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INFINITELY MANY PERIODIC SOLUTIONS FOR A SEMILINEAR WAVE EQUATION WITH X-DEPENDENT COEFFICIENTS*

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Abstract. This paper is devoted to the study of periodic (in time) solutions to an one-dimensional semilinear wave equation with x-dependent coefficients under various homogeneous boundary conditions. Such a model arises from the forced vibrations of a nonhomogeneous string and propagation of seismic waves in nonisotropic media. By combining variational methods with an approximation argument, we prove that there exist infinitely many periodic solutions whenever the period is a rational multiple of the length of the spatial interval. The proof is essentially based on the spectral properties of the wave operator with x-dependent coefficients.

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1. Introduction

In this paper, we are concerned with the existence of infinitely many periodic solutions to the wave equation with x-dependent coefficients

$$\rho(x)u_{tt} - (\rho(x)u_x)_x = \mu\rho(x)u + \rho(x)|u|^{p-1}u, \quad t \in \mathbb{R}, \quad 0 < x < \pi, \tag{1.1}$$

with the boundary conditions

$$a_1 u(t,0) + b_1 u_x(t,0) = 0, \quad a_2 u(t,\pi) + b_2 u_x(t,\pi) = 0, \; t \in \mathbb{R}, \tag{1.2}$$

and the periodic conditions

$$u(t+T,x) = u(t,x), \; u_t(t+T,x) = u_t(t,x), \quad t \in \mathbb{R}, \quad 0 < x < \pi. \tag{1.3}$$

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Here $a_i^2 + b_i^2 \neq 0$ for $i = 1, 2$, $0 < p < 1$ and $\mu > 0$ is a constant, and the period T is a rational multiple of π. For convenience, we write

$$T = 2\pi \frac{a}{b},$$

where a, b are relatively prime positive integers.

As stated in [2–4, 15–21, 26, 27], equation (1.1) is a mathematical model to account for the forced vibrations of a bounded nonhomogeneous string and the propagation of seismic waves in nonisotropic media. More precisely, the vertical displacement $u(t, z)$ at time t and depth z of a plane seismic wave is described by the equation

$$\omega(z)u_{tt} - (\nu(z)u_z)_z = 0 \tag{1.4}$$

with some initial conditions in t and boundary conditions in z, where $\omega(z)$ is the rock density and $\nu(z)$ is the elasticity coefficient. By the change of variable

$$x = \int_0^z \left(\frac{\omega(s)}{\nu(s)} \right)^{1/2} \mathrm{d}s,$$

equation (1.4) is transformed into

$$\rho(x)u_{tt} - (\rho(x)u_x)_x = 0,$$

where $\rho = (\omega\nu)^{1/2}$ denotes the acoustic impedance function.

It is well known that when $\rho(x)$ is a non-zero constant, it corresponds to the constant coefficients wave equation. A great deal of attention has been paid to find periodic solutions of nonlinear wave equations with constant coefficients since 1960s. The related results are [1, 5–7, 12, 13, 23–25, 30] and the references therein. In recent decades, the problem of finding infinitely many periodic solutions of wave equations with constant coefficients has captured much research interest. In [28], with the aid of variational methods, Tanaka proved that there exist infinitely many periodic solutions for the nonhomogeneous one-dimensional wave equation with the nonlinearity $|u|^{p-1}u$, where $1 < p < 1 + \sqrt{2}$. Later, he [29] extended the result to the case $p > 1$ by a delicate calculation. Employing variational methods together with an approximation argument, Ding *et al.* [14] obtained the existence of infinitely many periodic solutions to the one-dimensional homogeneous wave equation with the general nonlinearity satisfying superlinear or sublinear growth (see also [33] for the existence of infinitely many solutions of periodic Hamiltonian elliptic system). For higher dimensional case, Chen and Zhang [10, 11] considered the wave equation with the nonlinearity $|u|^{p-1}u$ in a ball in $\mathbb{R}^N$ and obtained the existence of infinitely many periodic solutions by using variational methods and an approximation argument, where $0 < p < 1$. These results are essentially based on the properties of the spectrum of the wave operator.

On the other hand, Barbu and Pavel [2–4] first studied the problem of finding periodic solutions to the nonlinear wave equations with x-dependent coefficients. For the nonlinear term satisfying Lipschitz continuous and sublinear growth, Barbu and Pavel [3] proved the existence and regularity of periodic solutions. Rudakov [26] considered the existence of periodic solutions for such wave equation with power-law growth nonlinearity. Ji and Li [20] obtained an existence result of periodic solution for the case that coefficients may not satisfy the restriction $\eta_\rho(x) > 0$, which actually solves an open problem posted in [3]. For the bounded nonlinearity, without the restriction $\eta_\rho(x) > 0$ on the coefficients, Ji [17] obtained the existence of periodic solutions for the Dirichlet-Neumann boundary value problem. Chen [9] applied a global inverse function theorem (see [22]) to prove the existence and uniqueness of periodic solutions for a system of nonlinear wave equation with x-dependent coefficients. These papers deal with the problem under the Dirichlet boundary conditions. Compared with the above works, Ji and his collaborators acquired some related results for the general Sturm-Liouville boundary value problem [15, 18], and periodic and anti-periodic boundary value problem [16, 19]. In [31], by using topological degree methods, Wang and An obtained an existence result on periodic solution of the problem with resonance and the sublinear nonlinearity under Dirichlet-Neumann boundary conditions. Recently, with the help of the Leray-Schauder degree theory, Ji *et al.* [21] obtained the existence and multiplicity of periodic solutions under the Dirichlet-Neumann boundary conditions. The restriction to such type of boundary value problem guarantees the compactness of the inverse operator on its range.

In comparison with these works, the aim of this paper is devoted to the existence of infinitely many periodic solutions for the problem (1.1)–(1.3). In order to solve this problem, we construct a suitable function space which is called working space here. By combining the $\mathbb{Z}_2$-index theory with minimax method for even functionals, we first work out the problem on some given subspaces of the working space. Then we obtain infinitely many periodic solutions of the problem (1.1)–(1.3) by a limiting process. Our method is based on a delicate analysis for the asymptotic character of the spectrum of the wave operator with x-dependent coefficients under various homogeneous boundary conditions, and the spectral properties play an essential role in the proof.

In this paper, we make the following assumptions:

(A1) $\rho(x) \in H^2(0,\pi)$ is such that $0 < \rho(x) \le \beta_0$ for any $x \in [0,\pi]$, and

$$\rho_0 = \operatorname{ess\,inf} \eta_\rho(x) > 0,$$

where

$$\eta_\rho(x) = \frac{1}{2}\frac{\rho''}{\rho} - \frac{1}{4}\left(\frac{\rho'}{\rho}\right)^2.$$

(A2) Set $\alpha_1 = a_1 - \frac{b_1}{2}\left(\frac{\rho'(0)}{\rho(0)}\right)$, $\alpha_2 = a_2 - \frac{b_2}{2}\left(\frac{\rho'(\pi)}{\rho(\pi)}\right)$, $\beta_1 = -b_1$, $\beta_2 = b_2$, and satisfy

$$\alpha_i \ge 0,\ \beta_i \ge 0 \text{ and } \alpha_i + \beta_i > 0,\ \text{for } i = 1,2.$$

The rest of this paper is organized as follows. In Section 2, we give some notations and preliminaries such as the definition of weak solution of problem (1.1)–(1.3), the asymptotic character of the spectrum of the wave operator with x-dependent coefficients under various homogeneous boundary conditions, and the definition of the working space, etc. Meanwhile, we characterize the solutions of problem (1.1)–(1.3) as the critical points of the corresponding variational problem, and state the main results. In Section 3, we prove the bounds of the corresponding functional on some spherical surfaces and some subspaces. In Section 4, with the aid of $\mathbb{Z}_2$-index theory and minimax method, we obtain a sequence of the critical points for the corresponding functional restricted on some given subspaces. In Section 5, by an approximation argument, we prove the main result.

2. Preliminaries and main results

Set $\Omega = (0,T) \times (0,\pi)$, and denote

$$\begin{aligned}\Psi = \{\psi \in C^\infty(\Omega) : a_1\psi(t,0) + b_1\psi_x(t,0) = 0,\ a_2\psi(t,\pi) + b_2\psi_x(t,\pi) = 0,\\ \psi(0,x) = \psi(T,x), \psi_t(0,x) = \psi_t(T,x)\}.\end{aligned}$$

Let

$$L^r(\Omega) = \left\{u : \|u\|^r_{L^r(\Omega)} = \int_\Omega |u(t,x)|^r \rho(x)\mathrm{d}t\mathrm{d}x < \infty\right\},\ r \ge 1.$$

It is well known that Ψ is dense in $L^r(\Omega)$ for any $r \geq 1$, and $L^2(\Omega)$ is a Hilbert space with the inner product

$$\langle u, v\rangle = \int_\Omega u(t,x)\overline{v(t,x)}\rho(x)\mathrm{d}t\mathrm{d}x,\ \forall u, v \in L^2(\Omega).$$

We rewrite (1.1)–(1.3) on Ω in the following form

$$\rho(x)u_{tt} - (\rho(x)u_x)_x = \mu\rho(x)u + \rho(x)|u|^{p-1}u,\ (t,x) \in \Omega, \tag{2.1}$$
$$a_1u(t,0) + b_1u_x(t,0) = 0,\ a_2u(t,\pi) + b_2u_x(t,\pi) = 0,\ t \in (0,T), \tag{2.2}$$
$$u(0,x) = u(T,x),\ u_t(0,x) = u_t(T,x),\ x \in (0,\pi). \tag{2.3}$$

Definition 2.1. A function $u \in L^r(\Omega)$ is said to be a weak solution to problem (2.1)–(2.3) if

$$\int_\Omega u(\rho\psi_{tt} - (\rho\psi_x)_x)\mathrm{d}t\mathrm{d}x - \int_\Omega (\mu u + |u|^{p-1}u)\psi\rho\mathrm{d}t\mathrm{d}x = 0,\ \forall \psi \in \Psi.$$

In order to look for the periodic solutions of problem (2.1)–(2.3), we need to use the complete orthonormal system of eigenfunctions $\{\phi_j(t)\varphi_k(x) : j \in \mathbb{Z}, k \in \mathbb{N}\}$ in $L^2(\Omega)$ (see [32]), where

$$\phi_j(t) = T^{-\frac{1}{2}}e^{i\nu_j t},\ \nu_j = 2j\pi T^{-1},\ j \in \mathbb{Z},$$

and λ_k, $\varphi_k(x)$ are given by the Sturm-Liouville problem

$$\begin{aligned} &-(\rho(x)\varphi_k'(x))' = \lambda_k^2\rho(x)\varphi_k(x),\ k \in \mathbb{N},\\ &a_1\varphi_k(0) + b_1\varphi_k'(0) = 0,\\ &a_2\varphi_k(\pi) + b_2\varphi_k'(\pi) = 0. \end{aligned} \tag{2.4}$$

It is known that λ_k^2 is increasingly convergent to $+\infty$ as k goes to $+\infty$. Set $z_k(x) = (\rho(x))^{1/2}\varphi_k(x)$, then (2.4) can be transformed into the following Sturm-Liouville problem

$$\begin{aligned} &z_k''(x) + (\lambda_k^2 - \eta_\rho(x))z_k(x) = 0,\ k \in \mathbb{N},\\ &\alpha_1 z_k(0) - \beta_1 z_k'(0) = 0,\\ &\alpha_2 z_k(\pi) + \beta_2 z_k'(\pi) = 0, \end{aligned} \tag{2.5}$$

where α_i, β_i satisfy the assumption (A2) for $i = 1, 2$. In this situation, it is more convenient to study the properties of the eigenvalues λ_k^2. Now, we divide (2.5) into the following several cases:

Case 1: $\alpha_1 > 0, \beta_1 = 0, \alpha_2 > 0, \beta_2 = 0$;
Case 2: $\alpha_1 > 0, \beta_1 = 0, \alpha_2 = 0, \beta_2 > 0$;
Case 3: $\alpha_1 = 0, \beta_1 > 0, \alpha_2 > 0, \beta_2 = 0$;
Case 4: $\alpha_1 = 0, \beta_1 > 0, \alpha_2 = 0, \beta_2 > 0$;
Case 5: $\alpha_1 > 0, \beta_1 > 0, \alpha_2 > 0, \beta_2 > 0$.

Here, Case 1 is called Dirichlet boundary conditions, Case 2 and Case 3 are called Dirichlet-Neumann boundary conditions, Case 4 is called Neumann boundary conditions and Case 5 is called general boundary conditions. In what follows, we shall deal with problem (2.1)–(2.3) according to the different types of boundary conditions. However, it is worthwhile to remark that, Case 2 and Case 3 are actually equivalent under the transformation $\tilde{x} = \pi - x$, so we only need to deal with Case 2 for these two cases.

Define the linear operator L_0 by

$$L_0\psi = \rho^{-1}\left(\rho\psi_{tt} - (\rho\psi_x)_x\right), \ \forall\psi \in \Psi,$$

and denote its extension in $L^2(\Omega)$ by L. It is known that L is a selfadjoint operator (see [3, 15]), and $u \in L^2(\Omega)$ is a weak solution of problem (2.1)–(2.3) if and only if

$$Lu = \mu u + |u|^{p-1}u.$$

Moreover, it is easy to see that the eigenvalues of L have the form $\lambda_{jk} = \lambda_k^2 - \nu_j^2$. Denote the set of eigenvalues of operator L by

$$\Lambda(L) = \{\lambda_{jk} : \lambda_{jk} = \lambda_k^2 - \nu_j^2\}.$$

To analyze $\Lambda(L)$ in great detail, we need to use the asymptotic formulas of λ_k, which are characterized by Barbu and Pavel [3] for Case 1, and Ji [15] for Cases 2–5, respectively.

Lemma 2.2 ([3])**.** *Assume that $\rho(x)$ satisfies (A1), then the eigenvalues of problem (2.5) with the Dirichlet boundary conditions (i.e., Case 1) have the form*

$$\lambda_k = k + \theta_k \text{ with } \theta_k \to 0 \text{ as } k \to \infty,$$

where

$$0 < \frac{\rho_2}{k} \le \sqrt{k^2+\rho_0} - k \le \theta_k \le \sqrt{k^2+\rho_1} - k \le \frac{\rho_1}{2k}, \quad for \quad k \ge 1, \tag{2.6}$$

with $\rho_1 = \frac{2}{\pi}\int_0^\pi \eta_\rho(x)\mathrm{d}x$ and $\rho_2 = \sqrt{\rho_0+1} - 1$.

By Lemma 2.2, we can easily obtain the following lemma.

Lemma 2.3. *Let assumption (A1) hold. Then, for Case 1 we have*

(i) L has at least one essential spectral point, and all of them belong to $[2\rho_2, \rho_1]$;
(ii) If $\lambda \in \Lambda(L)$ and $\lambda \notin [2\rho_2, \rho_1]$, then λ is isolated and its multiplicity is finite.

Proof. By Lemma 2.2, the eigenvalues of operator L can be rewritten as

$$\lambda_{jk} = \lambda_k^2 - \nu_j^2 = a^{-2}(ka + \theta_k a - jb)(ka + \theta_k a + jb).$$

Thus, when $ka \ne |j|b$, it is easy to verify that $|\lambda_{jk}| \to \infty$ as $j, k \to \infty$. On the other hand, when $ka = |j|b$, by (2.6) we have

$$2\rho_2 \leftarrow (\frac{\rho_2}{k})^2 + 2\rho_2 \le \lambda_{jk} = \theta_k(2k+\theta_k) \le \rho_1 + (\frac{\rho_1}{2k})^2 \to \rho_1,$$

as $k \to \infty$. Therefore, $\Lambda(L)$ has at least one accumulation point in $[2\rho_2, \rho_1]$. Moreover, λ is isolated and its multiplicity is finite when $\lambda \in \Lambda(L)$ and $\lambda \notin [2\rho_2, \rho_1]$. The proof is completed. □

For Case 1, we have the main result.

Theorem 2.4. *Assume that $0 < p < 1$, and $\mu \notin \Lambda(L)$ and satisfies $\mu > \rho_1$. If the assumption (A1) holds, then the problem (2.1)–(2.3) with Dirichlet boundary conditions (i.e. Case 1) has infinitely many periodic weak solutions u_l satisfying*

$$\int_\Omega |u_l|^{p+1} \rho \mathrm{d}t\mathrm{d}x \to 0, \quad as \quad l \to \infty.$$

Recalling that Case 2 is equivalent to Case 3, Ji [15] has given the asymptotic formula of λ_k for Case 2, and we state it as the following lemma.

Lemma 2.5 ([15])**.** *Assume that $\rho(x)$ satisfies (A1), then the eigenvalues of problem (2.5) with Dirichlet-Neumann boundary conditions (i.e., Case 2) have the form*

$$\lambda_k = k + 1/2 + \theta_k \ with \ \theta_k \to 0 \ as \ k \to \infty,$$

where

$$0 < \frac{\rho_3}{2k+1} \leq \theta_k \leq \frac{\rho_1}{\sqrt{(k+1/2)^2 + \rho_1} + k + 1/2} \leq \frac{\rho_1}{2k+1},$$

with $\rho_3 = \frac{\rho_0}{1+\rho_0}$ and $\rho_1 = \frac{2}{\pi}\int_0^\pi \eta_\rho(x)\mathrm{d}x$.

By a similar proof as that of Lemma 2.3, we can obtain the following Lemma 2.6, Lemma 2.9 and Lemma 2.12. Thus, their proofs are omitted here.

Lemma 2.6. *Let assumption (A1) hold. Then, for Case 2 we have*

(i) L has at least one essential spectral point, and all of them belong to $[\rho_3, \rho_1]$;
(ii) If $\lambda \in \Lambda(L)$ and $\lambda \notin [\rho_3, \rho_1]$, then λ is isolated and its multiplicity is finite.

Theorem 2.7. *Assume that $0 < p < 1$, and $\mu \notin \Lambda(L)$ and satisfies $\mu > \rho_1$. If the assumption (A1) holds, then the problem (2.1)–(2.3) with Dirichlet-Neumann boundary conditions (i.e., Case 2 or 3) has infinitely many periodic weak solutions u_l satisfying*

$$\int_\Omega |u_l|^{p+1} \rho \mathrm{d}t\mathrm{d}x \to 0, \quad as \quad l \to \infty.$$

Similarly, for Case 4, we have the following results.

Lemma 2.8 ([15])**.** *Assume that $\rho(x)$ satisfies (A1), then the eigenvalues of problem (2.5) with Neumann boundary conditions (i.e., Case 4) have the form*

$$\lambda_k = k + \theta_k \ with \ \theta_k \to 0 \ as \ k \to \infty,$$

where

$$0 < \frac{\rho_2}{k} \leq \theta_k \leq \sqrt{k^2 + \rho_1} - k \leq \frac{\rho_1}{2k}, \quad for \quad k \geq 1,$$

with $\rho_2 = \sqrt{1+\rho_0} - 1$ and $\rho_1 = \frac{2}{\pi}\int_0^\pi \eta_\rho(x)\mathrm{d}x$.

Lemma 2.9. *Let assumption (A1) hold. Then, for Case 4 we have*

(i) L has at least one essential spectral point, and all of them belong to $[2\rho_2, \rho_1]$;
(ii) If $\lambda \in \Lambda(L)$ and $\lambda \notin [2\rho_2, \rho_1]$, then λ is isolated and its multiplicity is finite.

Theorem 2.10. *Assume that $0 < p < 1$, and $\mu \notin \Lambda(L)$ and satisfies $\mu > \rho_1$. If the assumption $(A1)$ holds, then the problem (2.1)–(2.3) with Neumann boundary conditions (i.e., Case 4) has infinitely many periodic weak solutions u_l satisfying*

$$\int_\Omega |u_l|^{p+1}\rho \mathrm{d}t\mathrm{d}x \to 0, \quad as \quad l \to \infty.$$

For Case 5, we have the following results.

Lemma 2.11 ([15])**.** *Assume that $\rho(x)$ satisfies (A1), then there exists a constant $N_0 > 1$ such that the eigenvalues of problem (2.5) with the general boundary conditions (i.e., Case 5) have the form*

$$\lambda_k = k + \theta_k \text{ with } \theta_k \to 0 \text{ as } k \to \infty,$$

where

$$0 < \frac{\rho_2}{k} \leq \theta_k \leq \sqrt{k^2 + 2\rho_4} - k \leq \frac{\rho_4}{k}, \quad for \quad k \geq N_0,$$

with

$$\rho_2 = \sqrt{1+\rho_0} - 1, \quad \rho_4 = \frac{1}{\pi}\Big(\frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2} + 1 + \int_0^\pi \eta_\rho(x)\mathrm{d}x\Big).$$

Lemma 2.12. *Let assumption $(A1)$ hold. Then, for Case 5 we have*

(i) L has at least one essential spectral point, and all of them belong to $[2\rho_2, 2\rho_4]$;
(ii) If $\lambda \in \Lambda(L)$ and $\lambda \notin [2\rho_2, 2\rho_4]$, then λ is isolated and its multiplicity is finite.

Theorem 2.13. *Assume that $0 < p < 1$, and $\mu \notin \Lambda(L)$ and satisfies $\mu > 2\rho_4$. If the assumption $(A1)$ holds, then the problem (2.1)–(2.3) with the general boundary conditions (i.e., Case 5) has infinitely many periodic weak solutions u_l satisfying*

$$\int_\Omega |u_l|^{p+1}\rho \mathrm{d}t\mathrm{d}x \to 0, \quad as \quad l \to \infty.$$

Remark 2.14. In order to acquire the solutions of problem (2.1)–(2.3) *via* variational methods, we construct a function space E (which is called the working space here and will be given later) in which the critical point theory can be applied.

On the other hand, by a careful observation, the conditions and conclusions in Theorems 2.4, 2.7 and 2.10 are very similar, thus their proofs can be accomplished in a similar way. Moreover, we find the only difference between Theorem 2.4 and Theorem 2.13 lies in the range of the constant μ. More precisely, in Theorem 2.4 we require $\mu > \rho_1$, but $\mu > 2\rho_4$ in Theorem 2.13. The condition $\mu > \rho_1$ or $\mu > 2\rho_4$ can make sure that the working space E is well defined, which plays an essential role in the proof. Therefore, by above discussion we only give the proof of Theorem 2.4 here and by the same argument we can also prove Theorem 2.13.

In what follows, we always assume that $\mu \notin \Lambda(L)$ and $\mu > \rho_1$, then there exists a constant $\delta > 0$ such that

$$|\lambda_{jk} - \mu| \geq \delta > 0, \; j \in \mathbb{Z}, k \in \mathbb{N}^+. \tag{2.7}$$

For $u \in L^2(\Omega)$, we rewrite $u(t,x) = \sum\limits_{j,k} u_{jk}\phi_j(t)\varphi_k(x)$, where u_{jk} are the Fourier coefficients. Define the function space

$$E = \Big\{u \in L^2(\Omega) \mid \|u\|_E^2 = \sum_{j,k} |\lambda_{jk} - \mu||u_{jk}|^2 < \infty\Big\},$$

which is called the working space here. The estimate (2.7) shows that $\|\cdot\|_E$ is a norm on E. Furthermore, the space E is a Hilbert space equipped with the norm $\|\cdot\|_E$.

Since $\|u\|_{L^2(\Omega)}^2 = \sum\limits_{j,k} |u_{jk}|^2$, from (2.7), we have

$$\|u\|_{L^2(\Omega)}^2 \leq \delta^{-1}\|u\|_E^2, \tag{2.8}$$

which implies the continuous embedding $E \hookrightarrow L^2(\Omega)$. Moreover, for $1 \leq r \leq 2$, the continuous embedding $L^2(\Omega) \hookrightarrow L^r(\Omega)$ implies that there exists a constant $C = C(r)$ such that

$$\|u\|_{L^r(\Omega)} \leq C\|u\|_E. \tag{2.9}$$

Let $f(u) = |u|^{p-1}u$ and $F(u) = \int_0^u f(s)\mathrm{d}s = \frac{1}{p+1}|u|^{p+1}$. It is easy to see that if f is odd, then F is even. Define the functional

$$\Phi(u) = -\frac{1}{2}\langle (L-\mu)u, u\rangle + \int_\Omega F(u)\rho \mathrm{d}t\mathrm{d}x, \quad \forall u \in E. \tag{2.10}$$

Thus, Φ is an even C^1 functional on E and

$$\langle \Phi'(u), v\rangle = -\langle (L-\mu)u, v\rangle + \int_\Omega f(u)v\rho \mathrm{d}t\mathrm{d}x, \quad \forall u,\ v \in E. \tag{2.11}$$

Consequently, u is a weak solution of problem (2.1)–(2.3) if and only if $\Phi'(u) = 0$. Thus, we can characterize the solutions of problem (2.1)–(2.3) as the critical points of the functional Φ in E. In addition, from the proof of Lemma 2.3, it's not difficult to see that Φ is neither bounded from above nor from below, which shows that we can not obtain the critical points of Φ by a simple minimization or maximization. In what follows, by variational methods together with an approximation argument, we prove that there exist infinitely many periodic solutions for the problem (2.1)–(2.3).

3. Bounds of the functional Φ

As already noted in Remark 2.14, here we give all the details only of the proof of Theorem 2.4; so, we deal with Case 1 and consider Lemma 2.3 as starting point of our setting.

In this section, we consider the bounds of Φ on some spherical surfaces and some subspaces, which can help us to distinguish the critical points by the different critical values. Thus, we firstly need to decompose the space E into some suitable subspaces.

Recalling that $\mu \notin \Lambda(L)$ and $\mu > \rho_1$, the working space E can be decomposed as a direct sum $E = E^+ \oplus E^-$, where

$$E^+ = \Big\{u \in E \big| u = \sum_{\lambda_{jk} > \mu} u_{jk}\phi_j(t)\varphi_k(x), \quad j \in \mathbb{Z}, k \in \mathbb{N}^+\Big\},$$

$$E^- = \Big\{u \in E \big| u = \sum_{\lambda_{jk} < \mu} u_{jk}\phi_j(t)\varphi_k(x), \quad j \in \mathbb{Z}, k \in \mathbb{N}^+\Big\}.$$

Denote

$$E_0 = \Big\{ u \in L^2(\Omega) \big| u = \sum_{ka=|j|b} u_{jk}\phi_j(t)\varphi_k(x), \quad j \in \mathbb{Z}, k \in \mathbb{N}^+ \Big\}.$$

For any $u \in E^-$, we write $u = \sum\limits_{\lambda_{jk}<\mu} u_{jk}\phi_j\varphi_k$, then

$$\langle (L-\mu)u, u\rangle = -\sum_{\lambda_{jk}<\mu} |\lambda_{jk}-\mu||u_{jk}|^2 = -\|u\|_E^2. \tag{3.1}$$

Moreover, for any $u \in E^+$, by a similar calculation we have

$$\langle (L-\mu)u, u\rangle = \|u\|_E^2. \tag{3.2}$$

Remark 3.1. By Lemma 2.3, we have E_0 is an infinite dimensional space spanned by the eigenfunctions $\phi_j\varphi_k$ for $ka = |j|b$. Moreover, it is easy to see $\dim(E^+ \cap E_0) < \infty$ and $\dim(E^- \cap E_0) = \infty$.

For any $l, m \in \mathbb{N}^+$, let

$$W_m = \operatorname{span}\Big\{ \phi_j\varphi_k \mid -m \le j \le m,\ 1 \le k \le m \Big\},$$
$$E^m = \Big(W_m \cap E^- \Big) \oplus E^+, \quad E_l = E^- \oplus \Big(W_l \cap E^+ \Big),$$

and denote

$$E_l^m = E^m \cap E_l.$$

Obviously, $E^m \subset E^{m+1}$, $E = \bigcup_{m\in\mathbb{N}^+} E^m$, and E_l^m is a finite dimensional space.

Lemma 3.2. *There exist σ_l, $R_l > 0$ such that*

$$\Phi(u) \ge \sigma_l, \quad \forall u \in E_{l+1} \cap S_{R_l},$$

where $S_{R_l} = \{u \in E \mid \|u\|_E = R_l\}$.

Proof. For $u \in E_{l+1} = E^- \oplus (W_{l+1} \cap E^+)$, split $u = u^+ + u^-$, where $u^+ \in W_{l+1} \cap E^+$ and $u^- \in E^-$. Since $\dim(W_{l+1} \cap E^+) < \infty$, all norms of u^+ in $W_{l+1} \cap E^+$ are equivalent. Moreover, since the projection mapping $P : L^{p+1}(\Omega) \to W_{l+1} \cap E^+$ is bounded, then there exists a constant $C_0 > 0$ such that

$$\|u^+\|_E^2 \le C_0 \|u\|_{p+1}^2. \tag{3.3}$$

From (2.10), taking any $u \in E$ we have

$$\begin{aligned}\Phi(u) &= -\frac{1}{2}\langle (L-\mu)u, u\rangle + \frac{1}{p+1}\int_\Omega |u|^{p+1}\rho \mathrm{d}t\mathrm{d}x \\ &= -\frac{1}{2}\|u^+\|_E^2 + \frac{1}{2}\|u^-\|_E^2 + C_0\|u\|_{L^{p+1}(\Omega)}^2 + \frac{1}{p+1}\int_\Omega |u|^{p+1}\rho \mathrm{d}t\mathrm{d}x - C_0\|u\|_{L^{p+1}(\Omega)}^2 \\ &= I_1 + I_2,\end{aligned} \tag{3.4}$$

where

$$I_1 = -\frac{1}{2}\|u^+\|_E^2 + \frac{1}{2}\|u^-\|_E^2 + C_0\|u\|_{L^{p+1}(\Omega)}^2,$$

and

$$I_2 = \frac{1}{p+1}\int_\Omega |u|^{p+1}\rho \mathrm{d}t\mathrm{d}x - C_0\|u\|_{L^{p+1}(\Omega)}^2.$$

In what follows, we devote to estimate I_1 and I_2.

By (3.3), we obtain

$$I_1 \geq \frac{1}{2}\|u^+\|_E^2 + \frac{1}{2}\|u^-\|_E^2 = \frac{1}{2}\|u\|_E^2. \tag{3.5}$$

On the other hand, in virtue of $0 < p < 1$, for any $\kappa > 0$ large enough (which will be chosen later), there exists a constant $\delta_\kappa > 0$ small enough such that

$$|f(u)| = |u|^p \geq \kappa|u|, \quad \text{for} \quad |u| \leq \delta_\kappa. \tag{3.6}$$

Thus

$$F(u) = \frac{1}{p+1}|u|^{p+1} \geq \begin{cases} \frac{\kappa}{p+1}|u|^2, & \text{for} \quad |u| \leq \delta_\kappa, \\ \frac{1}{p+1}|u|^{p+1}, & \text{for} \quad |u| > \delta_\kappa. \end{cases}$$

Set

$$u_1 = \begin{cases} u, & \text{if } |u| \leq \delta_\kappa, \\ 0, & \text{if } |u| > \delta_\kappa, \end{cases} \qquad u_2 = \begin{cases} 0, & \text{if } |u| \leq \delta_\kappa, \\ u, & \text{if } |u| > \delta_\kappa. \end{cases}$$

Then $u = u_1 + u_2$ and $\|u\|_{L^{p+1}(\Omega)} \leq \|u_1\|_{L^{p+1}(\Omega)} + \|u_2\|_{L^{p+1}(\Omega)}$. Moreover, reviewing $\Omega = (0,T)\times(0,\pi)$ and $0 < \rho(x) \leq \beta_0$, by Hölder's inequality, we have

$$\|u_1\|_{L^{p+1}(\Omega)}^{p+1} = \int_\Omega |u_1|^{p+1}\rho \mathrm{d}t\mathrm{d}x \leq (\beta_0 T\pi)^{\frac{1-p}{2}}\|u_1\|_{L^2(\Omega)}^{p+1}.$$

Thus

$$\begin{aligned} I_2 &\geq \frac{\kappa}{p+1}\int_{\Omega_{\delta_\kappa}} |u_1|^2\rho \mathrm{d}t\mathrm{d}x + \frac{1}{p+1}\int_{\Omega\setminus\Omega_{\delta_\kappa}} |u_2|^{p+1}\rho \mathrm{d}t\mathrm{d}x - C_0\|u\|_{L^{p+1}(\Omega)}^2 \\ &= \frac{\kappa}{p+1}\|u_1\|_{L^2(\Omega)}^2 + \frac{1}{p+1}\|u_2\|_{L^{p+1}(\Omega)}^{p+1} - C_0\|u\|_{L^{p+1}(\Omega)}^2 \\ &\geq \left(\frac{\kappa}{p+1}(\beta_0 T\pi)^{\frac{p-1}{p+1}} - 2C_0\right)\|u_1\|_{L^{p+1}(\Omega)}^2 + \left(\frac{1}{p+1} - 2C_0\|u_2\|_{L^{p+1}(\Omega)}^{1-p}\right)\|u_2\|_{L^{p+1}(\Omega)}^{p+1}, \end{aligned}$$

where $\Omega_{\delta_\kappa} = \{(t,x)\in\Omega \mid |u| \leq \delta_\kappa\}$.

Now, take $\kappa > 0$ large enough such that $\frac{\kappa}{p+1}(\beta_0 T\pi)^{\frac{p-1}{p+1}} - 2C_0 > 0$, then fix δ_κ satisfying (3.6). In addition, by (2.9) we have $\|u_2\|_{L^{p+1}(\Omega)} \leq \|u\|_{L^{p+1}(\Omega)} \leq C\|u\|_E$. Take $R_l > 0$ small enough such that $\frac{1}{p+1} - 2C_0(CR_l)^{1-p} > 0$.

Thus, we obtain

$$I_2 > 0, \quad \text{for} \quad u \in E_{l+1} \cap S_{R_l}.$$

Consequently, by combining $I_2 > 0$ with (3.5), from (3.4) we obtain

$$\Phi(u) \geq \frac{1}{2}\|u\|_E^2 = \frac{1}{2}R_l^2, \quad \text{for} \quad u \in E_{l+1} \cap S_{R_l}.$$

Taking $\sigma_l = \frac{1}{2}R_l^2$, we arrive at the conclusion. □

Proposition 3.3. *$\zeta_l \to 0$ as $l \to \infty$, where*

$$\zeta_l = \sup_{u \in (E_l)^\perp \setminus \{0\}} \frac{\|u\|_{L^{p+1}(\Omega)}}{\|u\|_E}. \tag{3.7}$$

Proof. In view of $E_l = E^- \oplus (W_l \cap E^+)$, we have $(E_l)^\perp \subset E^+$. Therefore, by Lemma 2.3,

$$\lambda^+_{j_l k_l} = \min\{\lambda_{jk} \in \Lambda(L) \mid \lambda_{jk} > \mu \text{ for } k > l, \ \text{or } j > l, \text{or } j < -l\}$$

is well defined.

For $u \in (E_l)^\perp \setminus \{0\}$, we have

$$\|u\|_E^2 = \langle (L-\mu)u, u\rangle = \sum_{j,k} |\lambda_{jk} - \mu||u_{jk}|^2 \geq (\lambda^+_{j_l k_l} - \mu)\|u\|^2_{L^2(\Omega)}. \tag{3.8}$$

By (2.9) and (3.8), with the help of Hölder's inequality, we obtain

$$\|u\|_{L^{p+1}(\Omega)} \leq \|u\|^\theta_{L^2(\Omega)}\|u\|^{1-\theta}_{L^r(\Omega)} \leq \frac{C^{1-\theta}}{(\lambda^+_{j_l k_l} - \mu)^{\theta/2}}\|u\|_E,$$

where $r < p+1$, and $1/(p+1) = \theta/2 + (1-\theta)/r$.

Noting $u \in (E_l)^\perp \setminus \{0\}$, it is easy to see that $\lambda^+_{j_l k_l} \to \infty$ as $l \to \infty$. Consequently,

$$\zeta_l = \sup_{u \in (E_l)^\perp \setminus \{0\}} \frac{\|u\|_{L^{p+1}(\Omega)}}{\|u\|_E} \leq \frac{C^{1-\theta}}{(\lambda^+_{j_l k_l} - \mu)^{\theta/2}} \to 0, \quad \text{as} \quad l \to 0.$$

We arrive at the result. □

Lemma 3.4. *There exists a constant $\varrho_l > 0$ such that*

$$\Phi(u) \leq \varrho_l, \quad \forall u \in (E_l)^\perp.$$

In addition, $\varrho_l \to 0$ as $l \to \infty$.

Proof. Noting $(E_l)^\perp \subset E^+$, for $u \in (E_l)^\perp$, by (3.7) we have

$$\Phi(u) = -\frac{1}{2}\langle (L-\mu)u, u\rangle + \frac{1}{p+1}\int_\Omega |u|^{p+1}\rho \mathrm{d}t\mathrm{d}x \leq -\frac{1}{2}\|u\|_E^2 + \frac{1}{p+1}\zeta_l^{p+1}\|u\|_E^{p+1}.$$

Since $0 < p < 1$, a direct calculation yields

$$\Phi(u) \le -\frac{1}{2}\zeta_l^{\frac{2(p+1)}{1-p}} + \frac{1}{p+1}\zeta_l^{p+1}\zeta_l^{\frac{(p+1)^2}{1-p}} = \Big(\frac{1}{p+1} - \frac{1}{2}\Big)\zeta_l^{\frac{2(p+1)}{1-p}}.$$

Let $\varrho_l = \Big(\frac{1}{p+1} - \frac{1}{2}\Big)\zeta_l^{\frac{2(p+1)}{1-p}}$, by Proposition 3.3, we have $\varrho_l \to 0$ as $l \to \infty$. The proof is completed. □

4. Sequence of critical points for restricted functional

In this section, we intend to employ the $\mathbb{Z}_2$-index theory and minimax method to obtain a sequence of critical points for the functional Φ restricted on some subspaces of E.

At first, in order to make the following statement more precise, we make some notations. Denote

$$\Sigma = \{A \subset E^m \backslash \{0\} \mid A = \bar{A}, -A = A\}$$

which is made of closed symmetric subset of $E^m \backslash \{0\}$, where $\bar{A}$ denotes the closure of A. For any $c \in \mathbb{R}$, let $\Phi^c = \{u \in E \mid \Phi(u) \ge c\}$ denote the level set of Φ at c, and

$$K = \{u \in E^m \mid \Phi'_m(u) = 0\}$$

denote the critical points set of $\Phi_m = \Phi|_{E^m}$.

The mapping $\gamma : \Sigma \to \mathbb{N} \cup \{+\infty\}$ is called genus if it satisfies, for any $A \in \Sigma$,

$$\gamma(A) = \begin{cases} 0, \quad \text{if } A = \emptyset, \\ \inf\{N \in \mathbb{Z}_+ | \exists \text{ an odd mapping } h \in C(A, \mathbb{R}^N \backslash \{0\})\}, \\ +\infty, \quad \text{if no such odd mapping.} \end{cases}$$

In this paper, we need to use the following properties of genus (see [8]).

(G1) Assume $E = E_1 \oplus E_2$, $\dim E_1 = N$, and $\gamma(A) > N$ for $A \in \Sigma$, then $A \cap E_2 \ne \emptyset$.

(G2) Assume that $U \subset \mathbb{R}^N$ is an open bounded symmetric neighborhood of the origin in $\mathbb{R}^N$, then $\gamma(\partial U) = N$.

(G3) (Super-variant) For any continuous odd mapping $h : E^m \to E^m$, it holds $\gamma(A) \le \gamma(\overline{h(A)})$, $\forall A \in \Sigma$.

The following compact embedding plays an important role in this paper.

Proposition 4.1. *For any $r \in [1, 2]$, the embedding*

$$E \ominus E_0 \hookrightarrow L^r(\Omega), \tag{4.1}$$

is compact.

Proof. For $u \in E \ominus E_0$, we write $u = \sum\limits_{ka \ne |j|b} u_{jk}\phi_j\varphi_k$.

Noting $\|u\|_E = \big(\sum\limits_{j,k} |\lambda_{jk} - \mu||u_{jk}|^2\big)^{\frac{1}{2}}$, we have that the mapping

$$\tau_0 : u = \sum_{j,k} u_{jk}\phi_j\varphi_k \mapsto \{|\lambda_{jk} - \mu|^{\frac{1}{2}} u_{jk}\}$$

is continuous from $E \ominus E_0$ to l^2, where l^2 denotes the space of square summable sequences.

Observing that $|\lambda_{jk}-\mu| \to \infty$ as $j,k\to\infty$, it follows that the mapping

$$\tau_1 : \{|\lambda_{jk}-\mu|^{\frac{1}{2}} u_{jk}\} \mapsto \{u_{jk}\}$$

is compact from l^2 to l^2.

Since $\{\phi_j\varphi_k\}$ is a complete orthonormal sequence of $L^2(\Omega)$, then the mapping

$$\tau_2 : \{u_{jk}\} \mapsto u = \sum_{j,k} u_{jk}\phi_j\varphi_k$$

is continuous from l^2 to $L^2(\Omega)$.

Consequently, the mapping

$$\tau_2\tau_1\tau_0 : E\ominus E_0 \to L^2(\Omega)$$

is compact. Furthermore, for $1\le r\le 2$, the continuous embedding $L^2(\Omega)\hookrightarrow L^r(\Omega)$ implies that the embedding $E\ominus E_0\hookrightarrow L^r(\Omega)$ is compact. □

To acquire the critical points of $\Phi_m = \Phi|_{E^m}$ on subspaces E^m by variational methods, it needs to verify that Φ_m satisfies (PS) condition, which means that any sequence $\{u_i\}\subset E^m$ for which $\Phi_m(u_i)$ is bounded and $\Phi_m'(u_i)\to 0$ as $i\to\infty$ contains a convergent subsequence.

Lemma 4.2. *Φ_m satisfies (PS) condition.*

Proof. Assume $\{u_i\}\subset E^m$ is such that $\Phi_m(u_i)$ is bounded and $\Phi_m'(u_i)\to 0$ as $i\to\infty$. Split $u_i = u_i^+ + u_i^-$, where $u_i^+\in E^+$ and $u_i^-\in W_m\cap E^-$.

For $u_i^+\in E^+$, since $\Phi'(u_i)\to 0$ as $i\to\infty$, by (2.11) and (3.2), we have

$$\begin{aligned} o(1)\|u_i^+\|_E &\ge \langle -\Phi'(u_i), u_i^+\rangle = \langle (L-\mu)u_i^+, u_i^+\rangle - \int_\Omega |u_i|^{p-1}u_i u_i^+\rho \mathrm{d}t\mathrm{d}x \\ &\ge \|u_i^+\|_E^2 - \|u_i\|_{L^{p+1}(\Omega)}^p \|u_i^+\|_{L^{p+1}(\Omega)}. \end{aligned} \tag{4.2}$$

By combining (2.9) with (4.2), a direct calculation yields

$$\|u_i^+\|_E - C\|u_i\|_E^p \le o(1), \tag{4.3}$$

where the constant C is independent of i.

For $u_i^-\in W_m\cap E^-$, by (2.11) and (3.1) we have

$$\begin{aligned} o(1)\|u_i^-\|_E &\ge \langle \Phi'(u_i), u_i^-\rangle = -\langle (L-\mu)u_i^-, u_i^-\rangle + \int_\Omega |u_i|^{p-1}u_i u_i^-\rho \mathrm{d}t\mathrm{d}x \\ &\ge \|u_i^-\|_E^2 - \|u_i\|_{L^{p+1}(\Omega)}^p \|u_i^-\|_{L^{p+1}(\Omega)}. \end{aligned}$$

Similarly, we have

$$\|u_i^-\|_E - C\|u_i\|_E^p \le o(1). \tag{4.4}$$

Since $0<p<1$ and $\|u_i\|_E^2 = \|u_i^+\|_E^2 + \|u_i^-\|_E^2$, by (4.3) and (4.4), we have that there exists a constant $M>0$ independent of i such that

$$\|u_i\|_E \le M.$$

On the other hand, since E is a Hilbert space, we have $u_i \rightharpoonup u$ in E along with a subsequence as $i \to \infty$ for some $u \in E^m$. For the sake of convenience, we still use $\{u_i\}$ to denote the subsequence. Let u^- denote the weak limit of $\{u_i^-\}$, and decompose $u_i^+ = v_i + y_i$ and $u^+ = v + y$, where v, y are the weak limits of $\{v_i\}$, $\{y_i\}$ respectively, and $v_i, v \in E^+ \ominus E_0$, $y_i, y \in E^+ \cap E_0$.

In what follows, we devote to proving $u_i \to u$ in E^m.

In virtue of $\dim(W_m \cap E^-) < \infty$, it follows $u_i^- \to u^-$ in E.

Noting $v_i, v \in E^+ \ominus E_0$ and $u_i = v_i + y_i + u_i^-$, we have $u_i - v_i \in (E^+ \ominus E_0)^\perp$. Thus, we have

$$\langle (L-\mu)(u_i - v_i), v_i - v\rangle = 0.$$

Therefore, we have

$$\begin{aligned}\|v_i - v\|_E^2 &= \langle (L-\mu)(v_i - v), v_i - v\rangle \\ &= -\langle \Phi'(u_i), v_i - v\rangle + \int_\Omega |u_i|^{p-1} u_i (v_i - v)\rho \mathrm{d}t\mathrm{d}x - \langle (L-\mu)v, v_i - v\rangle \\ &\le o(1)\|v_i - v\|_E + \|u_i\|^p_{L^{p+1}(\Omega)} \|v_i - v\|_{L^{p+1}(\Omega)} + o(1). \end{aligned} \tag{4.5}$$

Since $v_i, v \in E^+ \ominus E_0$, by (4.1), we have $v_i \to v$ in $L^{p+1}(\Omega)$ as $i \to \infty$. Therefore, from (4.5), we obtain

$$\|v_i - v\|_E \to 0, \quad \text{as} \quad i \to 0.$$

Since $\dim(E^+ \cap E_0) < \infty$, it follows $y_i \to y$ in E. We complete the proof. □

Recalling $\dim E^m_{l+1} < \infty$, denote

$$\mathscr{F}_l^m = \{A \in \Sigma \mid A \subset E^m, \gamma(A) \ge \dim E^m_{l+1}\},$$

and

$$c_{lm} = \sup_{A \in \mathscr{F}_l^m} \inf_{u \in A} \Phi_m(u), \quad \forall m \in \mathbb{N}^+. \tag{4.6}$$

By the properties of genus and deformation lemma argument (see [8]), we have the following lemma.

Lemma 4.3. *c_{lm} is a critical value of Φ_m and satisfies*

$$\sigma_l \le c_{lm} \le \varrho_l.$$

Proof. Firstly, we prove that c_{lm} is a critical value of Φ_m by contradiction. If c_{lm} is not a critical value of Φ_m, then there exists $\bar\varepsilon > 0$ such that $\Phi_m^{-1}[c_{lm} - \bar\varepsilon, c_{lm} + \bar\varepsilon] \cap K = \emptyset$, provided by the (PS) condition.

By the definition of c_{lm}, for any $\varepsilon \in (0, \bar\varepsilon)$, there exists $A_0 \in \mathscr{F}_l^m$ such that $\inf_{u \in A_0} \Phi_m(u) \ge c_{lm} - \varepsilon$ which implies

$$A_0 \subset \Phi_m^{c_{lm} - \varepsilon}.$$

Since $\Phi_m \in C^1(E^m, \mathbb{R})$ is even and satisfies (PS) condition, then there exists an odd map $\eta \in C([0,1] \times E^m, E^m)$ satisfying:
(a) $\eta(t, \cdot)$ is a homeomorphism of E^m, $\forall t \in [0,1]$;
(b) $\eta(1, \Phi_m^{c_{lm} - \varepsilon}) \subset \Phi_m^{c_{lm} + \varepsilon}$.

By the properties (G3) and (a), we obtain $\gamma(A_0) \leq \gamma(\eta(1, A_0))$. Thus, we have $\eta(1, A_0) \in \mathscr{F}_l^m$. Noting $A_0 \subset \Phi_m^{c_{lm}-\varepsilon}$, by the property (b) it follows

$$c_{lm} \geq \inf_{u \in \eta(1,A_0)} \Phi_m(u) \geq c_{lm} + \varepsilon,$$

which is a contradiction. Therefore, c_{lm} is a critical value of Φ_m.

Secondly, since $\dim E_l^m < \dim E_{l+1}^m < \infty$, then for $A \in \mathscr{F}_l^m$, by the property (G1) of genus, we obtain $A \cap (E_l^m)^\perp \neq \emptyset$. Since $(E_l^m)^\perp = (E^m)^\perp \cup (E_l)^\perp$ and $A \subset E^m$, we have

$$A \cap (E_l)^\perp = A \cap (E_l^m)^\perp \neq \emptyset.$$

Thus, from Lemma 3.4, we have

$$\inf_{u \in A} \Phi_m(u) \leq \sup_{u \in A \cap (E_l)^\perp} \Phi_m(u) \leq \sup_{u \in (E_l)^\perp} \Phi_m(u) \leq \varrho_l, \quad \forall A \in \mathscr{F}_l^m.$$

Consequently, $c_{lm} = \sup_{A \in \mathscr{F}_l^m} \inf_{u \in A} \Phi_m(u) \leq \varrho_l$.

On the other hand, denote $S_{R_l}^m = E_{l+1}^m \cap S_{R_l}$, where R_l is the constant occurred in Lemma 3.2. Thus, by the property (G2) of genus, we have $\gamma(S_{R_l}^m) = \dim E_{l+1}^m$. Thus, $S_{R_l}^m \in \mathscr{F}_l^m$. By Lemma 3.2, we obtain

$$c_{lm} = \sup_{A \in \mathscr{F}_l^m} \inf_{u \in A} \Phi_m(u) \geq \inf_{u \in S_{R_l}^m} \Phi_m(u) \geq \inf_{u \in E_{l+1} \cap S_{R_l}} \Phi_m(u) \geq \sigma_l.$$

We complete the proof. □

Let $\{u_{lm}\}$ be the sequence of critical points of Φ_m corresponding to the critical values c_{lm}. Whereafter, we shall to prove that $\{u_{lm}\}$ contains a convergent subsequence as m goes to infinity.

5. The proof of Theorem 2.4

In this section, we first prove that for every $l \in \mathbb{N}^+$ there exists $u_l \in E$ such that $u_{lm} \to u_l$ in E as $m \to \infty$. Then we prove that u_l are the critical points of the functional Φ.

Lemma 5.1. *For any $l \in \mathbb{N}^+$ there exists a constant $M_l > 0$, independent of m, such that $\|u_{lm}\|_E \leq M_l$ for all $m \in \mathbb{N}^+$.*

Proof. Since u_{lm} are the critical points of Φ_m, we have $\Phi_m'(u_{lm}) = 0$. Thus,

$$\langle (L-\mu)u_{lm}, v\rangle = \int_\Omega |u_{lm}|^{p-1} u_{lm} v \rho \mathrm{d}t\mathrm{d}x, \quad \forall v \in E^m. \tag{5.1}$$

Taking $v = u_{lm}$ in (5.1), by Lemma 4.3, we have

$$\varrho_l \geq \Phi_m(u_{lm}) = \left(\frac{1}{p+1} - \frac{1}{2}\right) \int_\Omega |u_{lm}|^{p+1} \rho \mathrm{d}t\mathrm{d}x.$$

Thus, there exists a constant $C > 0$ independent of m such that

$$\|u_{lm}\|_{L^{p+1}(\Omega)} \leq C. \tag{5.2}$$

Split $u_{lm} = u_{lm}^+ + u_{lm}^-$, where $u_{lm}^+ \in E^m \cap E^+$ and $u_{lm}^- \in E^m \cap E^-$. Noting $\|u_{lm}^+\|_E^2 = \langle (L-\mu)u_{lm}^+, u_{lm}^+\rangle$ and taking $v = u_{lm}$ in (5.1), by (2.9) and (5.2), we obtain

$$\|u_{lm}^+\|_E^2 = \langle (L-\mu)u_{lm}, u_{lm}^+\rangle = \int_\Omega |u_{lm}|^{p-1} u_{lm} u_{lm}^+ \rho \mathrm{d}t\mathrm{d}x \le C\|u_{lm}^+\|_E.$$

Therefore, we obtain $\|u_{lm}^+\|_E \le C$. Similarly, $\|u_{lm}^-\|_E \le C$. Consequently, there exists a constant $M_l > 0$ independent of m such that $\|u_{lm}\|_E \le M_l$. The proof is completed. □

Since E is a Hilbert space, by Lemma 5.1, we have $u_{lm} \rightharpoonup u_l$ in E along with a subsequence as $m \to \infty$ for some $u_l \in E$. The following lemma shows that we can extract a subsequence of $\{u_{lm}\}$ which converges strongly to $u_l \in E$.

Lemma 5.2. *$u_{lm} \to u_l$ in E along with a subsequence as $m \to \infty$ for some $u_l \in E$.*

Proof. Decompose $u_{lm} = v_{lm} + y_{lm} + w_{lm} + z_{lm}$ and $u_l = v_l + y_l + w_l + z_l$, where v_l, y_l, w_l, z_l are the weak limits of $\{v_{lm}\}$, $\{y_{lm}\}$, $\{w_{lm}\}$, $\{z_{lm}\}$ respectively, and $v_{lm}, v_l \in E^+ \ominus E_0$, $y_{lm}, y_l \in E^+ \cap E_0$, $w_{lm}, w_l \in E^- \ominus E_0$, $z_{lm}, z_l \in E^- \cap E_0$.

Recalling $E^m \subset E^{m+1}$ and $E = \bigcup_{m\in\mathbb{N}} E^m$, we have

$$\|(id - P_m)u\|_E \to 0, \quad \text{as} \quad m \to \infty, \quad \forall u \in E,$$

where id denotes the identity mapping and $P_m : E \to E_m$ is the natural projection. In addition, in virtue of (2.9), we obtain

$$\|(id - P_m)u\|_{L_{p+1}(\Omega)} \to 0, \quad \text{as} \quad m \to \infty, \quad \forall u \in E. \tag{5.3}$$

(i) For $v_{lm}, v_l \in E^+ \ominus E_0$, by (3.2), we have

$$\|v_{lm} - v_l\|_E^2 = \langle (L-\mu)(v_{lm} - v_l), v_{lm} - v_l\rangle = \langle (L-\mu)v_{lm}, v_{lm} - v_l\rangle - \langle (L-\mu)v_l, v_{lm} - v_l\rangle. \tag{5.4}$$

In virtue of $v_{lm} \rightharpoonup v_l$ in E as $m \to \infty$ and $E \hookrightarrow L^2(\Omega)$, we have $v_{lm} \rightharpoonup v_l$ in $L^2(\Omega)$. Furthermore, with the aid of Proposition 4.1, we obtain $v_{lm} \to v_l$ in $L^{p+1}(\Omega)$ as $m \to \infty$. Thus, it follows

$$\langle (L-\mu)v_l, v_{lm} - v_l\rangle \to 0, \quad \text{as} \quad m \to \infty. \tag{5.5}$$

On the other hand, noting $v_{lm}, v_l \in E^+ \ominus E_0$ and $u_{lm} = v_{lm} + y_{lm} + w_{lm} + z_{lm}$, we have $u_{lm} - v_{lm} \in (E^+ \ominus E_0)^\perp$. Thus

$$\langle (L-\mu)(u_{lm} - v_{lm}), v_{lm} - v_l\rangle = 0.$$

Since $u_{lm} \in E^m$, and $(id - P_m)v_l \in (E^m)^\perp$ and $v_{lm} - P_m v_l \in E^m$, from (5.1) we have

$$\begin{aligned}
\langle (L-\mu)v_{lm}, v_{lm} - v_l\rangle &= \langle (L-\mu)u_{lm}, v_{lm} - v_l\rangle \\
&= \langle (L-\mu)u_{lm}, v_{lm} - P_m v_l\rangle + \langle (L-\mu)u_{lm}, P_m v_l - v_l\rangle \\
&= \int_\Omega |u_{lm}|^{p-1} u_{lm}(v_{lm} - P_m v_l)\rho \mathrm{d}t\mathrm{d}x \\
&\le \|u_{lm}\|_{L^{p+1}(\Omega)}^p \|v_{lm} - v_l\|_{L^{p+1}(\Omega)} + \|u_{lm}\|_{L^{p+1}(\Omega)}^p \|(id - P_m)v_l\|_{L^{p+1}(\Omega)} \\
&\le o(1),
\end{aligned} \tag{5.6}$$

where the last inequality is acquired by (5.3) and $v_{lm} \to v_l$ in $L^{p+1}(\Omega)$.

Inserting (5.5) and (5.6) into (5.4), we have

$$\|v_{lm} - v_l\|_E \to 0, \text{ as } m \to \infty. \tag{5.7}$$

(ii) Similarly, for $w_{lm}, w_l \in E^- \ominus E_0$, we have

$$\|w_{lm} - w_l\|_E^2 = -\langle (L-\mu)(w_{lm} - w_l), w_{lm} - w_l\rangle \to 0, \text{ as } m \to \infty. \tag{5.8}$$

(iii) For $y_{lm}, y_l \in E^+ \cap E_0$, Remark 3.1 shows $\dim(E^+ \cap E_0) < \infty$. Thus, $y_{lm} \rightharpoonup y_l$ in E implies

$$\|y_{lm} - y_l\|_E \to 0, \text{ as } m \to \infty. \tag{5.9}$$

(iv) For $z_{lm}, z_l \in E^- \cap E_0$, noting that $\dim(E^- \cap E_0) = \infty$ and Proposition 4.1 can not be applied, we will prove $z_{lm} \to z_l$ in E as $m \to \infty$ by the monotone method.

Since $z_{lm} \rightharpoonup z$ in E, we have

$$\begin{aligned}\|z_{lm} - z\|_E^2 &= -\langle (L-\mu)(z_{lm} - z_l), z_{lm} - z_l\rangle \\ &= -\langle (L-\mu)u_{lm}, z_{lm} - P_m z_l\rangle + \langle (L-\mu)z_l, z_{lm} - z_l\rangle \\ &\le -\langle f(u_{lm}), (id - P_m)z_l\rangle - \langle f(u_{lm}), z_{lm} - z_l\rangle + o(1).\end{aligned}$$

In virtue of (5.2) and (5.3), we have

$$|\langle f(u_{lm}), (id - P_m)z_l\rangle| \le \|u_{lm}\|^p_{L^{p+1}(\Omega)} \|(id - P_m)z_l\|_{L^{p+1}(\Omega)} \to 0, \quad \text{as } m \to \infty.$$

On the other hand, decompose

$$\langle f(u_{lm}), z_{lm} - z_l\rangle = \langle f(u_{lm}) - f(\tilde u_{lm} + z_l), z_{lm} - z_l\rangle + \langle f(\tilde u_{lm} + z_l) - f(u_l), z_{lm} - z_l\rangle + \langle f(u_l), z_{lm} - z_l\rangle,$$

where $\tilde u_{lm} = v_{lm} + y_{lm} + w_{lm}$.

Since $f(u) = |u|^{p-1}u$ is nondecreasing in u, then

$$\langle f(u_{lm}) - f(\tilde u_{lm} + z_l), z_{lm} - z_l\rangle \ge 0.$$

By (2.8) and (5.7)–(5.9), we have $\tilde u_{lm} \to \tilde u_l$ in $L^2(\Omega)$, where $\tilde u_l = v_l + y_l + w_l$. Moreover, since $f : u \mapsto |u|^{p-1}u$ is continuous from $L^2(\Omega)$ to $L^{2/p}(\Omega)$. Thus, we have

$$\langle f(\tilde u_{lm} + z_l) - f(u_l), z_{lm} - z_l\rangle \to 0, \quad \text{as } m \to \infty.$$

In addition, in virtue of $z_{lm} \rightharpoonup z_l$ in $L^2(\Omega, \rho)$, we have

$$\langle f(u_l), z_{lm} - z_l\rangle \to 0, \quad \text{as } m \to \infty.$$

Consequently,

$$\|z_{lm} - z_l\|_E^2 \to 0, \text{ as } m \to \infty. \tag{5.10}$$

Finally, from (5.7)–(5.10), we obtain

$$\|u_{lm} - u_l\|_E \to 0, \quad \text{as } m \to \infty.$$

The proof is completed. □

Now we are in the position to prove Theorem 2.4.

Proof of Theorem 2.4. Since u_{lm} are the critical points of Φ_m, a simple calculation yields

$$\langle (L-\mu)u_{lm}, u_{lm}-v\rangle = \langle f(u_{lm}), u_{lm}-P_m v\rangle, \quad \forall v \in E.$$

Thus, with the aid of monotonicity of $f(u)$, we have

$$\begin{aligned}\langle (L-\mu)u_{lm}, u_{lm}-v\rangle - \langle f(v), u_{lm}-v\rangle &= \langle f(u_{lm}), u_{lm}-P_m v\rangle - \langle f(v), u_{lm}-v\rangle \\ &\geq \langle f(u_{lm}), v-P_m v\rangle. \end{aligned} \tag{5.11}$$

By (5.2) and (5.3), we have

$$|\langle f(u_{lm}), v-P_m v\rangle| \leq \|u_{lm}\|^p_{L^{p+1}(\Omega)} \|(id-P_m)v\|_{L^{p+1}(\Omega)} \to 0, \quad \text{as} \quad m\to\infty.$$

Thus, noting $u_{lm}\to u_l$ in E as $m\to\infty$ and passing to the limit in (5.11), a simple calculation yields that for any $v\in E$,

$$\begin{aligned} 0 &= \lim_{m\to\infty} \langle f(u_{lm}), v-P_m v\rangle \\ &\leq \lim_{m\to\infty} \langle (L-\mu)u_{lm}, u_{lm}-v\rangle - \lim_{m\to\infty}\langle f(v), u_{lm}-v\rangle \\ &= \langle (L-\mu)u_l, u_l-v\rangle - \langle f(v), u_l-v\rangle. \end{aligned} \tag{5.12}$$

Taking $v = u_l - s\psi$ for $s>0$, $\psi\in E$ in (5.12), and dividing by s, we obtain

$$\langle (L-\mu)u_l, \psi\rangle - \langle f(u_l - s\psi), \psi\rangle \geq 0.$$

Now letting $s\to 0$ in above inequality, we have

$$\langle (L-\mu)u_l, \psi\rangle - \langle f(u_l), \psi\rangle \geq 0.$$

Since ψ is chosen arbitrarily, then

$$\langle (L-\mu)u_l, \psi\rangle - \langle f(u_l), \psi\rangle = 0,$$

i.e.,

$$\langle (L-\mu)u_l, \psi\rangle - \int_\Omega f(u_l)\psi\rho \mathrm{d}t\mathrm{d}x = 0, \quad \forall \psi\in E.$$

On the other hand, noting $u_{lm}\to u_l$ in E as $m\to\infty$ and by (2.9), we have

$$\Phi(u_{lm}) \to \Phi(u_l), \quad \text{as} \quad m\to\infty,$$

where $\Phi(u)$ is defined by (2.10). Therefore, by Lemma 4.3, we obtain

$$0 < \sigma_l \leq \Phi(u_l) \leq \varrho_l.$$

Since $\varrho_l \to 0$ as $l \to \infty$, then $\{u_l\}$ are infinitely many weak solutions of problem (2.1)–(2.3). In addition, since $\Phi'(u_l) = 0$ and $0 < p < 1$, by $\varrho_l \to 0$ as $l \to \infty$, a simple calculation yields

$$\int_{\Omega} |u_l|^{p+1} \rho \mathrm{d}t\mathrm{d}x = \left(\frac{1}{p+1} - \frac{1}{2}\right)^{-1} \Phi(u_l) \to 0, \quad \text{as} \quad l \to \infty.$$

The proof of Theorem 2.4 is completed. □

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