ALTERNATING AND VARIABLE CONTROLS FOR THE WAVE EQUATION[*]

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Abstract. The present article discusses the *exact observability* of the wave equation when the observation subset of the boundary is variable in time. In the one-dimensional case, we prove an equivalent condition for the exact observability, which takes into account only the location in time of the observation. To this end we use Fourier series. Then we investigate the two specific cases of single exchange of the control position, and of exchange at a constant rate. In the multi-dimensional case, we analyse sufficient conditions for the exact observability relying on the multiplier method. In the last section, the multi-dimensional results are applied to specific settings and some connections between the one and multi-dimensional case are discussed; furthermore some open problems are presented.

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1. INTRODUCTION AND FORMULATION OF THE MAIN RESULTS

In this work we study exact observability for the wave equation with Dirichlet boundary condition, *i.e.*

$$u_{tt} - \Delta u = 0, \qquad\qquad (x,t) \in \Omega \times (0,T), \qquad (1.1)$$

$$u(x,t) = 0, \qquad\qquad (x,t) \in \partial\Omega \times (0,T), \qquad (1.2)$$

$$u(x,0) = u_0(x), \qquad\qquad x \in \Omega, \qquad (1.3)$$

$$u_t(x,0) = u_1(x), \qquad\qquad x \in \Omega; \qquad (1.4)$$

when the subset of observation is not (in general) of the form $\Gamma \times (0,T) \subset \partial\Omega \times (0,T)$ for a fixed $\Gamma \subset \partial\Omega$.

Here Δ is the Laplace operator and Ω is a domain (*i.e.*, an open, connected and bounded set) in $\mathbb{R}^d$, which we consider of class C^2 *or* convex (for more information see [5]).

For the reader's convenience, we have gathered in Section 2 some known results, and some extensions we need, on basic existence and regularity theory for the problem above. In the following the trace of the normal derivative of weak solutions will be understood in the sense of Theorem 2.2.

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Since the treatment of the observability problem (see below for a definition or [7]) is different in the one dimensional and in the multi-dimensional case, due to the different topology of the subset of observation, we split the discussion into two subsections.

1.1. One-dimensional case ($d = 1$)

In this case, w.l.o.g. we will consider $\Omega = (0, \pi)$. We are interested in studying the controllability property of the one dimensional wave equation with Dirichlet boundary condition (*e.g.*, by a control at $x = 0$). It is known (see for instance [6] or [10]) that the exact controllability is equivalent to the observability inequalities such as

$$\int_0^T |u_x(0, t)|^2 \, \mathrm{d}t \asymp E_0. \tag{1.5}$$

Here and in the rest of the paper the symbol $A \asymp B$ means that $c^{-1}A < B < cA$ for some positive real c; where u solves (1.1)–(1.4) and E_0 is the energy of the initial data $(u_0, u_1) \in H_0^1(0, \pi) \times L^2(0, \pi)$; see (2.6). It is also clear that the observability inequality (1.6) remains valid if $u_x(0, t)$ is replaced by $u_x(\pi, t)$, which provides the controllability for a system controlled at $x = \pi$.

It is well known that (1.6) holds for any $T \geq 2\pi$, as a consequence of the Parseval equality, for a proof see [7]. Moreover, if the control is actuated on both the endpoints, the optimal control time T reduces to π.

Although both the inequalities require an investigation, for the study we are going to explain the so called *direct observability inequality* follows by hidden regularity property (see Thm. 2.2).

For any $T > 0$ and $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, the map

$$H_0^1(\Omega) \times L^2(\Omega) \to L^2(\partial\Omega \times (0, T)),$$
$$(u_0, u_1) \mapsto \partial_\nu u,$$

is well defined, continuous and coincides with the classical trace (in the sense of Sobolev spaces) if $(u_0, u_1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$.

In the paper, we concentrate our analysis on the *inverse observability inequality*

$$\int_0^T |u_x(0, t)|^2 \, \mathrm{d}t \geq cE_0. \tag{1.6}$$

For brevity, from now the inequality (1.6) will be called *observability inequality*.

In this work, we want to generalize the situation of having the observability set as $\{0\} \times (0, T)$ or $\{\pi\} \times (0, T)$; that is we want to alternate the spatial controllability point still maintaining the observability inequality true, for an optimal control time. We are not aware of specific contributions in this sense with the exception the recent paper [9], which mostly treats bulk control, but touches the variable boundary control; however the methods of [9] are completely different from ours. The motivation for this generalization came from typical engineering problems (see [3]) in which the control sets have to change in time, so that, also the observability set has to change in time.

Consider an arbitrary family of consecutive disjoint open intervals $\{I_k \,:\, k \in \mathcal{K}\}$ of $(0, \infty)$ with $\mathcal{K} \subset \mathbb{N}_0$ (possibly finite or infinite); we define the *observability sets*

$$\mathcal{F} := \{\{\lambda_k\} \times I_k \,:\, k \in \mathcal{K}\}, \tag{1.7}$$

where $\lambda_k = 0$ if k is even, otherwise $\lambda_k = \pi$. Without loss of generality we can suppose $\mathcal{K} = \{0, \ldots, N\}$ if $\mathcal{K}$ is finite or $\mathcal{K} = \mathbb{N}_0$ otherwise; furthermore we set $(t_{k-1}, t_k) := I_k$ and $t_{-1} = 0$.

We want to investigate whether the following property is satisfied:

Definition 1.1 ($\mathcal{F}$-observability). We say that the system (1.1)–(1.4) is $\mathcal{F}$-observable in time $T > 0$ if the inverse observability inequality holds true, that is

$$\sum_{k=0}^{n} \int_{I_k \cap (0,T)} |u_x(\lambda_k, t)|^2 \, \mathrm{d}t \geq c \, E_0, \tag{1.8}$$

for some $n \in \mathbb{N}$ and $c > 0$ independent of $(u_0, u_1) \in H_0^1(0, \pi) \times L^2(0, \pi)$. As above, u is the weak solution to (1.1)–(1.4) with initial data (u_0, u_1).

We shall say that the system (1.1)–(1.4) is $\mathcal{F}$-observable, if the system is $\mathcal{F}$-observable for some time $T > 0$.

In applications we are interested in knowing the least time T such that the $\mathcal{F}$-observability property holds; we make this more precise in the following definition.

Definition 1.2 (Optimal control time). Suppose that the system (1.1)–(1.4) is $\mathcal{F}$-observable in time T, but it is not $\mathcal{F}$-observable in any time $T' < T$. Then $T_{opt} := T$ will be called the *optimal control time.*

In the following the family of intervals $\{I_k : k \in \mathcal{K}\}$ is given. In order to state our result on the observability property, we have to introduce some notations. Let r be the quotient map

$$r : \mathbb{R} \to \mathbb{R}/(2\pi\mathbb{Z}) \approx S^1.$$

Lastly we define

$$\tilde{I}_k = \{r(t) : t \in I_k - \lambda_k\}, \qquad k \in \mathbb{N}_0, \tag{1.9}$$

where as above, $\lambda_k = 0$ if $k \in 2\mathbb{N}_0$, $\lambda_k = \pi$ otherwise. We are ready to state our main Theorem.

Theorem 1.3. *Let $n \in \mathbb{N}_0$. The system (1.1)–(1.4) is $\mathcal{F}$-observable in time $T = \sup(I_n)$ if and only if*

$$\left| S^1 \setminus \bigcup_{k=0}^{n} \tilde{I}_k \right| = 0. \tag{1.10}$$

Furthermore, let h be the least index n in $\mathcal{K}$ for which the condition (1.10) holds, then the optimal control time is the infimum over the values t for which

$$\left| S^1 \setminus \left(\bigcup_{k=0}^{h-1} \tilde{I}_k \cup r((t_{h-1} - \lambda_h, t - \lambda_h)) \right) \right| = 0.$$

If $\mathcal{K} = \{0\}$, then the family of subsets considered in Theorem 1.3 contains a single interval. In this case the content of Theorem 1.3 reduces to the one point observation result discussed at the beginning of the subsection. Again, for a complete proof in this special case we refer to [7]. Finally, we note that in this case, according to Definition 1.2, the optimal control time is $T_{opt} = 2\pi$.

Interestingly, one has the following refinement of Theorem 1.3.

Corollary 1.4. *If the condition (1.10) holds, then we can choose a family $\{\tilde{J}_k : k = 0, \ldots, n\}$ of disjoint subsets of S^1 (possibly empty), such that $\tilde{J}_k \subset \tilde{I}_k$ and*

$$\left| S^1 \setminus \bigcup_{k=0}^{n} \tilde{J}_k \right| = 0. \tag{1.11}$$

Furthermore, denote by I'_k the open subset of I_k, such that $\tilde{J}_k = \{r(t) : t \in I'_k - \lambda_k\} \subset \tilde{I}_k$; then the system is $\mathcal{F}'$-observable, with

$$\mathcal{F}' = \{\{\lambda_k\} \times I'_k : k \in \mathcal{K}\},$$

and

$$\sum_{k=0}^{n} |I'_k| = 2\pi.$$

A couple of helpful remarks may be in order.

Remark 1.5. Corollary 1.4 shows that $T_{opt} \geq 2\pi$ for any $\mathcal{F}$ as in (1.7).

Remark 1.6. By the proof of Theorem 1.3 it will be clear that one can generalize to the following situation:

 – The subsets I_k, with $k \in \mathcal{K}$, are not a covering of $(0, \infty)$.
 – We can observe on both the endpoints at the same time.

To state this result, let each of $\{I^0_{k'} : k' \in \mathcal{K}'\}$, $\{I^\pi_{k''} : k'' \in \mathcal{K}''\}$ be a finite collection of disjoint subintervals of $(0, \infty)$. We may assume that $\sup I^0_h < \inf I^0_m$ for $h < m$ and a similar assumption for the $I^\pi_{k''}$'s.

Then we can define the observability sets

$$\mathcal{B} := \{\{0\} \times I^0_{k'} : k' \in \mathcal{K}'\} \cup \{\{\pi\} \times I^\pi_{k''} : k'' \in \mathcal{K}''\}.$$

Furthermore, Definition 1.1 readily extends to this situation. Define,

$$\tilde{I}^0_{k'} := \{r(t) : t \in I^0_{k'}\}, \qquad\qquad\qquad k' \in \mathcal{K}',$$
$$\tilde{I}^\pi_{k''} := \{r(t) : t \in I^\pi_{k''} - \pi\}, \qquad\qquad\qquad k'' \in \mathcal{K}''.$$

Then, the system is (1.1)–(1.4) $\mathcal{B}$-observable if and only if

$$\left| S^1 \setminus \left(\bigcup_{k' \in \mathcal{K}', k'' \in \mathcal{K}''} \tilde{I}^0_{k'} \cup \tilde{I}^\pi_{k''} \right) \right| = 0.$$

We refrain from writing this as a first formulation since we are mainly interested in alternating observations.

Remark 1.7. Condition (1.10) should be compared to the one in Definition 4.1 of [9]. It can be seen as the requirements needed to observe the complete energy of any solution.

Next, we focus on the applications of Theorem 1.3 and its Corollary 1.4 in particular cases in order to show the potentiality of these results.

1.1.1. Single exchange of the control position

In this subsection we analyse the case $\mathcal{K} = \{0, 1\}$ and

$$I_0 = (0, T_0), \qquad I_1 = (T_0, +\infty), \tag{1.12}$$

where $T_0 \in (0, 2\pi)$.

Corollary 1.8. *In the previous situation, for all $\pi < T_0 < 2\pi$ the optimal control time is $T_{opt} = 3\pi$ and the system is $\mathcal{F}'$-observable with*

$$I_0' = (0, T_0), \qquad I_1' = (\pi + T_0, 3\pi). \tag{1.13}$$

For all $0 < T_0 \leq \pi$, the optimal control time is $T_{opt} = 2\pi + T_0$ and the system is $\mathcal{F}'$-observable with

$$I_0' = (0, T_0), \qquad I_1' = (2T_0, 2\pi + T_0), \tag{1.14}$$

or

$$I_0' = \emptyset, \qquad I_1' = (T_0, 2\pi + T_0). \tag{1.15}$$

1.1.2. Exchange of the control position at a constant time rate

Then we look at another application of Theorem 1.3. We suppose that $\mathcal{K} = \mathbb{N}_0$ and

$$I_k = (kT_0, (k+1)T_0), \qquad k \in \mathbb{N}_0,$$

where T_0 is a fixed number in $(0, 2\pi)$. So, in this case each interval I_k has the same length. Moreover, in this case

$$\mathcal{F} = \{\{\lambda_k\} \times (kT_0, (k+1)T_0) \,|\, k \in \mathbb{N}_0\}. \tag{1.16}$$

Lemma 1.9. *In the previous situation (i.e. $\mathcal{F}$ is as in (1.16)), suppose that $T_0 \in (\pi, 2\pi)$. Let $n \in \mathbb{N}$ be the unique integer such that*

$$\frac{n+2}{n+1}\pi \leq T_0 < \frac{n+1}{n}\pi; \tag{1.17}$$

then the system is $\mathcal{F}$-observable with

$$T_{opt} = (n+2)\pi. \tag{1.18}$$

By the previous Lemma one may infer that if T_0 approaches π from above then the optimal control time blows up. Indeed, as the next lemma shows, when $T_0 = \pi$ the system cannot be observed.

Lemma 1.10. *Let $\mathcal{F}$ be as in (1.16). Then:*

  (i) *if $T_0 = \pi/2m$, with $m \in \mathbb{N}$, then the system is $\mathcal{F}$-observable with $T_{opt} = 2\pi$;*
  (ii) *if $T_0 = \pi/(2n+1)$, with $n \in \mathbb{N}_0$ then the system is not $\mathcal{F}$-observable in any T;*
  (iii) *if $T_0 = 2\pi/(2h+1)$, with $h \in 2\mathbb{N}$, then the system is $\mathcal{F}$-observable with $T_{opt} = 3\pi - T_0/2$;*
  (iv) *if $T_0 = 2\pi/(2h+1)$, with $h \in 2\mathbb{N}_0 + 1$, then the system is $\mathcal{F}$-observable with $T_{opt} = 3\pi$.*

Before proceeding further in the investigation, we introduce some notations

$$N := \left\lfloor \frac{2\pi}{T_0} \right\rfloor, \qquad j := \left\lfloor \frac{\pi}{T_0} \right\rfloor; \tag{1.19}$$

here we denote by $\lfloor x \rfloor$ the integer part of $x \in (0, +\infty)$. Instead, $\lceil x \rceil$ denotes the least integer greater than or equal to $x \in (0, +\infty)$.

The following lemma, which complements Lemma 1.10, shows that the system is $\mathcal{F}$-observable for all T_0 not of the form $\pi/(2n+1)$. It can be easily checked that all the alternatives appearing in the statement are mutually excluding.

Lemma 1.11. *Let $\mathcal{F}$ be as in (1.16), suppose that $T_0 \in (0,\pi)$ and $2\pi/T_0 \notin \mathbb{N}$, then the system is $\mathcal{F}$-observable. Furthermore, we have the following:*
(i) if N, j are even, then $T_{opt} = 2\pi + jT_0$;
(ii) if N is even and j is odd, then:

  – *if $h^* = k^* + 1$, then $T_{opt} = \pi + T_0(Nk^* + 1)$,*
  – *if $h^* = k^*$, then $T_{opt} = \pi + T_0(Nk^* - j + 1)$,*

where

$$h^* = \left\lceil \frac{\pi - jT_0 + T_0}{2\pi - NT_0} \right\rceil, \qquad k^* = \left\lceil \frac{T_0}{2\pi - NT_0} \right\rceil;$$

(iii) if N is odd and j is even, then:

  – *if $l^* = q^*$, then $T_{opt} = 2\pi q^* + jT_0$,*
  – *if $q^* + 1 = l^*$, then $T_{opt} = 2\pi q^* + jT_0 + \pi$,*

where

$$l^* = \left\lceil \frac{T_0}{(N+1)T_0 - 2\pi} \right\rceil, \qquad q^* = \left\lceil \frac{\pi - jT_0}{(N+1)T_0 - 2\pi} \right\rceil;$$

(iv) if N, j are odd, then $T_{opt} = NT_0 + \pi$.

1.2. Multi-dimensional case ($d > 1$)

Now we turn to the multidimensional case. Since the Fourier method applied in the one dimensional case does not seem suitable for the multidimensional case (see [7]), a multiplier technique can instead be applied to prove observability (see [6] or [10]).

The multiplier method permits us only to formulate sufficient conditions in order to obtain observability, so we will not provide an equivalent condition for the observability as we have done in the one dimensional case. Instead, we will show that if the observability set can be written in a special form (see (1.23)) then the system is exactly observable.

Let $\Sigma \subset \partial\Omega \times (0,T)$ be an $\mathcal{H}^{d-1} \otimes \mathcal{L}^1$-measurable set (here $\mathcal{H}^{d-1}$, $\mathcal{L}^1$ are respectively the $d-1$ dimensional Hausdorff measure and the 1 dimensional Lebesgue measure). In analogy with the one-dimensional case we are interested in the following property.

Definition 1.12 (Σ-observability)**.** Suppose that $\overline{\Sigma} \cap (\partial\Omega \times \{T\}) \neq \emptyset$. We say that the system is Σ-observable in time T if there exists a positive constant C such that

$$\int_\Sigma |\partial_\nu u|^2 \, d(\mathcal{H}^{d-1} \otimes \mathcal{L}^1) \geq C \left(\|u_0\|^2_{H^1_0(\Omega)} + \|u_1\|^2_{L^2(\Omega)} \right); \tag{1.20}$$

for all $(u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega)$ and u is the unique weak solution of (1.1)–(1.4) for the initial data (u_0, u_1).
The set Σ in (1.20) is called the *observability set*.

Of course the validity of the property in Definition 1.12 depends strongly on the subset $\Sigma \subset \partial\Omega \times (0,T)$. However, in the following, we do not explicitly refer to Σ when it is clear from the context.

We remark that this definition is consistent with Definition 1.8, but we prefer to give it independently, to stress the topological differences of the observation subset between the one and the multi-dimensional case.

Next we introduce some notations. Let $\varphi : [0,T] \to \mathbb{R}^d$ be a piecewise continuously differentiable and continuous curve of finite length, *i.e.*,

$$L(\varphi) := \int_0^T |\varphi'(t)| \, dt \, < +\infty. \tag{1.21}$$

Then define

$$\Gamma_\varphi(t) = \{x \in \partial\Omega \,|\, (x - \varphi(t)) \cdot \nu > 0\}, \quad t \in (0,T), \tag{1.22}$$

$$\Sigma_\varphi = \bigcup_{t \in (0,T)} \Gamma_\varphi(t) \times \{t\}, \tag{1.23}$$

$$c_i = \max_{\overline{\Omega}} |x - \varphi(i)|, \qquad i = 0, T. \tag{1.24}$$

Furthermore, let $P = \{t_{-1}, \ldots, t_N : 0 = t_{-1} < t_0 < \cdots < t_{N-1} < t_N := T\}$ be a partition of the interval $[0,T]$ and we denote by $\delta(P) := \sup_{j=0,\ldots,N}\{|t_j - t_{j-1}|\}$ the amplitude of the partition P. Furthermore, we set

$$x_{\varphi,j} := \varphi(t_{j-1}), \tag{1.25}$$

$$\Gamma_{\varphi,j} := \{x \in \partial\Omega \,|\, (x - x_{\varphi,j}) \cdot \nu > 0\}, \tag{1.26}$$

for all $j = 0, \ldots, N$. For future convenience, for a partition P as above, we define

$$\Sigma_\varphi^P := \bigcup_{j=0}^N \Gamma_{\varphi,j} \times (t_{j-1}, t_j). \tag{1.27}$$

In the following we suppose that Σ_φ is $\mathcal{H}^{d-1} \otimes \mathcal{L}^1$-measurable.

Theorem 1.13 (Variable Support Observability)**.** *Under the above hypotheses, suppose there exists a sequence of partitions $\{P_k : k \in \mathbb{N}\}$ of the interval $[0,T]$, such that $\delta(P_k) \searrow 0$ as $k \nearrow \infty$ and*

$$\lim_{k\to\infty} \left(\mathcal{H}^{d-1} \otimes \mathcal{L}^1\right)\left(\Sigma_\varphi \Delta \Sigma_\varphi^{P_k}\right) = 0, \tag{1.28}$$

where $A\Delta B := (A \setminus B) \cup (B \setminus A)$. Furthermore, suppose that T verifies

$$T > c_0 + \int_0^T |\varphi'(t)| \, dt + c_T. \tag{1.29}$$

Then the system is Σ_φ-observable in time T (see Def. 1.12).

See Figures 19 and 20 in Section 6 for examples.

In Section 5 we will provide an extension of the previous Theorem, when φ has only a bounded variation on $[0,T]$, see Theorem 5.3. Indeed, Theorem 1.13 can be viewed as a corollary of Theorem 5.3 (see Rem. 5.4), but we prefer to state it independently since it is sufficient in many applications.

Remark 1.14. We do not address the question of the controllability for $T = c_0 + \int_0^T |\varphi'(t)| \, dt + c_T$. In general such a T is not expected to be optimal (see Sect. 6). Controllability for the threshold optimal time is a tough question even in the case of constant control set (see [2] or [7]).

1.3. Overview

The article is divided into five parts, in Section 3 we prove the results stated in Section 1.1; next we obtain from this the results stated in Sections 1.1.1–1.1.2.

In Section 4, with the results of Section 1.1.2, we analyse the dependence of the optimal control time T_{opt} on the length of the interval T_0, proving that the map $T_0 \mapsto T_{opt}$ has a countable set of discontinuity.

In Section 5 we prove the results of Section 1.2 and we give a refinement of Theorem 1.13 when φ is only a map of bounded variation on $[0, T]$ (see Thm. 5.3).

In the last section, we work out some applications of the theorems stated above and we discuss some problems and further developments.

As a conclusion to this section, we want to emphasize that to each observability result follows (by duality) a controllability theorem; indeed the approach of [7] or [10] extends to our setting.

It is worthwhile to give a flavour of this, we state explicitly the controllability result corresponding to Theorem 1.13 (recall that Σ_φ is assumed to be $\mathcal{H}^{d-1} \otimes \mathcal{L}^1$-measurable).

Theorem 1.15. *Let the hypotheses of Theorem 1.13 be satisfied. Then for all $(v_0, v_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ there exists a control $w \in L^2(\Sigma_\varphi)$ such that the weak solution v of the following differential problem*

$$
\begin{align}
v_{tt} - \Delta v = 0, && (x,t) \in \Omega \times (0,T), && (1.30) \\
v(x,t) = w(x,t), && (x,t) \in \Sigma_\varphi, && (1.31) \\
v(x,t) = 0, && (x,t) \in \partial\Omega \times (0,T) \setminus \Sigma_\varphi, && (1.32) \\
v(x,0) = v_0(x), && x \in \Omega, && (1.33) \\
v_t(x,0) = v_1(x), && x \in \Omega, && (1.34)
\end{align}
$$

satisfies

$$
v(x,T) = 0, \qquad v_t(x,T) = 0.
$$

For the weak formulation of the problem (1.30)–(1.34) one can use the same argument of ([7], p. 21); see [1] for an explicit formulation.

The proof of Theorem 1.15 consists in an application of the HUM method; see [7, 10] for details. Both formulation and proofs are in fact given in [7, 10] for observability sets of the form $\Sigma = \Gamma \times (0, T)$, but extend to our case in a straightforward way. The reader can readily deduce analogous results corresponding to Theorems stated above in the one dimensional case.

2. Preliminary results

We recall the following well posedness result which can be readily verified by semigroup methods (see *e.g.* [11]).

Proposition 2.1 (Well Posedness)**.** *With the notations introduced in Section 1, the following holds:*
(i) Given

$$
u_0 \in H_0^1(\Omega), \qquad u_1 \in L^2(\Omega), \tag{2.1}
$$

the problem (1.1)–(1.4) admits a unique weak solution satisfying

$$
u \in C([0,\infty); H_0^1(\Omega)) \cap C^1([0,\infty); L^2(\Omega)). \tag{2.2}
$$

(*ii*) *Given*

$$u_0 \in H^2(\Omega) \cap H_0^1(\Omega), \qquad u_1 \in H_0^1(\Omega), \tag{2.3}$$

the problem (1.1)–(1.4) admits a unique strong solution satisfying

$$u \in C([0, \infty); H^2(\Omega) \cap H_0^1(\Omega)) \cap C^2([0, \infty); L^2(\Omega)). \tag{2.4}$$

For any weak solution u as in (2.2), we define the energy E(t) at time t as

$$E(t) = \frac{1}{2} \int_\Omega (|\nabla u(x,t)|^2 + |u_t(x,t)|^2) \, \mathrm{d}x; \tag{2.5}$$

then E(t) is constant and is equal to the initial energy

$$E_0 = \frac{1}{2} \int_\Omega (|\nabla u_0(x)|^2 + |u_1(x)|^2) \, \mathrm{d}x. \tag{2.6}$$

We note that if u is a weak solution of problem (1.1)–(1.4) then the trace of the normal derivative $\partial_\nu u$ on $\partial\Omega$ is not well defined in the sense of traces of Sobolev functions (see [4] or [5]). For future discussions, the following result is very useful.

Theorem 2.2 (Hidden Regularity). *Let $\Omega \subset \mathbb{R}^d$ be a domain of class C^2 or convex. Then for any $T > 0$ and $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, the map*

$$H_0^1(\Omega) \times L^2(\Omega) \to L^2(\partial\Omega \times (0,T)),$$
$$(u_0, u_1) \mapsto \partial_\nu u,$$

is well defined, continuous and coincides with the classical trace (in the sense of Sobolev spaces) if $(u_0, u_1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ since u is as in (2.4).

In case Ω is a C^2 set, the proof can be found in [6], [8] or [10]; in the one dimensional case Theorem 2.2 can be proven also by the use of Fourier series (see [7], Chap. 1).

To our knowledge, no proof can be found in the case of convex domains Ω; so we will sketch the proof. In the C^2-case, the proof in [6] starts with using explicitly the smoothness assumption, and it does not extend to convex domains. Our approach is based on an identity proved by the authors in [1] and some well known regularity results for the Laplace operator on convex domains.

Proof of Theorem 2.2 when Ω is convex. Let us start with

$$u_0, u_1 \in \mathrm{Span}\{e_k \,|\, k \in \mathbb{N}_0\},$$

where e_k are the eigenfuctions of the Laplacian operator with Dirichlet boundary condition on Ω. Regularity theory for elliptic PDEs ensures that $e_k \in H^2(\Omega) \cap H_0^1(\Omega)$; see [5].

Thus, the strong solution u of (1.1)–(1.4) is as in (2.4) and $\partial_\nu u$ is well defined in the sense of the trace for Sobolev spaces; see [4]. Furthermore, since $u \in C([0, T]; H^2(\Omega) \cap H_0^1(\Omega))$ then $\partial_\nu u \in C([0, T]; L^2(\partial\Omega))$.

To prove the assertion we use the following equality (see [1], Lem. 1.8),

$$\frac{1}{2}\int_{\partial\Omega\times(0,T)}(x-x_0)\cdot\nu\,|\partial_\nu u(x,t)|^2 d\mathcal{H}^{d-1}(x)\otimes\mathrm{d}t$$

$$=\left[\int_\Omega u_t(x,t)\left(\nabla u(x,t)\cdot(x-x_0)+\frac{d-1}{2}u(x,t)\right)\mathrm{d}x\right]_0^T+T\,E_0; \tag{2.7}$$

where x_0 is an arbitrary point of $\mathbb{R}^d$.

Note that, since each convex function is locally lipschitz (see [4], Thm. 6.7), then the exterior normal derivative ν is well defined $\mathcal{H}^{d-1}$-a.e. on $\partial\Omega$. From the convexity, it follows that for each $x_0\in\Omega$ then $(x-x_0)\cdot\nu\geq\delta>0$ $\mathcal{H}^{d-1}$-a.e., for some δ positive. By (2.7) we have

$$\frac{\delta}{2}\int_0^T\int_{\partial\Omega}|\partial_\nu u(x,t)|^2 d\mathcal{H}^{d-1}(x)\,\mathrm{d}t\leq\left[\int_\Omega u_t(x,t)\left(\nabla u(x,t)\cdot(x-x_0)+\frac{d-1}{2}u(x,t)\right)\mathrm{d}x\right]_0^T+T\,E_0. \tag{2.8}$$

Using Poincaré's inequality, it is clear that the RHS of (2.8) can be dominated by $C\,E_0$, for some positive constant C; a standard density argument completes the proof. $\qquad\square$

3. One dimensional case

In the one dimensional case, as stated in Section 1.1, w.l.o.g. we set $\Omega=(0,\pi)$. It is well known that the family $\{\sin(n\,x)\,|\,n\in\mathbb{N}\}$ forms a complete orthogonal system of $L^2(0,\pi)$. Indeed, this family consists of eigenfunctions of the operator d^2/dx^2 with Dirichlet condition. Thus, the unique solution u for the initial data $(u_0,u_1)\in H_0^1(0,\pi)\times L^2(0,\pi)$ provided by Proposition 2.1 is given by

$$u(x,t)=\sum_{j=1}^\infty\left[\hat{u}_0(j)\cos(jt)+\frac{1}{j}\hat{u}_1(j)\sin(jt)\right]\sin(jx); \tag{3.1}$$

here

$$\hat{u}_h(j)=\frac{2}{\pi}\int_0^\pi u_h(x)\,\sin(j\,x)\,\mathrm{d}x,\qquad h=0,1;\,j\in\mathbb{N}. \tag{3.2}$$

By a standard approximation argument, we see that the set

$$\mathcal{H}=\{(u_0,u_1)\,|\,u_0,u_1\in\mathrm{Span}\,(\sin(jx)\,|\,j\in\mathbb{N})\,\},$$

is dense in $H_0^1(0,\pi)\times L^2(0,\pi)$. The proof of Theorem 1.3 relies on the following observation: if $(u_0,u_1)\in\mathcal{H}$ then

$$u_x(x,t)=\sum_{j=1}^N\left[\hat{u}_0(j)\cos(jt)+\frac{1}{j}\hat{u}_1(j)\sin(jt)\right]j\cos(jx), \tag{3.3}$$

and

$$u_x(\pi, t) = \sum_{j=1}^{N} j \left[\hat{u}_0(j) \cos(jt) \cos(j\pi) + \frac{1}{j} \hat{u}_1(j) \sin(jt) \cos(j\pi) \right]$$

$$= \sum_{j=1}^{N} j \left[\hat{u}_0(j) \cos(j(t-\pi)) + \frac{1}{j} \hat{u}_1(j) \sin(j(t-\pi)) \right] = u_x(0, t-\pi); \tag{3.4}$$

where $N \in \mathbb{N}$ is sufficiently large and the last equality in (3.4) follows by (3.3) with $x = 0$.

Now we recall the following result, which is an easy application of the Parseval identity in Hilbert spaces (in this case $L^2(0, \pi)$ and $L^2(0, 2\pi)$), for a complete proof we refer to [7].

Proposition 3.1. *Let u be the unique weak solution of (1.1)–(1.4) with $\Omega = (0, \pi)$, then*

$$\int_0^{2\pi} |u_x(0, t)|^2 \, \mathrm{d}t = 2(\|u_0'\|_{L^2(0,\pi)}^2 + \|u_1\|_{L^2(0,\pi)}^2); \tag{3.5}$$

here $(\cdot)' = d(\cdot)/\mathrm{d}x$.

Now we are ready to prove Theorem 1.3:

Proof of Theorem 1.3. We first prove that if the condition (1.10) holds then the system is $\mathcal{F}$-observable (see Def. 1.1) where $\mathcal{F}$ is as in (1.7); in turn we have to prove inequality (1.8).

Due to Theorem 2.2, we can suppose that $(u_0, u_1) \in \mathcal{H}$. Now, the left hand side of (1.8) is equal to

$$\sum_{k=0}^{n} \int_{I_k} |u_x(\lambda_k, t)|^2 \, \mathrm{d}t = \sum_{k=0}^{n} \int_{I_k} |u_x(0, t - \lambda_k)|^2 \, \mathrm{d}t$$

$$= \sum_{k=0}^{n} \int_{I_k - \lambda_k} |u_x(0, t)|^2 \, \mathrm{d}t = \sum_{k=0}^{n} \int_{\tilde{I}_k} |u_x(0, t)|^2 \, \mathrm{d}t \asymp \int_{\cup_{k=0}^{n} \tilde{I}_k} |u_x(0, t)|^2 \, \mathrm{d}t. \tag{3.6}$$

Here the first equality follows by (3.4), the second from a change of variable in the integrals. Then it follows by (1.10) that

$$\sum_{k=0}^{n} \int_{I_k} |u_x(\lambda_k, t)|^2 \, \mathrm{d}t \asymp \int_{\cup_{k=0}^{n} \tilde{I}_k} |u_x(0, t)|^2 \, \mathrm{d}t = \int_{S^1} |u_x(0, t)|^2 \, \mathrm{d}t = 4E_0,$$

where the last equality follows by Proposition 3.1.

Conversely, we show that if (1.10) does not hold then the observability inequality fails for some initial data. To do this we need the following:

Lemma 3.2. *For each open set $U \subset S^1$, there exists a pair $(u_0, u_1) \in H_0^1(0, \pi) \times L^2(0, \pi)$ different from the null pair, such that the corresponding solution u to the problem (1.1)–(1.4) satisfies*

$$supp \, u_x(0, \cdot) \subset U. \tag{3.7}$$

We postpone the proof of the Lemma 3.2 and continue with the previous discussion. Since, by assumption, the condition (1.10) fails, then there exists an open subset U contained in $S^1 \setminus \cup_{j=0}^{n} \tilde{I}_j$. Then choose $(u_0, u_1) \in$

$H_0^1(0, \pi) \times L^2(0, \pi)$ in accordance with Lemma 3.2. By (3.6), if the observability inequality (1.8) holds then we have

$$0 = \int_{\cup_{k=0}^n \tilde{I}_k} |u_x(0, t)|^2 \, \mathrm{d}t \asymp \sum_{k=0}^n \int_{I_k} |u_x(\lambda_k, t)|^2 \, \mathrm{d}t \geq CE_0 \neq 0,$$

which is evidently an inconsistency. The last assertion of Theorem 1.3 follows by the previous argument. □

Proof of Lemma 3.2. We can of course suppose that $\pi \notin U$ (otherwise we can replace U with $U \setminus \overline{V}$, where V is a sufficiently small neighborhood of π). Let $\psi \not\equiv 0$ be a C^∞-function on $(0, 2\pi)$ such that $\operatorname{supp} \psi \subset U$ and $\int_0^{2\pi} \psi(t) \, \mathrm{d}t = 0$; it is enough to prove that there exists a pair $(\tilde{u}_0, \tilde{u}_1) \in H_0^1(0, \pi) \times L^2(0, \pi)$ such that the weak solution of (1.1)–(1.4) $\tilde{u}$ satisfies $\tilde{u}_x(0, \cdot) = \psi$.

Firstly, by the Fourier series representation formula (3.1), we obtain

$$\tilde{u}_x(0, t) = \tilde{u}_0'(t) + \tilde{u}_1(t), \qquad t \in (0, \pi).$$

Instead for $t \in (\pi, 2\pi)$, set $t = 2\pi - \tau$ with $\tau \in (0, \pi)$ and using a simple reflection argument we obtain

$$\tilde{u}_x(0, 2\pi - \tau) = \tilde{u}_0'(\tau) - \tilde{u}_1(\tau), \qquad \tau \in (0, \pi).$$

Then $\tilde{u}_x(0, t) = \psi(t)$ is equivalent to

$$\psi(t) = \tilde{u}_0'(t) + \tilde{u}_1(t), \qquad \psi(2\pi - t) = \tilde{u}_0'(t) - \tilde{u}_1(t), \qquad t \in (0, \pi).$$

We immediately obtain

$$\tilde{u}_0'(t) = \frac{1}{2} \left(\psi(t) + \psi(2\pi - t) \right), \qquad t \in (0, \pi),$$

and

$$\tilde{u}_1(t) = \frac{1}{2} \left(\psi(t) - \psi(2\pi - t) \right), \qquad t \in (0, \pi). \tag{3.8}$$

Then

$$\tilde{u}_0(x) = \frac{1}{2} \int_0^x \left(\psi(t) + \psi(2\pi - t) \right) \, \mathrm{d}t, \qquad x \in (0, \pi). \tag{3.9}$$

Since by assumption ψ is a smooth function, $\int_0^{2\pi} \psi(t) \mathrm{d}t = 0$ and $\pi \notin U \supset \operatorname{supp} \psi$; then $\tilde{u}_0 \in H_0^1(0, \pi)$.

Thus, the functions $\tilde{u}_0 \in H_0^1(0, \pi)$ and $\tilde{u}_1 \in L^2(0, \pi)$, defined respectively in (3.9) and (3.8), provide the desired pair of initial data. □

Next we prove Corollary 1.4.

Proof of Corollary 1.4. It is sufficient to define recursively $\tilde{J}_0 := \tilde{I}_0$ and

$$\tilde{J}_k := \tilde{I}_k \setminus \bigcup_{j=0}^{k-1} \tilde{J}_j, \qquad k = 1, \dots, n.$$

The other statement in the corollary follows easily by this. □

Corollary 1.8 is an easy consequence of Theorem 1.3 or Corollary 1.4. Before proving the results of Section 1.1.2 we introduce some conventions.

With an abuse of notation, we denote the subset of S^1 mapped from the interval $(\alpha, \beta) \subset \mathbb{R}$ through the application r still as (α, β). So we will write

$$(\alpha, \beta) = (\gamma, \delta) \subset S^1;$$

if $r(\alpha) = r(\gamma)$, $r(\beta) = r(\delta)$. With this convention if $\alpha \in (0, 2\pi)$, $\beta \in (2\pi, 4\pi)$, we have

$$(\alpha, \beta) = (\alpha, 2\pi) \bigcup (0, \beta - 2\pi).$$

Moreover, in the following we identify the measure space $(S^1, |\cdot|)$ with the space $([0, 2\pi), \mathcal{L}^1)$.

In the proofs we use some figures to make them more clear. Indeed, in Figures 1, 3 and 8, we have represented in the first line (resp. second) line from the top the set, coming from intervals in $x = 0$ (resp. in $x = \pi$). If there is a third line, it shows the union of the two sets.

In Figures 4, 5, 7, 9, 10, 13, 12, 17 and 18 we are concerned with the calculation of T_{opt}; here the first line depicts the set still to be covered, the second line the observability interval covering such a missing set, and the third line an example of a partial observation.

In figures, we use the red color for the observability sets, the blue color to stress that this part of the observability set on S^1 has already been covered.

We begin with the proof of Lemma 1.10 which is the simplest one and permits us to explain in details some technique which will be used also to prove Lemma 1.9 and 1.11.

Proof of Lemma 1.10. (i) First of all, we note that if k is even then the observability subset $\tilde{I}_k$ comes from a subset of observation on the endpoint $x = 0$, so we obtain

$$\tilde{I}_0 = (0, T_0), \ \tilde{I}_2 = (2T_0, 3T_0), \dots, \tilde{I}_{2m-2} = (\pi - 2T_0, \pi - T_0),$$
$$\tilde{I}_{2m} = (\pi, \pi + T_0), \dots, \tilde{I}_{4m-2} = (2\pi - 2T_0, 2\pi - T_0);$$

instead, for odd indexes, we have

$$\tilde{I}_1 = (\pi + T_0, \pi + 2T_0), \dots, \tilde{I}_{2m-1} = (2\pi - T_0, 2\pi),$$
$$\tilde{I}_{2m+1} = (T_0, 2T_0), \tilde{I}_{2m+3} = (3T_0, 4T_0), \dots,$$
$$\tilde{I}_{4m-1} = ((4m - 1)T_0 + \pi, 4mT_0 + \pi) = (\pi - T_0, \pi).$$

It is easy to see that $(0, 2\pi)$ is covered by the following subsets, in the following order:

$$\tilde{I}_0, \ \tilde{I}_{2m+1}, \ \tilde{I}_2, \ \tilde{I}_{2m+3}, \ \dots, \ \tilde{I}_{4m-1}, \ \tilde{I}_{2m}, \ \tilde{I}_1, \ \tilde{I}_{2m+2}, \ \tilde{I}_3, \ \dots, \ \tilde{I}_{2m-1};$$

see Figure 1. The condition (1.10) in Theorem 1.3 is satisfied, then the system is observable in $T = 2\pi$.

Finally we note that $T = 2\pi$ is optimal, in fact $(\pi - T_0, \pi)$ is covered only by $\tilde{I}_{4m-1} = ((4m - 1)T_0, 4mT_0) = (2\pi - T_0, 2\pi)$ so the optimality follows by Theorem 1.3.

(ii) We proceed as in (i), by listing the subsets of observation coming, through the map r, from the time interval $(0, 2\pi)$. For even indexes we have

$$\tilde{I}_0 = (0, T_0), \ \tilde{I}_2 = (2T_0, 3T_0), \dots, \tilde{I}_{2n} = (\pi - T_0, \pi), \dots,$$
$$\tilde{I}_{2n+2} = (\pi + T_0, \pi + 2T_0), \dots,$$
$$\tilde{I}_{4n} = ((4n)T_0, (4n + 1)T_0) = (2\pi - 2T_0, 2\pi - T_0);$$
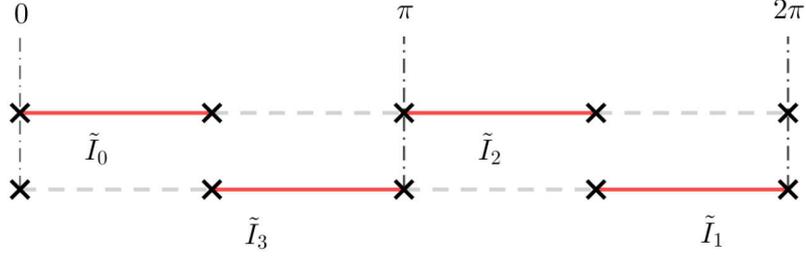
(a) *i*) Lemma 1.10. Case $T_0 = \pi/2$.
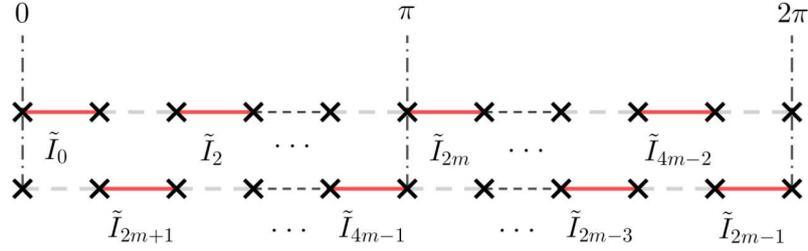
(b) *i*) Lemma 1.10. General case: $T_0 = \pi/(2m)$.

FIGURE 1. (*i*) Lemma 1.10. Two examples of T_0.
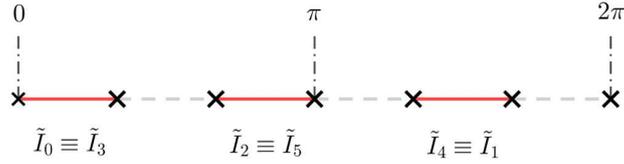
Figure 2. *ii*) *Lemma 1.10. Case $T_0 = \pi/3$.*

FIGURE 2. (*ii*) Lemma 1.10. Case $T_0 = \pi/3$.

instead for odd indexes we have

$$\tilde{I}_1 = (\pi + T_0, \pi + 2T_0), \ldots, \tilde{I}_{2n-1} = (2\pi - 2T_0, 2\pi - T_0), \ldots,$$
$$\tilde{I}_{2n+1} = (0, T_0), \ldots,$$
$$\tilde{I}_{4n+1} = ((4n+1)T_0 + \pi, (4n+2)T_0 + \pi) = (\pi - T_0, \pi).$$

It is clear that (see Fig. 2)

$$\tilde{I}_0 = \tilde{I}_{2n+1}, \ldots, \tilde{I}_{2n} = \tilde{I}_{4n+1}, \tilde{I}_{2n+2} = \tilde{I}_1, \ldots, \tilde{I}_{2n-1} = \tilde{I}_{4n}.$$

Thus, the following subsets

$$(T_0, 2T_0), \ldots, ((4n+1)T_0, (4n+2)T_0)$$

cannot be covered by other subsets since the subsets $\{\tilde{I}_k : k \in \mathbb{N}_0\}$ are 2π-periodic in time. In view of the necessary condition (1.10) in Theorem 1.3 the system cannot be observable in any time T.

(iii) Reasoning as in $(i)-(ii)$, we have for even indexes

$$\tilde{I}_0 = (0, T_0), \ldots, \tilde{I}_h = (hT_0, (h+1)T_0) = \left(\pi - \frac{T_0}{2}, \pi + \frac{T_0}{2}\right), \ldots,$$

$$\tilde{I}_{2h} = (2hT_0, (2h+1)T_0) = (2\pi - T_0, 2\pi),$$

where we use that $hT_0 = \pi - T_0/2$. Instead for odd indexes, we have

$$\tilde{I}_1 = (\pi + T_0, \pi + 2T_0), \ldots, \tilde{I}_{h-1} = (\pi + (h-1)T_0, \pi + hT_0) = \left(2\pi - \frac{3}{2}T_0, 2\pi - \frac{1}{2}T_0\right), \ldots,$$

$$\tilde{I}_{h+1} = \left(\frac{T_0}{2}, \frac{3T_0}{2}\right), \ldots, \tilde{I}_{2h-1} = ((2h-1)T_0 - \pi, 2hT_0 - \pi) = (\pi - 2T_0, \pi - T_0).$$

Note that the subset

$$\left(\pi - T_0, \pi - \frac{T_0}{2}\right) \tag{3.10}$$

is one of the subsets not covered by any $\tilde{I}_k$ for $k = 1, \ldots, 2h$ (see the last line in Fig. 3a). In particular, we will see that the subset in (3.10) would be the last to be covered among the intervals not yet covered.

We now proceed further listing the later subsets (in the index or chronological order). The observability subsets coming from the endpoint $x = 0$ are

$$\tilde{I}_{2h+2} = (2\pi + T_0, 2\pi + 2T_0) = (T_0, 2T_0), \ldots,$$

$$\tilde{I}_{3h} = ((3h)T_0, (3h+1)T_0) = (2\pi + (h-1)T_0, 2\pi + h\,T_0) = \left(3\pi - \frac{3T_0}{2}, 3\pi - \frac{T_0}{2}\right) = \left(\pi - \frac{3T_0}{2}, \pi - \frac{T_0}{2}\right);$$

instead, from the endpoint $x = \pi$ they are

$$\tilde{I}_{2h+1} = (\pi, \pi + T_0), \ldots, \tilde{I}_{3h-1} = \left(2\pi - \frac{5}{2}T_0, 2\pi - \frac{3}{2}T_0\right).$$

It is easy to see that $(0, 2\pi)$ is covered by the following subsets, which are ordered $e.g.$, according to their left endpoint as indicated below:

$$\tilde{I}_0, \tilde{I}_{2h+2}, \tilde{I}_2, \tilde{I}_{2h+4}, \ldots, \tilde{I}_{3h}, \tilde{I}_h, \tilde{I}_{2h+1}, \tilde{I}_1, \tilde{I}_{2h+3}, \tilde{I}_3, \ldots, \tilde{I}_{3h-1}, \tilde{I}_{h-1}, \tilde{I}_{2h}.$$

To compute the optimal control time, we note that the subset with the highest index in the previous sequence is $\tilde{I}_{3h} = r(I_{3h})$; then, by Theorem 1.3, $T_{opt} \leq (3h+1)T_0 = 3\pi - T_0/2$.

Moreover, the set

$$\tilde{I}_{3h} \setminus r((3hT_0, t)) \subset (\pi - T_0, \pi - T_0/2),$$

is not covered for any $t < (3h+1)T_0 = 3\pi - T_0/2$ (see Fig. 4). Thus, by Theorem 1.3, we have $T_{opt} \geq 3\pi - T_0/2$ and this concludes the proof.

(iv) The proof is similar to (iii). In this case, since h is odd, the observability sets $\tilde{I}_k$, for $k \leq 2h$, do not cover the subset

$$(hT_0, \pi),$$

(a) $[0, \pi] \subset [0, 2\pi)$.
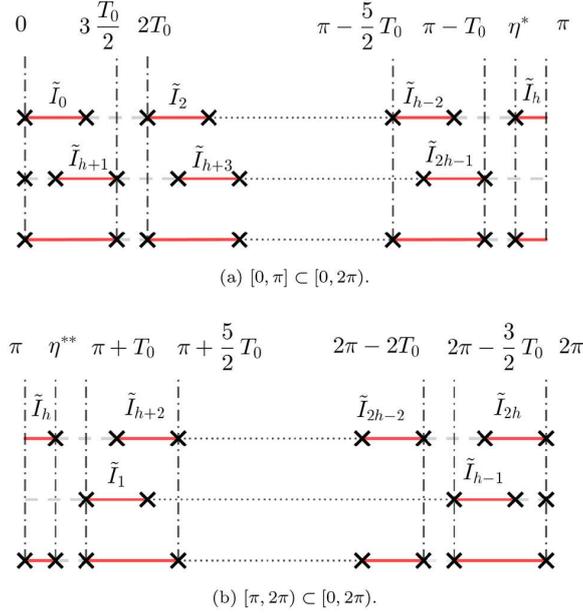

(b) $[\pi, 2\pi) \subset [0, 2\pi)$.

FIGURE 3. (*iii*) Lemma 1.10. We have represented separately the two halves of the interval $[0, 2\pi)$ and the observability sets $\tilde{I}_k$ for $k = 0, \ldots, N$.
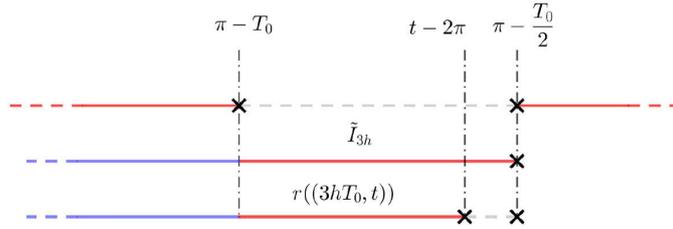

FIGURE 4. Computation of T_{opt} in (*iii*) of Lemma 1.10.

which is different from the set in (3.10) but plays the same role of that set in (*iii*).

Note that $\tilde{I}_{3h+1} \supset (hT_0, \pi)$. By Theorem 1.3, this implies

$$T_{opt} \leq (3h + 2)T_0 = 2\pi + \frac{h + 1}{2h + 1} 2\pi.$$

As Figure 5 shows, we can use only a portion of $\tilde{I}_{3h+1}$ to cover (hT_0, π) (the red part). Indeed,

$$\tilde{I}_{3h+1} \supset r((3h + 1)T_0, 3\pi)) = (hT_0, \pi).$$

Now the conclusion follows as in (*iii*).   □

The previous lemma shows that the $\mathcal{F}$-observability is not expected in general. The main idea is to "follow" the covering of S^1 with $\tilde{I}_k$ when k grows. This will be exploit in the proof of Lemma 1.9 below.

Proof of Lemma 1.9. Since $\tilde{I}_0 = (0, T_0)$, for the $\mathcal{F}$-observability it is sufficient to prove that the subsequent observation sets cover $(T_0, 2\pi)$.
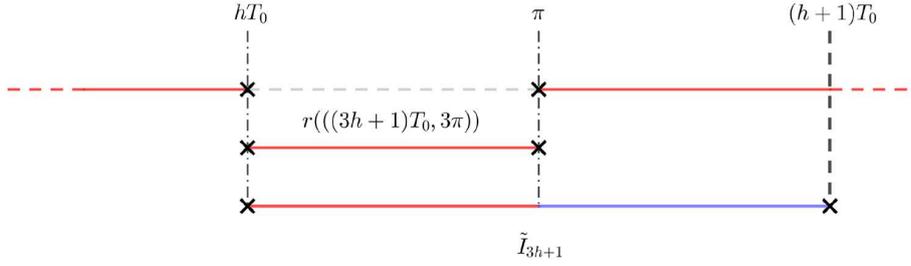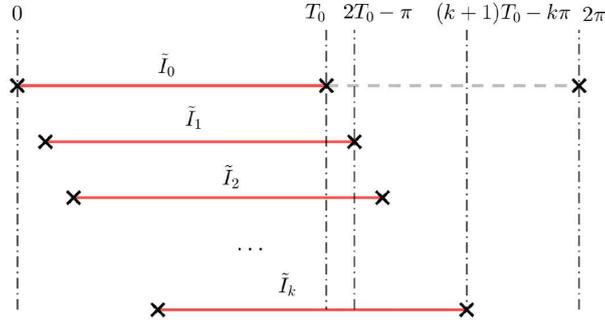
FIGURE 5. Computation of T_{opt} in (iv) of Lemma 1.10.

FIGURE 6. Lemma 1.9. We have represented the observations sets (to avoid confusion) on separate lines.

As Figure 6 shows, the subset $\tilde{I}_1$ is shifted towards the endpoint $t = 2\pi$ with a gap equal to $T_0 - \pi > 0$; more generally each $\tilde{I}_k$ has the same behaviour with a gap equal to $k(T_0 - \pi)$. Thus, we are able to cover the missing interval $(T_0, 2\pi)$ in a finite time.

Following our notation, we can check that for each $k \in \mathbb{N}$,

$$\tilde{I}_k = (kT_0 - k\pi, (k + 1)T_0 - k\pi).$$

For the optimality, we are interested in the first interval $\tilde{I}_k$ which "touches" the endpoint $t = 2\pi$. More precisely, we denote with n the least integer $k \geq 1$ such that,

$$(k + 1)T_0 - k\pi \geq 2\pi \;\Leftrightarrow\; T_0 \geq \frac{(k + 2)\pi}{(k + 1)}.$$

By minimality, we have $nT_0 - (n - 1)\pi < 2\pi$, *i.e.* $T_0 < \pi((n + 1)/n)$. In turn, n is the unique integer such that

$$\frac{n + 2}{n + 1}\pi \leq T_0 < \frac{n + 1}{n}\pi;$$

which is the condition (1.17).

We now turn to prove the formula for T_{opt}. As Figure 7 shows, we do not use all the subset $\tilde{I}_n$ to cover the missing part $(T_0, 2\pi)$, since the blue part is an extra observation set not useful to cover $(T_0, 2\pi)$. So, the optimal control time is the least $t \geq nT_0$ such that

$$r((nT_0, t)) = (nT_0 - n\pi, t - n\pi) \supset (T_0, 2\pi).$$
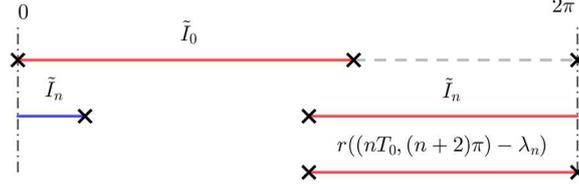
FIGURE 7. Computation of T_{opt} in Lemma 1.9.

This is equivalent to $t - n\pi \geq 2\pi$, *i.e.* $t \geq (n+2)\pi$; so $T_{opt} = (n+2)\pi$. $\square$

The proofs of Lemmas 1.9–1.10 contain all the ingredients needed to prove Lemma 1.11. Indeed, in (i) and (iv) we exploit the same strategy of Lemma 1.10, *i.e.* we list the subsets of observation $\tilde{I}_k$ when k grows and we follow the covering of the "missing subsets" of S^1. In such cases, the strategy works since we will discover that T_{opt} is "relatively small". More precisely, we will see $T_{opt} \leq 3\pi$.

Such strategy is no more efficient in cases (ii)-(iii). On the contrary, the proof is based on a covering analysis of the "missing sets" of S^1, as done in Lemma 1.9 for the set $(T_0, 2\pi)$. In Section 4, we will see that T_{opt} can be arbitrarily large provided T_0, such that $2\pi/T_0 \notin \mathbb{N}$, is suitable. Of course, such T_0 verifies (ii) or (iii).

Proof of Lemma 1.11. (i) By assumption, we have $N = 2j$ and $j \geq 2$. By Remark 1.5 we have $T_{opt} \geq 2\pi$. The observability sets for $t \leq 2\pi$ are

$$\tilde{I}_i, \qquad i = 0, \ldots, N.$$

With simple considerations, one can check that the above subsets do not cover S^1; see Figure 8. Namely the following subsets are not covered

$$J_k := ((j+2)T_0 - \pi + kT_0, (2+k)T_0), \tag{3.11}$$
$$J_{j+k} := ((j+1)T_0 + kT_0, \pi + (k+1)T_0); \tag{3.12}$$

where $k = 0, 2, \ldots, j - 2$.

So, we have to look at the subsequent observability set (in the index order) to show the $\mathcal{F}$-observability. Observe that, the first observability set $\tilde{I}_k$ with $k \geq 2j$ which intersects J_0 is $\tilde{I}_{N+2}$. Moreover,

$$\tilde{I}_{N+2} \cap J_0 = (\max\{NT_0 + 2T_0 - 2\pi, (j+2)T_0 - \pi\}, 2T_0) = ((j+2)T_0 - \pi, 2T_0) = J_0.$$

In fact the second last equality follows by

$$NT_0 + 2T_0 - 2\pi < (j+2)T_0 - \pi \Leftrightarrow (N-j)T_0 = jT_0 < \pi;$$

and the last inequality is verified (recall the definition (1.19)).

Owing to the periodical character of our setting, the same reasoning shows that

$$\tilde{I}_{N+2+m} \supset J_m; \qquad m = 0, 2, \ldots, j - 2.$$

Furthermore, the least time which allows us to cover the subsets J_m, for $m = 0, 2, \ldots, j - 2$, is the least one which ensures to cover J_{j-2} and it is the minimum t' such that

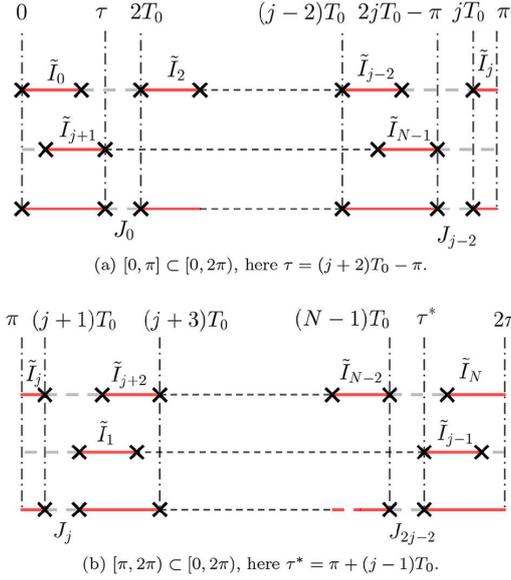$$r((3jT_0, t')) = (3jT_0 - 2\pi, t' - 2\pi) \supset J_{j-2} = (2jT_0 - \pi, jT_0).$$

(a) $[0, \pi] \subset [0, 2\pi)$, here $\tau = (j + 2)T_0 - \pi$.

(b) $[\pi, 2\pi) \subset [0, 2\pi)$, here $\tau^* = \pi + (j - 1)T_0$.

FIGURE 8. (i) Lemma 1.11. We have represented separately the two halves of the interval $[0, 2\pi)$ and the observability sets $\tilde{I}_k$ for $k = 0, \ldots, N$. The missing sets J_h are given in (3.11)–(3.12).
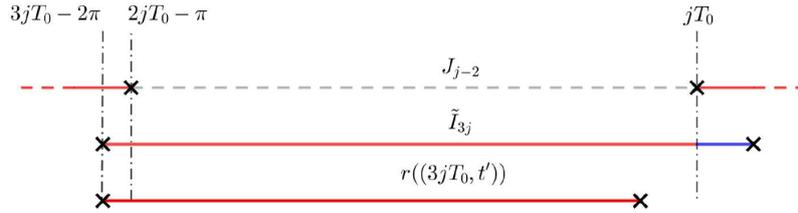

FIGURE 9. Computation of T_{opt} in (i) Lemma 1.11.

The previous condition is equivalent to $t' - 2\pi \geq jT_0$, then $2\pi + jT_0$ is the least time which allows us to cover J_m for $m = 0, 2, \ldots, j - 2$; see Figure 9.

Now, we show that we can cover the subsets J_m with $m = j, j + 2, \ldots, 2j - 2$.

Again by periodicity, we have only to prove that we can cover J_{2j-2}, the previous J_m's are then automatically covered by a simple translation argument. Moreover, it is easy to see that $\tilde{I}_{3j-1} \supset J_{2j-2}$ and the above argument shows that the least time which allows us to cover J_{2j-2} is $2\pi + (j - 1)T_0$; see Figure 10.

Finally, T_{opt} is the least time which allows us to cover *any* missing sets J_m's. By the previous considerations, we have $T_{opt} = 2\pi + jT_0$.

(ii) As in (i), we have $N = 2j$. Reasoning as in (i), the following subsets are not covered by $\tilde{I}_1, \ldots, \tilde{I}_N$:

$$L_h := ((h + 1)T_0, (j + 2 + h)T_0 - \pi), \tag{3.13}$$

$$L_{j-1} := (jT_0, (j + 1)T_0), \tag{3.14}$$

$$L_{j+1+h} := ((2 + h)T_0 + \pi, (j + h + 3)T_0); \tag{3.15}$$

where $h = 0, 2, \ldots, j - 3$. Here, we have assumed $j \geq 3$, the case $j = 1$ is simpler and follows by the same argument. Note that, in case $j = 1$ the only subset of S^1 not covered is $L_0 := (T_0, 2T_0)$.

The idea is to use an argument similar to the one used in Lemma 1.9 on each L_k. Roughly speaking, the
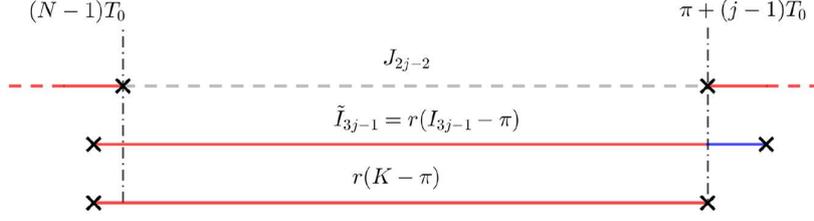
FIGURE 10. Computation of T_{opt} in (i) Lemma 1.11; here $K = ((3j-1)T_0, 2\pi + (j-1)T_0)$.
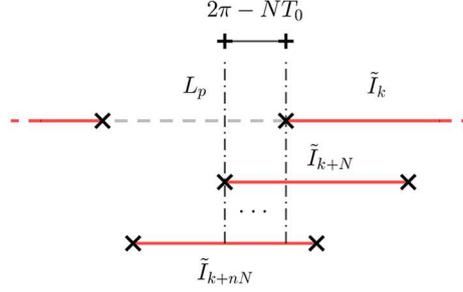

FIGURE 11. (ii) Lemma 1.11, with p depending on the choice of k.

clue is that for each $\tilde{I}_k$, "the corresponding one" in the next period (*i.e.* after a time 2π) is $\tilde{I}_{k+N}$ and it is "shifted" by a quantity $2\pi - NT_0$ towards the endpoint $t = 0$ with respect to $\tilde{I}_k$ (recall that we have identified the measure space $(S^1, |\cdot|)$ with $([0, 2\pi), \mathcal{L}^1)$). A similar argument holds for the subsequent time periods; indeed $\tilde{I}_{k+nN}$ is "shifted" by a quantity $n(2\pi - NT_0)$ towards the endpoint $t = 0$ with respect to $\tilde{I}_k$; see Figure 11. This fact will be used to prove that the missing sets in (3.13)–(3.15) can be covered in a finite time.

Furthermore, since N is even, if $\tilde{I}_k$ comes from an observation on the endpoint $x = 0$ (resp. $x = \pi$) then $\tilde{I}_{k+nN}$ still comes from an observation on $x = 0$ (resp. $x = \pi$); this will be useful in the following.

Again by periodicity as in the proof of point (i), by a shift argument, one can reduce the proof to show that the subsets L_{j-1} and L_{2j-2} are covered in finite time. Furthermore, the least time t which allows us to cover both L_{j-1} and L_{2j-2} will be the optimal control time.

For the reader's convenience, we divide the proof into two steps. In the first one, we determine the least $\Xi \in \mathbb{N}$ which allows us to cover both L_{j-1}, L_{2j-2} with subsets in $\{\tilde{I}_k : k \leq \Xi\}$. In the second one, we compute T_{opt} by means of Theorem 1.3.
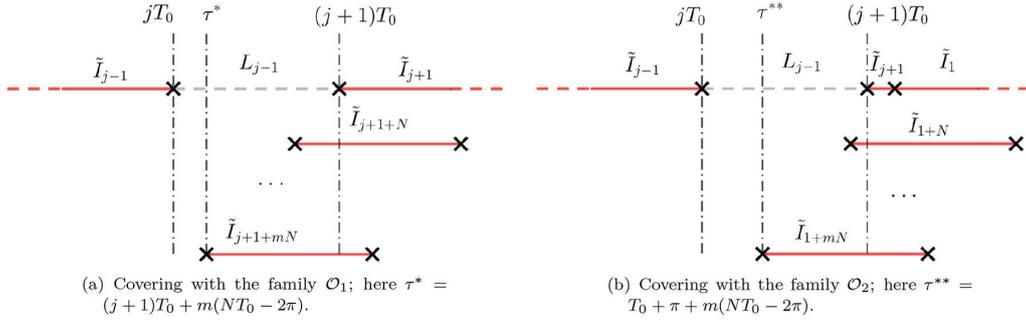
Step 1. We first analyse the covering of L_{j-1}. By the previous discussion, to cover L_{j-1} we can consider two families of subsets, namely $\mathcal{O}_1 := \{\tilde{I}_{mN+j+1} : m \in \mathbb{N}\}$ and $\mathcal{O}_2 := \{\tilde{I}_{1+mN} : m \in \mathbb{N}\}$; see Figure 12. Note that the intervals in $\mathcal{O}_1$ come from $x = 0$, while ones in $\mathcal{O}_2$ come from $x = \pi$. Let us compute the least $k_i \in \mathbb{N}$ (with $i = 1, 2$) such that the subsets in $\{\tilde{I}_k \in \mathcal{O}_i : k \leq k_i\}$ cover L_{j-1}.

First consider the family $\mathcal{O}_1$. As usual, we ask for the minimum $m^* > 1$ such that the following inclusion holds

$$\bigcup_{m=0}^{m^*} (\tilde{I}_{mN+j+1} \cap L_{j-1}) = (\max\{(m^*N + j + 1)T_0 - 2m^*\pi, jT_0\}, (j+1)T_0) \supset L_{j-1};$$

(*cf.* Fig. 12a). By (3.14), this occurs if and only if $(m^*N + j + 1)T_0 - 2m^*\pi \leq jT_0$. Thus $m^* = k^* := \lceil T_0/(2\pi - NT_0) \rceil$ and

$$k_1 = k^*N + j + 1;$$

(a) Covering with the family $\mathcal{O}_1$; here $\tau^* = (j+1)T_0 + m(NT_0 - 2\pi)$.

(b) Covering with the family $\mathcal{O}_2$; here $\tau^{**} = T_0 + \pi + m(NT_0 - 2\pi)$.

FIGURE 12. (ii) Lemma 1.11. Covering of L_{j-1} with the families $\mathcal{O}_1, \mathcal{O}_2$.
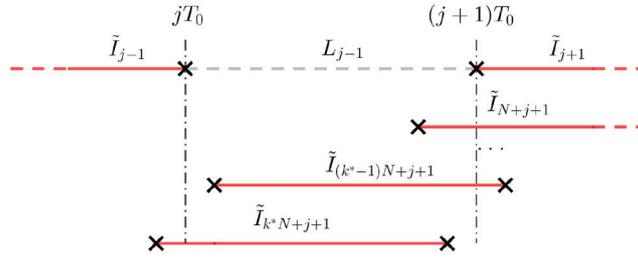
FIGURE 13. Covering of L_{j-1} with $\mathcal{O}_1$.

see Figure 13. A similar argument is valid for the family $\mathcal{O}_2$ (*cf.* Figure 12b). One discovers that $k_2 = 1 + h^*N$, where

$$h^* := \left\lceil \frac{\pi - jT_0 + T_0}{2\pi - NT_0} \right\rceil = \left\lceil \frac{1}{2} + \frac{T_0}{2\pi - NT_0} \right\rceil. \tag{3.16}$$

All together, $\tilde{k} := \min\{k_1, k_2\}$ is the least integer h such that $\{\tilde{I}_k \ : \ k \leq h\}$ covers L_{j-1}.

For L_{2j-2} a similar argument applies with respect to the families $\{\tilde{I}_{N+mN} \ : \ m \in \mathbb{N}\}$ and $\{\tilde{I}_{j+mN} \ : \ m \in \mathbb{N}\}$. One readily checks that, for any T_0, the family $\{\tilde{I}_k \ : \ k < \tilde{k}\}$ covers L_{2j-2}. This implies $\Xi = \tilde{k}$.

Step 2. We are now in a position to compute T_{opt}. Below k_i (with $i = 1, 2$) are as in **Step 1**. By (3.16) one has $k^* \leq h^* \leq k^* + 1$, thus we can divide the analysis into two cases:
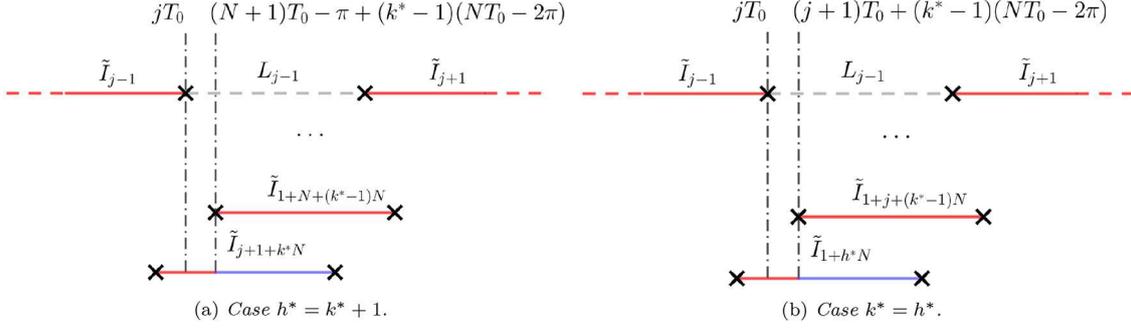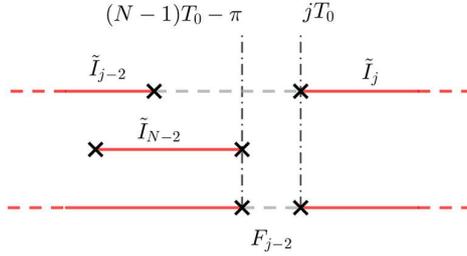
– If $h^* = k^* + 1$, then $k_1 < k_2$ and $\Xi = k_1$. One can check that (see Fig. 14a),

$$T_{opt} = 2k^*\pi + (N+1)T_0 - \pi + (k^* - 1)(NT_0 - 2\pi) = \pi + T_0(Nk^* + 1).$$

– If $k^* = h^*$, then $k_2 < k_1$ and $\Xi = k_2$. One can check that (see Fig. 14b) T_{opt} verifies

$$T_{opt} + \pi = 2\pi k^* + (j+1)T_0 + (k^* - 1)(NT_0 - 2\pi) = 2\pi + (Nk^* - j + 1)T_0.$$

This completes the proof of (ii).

FIGURE 14. Computation of T_{opt} in (ii).

FIGURE 15. (iii) Lemma 1.11. Covering of F_{j-2}, for time $t \leq 2\pi$.

(iii) Since N is odd, one has $N = 2j + 1$. Note that the observability sets which come from time less than 2π do not cover:

$$F_h := ((j+2)T_0 - \pi + hT_0, (h+2)T_0), \tag{3.17}$$

$$F_{j+h} := ((j+1+h)T_0, (h+1)T_0 + \pi), \tag{3.18}$$

$$F_{2j} := (NT_0, 2\pi). \tag{3.19}$$

where $h = 0, 2, \ldots, j - 2$ (see Fig. 15 for F_{j-2}).

We want to emulate the proof of (ii). We have only to pay attention to one issue: since N is odd, if $\tilde{I}_k$ comes from an observation at $x = 0$ (resp. $x = \pi$), then $\tilde{I}_{k+N+1}$ still comes from $x = 0$ (resp. $x = \pi$). The difference is that, $\tilde{I}_{k+N+1}$ is now "shifted" towards the endpoint $t = 2\pi$ by a quantity $(N+1)T_0 - 2\pi > 0$ with respect to $\tilde{I}_k$; (see Fig. 16a for $k = j - 2$ and Fig. 16b for $k = N - 2$).
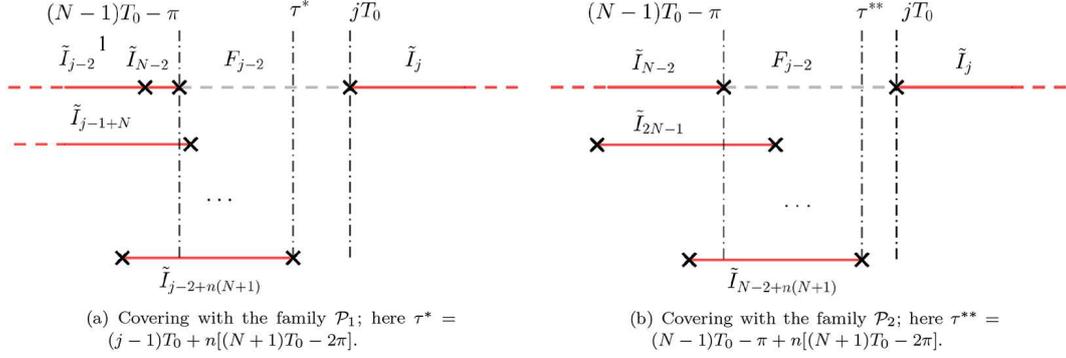
By periodicity, the problem of finding T_{opt} can be reduced to finding the minimum value of the time which allows us to cover $F_{j-2}, F_{2j-2}, F_{2j}$.

As in (ii), we split the proof into two steps. In the first one, we compute the least $\Theta \in \mathbb{N}$ such that F_{j-1}, F_{2j-2} and F_{2j} are covered by subsets in $\{\tilde{I}_k : k \leq \Theta\}$. In the second one, T_{opt} is found using Theorem 1.3.

Step 1. We first analyse the covering of F_{j-2}. By the initial discussion, we can only consider the families of subsets $\mathcal{P}_1 := \{\tilde{I}_{j-2+n(N+1)} : n > 0\}$ and $\mathcal{P}_2 := \{\tilde{I}_{N-2+n(N+1)} : n > 0\}$; see Figure 16. As the Figure 16 shows, each family provides an exhaustion of the missing set F_{j-2} "from the left to the right". Let us compute the least $h_i \in \mathbb{N}$ (with $i = 1, 2$) such that the subsets in $\{\tilde{I}_k \in \mathcal{P}_i : k \leq h_i\}$ cover F_{j-2}.

First consider the family $\mathcal{P}_1$. As usual, we ask for the minimum n^* such that the following inclusion holds

$$\bigcup_{n=0}^{n^*} (\tilde{I}_{j-2+n(N+1)} \cap F_{j-2}) = ((N-1)T_0 - \pi, \min\{(j-1)T_0 + n^*[(N+1)T_0 - 2\pi], jT_0\}) \supset F_{j-2}.$$

(a) Covering with the family $\mathcal{P}_1$; here $\tau^* = (j-1)T_0 + n[(N+1)T_0 - 2\pi]$.

(b) Covering with the family $\mathcal{P}_2$; here $\tau^{**} = (N-1)T_0 - \pi + n[(N+1)T_0 - 2\pi]$.

FIGURE 16. (iii) Lemma 1.11. Covering of F_{j-2} with the families $\mathcal{P}_1, \mathcal{P}_2$.

By (3.17), the previous inclusion holds if and only if $(j-1)T_0 + n^*[(N+1)T_0 - 2\pi] \geq jT_0$. Thus $n^* = l^* := \lceil T_0/((N+1)T_0 - 2\pi) \rceil$ and

$$h_1 = (j-2) + l^*(N+1).$$

Similar considerations hold for the family $\mathcal{P}_2$. One can readily check that

$$h_2 = N - 2 + q^*(N+1),$$

where

$$q^* = \left\lceil \frac{\pi - jT_0}{(N+1)T_0 - 2\pi} \right\rceil = \left\lceil \frac{T_0}{(N+1)T_0 - 2\pi} - \frac{1}{2} \right\rceil. \tag{3.20}$$

All together, $\tilde{h} := \min\{h_1, h_2\}$ is the least integer m such that $\{\tilde{I}_k \; : \; k \leq m\}$ covers F_{j-2}. For F_{2j-2} (resp. F_{2j}) a similar analysis applies with respect to the two families $\{\tilde{I}_{N-3+n(N+1)} \; : \; n \geq 1\}$, $\{\tilde{I}_{j-3+n(N+1)} \; : \; n \geq 1\}$ (resp. $\{\tilde{I}_{N-1+n(N+1)} \; : \; n \geq 1\}$, $\{\tilde{I}_{j-1+n(N+1)} \; : \; n \geq 1\}$). One can readily check that $\{\tilde{I}_k \; : \; k < \tilde{h}\}$ cover both F_{2j-2} and F_{2j}. This implies $\Theta = \tilde{h}$.

Step 2. We compute T_{opt}. By (3.20) one has $q^* \leq l^* \leq q^* + 1$, thus we have the following situations:

– If $l^* = q^*$, then $h_1 < h_2$ and $\Theta = h_1$. One can check that (cf. Fig. 17a)
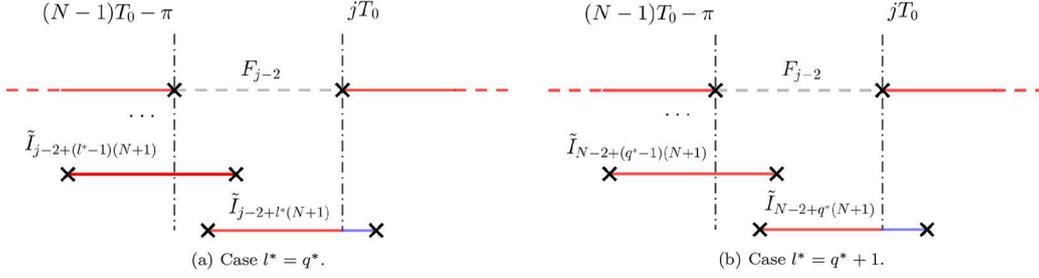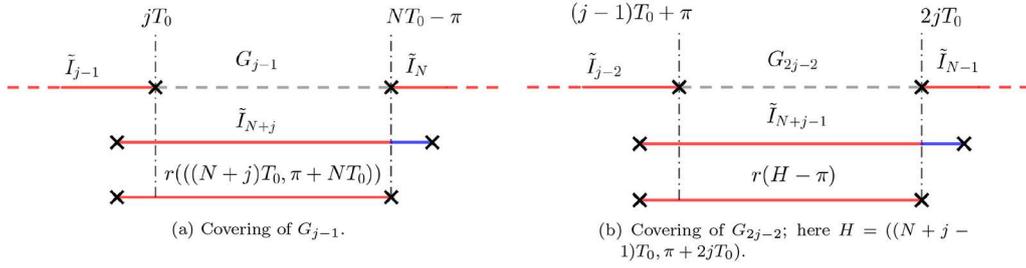
$$T_{opt} = 2\pi q^* + jT_0.$$

– If $q^* + 1 = l^*$, then $h_2 < h_1$ and $\Theta = h_2$. Thus, T_{opt} verifies (cf. Fig. 17b)

$$T_{opt} - \pi = 2\pi q^* + jT_0.$$

This closes the proof of (iii).

(iv) (Sketch). Since N is odd; then $N = 2j + 1$. Moreover, the sets $\tilde{I}_k$ with $k = 0, \ldots, N$, do not cover $[0, 2\pi)$. Namely the following subsets are not covered:

$$G_h := (T_0 + hT_0, (j+2)T_0 + hT_0 - \pi), \qquad\qquad h = 0, 2, \ldots, j-1;$$
$$G_{j+1+h'} := (\pi + 2T_0 + h'T_0, (j+3)T_0 + h'T_0), \qquad\qquad h' = 0, 2, \ldots, j-3.$$

FIGURE 17. Computation of T_{opt} in (iii).

FIGURE 18. Computation of T_{opt} in (iv) Lemma 1.11.

Note that $G_{j-1} = (jT_0, (2j+1)T_0 - \pi)$ and is nonempty. As in (ii) we have assumed that $j \geq 3$, in the case $j = 1$ the only subset to cover is $G_0 := (T_0, 3T_0 - \pi)$. Again, by a simple translation argument, we have only to study the covering of the sets G_{j-1}, G_{2j-2}. Moreover, the least time t which ensures that both sets are covered will provide the optimal control time T_{opt}. Lastly, $\pi + NT_0$ (resp. $\pi + 2jT_0$) is enough to cover G_{j-1} (resp. G_{2j-2}); see Figure 18a (resp. 18b). Since, $\pi + 2jT_0 < \pi + NT_0$, the claim is proven.

$\square$

4. DEPENDENCE OF T_{opt} ON T_0

In the situation of Section 1.1.2, by Lemmas 1.10–1.11, for each $T_0 \in (0, 2\pi) \setminus \{\frac{\pi}{2n+1} : n \in \mathbb{N}_0\}$, the optimal control time T_{opt} is well defined. Thus, we may regard T_{opt} as an application

$$T_{opt} : (0, 2\pi) \setminus \left\{ \frac{\pi}{2n+1} : n \in \mathbb{N}_0 \right\} \to \mathbb{R}, \qquad T_0 \mapsto T_{opt}, \tag{4.1}$$

and write it as $T_{opt}(T_0)$. Interestingly the formulas in Lemmas 1.9–1.11 permit us to study the behaviour of the map $T_{opt}(\cdot)$. Firstly, we consider the case $T_0 \in (\pi, 2\pi)$:

Proposition 4.1. *The map $T_{opt}(\cdot)$ on $(\pi, 2\pi)$ admits a countable set of discontinuities in*

$$\lambda_n := \frac{n+2}{n+1}\pi, \qquad n \in \mathbb{N}.$$

Moreover $\lim_{t \searrow \lambda_n} T_{opt}(t) = \pi(n+2)$ *and* $\lim_{t \nearrow \lambda_n} T_{opt}(t) = \pi(n+3)$.

Proof. The proof follows easily from Lemma 1.9. $\square$

Below, we will focus on the case $T_0 \in (0, \pi)$. We start the analysis of the map focusing on the values of $T_0 \in (0, \pi)$ for which $\lfloor \pi/T_0 \rfloor$ has a jump.

Proposition 4.2. *Let $T_0 = \pi/2n$ with $n \in \mathbb{N}$. Then $T_{opt}(T_0) = 2\pi$,*

$$\lim_{t \nearrow T_0} T_{opt}(t) = 3\pi, \tag{4.2}$$

$$\lim_{t \searrow T_0} T_{opt}(t) = 3\pi - \frac{\pi}{2n}. \tag{4.3}$$

Proof. The first assertion is the content of (i) in Lemma 1.10. Instead in order to prove (4.2), we note that if t belongs to a left neighborhood of T_0, we can write

$$t = \frac{\pi}{2n + \varepsilon},$$

for an appropriate $\varepsilon > 0$; moreover $t \nearrow T_0$ if and only if $\varepsilon \searrow 0$.

Note that

$$N(t) = 4n, \qquad j(t) = 2n,$$

where N, j are defined in (1.19). Applying (i) in Lemma 1.11, we obtain

$$T_{opt}(t) = 2\pi + 2n\frac{\pi}{2n + \varepsilon},$$

taking the limit for $\varepsilon \searrow 0$, one readily infers (4.2).

The limit in (4.3) is proved analogously. In fact let t be in a small right neighborhood of T_0. So we can write t as

$$t = \frac{\pi}{2n - \varepsilon},$$

for a small $\varepsilon > 0$; as before $t \searrow T_0$ if and only if $\varepsilon \searrow 0$. By (1.19) we have

$$N(t) = 4n - 1, \qquad j(t) = 2n - 1.$$

Since N, j are odd we can apply (iv) in Lemma 1.11. So we get

$$T_{opt}(t) = \pi + \frac{(4n-1)}{2n - \varepsilon}\pi \to \pi + \frac{(4n-1)}{2n}\pi,$$

as $\varepsilon \searrow 0$. $\qquad\square$

Lemma 1.11 and the argument used in the proof of Proposition 4.2 yield also the following results. The proof of Proposition 4.3 can be obtained easily by checking the behaviour of k^* and q^* in Lemma 1.11.

Proposition 4.3. *Let $T_0 = \pi/(2n + 1)$ with $n \in \mathbb{N}_0$. Then*

$$\lim_{t \nearrow T_0} T_{opt}(t) = +\infty, \tag{4.4}$$

$$\lim_{t \searrow T_0} T_{opt}(t) = +\infty. \tag{4.5}$$

Proposition 4.4. *Let* $T_0 = 2\pi/(2h+1)$ *with* $h \in \mathbb{N}$. *Then*
(i) if h *is even,*

$$\lim_{t \nearrow T_0} T_{opt}(t) = 3\pi + \frac{2h\pi}{2h+1}, \tag{4.6}$$

$$T_{opt}(T_0) = \lim_{t \searrow T_0} T_{opt}(t) = 2\pi + \frac{2h\pi}{2h+1}; \tag{4.7}$$

(ii) if h *is odd,*

$$T_{opt}(T_0) = \lim_{t \nearrow T_0} T_{opt}(t) = 3\pi, \tag{4.8}$$

$$\lim_{t \searrow T_0} T_{opt}(t) = 3\pi + \frac{2\pi h}{2h+1}. \tag{4.9}$$

Perhaps unexpectedly, there are other values of T_0 for which the map $T_0 \mapsto T_{opt}(T_0)$ is not continuous. For future convenience, we introduce the following notation

$$A_k = \left(\frac{\pi}{k + \frac{1}{2}}, \frac{\pi}{k}\right), \qquad B_k = \left(\frac{\pi}{k+1}, \frac{\pi}{k + \frac{1}{2}}\right), \qquad k \in \mathbb{N}.$$

Obviously

$$(0, \pi] = \left[\cup_{k \in \mathbb{N}} (A_k \cup B_k) \right] \bigcup \left[\cup_{k \in \mathbb{N}} \left\{ \frac{\pi}{k}, \frac{\pi}{k + \frac{1}{2}} \right\} \right]. \tag{4.10}$$

Note that the points in the set $\cup_{k \in \mathbb{N}} \left\{ \frac{\pi}{k}, \frac{\pi}{k + \frac{1}{2}} \right\}$ have been considered in Propositions 4.2–4.4. Next we look at the behaviour of the map $T_0 \mapsto T_{opt}(T_0)$ in the sets A_k, B_k.

Proposition 4.5. *For all* $k \in 2\mathbb{N}$ *(resp.* $k \in 2\mathbb{N}_0 + 1$), *the map* $T_0 \mapsto T_{opt}(T_0)$ *is continuous on each interval* A_k *(resp.* B_k) .

The proof of Proposition 4.5 follows straightforwardly from Lemma 1.11 when we keep in mind that: in A_k we have $N = 2k$, $j = k$, while in B_k we have $N = 2k+1$, $j = k$. Finally we explicitly list the discontinuities.

Proposition 4.6. *If* $k \in 2\mathbb{N}_0 + 1$, *in each interval* A_k, *the map* $T_{opt}(\cdot)$ *admits a countable set of discontinuities in*

$$\mu_m := \frac{\pi}{k + \left(\frac{1}{m+2}\right)}, \qquad m \in \mathbb{N}. \tag{4.11}$$

Furthermore, for each $m \in \mathbb{N}$,

$$\lim_{t \nearrow \mu_m} T_{opt}(t) = \pi + \mu_m(k(m+2) + 1),$$

$$\lim_{t \searrow \mu_m} T_{opt}(t) = \pi + \mu_m(k(m+3) + 1).$$

In particular, $\lim_{t \searrow \mu_m} T_{opt}(t) - \lim_{t \nearrow \mu_m} T_{opt}(t) = k\mu_m$.

Proposition 4.7. *For all $k \in 2\mathbb{N}$, the map $T_{opt}(\cdot)$ on each interval B_k admits a countable set of discontinuities in*

$$\xi_m := \frac{\pi}{k + \left(\frac{m+1}{m+2}\right)}, \qquad m \in \mathbb{N}. \tag{4.12}$$

Furthermore, for each $m \in \mathbb{N}$,

$$\lim_{t \nearrow \xi_m} T_{opt}(t) = \pi(m+3) + k\xi_m, \tag{4.13}$$

$$\lim_{t \searrow \xi_m} T_{opt}(t) = \pi(m+2) + k\xi_m. \tag{4.14}$$

In particular, $\lim_{t \nearrow \xi_m} T_{opt}(t) - \lim_{t \searrow \xi_m} T_{opt}(t) = \pi$.

We prove Proposition 4.7, the proof of Proposition 4.6 being similar.

Proof of Proposition 4.7. To begin, recall that $N(t) = 2k + 1$ and $j(t) = k$, for each $t \in B_k$. Thus N is odd and j is even and T_{opt} is provided by *(iii)* in Lemma 1.11. It is easy to see that the discontinuities of the map $B_k \ni t \mapsto T_{opt}(t)$ are the discontinuities of the maps

$$q^*(t) = \left\lceil \frac{t}{(2k+2)t - 2\pi} - \frac{1}{2} \right\rceil, \qquad l^*(t) = \left\lceil \frac{t}{(2k+2)t - 2\pi} \right\rceil,$$

where we have used (3.20). Thus, the discontinuities of the map $t \mapsto T_{opt}(t)$ on B_k are the solutions of the following equations

$$\frac{t}{(2k+2)t - 2\pi} = \frac{m}{2} + 1, \quad m \in \mathbb{N}. \tag{4.15}$$

With simple computations one obtains the sequence $\{\xi_m\}_{m \in \mathbb{N}}$ in (4.12). It is easy to see that $\{\xi_m\}_{m \in \mathbb{N}}$ is a decreasing sequence. To conclude, we consider two cases:

– If $m \in 2\mathbb{N}_0$ and $t \in (\xi_{m+1}, \xi_m)$ (where we set $\xi_0 := \pi/(k + \frac{1}{2})$), one has

$$\frac{m}{2} + 1 < \frac{t}{2(k+1)t - 2\pi} < \frac{m+1}{2} + 1 = \frac{m}{2} + \frac{3}{2}.$$

Thus $q^* = m/2 + 1$ and $l^* = m/2 + 2$. Lemma 1.11 *(iii)* yields

$$T_{opt}(t) = 2\pi \left(\frac{m}{2} + 1 \right) + jt + \pi = \pi(m+3) + jt.$$

– If $m \in 2\mathbb{N}_0 + 1$ and $t \in (\xi_{m+1}, \xi_m)$ we can argue as in the previous item. Set $m = 2n+1$ for some $n \in \mathbb{N}_0$. Note that

$$n + \frac{3}{2} < \frac{t}{2(k+1)t - 2\pi} < n + 2.$$

Thus $q^* = l^* = n + 2$ and Lemma 1.11 *(iii)* yields

$$T_{opt}(t) = 2\pi(n+2) + jt = \pi(m+3) + jt.$$

Combining the two cases above, one easily concludes the proof. □

5. Multidimensional case

In this section, we prove Theorem 1.13 and its extension to the BV class. We use a result proved by the authors in [1] (see Thm. 2.1 there); here we briefly recall it.

Let N be an integer and

- let $\{x_i\}_{i=0,\ldots,N}$ be an arbitrary family of points in $\mathbb{R}^d$;
- let $\{t_i\}_{i=-1,\ldots,N}$ be an increasing family of real numbers such that $t_{-1} = 0$, and define $T := t_N$;
- define

$$
\begin{aligned}
R_i &= \max\{|x - x_i| \,|\, x \in \overline{\Omega}\}, & i &= 0, \ldots, N, & &(5.1)\\
R_{i,i+1} &= |x_{i+1} - x_i|, & i &= 0, \ldots, N - 1, & &(5.2)\\
\Gamma_i &= \{x \in \partial\Omega \,|\, (x - x_i) \cdot \nu > 0\}, & i &= 0, \ldots, N. & &(5.3)
\end{aligned}
$$

Theorem 5.1. *For each real number T such that*

$$
T > R_N + \sum_{i=0}^{N-1} R_{i,i+1} + R_0, \tag{5.4}
$$

and each u weak solution to (1.1)–(1.4) with initial data $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ the following observability inequality holds

$$
\left(\max_{i=0,\ldots,N} R_i\right) \sum_{i=0}^{N} \int_{t_{i-1}}^{t_i} \int_{\Gamma_i} |\partial_\nu u|^2 d\mathcal{H}^{d-1} \, dt \geq C_T E_0, \tag{5.5}
$$

where $C_T = 2(T - R_N - \sum_{i=0}^{N-1} R_{i,i+1} - R_0)$.

Remark 5.2. Theorem 5.1 states that the system is Σ-observable (see Def. 1.12) in any time $T > R_N + \sum_{i=0}^{N} R_{i,i+1} + R_0$ with Σ as

$$
\bigcup_{i=0}^{N} \Gamma_i \times (t_{i-1}, t_i). \tag{5.6}
$$

In the proof of Theorem 1.13, we have to consider a sequence of partitions $\{P_k : k \in \mathbb{N}\}$ of the interval $[0, T]$, so we will add the upper index k in order to keep trace of the dependence on P_k; and for brevity we will set $\Sigma_\varphi^k := \Sigma_\varphi^{P_k}$.

With this preparation, we can prove Theorem 1.13.

Proof of Theorem 1.13. Let $\{t_{-1}^k, \ldots, t_{N_k}^k : 0 = t_{-1}^k < t_0^k < \ldots t_{N_k-1}^k < t_{N_k}^k = T\}$ be the partition P_k. We can apply Theorem 5.1 for $x_i = x_{\varphi,i}^k := \varphi(t_{i-1}^k)$ and $i = 0, \ldots, N_k$. Then the exact observability holds with $\Sigma = \Sigma_\varphi^k$ provided

$$
T > R_{N_k}^k + \sum_{i=0}^{N_k-1} R_{i+1,i}^k + R_0^k, \tag{5.7}
$$

where $R_i^k, R_{i+1,i}^k$ are as in (5.1)–(5.2) with $x_i = x_{\varphi,i}^k$ (*cf.* Rem. 5.2). We take a closer look at the middle term of the right hand side of (5.7):

$$\sum_{i=0}^{N_k-1} R_{i+1,i}^k = \sum_{i=0}^{N_k} |\varphi(t_i^k) - \varphi(t_{i-1}^k)| - |\varphi(t_{N_k}^k) - \varphi(t_{N_k-1}^k)|. \tag{5.8}$$

Since $\delta(P_k) \searrow 0$, invoking the smoothness hypotheses on φ, we see that the sum on the right hand side of (5.8) tends to the length of the curve

$$L(\varphi) := \int_0^T |\varphi'(t)| \, dt.$$

Furthermore, $R_0^k \equiv c_0$ since $t_{-1}^k \equiv 0$. By the continuity of φ at $t = T$, we have $R_{N_k}^k \to c_T$ and $|\varphi(t_{N_k}^k) - \varphi(t_{N_k-1}^k)| \searrow 0$, as $k \nearrow \infty$ (recall that $|t_{N_k}^k - t_{N_k-1}^k| \leq \delta(P_k) \searrow 0$).

By the previous discussion and (1.29), one sees that there exist an $M \in \mathbb{N}$ and a positive constant δ such that

$$T - R_{N_k}^k - \sum_{i=0}^{N_k-1} R_{i+1,i}^k - R_0^k \geq \delta > 0, \tag{5.9}$$

for all $k > M$. By inequality (5.5) in Theorem 5.1, we have

$$\left(\max_{i=0,\dots,N} R_i^k \right) \int_{(0,T)\times\partial\Omega} |\partial_\nu u|^2 \chi_{\Sigma_\varphi^k} \, d\mathcal{H}^{d-1} \otimes dt = \left(\max_{i=0,\dots,N} R_i^k \right) \sum_{j=0}^{N_k} \int_{t_{j-1}^k}^{t_j^k} \int_{\Gamma_{\varphi,j}} |\partial_\nu u|^2 \, d\mathcal{H}^{d-1} \, dt \geq 2\delta E_0, \tag{5.10}$$

where δ is as in (5.9) and χ_A is the indicator function of the subset $A \subset \partial\Omega \times (0,T)$. It is easy to see that

$$\max_{t\in[0,T],x\in\overline{\Omega}} |x - \varphi(t)| \geq \left(\max_{i=0,\dots,N} R_i^k \right). \tag{5.11}$$

By (1.28), we have at least for a subsequence

$$\chi_{\Sigma_\varphi^k}(x,t) \to \chi_{\Sigma_\varphi}(x,t), \qquad d\mathcal{H}^{d-1} \otimes dt - a.e. \tag{5.12}$$

By the estimate in (5.11), the uniformity of the estimate in (5.10), $\partial_\nu u \in L^2(\partial\Omega \times (0,T))$ (see Thm. 2.2) and the convergence in (5.12), the observability inequality (1.20) follows by Lebesgue dominated convergence theorem. $\qquad\square$

It is of some interest to weaken the hypotheses on the regularity of φ; see Remark 5.4.

For the reader's convenience, we recall that a function $\varphi : [0,T] \to \mathbb{R}^d$ is said to be of bounded variation (or briefly $\varphi \in BV[0,T]$) if

$$V_0^T(\varphi) := \sup \left\{ \sum_{j=0}^N |\varphi(t_j) - \varphi(t_{j-1})| \right\} < \infty, \tag{5.13}$$

where the sup is taken on all partitions $P := \{t_{-1}, \dots, t_N : 0 = t_{-1} < t_0 < \cdots < t_{N-1} < t_N := T\}$ of the interval $[0,T]$.

Theorem 5.3. *Let $\varphi \in BV[0,T]$ be left continuous at $t = T$. Suppose that there exists a sequence of partitions $\{P_k : k \in \mathbb{N}\}$ such that $\delta(P_k) \searrow 0$ as $k \nearrow \infty$ and (1.28) holds. Moreover, suppose that $T > 0$ satisfies*

$$T > c_0 + V_0^T(\varphi) + c_T. \tag{5.14}$$

Then the system is Σ_φ-observable, where Σ_φ is defined in (1.23).

Proof. The proof is an easy revision of the proof of Theorem 1.13. To avoid repetitions, we give only few remarks to stress the usefulness of the assumptions. To do so, we freely use the notation introduced in the proof of Theorem 1.13 above.

The left continuity at $t = T$ is required to ensure that $R_{N^k}^k \to c_T$ as $k \nearrow \infty$. Moreover, since $\varphi \in BV[0,T]$, one has

$$T - R_{N_k}^k - \sum_{i=0}^{N_k-1} R_{i+1,i}^k - R_0^k \geq T - R_{N_k}^k - V_0^T(\varphi) - c_0, \tag{5.15}$$

since $\sum_{i=0}^{N_k-1} R_{i+1,i}^k = \sum_{i=0}^{N_k-1} |\varphi(t_{i-1}^k) - \varphi(t_i^k)| \leq V_0^T(\varphi)$ for all $k \in \mathbb{N}$ and $c_0 \equiv R_0^k$ (recall that $t_{-1}^k \equiv 0$). By (5.14), (5.15) and $\lim_{k \to \infty} R_{N^k}^k = c_T$, then one sees that there exist $M \in \mathbb{N}$ and $\delta > 0$ such that

$$T - R_{N_k}^k - \sum_{i=0}^{N_k-1} R_{i+1,i}^k - R_0^k \geq \delta > 0, \quad \forall k > M.$$

By the previous considerations, the conclusion follows as in (5.10)–(5.12). □

We conclude this section with some remarks.

Remark 5.4. Note that Theorem 5.3 is an extension of both Theorems 1.13 and 5.1. Indeed, under the hypotheses of Theorem 1.13 (thus $\varphi \in C([0,T];\mathbb{R}^d) \cap C^1((0,T);\mathbb{R}^d)$ with finite length), we have $\varphi \in BV[0,T]$ and $V_0^T(\varphi) = \int_0^T |\varphi'(t)| \, dt$. Thus the conclusion of Theorem 1.13 follows by Theorem 5.3.

Next we show that Theorem 5.1 can be recovered from Theorem 5.3. Let $P := \{t_{-1}, \ldots, t_N : 0 = t_{-1} < t_0 < \cdots < t_{N-1} < t_N := T\}$ be as in Theorem 5.1. Define the piecewise constant function $\tilde{\varphi}(t) := x_i$ on $[t_{i-1}, t_i)$ for $i = 0, \ldots, N$ and $\tilde{\varphi}(T) = x_N$. Then $\tilde{\varphi}$ is left continuous at $t = T$ and $V_0^T(\tilde{\varphi}) = R_0 + \sum_{i=0}^{N-1} R_{i,i+1} + R_N$. Lastly, set $P_k \equiv P$ for each $k \in \mathbb{N}$. Then, the condition (1.9) is verified and the claim follows.

For a discussion of the optimality of the lower bound in (5.14) see Example 6.6 below.

Remark 5.5. Here we are not concerned with the optimal smoothness assumption on φ in Theorem 5.3. Our hypothesis $\varphi \in BV[0,T]$ is suggested by the lower bound in (5.4). Indeed, let $P = \{t_{-1}, \ldots, t_N : 0 = t_{-1} < t_0 < \cdots < t_{N-1} < t_N = T\}$ be a partition of the interval $[0,T]$ and let $x_j = \varphi(t_{j-1})$ for $j = 0, \ldots, N$ (*cf.* (1.25)). Then, the middle term on the RHS of (5.4) is naturally bounded by the total variation of the function φ in the interval $[0,T]$. This property has been exploited in the proofs of Theorems 1.13, 5.3.

6. APPLICATIONS AND FUTURE DIRECTIONS

We look here at some applications of Theorems 1.13 and 5.3. In order to state the first result we fix some notations:

– let $B(0,1) := \{x \in \mathbb{R}^3 : |x| < 1\}$ be the unit ball with center in the origin, and set $S^2 := \partial B(0,1) = \{x \in \mathbb{R}^3 : |x| = 1\}$;
– let $\Phi : [0, 2\pi] \times [0, \pi] \to S^2$ be the standard parametrization of the sphere; *i.e.*

$$\Phi(\theta, \theta') = (\cos\theta \sin\theta', \sin\theta \sin\theta', \cos\theta');$$
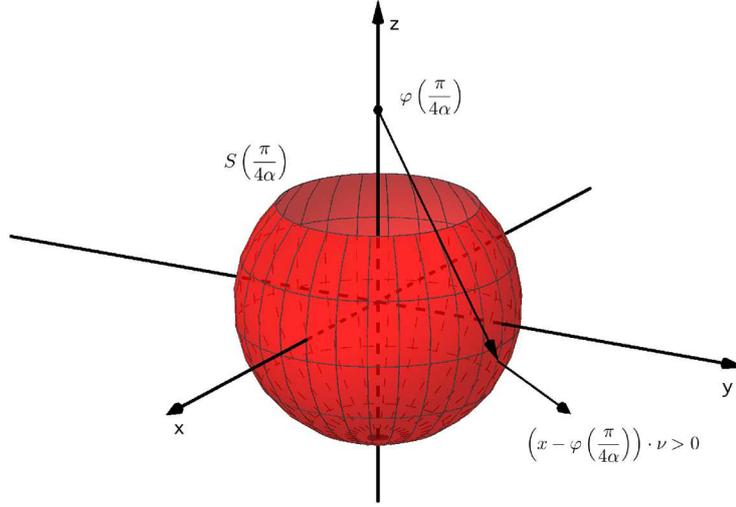
FIGURE 19. Construction of the set $S(\pi/(4\alpha))$.

– let η be a real number in $[0, 2\pi]$, then define

$$R(\eta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\eta & -\sin\eta \\ 0 & \sin\eta & \cos\eta \end{pmatrix}.$$

Corollary 6.1. *Let* $\Omega = B(0, 1)$ *and let* $\alpha \in \mathbb{R}^+$ *be a positive real number, such that*

$$\alpha < \frac{\pi}{2(1 + \sqrt{2}) + \pi\sqrt{2}}. \tag{6.1}$$

Then the system is $\mathcal{S}$*-observable in time* $T = \pi/\alpha$ *with* $\mathcal{S} = \cup_{t \in (0,T)} S(t) \times \{t\}$ *and*

$$S(t) = R\left(\alpha t - \frac{\pi}{4}\right) S, \tag{6.2}$$

$$S = \left\{ \Phi(\theta, \theta') \subset S^2 \,\middle|\, \theta \in (0, 2\pi), \, \theta' \in \left(\frac{\pi}{4}, \pi\right) \right\}. \tag{6.3}$$

Proof. Consider

$$\varphi(t) = \sqrt{2}(0, \cos(\pi/4 + \alpha t), \sin(\pi/4 + \alpha t)), \quad t \in [0, \pi/\alpha]. \tag{6.4}$$

One may check that $\Gamma_\varphi(t) = S(t)$ for each $t \in (0, T)$, where the $S(t)$ is defined in (6.2); see also Figure 19. By construction the length of φ is equal to $\sqrt{2}\pi$ (note that it is independent of α). Then Theorem 1.13 ensures the $\mathcal{S}$-exact observability provided:

$$\frac{\pi}{\alpha} = T > c_0 + L(\varphi) + c_T = 2(1 + \sqrt{2}) + \sqrt{2}\pi;$$

where we have used that $c_0 = c_T = 1 + \sqrt{2}$. Then the previous inequality is equivalent to (6.1). This concludes the proof. $\qquad\square$

Remark 6.2. We make some comments on Corollary 6.1 and we employ the notation of such corollary; in particular here φ is as in (6.4).

– The condition (6.1) can be interpreted as the requirement that the mapping "$t \mapsto \Gamma_\varphi(t)$" does not vary too fast in time. More specifically, α plays the role of the "angular velocity" of rotation of the subset S; thus small α implies small velocity of rotation. We are not aware if this result is sharp, but this behaviour seems reasonable in view of geometrical optics; see for instance [2, 9].
– It is routine to prove that for each $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$\mathcal{H}^{d-1}(\Gamma_\varphi(t)\Delta\Gamma_\varphi(t')) < \varepsilon, \qquad \text{if } |t - t'| < \delta(\varepsilon). \tag{6.5}$$

The condition in (6.5) can be viewed as a kind of continuity of the subset $\Gamma_\varphi(t)$ on the variable t, but this cannot be expected in cases when Ω is not smooth; see Example 6.4.

The second item in the previous remark gives us the opportunity of pointing out the next observation.

Remark 6.3. Let Ω and φ be such that the continuity condition (6.5) holds (*e.g.*. $B(0, 1) \subset \mathbb{R}^3$ and φ as in (6.4)), then there exists a sequence of partitions $\{P_k : k \in \mathbb{N}\}$ for which the condition (1.28) holds. To prove the claim, let $\varepsilon = 1/k$ and let $\delta_k > 0$ be such that the condition (6.5) holds true. Let $\mathbb{N} \ni K > 1/\delta_k$, define $t_j^k := (j+1)T/K$ for $j = -1, \ldots, K-1$ and $P_k := \{t_{-1}^k, \ldots, t_{K-1}^k\}$. Then, $\delta(P_k) \searrow 0$ and

$$\lim_{k\to\infty} (\mathcal{H}^{d-1} \otimes \mathcal{L}^1)(\Sigma_\varphi \Delta \Sigma_\varphi^{P_k})$$

$$\leq \lim_{k\to\infty} \sum_{i=0}^{K-1} (t_i^k - t_{i-1}^k) \sup_{t\in(t_{i-1}^k, t_i^k)} \mathcal{H}^{d-1}(\Gamma_\varphi(t)\Delta\Gamma_\varphi(t_{i-1}^k)) \leq \lim_{k\to\infty} \frac{1}{k}T = 0.$$

Therefore, in Theorem 5.3, assumption (6.5) can replace (1.28).

In the next example we study in some details an application in which (6.5) does not hold.

Example 6.4. Let $\Omega = (0, 1)^2$ be the square with unit side length and let β be a positive number; then consider the curve

$$\varphi(t) = \left(2, -\frac{1}{2} + \beta t\right), \qquad t \in \left[0, \frac{1}{\beta}\right]. \tag{6.6}$$

By definition (see (1.23) and Fig. 20) the observability set $\Sigma_\varphi \subset \partial(0,1)^2 \times (0, 1/\beta)$ can be written as

$$\Sigma_\varphi = \left(l_0 \times \left(0, \frac{1}{2\beta}\right]\right) \cup \left(l_1 \times \left(\frac{1}{2\beta}, \frac{1}{\beta}\right)\right), \tag{6.7}$$

where

$$l_0 = \{(x, y) \mid x = 0 \text{ and } y \in (0, 1), \text{ or } y = 1 \text{ and } x \in (0, 1)\},$$
$$l_1 = \{(x, y) \mid x = 0 \text{ and } y \in (0, 1), \text{ or } y \in \{0, 1\} \text{ and } x \in (0, 1)\}.$$

Furthermore, for any $t, t' \in (0, \pi)$ such that $t > 1/2 > t'$ we have

$$\mathcal{H}^{d-1}(\Gamma_\varphi(t)\Delta\Gamma_\varphi(t')) = \mathcal{H}^{d-1}(\{(x, y) \mid y = 1, x \in (0, 1)\}) = 1;$$
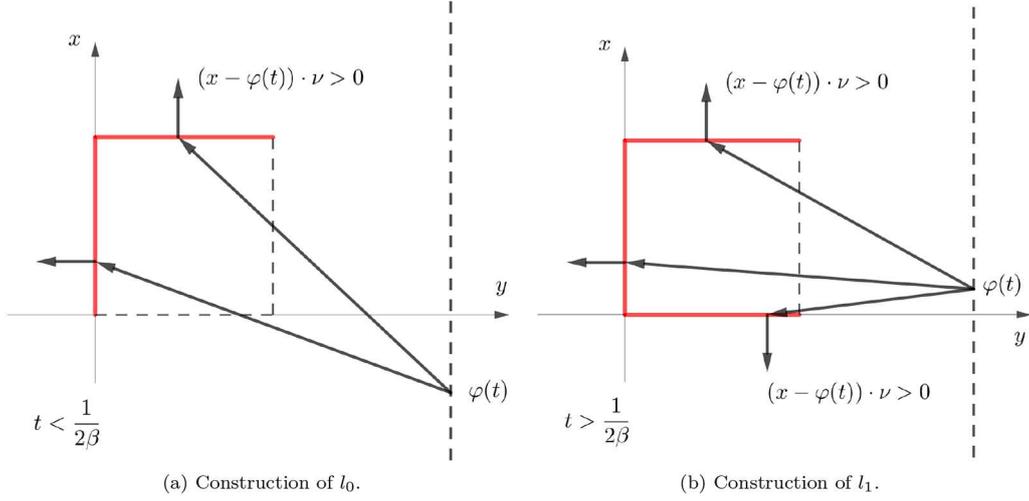
in this case we lose the continuity condition in (6.5).

(a) Construction of l_0. (b) Construction of l_1.

FIGURE 20. Construction of l_0, l_1. The red part of the boundary represents, at a certain time t (as indicated in the pictures), the set of all points x such that $(x - \varphi(t)) \cdot \nu > 0$.

Next, we note that Theorem 5.1 (or Thm. 5.3) implies the following result (for the case of a circle see [1], Sect. 4). See Figure 21 for an example of the observability set Σ in the next corollary.

Corollary 6.5 (*N-times alternating control*). *Let T and N be respectively a positive real and integer number, such that*

$$T > \sqrt{2}(N + 2), \tag{6.8}$$

and let $\{t_i\}_{i=0,\ldots,N-1}$ be any increasing finite sequence of $(0, T)$. Then the system is Σ-observable for $\Omega = (0, 1)^2$ and

$$\Sigma = \left(\bigcup_{i=0,\, i \in 2\mathbf{N}}^{N} d_0 \times (t_{j-1}, t_j) \right) \cup \left(\bigcup_{i=0,\, i \in 2\mathbf{N}+1}^{N} d_1 \times (t_{j-1}, t_j) \right),$$

where

$$d_0 = \left\{ (x, y) \in \partial\Omega \,\middle|\, x = 0 \text{ and } y \in (0, 1), \text{ or } y = 0 \text{ and } x \in (0, 1) \right\},$$
$$d_1 = \left\{ (x, y) \in \partial\Omega \,\middle|\, x = 1 \text{ and } y \in (0, 1), \text{ or } y = 1 \text{ and } x \in (0, 1) \right\},$$

Proof. Apply Theorem 5.1 with $x_i \equiv x_0 = (1, 1)$ if i is even or $x_i \equiv x_1 = (0, 0)$ otherwise. □

For a general discussion on the choice of the $\{t_i\}_{i=0,\ldots,N-1}$ see ([1], Sect. 2.1). Let us specialize Corollary 6.5 to the case
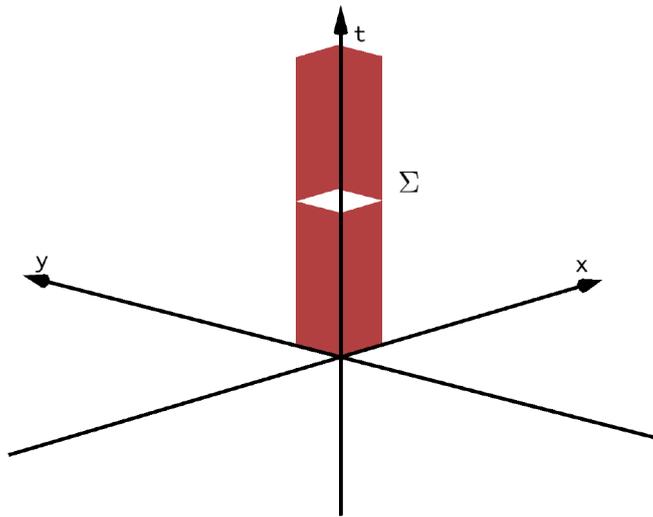
$$t_k = (k + 1)T_0, \qquad k = -1, \ldots, N, \tag{6.9}$$

FIGURE 21. Example of Σ as in Corollary 6.5 when $N = 1$, $t_0 = 7\sqrt{2}/4$ and $t_1 = 7\sqrt{2}/2$.

thus $T := t_N = (N + 1)T_0$. From (6.8) we obtain

$$(N + 1)T_0 > \sqrt{2}(N + 2),$$

that leads us to a lower bound for T_0

$$T_0 > \sqrt{2}\,\frac{N + 2}{N + 1} > \sqrt{2}. \tag{6.10}$$

In the situation just introduced, we note that:

- In the fixed case for the square (*e.g.*. the control acts only on the subset d_0 for each time) the controllability holds for any $T > 2\sqrt{2}$, by this for $\sqrt{2} < T_0 \leq 2\sqrt{2}$ Corollary 6.5 cannot be deduced by the fixed controllability case.
- The lower bound in (6.10) is inherent in the application of the multiplier method in the proof of the observability inequality. In the one dimensional case instead we can allow for arbitrarily small T_0 (see Sect. 1.1.2). The one dimensional case suggests that the limitation in (6.10) is due to methodology of proof and not to an intrinsic obstruction of the problem.

  It is of our interest to know if there exists a sequence $\{T_{0,h}\}_{h\in\mathbb{N}}$, such that $T_{0,h} \searrow 0$ as $h \nearrow \infty$ and for all $h \in \mathbb{N}$ the system of Corollary 6.5 is exactly observable.

Up to now we have not discussed the optimality of the lower bound in (1.29) provided in Theorem 1.13; or more generally the optimality of (5.14) in Theorem 5.3. The next example shows that the lower bound for T in (5.14) is not optimal in general.

Example 6.6. Consider $\Omega = (0, 1)^2$, take any $\varphi : [0, T] \to \mathbb{R}^2$ continuous and piecewise differentiable of *positive length*, such that $\varphi((0, T)) \subset \Omega$ and $\varphi(0) = \varphi(T) = 0$. From the convexity of Ω, it follows that

$$\Gamma_\varphi(t) \equiv \partial\Omega, \qquad \forall t \in (0, T).$$

So (5.14), or equivalently (1.29), provides

$$T > 2\sqrt{2} + L(\varphi). \tag{6.11}$$

It is known (see [6] or [10]) that the system is exactly observable for any $T > \sqrt{2}$, so the inequality in (6.11) is not optimal.

Besides this, it is unknown if the lower bound (5.14) in Theorem 5.3 is optimal for specific domain Ω and observability set Σ; *e.g.*. the ones given in Corollary 6.1.

7. Conclusion

In this article we extend some classical results concerning the exact boundary observability of the wave equation.

In the one dimensional case, we provide a complete treatment of the observability problem. Indeed, Theorem 1.3 and Corollary 1.4 completely characterize the boundary observability in terms on how the subsets of observation cover the one dimensional sphere S^1; see the condition in (1.10) or in (1.11).

In the multidimensional case, we work out sufficient conditions for exact boundary observability; see Theorem 1.13 or more generally Theorem 5.3. The proofs of these results are based on the multiplier method (see [10] or [6]), which is used to prove the observability in case of constant observation, *i.e.* when the observation subset is $\Sigma \equiv \Gamma \times (0, T)$ for some $\Gamma \subset \partial\Omega \times (0, T)$. Our results extends the classical ones even in the case of piecewise constant subset of observation (see Rem. 5.4).

The applicability of our results is exemplified in Section 6, where we also list some perspective developments. We recall here the issues of alternating control with arbitrarily small switch time T_0, and the optimality of the controllability time T, both in the multidimensional case.

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