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# CONTROLLABILITY OF LOW REYNOLDS NUMBERS SWIMMERS OF CILIATE TYPE

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**Abstract.** We study the locomotion of a ciliated microorganism in a viscous incompressible fluid. We use the Blake ciliated model: the swimmer is a rigid body with tangential displacements at its boundary that allow it to propel in a Stokes fluid. This can be seen as a control problem: using periodical displacements, is it possible to reach a given position and a given orientation? We are interested in the minimal dimension $d$ of the space of controls that allows the microorganism to swim. Our main result states the exact controllability with $d = 3$ generically with respect to the shape of the swimmer and with respect to the vector fields generating the tangential displacements. The proof is based on analyticity results and on the study of the particular case of a spheroidal swimmer.

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## 1. Introduction

The aim of this article is to analyze the controllability of a system associated with a model of micro-swimmers. The swimmers considered here are ciliated microorganisms immersed in a viscous incompressible fluid. We use the Blake ciliated model [4, 5]: we assume that the shape of the swimmer is fixed, and we replace the propelling mechanism of the cilia by time periodic tangential displacements. Due to the micro-scale of the swimmer (very low Reynolds numbers), the inertial forces are neglected and in particular, the fluid motion is governed by the steady-state Stokes system. For more details about this model, we refer the reader to [4, 5, 14, 26, 27, 37]. An important property of the corresponding system is that it can be rewritten as a finite-dimensional nonlinear control problem and this permits the use of the geometric controllability theory. Such an approach is classical and comes back to [2, 33]. In the case of very high Reynolds numbers, one can assume that the fluid is potential and this leads also to a finite-dimensional nonlinear controlled system that can again be studied with the geometric controllability theory: see [7] for one of the first results in that direction.

The study done here follows the works of San *et al.* [33], Sigalotti and Vivalda [36], where a similar model is considered. In this first model, the swimming mechanism is modeled by a tangential velocity which is unrelated to a tangential displacement. If we impose that this tangential velocity comes from a boundary displacement, the problem is more complicated and is only tackled in San *et al.* [34]. In this last work, only axi-symmetric

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swimmers are considered and the control problem corresponds to a motion of the swimmer along the axis of symmetry.

Let us mention several other classes of swimmers which have been tackled in the literature. Apart ciliated swimmers, let us mention, among them, the three link swimmer introduced in [32], the three sphere swimmer introduced in [31] and for which the controllability has been shown in [2] (its extension, the $n$-sphere swimmer has been first studied in [3]). Another swimming mechanism consists in small deformations at its boundary. Such a model is considered in [24, 25]. Let us also mention some other related works: the case where the fluid is inviscid and potential leads to a very close theory see [11–13, 29].

In this paper, we deal with swimmers of arbitrary shape and our aim is to control all the rigid motions of the swimmer, *i.e.* its position and orientation by using boundary displacements. Our first contribution is to give a model for such a swimming mechanism and to write the corresponding fluid-swimmer system as a suitable controlled system. Let us briefly explain how the boundary displacement of the swimmer is built. First of all, the shape of the swimmer is defined as the image of the unit sphere $\mathbb{S}^2$ of $\mathbb{R}^3$ by some diffeomorphism $\mathrm{Id} + \Psi_0$ of $\mathbb{R}^3$. The displacement on the boundary is obtained from $d$ vector fields $\delta_1, \cdots, \delta_d$ of $\mathbb{S}^2$ from which we build the map $\mathcal{X}_\delta(s) : y \in \mathbb{S}^2 \mapsto \exp_y\left(\sum_{i=1}^d s_i \delta_i(y)\right) \in \mathbb{S}^2$. For $s \in \mathbb{R}^d$ small enough, $\mathcal{X}_\delta(s)$ is a diffeomorphism of $\mathbb{S}^2$. The definition of the exponential map on manifolds can be found, for instance, in [22, 23, 28]. The main result (Thm. 2.8) states that for $d \geqslant 3$ and generically with respect to $\Psi_0$ and $\delta$, the swimmer is controllable, *i.e.* any rigid position of the swimmer can be tracked and reached. Let us emphasize that here, we only need $d = 3$ elementary deformations. This is a novelty compared to other controllability results, see for instance [3, 24], where four elementary deformations are required to fully control the rigid position of the swimmer. Let us also point out the works [1, 18] where less than four elementary deformations are required. Nevertheless, in these works, the fluid is only present on half of the space $\mathbb{R}^3$, and they enrich the reachable set using the boundary effects. Finally, let us also quote that due to the scallop theorem [32] (see also [30]), it is known that at least two elementary deformations are required to control the swimmer. We believe that our result holds true for $d = 2$ (see Rem. 4.6) but the proof should follow a different method. Indeed, we use here explicit formulas for the Stokes system in the exterior of a ball. Unfortunately because of symmetry properties of the sphere, it seems that for such a shape, we need $d \geqslant 3$. In order to reach $d = 2$, we would need to remove such a symmetry, but in that case the difficulty would be to compute the solutions of the Stokes system.

This article is organized as follows. In Section 2, we introduce the model corresponding to the ciliate locomotion. We introduce in particular the velocity fields $\delta = (\delta_1, \ldots, \delta_d)$ that generates the tangential displacement. The shape of the microorganism is parametrized by a transformation of the unit sphere of $\mathbb{R}^3$ through a diffeomorphism $\mathrm{Id}_{\mathbb{R}^3} + \Psi_0$. The corresponding fluid-structure system is written in (2.9). We also give the main result (Thm. 2.8), that is the exact controllability for $d \geqslant 3$ and generically with respect to $\Psi_0$ and $\delta$. We rewrite the fluid–structure system in Section 3 so that we can apply general results from the geometric controllability theory and in particular the Rashevsky–Chow theorem. In order to use such a theorem, we need to compute the Lie brackets associated with the controls. These Lie brackets involve in particular several Stokes systems with non-homogeneous Dirichlet boundary conditions. Consequently, to obtain explicit formulas we particularize the problem in Section 4 by considering the case where the shape of the swimmer is a ball. In that case, using a classical work of Brenner [6], we can consider particular cases for $\delta = (\delta_1, \ldots, \delta_d)$ and show that for $d = 3$, there exists a choice such that the system is controllable. Using analytical properties of the system, with respect to $\Psi_0$, $\delta$ and $s$, we can then prove the main result of this paper (Thm. 2.8).

## 2. The model and the main results

This section is organized as follows. First we introduce some notation in Section 2.1. Then we present in Section 2.2 the swimmer mechanism, that is how we model the boundary displacements as a diffeomorphism of the boundary of the swimmer. We couple this with the Stokes system and with the Newton laws to obtain our model in Section 2.3. Finally, we state the main results, in particular Theorem 2.8, in Section 2.4.

### 2.1. Notation

We first give some notations used throughout the article.

- $|\cdot|$ stands for the Euclidean norm on $\mathbb{R}^d$ or on $\mathcal{M}_3(\mathbb{R})$.
- Given $k \in \mathbb{N}$, $C_0^k(\mathbb{R}^3)$ is defined by

$$C_0^k(\mathbb{R}^3) = \left\{ f \in C^k(\mathbb{R}^3) \;\mid\; \lim_{|x|\to\infty} |\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3} f(x)| = 0\,,\ \forall\, \alpha_1,\alpha_2,\alpha_3 \in \mathbb{N} \text{ s.t. } \alpha_1+\alpha_2+\alpha_3 \leqslant k \right\}.$$

  This is a Banach space when endowed with the norm:

$$\|f\|_{C_0^k(\mathbb{R}^3)} = \sum_{\substack{\alpha_1,\alpha_2,\alpha_3\in\mathbb{N}\,,\\ \alpha_1+\alpha_2+\alpha_3\leqslant k}} \sup_{x\in\mathbb{R}^3} \left|\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3} f(x)\right|.$$

  We also set $C_0^\infty(\mathbb{R}^3) = \bigcap_{k=0}^{\infty} C_0^k(\mathbb{R}^3)$.
- Given $k \in \mathbb{N}^* \cup \{\infty\}$, $\mathcal{D}_0^k$ is the connected component of

$$\left\{ f \in C_0^k(\mathbb{R}^3)^3 \;\mid\; \mathrm{Id}_{\mathbb{R}^3} + f \text{ is a } C^1\text{-diffeomorphism of } \mathbb{R}^3 \right\}$$

  containing 0.
- For a $C^\infty$-manifold $\mathcal{M}$, $\mathrm{T}\mathcal{M}$ is the tangent bundle of $\mathcal{M}$ and $C^k(\mathcal{M},\mathrm{T}\mathcal{M})$ is the set of $k$-differentiable tangent vector fields of $\mathcal{M}$.

### 2.2. The swimmer mechanism

Before entering in the core of this section, let us say some words about the model construction. First of all, we assume that the shape of the swimmer is diffeomorphic (by a diffeomorphism $\mathrm{Id} + \Psi_0$) to the unit ball $B_0$ of $\mathbb{R}^3$, and its boundary is thus diffeomorphic to the unit sphere $\mathbb{S}^2$ of $\mathbb{R}^3$.

Then, we consider boundary displacements of the swimmer associated with a tangential velocity. To simplify, we consider the tangential velocity on $\mathbb{S}^2$ and construct the corresponding boundary displacements also on the sphere. Such a displacement should be a diffeomorphism and this leads to some constraints on the tangential velocity. Then we apply $\mathrm{Id} + \Psi_0$ to obtain a boundary displacement of the swimmer, and we immerse it in the Stokes fluid where it can move through rigid motions. This mechanism is summarized in Figure 1.

Note that in the case of an axi-symmetric swimmer moving along its axis of symmetry, a similar model is considered in [26] and [34]. Nevertheless, in these articles the diffeomorphism $\mathrm{Id} + \Psi_0$ is explicit (prolate spherical coordinates) and it is easier to write the boundary displacement from the tangential velocity.

For any $k \in \mathbb{N}^* \cup \{\infty\}$ and any $\Theta \in C^k(\mathbb{S}^2, \mathrm{T}\mathbb{S}^2)$, we can consider the mapping

$$\mathcal{X} : \mathbb{S}^2 \to \mathbb{S}^2, \quad y \mapsto \cos(|\Theta(y)|)y + \operatorname{sinc}(|\Theta(y)|)\Theta(y). \tag{2.1}$$

Here, we recall that sinc is the cardinal sine function. If $\Theta \equiv 0$, then $\mathcal{X} = \mathrm{Id}_{\mathbb{S}^2}$. Formula (2.1) comes from the exponential formula $\mathcal{X} = \exp(\Theta)$ in the case of $\mathbb{S}^2$ (see for instance [22, 23, 28]). Expanding the sine and cosine functions, one can see using the above expression and [38] that for every $k \in \mathbb{N}^*$, $\Theta \in C^k(\mathbb{S}^2, \mathrm{T}\mathbb{S}^2) \mapsto \mathcal{X} \in C^k(\mathbb{S}^2)$

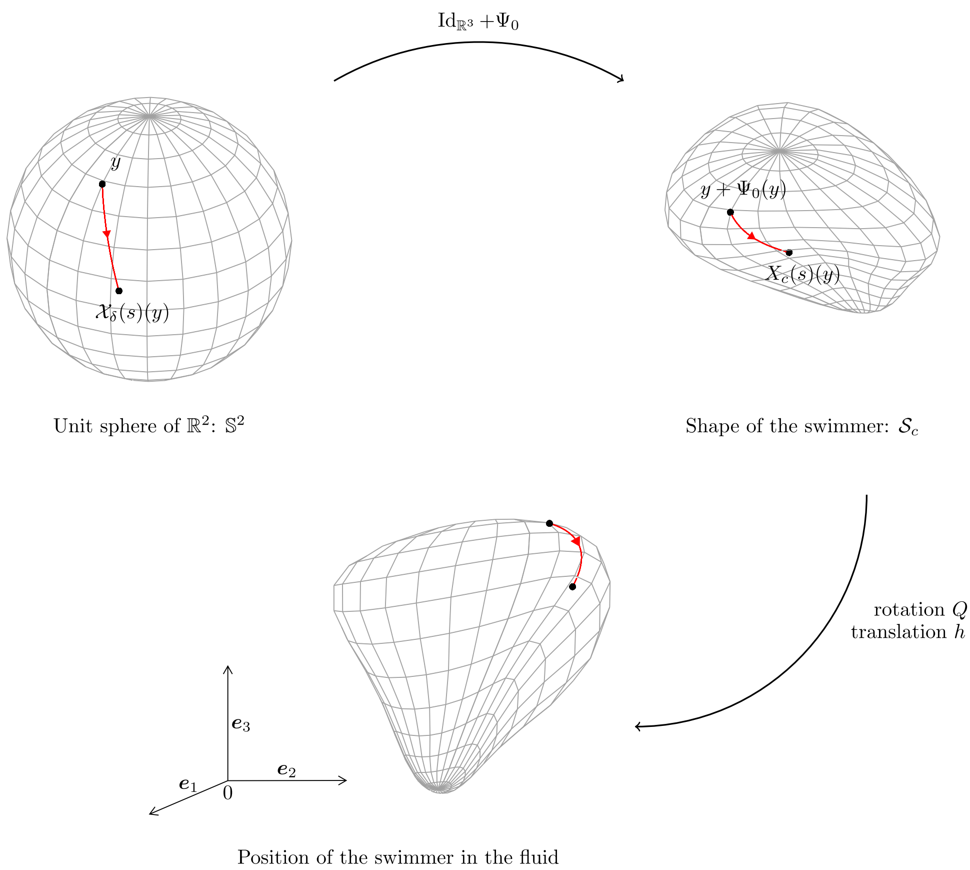


FIGURE 1. Swimmer model and swimming mechanism.

is analytic. In fact, we have,

$$\begin{aligned}\mathcal{X} = \cos|\Theta|\,\mathrm{Id}_{\mathbb{S}^2} + \sin|\Theta|\,\frac{\Theta}{|\Theta|} &= \sum_{n=0}^{\infty}(-1)^n\frac{|\Theta|^{2n}}{(2n)!}\,\mathrm{Id}_{\mathbb{S}^2} + \sum_{n=0}^{\infty}(-1)^n\frac{|\Theta|^{2n}}{(2n+1)!}\,\Theta \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left((2n+1)\,\mathrm{Id}_{\mathbb{S}^2} + \Theta\right)|\Theta|^{2n}\,. \end{aligned} \tag{2.2}$$

Let us now consider for $k \in \mathbb{N}^* \cup \{\infty\}$ and $d \in \mathbb{N}^*$, $\delta = (\delta_1, \ldots, \delta_d) \in C^k\left(\mathbb{S}^2, \mathrm{T}\mathbb{S}^2\right)^d$. For any $s = (s_1, \ldots, s_d) \in \mathbb{R}^d$, we write

$$\Theta_\delta(s) := \sum_{j=1}^{d} s_j\delta_j \tag{2.3}$$

and we consider the mapping $\mathcal{X}_\delta(s)$ obtained from (2.1) with $\Theta = \Theta_\delta(s)$:

$$\mathcal{X}_\delta(s) : \mathbb{S}^2 \to \mathbb{S}^2, \quad y \mapsto \cos\left(\left|\sum_{j=1}^{d} s_j\delta_j(y)\right|\right) y + \operatorname{sinc}\left(\left|\sum_{j=1}^{d} s_j\delta_j(y)\right|\right) \sum_{k=1}^{d} s_k\delta_k(y).$$

We denote by $\tilde{\mathcal{J}}(\delta)$ the set of $s \in \mathbb{R}^d$ such that $\mathcal{X}_\delta(s)$ is a diffeomorphism of $\mathbb{S}^2$ and by $\mathcal{J}(\delta)$ the connected component of $\tilde{\mathcal{J}}(\delta)$ containing 0. We have the following standard result which proof is postponed in Appendix B.

**Lemma 2.1.** *Given $k \in \mathbb{N}^* \cup \{\infty\}$, $d \in \mathbb{N}^*$ and $\delta = (\delta_1, \cdots, \delta_d) \in C^k(\mathbb{S}^2, \mathrm{T}\mathbb{S}^2)^d$, $\tilde{\mathcal{J}}(\delta)$ and $\mathcal{J}(\delta)$ are nonempty open subsets of $\mathbb{R}^d$.*

**Remark 2.2.** In particular, there exists $\varepsilon = \varepsilon(\delta) > 0$ such that for every $s \in \mathbb{R}^d$, with $|s| < \varepsilon$, $\mathcal{X}_\delta(s)$ is a $C^k$-diffeomorphism of $\mathbb{S}^2$.

For every $d \in \mathbb{N}^*$ and every $k \in \mathbb{N}^* \cup \{\infty\}$, we write (see Sect. 2.1 for the definition of $\mathcal{D}_0^k$)

$$\mathcal{SC}^k(d) = \mathcal{D}_0^k \times C^k(\mathbb{S}^2, \mathrm{T}\mathbb{S}^2)^d.$$

This is a subset of the Banach space $C_0^k(\mathbb{R}^3)^3 \times C^k(\mathbb{S}^2, \mathrm{T}\mathbb{S}^2)^d$. We use the notation $\|\cdot\|$ for the norm of $C_0^k(\mathbb{R}^3)^3 \times C^k(\mathbb{S}^2, \mathrm{T}\mathbb{S}^2)^d$:

$$\|(\Psi_0, \delta)\| = \|\Psi_0\|_{C_0^k(\mathbb{R}^3)^3} + \|\delta\|_{C^k(\mathbb{S}^2, \mathrm{T}\mathbb{S}^2)^d}.$$

The elements $c = (\Psi_0, \delta)$ of $\mathcal{SC}^k(d)$ characterize the swimmer: $\Psi_0$ corresponds to the shape of the swimmer

$$\mathcal{S}_c := \left(\mathrm{Id}_{\mathbb{R}^3} + \Psi_0\right)\left(\mathbb{S}^2\right) \tag{2.4}$$

and $\delta$ corresponds to the swimming mechanism. Our main result will be generic with respect to these *swimmer characteristics* in the topology of $C_0^k(\mathbb{R}^3)^3 \times C^k(\mathbb{S}^2, \mathrm{T}\mathbb{S}^2)^d$: for any given swimmer characteristics $\bar{c}$, one can find swimmer characteristics $c$ arbitrary close to $\bar{c}$ that allows the swimmer to control its trajectory.

For every $c = (\Psi_0, \delta) \in \mathcal{SC}^k(d)$, we define the global boundary displacement of the swimmer $X_c$ by

$$X_c(s) = \left(\mathrm{Id}_{\mathbb{R}^3} + \Psi_0\right) \circ \mathcal{X}_\delta(s) \qquad (s \in \mathcal{J}(\delta)). \tag{2.5}$$

For every $s \in \mathcal{J}(\delta)$, $X_c(s)$ is a $C^k$-diffeomorphism from $\mathbb{S}^2$ onto $\mathcal{S}_c$. Our aim is to find a time dependent function $s : \mathbb{R}^+ \to \mathcal{J}(\delta)$ so that the boundary displacement $X_c(s(t))$, $t \in \mathbb{R}^+$ can control the position of the swimmer.

In what follows, we add the following constraint on $s$: there exists $T > 0$ so that

$$s(kT) = s(0) \quad (k \in \mathbb{N}). \tag{2.6}$$

Such a constraint is natural for the swimming mechanism and allows us to focus on a "cycle", that is $t \in [0, T]$. We are thus lead to consider function $s$ such that $s(T) = s(0)$. We can see that this constraint does not play any role in our controllability result.

In what follows, in order to establish our generic result with respect to the characteristics of the swimmer, we need to introduce the subset of $\mathcal{SC}^k(d) \times \mathbb{R}^d$ corresponding to the points $(c, s)$ such that $X_c(s)$ is a $C^k$-diffeomorphism from $\mathbb{S}^2$ onto $\mathcal{S}_c$. To this end, we set

$$\mathcal{A}^k(d) = \left\{((\Psi_0, \delta), s) \in \mathcal{SC}^k(d) \times \mathbb{R}^d \mid s \in \mathcal{J}(\delta)\right\}, \tag{2.7}$$

where $d \in \mathbb{N}^*$ and where $k \in \mathbb{N}^* \cup \{\infty\}$. We have the following result on $\mathcal{A}^k(d)$ (we postpone the proof in Appendix B).

**Proposition 2.3.** *For any $d \in \mathbb{N}^*$ and $k \in \mathbb{N}^*$, $\mathcal{A}^k(d)$ is a connected open set of $C_0^k(\mathbb{R}^3)^3 \times C^k(\mathbb{S}^2, \mathrm{T}\mathbb{S}^2)^d \times \mathbb{R}^d$. The set $\mathcal{A}^\infty(d)$ is dense in $\mathcal{A}^k(d)$. Furthermore, $(c, s) \in \mathcal{A}^k(d) \mapsto X_c(s) \in C^k(\mathbb{S}^2, \mathbb{R}^3)$ is analytic.*

**Remark 2.4.** In particular, assume $k \in \mathbb{N}^* \cup \{\infty\}$, $d \in \mathbb{N}^*$, $\overline{c} \in \mathcal{SC}^k(d)$, $T > 0$ and that $\overline{s} \in C^0([0,T])^d$ satisfies $(\overline{c}, \overline{s}(t)) \in \mathcal{A}^k(d)$ for all $t \in [0,T]$. Then from the above proposition and from a compactness argument, there exists $\varepsilon > 0$ such that for every $c \in \mathcal{SC}^k(d)$ and for every $s \in C^0([0,T])^d$ satisfying

$$\|\overline{c} - c\| + \|\overline{s} - s\|_{C^0([0,T])^d} < \varepsilon,$$

we have $(c, s(t)) \in \mathcal{A}^k(d)$ for all $t \in [0,T]$.

### 2.3. Fluid–structure interactions and motion of the swimmer

Immersed into a viscous incompressible fluid, the swimmer described in the above section can translate and rotate. We write for $Q \in \mathcal{SO}(3)$ and $h \in \mathbb{R}^3$,

$$X^\dagger(h,Q,s)(y) := QX_c(s)(y) + h \quad (y \in \mathbb{S}^2) \quad \text{and} \quad \mathcal{S}^\dagger(h,Q) := Q\mathcal{S}_c + h.$$

We also denote by $\mathcal{F}_c \subset \mathbb{R}^3$ (respectively by $\mathcal{F}^\dagger(h,Q)$) the unbounded connected component of $\mathbb{R}^3 \setminus \mathcal{S}_c$ (respectively $\mathbb{R}^3 \setminus \mathcal{S}^\dagger(h,Q)$). These correspond to fluid domains.

A point on the surface of the swimmer can be parametrized as follows

$$x = X^\dagger(h,Q,s)(y) \qquad (y \in \mathbb{S}^2).$$

Assume that $(h,Q,s)$ is a $C^1$ function with respect to the time. Then the velocity of the above point $x$ is:

$$v^\dagger(t,x) = \dot{Q}Q^\top(x-h) + \dot{h} + Q\frac{\mathrm{d}\, X_c(s)}{\mathrm{d}t}\left(X_c(s)^{-1}\left(Q^\top(x-h)\right)\right).$$

Here and in what follows, $\cdot^\top$ denotes the matrix transposition and the dot above a function means its time derivative.

The system describing the motion of the swimmer is given by the following system:

$$-\Delta u^\dagger + \nabla p^\dagger = 0 \qquad \text{in } \mathcal{F}^\dagger(h,Q), \tag{2.8a}$$

$$\operatorname{div} u^\dagger = 0 \qquad \text{in } \mathcal{F}^\dagger(h,Q), \tag{2.8b}$$

$$u^\dagger(t,x) = v^\dagger(t,x) \qquad \text{on } \mathcal{S}^\dagger(h,Q), \tag{2.8c}$$

$$\lim_{|x|\to\infty} u^\dagger(t,x) = 0\,, \tag{2.8d}$$

$$\int_{\mathcal{S}^\dagger(h,Q)} \sigma(u^\dagger,p^\dagger)\mathbf{n}\,\mathrm{d}\Gamma = 0\,, \tag{2.8e}$$

$$\int_{\mathcal{S}^\dagger(h,Q)} (x-h) \times \sigma(u^\dagger,p^\dagger)\mathbf{n}\,\mathrm{d}\Gamma = 0\,, \tag{2.8f}$$

where $\mathbf{n}$ is the unit outer normal to $\partial\mathcal{F}^\dagger(h,Q)$ and where we have used the notation

$$\sigma(u^\dagger,p^\dagger) := 2\mathrm{D}(u^\dagger) - p^\dagger \mathrm{I}_3\,, \qquad \mathrm{D}(u^\dagger) := \frac{1}{2}\left(\nabla u + (\nabla u)^\top\right).$$

The functions $u^\dagger$ and $p^\dagger$ are respectively the velocity and the pressure of the fluid. Equations (2.8a) and (2.8b) are the Stokes system, (2.8c) corresponds to the no-slip boundary condition. Finally, (2.8e) and (2.8f) are the Newton laws with the hypotheses that the inertial effects can be neglected.

We then perform a change of variable to work in a referential attached to the swimmer: we set

$$u^\dagger(t,x) = Q(t)u\left(t, Q(t)^\top(x-h(t))\right), \qquad p^\dagger(t,x) = p\left(t, Q(t)^\top(x-h(t))\right),$$

$$\ell(t) = Q(t)^\top \dot{h}(t), \qquad Q(t)^\top \dot{Q}(t) = \mathbb{A}(\omega(t)),$$

where for any $\omega \in \mathbb{R}^3$,

$$\mathbb{A}(\omega) := \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}.$$

After some calculation (see, for instance, [35]), we obtain the following system:

$$-\Delta u + \nabla p = 0 \qquad \text{in } \mathcal{F}_c, \tag{2.9a}$$

$$\operatorname{div} u = 0 \qquad \text{in } \mathcal{F}_c, \tag{2.9b}$$

$$u(t,x) = \ell(t) + \omega(t) \times x + \frac{\mathrm{d}\, X_c(s)}{\mathrm{d}t}\left(X_c(s)^{-1}(x)\right) \qquad \text{on } \mathcal{S}_c, \tag{2.9c}$$

$$\lim_{|x|\to\infty} u(t,x) = 0\,, \tag{2.9d}$$

$$\int_{\mathcal{S}_c} \sigma(u,p)\mathbf{n}\,\mathrm{d}\Gamma = 0\,, \tag{2.9e}$$

$$\int_{\mathcal{S}_c} x \times \sigma(u,p)\mathbf{n}\,\mathrm{d}\Gamma = 0\,, \tag{2.9f}$$

$$\dot{h} = Q\ell\,, \tag{2.9g}$$

$$\dot{Q} = Q\mathbb{A}(\omega)\,. \tag{2.9h}$$

In what follows, we will rewrite the above system as a finite-dimensional dynamical system with state $(h, Q, s)$ and control input $\dot{s}$, see Section 3, equations (3.5) and (3.9).

## 2.4. Main results

In order to state our main result, we first give the definition of a solution:

**Definition 2.5.** Assume $T > 0$, $k \in \mathbb{N} \cup \{\infty\}$ with $k \geqslant 2$, $d \in \mathbb{N}^*$ and consider $c = (\Psi_0, \delta) \in \mathcal{SC}^k(d)$. Suppose also $s \in W^{1,1}(0,T)^d$ is such that $X_c(s(t))$ is invertible for all $t \in [0,T]$. We say that $(u, p, \ell, \omega, h, Q)$ is a solution of (2.9) if

$$(h, Q) \in W^{1,1}(0,T;\mathbb{R}^3 \times SO(3)), \quad (\ell, \omega) \in L^1(0,T;\mathbb{R}^3 \times \mathbb{R}^3),$$
$$(u,p) \in L^1\left(0,T; \left(D^{1,2}(\mathcal{F}_c)^3 \cap H^2_{loc}(\overline{\mathcal{F}_c})^3\right) \times \left(L^2(\mathcal{F}_c) \cap H^1_{loc}(\overline{\mathcal{F}_c})\right)\right)$$

and if they satisfy the equations of (2.9) almost everywhere or in the trace sense.

We have used the notation $D^{l,q}$ for the homogeneous Sobolev spaces (see [17]). In particular

$$D^{1,2}(\mathcal{F}_c) = \left\{ f \in L^2_{loc}(\mathcal{F}_c)\,,\ \nabla f \in L^2(\mathcal{F}_c)^3 \right\}.$$

We refer to Proposition 3.3 for the well-posedness of the system. Our main result (Thm. 2.8) ensures that generically with respect to the swimmer characteristics $c$, the system (2.9) is controllable. Let us precise our definition of controllability:

**Definition 2.6.** We say that (2.9) is controllable in time $T > 0$ for $c = (\Psi_0, \delta) \in \mathcal{SC}^k(d)$ if for any

$$h_0, h_1 \in \mathbb{R}^3, \quad Q_0, Q_1 \in SO(3),$$

there exists $s \in W^{1,1}(0,T)^d$ such that $X_c(s(t))$ is invertible for $t \in [0,T]$, such that

$$s(0) = s(T)$$

and such that the solution of (2.9) with

$$h(0) = h_0\,, \quad Q(0) = Q_0,$$

satisfies

$$h(T) = h_1\,, \quad Q(T) = Q_1.$$

**Remark 2.7.** This says in particular that for a swimmer of shape $\mathcal{S}_c$ given by (2.4), there exist periodic boundary displacements associated with the tangential velocities $\delta$ so that the swimmer can control its position $h$ and its orientation $Q$. The invertibility of $X_c(s(t))$ can be written as $s(t) \in \widetilde{\mathcal{J}}(\delta)$. The condition $s(0) = s(T)$ corresponds to the idea that the swimmer is repeating this mechanism periodically (see (2.6)).

The main result states a better property (tracking) that the controllability: for a given trajectory, there exist boundary displacements associated with the tangential velocities $\delta$ so that the position of the swimmer can remain arbitrary close to the trajectory.

**Theorem 2.8.** *Given $d \geqslant 3$, $\varepsilon, \eta > 0$, $\overline{c} = (\overline{\Psi}_0, \overline{\delta}) \in \mathcal{SC}^2(d)$, $T > 0$ and $(\overline{h}, \overline{Q}, \overline{s}) \in C^0\big([0,T]; \mathbb{R}^3 \times \mathcal{SO}(3) \times \mathcal{J}(\overline{\delta})\big)$.*

*There exists $c = (\Psi_0, \delta) \in \mathcal{SC}^\infty(d)$ such that*

$$\|c - \overline{c}\| < \varepsilon\,,$$

*and there exists $s \in C^\infty([0,T]; \mathbb{R}^d)$, with*

$$s(t) \in \mathcal{J}(\delta)\,, \qquad s(0) = \overline{s}(0)\,, \qquad s(T) = \overline{s}(T) \quad \textit{and} \quad |s(t) - \overline{s}(t)| \leqslant \eta \qquad (t \in [0,T]),$$

*such that the corresponding solution $(h, Q)$ of (2.9) with initial conditions*

$$h(0) = \overline{h}(0)\,, \quad Q(0) = \overline{Q}(0)$$

*satisfies*

$$\sup_{t \in [0,T]} \big(|h(t) - \overline{h}(t)| + |Q(t) - \overline{Q}(t)|\big) < \eta$$

*together with*

$$h(T) = \overline{h}(T)\,, \quad Q(T) = \overline{Q}(T).$$

The proof of this theorem in given in Section 4.5.

**Remark 2.9.** Let us point out that we obtain a tracking property (and the controllability) not only for the position $(h, Q)$ but also for the boundary displacement (that is for $s \in \mathcal{J}(\delta)$).

We have in particular the following corollary.

**Corollary 2.10.** *Assume $d \geqslant 3$, $\varepsilon > 0$, $\overline{c} \in \mathcal{SC}^2(d)$, and $T > 0$. There exists $c \in \mathcal{SC}^\infty(d)$ with*

$$\|c - \overline{c}\| < \varepsilon \,,$$

*such that (2.9) is controllable in time $T$ for the swimmer characteristics $c$.*

*In particular the set of swimmer characteristics $c$ such that the system (2.9) is controllable is an open dense set of $\mathcal{SC}^2(d)$.*

**Remark 2.11.** In our opinion, the most important point of our above results is that the controllability can be achieved with only $d = 3$ controls. As already mentioned in the introduction, the existing swimming controllability results were obtained for $d \geqslant 4$ controls (see for instance [3, 24]).

Based on the above theorem, we can also derive the existence of optimal controls. We refer to [13] for similar optimal control problems.

**Theorem 2.12.** *Given $d \geqslant 3$ and $c = (\Psi_0, \delta) \in \mathcal{SC}^2(d)$ such that the system (2.9) is controllable and set $\Lambda$ a compact of $\mathbb{R}^d$ containing $0$ in its interior and $K$ a compact set of $\mathcal{J}(\delta)$ which is connected by $C^1$-arcs and has a nonempty interior. Let $g \in C^0(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathcal{SO}(3) \times \mathbb{R}^d \times \mathbb{R}^d; \mathbb{R})$ such that $g$ is convex with respect to the fifth variable.*

*Given $(h^0, Q^0), (h^1, Q^1) \in \mathbb{R}^3 \times \mathcal{SO}(3)$ and $s^0$, $s^1 \in K$, we have:*

1. *there exists $T_* > 0$ such that for every $T > T_*$, the optimal control problem*

$$\min \int_0^T g\big(t, h(t), Q(t), s(t), \dot{s}(t)\big)\,\mathrm{d}t$$
$$\left|\begin{array}{lll} s \in W^{1,\infty}(0,T)^d, & & \\ s(t) \in K & (t \in [0,T]), & \\ \dot{s}(t) \in \Lambda & (t \in (0,T) \text{ a.e.}), & \\ (h,Q) \text{ solution of } (2.9), & & \\ h(0) = h^0, & Q(0) = Q^0, & s(0) = s^0, \\ h(T) = h^1, & Q(T) = Q^1, & s(T) = s^1. \end{array}\right. \tag{2.10}$$

   *admits a solution;*
2. *the time optimal control problem*

$$\min\ T$$
$$\left|\begin{array}{lll} T > 0\,, & & \\ s \in W^{1,\infty}(0,T)^d, & & \\ s(t) \in K & (t \in [0,T]), & \\ \dot{s}(t) \in \Lambda & (t \in (0,T) \text{ a.e.}), & \\ (h,Q) \text{ solution of } (2.9), & & \\ h(0) = h^0, & Q(0) = Q^0, & s(0) = s^0, \\ h(T) = h^1, & Q(T) = Q^1, & s(T) = s^1. \end{array}\right. \tag{2.11}$$

   *admits a solution.*

*Proof.* Let us scratch the proof for the first optimal control problem, that is (2.10).

We apply the Filippov theorem ([10], Thm. 9.3.i, p. 314) and its extension, see ([10], Sect. 9.5, p. 318 ). One can check that conditions ((a), (b), (c), p. 317 of [10]) are fulfilled with the above hypotheses.

The only hypothesis that needs to be checked carefully is the existence of an admissible control, *i.e.* that there exists a triplet $(h, Q, s)$ satisfying the constraints of (2.10). To this end, we are going to construct a trajectory on the time interval $[0, 1]$ satisfying the constraint on $s$. First of all, since $K$ is connected by $C^1$-arcs and since the interior of $K$ is nonempty, there exist a point $\overline{s}$ in the interior of $K$ and two $C^1$-arcs $\widehat{s}_0 : [0, 1/3] \to K$ and $\widehat{s}_1 : [2/3, 1] \to K$ such that

$$\widehat{s}_0(0) = s^0, \quad \widehat{s}_0(1/3) = \overline{s}, \quad \widehat{s}_1(2/3) = \overline{s}, \quad \widehat{s}_1(1) = s^1.$$

Let us then define $(\widehat{h}^0, \widehat{Q}^0) \in \mathbb{R}^3 \times \mathcal{SO}(3)$ the final value of the solution of (2.9) in $[0, 1/3]$ with initial condition $(h^0, Q^0)$ and control $\widehat{s}_0$. Similarly, we define $(\widehat{h}^1, \widehat{Q}^1) \in \mathbb{R}^3 \times \mathcal{SO}(3)$ the initial condition such that the solution of (2.9) in $[2/3, 1]$ with initial condition (at $2/3$) $(\widehat{h}^1, \widehat{Q}^1)$ and control $\widehat{s}_1$ reaches $(h^1, Q^1)$ at the final time (such a construction can be obtained by time reversion).
We conclude, using Theorem 2.8, together with the fact that $\overline{s}$ is in the interior of $K$, that there exists a control $\widehat{s}_{1/2} \in C^1([1/3, 2/3]; \mathbb{R}^d)$ steering $(\widehat{h}^0, \widehat{Q}^0)$ to $(\widehat{h}^1, \widehat{Q}^1)$ and such that $\widehat{s}_{1/2}(t) \in K$ for every $t \in [1/3, 2/3]$.
All in all, by concatenation of $\widehat{s}_0$, $\widehat{s}_{1/2}$ and $\widehat{s}_1$, we have built a control $\widehat{s} \in W^{1,\infty}(0, 1; \mathbb{R}^d)$ steering $(h^0, Q^0)$ to $(h^1, Q^1)$ and such that $\widehat{s}(t) \in K$ for every $t \in [0, 1]$.
Nevertheless, the property $\mathrm{d}\widehat{s}(t)/\mathrm{d}t \in \Lambda$ may not hold. For $T > 0$, we take the control $s(t) = \widehat{s}(t/T)$ and we see that $s(t) \in K$ for every $t \in [0, T]$ and this control steers $(h^0, Q^0)$ to $(h^1, Q^1)$ in time $T$. Furthermore, we have $\sup_{[0,T]} |\dot{s}| = \frac{1}{T} \sup_{[0,1]} |\mathrm{d}\widehat{s}/\mathrm{d}t|$. Since $\widehat{s} \in W^{1,\infty}(0, 1; \mathbb{R}^d)$ and since $0$ is an interior point of $\Lambda$, we conclude that for $T$ larger than some $T_*$ (depending on $\widehat{s}$ and $\Lambda$), $s$ is an admissible control.

For the time optimal control problem, namely (2.11), the proof is similar and relies on ([10], Thm. 9.2.i, p. 311) and its extensions. □

## 3. Rewriting the system

This section is devoted to rewrite system (2.9) as a nonlinear finite-dimensional control problem (system (3.9) below) and to compute Lie brackets that will be useful to apply the Rashevsky–Chow theorem.
From now on, we assume $k \geqslant 2$. It is used in the regularity of the solution of the Stokes system.

### 3.1. Decomposition of the system

In this paragraph, we follow the classical decomposition of low Reynolds number swimmers, see for instance the pioneer work [2]. Given $d \in \mathbb{N}^*$, $k \geqslant 2$ and $c \in \mathcal{SC}^k(d)$, let us first expand (2.9c). To this end, we define

$$D_c^i(s) = \partial_{s_i} X_c(s) \circ X_c(s)^{-1} \qquad (i \in \{1, \cdots, d\}, (c, s) \in \mathcal{A}^k(d)) \tag{3.1}$$

so that for any solution $(h, Q, s)$ of (2.1), the relation (2.9c) writes

$$u(t, x) = \sum_{i=1}^{3} \ell_i(t) \boldsymbol{e}_i + \sum_{i=1}^{3} \omega_i(t) \left(\boldsymbol{e}_i \times x\right) + \sum_{i=1}^{d} \dot{s}_i(t) D_c^i(s(t))(x) \qquad (x \in \mathcal{S}_c),$$

where $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ is the canonical basis of $\mathbb{R}^3$.
This leads us to consider the following Stokes systems

$$\begin{aligned} -\Delta u_c^i + \nabla p_c^i &= 0 && \text{in } \mathcal{F}_c,\\ \operatorname{div} u_c^i &= 0 && \text{in } \mathcal{F}_c,\\ u_c^i &= \boldsymbol{e}_i && \text{on } \mathcal{S}_c,\\ \lim_{|x|\to\infty} u_c^i(x) &= 0, \end{aligned} \qquad (i \in \{1,2,3\}), \tag{3.2a}$$

$$\begin{aligned} -\Delta u_c^i + \nabla p_c^i &= 0 && \text{in } \mathcal{F}_c,\\ \operatorname{div} u_c^i &= 0 && \text{in } \mathcal{F}_c,\\ u_c^i &= \boldsymbol{e}_{i-3} \times x && \text{on } \mathcal{S}_c,\\ \lim_{|x|\to\infty} u_c^i(x) &= 0, \end{aligned} \qquad (i \in \{4,5,6\}), \tag{3.2b}$$

$$\begin{aligned} -\Delta v_c^i + \nabla q_c^i &= 0 && \text{in } \mathcal{F}_c,\\ \operatorname{div} v_c^i &= 0 && \text{in } \mathcal{F}_c,\\ v_c^i &= D_c^i(s) && \text{on } \mathcal{S}_c,\\ \lim_{|x|\to\infty} v_c^i(x) &= 0, \end{aligned} \qquad (i \in \{1,\cdots,d\}). \tag{3.2c}$$

Notice that $v_c^i$ and $q_c^i$ are also functions of $s$. In (3.2), the pairs $(u_c^i, p_c^i)$ and $(v_c^i, q_c^i)$ are well-defined in $\left(D^{1,2}(\mathcal{F}_c)^3 \cap H^2_{loc}(\overline{\mathcal{F}_c})^3\right) \times \left(L^2(\mathcal{F}_c) \cap H^1_{loc}(\overline{\mathcal{F}_c})\right)$, where $D^{1,2}(\mathcal{F}_c) = \left\{f \in L^2_{loc}(\mathcal{F}_c)\,,\ \nabla f \in L^2(\mathcal{F}_c)^3\right\}$. We refer to ([17], Lem. V.1.1, p. 305, Thm. V.1.1, p. 306) for the well-posedness of the exterior Stokes problem.

Then

$$u = u(\ell, \omega, \lambda, s) := \sum_{i=1}^3 \ell_i u_c^i + \sum_{i=1}^3 \omega_i u_c^{i+3} + \sum_{i=1}^d \dot{s}_i v_c^i(s)$$

satisfies (2.9a)–(2.9c). In that case, (2.9e) and (2.9f) can also be rewritten. Indeed, after an integration by parts and using ([17], Thm. V.3.2, p. 314), we find

$$\int_{\mathcal{S}_c} \boldsymbol{e}_i \cdot \sigma(u,p)\mathbf{n}\,\mathrm{d}\Gamma = 2\int_{\mathcal{F}_c} \mathrm{D}(u) : \mathrm{D}(u_c^i)\,\mathrm{d}x$$

and

$$\int_{\mathcal{S}_c} (\boldsymbol{e}_i \times x) \cdot \sigma(u,p)\mathbf{n}\,\mathrm{d}\Gamma = 2\int_{\mathcal{F}_c} \mathrm{D}(u) : \mathrm{D}(u_c^{i+3})\,\mathrm{d}x,$$

where $\mathbf{n}$ is the unit outer normal to $\partial\mathcal{F}_c$.

We define the matrices $K_c \in \mathcal{M}_6(\mathbb{R})$ and $N_c(s) \in \mathcal{M}_{6,d}(\mathbb{R})$ by:

$$K_c = 2\left(\int_{\mathcal{F}_c} \mathrm{D}(u_c^i) : \mathrm{D}(u_c^j)\,\mathrm{d}x\right)_{i,j\in\{1,\ldots,6\}} \quad \text{and} \quad N_c(s) = -2\left(\int_{\mathcal{F}_c} \mathrm{D}(u_c^i) : \mathrm{D}(v_c^j)\,\mathrm{d}x\right)_{\substack{i\in\{1,\ldots,6\}\\ j\in\{1,\ldots,d\}}}, \tag{3.3}$$

so that relations (2.9e) and (2.9f) are equivalent to

$$K_c \begin{pmatrix} \ell \\ \omega \end{pmatrix} = N_c(s)\dot{s}\,. \tag{3.4}$$

The following result holds (see [24]).

**Lemma 3.1.** *Given $k \geqslant 2$, the mapping $(c, s) \in \mathcal{A}^k(d) \mapsto (K_c, N_c(s)) \in \mathcal{M}_6(\mathbb{R}) \times \mathcal{M}_{6,d}(\mathbb{R})$ is analytic and for every $c$, $K_c$ is positive definite.*

We recall that $\mathcal{A}^k(d)$ is defined by (2.7). We refer to [38] for the definitions and properties of analytic functions in Banach spaces.

Finally, (2.9) writes

$$\dot{h} = Q\ell, \tag{3.5a}$$
$$\dot{Q} = Q\mathbb{A}(\omega), \tag{3.5b}$$
$$\dot{s} = \lambda, \tag{3.5c}$$
$$\begin{pmatrix} \ell \\ \omega \end{pmatrix} = K_c^{-1} N_c(s)\lambda. \tag{3.5d}$$

In the above set of equations, we have introduced the new control variable $\lambda = \dot{s}$. In fact, since we also want to impose some conditions on $s$, we put $s$ in the state of the system and the control of this extended system is $\lambda$. This also shows that (2.9) is a finite-dimensional nonlinear dynamical system with control $\lambda$.

## 3.2. Formulation of the system in a Lie group

Let us define:

$$P(h, Q, s) = \begin{pmatrix} \begin{pmatrix} Q & h \\ 0 & 1 \end{pmatrix} & \begin{matrix} 0 & 0 \end{matrix} \\ \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} \mathrm{I}_d & s \\ 0 & 1 \end{matrix} \end{pmatrix} \in \mathcal{M}_{d+5}(\mathbb{R}) \qquad ((h, Q, s) \in \mathbb{R}^3 \times \mathcal{M}_3(\mathbb{R}) \times \mathbb{R}^d)$$

and

$$E(3, d) = \left\{ P(h, Q, s)\,,\ (h, Q, s) \in \mathbb{R}^3 \times \mathcal{SO}(3) \times \mathbb{R}^d \right\} \subset \mathcal{GL}_{d+5}(\mathbb{R}).$$

Notice that the map $P : \mathbb{R}^3 \times \mathcal{SO}(3) \times \mathbb{R}^d \to E(3, d)$ is a bijection. In addition, endowed with the matrix product, $E(3, d)$ is a Lie group whose Lie algebra is:

$$\mathfrak{e}(3, d) = \left\{ \mathfrak{p}(\ell, \omega, \lambda),\ (\ell, \omega, \lambda) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^d \right\},$$

with,

$$\mathfrak{p}(\ell, \omega, \lambda) = \mathfrak{q}(\ell, \mathbb{A}(\omega), \lambda)$$

and with $$\mathfrak{q}(\ell, M, s) = \begin{pmatrix} \begin{pmatrix} M & \ell \\ 0 & 0 \end{pmatrix} & \begin{matrix} 0 & 0 \end{matrix} \\ \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} 0 & \lambda \\ 0 & 0 \end{matrix} \end{pmatrix} \in \mathcal{M}_{d+5}(\mathbb{R}) \qquad ((\ell, \omega, \lambda) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^d,\ M \in \mathcal{M}_3(\mathbb{R})).$$

Clearly, $\mathfrak{p} : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^d \to \mathfrak{e}(3,d)$ is a bijection.

Let us finally define:

$$I(Q) = P(0,Q,0) \in E(3,d) \qquad (Q \in \mathcal{SO}(3)),$$

so that we have:

$$\mathrm{T}_{(h,Q,s)}\left(\mathbb{R}^3 \times \mathcal{SO}(3) \times \mathbb{R}^d\right) = \mathfrak{q}^{-1}\big(I(Q)\mathfrak{e}(3,d)\big)\,.$$

Let us define for every $j \in \{1,\dots,d\}$,

$$V_c^j(s) = \begin{pmatrix} \ell_c^j(s) \\ \omega_c^j(s) \end{pmatrix} = K_c^{-1} N_c(s) \boldsymbol{e}_j \qquad ((c,s) \in \mathcal{A}^k(d)), \tag{3.6}$$

the $j$th column of $K_c^{-1}N_c(s)$, with $\{\boldsymbol{e}_j\}_{j\in\{1,\dots,d\}}$ the canonical basis of $\mathbb{R}^d$. With such a notation, (3.5d) becomes,

$$\begin{pmatrix} \ell \\ \omega \end{pmatrix} = \sum_{j=1}^{d} \lambda_j \begin{pmatrix} \ell_c^j \\ \omega_c^j \end{pmatrix}.$$

Let us also define:

$$\tilde{\mathfrak{f}}_c^j(s) = \mathfrak{p}\left(\ell_c^j(s), \omega_c^j(s), \boldsymbol{e}_j\right) \in \mathfrak{e}(3,d) \qquad \text{and} \qquad f_c^j(h,Q,s) = I(Q)\tilde{\mathfrak{f}}_c^j(s) \in \mathrm{T}_{P(h,Q,s)}E(3,d), \tag{3.7}$$

with $\mathrm{T}_{(h,Q,s)}E(3,d)$, the tangent space of $E(3,d)$ at the point $P(h,Q,s)$.
Based on Lemma 3.1 we obtain the following lemma.

**Lemma 3.2.** *Given $k \geqslant 2$ and $d \in \mathbb{N}^*$, for every $j \in \{1,\cdots,d\}$, the map $((c,s),h,Q) \in \mathcal{A}^k(d) \times \mathbb{R}^3 \times \mathcal{SO}(3) \mapsto f_c^j(h,Q,s) \in \mathcal{M}_{d+5}(\mathbb{R})$ is analytic.*

*Proof.* From Lemma 3.1 and the definition (3.6), we deduce that

$$(c,s) \in \mathcal{A}^k(d) \mapsto (\ell_c^j(s), \omega_c^j(s)) \in \mathbb{R}^6$$

is analytic. Using that $\mathfrak{p}$ and $P$ are linear maps, we deduce the result. ☐

Relation (3.5) now reads:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} P(h,Q,s) &= \begin{pmatrix} \begin{pmatrix} \dot{Q} & \dot{h} \\ 0 & 0 \end{pmatrix} & 0 & 0 \\ 0 & 0 & \dot{s} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} Q\mathbb{A}(\omega) & Q\ell \\ 0 & 0 \end{pmatrix} & 0 & 0 \\ 0 & 0 & \lambda \\ 0 & 0 & 0 \end{pmatrix} = I(Q)\mathfrak{p}(\ell,\omega,\lambda) \\ &= I(Q)\sum_{j=1}^{d} \mathfrak{p}(\ell_c^j, \omega_c^j, \boldsymbol{e}_j)\lambda_j = \sum_{j=1}^{d} I(Q)\tilde{\mathfrak{f}}_c^j(s)\lambda_j = \sum_{j=1}^{d} f_c^j(h,Q,s)\lambda_j \end{aligned} \tag{3.8}$$

and can also be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} h \\ Q \\ s \end{pmatrix} = \sum_{j=1}^{d} \mathfrak{q}^{-1}\left(f_c^j(h,Q,s)\right)\lambda_j\,. \tag{3.9}$$

From [24], Proposition 1.6 (see also [16]), we deduce.

**Proposition 3.3.** *Let $k \in \mathbb{N}\cup\{\infty\}$ with $k \geqslant 2$, $d \in \mathbb{N}^*$, $(\psi_0, \delta) \in \mathcal{SC}^k(d)$, $T > 0$ and $\lambda \in L^1_{loc}(\mathbb{R}_+)^d$ (respectively $\lambda \in C^{p-1}(\mathbb{R}_+)^d$, $p \in \mathbb{N}^*$).*

*Then for every $(h_0, Q_0, s_0) \in \mathbb{R}^3 \times \mathcal{SO}(3) \times \mathcal{J}(\delta)$, the system (3.9) endowed with the initial condition $(h, Q, s)(0) = (h_0, Q_0, s_0)$ and control $\lambda$ admits a unique maximal solution $(h, Q, s)$ which is absolutely continuous (respectively of class $C^p$).*

*Furthermore, if for every $t \in [0, T]$, $s(t) = s_0 + \int_0^t \lambda(\tau)\,\mathrm{d}\tau$ belongs to $\mathcal{J}(\delta)$, then the solution $(h, Q, s)$ of (3.9) endowed with the initial condition $(h, Q, s)(0) = (h_0, Q_0, s_0)$ is well-defined on $[0, T]$.*

### 3.3. Lie brackets computations

Let us now compute the Lie brackets of the system (3.8). We have

$$\partial_h f^i_c(h, Q, s) \cdot \tilde{h} = 0\,, \quad \partial_s f^i_c(h, Q, s) \cdot \boldsymbol{e}_j = I(Q)\partial_{s_j}\tilde{\mathfrak{f}}^i_c(s) \quad \text{and} \quad \partial_Q f^i_c(h, Q, s) \cdot (Q\mathbb{A}(\omega)) = I(Q)\mathfrak{p}(0, \omega, 0)\tilde{\mathfrak{f}}^i_c(s).$$

In order to make relations shorter, we set $\partial_{i_1,\dots,i_n}$ for $\partial_{s_{i_1}} \cdots \partial_{s_{i_n}}$. This notation will be used all along the article.

For $i, j \in \{1, \cdots, d\}$, we have

$$\begin{aligned}
[f^j_c, f^i_c] &= \partial_{(h,Q,s)} f^i_c \cdot f^j_c - \partial_{(h,Q,s)} f^j_c \cdot f^i_c \\
&= I(Q) \left(\partial_j \tilde{\mathfrak{f}}^i_c - \partial_i \tilde{\mathfrak{f}}^j_c + \mathfrak{p}(0, \omega^j_c, 0)\tilde{\mathfrak{f}}^i_c - \mathfrak{p}(0, \omega^i_c, 0)\tilde{\mathfrak{f}}^j_c\right) \\
&= I(Q) \left(\mathfrak{p}\left(\partial_j \ell^i_c - \partial_i \ell^j_c, \partial_j \omega^i_c - \partial_i \omega^j_c, 0\right) + \begin{pmatrix} \mathbb{A}(\omega^j_c)\mathbb{A}(\omega^i_c) - \mathbb{A}(\omega^i_c)\mathbb{A}(\omega^j_c) & \mathbb{A}(\omega^j_c)\ell^i_c - \mathbb{A}(\omega^i_c)\ell^j_c & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) \\
&= I(Q) \left(\mathfrak{p}\left(\partial_j \ell^i_c - \partial_i \ell^j_c, \partial_j \omega^i_c - \partial_i \omega^j_c, 0\right) + \mathfrak{p}\left(\omega^j_c \times \ell^i_c - \omega^i_c \times \ell^j_c, \omega^j_c \times \omega^i_c, 0\right)\right) \\
&= I(Q)\, \tilde{\mathfrak{p}}\left(\partial_j V^i_c - \partial_i V^j_c + V^j_c \wedge V^i_c\right),
\end{aligned} \tag{3.10}$$

reminding that $V^i_c = \begin{pmatrix} \ell^i_c \\ \omega^i_c \end{pmatrix}$ and where we have defined

$$\begin{aligned}
\begin{pmatrix} \ell^j_c \\ \omega^j_c \end{pmatrix} \wedge \begin{pmatrix} \ell^i_c \\ \omega^i_c \end{pmatrix} &= \begin{pmatrix} \omega^j_c \times \ell^i_c - \omega^i_c \times \ell^j_c \\ \omega^j_c \times \omega^i_c \end{pmatrix} \in \mathbb{R}^3 \times \mathbb{R}^3 \qquad (\ell^i_c, \ell^j_c, \omega^i_c, \omega^j_c \in \mathbb{R}^3) \\
\text{and} \qquad \tilde{\mathfrak{p}}\left(\begin{pmatrix} \ell \\ \omega \end{pmatrix}\right) &= \mathfrak{p}(\ell, \omega, 0) \qquad (\ell, \omega \in \mathbb{R}^3).
\end{aligned} \tag{3.11}$$

Let us also express the third order Lie brackets which will be useful in the following. With a similar computation to the one done in (3.10), we have, for every $i, j, k \in \{1, \cdots, d\}$,

$$\begin{aligned}
[f^k_c, [f^j_c, f^i_c]] = I(Q)\, \tilde{\mathfrak{p}}\Big(&\partial_k\left(\partial_j V^i_c - \partial_i V^j_c\right) + \partial_k V^j_c \wedge V^i_c + V^j_c \wedge \partial_k V^i_c \\
&+ V^k_c \wedge \left(\partial_j V^i_c - \partial_i V^j_c + V^j_c \wedge V^i_c\right)\Big).
\end{aligned} \tag{3.12}$$

By induction, we can show that the Lie brackets of elements of $\left\{f^1_c, \cdots, f^d_c\right\}$ have the form

$$I(Q)\, \tilde{\mathfrak{p}}(V),$$

where $V$ depends only on $\left\{\tilde{\mathfrak{f}}^1_c, \cdots, \tilde{\mathfrak{f}}^d_c\right\}$. This implies two properties: the dimension of the Lie algebra generated by the family $\left\{f^1_c, \cdots, f^d_c\right\}$ does not depend on the position $(h, Q) \in \mathbb{R}^3 \times SO(3)$ of the swimmer. We can thus compute the Lie brackets only for $h = 0$ and $Q = I_3$. The second property comes from the definition (3.11) of $\tilde{\mathfrak{p}}$. We see that the Lie brackets are always included in the following subspace of $\mathfrak{e}(3,d)$:

$$\left\{\mathfrak{p}(\ell,\omega,0)\,,\ (\ell,\omega)\in\mathbb{R}^3\times\mathbb{R}^3\right\},$$

of dimension 6. If we can generate this subspace with the Lie brackets of $\left\{f_c^1,\cdots,f_c^d\right\}$, then, using the form (3.7) of $f_c^j$, we obtain that the Lie algebra generated by $\left\{f_c^1,\cdots,f_c^d\right\}$ is $\mathfrak{e}(3,d)$. In the following lemma, we summarize the above properties, and we particularize the second one in the case of Lie brackets of order 2 or 3.

**Lemma 3.4.** *Let $d \geqslant 2$, $(h,Q)\in\mathbb{R}^3\times\mathcal{SO}(3)$ and $(c,s)\in\mathcal{A}^\infty(d)$ then we have*

$$\dim \mathrm{Lie}_{(h,Q,s)}\left\{f_c^1,\cdots,f_c^d\right\} = \dim \mathrm{Lie}_{(0,\mathrm{I}_3,s)}\left\{f_c^1,\cdots,f_c^d\right\} \qquad ((h,Q)\in\mathbb{R}^3\times\mathcal{SO}(3)). \tag{3.13}$$

*In particular, if*

$$\dim \mathrm{Span}\left\{\tilde{\mathfrak{p}}^{-1}([f_c^j,f_c^i](0,\mathrm{I}_3,s))\,,\ i,j\in\{1,\cdots,d\}\right\} = 6 \tag{3.14}$$

*or if*

$$\dim\left(\mathrm{Span}\left\{\tilde{\mathfrak{p}}^{-1}([f_c^j,f_c^i](0,\mathrm{I}_3,s)),\tilde{\mathfrak{p}}^{-1}([f_c^k,[f_c^j,f_c^i]](0,\mathrm{I}_3,s)),\ i,j,k\in\{1,\cdots,d\}\right\}\right) = 6, \tag{3.15}$$

*then the Lie algebra generated by the family $\left\{f_c^1,\cdots,f_c^d\right\}$ is maximal:*

$$\mathrm{Lie}_{(0,\mathrm{I}_3,s)}\left\{f_c^1,\cdots,f_c^d\right\} = \mathfrak{e}(3,d).$$

Note that (3.13) ensures that the dimension of the Lie algebra evaluated at a point $(h,Q,s)$ is independent of $h\in\mathbb{R}^3$ and $Q\in\mathcal{SO}(3)$. In addition, using the analyticity of $\left\{f_c^1,\cdots,f_c^d\right\}$ with respect to $s$, we will also show in the following lemma that the dimension of the Lie algebra is also independent of $s$, that is to say that the map $(h,Q,s)\in\mathbb{R}^3\times\mathcal{SO}(3)\times\mathcal{J}(\delta)\mapsto \dim\mathrm{Lie}_{(h,Q,s)}\left\{f_c^1,\cdots,f_c^d\right\}\in\mathbb{N}$ is a constant map. In particular, it is sufficient to compute the dimension of the Lie algebra generated by the vector fields $\{f_c^1,\cdots,f_c^d\}$ at the point $(h,Q,s)=(0,\mathrm{I}_3,0)$. This is our strategy in what follows (see Sect. 4.4).

**Lemma 3.5.** *Let $d\geqslant 2$ and $c=(\Psi_0,\delta)\in\mathcal{SC}^\infty(d)$. Then for any $(h,Q,s)\in\mathbb{R}^3\times\mathcal{SO}(3)\times\mathcal{J}(\delta)$*

$$\dim \mathrm{Lie}_{(h,Q,s)}\left\{f_c^1,\cdots,f_c^d\right\} = \dim \mathrm{Lie}_{(0,\mathrm{I}_3,0)}\left\{f_c^1,\cdots,f_c^d\right\}.$$

*Proof.* Since $\mathcal{J}(\delta)$ is open and connected, there exists a $C^1$-path, $\overline{s}:[0,1]\to\mathcal{J}(\delta)$ joining 0 to $s$. By simply taking the control $\lambda = \mathrm{d}\overline{s}/\mathrm{d}t$, the solution of (3.9) with initial condition $(0,\mathrm{I}_3,0)$ is at time 1 at the position $(h_1,Q_1,s)$ for some $h_1\in\mathbb{R}^3$ and $Q_1\in\mathcal{SO}(3)$. Now, using the analyticity of $\left\{f_c^1,\cdots,f_c^d\right\}$ with respect to $(h,Q,s)$, we can apply Hermann–Nagano theorem ([20], Thm. 6, p. 48) and deduce that the dimension of the Lie algebra generated by $\left\{f_c^1,\cdots,f_c^d\right\}$ is constant in an orbit. This yields

$$\dim \mathrm{Lie}_{(0,\mathrm{I}_3,0)}\left\{f_c^1,\cdots,f_c^d\right\} = \dim \mathrm{Lie}_{(h_1,Q_1,s)}\left\{f_c^1,\cdots,f_c^d\right\}.$$

Finally, using (3.13), we deduce.

$$\dim \mathrm{Lie}_{(h_1,Q_1,s)}\left\{f_c^1,\cdots,f_c^d\right\} = \dim \mathrm{Lie}_{(h,Q,s)}\left\{f_c^1,\cdots,f_c^d\right\},$$

for any $(h,Q)\in\mathbb{R}^3\times\mathcal{SO}(3)$. This ends the proof. $\square$

To obtain the dimension of the Lie algebra generated by $f_c^1, \cdots, f_c^d$, we will compute the associated Lie brackets. To this end, one has to compute the derivatives of $s \mapsto V_c^i(s)$, where $V_c^i$ is defined by (3.6). That is to say that we have to compute:

$$\begin{aligned}\partial_s^\alpha N(s)\boldsymbol{e}_j &= -2\left(\partial_s^\alpha\left(\int_{\mathcal{F}_c} \mathrm{D}(u_c^i):\mathrm{D}\big(v_c^j(s)\big)\,\mathrm{d}x\right)\right)_{i\in\{1,\dots,6\}} \\ &= -2\left(\int_{\mathcal{F}_c} \mathrm{D}(u_c^i):\mathrm{D}\left(\partial_s^\alpha v_c^j(s)\right)\,\mathrm{d}x\right)_{i\in\{1,\dots,6\}} = -\begin{pmatrix}\displaystyle\int_{\mathcal{S}_c} \sigma\left(\partial_s^\alpha v_c^j(s), \partial_s^\alpha q_c^j(s)\right)\mathbf{n}\,\mathrm{d}x \\ \displaystyle\int_{\mathcal{S}_c} x\times\sigma\left(\partial_s^\alpha v_c^j(s), \partial_s^\alpha q_c^j(s)\right)\mathbf{n}\,\mathrm{d}x\end{pmatrix}\end{aligned} \tag{3.16}$$

for $j \in \{1, \cdots, d\}$ and for $\alpha \in \mathbb{N}^d$.
In the above expression, $v_c^j(s)$ and $q_c^j(s)$ are the solutions of (3.2c). In particular, $(\partial_s^\alpha v_c^j(s), \partial_s^\alpha q_c^j(s))$ is solution of the following system

$$\begin{aligned} -\Delta(\partial_s^\alpha v_j) + \nabla(\partial_s^\alpha q_j) &= 0 && \text{in } \mathcal{F}_c\,, \\ \operatorname{div}(\partial_s^\alpha v_j) &= 0 && \text{in } \mathcal{F}_c\,, \\ \partial_s^\alpha v_j(s) &= \partial_s^\alpha D_c^j(s) && \text{on } \mathcal{S}_c\,, \end{aligned} \tag{3.17}$$

with $D_c^i$ defined by (3.1). In general, it is not possible to obtain an explicit formula for $\partial_s^\alpha N(s)\boldsymbol{e}_j$, but this can be done in the case of the sphere and for particular boundary conditions (see Sect. 4).

## 4. The case of the unit sphere

In this section, we consider the situation where $\mathcal{S}_c = \mathbb{S}^2$ and namely the case where $\Psi_0 = 0$.

### 4.1. Derivation of boundary conditions

In this paragraph, we compute the expressions of $D_c^i$ given by (3.1) for $\Psi_0 = 0$ at $s = 0$. In that case, we have.

**Proposition 4.1.** *Let $d \geqslant 1$ and $c = (0, \delta) \in \mathcal{SC}^2(d)$. For $i, j, k \in \{1, \cdots, d\}$, we have*

$$\begin{aligned} D_c^i(0) &= \delta_i\,, \\ \partial_j D_c^i(0) &= -\,\mathrm{G}_\Gamma\,\delta_i\cdot\delta_j\,, \\ \partial_{k,j} D_c^i(0) &= \frac{1}{6}\big(-2\,\langle\delta_j,\delta_k\rangle\,\delta_i + \langle\delta_i,\delta_k\rangle\,\delta_j + \langle\delta_i,\delta_j\rangle\,\delta_k\big) \\ &\quad + \frac{1}{2}\big(\mathrm{G}_\Gamma\left(\mathrm{G}_\Gamma\,\delta_i\cdot\delta_k\right)\cdot\delta_j + \mathrm{G}_\Gamma\left(\mathrm{G}_\Gamma\,\delta_i\cdot\delta_j\right)\cdot\delta_k + \mathrm{G}_\Gamma\,\delta_i\cdot\left(\mathrm{G}_\Gamma\,\delta_j\cdot\delta_k + \mathrm{G}_\Gamma\,\delta_k\cdot\delta_j\right)\big). \end{aligned}$$

*In the above relations, the differential operator $\mathrm{G}_\Gamma$ is defined by*

$$\mathrm{G}_\Gamma\,u\cdot v := \nabla u\cdot v + \langle u, v\rangle\,\mathrm{Id}_{\mathbb{S}^2} \tag{4.1}$$

The proof of the above result is obtained by combining Lemmas 4.2 and 4.3.

**Lemma 4.2.** *Let $d \geqslant 1$ and $c = (0, \delta) \in \mathcal{SC}^2(d)$. For $i, j, k \in \{1, \cdots, d\}$, we have at $s = 0$*

$$\begin{aligned} D_c^i &= \partial_i\mathcal{X}_\delta\,, \\ \partial_j D_c^i &= \partial_{j,i}\mathcal{X}_\delta - \nabla D_c^i\cdot D_c^j\,, \\ \partial_{k,j} D_c^i &= \partial_{k,j,i}\mathcal{X}_\delta - \left(\nabla\partial_k D_c^i\cdot D_c^j + \nabla D_c^i\cdot\partial_k D_c^j + \nabla\partial_{i,j}\mathcal{X}_\delta\cdot D_c^k\right). \end{aligned}$$

*Proof.* Let us first notice that for $\Psi_0 = 0$, we have $D_c^i = \partial_i \mathcal{X}_\delta \circ \mathcal{X}_\delta^{-1}$ or equivalently,

$$\partial_i \mathcal{X}_\delta = D_c^i \circ \mathcal{X}_\delta$$

and hence, at $s = 0$ ($\mathcal{X}_\delta(0) = \mathrm{Id}_{\mathbb{S}^2}$),

$$D_c^i = \partial_i \mathcal{X}_\delta\,;$$

– 1st derivative:

$$\partial_{j,i} \mathcal{X}_\delta = \partial_j D_c^i \circ \mathcal{X}_\delta + \nabla D_c^i \circ \mathcal{X}_\delta \cdot \partial_j \mathcal{X}_\delta = \left(\partial_j D_c^i + \nabla D_c^i \cdot D_c^j\right) \circ \mathcal{X}_\delta$$

and hence, at $s = 0$,

$$\partial_j D_c^i = \partial_{j,i} \mathcal{X}_\delta - \nabla D_c^i \cdot D_c^j\,;$$

– 2nd derivative:

$$\partial_{k,j,i} \mathcal{X}_\delta = \left(\partial_{k,j} D_c^i + \nabla \partial_k D_c^i \cdot D_c^j + \nabla D_c^i \cdot \partial_k D_c^j + \nabla \left(\partial_j D_c^i + \nabla D_c^i \cdot D_c^j\right) \cdot D_c^k\right) \circ \mathcal{X}_\delta$$

and hence, at $s = 0$,

$$\partial_{k,j} D_c^i = \partial_{k,j,i} \mathcal{X}_\delta - \left(\nabla \partial_k D_c^i \cdot D_c^j + \nabla D_c^i \cdot \partial_k D_c^j + \nabla \partial_{i,j} \mathcal{X}_\delta \cdot D_c^k\right).$$

□

Let us now compute the derivatives of $\mathcal{X}_\delta$.

**Lemma 4.3.** *For $\delta = (\delta_1, \cdots, \delta_d) \in C^2(\mathbb{S}^2, \mathrm{T}\mathbb{S}^2)^d$, we have at $s = 0$, for $i, j, k \in \{1, \cdots, d\}$,*

$$\begin{aligned}
\mathcal{X}_\delta(s)\big|_{s=0} &= \mathrm{Id}_{\mathbb{S}^2},\\
\partial_i \mathcal{X}_\delta(s)\big|_{s=0} &= \delta_i,\\
\partial_{j,i} \mathcal{X}_\delta(s)\big|_{s=0} &= -\langle \delta_i, \delta_j \rangle \,\mathrm{Id}_{\mathbb{S}^2},\\
\partial_{k,j,i} \mathcal{X}_\delta(s)\big|_{s=0} &= \frac{-1}{3}\left(\langle \delta_j, \delta_k \rangle\, \delta_i + \langle \delta_i, \delta_k \rangle\, \delta_j + \langle \delta_i, \delta_j \rangle\, \delta_k\right).
\end{aligned}$$

*Proof.* To simplify the notation, we set $\mathcal{X} = \mathcal{X}_\delta$ and $\Theta = \Theta_\delta(s) = \sum_{i=1}^d s_i \delta_i$.
For every $n \in \mathbb{N}^*$, set $A_n = ((2n+1)\,\mathrm{Id}_{\mathbb{S}^2} + \Theta)$, so that, according to (2.2), we have $\mathcal{X} = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, A_n |\Theta|^{2n}$.
Then, for every $n \in \mathbb{N}^*$, we have:

– 1st derivative:

$$\partial_i \left(A_n |\Theta|^{2n}\right) = \delta_i |\Theta|^{2n} + 2n A_n \langle \delta_i, \Theta \rangle |\Theta|^{2n-2}$$

and hence,

$$\partial_i \left(A_n |\Theta|^{2n}\right)\big|_{s=0} = \begin{cases} \delta_i & \text{if } n = 0\,, \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \partial_i \mathcal{X}\big|_{s=0} = \delta_i;$$

– 2nd derivative:

$$\begin{aligned}\partial_{j,i}\left(A_n|\Theta|^{2n}\right) &= 2n\delta_i\left\langle\delta_j,\Theta\right\rangle|\Theta|^{2n-2} + 2n\delta_j\left\langle\delta_i,\Theta\right\rangle|\Theta|^{2n-2}\\ &\quad + 2nA_n\left\langle\delta_i,\delta_j\right\rangle|\Theta|^{2n-2} + 2n(2n-2)A_n\left\langle\delta_i,\Theta\right\rangle\left\langle\delta_j,\Theta\right\rangle|\Theta|^{2n-4}\\ &= 2n\left(\left(\delta_i\left\langle\delta_j,\Theta\right\rangle + \delta_j\left\langle\delta_i,\Theta\right\rangle + A_n\left\langle\delta_i,\delta_j\right\rangle\right)|\Theta|^{2n-2} + (2n-2)A_n\left\langle\delta_i,\Theta\right\rangle\left\langle\delta_j,\Theta\right\rangle|\Theta|^{2n-4}\right)\end{aligned}$$

and hence,

$$\partial_{j,i}\left(A_n|\Theta|^{2n}\right)\Big|_{s=0} = \begin{cases}6\left\langle\delta_i,\delta_j\right\rangle\mathrm{Id}_{\mathbb{S}^2} & \text{if } n=1\,,\\ 0 & \text{otherwise}\end{cases}\qquad\text{and}\qquad \partial_{j,i}\mathcal{X}\big|_{s=0} = -\left\langle\delta_i,\delta_j\right\rangle\mathrm{Id}_{\mathbb{S}^2};$$

– 3rd derivative:

$$\begin{aligned}\partial_{k,j,i}\left(A_n|\Theta|^{2n}\right) &= 2n\Big(\left(\delta_i\left\langle\delta_j,\delta_k\right\rangle + \delta_j\left\langle\delta_i,\delta_k\right\rangle + \delta_k\left\langle\delta_i,\delta_j\right\rangle\right)|\Theta|^{2n-2}\\ &\quad + (2n-2)\left(\delta_i\left\langle\delta_j,\Theta\right\rangle + \delta_j\left\langle\delta_i,\Theta\right\rangle + A_n\left\langle\delta_i,\delta_j\right\rangle\right)\left\langle\delta_k,\Theta\right\rangle|\Theta|^{2n-4}\\ &\quad + (2n-2)\delta_k\left\langle\delta_i,\Theta\right\rangle\left\langle\delta_j,\Theta\right\rangle|\Theta|^{2n-4}\\ &\quad + (2n-2)A_n\left\langle\delta_i,\delta_k\right\rangle\left\langle\delta_j,\Theta\right\rangle|\Theta|^{2n-4} + (2n-2)A_n\left\langle\delta_i,\Theta\right\rangle\left\langle\delta_j,\delta_k\right\rangle|\Theta|^{2n-4}\\ &\quad + (2n-2)(2n-4)A_n\left\langle\delta_i,\Theta\right\rangle\left\langle\delta_j,\Theta\right\rangle\left\langle\delta_k,\Theta\right\rangle|\Theta|^{2n-6}\Big)\\ &= 2n\Big(\left(\delta_i\left\langle\delta_j,\delta_k\right\rangle + \delta_j\left\langle\delta_i,\delta_k\right\rangle + \delta_k\left\langle\delta_i,\delta_j\right\rangle\right)|\Theta|^{2n-2}\\ &\quad + (2n-2)\big(\delta_i\left\langle\delta_j,\Theta\right\rangle\left\langle\delta_k,\Theta\right\rangle + \delta_j\left\langle\delta_i,\Theta\right\rangle\left\langle\delta_k,\Theta\right\rangle + \delta_k\left\langle\delta_i,\Theta\right\rangle\left\langle\delta_j,\Theta\right\rangle\\ &\qquad\qquad + A_n\left(\left\langle\delta_i,\delta_j\right\rangle\left\langle\delta_k,\Theta\right\rangle + \left\langle\delta_i,\delta_k\right\rangle\left\langle\delta_j,\Theta\right\rangle + \left\langle\delta_i,\Theta\right\rangle\left\langle\delta_j,\delta_k\right\rangle\right)\big)|\Theta|^{2n-4}\\ &\quad + (2n-2)(2n-4)A_n\left\langle\delta_i,\Theta\right\rangle\left\langle\delta_j,\Theta\right\rangle\left\langle\delta_k,\Theta\right\rangle|\Theta|^{2n-6}\Big)\end{aligned}$$

and hence,

$$\partial_{k,j,i}\left(A_n|\Theta|^{2n}\right)\Big|_{s=0} = \begin{cases}2\left(\delta_i\left\langle\delta_j,\delta_k\right\rangle + \delta_j\left\langle\delta_i,\delta_k\right\rangle + \delta_k\left\langle\delta_i,\delta_j\right\rangle\right) & \text{if } n=1\,,\\ 0 & \text{otherwise}\end{cases}$$

$$\text{and}\qquad \partial_{k,j,i}\mathcal{X}\big|_{s=0} = \frac{-1}{3}\left(\delta_i\left\langle\delta_j,\delta_k\right\rangle + \delta_j\left\langle\delta_i,\delta_k\right\rangle + \delta_k\left\langle\delta_i,\delta_j\right\rangle\right).$$

□

We are now in position to give the proof of Proposition 4.1.

*Proof of Proposition 4.1.* According to Lemmas 4.2 and 4.3 and (4.1), we deduce

$$D_c^i(0) = \delta_i \quad\text{and}\quad \partial_j D_c^i(0) = -\,\mathrm{G}_\Gamma\,\delta_i\cdot\delta_j.$$

We also have

$$\begin{aligned}\partial_{k,j}D_c^i(0) &= \frac{-1}{3}\left(\left\langle\delta_j,\delta_k\right\rangle\delta_i + \left\langle\delta_i,\delta_k\right\rangle\delta_j + \left\langle\delta_i,\delta_j\right\rangle\delta_k\right)\\ &\quad + \nabla\left(\mathrm{G}_\Gamma\,\delta_i\cdot\delta_k\right)\cdot\delta_j + \nabla\delta_i\cdot\left(\mathrm{G}_\Gamma\,\delta_j\cdot\delta_k\right) + \nabla\left(\left\langle\delta_i,\delta_j\right\rangle\mathrm{Id}_{\mathbb{S}^2}\right)\cdot\delta_k\\ &= \frac{-1}{3}\left(\left\langle\delta_j,\delta_k\right\rangle\delta_i + \left\langle\delta_i,\delta_k\right\rangle\delta_j + \left\langle\delta_i,\delta_j\right\rangle\delta_k\right)\\ &\quad - \left\langle\mathrm{G}_\Gamma\,\delta_i\cdot\delta_k,\delta_j\right\rangle\mathrm{Id}_{\mathbb{S}^2} + \mathrm{G}_\Gamma\left(\mathrm{G}_\Gamma\,\delta_i\cdot\delta_k\right)\cdot\delta_j\end{aligned}$$

$$
\begin{aligned}
&- \langle \delta_i, \mathrm{G}_\Gamma\, \delta_j \cdot \delta_k \rangle \,\mathrm{Id}_{\mathbb{S}^2} + \mathrm{G}_\Gamma\, \delta_i \cdot (\mathrm{G}_\Gamma\, \delta_j \cdot \delta_k) \\
&+ \langle \nabla \delta_i \cdot \delta_k, \delta_j \rangle \,\mathrm{Id}_{\mathbb{S}^2} + \langle \delta_i, \nabla \delta_j \cdot \delta_k \rangle \,\mathrm{Id}_{\mathbb{S}^2} + \langle \delta_i, \delta_j \rangle\, \delta_k \\
= &\, \frac{-1}{3}\big( \langle \delta_j, \delta_k \rangle\, \delta_i + \langle \delta_i, \delta_k \rangle\, \delta_j - 2\, \langle \delta_i, \delta_j \rangle\, \delta_k \big) + \mathrm{G}_\Gamma\, (\mathrm{G}_\Gamma\, \delta_i \cdot \delta_k) \cdot \delta_j + \mathrm{G}_\Gamma\, \delta_i \cdot (\mathrm{G}_\Gamma\, \delta_j \cdot \delta_k)\,.
\end{aligned}
$$

Symmetrizing this expression with respect to $j$ and $k$, we obtain the result. ☐

## 4.2. Stokes solutions on the exterior of a sphere

The results given here are borrowed from [6]. In this section, we use spherical coordinates $(r, \theta, \varphi) \in \mathbb{R}_+ \times [0, \pi] \times [0, 2\pi)$ which are recalled in Appendix A.

We recall that a spherical harmonic of degree $n \geqslant 0$ is defined by

$$
\begin{array}{rcl}
[0,\pi] \times [0, 2\pi] & \to & \mathbb{R} \\
(\theta, \varphi) & \mapsto & \displaystyle\sum_{m=-n}^{n} \gamma_n^m Y_n^m(\theta, \varphi)
\end{array}
\tag{4.2}
$$

and a rigid spherical harmonic of degree $-(n+1)$ by

$$
\begin{array}{rcl}
\mathbb{R}_+^* \times [0,\pi] \times [0, 2\pi] & \to & \mathbb{R} \\
(r, \theta, \varphi) & \mapsto & r^{-(n+1)} \displaystyle\sum_{m=-n}^{n} \gamma_n^m Y_n^m(\theta, \varphi),
\end{array}
\tag{4.3}
$$

with $\gamma_n^m$ any real coefficient, and where $Y_n^m$ is defined by

$$
\begin{cases}
Y_n^m(\theta, \varphi) = c_n^m \cos(m\varphi) P_n^m(\cos\theta), \\
Y_n^0(\theta, \varphi) = c_n^0 P_n^0(\cos\theta)\,, \\
Y_n^{-m}(\theta, \varphi) = c_n^m \sin(m\varphi) P_n^m(\cos\theta),
\end{cases}
\qquad (n \in \mathbb{N}\,,\ 0 \leqslant m \leqslant n\,,\ (\theta, \varphi) \in [0,\pi] \times [0, 2\pi]),
\tag{4.4}
$$

with

$$
c_n^m = \begin{cases}
\sqrt{\dfrac{(2n+1)}{4\pi}} & \text{if } m = 0\,, \\
\sqrt{\dfrac{(2n+1)(n-m)!}{2\pi(n+m)!}} & \text{if } m > 0
\end{cases}
\qquad (n \in \mathbb{N}\,, 0 \leqslant m \leqslant n)
$$

and with $P_n^m$ is the associated Legendre polynomial of degree $n$ and order $m$, that is to say that

$$
P_n^m(x) = \frac{1}{2^n n!} \left(1 - x^2\right)^{\frac{m}{2}} \frac{\mathrm{d}^{n+m}}{\mathrm{d}x^{n+m}} \left((x^2 - 1)^n\right) \qquad (x \in [-1, 1]\,,\ n \in \mathbb{N}\,,\ m \in \{0, \cdots, n\}).
$$

We recall that the family $\{Y_n^m\}_{\substack{n \in \mathbb{N},\\ m \in \{-n, \cdots, n\}}}$ forms an orthonormal basis of $L^2(\partial \mathcal{S}_0)$, see for instance ([15], Chap. VII, Sect. 5.3, p. 513). More precisely, this family is orthonormal for the scalar product

$$
\langle \zeta, \Upsilon \rangle = \int_0^{2\pi} \int_0^{\pi} \zeta(\theta, \varphi) \Upsilon(\theta, \varphi) \sin\theta \, \mathrm{d}\theta \mathrm{d}\varphi\,.
$$

**Remark 4.4.** Let us mention that,

$$\begin{cases} P_1^0(x) = \dfrac{1}{2}\dfrac{\mathrm{d}^1}{\mathrm{d}x^1}\left(x^2-1\right) = x\,, \\ P_1^1(x) = \dfrac{1}{2}\left(1-x^2\right)^{\frac{1}{2}}\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}\left(x^2-1\right) = \left(1-x^2\right)^{\frac{1}{2}} \end{cases} \qquad (x\in[-1,1]).$$

and hence,

$$\begin{cases} Y_1^1(\theta,\varphi) = \sqrt{\dfrac{3}{4\pi}}\cos\varphi\sin\theta\,, \\ Y_1^0(\theta,\varphi) = \sqrt{\dfrac{3}{4\pi}}\cos\theta\,, \\ Y_1^{-1}(\theta,\varphi) = \sqrt{\dfrac{3}{4\pi}}\sin\varphi\sin\theta\,, \end{cases} \qquad ((\theta,\varphi)\in[0,\pi]\times[0,2\pi]).$$

According to Lamb [21], see also Brenner ([6], Eq. 2.13), the solution $(v,q)$ of the Stokes equation in an exterior domain can be expressed in spherical coordinates (see Appendix A for the related definition of spherical coordinates and expression of the usual operators $\nabla$, div and rot) as

$$v = \sum_{n=1}^{\infty}\left(\operatorname{rot}\left(\chi_{-(n+1)}r\boldsymbol{e}_r\right) + \nabla\phi_{-(n+1)} - \frac{n-2}{2n(2n-1)}r^2\nabla p_{-(n+1)} + \frac{n+1}{n(2n-1)}p_{-(n+1)}r\boldsymbol{e}_r\right) \tag{4.5a}$$

$$q = \sum_{n=1}^{\infty} p_{-(n+1)} \tag{4.5b}$$

where $\chi_{-(n+1)}, \phi_{-(n+1)}$ and $p_{-(n+1)}$ are rigid spherical harmonics of degree $-(n+1)$ defined as in (4.3). Furthermore, the drag and torque exerted on the immersed domain by the fluid can be expressed as

$$F = -4\pi\nabla\left(r^3p_{-2}\right), \tag{4.6a}$$

$$T = -8\pi\nabla\left(r^3\chi_{-2}\right). \tag{4.6b}$$

Let us mention that $F$ and $T$ are constant vectors of $\mathbb{R}^3$. In fact, we have,

$$\nabla\left(r^3\left(r^{-2}Y_1^0(\theta,\varphi)\right)\right) = \sqrt{\frac{3}{4\pi}}\nabla\left(r\cos\theta\right) = \sqrt{\frac{3}{4\pi}}\nabla z = \sqrt{\frac{3}{4\pi}}\begin{pmatrix}0\\0\\1\end{pmatrix},$$

$$\nabla\left(r^3\left(r^{-2}Y_1^1(\theta,\varphi)\right)\right) = \sqrt{\frac{3}{4\pi}}\nabla\left(r\cos\varphi\sin\theta\right) = \sqrt{\frac{3}{4\pi}}\nabla x = \sqrt{\frac{3}{4\pi}}\begin{pmatrix}1\\0\\0\end{pmatrix},$$

$$\nabla\left(r^3\left(r^{-2}Y_1^{-1}(\theta,\varphi)\right)\right) = \sqrt{\frac{3}{4\pi}}\nabla\left(r\sin\varphi\sin\theta\right) = \sqrt{\frac{3}{4\pi}}\nabla y = \sqrt{\frac{3}{4\pi}}\begin{pmatrix}0\\1\\0\end{pmatrix}.$$

When the exterior domain is the exterior of the unit ball of $\mathbb{R}^3$, $v \cdot \boldsymbol{e}_r$, $\mathrm{div}_\Gamma\, v$ and $\mathrm{rot}_\Gamma\, v$ for $r = 1$ can be expressed as a sum of spherical harmonics (see (A.1) for the definitions of $\mathrm{div}_\Gamma$ and $\mathrm{rot}_\Gamma$),

$$v \cdot \boldsymbol{e}_r = \sum_{n=0}^{\infty} X_n, \tag{4.7a}$$

$$\mathrm{div}_\Gamma\, v = \sum_{n=0}^{\infty} Y_n, \tag{4.7b}$$

$$\mathrm{rot}_\Gamma\, v = \sum_{n=0}^{\infty} Z_n, \tag{4.7c}$$

with $X_n, Y_n$ and $Z_n$ spherical harmonics of degree $n$.

According to [6], $\chi_{-(n+1)}, \phi_{-(n+1)}$ and $p_{-(n+1)}$ are related to $X_n, Y_n$ and $Z_n$ by

$$\chi_{-(n+1)}(r,\theta,\varphi) = \frac{r^{-(n+1)}}{n(n+1)} Z_n(\theta,\varphi), \tag{4.8a}$$

$$\phi_{-(n+1)}(r,\theta,\varphi) = \frac{r^{-(n+1)}}{2(n+1)} (nX_n(\theta,\varphi) + Y_n(\theta,\varphi)), \tag{4.8b}$$

$$p_{-(n+1)}(r,\theta,\varphi) = r^{-(n+1)} \frac{2n-1}{n+1} \left((n+2)X_n(\theta,\varphi) + Y_n(\theta,\varphi)\right), \tag{4.8c}$$

for every $n \in \mathbb{N}^*$.

Using the decomposition (4.7) for (3.17) we obtain

$$\partial_s^\alpha N(s)\boldsymbol{e}_j = -\begin{pmatrix} 4\pi\nabla\left(r^3 p_{-2}\right) \\ 8\pi\nabla\left(r^3\chi_{-2}\right) \end{pmatrix}.$$

Since $\partial_s^\alpha D_c^j(s)$ is a tangential field, $p_{-2}$ and $\chi_{-2}$ are given by (4.8) with $X = 0$, *i.e.*

$$p_{-2}(r,\theta,\varphi) = \frac{r^{-2}}{2} Y_1(\theta,\varphi) \qquad \text{and} \qquad \chi_{-2}(r,\theta,\varphi) = \frac{r^{-2}}{2} Z_1(\theta,\varphi),$$

where $Y_1$ and $Z_1$ are defined from (4.7) with $v = \partial_s^\alpha D_c^j(s)$. More precisely, we obtain

$$\partial_s^\alpha N(s)\boldsymbol{e}_j = -\sqrt{3\pi}\begin{pmatrix} \langle \mathrm{div}_\Gamma\, \partial_s^\alpha D_c^j(s), Y_1^1\rangle \\ \langle \mathrm{div}_\Gamma\, \partial_s^\alpha D_c^j(s), Y_1^{-1}\rangle \\ \langle \mathrm{div}_\Gamma\, \partial_s^\alpha D_c^j(s), Y_1^0\rangle \\ 2\langle \mathrm{rot}_\Gamma\, \partial_s^\alpha D_c^j(s), Y_1^1\rangle \\ 2\langle \mathrm{rot}_\Gamma\, \partial_s^\alpha D_c^j(s), Y_1^{-1}\rangle \\ 2\langle \mathrm{rot}_\Gamma\, \partial_s^\alpha D_c^j(s), Y_1^0\rangle \end{pmatrix}.$$

Let us also recall that for a spherical body, the matrix $K_c$ introduced in (3.3) is (see Sects. 5.2 and 5.3 of [19]) $2\pi\begin{pmatrix}3I_3 & 0\\ 0 & 4I_3\end{pmatrix}$ and hence,

$$\partial_s^\alpha V_c^j(s) = -\sqrt{\frac{3}{4\pi}}\begin{pmatrix}\frac{1}{3}I_3 & 0\\ 0 & \frac{1}{2}I_3\end{pmatrix}\begin{pmatrix}\langle \operatorname{div}_\Gamma \partial_s^\alpha D_c^j(s), Y_1^1\rangle\\ \langle \operatorname{div}_\Gamma \partial_s^\alpha D_c^j(s), Y_1^{-1}\rangle\\ \langle \operatorname{div}_\Gamma \partial_s^\alpha D_c^j(s), Y_1^0\rangle\\ \langle \operatorname{rot}_\Gamma \partial_s^\alpha D_c^j(s), Y_1^1\rangle\\ \langle \operatorname{rot}_\Gamma \partial_s^\alpha D_c^j(s), Y_1^{-1}\rangle\\ \langle \operatorname{rot}_\Gamma \partial_s^\alpha D_c^j(s), Y_1^0\rangle\end{pmatrix}. \tag{4.9}$$

### 4.3. Particular choices of $\delta$

In order to fully define the swimmer configuration, $c = (0, \delta)$, we introduce some explicit choices of $\delta_i$'s. The first type of $\delta_i$ that we consider is

$$\zeta_n^m(\theta,\varphi) = \partial_\theta Y_n^m(\theta,\varphi)\boldsymbol{e}_\theta + \partial_\varphi Y_n^m(\theta,\varphi)\frac{\boldsymbol{e}_\varphi}{\sin\theta} \tag{4.10}$$

and the second type is

$$\xi_n^m(\theta,\varphi) = \partial_\varphi Y_n^m(\theta,\varphi)\frac{\boldsymbol{e}_\theta}{\sin\theta} - \partial_\theta Y_n^m(\theta,\varphi)\boldsymbol{e}_\varphi\,, \tag{4.11}$$

with $n \in \mathbb{N}$ and $m \in \{-n, \cdots, n\}$.

Let us remind that, according to Proposition 4.1, we have

$$D_c^i(0)(\theta,\varphi) = \delta_i(\theta,\varphi).$$

Let us then compute $V_c^i(s)$ given by (4.9) at $s = 0$ for the two possible choices of $\delta_i$ given by (4.10) and (4.11).

– If $\delta_i = \zeta_n^m$. Assume $m \geqslant 0$, the case $m \leqslant 0$ is similar. In order to compute $V_c^i(0)$, one have to compute the solution $v = v_r\boldsymbol{e}_r + v_\theta\boldsymbol{e}_\theta + v_\varphi\boldsymbol{e}_\varphi$ of the Stokes equation set on the exterior of the unit ball with the Dirichlet boundary condition $v = \delta_i$, that is to say that at $r = 1$, $v_r$, $v_\theta$ and $v_\varphi$ shall satisfy

$$v_r(\theta,\varphi) = 0\,, \qquad v_\theta(\theta,\varphi) = \partial_\theta Y_n^m(\theta,\varphi) = -c_n^m \sin\theta (P_n^m)'(\cos\theta)\cos(m\varphi)$$
$$\text{and} \qquad v_\varphi(\theta,\varphi) = \frac{1}{\sin\theta}\partial_\varphi Y_n^m(\theta,\varphi) = \frac{-mc_n^m}{\sin\theta}P_n^m(\cos\theta)\sin(m\varphi).$$

In order to express the solution in a sum of rigid spherical harmonics, we compute the decomposition in spherical harmonics in (4.7),

$$\begin{aligned}
&\boldsymbol{e}_r\cdot v = 0\,,\\
&\operatorname{div}_\Gamma v = \frac{-1}{\sin\theta}\left(\partial_\theta(v_\theta\sin\theta) + \partial_\varphi v_\varphi\right)\\
&\qquad = \frac{-c_n^m}{\sin\theta}\left(-\partial_\theta(\sin^2\theta(P_n^m)'(\cos\theta)) - \frac{m^2}{\sin\theta}P_n^m(\cos\theta)\right)\cos(m\varphi)\\
&\qquad = -c_n^m\left(\frac{-1}{\sin\theta}\partial_\theta((1-\cos^2\theta)(P_n^m)'(\cos\theta)) - \frac{m^2}{1-\cos^2\theta}P_n^m(\cos\theta)\right)\cos(m\varphi)
\end{aligned}$$

$$
\begin{aligned}
&= c_n^m n(n+1) P_n^m(\cos\theta)\cos(m\varphi) \\
&= n(n+1) Y_n^m\,, \\
\operatorname{rot}_\Gamma v &= \frac{1}{\sin\theta}\left(\partial_\theta(v_\varphi \sin\theta) - \partial_\varphi v_\theta\right) \\
&= \frac{c_n^m}{\sin\theta}\left(-m\partial_\theta(P_n^m(\cos\theta)) - m\sin\theta (P_n^m)'(\cos\theta)\right)\sin(m\varphi) \\
&= 0\,.
\end{aligned}
$$

In the above relations, we have used the property of the associated Legendre polynomials, see for instance ([15], Chap. V, Sect. 10.3, p. 327)

$$
\frac{\mathrm{d}}{\mathrm{d}\zeta}\left((1-\zeta^2)(P_n^m)'(\zeta)\right) - \frac{m^2}{1-\zeta^2}P_n^m(\zeta) = -n(n+1)P_n^m(\zeta).
$$

Consequently, by orthogonality of spherical harmonics, we obtain $V_c^i(0) = 0$, for $n \geqslant 2$.

– If $\delta_i = \xi_n^m$. Assume $m \geqslant 0$, the case $m \leqslant 0$ is similar. Similarly, we have to compute the solution $v = v_r \boldsymbol{e}_r + v_\theta \boldsymbol{e}_\theta + v_\varphi \boldsymbol{e}_\varphi$ of the Stokes equation set on the exterior of the unit ball with the Dirichlet boundary condition:

$$
v_r = 0\,, \qquad v_\theta = \frac{1}{\sin\theta}\partial_\varphi Y_n^m(\theta,\varphi) = \frac{-m c_n^m}{\sin\theta} P_n^m(\cos\theta)\sin(m\varphi)
$$
$$
\text{and} \qquad v_\varphi = -\partial_\theta Y_n^m(\theta,\varphi) = c_n^m \sin\theta (P_n^m)'(\cos\theta)\cos(m\varphi)
$$

and similarly to the previous case, we obtain

$$
\boldsymbol{e}_r \cdot v = 0\,, \quad \operatorname{div}_\Gamma v = 0 \quad \text{and} \quad \operatorname{rot}_\Gamma v = -n(n+1)P_n^m(\cos\theta)e^{im\varphi}.
$$

Consequently, for $n \geqslant 2$, we have $V_c^i(0) = 0$.

**Lie brackets at** $s = 0$**.** Due to the choice of the $\delta_i$'s given by (4.10) and (4.11), we obtain (choosing $n \geqslant 2$) $V_c^i(0) = 0$ for every $i \in \{1, \cdots, d\}$. Consequently, at $s = 0$ the expression of the Lie brackets given in (3.10) and (3.12) are

$$
f_c^i(0, \mathrm{I}_3, 0) = \mathfrak{p}\,(0, 0, \boldsymbol{e}_i)\,, \tag{4.12a}
$$
$$
[f_c^j, f_c^i](0, \mathrm{I}_3, 0) = \tilde{\mathfrak{p}}\left(\partial_j V_c^i - \partial_i V_c^j\right), \tag{4.12b}
$$
$$
[f_c^k, [f_c^j, f_c^i]](0, \mathrm{I}_3, 0) = \tilde{\mathfrak{p}}\left(\partial_k\left(\partial_j V_c^i - \partial_i V_c^j\right)\right). \tag{4.12c}
$$

## 4.4. Explicit computations

In this section, we combine (4.9) and 4.1 in order to compute explicitly (4.12). This computation has been made by using the computer algebra system `maxima`.

**Case** $d = 4$**.** In this case, we consider $\delta = (\delta_1, \cdots, \delta_4)$, with

$$
\delta_1 = \zeta_4^1\,, \qquad \delta_2 = \xi_4^0\,, \qquad \delta_3 = \zeta_3^0 \quad \text{and} \quad \delta_4 = \zeta_3^1\,.
$$

Setting $\Delta_{i,j} = \partial_j V_c^i(0) - \partial_i V_c^j(0)$ the $6 \times 6$ matrix $(\Delta_{12} \mid \Delta_{1,3} \mid \Delta_{1,4} \mid \Delta_{2,3} \mid \Delta_{2,4} \mid \Delta_{3,4})$ is

$$\begin{pmatrix} 0 & \frac{2^{5/2}\,5^{3/2}}{\sqrt{7}\,\pi} & 0 & 0 & 0 & 0 \\ -\frac{2^{3/2}\,5^{3/2}}{\pi} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s\,\frac{4\,\sqrt{3}\,5^{3/2}}{\sqrt{7}\,\pi} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{5\,3^{5/2}}{\sqrt{2}\,\sqrt{7}\,\pi} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{11\,3^{3/2}}{\sqrt{2}\,\pi} \\ 0 & 0 & 0 & \frac{90}{\sqrt{7}\,\pi} & 0 & 0 \end{pmatrix}.$$

This, together with Lemma 3.4, ensures that the dimension of the Lie algebra generated by $f_c^1, \dots, f_c^4$ is of maximal dimension ($i.e.$ $6+4=10$) for $s=0$.

**Case $d = 3$.** In this case, we consider $\delta = (\delta_1, \delta_2, \delta_3)$, with

$$\delta_1 = \zeta_4^1\,, \qquad \delta_2 = \xi_4^0 \quad \text{and} \quad \delta_3 = \zeta_3^0\,.$$

Setting $\Delta_{i,j} = \partial_j V_c^i(0) - \partial_i V_c^j(0)$ and $\Delta_{i,j}^k = \partial_k \left(\partial_j V_c^i(s) - \partial_i V_c^j(s)\right)\big|_{s=0}$ the $6 \times 3$ matrix $(\Delta_{12} \mid \Delta_{13} \mid \Delta_{23})$ is

$$\begin{pmatrix} 0 & \frac{2^{5/2}\,5^{3/2}}{\sqrt{7}\,\pi} & 0 \\ -\frac{2^{3/2}\,5^{3/2}}{\pi} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{90}{\sqrt{7}\,\pi} \end{pmatrix}$$

and the $6 \times 9$ matrix $(\Delta_{12}^1 \mid \Delta_{12}^2 \mid \Delta_{12}^3 \mid \Delta_{13}^1 \mid \Delta_{13}^2 \mid \Delta_{13}^3 \mid \Delta_{23}^1 \mid \Delta_{23}^2 \mid \Delta_{23}^3)$ is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{8949}{286\,\sqrt{7}\,\pi^{3/2}} & 0 & 0 & 0 & -\frac{2360\,\sqrt{7}}{143\,\pi^{3/2}} & 0 \\ 0 & 0 & \frac{249\ 5^{7/2}}{143\ 2^{3/2}\,\sqrt{7}\,\pi^{3/2}} & 0 & -\frac{3879\,\sqrt{5}}{13\ 2^{3/2}\,\sqrt{7}\,\pi^{3/2}} & 0 & \frac{753\,\sqrt{5}\,7^{3/2}}{143\,\sqrt{2}\,\pi^{3/2}} & 0 & 0 \\ 0 & \frac{65\,367\ 5^{3/2}}{1001\ 2^{3/2}\,\pi^{3/2}} & 0 & 0 & 0 & \frac{1623\,\sqrt{5}}{11\ 2^{3/2}\,\pi^{3/2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This, together with Lemma 3.4, ensures that the dimension of the Lie algebra generated by $f_c^1, f_c^2, f_c^3$ is of maximal dimension ($i.e.$ $6+3=9$) for $s=0$.

In conclusion of the above computations, we have the following results.

**Lemma 4.5.** *For every $d \geqslant 2$ and every $(c,s) \in \mathcal{A}^2(d)$, we set $\Delta_{i,j}(s) = \partial_j V_c^i(s) - \partial_i V_c^j(s)$ and $\Delta_{i,j}^k(s) = \partial_k \Delta_{i,j}(s)$.*

– *For every $d \geqslant 4$, the analytic maps*

$$\begin{array}{rcl} \mathcal{A}^2(d) & \to & \mathbb{R} \\ (c,s) & \mapsto & \det\left(\Delta_{1,2}(s) \mid \Delta_{1,3}(s) \mid \Delta_{1,4}(s) \mid \Delta_{2,3}(s) \mid \Delta_{2,4}(s) \mid \Delta_{3,4}(s)\right) \end{array}$$

*and (at $s=0$)*

$$\begin{array}{rcl} \mathcal{SC}^2(d) & \to & \mathbb{R} \\ c & \mapsto & \det\left(\Delta_{1,2}(0) \mid \Delta_{1,3}(0) \mid \Delta_{1,4}(0) \mid \Delta_{2,3}(0) \mid \Delta_{2,4}(0) \mid \Delta_{3,4}(0)\right) \end{array}$$

*are non-identically* 0.

– *For every $d \geqslant 3$, the analytic maps*

$$\begin{array}{rcl} \mathcal{A}^2(d) & \to & \mathbb{R} \\ (c,s) & \mapsto & \det\big(\Delta_{1,2}(s) \mid \Delta_{1,3}(s) \mid \Delta_{2,3}(s) \mid \Delta^2_{1,2}(s) \mid \Delta^3_{1,2}(s) \mid \Delta^1_{1,3}(s)\big) \end{array}$$

*and (at $s=0$)*

$$\begin{array}{rcl} \mathcal{SC}^2(d) & \to & \mathbb{R} \\ c & \mapsto & \det\big(\Delta_{1,2}(0) \mid \Delta_{1,3}(0) \mid \Delta_{2,3}(0) \mid \Delta^2_{1,2}(0) \mid \Delta^3_{1,2}(0) \mid \Delta^1_{1,3}(0)\big) \end{array}$$

*are non identically* 0.

**Remark 4.6.** We tried to prove a similar result for $d=2$ but our numerical simulations seem to indicate that it is not possible. More precisely, we went up to the computation of Lie brackets of fifth order. In all the computations, we performed the maximal rank of the Lie algebra evaluated at $s=0$ was 3. In these computations, we have considered all possible choices of $\delta$ given by (4.10) and (4.11) up to spherical harmonics of order 6. We have also taken the parameters $m$ and $n$ in (4.10) and (4.11) randomly, using a Poisson law for $n$, and again the maximal rank obtained was 3.
However, we believe that the generic result, Theorem 2.8 is still valid for $d=2$ but probably the spherical swimmers are too symmetric to be controllable with only two elementary deformations.

From Lemma 4.5, we deduce.

**Proposition 4.7.** *Given $d \geqslant 3$, $\varepsilon > 0$ and $\overline{c} = (\overline{\Psi}_0, \overline{\delta}) \in \mathcal{SC}^2(d)$, there exists $c = (\Psi_0, \delta) \in \mathcal{SC}^\infty(d)$ such that*

$$\|c - \overline{c}\| < \varepsilon$$

*and*

$$\dim \mathrm{Lie}_{(h,Q,s)}\left\{f^1_c, \cdots, f^d_c\right\} = d + 6 \qquad ((h,Q,s) \in \mathbb{R}^3 \times \mathcal{SO}(3) \times \mathcal{J}(\delta)). \tag{4.13}$$

*Furthermore, for almost every $s \in \mathcal{J}(\delta)$ and every $(h,Q) \in \mathbb{R}^3 \times \mathcal{SO}(3)$, we have*

– *for $d=4$,*

$$\begin{aligned} &\mathrm{Lie}_{(h,Q,s)}\left\{f^1_c, \cdots, f^d_c\right\} \\ &\qquad = \mathrm{Span}\left(\left\{f^1_c(h,Q,s), \cdots, f^d_c(h,Q,s),\right\} \cup \left\{[f^j_c, f^i_c](h,Q,s)\,,\ i,j \in \{1,\cdots,d\}\right\}\right); \end{aligned} \tag{4.14}$$

– *for $d=3$,*

$$\begin{aligned} &\mathrm{Lie}_{(h,Q,s)}\left\{f^1_c, \cdots, f^d_c\right\} \\ &\qquad = \mathrm{Span}\left(\left\{f^1_c(h,Q,s), \cdots, f^d_c(h,Q,s)\right\} \cup \left\{[f^j_c, f^i_c](h,Q,s)\,,\ i,j \in \{1,\cdots,d\}\right\}\right. \\ &\qquad\qquad \left.\cup \left\{[f^k_c, [f^j_c, f^i_c]](h,Q,s)\,,\ i,j,k \in \{1,\cdots,d\}\right\}\right). \end{aligned} \tag{4.15}$$

*Proof.* Let us sketch the proof for $d \geqslant 4$. The proof in the case $d = 3$ is similar.

The analyticity of the map

$$F : c \in \mathcal{SC}^2(d) \mapsto \det\left(\Delta_{1,2}(0) \mid \Delta_{1,3}(0) \mid \Delta_{1,4}(0) \mid \Delta_{2,3}(0) \mid \Delta_{2,4}(0) \mid \Delta_{3,4}(0)\right)$$

given in Lemma 4.5 and its non-nullity ensure that for every $\overline{c}$ there exists $c$ such that $\|c - \overline{c}\| < \varepsilon$ and $F(c) \neq 0$. This together with Lemma 3.5 gives (4.13).

In addition, using the analyticity of

$$(c, s) \in \mathcal{A}(d) \mapsto \det\left(\Delta_{1,2}(s) \mid \Delta_{1,3}(s) \mid \Delta_{1,4}(s) \mid \Delta_{2,3}(s) \mid \Delta_{2,4}(s) \mid \Delta_{3,4}(s)\right),$$

we obtain (4.14). □

### 4.5. Proof of Theorem 2.8

In this paragraph, we prove Theorem 2.8 using Proposition 4.7 together with Lemma 3.5 and a standard consequence of Rashevsky–Chow theorem (see [20], Thm. 3 and Cor., p. 109).

*Proof of Theorem 2.8.* Assume $d \geqslant 3$, $\varepsilon > 0$, $\overline{c} = (\overline{\Psi}_0, \overline{\delta}) \in \mathcal{SC}^2(d)$. From Proposition 4.7, there exists $c = (\Psi_0, \delta) \in \mathcal{SC}^\infty(d)$ such that

$$\|c - \overline{c}\| < \varepsilon\,,$$

and

$$\dim \operatorname{Lie}_{(h,Q,s)}\left\{f_c^1, \cdots, f_c^d\right\} = d + 6 \qquad ((h, Q, s) \in \mathbb{R}^3 \times \mathcal{SO}(3) \times \mathcal{J}(\delta)).$$

We then apply (Cor., p. 109 of [20]). □

## 5. Conclusion and perspectives

We have proposed a new model for the swimming of ciliate micro-organisms, extending the model in [34] in the case where the geometry is not axisymmetric. The swimming mechanism consists in boundary displacements associated with tangential velocities. We have shown that generically with respect to the shape of the swimmer and to the tangential velocities, our swimmer can move into a Stokes fluid for at least $d = 3$ tangential velocities. It is already known since [32], that $d = 1$ is not sufficient (see also [30]). It is still an open question to know if $d = 2$ is sufficient (generically). From our numerical investigation, for a spherical swimmer it seems that $d = 2$ is not enough. One could also try to adapt the method developed here to other models such as the one in [25] but in that case the Lie brackets are more complicated to compute. We should also expect the same kind of result in the case of low Reynolds numbers (potential fluid) as in [11, 12].

We would like also to extend our results to a Navier–Stokes fluid and for a fluid–structure system confined in a bounded domain. All the methodology used here does not work anymore. For a Navier–Stokes fluid, the system is of infinite dimension and for a bounded domain, we see that the dimension of the Lie algebra generated by the tangential velocities may depend on the position of the swimmer.

## Appendix A. Formula in spherical coordinates

These results are borrowed from (Sect. A.15 of [19]) and are recalled here for the sake of completeness.

Consider the spherical coordinates:

$$x = r \sin\theta \cos\varphi\,, \quad y = r \sin\theta \sin\varphi \quad \text{and} \quad z = r\cos\theta\,,$$

with $(r,\theta,\varphi)\in\mathbb{R}_+\times[0,\pi]\times[0,2\pi]$. We have:

$$\begin{aligned}
\boldsymbol{e}_r &= \sin\theta\cos\varphi\boldsymbol{e}_1+\sin\theta\sin\varphi\boldsymbol{e}_2+\cos\theta\boldsymbol{e}_3,\\
\boldsymbol{e}_\theta &= \cos\theta\cos\varphi\boldsymbol{e}_1+\cos\theta\sin\varphi\boldsymbol{e}_2-\sin\theta\boldsymbol{e}_3,\\
\boldsymbol{e}_\varphi &= -\sin\varphi\boldsymbol{e}_1+\cos\varphi\boldsymbol{e}_2.
\end{aligned}$$

Let $f,v_r,v_\theta,v_\varphi$ be scalar functions and set $v=v_r\boldsymbol{e}_r+v_\theta\boldsymbol{e}_\theta+v_\varphi\boldsymbol{e}_\varphi$, then we have

$$\begin{aligned}
\nabla f &= \partial_r f\boldsymbol{e}_r+\frac{1}{r}\partial_\theta f\boldsymbol{e}_\theta+\frac{1}{r\sin\theta}\partial_\varphi f\boldsymbol{e}_\varphi,\\
\operatorname{div} v &= \frac{1}{r^2}\partial_r(r^2v_r)+\frac{1}{r\sin\theta}\partial_\theta(v_\theta\sin\theta)+\frac{1}{r\sin\theta}\partial_\varphi v_\varphi,\\
\operatorname{rot} v &= \frac{1}{r\sin\theta}\left(\partial_\theta(v_\varphi\sin\theta)-\partial_\varphi v_\theta\right)\boldsymbol{e}_r+\frac{1}{r}\left(\frac{1}{\sin\theta}\partial_\varphi v_r-\partial_r(rv_\varphi)\right)\boldsymbol{e}_\theta+\frac{1}{r}\left(\partial_r(rv_\theta)-\partial_\theta v_r\right)\boldsymbol{e}_\varphi.
\end{aligned}$$

and we define

$$\begin{aligned}
\operatorname{div}_\Gamma v &:= r\boldsymbol{e}_r\cdot\nabla(\boldsymbol{e}_r\cdot v)-r\operatorname{div} v\\
&= r\left(\partial_r v_r-\frac{1}{r^2}\partial_r(r^2v_r)-\frac{1}{r\sin\theta}\partial_\theta(v_\theta\sin\theta)-\frac{1}{r\sin\theta}\partial_\varphi v_\varphi\right)\\
&= -2v_r-\frac{1}{\sin\theta}\left(\partial_\theta(v_\theta\sin\theta)+\partial_\varphi v_\varphi\right), \qquad \text{(A.1a)}\\
\operatorname{rot}_\Gamma v &:= r\boldsymbol{e}_r\cdot\operatorname{rot} v\\
&= \frac{1}{\sin\theta}\left(\partial_\theta(v_\varphi\sin\theta)-\partial_\varphi v_\theta\right). \qquad \text{(A.1b)}
\end{aligned}$$

Let $u=u_r\boldsymbol{e}_r+u_\theta\boldsymbol{e}_\theta+u_\varphi\boldsymbol{e}_\varphi$ and $v=v_r\boldsymbol{e}_r+v_\theta\boldsymbol{e}_\theta+v_\varphi\boldsymbol{e}_\varphi$. Then we have:

$$\begin{aligned}
\nabla u &= \partial_r u_r\boldsymbol{e}_r\boldsymbol{e}_r^\top+\partial_r u_\theta\boldsymbol{e}_\theta\boldsymbol{e}_r^\top+\partial_r u_\varphi\boldsymbol{e}_\varphi\boldsymbol{e}_r^\top\\
&\quad+\frac{1}{r}\left(\partial_\theta u_r-u_\theta\right)\boldsymbol{e}_r\boldsymbol{e}_\theta^\top+\frac{1}{r}\left(\partial_\theta u_\theta+u_r\right)\boldsymbol{e}_\theta\boldsymbol{e}_\theta^\top+\frac{1}{r}\partial_\theta u_\varphi\boldsymbol{e}_\varphi\boldsymbol{e}_\theta^\top\\
&\quad+\frac{1}{r}\left(\frac{1}{\sin\theta}\partial_\varphi u_r-u_\varphi\right)\boldsymbol{e}_r\boldsymbol{e}_\varphi^\top+\frac{1}{r}\left(\frac{1}{\sin\theta}\partial_\varphi u_\theta-\operatorname{cotan}\theta u_\varphi\right)\boldsymbol{e}_\theta\boldsymbol{e}_\varphi^\top\\
&\quad+\frac{1}{r}\left(\frac{1}{\sin\theta}\partial_\varphi u_\varphi+u_r+\operatorname{cotan}\theta u_\theta\right)\boldsymbol{e}_\varphi\boldsymbol{e}_\varphi^\top.
\end{aligned}$$

So that we have,

$$\begin{aligned}
\nabla u\cdot v &= v_r\left(\partial_r u_r\boldsymbol{e}_r+\partial_r u_\theta\boldsymbol{e}_\theta+\partial_r u_\varphi\boldsymbol{e}_\varphi\right)+v_\theta\left(\frac{1}{r}\left(\partial_\theta u_r-u_\theta\right)\boldsymbol{e}_r+\frac{1}{r}\left(\partial_\theta u_\theta+u_r\right)\boldsymbol{e}_\theta+\frac{1}{r}\partial_\theta u_\varphi\boldsymbol{e}_\varphi\right)\\
&\quad+v_\varphi\left(\frac{1}{r}\left(\frac{1}{\sin\theta}\partial_\varphi u_r-u_\varphi\right)\boldsymbol{e}_r+\frac{1}{r}\left(\frac{1}{\sin\theta}\partial_\varphi u_\theta-\operatorname{cotan}\theta u_\varphi\right)\boldsymbol{e}_\theta\right.\\
&\quad\left.+\frac{1}{r}\left(\frac{1}{\sin\theta}\partial_\varphi u_\varphi+u_r+\operatorname{cotan}\theta u_\theta\right)\boldsymbol{e}_\varphi\right)\\
&= \left(v_r\partial_r u_r+\frac{v_\theta}{r}\left(\partial_\theta u_r-u_\theta\right)+\frac{v_\varphi}{r}\left(\frac{1}{\sin\theta}\partial_\varphi u_r-u_\varphi\right)\right)\boldsymbol{e}_r
\end{aligned}$$

$$+\left(v_r\partial_r u_\theta + \frac{v_\theta}{r}\left(\partial_\theta u_\theta + u_r\right) + \frac{v_\varphi}{r}\left(\frac{1}{\sin\theta}\partial_\varphi u_\theta - \operatorname{cotan}\theta u_\varphi\right)\right)\boldsymbol{e}_\theta$$
$$+\left(v_r\partial_r u_\varphi + \frac{v_\theta}{r}\partial_\theta u_\varphi + \frac{v_\varphi}{r}\left(\frac{1}{\sin\theta}\partial_\varphi u_\varphi + u_r + \operatorname{cotan}\theta u_\theta\right)\right)\boldsymbol{e}_\varphi$$

and in particular, for $u = u_\theta(\theta,\varphi)\boldsymbol{e}_\theta + u_\varphi(\theta,\varphi)\boldsymbol{e}_\varphi$ and $v = v_\theta(\theta,\varphi)\boldsymbol{e}_\theta + v_\varphi(\theta,\varphi)\boldsymbol{e}_\varphi$, we have at $r = 1$,

$$\begin{aligned}\nabla u\cdot v &= -\left(v_\theta u_\theta + v_\varphi u_\varphi\right)\boldsymbol{e}_r + \left(v_\theta\partial_\theta u_\theta + v_\varphi\left(\frac{1}{\sin\theta}\partial_\varphi u_\theta - \operatorname{cotan}\theta u_\varphi\right)\right)\boldsymbol{e}_\theta \\ &\quad + \left(v_\theta\partial_\theta u_\varphi + v_\varphi\left(\frac{1}{\sin\theta}\partial_\varphi u_\varphi + \operatorname{cotan}\theta u_\theta\right)\right)\boldsymbol{e}_\varphi \\ &= -\langle u, v\rangle\,\boldsymbol{e}_r + \mathrm{G}_\Gamma\, u\cdot v,\end{aligned} \tag{A.2}$$

where we have used the notation (4.1) for $\mathrm{G}_\Gamma$. This yields the following expression

$$\mathrm{G}_\Gamma\, u\cdot v = \left(v_\theta\partial_\theta u_\theta + v_\varphi\left(\frac{1}{\sin\theta}\partial_\varphi u_\theta - \operatorname{cotan}\theta u_\varphi\right)\right)\boldsymbol{e}_\theta + \left(v_\theta\partial_\theta u_\varphi + v_\varphi\left(\frac{1}{\sin\theta}\partial_\varphi u_\varphi + \operatorname{cotan}\theta u_\theta\right)\right)\boldsymbol{e}_\varphi. \tag{A.3}$$

## Appendix B. Proofs of some technical results

We gather here proofs of some results stated in this article.

*Proof of Lemma 2.1.* In order to prove that $\tilde{\mathcal{J}}(\delta)$ is a nonempty open set of $\mathbb{R}^d$, we first recall (see [8], Prop. 2, p. 287) and ([9], p. 1) that the set $\mathcal{O}$ of the $C^k$-diffeomorphisms is a nonempty open set of $C^k(\mathbb{S}^2)$.

Moreover, from (2.2) and (2.3), we have for any $(\delta, s) \in C^k(\mathbb{S}^2, \mathrm{T}\mathbb{S}^2) \times \mathbb{R}^d$

$$\left[\mathcal{X}_\delta(s)\right](y) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left((2n+1)y + \sum_{j=1}^d s_j\delta_j(y)\right)\left|\sum_{j=1}^d s_j\delta_j(y)\right|^{2n} \tag{B.1}$$

and thus the map $\Xi : s \in \mathbb{R}^d \mapsto \mathcal{X}_\delta(s) \in C^k(\mathbb{S}^2)$ is continuous (analytic). Finally, using that $\tilde{\mathcal{J}}(\delta) = \Xi^{-1}(\mathcal{O})$ we deduce that $\tilde{\mathcal{J}}(\delta)$ is an open set of $\mathbb{R}^d$.

Since $\mathcal{J}(\delta)$ is a connected component of an open set, $\mathcal{J}(\delta)$ is open.

Finally, since $\{0\} \subset \mathcal{J}(\delta) \subset \tilde{\mathcal{J}}(\delta)$, the sets $\mathcal{J}(\delta)$ and $\tilde{\mathcal{J}}(\delta)$ are nonempty. □

*Proof of Proposition 2.3.* In order to prove the connectivity of $\mathcal{A}^k(d)$, let us consider $f : \mathcal{A}^k(d) \to \{0,1\}$ a continuous function. For every $\Psi_0 \in \mathcal{D}_0^k$ and every $\delta \in C^k(\mathbb{S}^2, \mathrm{T}\mathbb{S}^2)^d$, we have by definition that $\mathcal{J}(\delta)$ is connected and contains 0. Consequently,

$$f(\Psi_0, \delta, s) = f(\Psi_0, \delta, 0) \quad (s \in \mathcal{J}(\delta)).$$

Now, using the continuity of $(\Psi_0, \delta) \mapsto f(\Psi_0, \delta, 0)$ and the connectivity of $\mathcal{SC}^k(d)$, we conclude that

$$f(\Psi_0, \delta, 0) = f(0, 0, 0) \quad ((\Psi_0, \delta) \in \mathcal{SC}^k(d)).$$

This yields the connectivity of $\mathcal{A}^k(d)$.

In order to prove that $\mathcal{A}^k(d)$ is open in $C_0^k(\mathbb{R}^3)^3 \times C^k(\mathbb{S}^2, \mathrm{T}\mathbb{S}^2)^d \times \mathbb{R}^d$, we first recall (see [24]) that $\mathcal{D}_0^k$ is an open set of $C_0^k(\mathbb{R}^3)^3$. Consequently, since

$$\mathcal{A}^k(d) = \mathcal{D}_0^k \times \left\{(\delta, s) \in C^k(\mathbb{S}^2, \mathrm{T}\mathbb{S}^2)^d \times \mathbb{R}^d \ \mid \ s \in \mathcal{J}(\delta)\right\},$$

it is enough to prove that $\left\{(\delta, s) \in C^k(\mathbb{S}^2, \mathrm{T}\mathbb{S}^2)^d \times \mathbb{R}^d \ \mid \ s \in \mathcal{J}(\delta)\right\}$ is open in $C^k(\mathbb{S}^2, \mathrm{T}\mathbb{S}^2)^d \times \mathbb{R}^d$.

Assume $(\overline{\delta}, \overline{s}) \in C^k(\mathbb{S}^2, \mathrm{T}\mathbb{S}^2)^d \times \mathbb{R}^d$ with $\overline{s} \in \mathcal{J}(\overline{\delta})$. We want now to show that if $(\delta, s) \in C^k(\mathbb{S}^2, \mathrm{T}\mathbb{S}^2)^d \times \mathbb{R}^d$ is close to $(\overline{\delta}, \overline{s})$, then $s \in \mathcal{J}(\delta)$. Since $\mathcal{J}(\overline{\delta})$ is open and connected, it is path-connected. Thus, we can consider a continuous map $\overline{\gamma} : [0,1] \to \mathcal{J}(\overline{\delta})$ with $\overline{\gamma}(0) = 0$ and $\overline{\gamma}(1) = \overline{s}$. Now, we use the continuous (analytic) map (see (B.1))

$$\widetilde{\Xi} : (\delta, s) \in C^k(\mathbb{S}^2, \mathrm{T}\mathbb{S}^2)^d \times \mathbb{R}^d \mapsto \mathcal{X}_\delta(s) \in C^k(\mathbb{S}^2)$$

and the open set $\mathcal{O}$ of the $C^k$-diffeomorphisms in $C^k(\mathbb{S}^2)$ to deduce as in the above proof that

$$\widetilde{\Xi}^{-1}(\mathcal{O}) = \left\{(\delta, s) \in C^k(\mathbb{S}^2, \mathrm{T}\mathbb{S}^2)^d \times \mathbb{R}^d \ \mid \ s \in \widetilde{\mathcal{J}}(\delta)\right\}$$

is an open set. Thus, there is an open neighborhood of the compact set $\{\overline{\delta}\} \times \mathrm{Im}(\overline{\gamma})$ in $\widetilde{\Xi}^{-1}(\mathcal{O})$. In particular, there exists $\varepsilon > 0$ such that if

$$\|\overline{\delta} - \delta\|_{C^k(\mathbb{S}^2, \mathrm{T}\mathbb{S}^2)^d} + |\overline{\gamma}(t) - s| < \varepsilon, \tag{B.2}$$

for some $t \in [0,1]$, then $s \in \widetilde{\mathcal{J}}(\delta)$. Assume

$$\|\overline{\delta} - \delta\|_{C^k(\mathbb{S}^2, \mathrm{T}\mathbb{S}^2)^d} + |\overline{s} - s| < \varepsilon,$$

then if we extend the path $\overline{\gamma}$ by a straight line joining $\overline{s}$ to $s$, the corresponding path $\gamma$ satisfies $\{\delta\} \times \mathrm{Im}(\gamma) \subset \widetilde{\Xi}^{-1}(\mathcal{O})$. By definition, all the points of $\mathrm{Im}(\gamma)$ belong to $\mathcal{J}(\delta)$ and thus $s \in \mathcal{J}(\delta)$.

The density of $\mathcal{A}^\infty(d)$ in $\mathcal{A}^k(d)$ follows from the density of $C_0^\infty(\mathbb{R}^3)^3 \times C^\infty(\mathbb{S}^2, \mathrm{T}\mathbb{S}^2)^d \times \mathbb{R}^d$ into $C_0^k(\mathbb{R}^3)^3 \times C^k(\mathbb{S}^2, \mathrm{T}\mathbb{S}^2)^d \times \mathbb{R}^d$ and from the fact that $\mathcal{A}^k(d)$ is an open set.

Finally, the analyticity of $(c, s) \in \mathcal{A}^k(d) \mapsto X_c(s) \in C^k(\mathbb{S}^2, \mathbb{R}^3)$ comes from (2.5) and the expression of $\mathcal{X}_\delta(s)$ (see (B.1)).

□

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