\frac{1}{2} - \frac{|s|}{4} - 2\varepsilon, \frac{1}{64}\}$.

Proof. The proof is divided into several steps.
Step 1. We prove that

$$\left\|\int_0^t W_c(t-\tau)u_{xx}(\tau)\mathrm{d}\tau\right\|_{C([0,T];H^s(0,L))} \leq CT^{\frac{1}{2}-\frac{|s|}{4}-2\varepsilon}\|u\|_{\mathcal{X}_{s,T}}. \tag{3.1}$$

By $L^2(0,L) \hookrightarrow H^s(0,L)$ and the Minkowski's inequality, we see that for any $t \in [0,T]$,

$$\begin{aligned} &\left\|\int_0^t W_c(t-\tau)u_{xx}(\tau)\mathrm{d}\tau\right\|_{H^s(0,L)} \\ &\leq C\left\|\int_0^t W_c(t-\tau)u_{xx}(\tau)\mathrm{d}\tau\right\|_{L^2(0,L)} \\ &\leq C\int_0^t \|W_c(t-\tau)u_{xx}(\tau)\|_{L^2(0,L)}\,\mathrm{d}\tau \\ &\leq C\int_0^t \tau^{-\frac{|s|}{4}-\varepsilon}(t-\tau)^{-\frac{1}{2}-\varepsilon}\left\|(t-\tau)^{\frac{2}{4}+\varepsilon}W_c(t-\tau)(\tau^{\frac{|s|}{4}+\varepsilon}u(\tau))_{xx}\right\|_{L^2(0,L)}\mathrm{d}\tau. \end{aligned} \tag{3.2}$$

From Proposition 2.1 and (3.2), one gets that for any $t \in [0,T]$,

$$\begin{aligned}\left\|\int_0^t W_c(t-\tau)u_{xx}(\tau)\mathrm{d}\tau\right\|_{H^s(0,L)} &\le C\int_0^t \tau^{-\frac{|s|}{4}-\varepsilon}(t-\tau)^{-\frac{1}{2}-\varepsilon}\left\|(\tau^{\frac{|s|}{4}+\varepsilon}u(\tau))_{xx}\right\|_{H^{-2}(0,L)}\mathrm{d}\tau\\ &\le C\int_0^t \tau^{-\frac{|s|}{4}-\varepsilon}(t-\tau)^{-\frac{1}{2}-\varepsilon}\|\tau^{\frac{|s|}{4}+\varepsilon}u(\tau)\|_{L^2(0,L)}\mathrm{d}\tau.\\ &\le C\sup_{\tau\in[0,T]}\|\tau^{\frac{|s|}{4}+\varepsilon}u(\tau)\|_{L^2(0,L)}\int_0^t \tau^{-\frac{|s|}{4}-\varepsilon}(t-\tau)^{-\frac{1}{2}-\varepsilon}\mathrm{d}\tau.\end{aligned}$$

When $0<t\le T$, put $\varsigma := \dfrac{\tau}{t}$, it has

$$\begin{aligned}\int_0^t \tau^{-\frac{|s|}{4}-\varepsilon}(t-\tau)^{-\frac{1}{2}-\varepsilon}\mathrm{d}\tau &= \int_0^1 (t\varsigma)^{-\frac{|s|}{4}-\varepsilon}(t-t\varsigma)^{-\frac{1}{2}-\varepsilon}t\mathrm{d}\varsigma\\ &= t^{1-\frac{|s|}{4}-\varepsilon-\frac{1}{2}-\varepsilon}\int_0^1 \varsigma^{-\frac{|s|}{4}-\varepsilon}(1-\varsigma)^{-\frac{1}{2}-\varepsilon}\mathrm{d}\varsigma\\ &\le CT^{\frac{1}{2}-\frac{|s|}{4}-2\varepsilon}.\end{aligned}$$

Combing the above two formulae, we arrive at estimate (3.1).

Step 2. We claim that

$$\left\|t^{\frac{|s|}{4}+\varepsilon}\int_0^t W_c(t-\tau)u_{xx}(\tau)\mathrm{d}\tau\right\|_{C([0,T];L^2(0,L))} \le CT^{\frac{1}{64}}\|u\|_{\mathcal{X}_{s,T}}. \tag{3.3}$$

In fact, by the Minkowski's inequality, we see that for any $t\in[0,T]$,

$$\begin{aligned}&\left\|t^{\frac{|s|}{4}+\varepsilon}\int_0^t W_c(t-\tau)u_{xx}(\tau)\mathrm{d}\tau\right\|_{L^2(0,L)} \le C\int_0^t t^{\frac{|s|}{4}+\varepsilon}\left\|W_c(t-\tau)u_{xx}(\tau)\right\|_{L^2(0,L)}\mathrm{d}\tau\\ &\le C\int_0^t\left[\tau^{\frac{|s|}{4}+\varepsilon}+(t-\tau)^{\frac{|s|}{4}+\varepsilon}\right]\left\|W_c(t-\tau)u_{xx}(\tau)\right\|_{L^2(0,L)}\mathrm{d}\tau\\ &\le C\int_0^t\left[(t-\tau)^{-\frac{1}{2}-\varepsilon}+\tau^{-\frac{|s|}{4}-\varepsilon}(t-\tau)^{\frac{|s|-2}{4}}\right]\left\|(t-\tau)^{\frac{1}{2}+\varepsilon}W_c(t-\tau)(\tau^{\frac{|s|}{4}+\varepsilon}u(\tau))_{xx}\right\|_{L^2(0,L)}\mathrm{d}\tau.\end{aligned}$$

From Proposition 2.1, the above inequality shows that for any $t\in[0,T]$,

$$\begin{aligned}&\left\|t^{\frac{|s|}{4}+\varepsilon}\int_0^t W_c(t-\tau)u_{xx}(\tau)\mathrm{d}\tau\right\|_{L^2(0,L)}\\ &\le C\int_0^t\left[(t-\tau)^{-\frac{1}{2}-\varepsilon}+\tau^{-\frac{|s|}{4}-\varepsilon}(t-\tau)^{\frac{|s|-2}{4}}\right]\left\|(\tau^{\frac{|s|}{4}+\varepsilon}u(\tau))_{xx}\right\|_{H^{-2}(0,L)}\mathrm{d}\tau\\ &\le C\int_0^t\left[(t-\tau)^{-\frac{1}{2}-\varepsilon}+\tau^{-\frac{|s|}{4}-\varepsilon}(t-\tau)^{\frac{|s|-2}{4}}\right]\|\tau^{\frac{|s|}{4}+\varepsilon}u(\tau)\|_{L^2(0,L)}\mathrm{d}\tau\end{aligned}$$

$$
\begin{aligned}
&\le C \sup_{\tau\in[0,T]} \|\tau^{\frac{|s|}{4}+\varepsilon}u(\tau)\|_{L^2(0,L)} \int_0^t \left[(t-\tau)^{-\frac{1}{2}-\varepsilon} + \tau^{-\frac{|s|}{4}-\varepsilon}(t-\tau)^{\frac{|s|-2}{4}}\right] \mathrm{d}\tau \\
&\le CT^{\frac{1}{2}-\varepsilon}\|u\|_{\mathcal{X}_{s,T}} \\
&\le CT^{\frac{1}{64}}\|u\|_{\mathcal{X}_{s,T}}
\end{aligned}
$$

when $0<\varepsilon<\frac{1}{80}$. This gives estimate (3.3).

Step 3. We claim that

$$
\left\|\int_0^t W_c(t-\tau)(uv)_x(\tau)\mathrm{d}\tau\right\|_{C([0,T];H^s(0,L))} \le CT^{\min\{\frac{1}{2}-\frac{|s|}{4}-2\varepsilon,\frac{1}{64}\}}\|u\|_{\mathcal{X}_{s,T}}\|v\|_{\mathcal{X}_{s,T}}. \tag{3.4}
$$

The proof is divided into three cases: $-2<s<-\frac{3}{2}$, $-\frac{3}{2}\le s<-1$ and $-1\le s\le 0$.

Case 1. $-2<s<-\frac{3}{2}$. By the Minkowski's inequality, we see that for any $t\in[0,T]$,

$$
\begin{aligned}
&\left\|\int_0^t W_c(t-\tau)(uv)_x(\tau)\mathrm{d}\tau\right\|_{H^s(0,L)} \le C\int_0^t \|W_c(t-\tau)(uv)_x(\tau)\|_{H^s(0,L)}\,\mathrm{d}\tau \\
&\le C\int_0^t \tau^{-\frac{|s|}{2}-2\varepsilon}\left\|W_c(t-\tau)\left[(\tau^{\frac{|s|}{4}+\varepsilon}u)(\tau^{\frac{|s|}{4}+\varepsilon}v)\right]_x(\tau)\right\|_{H^s(0,L)}\mathrm{d}\tau.
\end{aligned} \tag{3.5}
$$

From Proposition 2.1, then (3.5) implies that for any $t\in[0,T]$,

$$
\begin{aligned}
&\left\|\int_0^t W_c(t-\tau)(uv)_x(\tau)\mathrm{d}\tau\right\|_{H^s(0,L)} \\
&\le C\int_0^t \tau^{-\frac{|s|}{2}-2\varepsilon}\left\|\left[(\tau^{\frac{|s|}{4}+\varepsilon}u)(\tau^{\frac{|s|}{4}+\varepsilon}v)\right]_x(\tau)\right\|_{H^s(0,L)}\mathrm{d}\tau \\
&\le C\int_0^t \tau^{-\frac{|s|}{2}-2\varepsilon}\left\|(\tau^{\frac{|s|}{4}+\varepsilon}u)(\tau^{\frac{|s|}{4}+\varepsilon}v)(\tau)\right\|_{H^{s+1}(0,L)}\mathrm{d}\tau.
\end{aligned} \tag{3.6}
$$

By $L^1(0,L)\hookrightarrow H^{s+1}(0,L)$ when $s<-\frac{3}{2}$, and from the Hölder inequality, then (3.6) shows that for any $t\in[0,T]$,

$$
\begin{aligned}
&\left\|\int_0^t W_c(t-\tau)(uv)_x(\tau)\mathrm{d}\tau\right\|_{H^s(0,L)} \\
&\le C\int_0^t \tau^{-\frac{|s|}{2}-2\varepsilon}\left\|(\tau^{\frac{|s|}{4}+\varepsilon}u)(\tau^{\frac{|s|}{4}+\varepsilon}v)(\tau)\right\|_{L^1(0,L)}\mathrm{d}\tau \\
&\le C\sup_{\tau\in[0,T]}\|\tau^{\frac{|s|}{4}+\varepsilon}u(\tau)\|_{L^2(0,L)}\sup_{\tau\in[0,T]}\|\tau^{\frac{|s|}{4}+\varepsilon}v(\tau)\|_{L^2(0,L)}\int_0^T \tau^{-\frac{|s|}{2}-2\varepsilon}\mathrm{d}\tau \\
&\le CT^{1-\frac{|s|}{2}-2\varepsilon}\|u\|_{\mathcal{X}_{s,T}}\|v\|_{\mathcal{X}_{s,T}} \\
&\le CT^{\frac{1}{2}-\frac{|s|}{4}-2\varepsilon}\|u\|_{\mathcal{X}_{s,T}}\|v\|_{\mathcal{X}_{s,T}}.
\end{aligned}
$$

Here, we have used the fact $\frac{1}{2}-\frac{|s|}{4}-2\varepsilon<1-\frac{|s|}{2}-2\varepsilon$ in the last step.

Case 2. $-\frac{3}{2} \le s < -1$. By $H^{-1}(0,L) \hookrightarrow H^s(0,L)$ and the Minkowski's inequality, we see that for some $0 < \sigma \ll 1$ and any $t \in [0,T]$,

$$\begin{aligned}\left\|\int_0^t W_c(t-\tau)(uv)_x(\tau)\mathrm{d}\tau\right\|_{H^s(0,L)} &\le C\int_0^t \|W_c(t-\tau)(uv)_x(\tau)\|_{H^{-1}(0,L)}\,\mathrm{d}\tau \\ &\le C\int_0^t \tau^{-\frac{|s|}{2}-2\varepsilon}\left\|W_c(t-\tau)\left[(\tau^{\frac{|s|}{4}+\varepsilon}u)(\tau^{\frac{|s|}{4}+\varepsilon}v)\right]_x(\tau)\right\|_{H^{-1}(0,L)}\mathrm{d}\tau.\end{aligned}$$

Note that $\|w\|_{H^{-1}(0,L)} \le C\|w\|^{\frac{1}{2}}_{H^{-2}(0,L)}\|w\|^{\frac{1}{2}}_{L^2(0,L)}$ for any $w \in L^2(0,L)$. From Proposition 2.1 (take $\epsilon = \varepsilon$ and $s = -\frac{3}{2} - \sigma$ in (2.2)), the above inequality implies that for any $t \in [0,T]$,

$$\begin{aligned}&\left\|\int_0^t W_c(t-\tau)(uv)_x(\tau)\mathrm{d}\tau\right\|_{H^s(0,L)} \\ &\le C\int_0^t \tau^{-\frac{|s|}{2}-2\varepsilon}\left\|W_c(t-\tau)\left[(\tau^{\frac{|s|}{4}+\varepsilon}u)(\tau^{\frac{|s|}{4}+\varepsilon}v)\right]_x(\tau)\right\|^{\frac{1}{2}}_{H^{-2}(0,L)} \\ &\quad\cdot\left\|W_c(t-\tau)\left[(\tau^{\frac{|s|}{4}+\varepsilon}u)(\tau^{\frac{|s|}{4}+\varepsilon}v)\right]_x(\tau)\right\|^{\frac{1}{2}}_{L^2(0,L)}\mathrm{d}\tau \\ &\le C\int_0^t \tau^{-\frac{|s|}{2}-2\varepsilon}(t-\tau)^{-\left(\frac{\frac{3}{2}+\sigma}{4}+\varepsilon\right)\times\frac{1}{2}}\left\|W_c(t-\tau)\left[(\tau^{\frac{|s|}{4}+\varepsilon}u)(\tau^{\frac{|s|}{4}+\varepsilon}v)\right]_x(\tau)\right\|^{\frac{1}{2}}_{H^{-2}(0,L)} \\ &\quad\cdot\left\|(t-\tau)^{\frac{\frac{3}{2}+\sigma}{4}+\varepsilon}W_c(t-\tau)\left[(\tau^{\frac{|s|}{4}+\varepsilon}u)(\tau^{\frac{|s|}{4}+\varepsilon}v)\right]_x(\tau)\right\|^{\frac{1}{2}}_{L^2(0,L)}\mathrm{d}\tau \\ &\le C\int_0^t \tau^{-\frac{|s|}{2}-2\varepsilon}(t-\tau)^{-\frac{\frac{3}{2}+\sigma}{8}-\frac{\varepsilon}{2}}\left\|W_c(t-\tau)\left[(\tau^{\frac{|s|}{4}+\varepsilon}u)(\tau^{\frac{|s|}{4}+\varepsilon}v)\right]_x(\tau)\right\|^{\frac{1}{2}}_{H^{-\frac{3}{2}-\sigma}(0,L)} \\ &\quad\cdot\left\|\left[(\tau^{\frac{|s|}{4}+\varepsilon}u)(\tau^{\frac{|s|}{4}+\varepsilon}v)\right]_x(\tau)\right\|^{\frac{1}{2}}_{H^{-\frac{3}{2}-\sigma}(0,L)}\mathrm{d}\tau \\ &\le C\int_0^t \tau^{-\frac{|s|}{2}-2\varepsilon}(t-\tau)^{-\frac{\frac{3}{2}+\sigma}{8}-\frac{\varepsilon}{2}}\left\|\left[(\tau^{\frac{|s|}{4}+\varepsilon}u)(\tau^{\frac{|s|}{4}+\varepsilon}v)\right]_x(\tau)\right\|_{H^{-\frac{3}{2}-\sigma}(0,L)}\mathrm{d}\tau \\ &\le C\int_0^t \tau^{-\frac{|s|}{2}-2\varepsilon}(t-\tau)^{-\frac{\frac{3}{2}+\sigma}{8}-\frac{\varepsilon}{2}}\left\|(\tau^{\frac{|s|}{4}+\varepsilon}u)(\tau^{\frac{|s|}{4}+\varepsilon}v)(\tau)\right\|_{H^{-\frac{1}{2}-\sigma}(0,L)}\mathrm{d}\tau.\end{aligned} \tag{3.7}$$

By $L^1(0,L) \hookrightarrow H^{-\frac{1}{2}-\sigma}(0,L)$ when $0 < \sigma \ll 1$, and by the Hölder inequality, (3.7) yields that for any $t \in [0,T]$,

$$\begin{aligned}&\left\|\int_0^t W_c(t-\tau)(uv)_x(\tau)\mathrm{d}\tau\right\|_{H^s(0,L)} \\ &\le C\int_0^t \tau^{-\frac{|s|}{2}-2\varepsilon}(t-\tau)^{-\frac{\frac{3}{2}+\sigma}{8}-\frac{\varepsilon}{2}}\left\|(\tau^{\frac{|s|}{4}+\varepsilon}u)(\tau^{\frac{|s|}{4}+\varepsilon}v)(\tau)\right\|_{L^1(0,L)}\mathrm{d}\tau \\ &\le C\sup_{\tau\in[0,T]}\|\tau^{\frac{|s|}{4}+\varepsilon}u(\tau)\|_{L^2(0,L)}\sup_{\tau\in[0,T]}\|\tau^{\frac{|s|}{4}+\varepsilon}v(\tau)\|_{L^2(0,L)}\int_0^t \tau^{-\frac{|s|}{2}-2\varepsilon}(t-\tau)^{-\frac{\frac{3}{2}+\sigma}{8}-\frac{\varepsilon}{2}}\mathrm{d}\tau \\ &\le CT^{\frac{13}{16}-\frac{|s|}{2}-\frac{5}{2}\varepsilon-\frac{\sigma}{8}}\|u\|_{\mathcal{X}_{s,T}}\|v\|_{\mathcal{X}_{s,T}} \\ &\le CT^{\frac{1}{64}}\|u\|_{\mathcal{X}_{s,T}}\|v\|_{\mathcal{X}_{s,T}}\end{aligned}$$

when $0<\varepsilon<\frac{1}{80}$ and $0<\sigma<\frac{1}{8}$.
Case 3. $-1\le s\le 0$. By $L^2(0,L)\hookrightarrow H^s(0,L)$ and the Minkowski's inequality, we see that for any $t\in[0,T]$,

$$\begin{aligned}&\left\|\int_0^t W_c(t-\tau)(uv)_x(\tau)\mathrm{d}\tau\right\|_{H^s(0,L)}\le C\int_0^t \|W_c(t-\tau)(uv)_x(\tau)\|_{L^2(0,L)}\,\mathrm{d}\tau\\&\le C\int_0^t \tau^{-\frac{|s|}{2}-2\varepsilon}(t-\tau)^{-\frac{\frac32+\sigma}{4}-\varepsilon}\left\|(t-\tau)^{\frac{\frac32+\sigma}{4}+\varepsilon}W_c(t-\tau)\left[(\tau^{\frac{|s|}{4}+\varepsilon}u)(\tau^{\frac{|s|}{4}+\varepsilon}v)\right]_x(\tau)\right\|_{L^2(0,L)}\mathrm{d}\tau,\end{aligned}$$

where $0<\sigma\ll 1$. By Proposition 2.1, the above inequality implies that for any $t\in[0,T]$,

$$\begin{aligned}&\left\|\int_0^t W_c(t-\tau)(uv)_x(\tau)\mathrm{d}\tau\right\|_{H^s(0,L)}\\&\le C\int_0^t \tau^{-\frac{|s|}{2}-2\varepsilon}(t-\tau)^{-\frac{\frac32+\sigma}{4}-\varepsilon}\left\|\left[(\tau^{\frac{|s|}{4}+\varepsilon}u)(\tau^{\frac{|s|}{4}+\varepsilon}v)\right]_x(\tau)\right\|_{H^{-\frac32-\sigma}(0,L)}\mathrm{d}\tau\\&\le C\int_0^t \tau^{-\frac{|s|}{2}-2\varepsilon}(t-\tau)^{-\frac{\frac32+\sigma}{4}-\varepsilon}\left\|(\tau^{\frac{|s|}{4}+\varepsilon}u)(\tau^{\frac{|s|}{4}+\varepsilon}v)(\tau)\right\|_{H^{-\frac12-\sigma}(0,L)}\mathrm{d}\tau.\end{aligned}$$

Similar to the proof of Case 2, we deduce that

$$\left\|\int_0^t W_c(t-\tau)(uv)_x(\tau)\mathrm{d}\tau\right\|_{H^s(0,L)}\le CT^{\frac58-\frac{|s|}{2}-3\varepsilon-\frac{\sigma}{4}}\|u\|_{\mathcal{X}_{s,T}}\|v\|_{\mathcal{X}_{s,T}}\le CT^{\frac{1}{32}}\|u\|_{\mathcal{X}_{s,T}}\|v\|_{\mathcal{X}_{s,T}}$$

when $0<\varepsilon<\frac{1}{80}$ and $0<\sigma<\frac{1}{8}$.
Combing the results obtained in Case 1–Case 3, we know that (3.4) is true.

Step 4. We claim that

$$\left\|t^{\frac{|s|}{4}+\varepsilon}\int_0^t W_c(t-\tau)(uv)_x(\tau)\mathrm{d}\tau\right\|_{C([0,T];L^2(0,L))}\le CT^{\frac{1}{16}}\|u\|_{\mathcal{X}_{s,T}}\|v\|_{\mathcal{X}_{s,T}}. \tag{3.8}$$

In fact, by the Minkowski's inequality, we see that for some $0<\sigma\ll 1$ any $t\in[0,T]$,

$$\begin{aligned}&\left\|t^{\frac{|s|}{4}+\varepsilon}\int_0^t W_c(t-\tau)(uv)_x(\tau)\mathrm{d}\tau\right\|_{L^2(0,L)}\le C\int_0^t t^{\frac{|s|}{4}+\varepsilon}\|W_c(t-\tau)(uv)_x(\tau)\|_{L^2(0,L)}\,\mathrm{d}\tau\\&\le C\int_0^t\left[(t-\tau)^{\frac{|s|}{4}+\varepsilon}+\tau^{\frac{|s|}{4}+\varepsilon}\right]\|W_c(t-\tau)(uv)_x(\tau)\|_{L^2(0,L)}\,\mathrm{d}\tau\\&\le C\int_0^t \tau^{-\frac{|s|}{2}-2\varepsilon}(t-\tau)^{-\frac{s+\frac32+\sigma}{4}}\left\|(t-\tau)^{\frac{\frac32+\sigma}{4}+\varepsilon}W_c(t-\tau)\left[(\tau^{\frac{|s|}{4}+\varepsilon}u)(\tau^{\frac{|s|}{4}+\varepsilon}v)\right]_x(\tau)\right\|_{L^2(0,L)}\mathrm{d}\tau\\&\quad+C\int_0^t \tau^{-\frac{|s|}{4}-\varepsilon}(t-\tau)^{-\frac{\frac32+\sigma}{4}-\varepsilon}\left\|(t-\tau)^{\frac{\frac32+\sigma}{4}+\varepsilon}W_c(t-\tau)\left[(\tau^{\frac{|s|}{4}+\varepsilon}u)(\tau^{\frac{|s|}{4}+\varepsilon}v)\right]_x(\tau)\right\|_{L^2(0,L)}\mathrm{d}\tau.\end{aligned}$$

Again by Proposition 2.1, the above inequality implies that for any $t \in [0,T]$,

$$\begin{aligned}
&\left\| t^{\frac{|s|}{4}+\varepsilon} \int_0^t W_c(t-\tau)(uv)_x(\tau)\mathrm{d}\tau \right\|_{L^2(0,L)} \\
&\le C\int_0^t \tau^{-\frac{|s|}{2}-2\varepsilon}(t-\tau)^{-\frac{s+\frac{3}{2}+\sigma}{4}} \left\| \left[(\tau^{\frac{|s|}{4}+\varepsilon}u)(\tau^{\frac{|s|}{4}+\varepsilon}v)\right]_x(\tau)\right\|_{H^{-\frac{3}{2}-\sigma}(0,L)}\mathrm{d}\tau \\
&\quad + C\int_0^t \tau^{-\frac{|s|}{4}-\varepsilon}(t-\tau)^{-\frac{\frac{3}{2}+\sigma}{4}-\varepsilon} \left\| \left[(\tau^{\frac{|s|}{4}+\varepsilon}u)(\tau^{\frac{|s|}{4}+\varepsilon}v)\right]_x(\tau)\right\|_{H^{-\frac{3}{2}-\sigma}(0,L)}\mathrm{d}\tau.
\end{aligned}$$

Similar to the proof of Case 2, we obtain that

$$\begin{aligned}
&\left\| t^{\frac{|s|}{4}+\varepsilon} \int_0^t W_c(t-\tau)(uv)_x(\tau)\mathrm{d}\tau \right\|_{C([0,T];L^2(0,L))} \\
&\le CT^{\frac{5}{8}-\frac{|s|}{4}-2\varepsilon-\frac{\sigma}{4}}\|u\|_{\mathcal{X}_{s,T}}\|v\|_{\mathcal{X}_{s,T}} \\
&\le CT^{\frac{1}{16}}\|u\|_{\mathcal{X}_{s,T}}\|v\|_{\mathcal{X}_{s,T}}
\end{aligned}$$

when $0<\varepsilon<\frac{1}{80}$ and $0<\sigma<\frac{1}{8}$.

From (3.1), (3.3), (3.4) and (3.8), we obtain the desired result of Lemma 3.1. □

With a slight modification of the original proof of Lemma 3.1, we can obtain the following "stronger" estimate in some sense.

Lemma 3.2. *Let $L>0$, $0<T\le 1$. For any given $-2<s<0$, choose $0<\varepsilon<\min\{\frac{2+s}{8},\frac{1}{80}\}$ and $0<\kappa<\min\{|s|,\frac{1}{64}\}$. Then there exists a constant $C>0$ such that for any u, $v\in\mathcal{X}^{\varepsilon}_{s,T}$, it holds*

$$\left\| \int_0^t W_c(t-\tau)\left(u_{xx}+uv_x\right)(\tau)\mathrm{d}\tau \right\|_{C([0,T];H^{s+\kappa}(0,L))} \le CT^{\alpha(s,\varepsilon)}\left(\|u\|_{\mathcal{X}_{s,T}}+\|u\|_{\mathcal{X}_{s,T}}\|v\|_{\mathcal{X}_{s,T}}\right),$$

where $\alpha(s,\varepsilon)=\min\{\frac{1}{2}-\frac{|s|}{4}-2\varepsilon,\frac{1}{64}\}$.

Proof. The proof is divided into two steps.
Step 1. We prove that

$$\left\| \int_0^t W_c(t-\tau)u_{xx}(\tau)\mathrm{d}\tau \right\|_{C([0,T];H^{s+\kappa}(0,L))} \le CT^{\frac{1}{2}-\frac{|s|}{4}-2\varepsilon}\|u\|_{\mathcal{X}_{s,T}}. \tag{3.9}$$

Noting that $L^2(0,L)\hookrightarrow H^{s+\kappa}(0,L)$, we have

$$\left\| \int_0^t W_c(t-\tau)u_{xx}(\tau)\mathrm{d}\tau \right\|_{H^{s+\kappa}(0,L)} \le C\left\| \int_0^t W_c(t-\tau)u_{xx}(\tau)\mathrm{d}\tau \right\|_{L^2(0,L)}.$$

Along the same lines of the proof in Step 1, Lemma 3.1, one can arrive at estimate (3.9).

Step 2. We claim that

$$\left\| \int_0^t W_c(t-\tau)(uv)_x(\tau)\mathrm{d}\tau \right\|_{C([0,T];H^{s+\kappa}(0,L))} \le CT^{\min\{\frac{1}{2}-\frac{|s|}{4}-2\varepsilon,\frac{1}{64}\}}\|u\|_{\mathcal{X}_{s,T}}\|v\|_{\mathcal{X}_{s,T}}. \tag{3.10}$$

The proof is divided into three cases: $-2 < s < -\frac{3}{2} - \kappa$, $-\frac{3}{2} - \kappa \leq s < -1 - \kappa$ and $-1 - \kappa \leq s < -\kappa$.

Case 1. $-2 < s < -\frac{3}{2} - \kappa$. Similar to the proof in Lemma 3.1, it has

$$\begin{aligned}
&\left\| \int_0^t W_c(t-\tau)(uv)_x(\tau) \mathrm{d}\tau \right\|_{H^{s+\kappa}(0,L)} \\
&\leq C \int_0^t \|W_c(t-\tau)(uv)_x(\tau)\|_{H^{s+\kappa}(0,L)} \mathrm{d}\tau \\
&\leq C \int_0^t \tau^{-\frac{|s|}{2} - 2\varepsilon} \left\| W_c(t-\tau) \left[(\tau^{\frac{|s|}{4}+\varepsilon} u)(\tau^{\frac{|s|}{4}+\varepsilon} v) \right]_x (\tau) \right\|_{H^{s+\kappa}(0,L)} \mathrm{d}\tau \\
&\leq C \int_0^t \tau^{-\frac{|s|}{2} - 2\varepsilon} \left\| \left[(\tau^{\frac{|s|}{4}+\varepsilon} u)(\tau^{\frac{|s|}{4}+\varepsilon} v) \right]_x (\tau) \right\|_{H^{s+\kappa}(0,L)} \mathrm{d}\tau \\
&\leq C \int_0^t \tau^{-\frac{|s|}{2} - 2\varepsilon} \left\| (\tau^{\frac{|s|}{4}+\varepsilon} u)(\tau^{\frac{|s|}{4}+\varepsilon} v)(\tau) \right\|_{H^{s+\kappa+1}(0,L)} \mathrm{d}\tau.
\end{aligned}$$

By $L^1(0,L) \hookrightarrow H^{s+\kappa+1}(0,L)$ when $s < -\frac{3}{2} - \kappa$, and from the Hölder inequality, then the above formula shows that for any $t \in [0,T]$,

$$\begin{aligned}
&\left\| \int_0^t W_c(t-\tau)(uv)_x(\tau) \mathrm{d}\tau \right\|_{H^{s+\kappa+1}(0,L)} \\
&\leq C \int_0^t \tau^{-\frac{|s|}{2} - 2\varepsilon} \left\| (\tau^{\frac{|s|}{4}+\varepsilon} u)(\tau^{\frac{|s|}{4}+\varepsilon} v)(\tau) \right\|_{L^1(0,L)} \mathrm{d}\tau \\
&\leq C T^{1-\frac{|s|}{2}-2\varepsilon} \|u\|_{\mathcal{X}_{s,T}} \|v\|_{\mathcal{X}_{s,T}} \\
&\leq C T^{\frac{1}{2}-\frac{|s|}{4}-2\varepsilon} \|u\|_{\mathcal{X}_{s,T}} \|v\|_{\mathcal{X}_{s,T}}.
\end{aligned}$$

Case 2. $-\frac{3}{2} - \kappa \leq s < -1 - \kappa$. By $H^{-1}(0,L) \hookrightarrow H^{s+\kappa}(0,L)$, we see that

$$\left\| \int_0^t W_c(t-\tau)(uv)_x(\tau) \mathrm{d}\tau \right\|_{H^{s+\kappa}(0,L)} \leq C \int_0^t \|W_c(t-\tau)(uv)_x(\tau)\|_{H^{-1}(0,L)} \mathrm{d}\tau.$$

Then, along the same lines of the proof in Lemma 3.1, one can arrive at

$$\left\| \int_0^t W_c(t-\tau)(uv)_x(\tau) \mathrm{d}\tau \right\|_{H^{s+\kappa}(0,L)} \leq C T^{\frac{13}{16} - \frac{|s|}{2} - \frac{5}{2}\varepsilon - \frac{\sigma}{8}} \|u\|_{\mathcal{X}_{s,T}} \|v\|_{\mathcal{X}_{s,T}} \leq C T^{\frac{1}{64}} \|u\|_{\mathcal{X}_{s,T}} \|v\|_{\mathcal{X}_{s,T}}$$

when $0 < \varepsilon < \frac{1}{80}$, $0 < \kappa < \frac{1}{64}$ and $0 < \sigma < \frac{1}{32}$.

Case 3. $-1 - \kappa \leq s < -\kappa$. By $L^2(0,L) \hookrightarrow H^{s+\kappa}(0,L)$, one has

$$\left\| \int_0^t W_c(t-\tau)(uv)_x(\tau) \mathrm{d}\tau \right\|_{H^{s+\kappa}(0,L)} \leq C \int_0^t \|W_c(t-\tau)(uv)_x(\tau)\|_{L^2(0,L)} \mathrm{d}\tau.$$

Similar to the proof in Lemma 3.1, one can obtain that

$$\left\| \int_0^t W_c(t-\tau)(uv)_x(\tau) \mathrm{d}\tau \right\|_{H^{s+\kappa}(0,L)} \leq C T^{\frac{5}{8} - \frac{|s|}{2} - 3\varepsilon - \frac{\sigma}{4}} \|u\|_{\mathcal{X}_{s,T}} \|v\|_{\mathcal{X}_{s,T}} \leq C T^{\frac{1}{32}} \|u\|_{\mathcal{X}_{s,T}} \|v\|_{\mathcal{X}_{s,T}}$$

when $0<\varepsilon<\frac{1}{80}$, $0<\kappa<\frac{1}{64}$ and $0<\sigma<\frac{1}{32}$.

Combing the results obtained in Case 1–Case 3 leads to (3.10). Finally, (3.9) and (3.10) together imply that Lemma 3.2 holds. □

Proposition 3.3. *Let $\delta\in\mathbb{R}$, $L>0$, $0<T\le 1$. For any given $-2<s\le 0$, choose $0<\varepsilon<\min\{\frac{2+s}{8},\frac{1}{80}\}$. Assume that $\phi\in H^s(0,L)$, $\vec{h}\in\mathcal{H}^s(0,T)$ and $t^{\frac{|s|}{4}+\varepsilon}\vec{h}\in\mathcal{H}^0(0,T)$. Then there exists a $T_*\in(0,T]$ depending on $\|\phi\|_{H^s(0,L)}+\|\vec{h}\|_{\mathcal{H}^s(0,T)}+\|t^{\frac{|s|}{4}+\varepsilon}\vec{h}\|_{\mathcal{H}^0(0,T)}$, such that IBVP (1.1) admits a unique solution $u\in\mathcal{X}_{s,T_*}$. Moreover, the corresponding solution map from the space of initial and boundary data to the solution space is continuous.*

Proof. The solution of IBVP (1.1) can be written in the form

$$u(t)=W_c(t)\phi+W_{bdr}(t)\vec{h}-\int_0^t W_c(t-\tau)\left(u_{xx}+uu_x\right)(\tau)\mathrm{d}\tau. \tag{3.11}$$

For $w\in\mathcal{X}_{s,T_*}(d):=\left\{w\in\mathcal{X}_{s,T_*}\ \middle|\ \|w\|_{\mathcal{X}_{s,T_*}}\le d\right\}$, define

$$\Gamma(w)=W_c(t)\phi+W_{bdr}(t)\vec{h}-\int_0^t W_c(t-\tau)\left(w_{xx}+ww_x\right)(\tau)\mathrm{d}\tau. \tag{3.12}$$

Then by Propositions 2.1–2.2, Lemma 3.1 and (3.12), we obtain that for any w, w_1, $w_2\in\mathcal{X}_{s,T_*}(d)$,

$$\begin{aligned}&\|\Gamma(w)\|_{\mathcal{X}_{s,T_*}}\\ &\le C\left(\|\phi\|_{H^s(0,L)}+\|\vec{h}\|_{\mathcal{H}^s(0,T)}+\|t^{\frac{|s|}{4}+\varepsilon}\vec{h}\|_{\mathcal{H}^0(0,T)}\right)+CT_*^{\alpha(s,\varepsilon)}\left(\|w\|_{\mathcal{X}_{s,T_*}}+\|w\|^2_{\mathcal{X}_{s,T_*}}\right)\end{aligned}$$

and

$$\begin{aligned}&\|\Gamma(w_1)-\Gamma(w_2)\|_{\mathcal{X}_{s,T_*}}\\ &\le CT_*^{\alpha(s,\varepsilon)}\left(\|w_1-w_2\|_{\mathcal{X}_{s,T_*}}+\|w_1+w_2\|_{\mathcal{X}_{s,T_*}}\|w_1-w_2\|_{\mathcal{X}_{s,T_*}}\right).\end{aligned}$$

Similar to the proof of Proposition 4.1 in [16], through a suitable choice of T_*, we can show that Γ is a contraction map in $\mathcal{X}_{s,T_*}(d)$. The fixed point theorem guarantees that IBVP (1.1) admits a unique solution $u=\Gamma(u)$ in $\mathcal{X}_{s,T_*}(d)$. □

4. Global well-posedness

In Section 3, we have obtained the local well-posedness for IBVP (1.1). In order to derive the global well-posedness, we usually need to construct a priori estimates for all s lies in $(-2,0]$. However, by virtue of Lemma 3.2, we only need to establish a prior estimate for $s=0$.

4.1. *A priori* estimate for $s=0$

Similar to Subsection 4.1 of [16], we can prove that IBVP (1.1) is actually locally well-posed in $C([0,T_*;L^2(0,L))\bigcap L^2(0,T_*;H^2(0,L))$ under the condition $(\phi,\vec{h})\in L^2(0,L)\times\mathcal{H}^0(0,T)$. Our rest proof is along the same lines as the proof for a priori estimate (4.9) in [16]. For any given $T>0$, we claim that there exists a continuous nondecreasing function $\gamma:\mathbb{R}^+\to\mathbb{R}^+$ such that for any solution $u\in C([0,T;L^2(0,L))\bigcap L^2(0,T;H^2(0,L))$ of IBVP (1.1), it holds

$$\|u\|_{C([0,T];L^2(0,L))}\le\gamma\left(\|\phi\|_{L^2(0,L)}+\|\vec{h}\|_{\mathcal{H}^0(0,T)}\right). \tag{4.1}$$

Let y solve

$$\begin{cases} y_t + y_{xxxx} + \delta y_{xxx} = 0, & (x,t) \in (0,L) \times (0,T), \\ y(x,0) = 0, & x \in (0,L), \\ y(0,t) = h_1(t),\ y(L,t) = h_2(t),\ y_x(0,t) = h_3(t),\ y_x(L,t) = h_4(t), & t \in (0,T) \end{cases} \tag{4.2}$$

and z solve

$$\begin{cases} z_t + z_{xxxx} + \delta z_{xxx} + z_{xx} + z z_x = -(zy)_x - y y_x - y_{xx}, & (x,t) \in (0,L) \times (0,T), \\ z(x,0) = \phi(x), & x \in (0,L), \\ z(0,t) = 0,\ z(L,t) = 0,\ z_x(0,t) = 0,\ z_x(L,t) = 0, & t \in (0,T). \end{cases} \tag{4.3}$$

Then it is easy to check that $u = y + z$.

Multiplying both side of the first equation in (4.3) by z and integrating by parts, we get

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z\|^2_{L^2(0,L)} + \|z_{xx}\|^2_{L^2(0,L)} \\ &= \|z_x\|^2_{L^2(0,L)} + \int_0^L z_x y z \mathrm{d}x + \frac{1}{2}\int_0^L z_x y^2 \mathrm{d}x - \int_0^L z_{xx} y \mathrm{d}x. \end{aligned} \tag{4.4}$$

Now we estimate the four terms on the right hand side of (4.4). The Gagliardo–Nirenberg's inequality and the Young's inequality imply that for some $0 < \widetilde{\varepsilon} \ll 1$,

$$\begin{aligned} &\|z_x\|^2_{L^2(0,L)} \le C\left(\|z_{xx}\|^{\frac{1}{2}}_{L^2(0,L)}\|z\|^{\frac{1}{2}}_{L^2(0,L)} + \|z\|_{L^2(0,L)}\right)^2 \\ &\le C\|z_{xx}\|_{L^2(0,L)}\|z\|_{L^2(0,L)} + C\|z\|^2_{L^2(0,L)} \\ &\le \widetilde{\varepsilon}^2\|z_{xx}\|^2_{L^2(0,L)} + \left(\tfrac{C}{\widetilde{\varepsilon}^2} + C\right)\|z\|^2_{L^2(0,L)}. \end{aligned} \tag{4.5}$$

From the Gagliardo–Nirenberg's inequality, the Hölder's inequality, the Young's inequality and (4.5), we get

$$\begin{aligned} &\int_0^L z_x y z \mathrm{d}x \le \|z_x\|_{L^\infty(0,L)}\|yz\|_{L^1(0,L)} \le \|z_x\|^2_{L^\infty(0,L)} + \|yz\|^2_{L^1(0,L)} \\ &\le C\left(\|z_{xx}\|^{\frac{1}{2}}_{L^2(0,L)}\|z_x\|^{\frac{1}{2}}_{L^2(0,L)} + \|z_x\|_{L^2(0,L)}\right)^2 + \|y\|^2_{L^2(0,L)}\|z\|^2_{L^2(0,L)} \\ &\le C\|z_{xx}\|_{L^2(0,L)}\|z_x\|_{L^2(0,L)} + C\|z_x\|^2_{L^2(0,L)} + \|y\|^2_{L^2(0,L)}\|z\|^2_{L^2(0,L)} \\ &\le \widetilde{\varepsilon}\|z_{xx}\|^2_{L^2(0,L)} + \left(\tfrac{C}{\widetilde{\varepsilon}} + C\right)\|z_x\|^2_{L^2(0,L)} + \|y\|^2_{L^2(0,L)}\|z\|^2_{L^2(0,L)} \\ &\le \left(C\widetilde{\varepsilon} + C\widetilde{\varepsilon}^2\right)\|z_{xx}\|^2_{L^2(0,L)} + \left(\tfrac{C}{\widetilde{\varepsilon}^2} + C\right)\left(\tfrac{C}{\widetilde{\varepsilon}} + C\right)\|z\|^2_{L^2(0,L)} + \|y\|^2_{L^2(0,L)}\|z\|^2_{L^2(0,L)}. \end{aligned} \tag{4.6}$$

Similar to (4.6), we acquire

$$\begin{aligned} &\frac{1}{2}\int_0^L z_x y^2 \mathrm{d}x \le \|z_x\|_{L^\infty(0,L)}\|y\|^2_{L^2(0,L)} \le \|z_x\|^2_{L^\infty(0,L)} + \|y\|^4_{L^2(0,L)} \\ &\le \left(C\widetilde{\varepsilon} + C\widetilde{\varepsilon}^2\right)\|z_{xx}\|^2_{L^2(0,L)} + \left(\tfrac{C}{\widetilde{\varepsilon}^2} + C\right)\left(\tfrac{C}{\widetilde{\varepsilon}} + C\right)\|z\|^2_{L^2(0,L)} + \|y\|^4_{L^2(0,L)}. \end{aligned} \tag{4.7}$$

Again, by the Young's inequality, it holds

$$-\int_0^L z_{xx} y \mathrm{d}x \le \|z_{xx}\|_{L^2(0,L)} \|y\|_{L^2(0,L)} \le \widetilde{\varepsilon}\|z_{xx}\|^2_{L^2(0,L)} + \frac{1}{4\widetilde{\varepsilon}}\|y\|^2_{L^2(0,L)}. \tag{4.8}$$

Choosing $\widetilde{\varepsilon}$ small enough, from (4.4)–(4.8), we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|z\|^2_{L^2(0,L)} + \|z_{xx}\|^2_{L^2(0,L)} \le \left(C + \|y\|^2_{L^2(0,L)}\right)\|z\|^2_{L^2(0,L)} + \|y\|^4_{L^2(0,L)} + C\|y\|^2_{L^2(0,L)}. \tag{4.9}$$

Proposition 2.2 implies that

$$\|y\|_{C([0,T];L^2(0,L))} \le C\|\vec{h}\|_{\mathcal{H}^0(0,T)}. \tag{4.10}$$

From (4.9) and (4.10), for any $t \in [0,T]$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|z(\cdot,t)\|^2_{L^2(0,L)} \le C\left(1 + \|\vec{h}\|^2_{\mathcal{H}^0(0,T)}\right)\|z(\cdot,t)\|^2_{L^2(0,L)} + C\left(\|\vec{h}\|^4_{\mathcal{H}^0(0,T)} + \|\vec{h}\|^2_{\mathcal{H}^0(0,T)}\right). \tag{4.11}$$

Applying the Gronwall's inequality to (4.11), one gets

$$\begin{aligned} &\sup_{t\in[0,T]}\|z(\cdot,t)\|^2_{L^2(0,L)} \\ &\le e^{CT\left(1+\|\vec{h}\|^2_{\mathcal{H}^0(0,T)}\right)}\left[\|\phi\|^2_{L^2(0,L)} + CT\left(\|\vec{h}\|^4_{\mathcal{H}^0(0,T)} + \|\vec{h}\|^2_{\mathcal{H}^0(0,T)}\right)\right]. \end{aligned} \tag{4.12}$$

This implies that there exists a continuous nondecreasing function $\gamma_* : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|z\|_{C(0,T;L^2(0,L))} \le \gamma_*\left(\|\phi\|_{L^2(0,L)} + \|\vec{h}\|_{\mathcal{H}^0(0,T)}\right). \tag{4.13}$$

From (4.10), (4.13) and $u = y + z$, one concludes that (4.1) is valid. □

4.2. Proof of Theorem 1.2

Although we will take the same strategy as [16] to yield the global well-posedness, we still give a detailed proof for Theorem 1.2 for the reader's convenience.

We only need to consider the case when $-2 < s < 0$. In fact, if $s = 0$, combining the local well-posedness stated in Proposition 3.3 and a priori estimate (4.1) in Section 4.1, we immediately obtain the global well-posedness of IBVP (1.1) when $s = 0$.

Now assume that $-2 < s < 0$. Let $\delta \in \mathbb{R}$, $\phi \in H^s(0,L)$, $\vec{h} \in \mathcal{H}^s(0,T)$ and $t^{\frac{|s|}{4}+\varepsilon}\vec{h} \in \mathcal{H}^0(0,T)$ for some $0 < \varepsilon \ll 1$. From Proposition 3.3, we know that there exists a $T_* \in (0,T]$ depending on $\|\phi\|_{H^s(0,L)}$, $\|\vec{h}\|_{\mathcal{H}^s(0,T)}$ and $\|t^{\frac{|s|}{4}+\varepsilon}\vec{h}\|_{\mathcal{H}^0(0,T)}$, such that IBVP (1.1) admits a local solution $u \in \mathcal{X}_{s,T_*}$. And we will consider the solution in the time intervals $[0,T_*]$ and $[T_*,T]$ separately.

First, we consider the solution of (1.1)

$$u(\cdot,t) = W_c(t)\phi(\cdot) + W_{bdr}(t)\vec{h} - \int_0^t W_c(t-\tau)\left(u_{xx} + uu_x\right)(\cdot,\tau)\mathrm{d}\tau$$

in the time interval $[0,T_*]$. In the right hand side, there are three terms. For the first term, there exists $0 < T_1 < T_*$ such that $W_c(t)\phi \in C([T_1,T_*];H^\infty(0,L))$. For the second term, recalling $t^{\frac{|s|}{4}+\varepsilon}\vec{h} \in \mathcal{H}^0(0,T) =$

$H^{\frac{3}{8}}(0,T) \times H^{\frac{3}{8}}(0,T) \times H^{\frac{1}{8}}(0,T) \times H^{\frac{1}{8}}(0,T)$, we know that there exists $0 < T_2 < T_*$ such that $W_{bdr}(t)\vec{h} \in C([T_2, T_*];\, L^2(0,L))$ by Proposition 2.2. For the third term, by Lemma 3.2, there exists a small $\kappa > 0$ such that

$$\int_0^t W_c(t-\tau)(u_{xx} + uu_x)(\tau)\mathrm{d}\tau \in C([0,T_*]; H^{s+\kappa}(0,L)).$$

By iterating this argument, we can obtain that

$$\int_0^t W_c(t-\tau)(u_{xx} + uu_x)(\tau)\mathrm{d}\tau \in C([T_3,T_*]; L^2(0,L)).$$

for some $0 < T_3 < T_*$. Take $T_4 = \max\{T_1, T_2, T_3\}$, then the solution u of (1.1) in the time interval $[0, T_*]$ satisfies

$$u \in C([0,T_*]; H^s(0,L)), \quad u \in C([T_4,T_*]; L^2(0,L)).$$

Next, we consider the solution of equation (1.1) in the time interval (T_*, T). Denote $U(x,t) = u(x, t - T_*)$, then U satisfies:

$$\begin{cases} U_t + U_{xxxx} + \delta U_{xxx} + U_{xx} + UU_x = 0, & (x,t) \in (0,L) \times (0, T-T_*), \\ U(x,0) = u(x,T_*), & x \in (0,L), \\ U(0,t) = h_1(T_*+t),\ U(L,t) = h_2(T_*+t), & t \in (0,T-T_*), \\ U_x(0,t) = h_3(T_*+t),\ U_x(L,t) = h_4(T_*+t), & t \in (0,T-T_*), \end{cases}$$

where $u(x,T_*) \in L^2(0,L)$ and $(h_1(T_*+t), h_2(T_*+t), h_3(T_*+t), h_4(T_*+t)) \in \mathcal{H}^0(0,T-T_*) = H^{\frac{3}{8}}(0,T-T_*) \times H^{\frac{3}{8}}(0,T-T_*) \times H^{\frac{1}{8}}(0,T-T_*) \times H^{\frac{1}{8}}(0,T-T_*)$. By a priori estimate (4.1) for $s = 0$ obtained in Section 4.1, we conclude that U exists globally, and $U \in C([0,T-T_*]; L^2(0,L))$. This implies that $U \in C([0,T-T_*]; H^s(0,L))$. Hence $u \in C([T_*,T]; H^s(0,L))$.

Finally, combining the well-posedness results on two intervals $[0,T_*]$ and $[T_*,T]$ together, we have $u \in C([0,T]; H^s(0,L))$. □

5. Ill-posedness: Proof of Theorem 1.3

The proof strategy is inspired by [10, 17, 21], which concerning the ill-posedness issues for the Cauchy problem of several kinds of nonlinear dispersive equations.

Proposition 5.1. *Let $T > 0$ and $s < -2$. Then there does not exist any subspace $\mathscr{X}_T^s$ of $C([0,T]; H^s(0,L))$ such that the following three items all hold for IBVP (1.4):*

(i) $\|u\|_{C([0,T];H^s(0,L))} \le C\|u\|_{\mathscr{X}_T^s}, \quad \forall\, u \in \mathscr{X}_T^s$;
(ii) $\|W_c(t)\phi\|_{\mathscr{X}_T^s} \le C\|\phi\|_{H^s(0,L)}, \quad \forall\, \phi \in H^s(0,L)$;
(iii) $\left\| \int_0^t W_c(t-\tau)(uv_x)(\tau)\mathrm{d}\tau \right\|_{\mathscr{X}_T^s} \le C\|u\|_{\mathscr{X}_T^s}\|v\|_{\mathscr{X}_T^s}, \quad \forall\, u, v \in \mathscr{X}_T^s.$

Proof. We give the proof by contradiction argument. Assume that there exists such a space $\mathscr{X}_T^s$ as stated in Proposition 5.1. For any ϕ, $\psi \in H^s(0,L)$, take $u(t) = W_c(t)\phi$ and $v(t) = W_c(t)\psi$. By our hypothesis of (i), (ii) and (iii), one can obtain that

$$\begin{aligned}
&\left\|\int_0^t W_c(t-\tau)\big(W_c(\tau)\phi(x)\big)\partial_x\big(W_c(\tau)\psi(x)\big)\mathrm{d}\tau\right\|_{H^s(0,L)}\\
&\leq C\left\|\int_0^t W_c(t-\tau)\big(W_c(\tau)\phi(x)\big)\partial_x\big(W_c(\tau)\psi(x)\big)\mathrm{d}\tau\right\|_{\mathscr{X}_T^s}\\
&\leq C\|W_c(\tau)\phi(x)\|_{\mathscr{X}_T^s}\|W_c(\tau)\psi(x)\|_{\mathscr{X}_T^s}\\
&\leq C\|\phi(x)\|_{H^s(0,L)}\|\psi(x)\|_{H^s(0,L)}, \quad \forall\, 0<t<T.
\end{aligned} \tag{5.1}$$

We claim that when $s<-2$, for any positive constant C and $0<t<T$, there exist suitably selected ϕ and ψ such that

$$\left\|\int_0^t W_c(t-\tau)\big(W_c(\tau)\phi(x)\big)\partial_x\big(W_c(\tau)\psi(x)\big)\mathrm{d}\tau\right\|_{H^s(0,L)} > C\|\phi(x)\|_{H^s(0,L)}\|\psi(x)\|_{H^s(0,L)}, \tag{5.2}$$

which contradicts (5.1).

In fact, we choose ϕ and ψ as below

$$\phi(x)=N^{-s-\frac{1}{2}}\sin\frac{\pi(N+1)x}{L}, \quad \psi(x)=N^{-s-\frac{1}{2}}\sin\frac{\pi N x}{L}, \quad x\in(0,L).$$

Here N is a positive integer and we will let N tend to infinity later. One can check that $\|\phi\|_{H^s(0,L)}\sim 1$ and $\|\psi\|_{H^s(0,L)}\sim 1$. Define $p(N):=-(\frac{\pi N}{L})^4+(\frac{\pi N}{L})^2$. Recall the definition of $W_c(t)$, we have

$$\begin{aligned}
&\big(W_c(\tau)\phi(x)\big)\partial_x\big(W_c(\tau)\psi(x)\big)\\
&=\left(e^{p(N+1)\tau}N^{-s-\frac{1}{2}}\sin\frac{\pi(N+1)x}{L}\right)\partial_x\left(e^{p(N)\tau}N^{-s-\frac{1}{2}}\sin\frac{\pi N x}{L}\right)\\
&=\left(e^{p(N+1)\tau}N^{-s-\frac{1}{2}}\sin\frac{\pi(N+1)x}{L}\right)\left(\frac{\pi N}{L}e^{p(N)\tau}N^{-s-\frac{1}{2}}\cos\frac{\pi N x}{L}\right)\\
&=\frac{\pi}{2L}N^{-2s}e^{p(N+1)\tau+p(N)\tau}\left(\sin\frac{\pi x}{L}+\sin\frac{\pi(2N+1)x}{L}\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_0^t W_c(t-\tau)\big(W_c(\tau)\phi(x)\big)\partial_x\big(W_c(\tau)\psi(x)\big)\\
&=\int_0^t e^{p(1)(t-\tau)}\frac{\pi}{2L}N^{-2s}e^{p(N+1)\tau+p(N)\tau}\sin\frac{\pi x}{L}\mathrm{d}\tau\\
&\quad+\int_0^t e^{p(2N+1)(t-\tau)}\frac{\pi}{2L}N^{-2s}e^{p(N+1)\tau+p(N)\tau}\sin\frac{\pi(2N+1)x}{L}\mathrm{d}\tau\\
&=\left[\frac{\pi}{2L}N^{-2s}e^{p(1)t}\sin\frac{\pi x}{L}\frac{1}{p(N+1)+p(N)-p(1)}e^{p(N+1)\tau+p(N)\tau-p(1)\tau}\right]_0^t
\end{aligned}$$

$$+\left[\frac{\frac{\pi}{2L}N^{-2s}e^{p(2N+1)t}}{p(N+1)+p(N)-p(2N+1)}\sin\frac{\pi(2N+1)x}{L}e^{p(N+1)\tau+p(N)\tau-p(2N+1)\tau}\right]_0^t$$

$$=\frac{\pi}{2L}\frac{N^{-2s}}{p(N+1)+p(N)-p(1)}e^{p(1)t}\left(e^{p(N+1)t+p(N)t-p(1)t}-1\right)\sin\frac{\pi x}{L}$$

$$+\frac{\pi}{2L}\frac{N^{-2s}e^{p(2N+1)t}}{p(N+1)+p(N)-p(2N+1)}\left(e^{p(N+1)t+p(N)t-p(2N+1)t}-1\right)\sin\frac{\pi(2N+1)x}{L}.$$

For any fixed $0<t<T$, the above equality shows that

$$\int_0^t W_c(t-\tau)\big(W_c(\tau)\phi(x)\big)\partial_x\big(W_c(\tau)\psi(x)\big)\mathrm{d}\tau\sim O(N^{-2s-4})\sin\frac{\pi x}{L}+o(1)\sin\frac{\pi(2N+1)x}{L}$$

as $N\to+\infty$. Therefore,

$$\left\|\int_0^t W_c(t-\tau)\big(W_c(\tau)\phi(x)\big)\partial_x\big(W_c(\tau)\psi(x)\big)\mathrm{d}\tau\right\|_{H^s(0,L)}\sim O(N^{-2s-4}).$$

Note that $\|\phi\|_{H^s(0,L)}\sim 1$ and $\|\psi\|_{H^s(0,L)}\sim 1$, the above inequality yields that (5.2) holds for our specially selected ϕ and ψ when $s<-2$. □

Proof of Theorem 1.3. Let $s<-2$. Assume that there exists $\widehat{T}>0$ such that IBVP (1.4) admits a local solution $u\in C([0,\widehat{T}];H^s(0,L))$, and the flow map

$$\Xi: H^s(0,L)\to C([0,\widehat{T}];H^s(0,L)),\quad \phi\mapsto u(t)$$

is C^2 at the origin. Then the solution of IBVP (1.4) can be written as follows

$$(\Xi(\phi))(t)=W_c(t)\phi-\int_0^t W_c(t-\tau)\mathcal{B}((\Xi(\phi))(\tau),(\Xi(\phi))(\tau))\mathrm{d}\tau,$$

where $\mathcal{B}(\varphi,\psi)=\frac{1}{2}\partial_x(\varphi\psi)$.

Along the same lines as [10, 21], we can compute the first and second Fréchet derivative of Ξ at the origin to obtain

$$d_0(\Xi(\phi))(t)=W_c(t)\phi$$

and

$$d_0^2(\Xi(\phi,\psi))(t)=-\int_0^t W_c(t-\tau)\mathcal{B}(W_c(\tau)\phi,W_c(\tau)\psi)\mathrm{d}\tau.$$

From the assumption on the smoothness of Ξ at the origin, we conclude that

$$d_0^2(\Xi)\in\mathcal{L}(H^s(0,L)\times H^s(0,L),H^s(0,L)).$$

Hence,

$$\left\|d_0^2(\Xi(\phi,\psi))(t)\right\|_{H^s(0,L)}\le C\|\phi\|_{H^s(0,L)}\|\psi\|_{H^s(0,L)}$$

for all $\phi, \psi \in H^s(0, L)$. The above estimate is equivalent to (5.1). From Proposition 5.1, we know that Theorem 1.3 is valid. □

Acknowledgements. The authors would like to thank the anonymous referee for his valuable suggestions, which improved a lot the manuscript.

References

[1] A. Armaou and P.D. Christofides, Feedback control of the Kuramoto-Sivashinsky equation. *Physica D* **137** (2000) 49–61.
[2] J.L. Bona, S. Sun and B.-Y. Zhang, A non-homogeneous boundary-value problem for the Korteweg-de Vries equation in a quarter plane. *Trans. Am. Math. Soc.* **354** (2001) 427–490.
[3] J.L. Bona, S. Sun and B.-Y. Zhang, A nonhomogeneous boundary-value problem for the Korteweg-de Vries equation posed on a finite domain. *Commun. Partial Differ. Equ.* **28** (2003) 1391–1436.
[4] J.L. Bona, S. Sun and B.-Y. Zhang, A non-homogeneous boundary-value problem for the Korteweg-de Vries equation posed on a finite domain II. *J. Differ. Equ.* **247** (2009) 2558–2596.
[5] E. Cerpa, P. Guzmán and A. Mercado, On the control of the linear Kuramoto-Sivashinsky equation. *ESAIM: COCV* **23** (2017) 165–194.
[6] E. Cerpa and A. Mercado, Local exact controllability to the trajectories of the 1-D Kuramoto-Sivashinsky equation. *J. Differ. Equ.* **250** (2011) 2024–2044.
[7] P. Collet, J.-P. Eckmann, H. Epstein and J. Stubbe, A global attracting set for the Kuramoto-Sivashinsky equation. *Commun. Math. Phys.* **152** (1993) 203–214.
[8] A.T. Cousin and N.A. Larkin, Kuramoto-Sivashinsky equation in domains with moving boundaries. *Port. Math.* **59** (2002) 335–349.
[9] A. Demirkaya, The existence of a global attractor for a Kuramoto-Sivashinsky type equation in 2D. *Discrete Contin. Dyn. Syst.* (2009) 198–207.
[10] A. Esfahani, Sharp well-posedness of the Ostrovsky, Stepanyams and Tsimring equation. *Math. Commun.* **18** (2013) 323–335.
[11] L. Giacomelli and F. Otto, New bounds for the Kuramoto-Sivashinsky equation. *Commun. Pure Appl. Math.* **58** (2005) 297–318.
[12] J.S. Il'yashenko, Global Analysis of the Phase Portrait for the Kuramoto-Sivashinsky Equation. *J. Dyn. Differ. Equ.* **4** (1992) 585–615.
[13] Y. Kuramoto, On the formation of dissipative structures in reaction-diffusion systems. *Prog. Theor. Phys. Suppl.* **64** (1978) 346–367.
[14] Y. Kuramoto, Instability and turbulence of wavefronts in reaction-diffusion systems. *Prog. Theor. Phys.* **63** (1980) 1885–1903.
[15] J.L. Lions and E. Magenes, Non-Homogeneous Boundary Value Problems and Applications. Vol. II. Springer-Verlag, Berlin-Heidelberg-New York (1972).
[16] J. Li, B.-Y. Zhang and Z.X. Zhang, A nonhomogeneous boundary value problem for the Kuramoto-Sivashinsky equation in a quarter plane. *Math. Meth. Appl. Sci.* **40** (2017) 5619–5641.
[17] L. Molinet, F. Ribaud and A. Youssfi, Ill-posedness issues for a class of parabolic equations. *Proc. R. Soc. Edinburgh* **132** (2002) 1407–1416.
[18] B. Nicolaenko and B. Scheurer, Remarks on the Kuramoto-Sivashinsky equation. *Physica D* **12** (1984) 391–395.
[19] B. Nicolaenko, B. Scheurer and R. Temam, Some global dynamical properties of the Kuramoto-Sivashinsky equations: nonlinear stability and attractors. *Physica D* **16** (1985) 155–183.
[20] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations. *Applied Mathematical Sciences*, Vol. 44. Springer-Verlag, New York (1983).
[21] D. Pilod, Sharp well-posedness results for the Kuramoto-Velarde equation. *Commun. Pure Appl. Anal.* **7** (2008) 867–881.
[22] G.I. Sivashinsky, Nonlinear analysis of hydrodynamic instability in laminar flames–I. Derivation of basic equations. *Acta Astronaut.* **4** (1977) 1177–1206.
[23] G.I. Sivashinsky, On flame propagation under conditions of stoichiometry. *SIAM J. Appl. Math.* **39** (1980) 67–82.
[24] E. Tadmor, The well-posedness of the Kuramoto-Sivashinsky equation. *SIAM J. Math. Anal.* **17** (1986) 884–893.