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OPTIMAL CONTROL OF HIGHER ORDER DIFFERENTIAL INCLUSIONS WITH FUNCTIONAL CONSTRAINTS

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Abstract. The present paper studies the Mayer problem with higher order evolution differential inclusions and functional constraints of optimal control theory (P_{FC}); to this end first we use an interesting auxiliary problem with second order discrete-time and discrete approximate inclusions (P_{FD}). Are proved necessary and sufficient conditions incorporating the Euler–Lagrange inclusion, the Hamiltonian inclusion, the transversality and complementary slackness conditions. The basic concept of obtaining optimal conditions is locally adjoint mappings and equivalence results. Then combining these results and passing to the limit in the discrete approximations we establish new sufficient optimality conditions for second order continuous-time evolution inclusions. This approach and results make a bridge between optimal control problem with higher order differential inclusion (P_{FC}) and constrained mathematical programming problems in finite-dimensional spaces. Formulation of the transversality and complementary slackness conditions for second order differential inclusions play a substantial role in the next investigations without which it is hardly ever possible to get any optimality conditions; consequently, these results are generalized to the problem with an arbitrary higher order differential inclusion. Furthermore, application of these results is demonstrated by solving some semilinear problem with second and third order differential inclusions.

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1. Introduction

Optimal control of discrete-differential inclusions with lumped and distributed parameters has been expanding in all directions at an astonishing rate during the last few decades. Note that the differential inclusions are not only models for many dynamical processes but they also provide a powerful tool for various branches of mathematical analysis; see more discussions and comments in the relatively recent publications [1, 3, 5, 6, 9, 15–17, 28] and the references therein. First order differential inclusions play a crucial role in the mathematical theory of optimal processes given in the next papers [8–12, 26, 27].

The paper [7] introduces a new class of variational problems for differential inclusions, motivated by the control of forest fires. The area burned by the fire at time $t > 0$ is modelled as the reachable set for a differential inclusion $x' \in F(x)$, starting from an initial set. To block the fire, a wall can be constructed progressively in

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time, at a given speed. In this paper is studied the possibility of constructing a wall which completely encircles the fire; are shown that the search for blocking strategies and for optimal strategies can be reduced to a problem involving one single admissible rectifiable set. In the paper [5] are derived a variant of the nonsmooth maximum principle for optimal control problems with both pure and mixed states and control constraints. A notable feature is that a well known penalization technique is used for state constrained problem together with an appeal to a recent nonsmooth maximum principle for problems with mixed constraints.

Closely related optimality problems for a first order differential inclusions were considered by Loewen and Rockafellar [13] and Mordukhovich [27], and the present study is an important generalization of their works to the problem with higher order differential inclusions. In the paper [13] are considered a Mayer problem of optimal control, whose dynamic constraint is given by a convex-valued first order differential inclusion. Both state and endpoint constraints are involved. Are proved necessary conditions incorporating the Hamiltonian inclusion, the Euler–Lagrange inclusion, and the Weierstrass–Pontryagin maximum condition. The paper [27] is devoted to the study of a Mayer-type optimal control problem for semilinear unbounded evolution inclusions in reflexive and separable Banach spaces subject to endpoint constraints described by finitely many Lipschitzian equalities and inequalities. First are constructed a sequence of discrete approximations to the optimal control problem for evolution inclusions and proved that optimal solutions to discrete approximation problems uniformly converge to a given optimal solution for the original continuous-time problem.

In fact, the difficulty in the problems with second order and especially as more for higher order ordinary differential inclusions is rather to construct the Euler–Lagrange type second order adjoint inclusions and the suitable transversality conditions. In general, a convenient procedure for eliminating this complication in optimal control theory involving higher order derivatives is a formal reduction of the problems by substitution to the system of first order differential inclusions or equations. But in practice returning to the original "higher order problem" and expressing the arising optimality conditions in terms of the original problem data in general is very difficult. That is why on the whole in literature only the qualitative properties of second order differential inclusions are investigated (see [2, 4, 14, 25] and references therein). The paper [4] gives necessary and sufficient conditions ensuring the existence of solutions to the second order differential inclusions with state of constraints. Furthermore, the second order interior tangent sets are introduced and studied to obtain such conditions. In the work [14] are proved the existence of viable solutions for an autonomous second-order functional differential inclusions, where the multifunction that define the inclusion is upper semicontinuous, compact valued and contained in the subdifferential of a proper lower semicontinuous convex function. In the paper [25] by using the methodology of the viability theory the existence of Lyapunov functions for second-order differential inclusions is analyzed. A necessary assumption on the initial states and sufficient conditions for the existence of local and global Lyapunov functions are obtained.

In spite of the presence of the above mentioned qualitative studies, optimal control of problems with the higher order differential inclusions has not been studied in the literature; up to our best knowledge, there a few papers of Mahmudov [18–24] devoted to such studies. The paper [19] concerns itself with the second order polyhedral optimization described by ordinary discrete and differential inclusions. The stated second order discrete problem is reduced to the polyhedral minimization problem with polyhedral geometric constraints and necessary and sufficient conditions of optimality are derived in terms of the polyhedral Euler–Lagrange inclusions. The paper [20] is devoted to the study of optimal control theory with higher order differential inclusions and a varying time interval, under some attainability condition are established sufficient conditions of optimality. The paper [21] studies the Lagrange and Bolza types problems of optimal control theory with boundary value conditions given by second order differential inclusions. Mainly our purpose is to derive sufficient optimality conditions for mentioned problems with second order differential inclusions.

The paper [22] deals with a Bolza problem of optimal control theory given by second order convex differential inclusions with second order state variable inequality constraints. Necessary and sufficient conditions of optimality including distinctive transversality condition are proved in the form of Euler–Lagrange inclusions. The paper [23] studies a new class of problems of optimal control theory with Sturm–Liouville type differential inclusions involving second order linear self-adjoint differential operators. Necessary and sufficient conditions, containing both the Euler–Lagrange and Hamiltonian type inclusions and "transversality" conditions, are derived. The result strengthens and generalizes the problem with a second order non-self-adjoint differential operator; a suitable choice of coefficients then transforms this operator to the desired Sturm–Liouville type problem.

The present paper is dedicated to one of the difficult and interesting fields optimization of the second order ordinary differential inclusions with the functional constraints and its generalization to the case of $m(m \geq 3)$th-order differential inclusions. The posed problems and the corresponding optimality conditions are new. In this paper we consider a Mayer problem, although all results are valid even for a Bolza problem. The paper is organized in the following order:

In Section 2, the needed facts and supplementary results from the book of Mahmudov [17] are given; Hamiltonian function H and argmaximum sets of a set-valued mapping F, the locally adjoint mapping (LAM), local tent is introduced and the initial-value problems for second order discrete and differential inclusions and then mth-order differential inclusions with functional constraints are formulated. In Section 3, the optimality problem (P_{FD}) for the second order discrete inclusions with inequality type of endpoint constraints given in Section 2 is reduced to the problem by finite number of geometric constraints. For such problems, we use constructions of convex and non-smooth analysis; and here then, in terms of convex upper approximations, local tents and LAMs prove necessary and sufficient conditions of optimality including the suitable complementary slackness conditions. In Section 4, the problem for second order continuous-time evolution inclusions (P_{FC}) is approximated with associated discrete-approximation problem (P_{FDA}) by using the first and the second order difference operators and auxiliary set-valued mapping. It is noted that, transition to the problem (P_{FDA}) requires special equivalence Theorem 2.5 of an LAMs F^* and Q^* connecting the main results of discrete (P_{FD}) and discrete-approximate (P_{FDA}) problems. For construction of the transversality condition for a Mayer problem in its general form, both Chain rule for subdifferentiation of composition of convex functions and positive-homogeneity of LAM on the first argument skilfully are used. Consequently, the key to our success is the approximation and formulation of the equivalence Theorem 2.5. For a non-convex problem, the analogical results are proved by the help of local tents. It is obvious that this method, which is certainly of independent interest from qualitative viewpoint, can play an important role in numerical procedures as well. Section 5 is devoted to derivation of sufficient optimality conditions for second order differential inclusions with the inequality type of endpoint conditions. Construction of the Euler–Lagrange and transversality inclusions and the complementary slackness conditions are based on passing to the limit in the optimality conditions of discrete-approximate problem (P_{FDA}). At the end of this section is considered a Mayer problem (P_{FC}) with so-called semilinear second order continuous-time evolution inclusions and in the Weierstrass–Pontryagin form the optimality conditions are formulated. In Section 6 sufficient conditions of optimality for mth-order differential inclusions (P_H) with functional constraints are deduced; the basic idea is again to replace the continuous problem (P_H) by the discrete-approximation problem and according to (P_H) we associate the corresponding mth-order discrete approximate problem. In this paper, the establishment of these conditions together with the detailed transversality condition is omitted and we start our discussion with a presentation and study of sufficient optimality conditions for problem (P_H). The results of Corollaries 6.2 and 6.3 regarding the optimality conditions to a third order differential inclusions problems are necessary for a comprehensive and brief presentation of the main results. In Section 7 sufficient conditions of optimality for second order differential inclusions with non-functional initial and endpoint constraints are derived. In particular, the constructed higher order Mahmudov's inclusion coincides with the Euler–Lagrange inclusion for first order differential inclusions.

2. Needed facts and problem statement

Necessary notions can be found in [17]. Let $\mathbb{R}^n$ be n-dimensional Euclidean space, $\langle x, y\rangle$ be an inner product of elements $x, y \in \mathbb{R}^n, (x, y)$ be a pair of x, y. Assume that $F(\cdot, t) : \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n, t \in [t_0, t_1]$ is a multivalued (set-valued) mapping from $\mathbb{R}^{2n}$ into $\mathbb{R}^n$. Then $F(\cdot, t)$ is convex, if its graph $gphF(\cdot, t) = \{(x, y, z,) : z \in F(x, y, t)\}$ is a convex subset of $\mathbb{R}^{3n}$. The multivalued mapping $F(\cdot, t)$ is convex closed if its graph is a convex closed set in $\mathbb{R}^{3n}$. The domain of $F(\cdot, t)$ is denoted by $domF(\cdot, t)$ and is defined as follows $domF(\cdot, t) = \{(x, y) : F(x, y, t) \neq \varnothing\}$. $F(\cdot, t)$ is convex-valued, if $F(x, y, t)$ is a convex set for $(x, y) \in domF(\cdot, t)$. Let us introduce the Hamiltonian function and argmaximum set for multivalued mapping $F(\cdot,t)$

$$H_{F(\cdot,t)}(x,y,z^*) = \sup\{\langle z,z^*\rangle : z \in F(x,y,t)\}, z^* \in \mathbb{R}^n,$$
$$F_{Arg}(x,y;z^*,t) = F_A(x,y;z^*,t) = \{z \in F(x,y,t) : \langle z,z^*\rangle = H_{F(\cdot,t)}(x,y,z^*)\},$$

respectively. For convex $F(\cdot,t)$ we set $H_{F(\cdot,t)}(x,y,z^*) = -\infty$ if $F(x,y,t) = \varnothing$.
Let $intA$ be the interior of the set $A \subset \mathbb{R}^{3n}$ and riA be the relative interior of the set A, *i.e.* the set of interior points of A with respect to its affine hull $AffA$.
The convex cone $K_A(x_0,y_0,z_0)$ is called the cone of tangent directions at a point $(x_0,y_0,z_0) \in A$ to the set A if from $(\bar{x},\bar{y},\bar{z}) \in K_A(x_0,y_0,z_0)$ it follows that $(\bar{x},\bar{y},\bar{z})$ is a tangent vector to the set A at point $(x_0,y_0,z_0) \in A$, *i.e.*, there exists function $\varphi(\lambda) \in \mathbb{R}^{3n}$ such that $(x_0,y_0,z_0) + \lambda(\bar{x},\bar{y},\bar{z}) + \varphi(\lambda) \in A$ for sufficiently small $\lambda > 0$ and $\lambda^{-1}\varphi(\lambda) \to 0$, as $\lambda \downarrow 0$.

Definition 2.1. The cone $K_A(x_0,y_0,z_0)$ is called the local tent if for any $(\bar{x}_0,\bar{y}_0,\bar{z}_0) \in riK_A(x_0,y_0,z_0)$ there exists a convex cone $K \subseteq K_A(x_0,y_0,z_0)$ and a continuous mapping $\psi(\bar{x},\bar{y},\bar{z})$ defined in the neighbourhood of the origin such that

(1) $(\bar{x}_0,\bar{y}_0,\bar{z}_0) \in riK, LinK = LinK_A(x_0,y_0,z_0),$ where $LinK$ is a linear hull for a set$K$,
(2) $\psi(\bar{x},\bar{y},\bar{z}) = (\bar{x},\bar{y},\bar{z}) + r(\bar{x},\bar{y},\bar{z}), r(\bar{x},\bar{y},\bar{z})||(\bar{x},\bar{y},\bar{z})||^{-1} \to 0$ as $(\bar{x},\bar{y},\bar{z}) \to 0$,
(3) $(x_0,y_0,z_0) + \psi(\bar{x},\bar{y},\bar{z}) \in A, (\bar{x},\bar{y},\bar{z}) \in K \cap S_\epsilon(0)$ for some $\epsilon > 0,$ where $S_\epsilon(0)$ is the ball of radius ϵ.

For a convex mapping F a multifunction defined by

$$F^*\big(z^*;(x,y,z),t\big) := \{(x^*,y^*) : (x^*,y^*,-z^*) \in K^*_{gphF(\cdot,t)}(x,y,z)\},$$
$$K_{gphF(\cdot,t)}(x,y,z) = cone[gphF(\cdot,t) - (x,y,z)]$$

is called a locally adjoint multifunction (LAM) to F at a point $(x,y,z) \in gphF$, where $K^*_{gphF(\cdot,t)}(x,y,z)$ is the dual to the cone $K_{gphF(\cdot,t)}(x,y,z) = cone[gphF(\cdot,t) - (x,y,z)]$. The LAM to non-convex mapping F is defined as follows

$$F^*\big(z^*;(x,y,z),t\big) := \{(x^*,y^*) : H_{F(\cdot,t)}(x_1,y_1,z^*) - H_{F(\cdot,t)}(x,y,z^*) \le \langle x^*,x_1 - x\rangle$$
$$+ \langle y^*,y_1 - y\rangle, \forall(x_1,y_1) \in \mathbb{R}^{2n}\},\ (x,y,z) \in gphF(\cdot,t),\ z \in F_A(x,y;z^*,t).$$

Clearly for the convex mapping $H_{F(\cdot,t)}(\cdot,z^*)$ is concave and the latter definition of LAM coincide with the previous definition of LAM. We note that the coderivative concept of Mordukhovich [27] is essentially different for nonconvex mappings. Here, the principal difference is that the coderivative is not tangentially generated and uses the nonconvex normal cone and allows to avoid really restrictive assumptions for necessary optimality conditions for discrete and continuous systems.

Definition 2.2. With respect to [15] $h(\bar{x},x)$ is called a CUA of the function $g : \mathbb{R}^n \to \mathbb{R}^1 \cup \{\pm\infty\}$ at a point $x \in domg = \{x : |g(x) < +\infty\}$ if $h(\bar{x},x) \ge V(\bar{x},x)$ for all $\bar{x} \ne 0$ and $h(\cdot,x)$ is a convex closed positive homogeneous function, where

$$V(\bar{x},x) = \sup_{r(\cdot)} \limsup_{\alpha\downarrow 0} (1/\alpha)[g(x + \alpha\bar{x} + r(\alpha)) - g(x)],\ \alpha^{-1}r(\alpha) \to 0.$$

Here, the exterior supremum is taken on all $r(\alpha)$ such that $\alpha^{-1}r(\alpha) \to 0$ as $\alpha \downarrow 0$.

Definition 2.3. A set defined as follows $\partial h(0,x) = \{x^* \in \mathbb{R}^n : h(\bar{x},x) \geq \langle x^*, \bar{x}\rangle, \bar{x} \in \mathbb{R}^n\}$ is called a subdifferential of the function g at a point x and is denoted by $\partial g(x)$. The main advantage of this definition is its simplicity.

It should be noted that $h(\bar{x},x)$ is the support function of $\partial g(x)$. Thus, $h(\bar{x},x)$ and $\partial g(x)$ determine each other one to one and the function $h(\cdot,x)$ defined by equation

$$\partial h(\bar{x},x) = \sup_{x^*}\{\langle \bar{x},x\rangle : x^* \in \partial h(0,x)\} = \sup_{x^*}\{\langle \bar{x},x\rangle : x^* \in \partial g(x)\}$$

must be a CUA of f at x. Besides, if h_1 and h_2 are CUAs for the function g at a point x and $h_1 \geq h_2$ then $\partial_1 g(x) \supseteq \partial_2 g(x)$, where $\partial_1 g(x)$ and $\partial_2 g(x)$ are the subdifferentials defined by h_1 and h_2, respectively [17]. However, it is clear that h_1 is a worse approximation than h_2 in some neighbourhood of x.

Definition 2.4. For an arbitrary nonempty subset $P \subset \mathbb{R}^{3n}$ the cone defined by

$$coneP = \{(\bar{x},\bar{y},\bar{z}) : \bar{x} = \alpha x, \bar{y} = \alpha y, \bar{z} = \alpha z, (x,y,z) \in P, \alpha > 0\}$$

is called the cone generated by P. Let us now recall the next Theorem 2.1 [22] crucial in what follows.

Theorem 2.5. *Suppose that the convex-valued mapping $F(\cdot,t) : \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is so that the cone $K_{gphQ(\cdot,t)}(x,y,z), (x,y,z) \in gphQ(\cdot,t$ of tangent directions for the mapping $Q(x,y,t) = 2y - x + h^2 F(x,(y-x)/h,t)$ determines a local tent, where h is a fixed positive number. Then, the following inclusions are equivalent:*

$$(1)\ (x^*,y^*) \in Q^*(z^*;(x,y,z),t),\ z \in Q_A(x,y,;z^*,t),$$
$$(2)\ \big((x^*+y^*-z^*)/h^2,(y^*-2z^*)/h\big) \in F^*\Big(z^*;\big(x,(y-x)/h,(z-2y+x)/h^2\big),t\Big),$$
$$(z-2y+x)/h^2 \in F_A\big(x,(y-x)/h;z^*,t\big),\ z^* \in \mathbb{R}^n,$$

where $Q_A(x,y;z^*,t)$ is the argmaximum set for mapping $Q(\cdot,t)$. Let us mention the following useful proposition from [22].

Proposition 2.6. *For a proper convex function defined by $g(x,y) \equiv \varphi\big(x,(y-x)/h\big)$ the following inclusions are equivalent:*

$$(a)\ \big(\bar{x}^*,\bar{y}^*\big) \in \partial_z g(z), z = (x,y) \in domg;\ (b)\ \big(\bar{x}^*+\bar{y}^*,h\bar{y}^*\big) \in \partial\varphi\big(x,(y-x)/h\big).$$

Clearly for the convex mapping $H_F(\cdot,\cdot,y^*)$ is concave and the latter definition of LAM coincide with the previous definition of LAM.

Section 2 deals with the following second order discrete model labelled as (P_{FD}):

$$\text{minimize } g_0(x_{N-1},x_N), \tag{2.1}$$
$$(P_{FD}) \qquad x_{t+2} \in F(x_t,x_{t+1},t),\ t = 0,\dots,N-2;\ x_0 = \tilde{\alpha}_0,\ x_1 = \tilde{\beta}_1, \tag{2.2}$$
$$g_k(x_{N-1},x_N) \leq 0,\ k = 1,\dots,r, \tag{2.3}$$

where $x_t \in \mathbb{R}^n, g_k(\cdot), k = 1,\dots,r$ are real-valued functions, $g_k(\cdot) : \mathbb{R}^n \to \mathbb{R}^1 \cup \{\pm\infty\}, F(\cdot,t) : \mathbb{R}^{2n} \rightrightarrows \mathbb{R}^n$ is an evolution multivalued mapping and N is fixed natural numbers, $\tilde{\alpha}_0, \tilde{\beta}_1$ are fixed vectors. Conditions (2.3) are endpoint constraints for second order discrete-time problem (P_{FD}). A sequence $\{x_t\}_{t=0}^N = \{x_t : t = 0,1,\dots,N\}$ is called the feasible trajectory for the stated problem (2.1)–(2.3). It is required find a solution $\{x_t\}_{t=0}^N$ to a problem (P_{FD}) for the second discrete-time problem, satisfying (2.2), (2.3) and minimizing $g_0(x_{N-1},x_N)$. The problem (2.1)–(2.3) is convex if the $F(\cdot,t)$ and $g_k(\cdot), k = 0,1,\dots,r$ are convex multivalued function and convex proper function, respectively.

In Section 4, we deal with the approximation of convex problem for second order differential inclusions with functional constraints:

$$\begin{aligned} &\text{minimize } \varphi_0(x(t_1), x'(t_1)), && (2.4)\\ (P_{FC}) \qquad & x''(t) \in F(x(t), x'(t), t), \text{ a.e. } t \in [t_0, t_1], && (2.5)\\ & \varphi_k(x(t_1), x'(t_1)) \le 0,\ k = 1, \dots, r;\ x(t_0) = \alpha_0, x'(t_0) = \beta_1. && (2.6) \end{aligned}$$

Here, $F(\cdot, t) : \mathbb{R}^{2n} \rightrightarrows \mathbb{R}^n$ is an evolution convex mapping, $\varphi_k(\cdot) : \mathbb{R}^n \to \mathbb{R}^1, k = 0, \dots, r$ are convex continuous functions, t_0, t_1 and α_0, β_1 are fixed real numbers and vectors, correspondingly. It is required find a solution $\tilde{x}(\cdot)$ to a problem (2.4)–(2.6) for the second order differential inclusions, satisfying (2.5) almost everywhere (a.e.) on $[t_0, t_1]$, functional constraints and the initial conditions (2.6) that minimize the Mayer functional $\varphi_0\big(x(t_1), x'(t_1)\big)$. We label this problem with second order continuous-time evolution inclusions as (P_{FC}). Here, a feasible trajectory $x(\cdot)$ is an absolutely continuous function on a time interval $[t_0, t_1]$ together with the first-order derivative whose second derivative belongs to the space L_1^n. Notice that such class of functions is a Banach space, endowed with the different equivalent norms. For example,

$||x(\cdot)|| = |x(t_0)| + |x'(t_0)| + ||x''(\cdot)||_1$ or $||x(\cdot)|| = \sum_{k=0}^{2} ||x^{(k)}(\cdot)||_1$, where $||x^{(k)}(\cdot)||_1 = \int_{t_0}^{t_1} |x^{(k)}(t)| \mathrm{d}t$, and $|x|$ is an Euclidean norm in $\mathbb{R}^n$.

3. Necessary and sufficient conditions for second order discrete inclusions

At first we consider the convex problem (2.1)–(2.3). Let us introduce a vector $u = (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n(N+1)}$ and define in the space $\mathbb{R}^{n(N+1)}$ the following convex sets

$$M_t = \{u = (x_0, \dots, x_N) : (x_t, x_{t+1}, x_{t+2}) \in gphF(\cdot, t)\};\ D_k = \{u = (x_0, \dots, x_N) : g_k(x_{N-1}, x_N) \le 0\}$$
$$P_0 = \{u = (x_0, \dots, x_N) : x_0 = \tilde{\alpha}_0\},\ P_1 = \{u = (x_0, \dots, x_N) : x_1 = \tilde{\beta}_1\},\ t = 0, 1, \dots, N-2;\ k = 1, \dots, r.$$

Now, denoting $f_0(u) = g_0(x_{N-1}, x_N)$ we will reduce this problem to the problem with geometric constraints. Indeed, it can easily be seen that our basic problem (2.1)–(2.3) is equivalent to the following one

$$\text{minimize} f_0(u) \text{ subject to } S = \left(\bigcap_{k=1}^{r} D_k \right) \cap \left(\bigcap_{t=0}^{N-2} M_t \right) \cap P_0 \cap P_1, \tag{3.1}$$

where S is a convex set.

In the sense of the terminology of first order discrete inclusions [17, 27] we are ready to give the necessary and sufficient conditions for the problem (2.1)–(2.3).

Definition 3.1. Let us say that for the convex problem (2.1)–(2.3) the regularity condition is satisfied if there exist points $x_t \in \mathbb{R}^n (x_0 = \tilde{\alpha}_0, x_1 = \tilde{\beta}_1)$ such that one of the following two conditions holds:

$$(i)\ (x_t, x_{t+1}, x_{t+2}) \in ri(gphF(\cdot, t)),\ t = 0, \dots, N-2,\ (x_{N-1}, x_N) \in ridomg_0 \cap riB_k,\ k = 1 \dots, r;$$
$$(ii)\ (x_t, x_{t+1}, x_{t+2}) \in int(gphF(\cdot, t)),\ t = 0, \dots, N-2,\ (x_{N-1}, x_N) \in intB_k,\ k = 1 \dots, r;$$

(with the possible exception of one fixed t), and g_0 are continuous at (x_{N-1}, x_N), where $B_k = \{(x_{N-1}, x_N) : g_k(x_{N-1}, x_N) \le 0\}, k = 1, \dots, r$. It follows from the regularity condition that if $\{\tilde{x}_t\}_{t=0}^N$ is the optimal trajectory in the problem (P_{FD}), then the cones of tangent directions $K_{gphF(\cdot,t}(\tilde{x}_t, \tilde{x}_{t+1}, \tilde{x}t + 2)$ are not separable and consequently the condition of Theorem 3.2 ([17], p. 98) is satisfied.

Condition T. Suppose that in the problem (2.1)–(2.3) the mapping $F(\cdot,t)$ is such that the cones of tangent directions $K_{gphF(\cdot,t)}(\tilde{x}_t,\tilde{x}_{t+1},\tilde{x}_{t+2})$ are local tents, where $\tilde{x}_t$ are the points of the optimal trajectory $\{\tilde{x}_t\}_{t=0}^T$. Suppose, moreover, that the functions $g_k(\cdot), k=0,\dots,r$ admit a continuous CUA $h_k(\cdot,(\tilde{x}_{N-1},\tilde{x}_N))$ ([17], p. 122) at the points $(\tilde{x}_{N-1},\tilde{x}_N)$, which ensures that the subdifferentials $\partial_{(x,y)}g_k(\tilde{x}_{N-1},\tilde{x}_N)=\partial_{(x,y)}h_k(0,\tilde{x}_{N-1},\tilde{x}_N)$ are defined, where

$$\partial_{(x,y)}g_k(\tilde{x}_{N-1},\tilde{x}_N)=\{(x^*,y^*):g_k(\tilde{x}_{N-1},\tilde{x}_N)\geq\langle x-\tilde{x}_{N-1},x^*\rangle+\langle y-\tilde{x}_N,y^*\rangle,\ \forall(x,y)\}.$$

Remember that non-smooth and non-convex function cannot be approximated in a neighbourhood of some point with positively homogeneous functions, *i.e.*, directional derivatives. Just for such a class of functions, we have introduced the concept of CUAs.

Theorem 3.2. *Let $F(\cdot,t)$ be an evolution convex set-valued mapping and $g_k(\cdot),k_0,1,\dots,r$ be convex continuous functions. Let $\{\tilde{x}\}_{t=0}^N$ be an optimal trajectory to the discrete-time problem with functional constraints (P_{FD}). Then there exist the Lagrange multipliers $\lambda_k\geq 0,k=0,1,\dots,r$ and vectors $x_t^*,x_n^*,\mu_t^*(\mu_0^*=0),t=0,\dots,N-1$ simultaneously not all zero, such that the discrete Euler–Lagrange (i), transversality (ii) inclusions and the complementary slackness conditions (iii) hold:*

$$(i)\ \left(x_t^*-\mu_t^*,\mu_{t+1}^*\right)\in F^*\left(x_{t+2}^*(\tilde{x}_t,\tilde{x}_{t+1},\tilde{x}_{t+2}),t\right),\ t=0,1,\dots,N-2,$$
$$(ii)\ \left(\mu_{N-1}^*-x_{N-1}^*,-x_N^*\right)\in\sum_{k=0}^{r}\lambda_k\partial g_k(\tilde{x}_{N-1},\tilde{x}_N),$$
$$(iii)\ \lambda_k g_k(\tilde{x}_{N-1},\tilde{x}_N)=0,\ \lambda_k\geq 0,\ k=1,\dots,r.$$

Moreover, if the regularity condition is satisfied, then these conditions are sufficient for optimality of the trajectory of $\{\tilde{x}_t\}_{t=0}^N$.

Proof. Since $\{\tilde{x}_t\}_{t=0}^n$ is an optimal trajectory, it follows that, $\tilde{u}=(\tilde{x}_0,\dots,\tilde{x}_N)$ is a solution of the problem (3.1). For this convex mathematical programming we apply necessary optimality conditions from [12, 17]. Consequently, if $\tilde{u}=(\tilde{x}_0,\dots,\tilde{x}_N)$ is a point of minimum of problem (3.1). Then by Theorem 3.2 ([17], p. 98) for a function $f_0(\cdot)$ there exist not all zero vectors $u^{0*}\in\partial_u f_0(\tilde{u}),u^*(t)\in K_{M_t}^*(\tilde{u}),t=0,1,\dots,N-2,\bar{u}_k^*\in K_{D_k}^*(\tilde{u}),u_0^*\in K_{P_0}^*(\tilde{u}),u_1^*\in K_{P_1}^*(\tilde{u})$ and the number $\lambda_0\geq 0$, such that

$$\sum_{t=0}^{N-2}u^*(t)+\sum_{k=1}^{r}\bar{u}_k^*+u_0^*+u_1^*=\lambda_0 u^{0*}. \tag{3.2}$$

Hence, we should compute the cones appearing here. If $K_{gphF(\cdot,t)}(\tilde{x}_t,\tilde{x}_{t+1},\tilde{x}_{t+2}),(\tilde{x}_t,\tilde{x}_{t+1},\tilde{x}_{t+2})\in gphF(\cdot,t)$ is a cone of tangent directions then by Proposition 3.1 [23]

$$K_{M_t}^*(\tilde{u})=\{u^*=(x_0^*,\dots,x_N^*):(x_t^*,(x_{t+1}^*,x_{t+2}^*)\in K_{gphF(\cdot,t)}^*(\tilde{x}_t,\tilde{x}_{t+1},\tilde{x}_{t+2}),\ x_t^*=0,i\neq t,t+1,t+2\}.$$

Moreover, $x_0=\tilde{\alpha}_0,\tilde{u}+\lambda\bar{u}\in P_0$ if and only if $\bar{x}_0=0$. By analogy for $x_1=\tilde{\beta}_1$ we have $\bar{x}_1=0$. As a result, we derive

$$K_{P_0}^*(\tilde{u})=\{u^*=(x_0^*,\dots,x_N^*):x_t^*=0,t=1\dots,N\},\ K_{P_1}^*(\tilde{u})=\{u^*=(x_0^*,\dots,x_N^*):x_t^*=0,t\neq 1\}. \tag{3.3}$$

Denoting $f_k(u)\equiv g_k(x_{N-1},x_N),k=1,\dots,r$ and using Theorem 1.34 ([17], p. 50) we compute the dual cone $K_{D_k}^*(\tilde{u})$; it is not hard to see that there are two cases: (1) there no points u for which $f_k(u)<0$. Then since $\tilde{u}\in\cap_{k=1}^r D_k$ and $0=f_k(\tilde{u})\leq f_k(u)$, by Theorem 3.1 ([17], p. 97) $0\in\partial_u f_k(\tilde{u})$ *i.e.* $\tilde{u}$ is a point of m?nimum of $f_k(u)$; (2) there exists a point u such that $f_k(u)<0$. Then by the above mentioned theorem it is easy to see that

$$K^*_{D_k}(\tilde u)=\begin{cases}(0,\dots,0)\in\mathbb{R}^{n(N+1)}, & \text{if}\quad f_k(\tilde u)<0,\\ -cone\partial_u f_k(\tilde u), & \text{if}\quad f_k(\tilde u)=0.\end{cases}$$

Then taking into account that $cone\partial_u f_k(\tilde u)=\underbrace{(0,\dots,0)}_{n(N-1)}\times\{cone\partial_{(x,y)}g_k(\tilde x_{N-1},\tilde x_N)\}$, with the preceding notations we deduce that

$$K^*_{D_k}(\tilde u)=\begin{cases}(0,\dots,0)\in\mathbb{R}^{n(N+1)}, & \text{if}\quad g_k(\tilde x_{N-1},\tilde x_N)<0,\\ \underbrace{(0,\dots,0)}_{n(N-1)}\times\{cone\partial_{(x,y)}g_k(\tilde x_{N-1},\tilde x_N)\}, & \text{if}\quad g_k(\tilde x_{N-1},\tilde x_N)=0.\end{cases}\tag{3.4}$$

From definition of the function f_0 it is easy to see that the vector $u^{0*}\in\partial_u f_0(\tilde u)$ has a form $u^{0*}=(x_0^{0*},x_1^{0*},\dots,x_N^{0*}),x_t^{0*}=0,t=0,\dots,N-2;(x_{N-1}^{0*},x_N^{0*})\in\partial_{(x,y)}g_0(\tilde x_{N-1},\tilde x_N)$. Furthermore, according to the form of $K^*_{M_t}(\tilde u)$ and formula (3.3) we have

$$\begin{aligned}&\bar u_k^*=(0,\dots,0,\bar x^*_{(N-1)k},\bar x^*_{Nk}),(\bar x^*_{(N-1)k},\bar x^*_{Nk})\in\partial_{(x,y)}g_k(\tilde x_{N-1},\tilde x_N),k=1,\dots,r,\\ &u^*(0)=\big(x_0^*(0),x_1^*(0),0,\dots,0\big);\ u^*(1)=\big(0,x_1^*(1),x_2^*(1),0,\dots,0\big),\\ &u^*(N-4)=\big(0,\dots,0,x^*_{N-4}(N-4),x^*_{N-3}(N-4),x^*_{N-2}(N-4),0,0\big),\\ &u^*(N-3)=\big(0,\dots,0,x^*_{N-3}(N-3),x^*_{N-2}(N-3),x^*_{N-1}(N-3),0\big),\\ &u^*(N-2)=\big(0,\dots,0,x^*_{N-2}(N-2),x^*_{N-1}(N-2),x^*_N(N-2)\big),\\ &\big(x_t^*(t),x^*_{t+1}(t),x^*_{t+2}(t)\big)\in K^*_{gphF}\big(\tilde x_t,\tilde x_{t+1},\tilde x_{t+2}\big),\ t=2,\dots,N-3,\\ &w_0^*=\big(x_{00}^*,0,\dots,0\big),\ w_1^*=\big(0,x_{11}^*,0,\dots,0\big),\end{aligned}\tag{3.5}$$

where $x_{00}^*,x_{11}^*,\hat x_1^*$ are arbitrary vectors.

Now if $g_k(x_{N-1},x_N)<0,k=1,\dots,r$ for a point (x_{N-1},x_N), then the formula (3.4) is true. Let us denote

$$\begin{aligned}&\bar x^*_{(N-1)k}=-\lambda_k x^*_{(N-1)k},\ \bar x^*_{Nk}=-\lambda_k x^*_{Nk},\lambda_k\ge 0,\\ &x^*_{(N-1)0}=x^{0*}_{N-1},x^*_{N0}=x^{0*}_N,\ \big(x^*_{(N-1)k},x^*_{Nk}\big)\in\partial_{(x,y)}g_k(\tilde x_{N-1},\tilde x_N),k=1,\dots,r.\end{aligned}\tag{3.6}$$

Here $\lambda_k>0$, if $g_k(\tilde x_{N-1},\tilde x_N)=0$, and $\lambda_k=0$, if $g_k(\tilde x_{N-1},\tilde x_N)<0$, that is the complementary slackness conditions are satisfied

$$\lambda_k g_k(\tilde x_{N-1},\tilde x_N)=0,k=1,\dots,r.\tag{3.7}$$

We note that not all λ_k is equal to zero, because not all $u^*(t)(t=0,\dots,N-2),\bar u_k^*(k=1,\dots,r),u_0^*,u_1^*,\lambda_0$ are zero. Moreover, if for some k_0 the points (x_{N-1},x_N) satisfying $g_{k_0}(x_{N-1},x_N)<0$ do not exist, then as is shown above $0\in\partial_{(x,y)}g_{k_0}(x_{N-1},x_N)$. Setting $(x_{(N-1)k_0},x_{Nk_0})=0,\lambda_{k_0}=1$ and $\lambda_k=0,k\ne k_0$, we obtain that (3.7) is justified.

Now using the component-wise representation of (3.2) we deduce that

$$\begin{aligned}&x_{00}^*+x_0^*(0)=0,\ x_{11}^*+x_1^*(1)+x_1^*(0)=0,\\ &x_t^*(t)+x_t^*(t-1)+x_t^*(t-2)=0,\ t=2,\dots,N-2\end{aligned}\tag{3.8}$$

and then from the second formula of (3.5) by definition of LAM we derive that

$$\big(x_t^*(t), x_{t+1}^*(t)\big) \in F^*\big(-x_{t+2}^*(t); (\tilde{x}_t, \tilde{x}_{t+1}, \tilde{x}_{t+2}), t\big),\ t = 2, \dots, N-2. \tag{3.9}$$

Introducing the new notations $x_{t+1}^*(t) \equiv \mu_{t+1}^*, -x_{t+2}^*(t) \equiv x_{t+2}^*, t = 1, \dots, N-2$ in the third formula of (3.8) we obtain by (3.9) that

$$\big(x_t^*, -\mu_t^*, \mu_{t+1}^*\big) \in F^*\big(x_{t+2}^*; (\tilde{x}_t, \tilde{x}_{t+1}, \tilde{x}_{t+2}), t\big),\ t = 2, \dots, N-2. \tag{3.10}$$

On the other hand it is easy to see that setting, $x_{00}^* = -x_0^*, \mu_0^* \equiv 0$ and $x_{11}^* = -x_1^*$ in the first and second equalities, respectively, we can generalize the formula (3.10) to the case $t = 0, 1$. Finally, for $t = N-1$ we have

$$x_{N-1}^*(N-2) + x_{N-1}^*(N-3) + \textstyle\sum_{i=1}^r \bar{x}_{(N-1)i}^* = \lambda_0 x_{N-1}^{0*}.$$

Obviously, for $t = N$ we can write

$$x_N^*(N-2) + \textstyle\sum_{k=1}^r \bar{x}_{Nk}^* = \lambda_0 x_N^{0*}.$$

Thus, with the preceding notations (3.6) we have

$$\mu_{N-1}^* - x_{N-1}^* = \sum_{k=0}^r \lambda_k x_{(N-1)k}^*;\ -x_N^* = \sum_{k=0}^r \lambda_k x_{Nk}^*. \tag{3.11}$$

Combining the formulas of (3.11) we deduce that

$$\big(\mu_{N-1}^* - x_{N-1}^*, -x_N^*\big) \in \sum_{k=0}^r \lambda_k \partial_{(x,y)} g_k(\tilde{x}_{N-1}, \tilde{x}_N). \tag{3.12}$$

Now the necessary condition of theorem follows from the formulas (3.10), (3.11), (3.12). On the other hand, it is clear that by Theorems 1.30 ([9], p. 47) and 3.3 ([17], p. 98), under the regularity condition, the point $u^{0*} \in \partial_u f(\tilde{u}) \cap K_s^*(\tilde{u})$, which exists by Theorem 3.2 ([17], p. 98), admits a decomposition (3.2) with parameter $\lambda_0 = 1$ and by Theorem 3.4 ([17], p. 99) we have the desired result.

Theorem 3.3. *Suppose that the Condition T for the non-convex problem (2.1)–(2.3) is satisfied. Then for optimality of the trajectory $\{\tilde{x}_t\}_{t=0}^N$ in the non-convex problem, it is necessary that there exist numbers $\lambda_k \geq 0, k = 0, \dots, r$ and pair of vectors $\{x_t^*\}, \{\mu_t^*\}$ not all zero, to satisfy the conditions of Theorem 3.2 written out for non-convex problem.*

Proof. In this case the Condition T ensures the conditions of Theorem 3.24 ([17], p. 133) for the non-convex problem (3.1). Hence, according to this theorem, we get the necessary condition as in Theorem 3.2 by starting from the relation (3.2), written out for non-convex problem.

4. Necessary and sufficient conditions of optimality for discrete-approximate problems

Assume that h is a step on the t-axis and $x(t) \equiv x_h(t)$ is a grid function on a uniform grid on $[t_0, t_1]$. We introduce the following first and second order difference operators (forward and backward difference approximations $\Delta_+ x(t) \equiv \Delta x(t)$ and $\Delta_- x(t), t = t_0, t_0 + h, \dots, t_1 - 2$)

$$\Delta x(t) = \frac{1}{h}[x(t+h) - x(t)],\ \Delta_- x(t) = \frac{1}{h}[x(t) - x(t-h)], \Delta^2 x(t) = \frac{1}{h}[\Delta x(t+h) - \Delta x(t)]$$

and associate with the problem (2.4)–(2.6) the following second order discrete-approximate evolution problem

$$\text{minimize } \varphi_0\big(x(t_1-h),\Delta_- x(t_1)\big), \tag{4.1}$$
$$\Delta^2 x(t)\in F\big(x(t),\Delta x(t),t\big), t=t_0,t_0+h,\dots,t_1-2h, \tag{4.2}$$
$$\varphi_k\big(x(t_1-h),\Delta_- x(t_1)\big)\le 0,\ k=1,\dots,r; \tag{4.3}$$
$$x(t_0)=\alpha_0,\ \Delta x(t_0)=\beta_1.$$

First of all reduce the problem (4.1)–(4.3) to a problem of the form (2.1)–(2.3) with endpoints constraints. Introducing a new set-valued mapping $Q(\cdot,t):\mathbb{R}^n\times\mathbb{R}^n\rightrightarrows\mathbb{R}^n$ and functions $g_k, k=0,\dots,r$

$$Q(x,y,t)=2y-x+h^2F\big(x,(y-x)/h,t\big)$$
$$g_k\big(x(t_1-h),x(t_1)\big)\equiv\varphi_k\big(x(t_1),\Delta_- x(t_1)\big), k=0,1,\dots,r \tag{4.4}$$

we rewrite the problem (4.1)–(4.3) as follows:

$$\text{minimize } g_0\big(x(t_1-h),x(t_1)\big), \tag{4.5}$$
$$(P_{FDA})\qquad x(t+2h)\in Q\big(x(t),x(t+h),t\big), \tag{4.6}$$
$$g_k\big(x(t_1-h),x(t_1)\big)\le 0,\ k=1,\dots,r, \tag{4.7}$$
$$x(t_0)=\alpha_0,\ x(t_0+h)=\alpha_0+h\beta_1, t=t_0,t_0+h,\dots,t_1-2h. \tag{4.8}$$

By Theorem 3.2 for optimality of the trajectory $\{\tilde x(t)\}:=\{\tilde x(t):t=t_0,t_0+h,\dots,t_1\}$ in problem (4.5)–(4.8) it is necessary that there exist a pair of vectors $\{\bar\mu^*(t)\},\{\bar x^*(t)\}(\bar\mu^*(t_0)=0)$ and a number $\lambda_k=\lambda_{kh}\ge 0, k=0,\dots,r$, not all zero, such that the discrete Euler–Lagrange, transversality inclusions and the complementary slackness conditions are satisfied:

$$\big(\bar x^*(t)-\bar\mu^*(t)\bar\mu^*(t+h)\big)\in Q^*\big(\bar x^*(t+2h);(\tilde x(t),\tilde x(t+h),\tilde x(t+2h)),t\big), \tag{4.9}$$
$$\tilde x(t+2h)\in Q_A\big(\tilde x(t),\tilde x(t+h);\bar x^*(t+2h),t\big),\ t=t_0,t_0+h\dots,t_1-2h,$$
$$\big(\bar\mu(t_1-h)-\bar x^*(t_1-h),-\bar x^*(t_1)\big)\in\sum_{k=0}^{r}\lambda_k\partial g_k\big(\tilde x(t_1-h),\tilde x(t_1)\big), \tag{4.10}$$
$$\lambda_k g_k\big(\tilde x(t_1-h),\tilde x(t_1)\big)=0,\ \lambda_k\ge 0, k=1,\dots,r. \tag{4.11}$$

Theorem 4.1. *(Approximate Euler–Lagrange conditions for second order discrete approximations of constrained evolution inclusions) Suppose that $F(\cdot,t)$ be a convex set-valued mapping, and $g_k(\cdot,\cdot), k=0,1,\dots,r$ be convex continuous functions. Then for optimality of the trajectory $\{\tilde x(t)\}$ in the second order discrete approximation problem (4.1)–(4.3), it is necessary that there exist the Lagrange multipliers $\lambda_k\ge 0, k=0,\dots,r$ and a pair of vectors $\{x^*(t),v^*(t)\}$ simultaneously not all zero, such that the second order approximate Euler–Lagrange (1), transversality inclusions (2) and the complementary slackness conditions (3) hold:*

(1) $\big(\Delta^2x^*(t)+\Delta v^*(t),v^*(t)\big)\in F^*\big(x^*(t+2h);(\tilde x(t),\Delta\tilde x(t),\Delta^2\tilde x(t)),t\big)$,
$\Delta^2\tilde x(t)\in F_A\big(\tilde x(t),\Delta\tilde x(t);x^*(t+2h),t\big);\ t=t_0,t_0+h\dots,t_1-2h,$

(2) $\big(v^*(t_1-h)+\Delta_- x(t_1),-x^*(t_1)\big)\in\sum_{k=0}^{r}\lambda_k\partial\varphi_k(\tilde x(t_1-h),\Delta\tilde x(t)),$

(3) $\lambda_k g_k(\tilde x(t_1-h),\tilde x(t_1))=0,\ \lambda_k\ge 0,\ k=1,\dots,r.$

And under the regularity condition, these conditions are also sufficient for optimality of $\{\tilde{x}(t)\}$.

Proof. In the above relationship (4.9) we should express the LAM Q^* in term of LAM F^*. Using the condition (2) of Theorem 2.5 we obtain immediately

$$\begin{aligned}&\Big[\big(\bar{x}^*(t)-\bar{\mu}^*(t)+\bar{\mu}^*(t+h)-\bar{x}^*(t+2h)\big)/h^2,\ \big(\bar{\mu}^*(t+h)-2\bar{x}^*(t+2h)\big)/h\Big]\\&\in F^*\big(\bar{x}^*(t+2h);\big(\tilde{x}(t),\Delta\tilde{x}(t),\Delta^2\tilde{x}(t)\big),t\big),\ t=t_0,t_0+h,\dots,t_1-2h,\\&\Delta^2\tilde{x}(t)\in F_A\big(\tilde{x}(t),\Delta\tilde{x}(t);x^*(t+2h),t\big).\end{aligned}$$

Hence, denoting here $\bar{v}^*(t)=\big[\bar{\mu}(t)-2\bar{x}^*(t+h)\big]/h$ we define $\bar{\mu}^*(t)=h\bar{v}^*(t)+2\bar{x}^*(t+h)$ and then we derive converted second order approximate Euler–Lagrange inclusions as follows

$$\big(\Delta^2\bar{x}^*(t)+\Delta\bar{v}^*(t),\bar{v}^*(t+h)\big)\in F^*\big(\bar{x}^*(t+2h);\big(\tilde{x}(t),\Delta\tilde{x}(t),\Delta^2\tilde{x}(t)\big),t\big). \tag{4.12}$$

In the next step we will prove the transversality condition (2). Since $\bar{\mu}^*(t_1-h)=h\bar{v}^*(t_1-h)+2\bar{x}^*(t_1)$ and $g_k\big(\tilde{x}(t_1-h),\tilde{x}(t_1)\big)\equiv\varphi_k\big(\tilde{x}(t_1),\Delta_-\tilde{x}(t_1)\big)$ (see (4.4)) with regard to Proposition 2.6 and backward difference approximation we obtain

$$\big(h\bar{v}^*(t_1-h)+\bar{x}^*(t_1)-\bar{x}^*(t_1-h),-h\bar{x}^*(t_1)\big)\in\sum_{k=0}^{r}\lambda_k\partial\varphi_k\big(\tilde{x}(t_1),\Delta_-\tilde{x}(t_1)\big). \tag{4.13}$$

Now we observe that the LAM F^* is positive homogeneous on the first argument that is, the inclusion (4.12) can be rewritten as follows

$$\big(\Delta^2h\bar{x}^*(t)+\Delta h\bar{v}^*(t),h\bar{v}^*(t+h)\big)\in F^*\big(h\bar{x}^*(t+2h);\big(\tilde{x}(t),\Delta\tilde{x}(t),\Delta^2\tilde{x}(t)\big),t\big). \tag{4.14}$$

Consequently, setting $x^*(t)=h\bar{x}^*(t),v^*(t)=h\bar{v}^*(t),t=t_0,t_0+h,\dots,t_1$ in (4.13), (4.14) we have the desired result.

Later on we need the following useful result, which is implied from Theorem 2.5.

Proposition 4.2. *Suppose that the convex-valued mapping $F_0(\cdot,t):\mathbb{R}^n\rightrightarrows\mathbb{R}^n$ is so that the cone $K_{gphQ(\cdot,t)}(x,y,z),(x,y,z)\in gphQ(\cdot,t)$ of tangent directions for the mapping $Q(x,y,z)=2y-x+h^2F_0(x,t)$ determines a local tent. Then, the following inclusions under the condition that $y^*=2z^*$ are equivalent*

(1) $(x^*,y^*)\in Q^*\big(z^*;(x,y,z),t\big);\ z\in Q_A(x,y;z^*,t),\ z^*\in\mathbb{R}^n$,

(2) $(x^*+z^*)/h^2\in F_0^*\big(z^*;\big(x,(z-2y+x)/h^2\big),t\big);\ (z-2y+x)/h^2\in F_{0A}(x;z^*,t)$.

Proof. In the present case $domF(\cdot,t)=domF_0(\cdot,t)\times\mathbb{R}^n$ and according to Theorem 2.5 $\big((x^*+y^*-z^*)/h^2,(y^*-2z^*)/h\big)\in F_0^*\big(z^*;\big(x,(z-2y+x)/h^2\big),t\big)\times\{0\}$, whence we have $(y^*-2z^*)/h=0$. It means that $y^*=2z^*$ and then $(x^*+y^*-z^*)/h^2=(x^*+z^*)/h^2\in F_0^*\big(z^*;\big(x,(z-2y+x)/h^2\big),t\big)$. Analogously, using the Theorem 4.1 we have.

Theorem 4.3. *Suppose that the Condition T is satisfied for the non-convex problem (4.1)–(4.3), i.e. the mapping $F(\cdot,t)$ is such that the cones of tangent directions $K_{gphF(\cdot,t)}(\tilde{x}(t),\Delta\tilde{x}(t),\Delta^2\tilde{x}(t))$ are local tents and that the functions $\varphi_k(\cdot),k=0,\dots,r$ admit a continuous CUA $h_k(\cdot,(\tilde{x}(t_1-h),\Delta_-\tilde{x}(t_1))$ at the points $(\tilde{x}(t_1-h),\Delta_-\tilde{x}(t_1)$. Then, for optimality of the trajectory $\{\tilde{x}(t)\}$, it is necessary that there exist nonnegative numbers $\lambda_k\geq0,k=0,\dots,r$ and a pair of vectors $\{x^*(t),v^*(t)\}$ not all zero, satisfying the conditions (1)–(3) of Theorem 4.1 written out for non-convex problem.*

Let us consider the following simplified version of Theorem 4.1.

Theorem 4.4. *Suppose that in the discrete-approximation problem* (P_{FD}) $F(x,(y-x)/h,t)\equiv F_0(x,t)$, *i.e. a mapping* $F_0(\cdot,t)$ *depends only on* x. *Then, the adjoint second order Euler–Lagrange discrete inclusions (1), transversality condition (2) and the complementary slackness conditions (3) of Theorem 4.1 have the forms*

$$\Delta^2 x^*(t)\in F_0^*\big(x^*(t+2h);\big(\tilde{x}(t),\Delta^2\tilde{x}(t)\big)\big),\ t=t_0,t_0+h,\dots,t_1-2h;$$
$$\big(\Delta_- x^*(t_1),-x^*(t_1)\big)\in\sum_{k=0}^{r}\lambda_k\partial\varphi_k\big(\tilde{x}(t_1-h),\Delta_-\tilde{x}(t_1)\big),$$
$$\lambda_k\varphi_k\big(\tilde{x}(t_1-h),\Delta_-\tilde{x}(t_1)\big)=0,\ \lambda_k\geq 0,\ k=1,\dots,r,$$

correspondingly.

Proof. By analogy with the proof of Theorem 4.1 let us return to the condition (4.9):

$$\big(\bar{x}^*(t)-\bar{\mu}^*(t),\bar{\mu}^*(t+h)\big)\in Q^*\big(\bar{x}^*(t+2h);\big(\tilde{x}(t),\tilde{x}(t+h),\tilde{x}(t+2h)\big),t\big).$$

Then by using the condition (2) of Proposition 4.2 we have the following inclusion

$$\big(\bar{x}^*(t)-\bar{\mu}^*(t)+\bar{x}^*(t+2h)\big)/h^2\in F^*\big(\bar{x}^*(t+2h);\big(\tilde{x}(t),\Delta^2\tilde{x}(t)\big),t\big),$$
$$\Delta^2\tilde{x}(t)\in F_{0A}\big(\tilde{x}(t);x^*(t+2h),t\big),\ t=t_0,t_0+h,\dots,t_1-2h.$$

On the other hand the condition $y^*=2z^*$ of Proposition 4.2 implies that $\mu^*(t)=2x^*(t+h)$. Then, by substitution of $\mu^*(t)=2x^*(t+h)$ into the last inclusion, we get

$$\Delta^2\bar{x}^*(t)\in F_0^*\big(\bar{x}^*(t+2h);\big(\tilde{x}(t),\Delta^2 tildex(t)\big),t\big).$$

The remaining discussion is similar to the proof of Theorem 4.1. The proof of theorem is complete.

5. Sufficient conditions of optimality for second order differential inclusions

In this section, the optimality conditions are given by using so-called second order adjoint differential inclusions, which are expressed in terms of LAM. Construction of the Euler–Lagrange and transversality inclusions and the complementary slackness conditions are based on passing to the limit in optimality conditions of Theorems 4.1–4.4. These conditions involve useful forms of the Weierstrass–Pontryagin condition and second order Euler–Lagrange type adjoint inclusions.

The adjoint Euler–Lagrange type differential inclusion

$$(a)\ \big(x^{*''}(t)+v^{*'}(t),v^*(t)\big)\in F^*\big(x^*(t);\big(\tilde{x}(t),\tilde{x}'(t),\tilde{x}''(t)\big),t\big),\ \text{a.e. } t\in[t_0,t_1].$$

The transversality condition at the endpoint $t=t_1$

$$(b)\ \big(v^*(t_1)+x^{*'}(t_1),-x^*(t_1)\big)\in\partial\varphi_0\big(\tilde{x}(t_1),\tilde{x}'(t_1)\big)+\sum_{k=1}^{r}\lambda_k\partial\varphi_k\big(\tilde{x}(t_1),\tilde{x}'(t_1)\big).$$

The complementary slackness conditions at $t=t_1$

$$(c)\ \lambda_k\varphi_k\big(\tilde{x}(t_1),\tilde{x}'(t_1)\big)=0,\ \lambda_k\geq 0,\ k=1,\dots,r.$$

In what follows we assume that $x^*(t), t \in [t_0, t_1]$ is absolutely continuous function together with the first order derivatives and $x^{*''}(\cdot) \in L_1^n([t_0, t_1))$. Besides $v^*(\cdot)$ is absolutely continuous and $v^{*'}(\cdot) \in L_1^n([t_0, t_1))$. The following condition guarantees that the LAM $F^*(\cdot, t)$ is nonempty at a given point:

$$(d)\ \tilde{x}''(t) \in F_A\big(\tilde{x}(t), \tilde{x}'(t); x^*(t), t\big),\ \text{a.e. } t \in [t_0, t_1].$$

In what follows, Theorem 5.1 is very important.

Theorem 5.1. *(Sufficient conditions of optimality for second order evolution differential inclusions with functional constraints) Suppose that $\varphi_k(\cdot), k = 0, 1, \dots, r$ are convex continuous functions, $F(\cdot, t)$ is an evolution convex set-valued mapping. Then for optimality of the trajectory $\tilde{x}(\cdot)$ in the problem (P_{FC}) it is sufficient that there exists a pair of absolutely continuous functions $\{x^*(t), v^*(t), \}, t \in [t_0, t_1]$ and Lagrange multipliers $\lambda_k \geq 0, k = 0, \dots, r(\lambda_0 = 1)$, satisfying a.e. the second order Euler–Lagrange inclusion (a), (d), the transversality condition (b) and the complementary slackness conditions (c) at $t = t_1$.*

Proof. By Theorem 2.1 ([17], p. 62) we can write $F^*\big(z^*; (x, y, z), t\big) = \partial_{(x,y)} H_{F(\cdot,t)}(x, y, z^*), z \in F_A(x, y; z^*, t)$. Then using the Moreau–Rockafellar Theorem 1.29 ([17], p. 46) from condition (a) we obtain

$$\Big(x^{*''}(t) + v^{*'}(t), v^*(t)\Big) \in \partial_{(x,y)} H_{F(\cdot,t)}\big(\tilde{x}(t), \tilde{x}'(t), x^*(t)\big),\ t \in [t_0, t_1].$$

In turn on the definition of subdifferential set of the Hamiltonian function for all feasible trajectory $x(t), t \in [t_0, t_1]$ we rewrite the last relation in the equivalent form:

$$\begin{aligned} &H_{F(\cdot,t)}\big(x(t), x'(t), x^*(t)\big) - H_{F(\cdot,t)}\big(\tilde{x}(t), \tilde{x}'(t), x^*(t)\big) \\ &\leq \big\langle x^{*''}(t) + v^{*'}(t), x(t) - \tilde{x}(t)\big\rangle + \langle v^*(t), x'(t) - \tilde{x}'(t)\rangle. \end{aligned} \tag{5.1}$$

Now by using definition of the Hamiltonian function, the inequality (5.1) can be reduced to the inequality

$$\langle x''(t), x^*(t)\rangle - \langle \tilde{x}''(t), x^*(t)\rangle \leq \big\langle x^{*''}(t), x(t) - \tilde{x}(t)\big\rangle + \frac{\mathrm{d}}{\mathrm{d}t}\langle v^*(t), x(t) - \tilde{x}(t)\rangle.$$

which implies that

$$\big\langle \big(x(t) - \tilde{x}(t)\big)'', x^*(t)\big\rangle - \big\langle x^{*''}(t), x(t) - \tilde{x}(t)\big\rangle - \frac{\mathrm{d}}{\mathrm{d}t}\langle v^*(t), x(t) - \tilde{x}(t)\rangle \leq 0. \tag{5.2}$$

Integrating of the inequality (5.2) over the interval $[t_0, t_1]$ we can write

$$\int_{t_0}^{t_1} \big[\big\langle \big(x(t) - \tilde{x}(t)\big)'', x^*(t)\big\rangle - \big\langle x^{*''}(t), x(t) - \tilde{x}(t)\big\rangle\big] \mathrm{d}t + \langle v^*(t_0), x(t_0) - \tilde{x}(t_0)\rangle - \langle v^*(t_1), x(t_1) - \tilde{x}(t_1)\rangle \leq 0. \tag{5.3}$$

We transform the expression in the square parentheses on the left hand side of (5.3) as follows

$$\big\langle \big(x(t) - \tilde{x}(t)\big)'', x^*(t)\big\rangle - \big\langle x^{*''}(t), x(t) - \tilde{x}(t)\big\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\big\langle \big(x(t) - \tilde{x}(t)\big)', x^*(t)\big\rangle - \frac{\mathrm{d}}{\mathrm{d}t}\big\langle x^{*'}(t), x(t) - \tilde{x}(t)\big\rangle.$$

Now by elementary property of the definite integrals we can compute the integral on the left hand side of (5.3)

$$\begin{aligned} \int_{t_0}^{t_1} \big[\big\langle \big(x(t) - \tilde{x}(t)\big)'', x^*(t)\big\rangle - \big\langle x^{*''}(t), x(t) - \tilde{x}(t)\big\rangle\big] \mathrm{d}t &= \langle x'(t_1) - \tilde{x}'(t_1), x^*(t_1)\rangle \\ -\langle x'(t_0) - \tilde{x}'(t_0), x^*(t_0)\rangle - \big\langle x^{*'}(t_1), x(t_1) - \tilde{x}(t_1)\big\rangle &+ \big\langle x^{*'}(t_0), x(t_0) - \tilde{x}(t_0)\big\rangle. \end{aligned} \tag{5.4}$$

Then substituting (5.4) into (5.3), we have

$$\langle x'(t_1)-\tilde{x}'(t_1),x^*(t_1)\rangle-\langle x'(t_0)-\tilde{x}'(t_0),x^*(t_0)\rangle \\ -\langle v^*(t_1)+x^{*'}(t_1),x(t_1)-\tilde{x}(t_1)\rangle+\langle v^*(t_0)+x^{*'}(t_0),x(t_0)-\tilde{x}(t_0)\rangle\le 0. \tag{5.5}$$

Now, remember that by transversality condition (b) of Theorem 4.1 and by Moreau–Rockafellar Theorem 1.29 ([17], p. 46) for all feasible trajectories $x(t), t\in[t_0,t_1]$ we can write

$$\varphi_0\big(x(t_1),x'(t_1)\big)+\sum_{k=1}^{r}\lambda_k\varphi_k\big(x(t_1),x'(t_1)\big)-\varphi_0\big(\tilde{x}(t_1),\tilde{x}'(t_1)\big)-\sum_{k=1}^{r}\lambda_k\varphi_k\big(\tilde{x}(t_1),\tilde{x}'(t_1)\big) \\ \ge\langle v^*(t_1)+x^{*'}(t_1),x(t_1)-\tilde{x}(t_1)\rangle-\langle x^*(t_1),x'(t_1)-\tilde{x}'(t_1)\rangle. \tag{5.6}$$

Recall that $x(t),\tilde{x}(t),t\in[t_0,t_1]$ are feasible solutions, that is $x(t_0)=\tilde{x}(t_0)=\alpha_0; x'(t_0)=\tilde{x}'(t_0)=\beta_1$ and from (5.5) we have

$$\langle x^*(t_1),x'(t_1)-\tilde{x}'(t_1)\rangle-\langle v^*(t_1)+x^{*'}(t_1),x(t_1)-\tilde{x}(t_1)\rangle\le 0. \tag{5.7}$$

Then by virtue of (5.6) and (5.7) we obtain

$$\textstyle\sum_{k=0}^{r}\lambda_k\varphi_k\big(x(t_1),x'(t_1)\big)\ge\sum_{k=0}^{r}\lambda_k\varphi_k\big(\tilde{x}(t_1),\tilde{x}'(t_1)\big),\lambda_k\ge 0(\lambda_0=1),\ k=1,\dots,r,\ \forall x(t_1).$$

Thus, using the complementary slackness conditions (c) at $t=t_1$ for an arbitrary feasible solution $x(\cdot)$ we deduce from this inequality that

$$\lambda_0\varphi_0\big(x(t_1),x'(t_1)\big)+\sum_{k=1}^{r}\lambda_k\varphi_k\big(x(t_1),x'(t_1)\big)\ge\lambda_0\varphi_0\big(\tilde{x}(t_1),\tilde{x}'(t_1)\big),\lambda_k\ge 0,\ k=1,\dots,r. \tag{5.8}$$

Note that, $x(\cdot)$ is a feasible solution and $\lambda_k\ge 0,k=1,\dots,r$ that is, $\sum_{k=1}^{r}\lambda_k\varphi_k\big(x(t_1),x'(t_1)\big)\le 0$. Then from (5.8) we derive that $\varphi_0\big(x(t_1),x'(t_1)\big)\ge\varphi_0\big(\tilde{x}(t_1),\tilde{x}'(t_1)\big),\forall x(\cdot)$ that is, $\tilde{x}(\cdot)$ is optimal.

Corollary 5.2. *In addition to assumptions of Theorem 5.1 let $F(\cdot,t)$ be a closed set-valued mapping. Then the conditions (a), (d) of Theorem 5.1 can be rewritten in term of Hamiltonian function as follows*

$$\Big(x^{*''}(t)+v^{*'}(t),v^*(t)\Big)\in\partial_{(x,y)}H_{F(\cdot,t)}\big(\tilde{x}(t),\tilde{x}'(t),x^*(t)\big);\ \tilde{x}''(t)\in\partial_{z^*}H_{F(\cdot,t)}\big(\tilde{x}(t),\tilde{x}'(t),x^*(t)\big).$$

Proof. In fact, the validity of the first relation we have seen in the proof of Theorem 5.1. On the other hand by Lemma 5.1 of [23] $\partial_{z^*}H_{F(\cdot,t)}(x,y,z^*)=F_A(x,y,z^*,t)$ and the assertions of corollary are equivalent with the conditions (a), (d) of Theorem 5.1.

Below we prove that if a multivalued mapping $F(\cdot,t)$ depends only on x, then the adjoint inclusion involves only one conjugate variable.

Corollary 5.3. *Suppose that for the convex problem with second order continuous-time evolution inclusions (P_{FC}) $f(x,x',t)\equiv F_0(x,t)$, and that the conditions of Theorem 5.1 are satisfied. Then the second order Euler–Lagrange differential inclusion (a), transversality condition (b) and the complementary slackness conditions (c) at $t=t_1$ of Theorem 5.1 consist of the following*

$$(i)\ \ x^{*''}(t)\in F_0^*\big(x^*(t);(\tilde{x}(t),\tilde{x}''(t)),t\big);\ \tilde{x}''(t)\in F_{0A}\big(\tilde{x}(t);,x^*(t),t\big),\ a.e.\ t\in[t_0,t_1].$$

$$(ii)\ \big(x^{*''}(t_1),-x^*(t_1)\big)\in\partial\varphi_0(\tilde{x}(t_1),\tilde{x}'(t_1))+\sum_{k=1}^{r}\lambda_k\partial\varphi_k(\tilde{x}(t_1),\tilde{x}'(t_1)),\ \lambda_k\varphi_k(\tilde{x}(t_1),\tilde{x}'(t_1))=0.$$

Proof. In the present case formulation of conditions (i), (ii) based on limiting procedure in the conditions of Theorem 4.4. Then by analogy with the proof of Theorem 5.1 sufficient conditions of optimality can be obtained.

Corollary 5.4. *The results of Theorem 5.1 remain true for a Bolza problem with functional* $J[x(t)] = \int_{t_0}^{t_1} g(x(t),t)dt + \varphi(x(t_1), x'(t_1))$ *and in this case the condition (a) consists of the following*

$$\left(\frac{d^2x^*(t)}{dt^2} + \frac{dv^*(t)}{dt}, v^*(t)\right) \in F^*\left(x^*(t); \left(\tilde{x}(t), \tilde{x}'(t), \tilde{x}''(t)\right), t\right) - \{\partial g(\tilde{x}(t),t)\} \times \{0\}.$$

Proof. It remains to note that in the proof of this corollary on the right-hand side of inequality (5.3) and, consequently, (5.5) zero will simply be replaced by the integral of the difference $g(x(t),t) - g(\tilde{x}(t),t)$ over the interval $[t_0, t_1]$.

Below nonconvexity of problem (P_{FC}) means that the Hamilton function $H_{F(P\cdot,t)}$, in general, is nonconcave and functions $\varphi_k, k = 0,\dots,r$ are nonconvex functions satisfying the conditions (i), (ii).

Corollary 5.5. *Suppose that* $\varphi_k : \mathbb{R}^{2n} \to \mathbb{R}^1 (k = 0,\dots,r)$ *are nonconvex functions and* $F(\cdot,t)$ *is a nonconvex set-valued mapping. Then for optimality of the arc* $\tilde{x}(t), t \in [t_0,t_1]$ *among all feasible solutions in such a nonconvex problem* (P_{FC})*, it is sufficient that there exists a pair of absolutely continuous functions* $\{x^*(t), v^*(t)\}, t \in [t_0,t_1]$ *satisfying a.e. the second order Euler–Lagrange inclusion (a), (d) of Theorem 5.1 in the nonconvex case and Lagrange multipliers* $\lambda_k \geq 0, k = 0,\dots,r (\lambda_0 = 1)$*, and the transversality condition (i) and the complementary slackness conditions (ii) at* $t = t_1$*:*

$$\begin{aligned}
&(i)\ \varphi_0(x,y) - \varphi_0\left(\tilde{x}(t_1), \tilde{x}'(t_1)\right) + \sum_{k=1}^{r} \lambda_k [\varphi_k(x,y) - \varphi_k\left(\tilde{x}(t_1), \tilde{x}'(t_1)\right)] \\
&\quad \geq \left\langle v^*(t_1) + x^{*'}(t_1), x - \tilde{x}(t_1)\right\rangle - \langle x^*(t_1), y - \tilde{x}'(t_1)\rangle, \\
&(ii)\ \lambda_k \varphi_k\left(\tilde{x}(t_1), \tilde{x}'(t_1)\right) = 0,\ \lambda_k \geq 0,\ k = 1,\dots,r.
\end{aligned}$$

Proof. On the one hand by condition (a) of Theorem 5.1 and definition of LAM in the nonconvex case we have the inequality (5.5). On the other hand, taking $x = x(t)$ and $y = x'(t)$ in the condition (ii) the validity of the inequality (5.6) is justified. Thus, the furthest proof of theorem is similar to the one for Theorem 5.1. To illustrate how to apply the results of Theorem 5.1 consider the example; suppose now we have Mayer problem (P_{FC}) linear second order continuous-time evolution inclusions:

$$\begin{aligned}
\text{minimize}\quad & \varphi_0(x(t_1), x'(t_1)), \\
& x''(t) \in F(x(t), x'(t)),\ \text{a.e. } t \in [t_0,t_1],\ F(x,y) = A_1x + A_2y + BU, \\
& \varphi_k(x(t_1), x'(t_1)) \leq 0,\ k = 1,\dots,r;\ x(t_0) = \alpha_0, x'(t_0) = \beta_1,
\end{aligned} \tag{5.9}$$

where $\varphi_k, k = 0,\dots,r$ are convex continuous functions, $A_i (i = 1,2), B$ are $n \times n$ and $n \times r$ matrices, respectively, U is a convex closed subset of $\mathbb{R}^r$. It is required to find a controlling parameter $\tilde{u}(t) \in U$ such that the trajectory $\tilde{x}(\cdot)$ corresponding to it minimizes $\varphi_0(x(t_1), x'(t_1))$.

Theorem 5.6. *The feasible trajectory* $\tilde{x}(\cdot)$ *of problem (5.9) corresponding to the control function* $\tilde{u}(\cdot)$ *minimizes* $\varphi_0(x(t_1), x'(t_1))$ *in the second order convex differential inclusions with endpoint constraints if there exists an absolutely continuous function* $x^*(\cdot)$ *satisfying the complementary slackness conditions (c) at* $t = t_1$*, the following second order adjoint differential equation, the transversality condition and Weierstrass–Pontryagin condition:*

$$x^{*''}(t) = A_1^* x^*(t) - A_2^* x^{*'}(t),\ a.e.\ t \in [t_0,t_1],\ \langle B\tilde{u}(t), x^*(t)\rangle = \sup_{u \in U} \langle Bu, x^*(t)\rangle,$$

$$\left(A_2^* x^*(t_1) + x^{*'}(t_1), -x^*(t_1)\right) \in \sum_{k=0}^{r} \lambda_k \partial\varphi_k\left(\tilde{x}(t_1), \tilde{x}'(t_1)\right),\ \lambda_k \geq 0 (\lambda_0 = 1),\ k = 1,\dots,r.$$

Proof. Obviously, if for a convex map $F(x,y) = A_1x + A_2y + BU$ the LAM is nonempty, then $F^*(z^*;(x,y,z)) = (A_1^*z^*, A_2^*z^*)$ as $-B^*z^* \in K_U^*(u)$. Then according to Theorem 5.1 the Euler–Lagrange inclusion is

$$x^{*''}(t) + v^{*'}(t) = A_1^*x^*(t),\ v^*(t) = A_2^*x^*(t). \tag{5.10}$$

Differentiating second equation of (5.10) and substituting into first equation we immediately have the needed second order adjoint equation. The other conditions of theorem is an immediate consequence of the conditions (b)–(d) of Theorem 5.1. The proof is completed.

6. Sufficient conditions of optimality for mth-order differential inclusions with functional constraints

In this section, the results obtained in Section 5 are generalized to the higher order (say mth-order) differential inclusions. Assume that $F(\cdot,t) : (\mathbb{R}^n)^m \rightrightarrows \mathbb{R}^n, t \in [t_0,t_1]$ is a multivalued (set-valued) mapping from $(\mathbb{R}^n)^m = \underbrace{\mathbb{R}^n \times \ldots \times \mathbb{R}^n}_{m}$ into $\mathbb{R}^n$. In what follows taking into account that $y = (y_1,\ldots,y_{m-1})$ the Hamiltonian function, argmaximum set and LAMs are defined completely by analogy. Now we deal with the convex optimal control problem for mth-order differential inclusions with the functional constraints:

$$\text{minimize } \varphi_0\big(x(t_1), x'(t_1), \ldots, x^{(m-1)}(t_1)\big), \tag{6.1}$$

$$(P_H) \qquad \frac{\mathrm{d}^m x(t)}{\mathrm{d}t^m} \in F\big(x(t), x'(t), \ldots, x^{(m-1)}(t), t\big),\ \text{a.e. } t \in [t_0,t_1], \tag{6.2}$$

$$\varphi_k\big(x(t_1), x'(t_1), \ldots, x^{(m-1)}(t_1)\big) \le 0,\ k = 1,\ldots,r, \tag{6.3}$$

$$x(t_0) = \alpha_0, x'(t_0) = \alpha_1, \ldots, x^{(m-1)}(t_0) = \alpha_{m-1}, \tag{6.4}$$

where $F(\cdot,t) : \big(\mathbb{R}^n\big)^m \rightrightarrows \mathbb{R}^n$ is convex mapping, $\varphi_k : \big(\mathbb{R}^n\big)^m \to \mathbb{R}^1, k = 0,1,\ldots,r$ are convex continuous functions and $\alpha_k, k = 0,1,\ldots,m-1$ are fixed vectors. We label this problem as (P_H). The problem is to find an arc $\tilde{x}(\cdot)$ of the problem (6.1)–(6.4) satisfying almost everywhere (a.e.) on $[t_0,t_1]$ the mth-order differential inclusions (6.2), the functional constraints (6.3) and the initial conditions (6.4) that minimizes the Mayer functional $\varphi_0\big(x(t_1), x'(t_1), \ldots, x^{(m-1)}(t_1)\big)$. Here, a feasible solution $x(\cdot)$ is an absolutely continuous function on a time interval $[t_0,t_1]$ together with the higher order derivatives until $m-1$, and $x^{(m)}(\cdot) \in L_1^n\big([t_0,t_1]\big)$. Obviously, such class of functions is a Banach space, endowed with the different equivalent norms. For example, $||x(\cdot)|| = \sum_{k=0}^{m-1} |x^k(t_0)| + ||x^{(m)}(\cdot)||_1$ or $||x(\cdot)|| = \sum_{k=0}^{m} ||x^{(k)}(\cdot)||_1$, where $||x^{(k)}(\cdot)||_1 = \int_{t_0}^{t_1} |x^{(k)}(t)|\mathrm{d}t$, and $|x|$ is an Euclidean norm in $\mathbb{R}^n$.

Notice that the principal method here is a direct method based on discrete approximations; we replace the continuous problem (P_H) by the discrete-approximate problem and then by passing to the formal limit we formulate sufficient conditions of optimality for the original problem with mth-order derivatives:

$$\begin{aligned} &\text{minimize} \quad \varphi_0\big(x(t_1), \Delta x(t_1), \ldots, \Delta^{(m-1)}x(t_1)\big), \\ &\Delta^m x(t) \in F\big(x(t), \Delta x(t), \ldots, \Delta^{(m-1)}x(t), t\big),\ t = t_0,\ldots,t_1 - mh;\ x(t_0) = \alpha_0, \ldots, \Delta^{(m-1)}x(t_0) = \alpha_{m-1}, \end{aligned} \tag{6.5}$$

where $\Delta^k, k = 1,\ldots,m$ are kth-order difference operators. In this section, establishment of these conditions is omitted and we start our discussion with a presentation and study of sufficient optimality conditions for problem (P_H). As a result of approximation method described above we establish the so-called mth-order Euler–Lagrange differential inclusion

$$\begin{aligned} (i)\ &\Big((-1)^m x^{*(m)}(t) + v_{m-1}^{*'}(t), v_{m-1}^*(t) + v_{m-2}^{*'}(t), \ldots, v_2^*(t) + v_1^{*'}(t), v_1^*(t)\Big) \\ &\in F^*\big(x^*(t); (\tilde{x}(t), \tilde{x}'(t), \ldots, \tilde{x}^{(m)}(t)), t\big),\ \text{a.e. } t \in [t_0,t_1] \end{aligned}$$

the transversality condition

$$(ii)\ \Big((-1)^m x^{*(m-1)}(t_1)+v^*_{m-1}(t_1),(-1)^{m-1}x^{*m-2}(t_1)+v^*_{m-2}(t_1),\dots,x^{*'}(t_1)+v^*_1(t_1),-x^*(t_1)\Big)$$
$$\in\sum_{k=0}^{r}\lambda_k\partial\varphi_k\big(\tilde{x}(t_1),\tilde{x}'(t_1),\dots,\tilde{x}^{(m-1)}(t_1)\big),\ \lambda_0=1$$

and the complementary slackness conditions at $t=t_1$:

$$(iii)\ \lambda_k\varphi_k\big(\tilde{x}(t_1),\tilde{x}'(t_1),\dots,\tilde{x}^{(m-1)}(t_1)\big)=0,\ \lambda_k\geq 0,\ k=1,\dots,r.$$

In what follows we suppose that $x^*(t), t\in[t_0,t_1]$, is absolutely continuous function with the higher order derivatives until $m-1$, and $x^{*(m)}(\cdot)\in L^n_1\big([t_0,t_1]\big)$. Besides $v^*_k(t), k=1,\dots,m-1, t\in[t_0,t_1]$ are absolutely continuous and $v^{*'}_k(\cdot)\in L^n_1\big([t_0,t_1]\big), k=1,\dots,m-1$.

Moreover in terms of argmaximum set we formulate the following condition guaranteeing that the LAM F^* is nonempty at a given point:

$$(iv)\ \tilde{x}^{(m)}(t)\in F\big(\tilde{x}(t),\tilde{x}'(t),\dots,\tilde{x}^{(m-1)}(t);x^*(t),t\big),\ \text{a.e. } t\in[t_0,t_1].$$

It turns out that the following assertion is true.

Theorem 6.1. *(Sufficient conditions of optimality for mth-order evolution differential inclusions with functional constraints) Suppose that $\varphi_k:\big(\mathbb{R}^n\big)^m\to\mathbb{R}^1, k=0,1,\dots,r$ are convex continuous functions, $F(\cdot,t):\big(\mathbb{R}^n\big)^m\rightrightarrows\mathbb{R}^n$ is an evolution convex set-valued mapping. Then for optimality of the trajectory $\tilde{x}(\cdot)$ in the problem (6.1)–(6.4) it is sufficient that there exist a collection of absolutely continuous functions $\{x^*(t),v^*_k(t),k=1,\dots,m-1\}, t\in[t_0,t_1]$ and Lagrange multipliers $\lambda_k\geq 0(\lambda_0=1), k=0,\dots,r$, satisfying a.e. the mth-order Euler–Lagrange inclusion (i), transversality condition (ii) and the complementary slackness conditions (iii) at $t=t_1$.*

Proof. Since $F^*\big(z^*;(x,y_1,y_2,\dots,y_{m-1},z),t\big)=\partial(x,y)H_{F(\cdot,t)}(x,y_1,y_2,\dots,y_{m-1},z^*), y=(y_1,y_2,\dots,y_{m-1}), z\in F(x,y_1,y_2,\dots,y_{m-1};z^*,t)$, by using the Moreau–Rockafellar theorem [12, 17] from the condition (i) and (iv) we obtain the mth-order adjoint differential inclusion

$$\Big((-1)^m x^{*(m)}(t)+v^{*'}_{m-1}(t),v^*_{m-1}(t)+v^{*'}_{m-2}(t),\dots,v^*_2(t)+v^{*'}_1(t),v^*_1(t)\Big)$$
$$\in\partial_{(x,y)}\big[H_{F(\cdot,t)}\big(\tilde{x}(t),\tilde{x}'(t),\dots,\tilde{x}^{(m-1)}(t),x^*(t)\big)\big],\ t\in[t_0,t_1].$$

On the definition of sudifferential set of the Hamiltonian function $H_{F(\cdot,t)}$ we rewrite the last relation in the form:

$$\begin{aligned}&H_{F(\cdot,t)}\big(x(t),x'(t),\dots,x^{(m-1)}(t),x^*(t)\big)-H_{F(\cdot,t)}\big(\tilde{x}(t),\tilde{x}'(t),\dots,\tilde{x}^{(m-1)}(t),x^*(t)\big)\\&\leq\Big\langle(-1)^m x^{*(m)}(t)+v^{*'}_{m-1}(t),x(t)-\tilde{x}(t)\Big\rangle+\Big\langle v^*_{m-1}(t)+v^{*'}_{m-2}(t),x'(t)-\tilde{x}'(t)\Big\rangle\\&\quad+\Big\langle v^*_{m-2}(t)+v^{*'}_{m-3}(t),x''(t)-\tilde{x}''(t)\Big\rangle+\ldots+\Big\langle v^*_2(t)+v^{*'}_1(t),x^{(m-2)}(t)-\tilde{x}^{(m-2)}(t)\Big\rangle\\&\quad+\Big\langle v^*_1(t),x^{(m-1)}(t)-\tilde{x}^{(m-1)}(t)\Big\rangle.\end{aligned}\tag{6.6}$$

In turn by using the definition of the Hamiltonian function, (6.6) can be converted to the following inequality

$$\left\langle x^{(m)}(t), x^*(t)\right\rangle - \left\langle \tilde{x}^{(m)}(t), x^*(t)\right\rangle \le \left\langle (-1)^m x^{*(m)}(t), x(t)-\tilde{x}(t)\right\rangle + \frac{\mathrm{d}}{\mathrm{d}t}\left\langle v^*_{m-1}(t), x(t)-\tilde{x}(t)\right\rangle$$
$$+\frac{\mathrm{d}}{\mathrm{d}t}\left\langle v^*_{m-2}(t), x'(t)-\tilde{x}'(t)\right\rangle + \cdots + \frac{\mathrm{d}}{\mathrm{d}t}\left\langle v^*_1(t), x^{(m-2)}(t)-\tilde{x}^{(m-2)}(t)\right\rangle.$$

In turn, let us rewrite this inequality as follows

$$\left\langle x^{(m)}(t)-\tilde{x}^{(m)}(t), x^*(t)\right\rangle + \left\langle (-1)^{m+1} x^{*(m)}(t), x(t)-\tilde{x}(t)\right\rangle$$
$$-\sum_{k=1}^{m-1}\frac{\mathrm{d}}{\mathrm{d}t}\left\langle v^*_k(t), x^{(m-k-1)}(t)-\tilde{x}^{(m-k-1)}(t)\right\rangle \le 0. \tag{6.7}$$

Now, integrating (6.7) over the interval $[t_0, t_1]$ we have

$$\int_{t_0}^{t_1}\left[\left\langle x^{(m)}(t)-\tilde{x}^{(m)}(t), x^*(t)\right\rangle + \left\langle (-1)^{m+1} x^{*(m)}(t), x(t)-\tilde{x}(t)\right\rangle\right]\mathrm{d}t$$
$$+\sum_{k=1}^{m-1}\left\langle v^*_k(t_0), x^{(m-k-1)}(t_0)-\tilde{x}^{(m-k-1)}(t_0)\right\rangle - \sum_{k=1}^{m-1}\left\langle v^*_k(t_1), x^{(m-k-1)}(t_1)-\tilde{x}^{(m-k-1)}(t_1)\right\rangle \le 0.$$

Consequently, taking into account that $x(\cdot), \tilde{x}(\cdot)$ are feasible $(x^{(k)}(t_0) = \tilde{x}^{(k)}(t_0) = \alpha_k, k = 0, \ldots, m-1)$ we can write

$$\int_{t_0}^{t_1}\left[\left\langle x^{(m)}(t)-\tilde{x}^{(m)}(t), x^*(t)\right\rangle + \left\langle (-1)^{m+1} x^{*(m)}(t), x(t)-\tilde{x}(t)\right\rangle\right]\mathrm{d}t$$
$$-\sum_{k=1}^{m-1}\left\langle v^*_k(t_1), x^{(m-k-1)}(t_1)-\tilde{x}^{(m-k-1)}(t_1)\right\rangle \le 0. \tag{6.8}$$

Denoting the expression in the square parentheses on the left hand side of (6.8) by M

$$M = \left\langle x^{(m)}(t)-\tilde{x}^{(m)}(t), x^*(t)\right\rangle + \left\langle (-1)^{m+1} x^{*(m)}(t), x(t)-\tilde{x}(t)\right\rangle$$

it is not hard to see that the first term on the right hand side of M can be converted as follows

$$\left\langle x^{(m)}(t)-\tilde{x}^{(m)}(t), x^*(t)\right\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\left\langle x^{(m-1)}(t)-\tilde{x}^{(m-1)}(t), x^*(t)\right\rangle - \frac{\mathrm{d}}{\mathrm{d}t}\left\langle x^{(m-2)}(t)-\tilde{x}^{(m-2)}(t), x^{*'}(t)\right\rangle$$
$$+\frac{\mathrm{d}}{\mathrm{d}t}\left\langle x^{(m-3)}(t)-\tilde{x}^{(m-3)}(t), x^{*''}(t)\right\rangle - \ldots - \left\langle x'(t)-\tilde{x}'(t), (-1)^m x^{*(m-1)}(t)\right\rangle. \tag{6.9}$$

On the other hand the second term on the right hand side of M implies that

$$\left\langle (-1)^{m+1} x^{*(m)}(t), x(t)-\tilde{x}(t)\right\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\left\langle (-1)^{m+1} x^{*(m-1)}(t), x(t)-\tilde{x}(t)\right\rangle$$
$$\left\langle (-1)^m x^{*(m-1)}(t), x'(t)-\tilde{x}'(t)\right\rangle. \tag{6.10}$$

Then from (6.9) and (6.10) we derive

$$\begin{aligned} M &= \frac{\mathrm{d}}{\mathrm{d}t}\Big\langle x^{(m-1)}(t) - \tilde{x}^{(m-1)}(t), x^*(t)\Big\rangle - \frac{\mathrm{d}}{\mathrm{d}t}\Big\langle x^{(m-2)}(t) - \tilde{x}^{(m-2)}(t), x^{*'}(t)\Big\rangle \\ &\quad + \frac{\mathrm{d}}{\mathrm{d}t}\Big\langle x^{(m-3)}(t) - \tilde{x}^{(m-3)}(t), x^{*''}(t)\Big\rangle - \ldots - \frac{\mathrm{d}}{\mathrm{d}t}\Big\langle \big(x(t) - \tilde{x}(t)\big), (-1)^m x^{*(m-1)}(t)\Big\rangle. \end{aligned}$$

Then we can compute the integral on the left hand side of (6.8) as follows:

$$\begin{aligned} \int_{t_0}^{t_1} M\mathrm{d}t &= \Big\langle x^{(m-1)}(t_1) - \tilde{x}^{(m-1)}(t_1), x^*(t_1)\Big\rangle - \Big\langle x^{(m-1)}(t_0) - \tilde{x}^{(m-1)}(t_0), x^*(t_0)\Big\rangle \\ &+ \Big\langle x^{(m-2)}(t_0) - \tilde{x}^{(m-2)}(t_0), x^{*'}(t_0)\Big\rangle - \Big\langle x^{(m-2)}(t_1) - \tilde{x}^{(m-2)}(t_1), x^{*'}(t_1)\Big\rangle \\ &+ \Big\langle x^{(m-3)}(t_1) - \tilde{x}^{(m-3)}(t_1), x^{*''}(t_1)\Big\rangle - \Big\langle x^{(m-3)}(t_0) - \tilde{x}^{(m-3)}(t_0), x^{*''}(t_0)\Big\rangle \\ &- \ldots - \Big\langle \big(x(t_0) - \tilde{x}(t_0)\big)(-1)^{m-1}, x^{*(m-1)}(t_0)\Big\rangle - \Big\langle \big(x(t_1) - \tilde{x}(t_1)\big)(-1)^m, x^{*(m-1)}(t_1)\Big\rangle. \end{aligned} \tag{6.11}$$

Since $x(\cdot), \tilde{x}(\cdot)$ are feasible ($x^{(k)}(t_0) = \tilde{x}^{(k)}(t_0) = \alpha_k, k = 0, \ldots, m-1$) from (6.11) we deduce that

$$\begin{aligned} \int_{t_0}^{t_1} M\mathrm{d}t &= \Big\langle x^{(m-1)}(t_1) - \tilde{x}^{(m-1)}(t_1), x^*(t_1)\Big\rangle - \Big\langle x^{(m-2)}(t_1) - \tilde{x}^{(m-2)}(t_1), x^{*'}(t_1)\Big\rangle \\ &+ \Big\langle x^{(m-3)}(t_1) - \tilde{x}^{(m-3)}(t_1), x^{*''}(t_1)\Big\rangle - \ldots - \Big\langle \big(x(t_1) - \tilde{x}(t_1)\big)(-1)^m, x^{*(m-1)}(t_1)\Big\rangle. \end{aligned} \tag{6.12}$$

Substituting (6.12) into (6.8) we derive

$$\begin{aligned} &\Big\langle x^{(m-1)}(t_1) - \tilde{x}^{(m-1)}(t_1), x^*(t_1)\Big\rangle - \Big\langle x^{(m-2)}(t_1) - \tilde{x}^{(m-2)}(t_1), x^{*'}(t_1)\Big\rangle + \Big\langle x^{(m-3)}(t_1) - \tilde{x}^{(m-3)}(t_1), x^{*''}(t_1)\Big\rangle \\ &- \ldots - \Big\langle \big(x(t_1) - \tilde{x}(t_1)\big), (-1)^m x^{*(m-1)}(t_1)\Big\rangle - \sum_{k=1}^{m-1} \Big\langle v_k^*(t_1), x^{(m-k-1)}(t_1) - \tilde{x}^{(m-k-1)}(t_1)\Big\rangle \leq 0. \end{aligned} \tag{6.13}$$

Now by the transversality condition (ii) for all feasible trajectories we have

$$\begin{aligned} &\sum_{k=0}^{m-1} \Big\langle (-1)^{k+1} x^{*(k)}(t_1) + v_k^*(t_1), x^{(m-k-1)}(t_1) - \tilde{x}^{(m-k-1)}(t_1)\Big\rangle \leq \sum_{k=1}^{r} \lambda_k \varphi_k\big(x(t_1), x'(t_1), \ldots, x^{(m-1)}(t_1)\big) \\ &\quad - \sum_{k=1}^{r} \lambda_k \varphi_k\big(\tilde{x}(t_1), \tilde{x}'(t_1), \ldots, \tilde{x}^{(m-1)}(t_1)\big), \big(v_0^*(t_1) \equiv 0\big). \end{aligned} \tag{6.14}$$

Finally, summing the inequalities (6.13) and (6.14) for all feasible solutions we derive

$$\sum_{k=0}^{r} \lambda_k \varphi_k\big(x(t_1), x'(t_1), \ldots, x^{(m-1)}(t_1)\big) - \sum_{k=0}^{r} \lambda_k \varphi_k\big(\tilde{x}(t_1), \tilde{x}'(t_1), \ldots, \tilde{x}^{(m-1)}(t_1)\big) \geq 0, \ \forall x(\cdot).$$

In this inequality using the complementary slackness conditions (iii) at $t = t_1$ we are convinced that the inequality is justified

$$\varphi_0\big(x(t_1), x'(t_1), \dots, x^{(m-1)}(t_1)\big) + \sum_{k=1}^{r} \lambda_k \varphi_k\big(x(t_1), x'(t_1), \dots, x^{(m-1)}(t_1)\big)$$
$$\geq \varphi_0\big(\tilde{x}(t_1), \tilde{x}'(t_1), \dots, \tilde{x}^{(m-1)}(t_1)\big),\ \lambda_k \geq 0,\ k = 1, \dots, r. \tag{6.15}$$

Notice that $x(\cdot)$ is an arbitrary feasible trajectory and $\lambda_k \geq 0, k = 1, \dots, r$. Then in the inequality (6.15) it is obvious that $\sum_{k=1}^{r} \lambda_k \varphi_k\big(x(t_1), x'(t_1), \dots, x^{(m-1)}(t_1)\big) \leq 0$, which implies that

$$\varphi_0\big(x(t_1), x'(t_1), \dots, x^{(m-1)}(t_1)\big) \geq \varphi_0\big(\tilde{x}(t_1), \tilde{x}'(t_1), \dots, \tilde{x}^{(m-1)}(t_1)\big),\ \forall x(\cdot)$$

and $\tilde{x}(t), t \in [t_0, t_1]$ is optimal.

The furthest proof of Corollary 6.2 below is similar to the one for Theorem 6.1.

Corollary 6.2. *Let us consider the nonconvex problem (6.1)–(6.4) that is* $\varphi_k : \big(\mathbb{R}^n\big)^m \to \mathbb{R}^1, k = 0, 1, \dots, r$ *are nonconvex functions and* $F(\cdot, t)$ *is a nonconvex set-valued mapping. Then for optimality of the arc* $\tilde{x}(t), t \in [t_0, t_1]$, *among all feasible solutions of the problem* (P_H) *it is sufficient that there exists a collection of absolutely continuous functions* $\{x^*(t), v_k^*(t), k = 1 \dots, m-1\}, t \in [t_0, t_1]$ *satisfying the complementary slackness conditions (iii) of Theorem 6.1 and a new Euler–Lagrange and transversality inclusions:*

$$(a)\ \Big((-1)^m x^{*(m)}(t) + v_{m-1}^{*'}(t) + x^*(t), v_{m-1}^*(t) + v_{m-2}^{*'}(t), \dots, v_2^*(t) + v_1^{*'}(t), v_1^*(t)\Big)$$
$$\in F^*\big(x^*(t); (\tilde{x}(t), \tilde{x}'(t), \dots, \tilde{x}^{(m)}(t)), t\big),\ a.e.\ t \in [t_0, t_1]$$
$$(b)\ \sum_{k=0}^{r} \lambda_k \Big[\varphi_k\big(x, y_1, \dots, y_{m-1}\big) - \varphi_k\big(\tilde{x}(t_1), \dots, \tilde{x}^{(m-1)}(t_1)\big)\Big] \geq \Big\langle (-1)^m x^{*(m-1)}(t_1) + v_{m-1}^*(t_1), x - \tilde{x}(t_1)\Big\rangle$$
$$\sum_{k=0}^{m-2} \Big\langle (-1)^{k+1} x^{*(k)}(t_1) + v_k^*(t_1), y_{m-k-1} - \tilde{x}^{m-k-1}(t_1)\Big\rangle \big(v_0^*(t_1) \equiv 0\big),\ \forall x, y_k \in \mathbb{R}^n,\ k = 1, \dots, m-1.$$

Now we apply Theorem 6.1 to get sufficient conditions of optimality for the following third order differential inclusions with functional constraints

$$\begin{aligned} \text{minimize}\quad & \varphi_0\big(x(t_1), x'(t_1), x''(t_1)\big), \\ & \frac{\mathrm{d}^3 x(t)}{\mathrm{d}t^3} \in G_0\big(x(t), t\big),\ \text{a.e. } t \in [t_0, t_1],\ x(t_0) = \alpha_0, x'(t_0) = \alpha_1, x''(t_0) = \alpha_2, \\ & \varphi_k\big(x(t_1), x'(t_1), x''(t_1)\big) \leq 0,\ k = 1, \dots, r, \end{aligned} \tag{6.16}$$

whereas usually $\varphi_k : \mathbb{R}^{3n} \to \mathbb{R}^1, k = 0, 1, \dots, r$ are convex continuous functions, $G_0(\cdot, t) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is an evolution convex set-valued mapping.

Corollary 6.3. *For optimality of the trajectory* $\tilde{x}(\cdot)$ *in the problem (6.16) it is sufficient that there exists an absolutely continuous functions* $x^*(t), t \in [t_0, t_1]$, *and Lagrange multipliers* $\lambda_k \geq 0 (\lambda_0 = 1), k = 0, \dots, r$, *satisfying a.e. the following third order Euler–Lagrange and transversality inclusions and the complementary* *slackness conditions at* $t = t_1$*:*

$$-x^{*'''}(t) \in G_0^*\Big(x^*(t); \big(\tilde{x}(t), \tilde{x}'''(t)\big), t\Big);\ \lambda_k \varphi_k\big(\tilde{x}(t_1), \tilde{x}'(t_1), \tilde{x}''(t_1)\big) = 0,\ \lambda_k \geq 0,\ k = 1, \dots, r;$$

$$\Big(-x^{*''}(t_1), x^{*'}(t_1), -x^*(t_1)\Big) \in \sum_{k=0}^{r} \lambda_k \partial \varphi_k\big(\tilde{x}(t_1), \tilde{x}'(t_1), \tilde{x}''(t_1)\big).$$

Proof. It should be pointed out that in the Euler–Lagrange inclusion (i) of Theorem 6.1 one has

$$\Big(-x^{*'''}(t) + v_2^{*'}(t), v_2^*(t) + v_1^{*'}(t), v_1^*(t)\Big) \in G_0^*\Big(x^*(t); \big(\tilde{x}(t), \tilde{x}'''(t)\big), t\Big) \times \{0\} \times \{0\},$$

whence $-x^{*'''}(t) + v_2^{*'}(t) \in G_0^*\big(x^*(t); \big(\tilde{x}(t), \tilde{x}'''(t)\big), t\big), v_2^*(t) + v_1^{*'}(t) \equiv 0, v_1^*(t) \equiv 0$. Then taking into account these relations we have the desired result.

7. Sufficient conditions of optimality for second order differential inclusions with non-functional initial and endpoint constraints

Note that in this section the optimality conditions are given for second order convex differential inclusions (P_Q) with convex initial point and endpoint nonfunctional constraints. These conditions are more precise than any previously published ones since they involve useful forms of the Weierstrass–Pontryagin condition and second order Euler–Lagrange type adjoint inclusions. In the reviewed results this effort culminates in Theorem 7.1.

$$(P_Q) \qquad \begin{aligned} &\text{minimize } \varphi_0\big(x(t_1), x'(t_1)\big), \\ &x''(t) \in F\big(x(t), x'(t), t\big),\ \text{a.e. } t \in [t_0, t_1], \\ &x(t_0) \in Q_0, x'(t_0) \in Q_1;\ x(t_1) \in M_0,\ x'(t_1) \in M_1, \end{aligned}$$

where φ_0 is a convex continuous function, $F(\cdot, t) : \mathbb{R}^{2n} \rightrightarrows \mathbb{R}^n$ is convex set-valued mapping and $Q_i, M_i \subseteq \mathbb{R}^n (i = 0, 1)$ are convex sets.

In Section 5, we have seen that the following adjoint inclusion is the second order Euler–Lagrange type inclusion for the problem (P_Q)

$$(a_1)\ \Big(x^{*''}(t) + v^{*'}(t), v^*(t)\Big) \in F^*\big(x^*(t); (\tilde{x}(t), \tilde{x}'(t), \tilde{x}''(t)), t\big),\ \text{a.e. } t \in [t_0, t_1],$$

where

$$(b_1)\ \tilde{x}^{''}(t) \in F_A\big(\tilde{x}(t), \tilde{x}'(t); x^*(t), t\big),\ \text{a.e. } t \in [t_0, t_1].$$

The transversality conditions at the endpoints $t = t_0$ and $t = t_1$ consist of the following

$$(c_1)\ \Big(v^*(t_0) + x^{*'}(t_0), -x^*(t_0)\Big) \in K^*_{Q_0}\big(\tilde{x}(t_0)\big) \times K^*_{Q_1}\big(\tilde{x}'(t_0)\big),$$

$$(d_1)\ \Big(v^*(t_1) + x^{*'}(t_1), -x^*(t_1)\Big) \in \partial_{(x,y)}\varphi_0\big(\tilde{x}(t_1), \tilde{x}'(t_1) - K^*_{M_0}\big(\tilde{x}(t_1)\big) \times K^*_{M_1}\big(\tilde{x}'(t_1)\big),$$

respectively. Now we are ready formulate the following theorem of optimality.

Theorem 7.1. *Suppose that φ_0 is a continuous and convex function, $F(\cdot, t)$ is a convex set-valued mapping and $Q_i, M_i (i = 0, 1)$ are convex sets. Then for optimality of the feasible trajectory $\tilde{x}(t)$ in the problem (P_Q) it is* *sufficient that there exists a pair of absolutely continuous functions* $\{x^*(t), v^*(t)\}, t \in [t_0, t_1]$*, satisfying a.e. the second order Euler–Lagrange type inclusion* (a_1),(b_1) *and the transversality conditions* (c_1), (d_1) *at the initial point* $t = t_0$ *and endpoint* $t = t_1$*, respectively.*

Proof. By the proof idea of Theorem 5.1 (see (5.5)) from (a_1), (b_1) we derive the following inequality

$$0 \geq \left\langle \frac{\mathrm{d}(x(t_1) - \tilde{x}(t_1))}{\mathrm{d}t}, x^*(t_1) \right\rangle - \left\langle \frac{\mathrm{d}(x(t_0) - \tilde{x}(t_0))}{\mathrm{d}t}, x^*(t_0) \right\rangle - \left\langle v^*(t_1) + \frac{\mathrm{d}x^*(t_1)}{\mathrm{d}t}, x(t_1) - \tilde{x}(t_1) \right\rangle + \left\langle v^*(t_0) + \frac{\mathrm{d}x^*(t_0)}{\mathrm{d}t}, x(t_0) - \tilde{x}(t_0) \right\rangle. \tag{7.1}$$

Now, by definition of dual cones $K^*_{Q_0}(\tilde{x}(t_0)), K^*_{Q_1}(\tilde{x}'(t_0))$ from the transversality condition (c_1) we deduce that

$$-\left\langle \frac{\mathrm{d}(x(t_0) - \tilde{x}(t_0))}{\mathrm{d}t}, x^*(t_0) \right\rangle + \left\langle v^*(t_0) + \frac{\mathrm{d}x^*(t_0)}{\mathrm{d}t}, x(t_0) - \tilde{x}(t_0) \right\rangle \geq 0, \ \forall x(t_0) \in Q_0; \ \forall x'(t_0) \in Q_1. \tag{7.2}$$

Then it follows from (7.1) and (7.2) that

$$0 \geq \left\langle \frac{\mathrm{d}(x(t_1) - \tilde{x}(t_1))}{\mathrm{d}t}, x^*(t_1) \right\rangle - \left\langle v^*(t_1) + \frac{\mathrm{d}x^*(t_1)}{\mathrm{d}t}, x(t_1) - \tilde{x}(t_1) \right\rangle. \tag{7.3}$$

Now, it can easily be seen that the transversality conditions (d_1) at the endpoint $t = t_1$, can be rewritten as follows

$$\varphi_0(x(t_1), x'(t_1)) - \varphi_0(\tilde{x}(t_1), \tilde{x}'(t_1)) \geq \left\langle v^*(t_1) + \frac{\mathrm{d}x^*(t_1)}{\mathrm{d}t} + x^*(t_1), x(t_1) - \tilde{x}(t_1) \right\rangle + \left\langle \frac{\mathrm{d}x^*(t_1)}{\mathrm{d}t} - x^*(t_1), \frac{\mathrm{d}(x(t_1) - \tilde{x}(t_1))}{\mathrm{d}t} \right\rangle; x^*(t_1) \in K^*_{M_0}(\tilde{x}(t_1)),\ x^{*'}(t_1) \in K^*_{M_1}(\tilde{x}'(t_1))$$

or, equivalently,

$$\varphi_0(x(t_1), x'(t_1)) - \varphi_0(\tilde{x}(t_1), \tilde{x}'(t_1)) \geq \left\langle v^*(t_1) + \frac{\mathrm{d}x^*(t_1)}{\mathrm{d}t}, x(t_1) - \tilde{x}(t_1) \right\rangle - \left\langle x^*(t_1), \frac{\mathrm{d}(x(t_1) - \tilde{x}(t_1))}{\mathrm{d}t} \right\rangle. \tag{7.4}$$

Thus, summing the inequalities (7.3), (7.4) for all feasible trajectories $x(\cdot)$ satisfying the initial conditions $x(t_0) \in Q_0, x'(t_0) \in Q_1$ and endpoint constraints $x(t_1) \in M_0, x'(t_1) \in M_1$ we have the needed inequality:

$$\varphi_0(x(t_1), x'(t_1)) - \varphi_0(\tilde{x}(t_1), \tilde{x}'(t_1)) \geq 0 \text{ or } \varphi_0(x(t_1), x'(t_1)) \geq \varphi_0(\tilde{x}(t_1), \tilde{x}'(t_1)).$$

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