

MODELING AND STABILIZATION OF CURRENT-CONTROLLED PIEZO-ELECTRIC BEAMS WITH DYNAMIC ELECTROMAGNETIC FIELD*

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Abstract. Piezoelectric materials can be controlled with current (or charge) as the electrical input, instead of voltage. The main purpose of this paper is to derive the governing equations for a current-controlled piezo-electric beam and to investigate stabilizability. The magnetic permeability in piezo-electric materials is generally neglected in models. However, it has a significant qualitative effect on properties of the control system such as stabilizability. Besides the consideration of current control, there are several new aspects to the model. Most importantly, a fully dynamic magnetic model is included. Also, electrical potential and magnetic vector potential are chosen to be quadratic-through thickness to include the induced effects of the electromagnetic field. Hamilton's principle is used to derive a boundary value problem that models a single piezo-electric beam actuated by a current (or charge) source at the electrodes. Two sets of decoupled system of partial differential equations are obtained; one for stretching of the beam and another one for bending motion. Since current (or charge) controller only affects the stretching motion, attention is focused on control of the stretching equations in this paper. It is shown that the Lagrangian of the beam is invariant under certain transformations. A Coulomb type gauge condition is used. This gauge condition decouples the electrical potential equation from the equations of the magnetic potential. A semigroup approach is used to prove that the Cauchy problem is well-posed. Unlike voltage actuation, a bounded control operator in the natural energy space is obtained. The paper concludes with analysis of stabilizability and comparison with other actuation approaches and models.

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1. INTRODUCTION

Piezoelectric materials are elastic beam/plates covered by electrodes at the top and bottom surfaces, insulated at the edges (to prevent fringing effects), and connected to an external electric circuit (see Fig. 1). They convert mechanical energy to electro-magnetic energy, and vice versa. These materials are widely used in civil,

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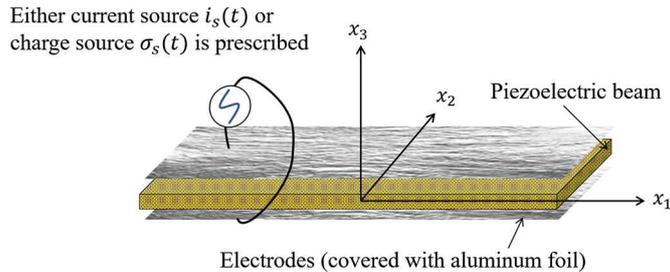


FIGURE 1. When either current $i_s(t)$ or charge $\sigma_s(t)$ is prescribed at the electrodes, an electric field is created between the electrodes, and the beam/plate either shrinks or extends. Experiments show that the input–output hysteresis is substantially less than when voltage actuation is used, *i.e.* see [11, 24].

aeronautic and space structures due to their small size and high power density. They can be actuated by either external mechanical forces or electrical forces. There are mainly three ways to (electrically) actuate piezo-electric materials: voltage, current or charge. Piezoelectric materials have been traditionally activated by a voltage source [2, 3, 8, 26, 28, 31, 33, 35, 36, 38]. It is well-known that the control operator is unbounded in the energy space if the piezo-electric structure is controlled by a voltage or a charge source, for instance see [8, 17, 22, 28, 33].

Hysteresis occurs in the voltage-strain relationship for piezo-electric structures; see for instance [33]. This complicates control of these materials. Therefore controller design needs to consider hysteresis in order to obtain maximum accuracy and effectiveness. Some approaches are passivity [15] and inverse compensation [34]. Another approach is to avoid incurring hysteresis is by applying only low voltages, but this prevents these structures from being used at their maximum potential. Another approach is current or charge actuation since there is evidence that these types of actuation lead to considerably less hysteresis; see for instance [5, 11, 16, 20, 21, 24]. Existing models for current control use electrostatic or quasi-static assumptions, and therefore, only circuit equations are attached to the standard beam equations. This assumption is based on the fact that acoustic wave speed is much smaller than the speed of light [39]. However, it has been shown [22, 27] that voltage-controlled beams, with electro-magnetic dynamics, are not controllable and moreover can not even be asymptotically stabilized for certain combinations of material parameters. Explicit polynomial estimates of asymptotic stability for certain combinations of parameters have been obtained [25]. Thus, controllability and stabilizability of piezo-electric beams are affected by magnetic effects. The stabilization problem is recently analyzed for the viscously damped voltage-controlled fully-magnetic model in the existence of no feedback controller. It is shown that the model is exponentially stable [30].

In this paper, dynamic magnetic effects are included in the derivation of a model for piezo-electric beams actuated by a current (or charge) source. The electromagnetic field is described in terms of scalar electric potential and magnetic vector potential. After deriving expressions of various contributions to the energy, Hamilton's Principle is used to derive a system of partial differential equations modelling the coupling between the mechanical and the electro-magnetic dynamics. Because the Lagrangian corresponding to Maxwell's equations is invariant under certain transformations; for instance see ([7], p. 80), these equations do not have a unique solution and the potentials are not uniquely determined. Obtaining a system of equations with a unique solution requires an appropriate gauge condition. Usually a gauge is chosen to decouple electrical equations from magnetic equations. A number of gauges are possible to obtain a unique solution such as Coulomb or Lorenz [7]. A Coulomb type of gauge is used here. Implementation of this gauge simplifies the equations considerably. The original highly coupled system of equations becomes a system of equations where the equations corresponding to the electrical variables are decoupled from the ones involving magnetic potential variables. Well-posedness of the model in a Hilbert space is then established where the norm corresponds to the energy of the system. It is shown that the spectrum of the generator consists entirely of imaginary eigenvalues. For some parameter values the system

cannot even be asymptotically stabilized. Stabilizability of the model is compared to voltage control, as well as to the case where magnetic effects are neglected.

Indicating the system description by (A, B, C) , it is well-known that if A is dissipative and $C = B^*$ then constant feedback is stabilizing. For a simple voltage-actuated piezo-electric beam with quasi-static or static electrical model, the “ B^* ” output measurement is the beam velocity. This is difficult to accomplish in practice due to a lack of inexpensive velocity sensors (laser vibrometers) ([32], p. 175); typically velocity is reconstructed from position or acceleration measurements if a laser vibrometer is not used. But for voltage-controlled piezo-electric beams, B^* is the total induced current at the electrodes of the piezo-electric beam [22]. This electrical measurement can be implemented. There are results in the engineering literature proposing voltage input-induced charge output. In [12], there is a design of measuring induced charge at the electrodes of piezo-electric beam. The advantage of this input-output system is to actuate the piezo-electric beam at high frequencies. In this paper it is shown that the control operator B is bounded in the case of current control with magnetic effects and also that the B^* output measurement is electro-magnetic, not mechanical, and can be implemented.

In the conference paper [23], preliminary versions of Theorem 3.1, Lemma 4.2, and Theorem 5.2 were reported for different choices of states and state space. In this manuscript, the states and the state space are based on the energy of the beam and full proofs are included for all results.

2. PHYSICAL ASSUMPTIONS

Let x_1, x_2 be the longitudinal directions, and x_3 be the transverse directions (see Fig. 1). Assume that the piezo-electric beam occupies the region $\Omega = [0, L] \times [-r, r] \times [-\frac{h}{2}, \frac{h}{2}]$ with the boundary $\partial\Omega$, the electroded region and the insulated region, where $L \gg h$. Throughout this paper, dots denote differentiation with respect to time, that is $\dot{x}(t) = \frac{dx}{dt}$.

A very widely-used linear constitutive relationship [36] for piezo-electric beams is

$$\begin{pmatrix} T \\ D \end{pmatrix} = \begin{bmatrix} c & -\gamma^T \\ \gamma & \varepsilon \end{bmatrix} \begin{pmatrix} S \\ E \end{pmatrix} \quad (2.1)$$

where $T = (T_{11}, T_{22}, T_{33}, T_{23}, T_{13}, T_{12})^T$ is the stress vector, $S = (S_{11}, S_{22}, S_{33}, S_{23}, S_{13}, S_{12})^T$ is the strain vector, $D = (D_1, D_2, D_3)^T$ and $E = (E_1, E_2, E_3)^T$ are the electric displacement and the electric field vectors, respectively, and moreover, the matrices $[c], [\gamma], [\varepsilon]$ are the matrices with elastic, electro-mechanic and dielectric constant entries (for more details see [36]). Under the assumption of transverse isotropy and polarization in x_3 -direction:

$$c_{11} = c_{22}, \quad c_{44} = c_{55}, \quad c_{13} = c_{23}, \quad c_{66} = \frac{1}{2}(c_{11} - c_{12}), \quad \gamma_{24} = \gamma_{15}, \quad \gamma_{31} = \gamma_{32}, \quad \varepsilon_{11} = \varepsilon_{22}$$

these matrices reduce to

$$c = \begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & 0 & 0 & 0 & \gamma_{15} & 0 \\ 0 & 0 & 0 & \gamma_{15} & 0 & 0 \\ \gamma_{31} & \gamma_{31} & \gamma_{33} & 0 & 0 & 0 \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{pmatrix}$$

TABLE 1. Notation

A	Magnetic potential vector	ρ	Mass density per unit volume
B	Magnetic flux density vector	n	Surface unit outward normal vector
c, α	Elastic stiffness coefficients	σ_s	Surface charge density
D	Electric displacement vector	σ_b	Volume charge density
E	Electric field intensity vector	S	Strain tensor
ε	Permittivity coefficients	T	Stress tensor
h	Thickness of the beam	U_1	x_1 component of the displacement field
i_b	Volume current density	U_3	x_3 component of the displacement field
i_s	Surface current density	v	Longitudinal disp. of the centerline of the beam
ϕ	Electric potential	V	Voltage
γ	Piezoelectric coefficients	w	Transverse displacement of the beam
μ	Magnetic permeability of beams	$\mathbb{1}_{(0,L)}(x)$	Characteristic function of the interval (0.1)

where $c_{66} = \frac{1}{2}(c_{11} - c_{12})$. All forces acting in the x_2 direction are assumed negligible which leads to a beam model. Moreover, T_{33} is also assumed to be zero. Therefore

$$T = (T_{11}, T_{13})^T, \quad S = (S_{11}, S_{13})^T, \quad D = (D_1, D_3)^T, \quad E = (E_1, E_3)^T$$

and (2.1) reduces to

$$\begin{pmatrix} T_{11} \\ T_{13} \\ D_1 \\ D_3 \end{pmatrix} = \begin{pmatrix} c_{11} & 0 & 0 & -\gamma_{31} \\ 0 & c_{44} & -\gamma_{15} & 0 \\ 0 & \gamma_{15} & \varepsilon_{11} & 0 \\ \gamma_{31} & 0 & 0 & \varepsilon_{33} \end{pmatrix} \begin{pmatrix} S_{11} \\ S_{13} \\ E_1 \\ E_3 \end{pmatrix}.$$

Let (U_1, U_3) denote the displacement field vector of a point (x_1, x_3) . Continuing with the Euler-Bernoulli beam small-displacement assumptions, the displacement field is

$$U_1 = v - x_3 \frac{\partial w}{\partial x_1}, \quad U_3 = w \quad (2.2)$$

where $v = v(x_1)$ and $w = w(x_1)$ denote the longitudinal displacement of the center line in x_1 direction, and transverse displacement of the beam, respectively. Since $S_{13} = \frac{1}{2} \left(\frac{\partial U_1}{\partial x_3} + \frac{\partial U_3}{\partial x_1} \right) = 0$, the only nonzero strain component is given by

$$S_{11} = \frac{\partial U_1}{\partial x_1} = \frac{\partial v}{\partial x_1} - x_3 \frac{\partial^2 w}{\partial x_1^2}. \quad (2.3)$$

To keep the notation simple let

$$\alpha = c_{11}, \quad \gamma = \gamma_{31}, \quad \gamma_1 = \gamma_{15}, \quad \varepsilon_1 = \varepsilon_{11}, \quad \varepsilon_3 = \varepsilon_{33}. \quad (2.4)$$

With the new notation, the linear constitutive equations for an Euler-Bernoulli piezo-electric beam are

$$\begin{cases} T_{11} = \alpha S_{11} - \gamma E_3 \\ T_{13} = -\gamma_1 E_1 \\ D_1 = \varepsilon_1 E_1 \\ D_3 = \gamma S_{11} + \varepsilon_3 E_3 \end{cases} \quad (2.5)$$

Let \mathbf{K} , \mathbf{P} , \mathbf{E} and \mathbf{B} be kinetic, potential, electrical, and magnetic energies of the beam, respectively, and let \mathbf{W} be the work done by the external forces (charge or current, see (2.22)). For modelling of charge or current-actuated piezo-electric beams the following Lagrangian

$$\mathbf{L} = \int_0^T [\mathbf{K} - (\mathbf{P} - \mathbf{E} + \mathbf{B}) + \mathbf{W}] dt \quad (2.6)$$

is appropriate [19]. The constitutive equations (2.5), where the pair (S, E) belongs to the set of independent variables, are used. In the above, $\mathbf{P} - \mathbf{E} + \mathbf{B}$ is called electrical enthalpy.

In modeling piezo-electric beams with voltage-actuated electrodes, a different Lagrangian

$$\mathbf{L} = \int_0^T [\mathbf{K} - (\mathbf{P} + \mathbf{E}) + \mathbf{B} + \mathbf{W}] dt \quad (2.7)$$

is appropriate [13] since the input for \mathbf{W} is voltage. In this case, the constitutive equations (2.5) are re-written in terms of the independent variables (S, D) and an appropriate Legendre transformation is applied allowing us to change the independent variables (S, E) in (2.6) to (S, D) in (2.7). Here $\mathbf{P} + \mathbf{E}$ denotes the total stored energy of the beam, and \mathbf{B} acts as the electrical kinetic energy of the beam. The reader should refer to [22] for more details.

The full set of Maxwell's equations is ([9], p. 332), letting B denote the magnetic field vector, and $\sigma_b, i_b, \sigma_s, i_s, V, \mu, n$ denote body charge density, body current density, surface charge density, surface current density, voltage, magnetic permeability, and unit normal vector to the surface $\partial\Omega$, respectively,

$$\begin{aligned} \nabla \cdot D &= \sigma_b & \text{in } \Omega \times \mathbb{R}^+ & \quad (\text{Electric Gauss's law}) \\ \nabla \cdot B &= 0 & \text{in } \Omega \times \mathbb{R}^+ & \quad (\text{Gauss's law of magnetism}) \\ \nabla \times E &= -\dot{B} & \text{in } \Omega \times \mathbb{R}^+ & \quad (\text{Faraday's law}) \\ \frac{1}{\mu}(\nabla \times B) &= i_b + \dot{D} & \text{in } \Omega \times \mathbb{R}^+ & \quad (\text{Ampère-Maxwell law}) \end{aligned} \quad (2.8)$$

The essential electric boundary conditions prescribed on the electrodes are

$$-D \cdot n = \sigma_s(t) \quad \text{on } \partial\Omega \times \mathbb{R}^+ \quad (\text{Charge}) \quad (2.9)$$

$$\frac{1}{\mu}(B \times n) = i_s(t) \quad \text{on } \partial\Omega \times \mathbb{R}^+ \quad (\text{Current}) \quad (2.10)$$

$$\phi = V(t) \quad \text{on } \partial\Omega \times \mathbb{R}^+ \quad (\text{Voltage}) \quad (2.11)$$

and appropriate mechanical boundary conditions are needed at the left and right end points $x = 0, L$ of the beam (the beam is clamped, hinged, free, etc.). In previous work, voltage-driven piezo-electric beams were considered [22]. In this paper, current and charge-driven electrodes are considered; that is, (2.11) is not used. There are three main approaches to modelling of electric and magnetic effects in piezo-electric beams [36]:

- i) **Electrostatic electric field:** This is the most widely-used approach in the literature. It completely ignores magnetic effects: $B = \dot{D} = i_b = \sigma_b = 0$. Maxwell's equations (2.8) reduce to $\nabla \cdot D = 0$ and $\nabla \times E = 0$. Therefore, there exists a scalar electric potential such that $E = -\nabla\phi$ and ϕ is determined up to a constant.
- ii) **Quasi-static electric field:** This approach includes some magnetic effects [36]. Although still $\sigma_b = i_b = 0$, \dot{D} and B are non-zero. Maxwell's equations (2.8) become

$$\nabla \cdot D = 0, \quad \nabla \cdot B = 0, \quad \dot{B} = -\nabla \times E, \quad \dot{D} = \frac{1}{\mu}(\nabla \times B).$$

The equation $\nabla \cdot B = 0$ implies that there exists a vector A , the *magnetic potential*, such that $B = \nabla \times A$. Also, from substituting B into $\dot{B} = -\nabla \times E$ that, *i.e.* $\nabla \times (\dot{A} + E) = 0$, there exists a scalar electric potential ϕ such that

$$E = -\dot{A} - \nabla\phi. \quad (2.12)$$

since the curl grad of any scalar function ϕ is zero. The time-varying magnetic effects \dot{A} affect the induced electric field. One simplification to this approach is to ignore A and \dot{A} since $A, \dot{A} \ll \phi$. With this assumption \dot{D} may be non-zero. However, \ddot{D} is zero since $\dot{A} = 0$.

iii) Fully dynamic electric field: Unlike the quasi-static assumption, A and \dot{A} are not assumed zero. Depending on the type of material, body charge density σ_b and body current density i_b may also be non-zero.

Since piezo-electric materials are not perfectly insulated, the electric field E causes current to flow. An electro-static approach is not appropriate, and the time-dependent equation of the continuity of electric charge (Ampère-Maxwell Law in (2.8)) must be included. In this paper, a fully dynamic approach that includes magnetic effects in order to model current- and charge-controlled piezo-electric beams is used. Taking the divergence of both sides of Ampère-Maxwell equation (2.8) leads to

$$\frac{1}{\mu} \nabla \cdot (\nabla \times B) = \nabla \cdot i_b + \nabla \cdot \dot{D}. \quad (2.13)$$

The term on the left hand side of the equation above is zero, and therefore by using Gauss's law (2.8), the following electric continuity condition

$$\dot{\sigma}_b + \nabla \cdot i_b = 0 \quad \text{in } \Omega \quad (2.14)$$

is obtained. The physical interpretation of (2.14) is the local conservation of electrical charge. From (2.8)

$$\frac{1}{\mu} \int_{\partial\Omega} (\nabla \times B) \cdot n \, dS = \int_{\partial\Omega} (i_b \cdot n + \dot{D} \cdot n) \, dS$$

where $i_b = (i_b^1, i_b^2, i_b^3)$ and $n = (0, 0, \pm 1)$. Using the charge boundary conditions (2.9) with $i_s(x, t) \equiv 0$,

$$0 = \frac{1}{\mu} \int_{\partial\Omega} \nabla \times B \cdot n \, dS = \int_{\partial\Omega} (i_b^3 - \dot{\sigma}_s) \, dS.$$

Alternatively, use the current boundary condition (2.10) with $\sigma_s \equiv 0$

$$\frac{1}{\mu} \int_{\partial\Omega} \nabla \times B \cdot n \, dS = \int_{\partial\Omega} i_b^3 \, dS$$

where n is the outward unit normal vector on $\partial\Omega$. This leads to the surface electric continuity conditions (or compatibility conditions)

$$\dot{\sigma}_s - i_b^3 = 0, \quad \text{or,} \quad \frac{di_s}{dx} - i_b^3 = 0 \quad \text{on } \partial\Omega. \quad (2.15)$$

For details, see ([10], Sect. 3.9).

Henceforth, to simplify the notation, $x = x_1$ and $z = x_3$. Since beam theory is used for the mechanical behaviour, assume that the magnetic vector potential A and other electro-magnetic variables have nonzero

components only in the x and z directions. The piezo-electric effect is the relationship between external electric field leads and strain. Upon production of an external electric field, the beam/plate extends or shrinks. This is known as the direct piezo-electric effect. On the other hand, when the plate/beam extends and shrink, it produces electric voltage; this is the inverse or indirect effect.

It is common to assume that the electric potential is linear through-thickness: $\phi(x, z) = \phi^0(x) + z\phi^1(x)$. This affects the induced potential effect since ϕ is completely known as a function of voltage. For example, when the voltage is prescribed at the electrodes, *i.e.* $\phi\left(\frac{h}{2}\right) = V$ and $\phi\left(-\frac{h}{2}\right) = 0$,

$$\phi = \frac{V}{2} + z\frac{V}{h},$$

and therefore the electric field component in the transverse direction E_3 becomes uniform in the transverse direction, that is, $E_3 = -\phi^1 = -\frac{V}{h}$ for electrostatic and quasi-static assumptions. Therefore the inverse effect does not exist if a linear-through thickness assumption is used. In this paper, a quadratic-through thickness potential distribution

$$\phi(x, z) = \phi^0(x) + z\phi^1(x) + \frac{z^2}{2}\phi^2(x). \quad (2.16)$$

is used. This reproduces the induced effect and improves the modeling accuracy without moving to the complexity of a fully two-dimensional description.

To keep the consistency with ϕ , assume that A is quadratic through-thickness as well:

$$A(x, z) = \begin{pmatrix} A_1(x, z) \\ 0 \\ A_3(x, z) \end{pmatrix} = \begin{pmatrix} A_1^0(x) + zA_1^1(x) + \frac{z^2}{2}A_1^2(x) \\ 0 \\ A_3^0(x) + zA_3^1(x) + \frac{z^2}{2}A_3^2(x) \end{pmatrix}. \quad (2.17)$$

By (2.12)

$$\begin{aligned} E_1 &= -\left(\dot{A}_1^0 + z\dot{A}_1^1 + \frac{z^2}{2}\dot{A}_1^2\right) - \left((\phi^0)_x + z(\phi^1)_x + \frac{z^2}{2}(\phi^2)_x\right), \\ E_3 &= -\left(\dot{A}_3^0 + z\dot{A}_3^1 + \frac{z^2}{2}\dot{A}_3^2\right) - (\phi^1 + z\phi^2). \end{aligned} \quad (2.18)$$

Now use the constitutive equations (2.5) along with (2.2), (2.3), and (2.16)–(2.18) to write

$$\mathbf{K} = \frac{\rho}{2} \int_{\Omega} \left(\dot{U}_1^2 + \dot{U}_3^2\right) dX = \frac{\rho h}{2} \int_0^L \left(\dot{v}^2 + \dot{w}^2 + \frac{h^2}{12}\dot{w}_x^2\right) dx, \quad (2.19)$$

$$\begin{aligned} \mathbf{E} - \mathbf{P} &= \frac{1}{2} \int_{\Omega} (D_1 E_1 + D_3 E_3 - T_{11} S_{11} - T_{13} S_{13}) dX \\ &= \frac{1}{2} \int_{\Omega} (-\alpha S_{11}^2 + 2\gamma S_{11} E_3 + \varepsilon_1 E_1^2 + \varepsilon_3 E_3^2) dX \\ &= \frac{1}{2} \int_0^L \left[-\alpha h \left(v_x^2 + \frac{h^2}{12} w_{xx}^2 \right) - 2\gamma h \left(\left(\phi^1 + \dot{A}_3^0 + \frac{h^2}{24} \dot{A}_3^2 \right) v_x - \frac{h^2}{12} w_{xx} \left(\phi^2 + \dot{A}_3^1 \right) \right) \right. \\ &\quad \left. + \varepsilon_1 h \left((\phi_x^0)^2 + \frac{h^2}{12} (\phi_x^1)^2 + \frac{h^4}{320} (\phi_x^2)^2 + (\dot{A}_1^0)^2 + \frac{h^2}{12} (\dot{A}_1^1)^2 + \frac{h^4}{320} (\dot{A}_1^2)^2 \right) \right] dx \end{aligned}$$

$$\begin{aligned}
& +\varepsilon_3 h \left((\phi^1)^2 + \frac{h^2}{12} (\phi^2)^2 + (A_3^0)^2 + \frac{h^2}{12} (\dot{A}_3^1)^2 + \frac{h^4}{320} (\dot{A}_3^2)^2 \right) \\
& + 2\varepsilon_1 h \left((\phi^0)_x \dot{A}_1^0 + \frac{h^2}{24} (\phi^0)_x (\phi^2)_x + \frac{h^2}{24} \dot{A}_1^0 \dot{A}_1^2 + \frac{h^2}{24} (\phi^0)_x \dot{A}_1^2 + \frac{h^2}{24} (\phi^2)_x \dot{A}_1^0 + \frac{h^2}{12} (\phi^1)_x \dot{A}_1^1 + \frac{h^4}{320} (\phi^2)_x \dot{A}_1^2 \right) \\
& + 2\varepsilon_3 h \left(\phi^1 \dot{A}_3^0 + \frac{h^2}{24} \dot{A}_3^0 \dot{A}_3^2 + \frac{h^2}{24} \phi^1 \dot{A}_3^2 + \frac{h^2}{12} \phi^2 \dot{A}_3^0 \right) \Big] dx, \tag{2.20}
\end{aligned}$$

$$\begin{aligned}
\mathbf{B} &= \frac{\mu}{2} \int_{\Omega} (\nabla \times A) \cdot (\nabla \times A) dX \\
&= \frac{\mu}{2} \int_0^L \int_{-h/2}^{h/2} \left(A_1^1 + z A_1^2 - (A_3^0)_x - z (A_3^1)_x - \frac{z^2}{2} (A_3^2)_x \right)^2 dz dx \\
&= \frac{\mu h}{2} \int_0^L \left[(A_1^1)^2 + \frac{h^2}{12} (A_1^2)^2 + ((A_3^0)_x)^2 + \frac{h^2}{12} ((A_3^1)_x)^2 + \frac{h^4}{320} ((A_3^2)_x)^2 \right. \\
&\quad \left. - 2 \left(A_1^1 (A_3^0)_x + \frac{h^2}{24} A_1^1 (A_3^2)_x - \frac{h^2}{12} A_1^2 (A_3^1)_x - \frac{h^2}{24} (A_3^0)_x (A_3^2)_x \right) \right] dx. \tag{2.21}
\end{aligned}$$

Now define the work \mathbf{W} done by the external forces. The body force resultants $i_b, \sigma_b, i_s, \sigma_s$ are [18]:

$$i_b = \int_{-h/2}^{h/2} \tilde{i}_b dz, \quad \sigma_b = \int_{-h/2}^{h/2} \tilde{\sigma}_b dz, \quad i_s = \int_{-h/2}^{h/2} \tilde{i}_s dz, \quad \sigma_s = \int_{-h/2}^{h/2} \tilde{\sigma}_s dz.$$

In the above the surface charge density $\tilde{\sigma}_s$ and surface current density \tilde{i}_s are independent of z since they are prescribed at the electrodes. For the Euler-Bernoulli beam, it is appropriate to assume that body charge $\tilde{\sigma}_b$ and body current \tilde{i}_b are independent of z ,

$$i_b = \tilde{i}_b h, \quad \sigma_b = \tilde{\sigma}_b h, \quad i_s = \tilde{i}_s h, \quad \sigma_s = \tilde{\sigma}_s h.$$

Surface charge σ_s or i_s may be non-zero, and either i_b or σ_b may be nonzero, depending on the type of actuation. Assume that there are no mechanical external forces acting on the beam. The work done by the electrical external forces is [19]

$$\begin{aligned}
\mathbf{W} &= \int_{\Omega} (-\tilde{\sigma}_b \phi + \tilde{i}_b \cdot A) dX + \int_{\partial\Omega} (-\tilde{\sigma}_s \phi + \tilde{i}_s \cdot A) dX \\
&= \int_{\Omega} (-\tilde{\sigma}_b \phi + \tilde{i}_b^1 A_1) dX + \int_{\partial\Omega} (-\tilde{\sigma}_s \phi + \tilde{i}_s^1 \cdot A_1) dX \\
&= - \int_0^L \int_{-h/2}^{h/2} \tilde{\sigma}_b \left(\phi^0(x) + z \phi^1(x) + \frac{z^2}{2} \phi^2(x) \right) dz dx + \int_0^L \int_{-h/2}^{h/2} \tilde{i}_b^1 \left(A_1^0(x) + z A_1^1(x) + \frac{z^2}{2} A_1^2(x) \right) dz dx \\
&\quad + \int_0^L (-\tilde{\sigma}_s (\phi(h/2) - \phi(-h/2)) + \tilde{i}_s^1 (A_1(h/2) - A_1(-h/2))) dx \\
&= \int_0^L \left(-\sigma_b \left(\phi^0 + \frac{h^2}{24} \phi^2 \right) + i_b^1 \left(A_1^0 + \frac{h^2}{24} A_1^2 \right) - \sigma_s \phi^1 + i_s^1 A_1^1 \right) dx \tag{2.22}
\end{aligned}$$

where $i_s(x, t) = (i_s^1(x, t), 0, 0)$, and $i_b(x, t) = (i_b^1(x, t), 0, 0)$. In the above i_s has only one nonzero component since $i_s \perp B$, and $i_s \perp n$ by (2.10). Moreover, i_b has only one nonzero component since we assumed that there is no force acting in the x_2 and x_3 directions.

If the magnetic effects are neglected, a variational approach cannot be used in the case of current actuation since $A \equiv 0$ and so $\mathbf{W} \equiv 0$. This is very different from the charge and voltage actuation cases since for charge and voltage actuation \mathbf{W} is not a function of A .

3. DERIVATION OF GOVERNING EQUATIONS

The application of Hamilton's principle, setting the variation of Lagrangian \mathbf{L} in (2.6) with respect to admissible displacements

$$\{v, w, \phi^0, \phi^1, \phi^2, A_1^0, A_1^1, A_1^2, A_3^0, A_3^1, A_3^2\}$$

to zero, yields two sets of partial differential equations. The first set relate to stretching of the beam and involve only the variables $v, \phi^1, A_1^0, A_3^0, A_3^1$:

$$\left\{ \begin{array}{l} \rho h \ddot{v} - \alpha h v_{xx} - \gamma h \left((\phi^1)_x + (\dot{A}_3^0)_x + \frac{h^2}{24} (\dot{A}_3^2)_x \right) = 0 \\ -\varepsilon_1 \frac{h^3}{12} \left((\phi^1)_{xx} + (\dot{A}_1^1)_x \right) + \varepsilon_3 h \left(\dot{A}_3^0 + \frac{h^2}{24} \dot{A}_3^2 + \phi^1 \right) - \gamma h v_x = \sigma_s \\ \frac{\varepsilon_1 h^3}{12} \ddot{A}_1^1 + \frac{\varepsilon_1 h^3}{12} (\dot{\phi}^1)_x - \mu h \left((A_3^0)_x + \frac{h^2}{24} (A_3^2)_x - A_1^1 \right) = i_s^1 \\ \varepsilon_3 h \left(\ddot{A}_3^0 + \frac{h^2}{24} \ddot{A}_3^2 + \dot{\phi}^1 \right) - \mu h \left((A_3^0)_{xx} + \frac{h^2}{24} (A_3^2)_{xx} - (A_1^1)_x \right) - \gamma h \dot{v}_x = 0 \\ \frac{\varepsilon_3 h^3}{24} \left(\ddot{A}_3^0 + \dot{\phi}^1 \right) + \frac{\varepsilon_3 h^5}{320} \ddot{A}_3^2 - \mu h^3 \left(\frac{h^2}{24} (A_3^0)_{xx} + \frac{h^2}{320} (A_3^2)_{xx} - \frac{1}{24} (A_1^1)_x \right) - \frac{\gamma h^3}{24} \dot{v}_x = 0 \end{array} \right. \quad (3.1)$$

The second set related to bending of the beam and involve only the variables $w, \phi^0, \phi^2, A_1^0, A_1^2, A_3^1$:

$$\left\{ \begin{array}{l} \rho h \ddot{w} - \frac{\rho h^3}{12} \ddot{w}_{xx} + \frac{\alpha h^3}{12} w_{xxxx} - \frac{\gamma h^3}{12} \left((\phi^2)_{xx} + (\dot{A}_3^1)_{xx} \right) = 0 \\ -\varepsilon_1 h \left((\phi^0)_{xx} + \frac{h^2}{24} (\phi^2)_{xx} + (\dot{A}_1^0)_x + \frac{h^2}{24} (\dot{A}_1^2)_x \right) = \sigma_b \\ -\frac{\varepsilon_1 h^3}{24} \left((\phi^0)_{xx} + \frac{h^2}{24} (\phi^2)_{xx} + (\dot{A}_1^0)_x + \frac{h^2}{24} (\dot{A}_1^2)_x \right) + \frac{\gamma h^3}{24} w_{xx} - \frac{\varepsilon_1 h^5}{720} (\phi^2)_{xx} - \frac{\varepsilon_1 h^5}{720} (\dot{A}_1^2)_x + \frac{\varepsilon_3 h^3}{12} \left(\phi^2 + \dot{A}_3^1 \right) = \frac{h^2 \sigma_b}{24} \\ \varepsilon_1 h \left(\ddot{A}_1^0 + \frac{h^2}{24} \ddot{A}_1^2 + (\phi^0)_x + \frac{h^2}{24} (\phi^2)_x \right) = i_b^1 \\ \frac{\varepsilon_1 h^3}{24} \left(\ddot{A}_1^0 + \frac{h^2}{24} \ddot{A}_1^2 \right) + (\phi^0)_x + \frac{h^2}{24} (\phi^2)_x + \frac{\varepsilon_1 h^5}{720} \ddot{A}_1^2 + \frac{\varepsilon_1 h^5}{720} (\dot{\phi}^2)_x + \frac{h^3 \mu}{12} (A_1^2 - (A_3^1)_x) = \frac{h^2 i_b^1}{24} \\ \frac{\varepsilon_3 h^3}{12} \left(\ddot{A}_3^1 - \frac{\mu}{\varepsilon_3} (A_3^1)_{xx} \right) + \frac{\varepsilon_3 h^3}{12} \dot{\phi}^2 + \frac{h^3 \mu}{12} (A_1^2)_x + \frac{\gamma h^3}{12} \dot{w}_{xx} = 0 \end{array} \right. \quad (3.2)$$

The bending of the beam $w(x, t)$ is described by the Rayleigh beam equation coupled to the electromagnetic equations. If the rotational inertia of the cross section of the beam is ignored, the terms \ddot{w}_{xx} in (3.2) and \ddot{w}_x in (3.3) disappear.

The boundary conditions depend on whether any of the variables have imposed (or essential) conditions at $x = 0, L$. The following is a list of all natural boundary conditions. The corresponding variable and the physical interpretation is indicated in brackets for each.

$$\left\{ \begin{array}{ll} \alpha h v_x + \gamma h \left(\phi^1 + \dot{A}_3^0 + \frac{h^2}{24} \dot{A}_3^2 \right) = 0 & \text{(Lateral force)} \\ \frac{\varepsilon_1 h^3}{12} \left(\dot{A}_1^1 + (\phi^1)_x \right) = 0 & \text{(First charge moment)} \\ \mu h \left(A_1^1 - (A_3^0)_x - \frac{h^2}{24} (A_3^2)_x \right) = 0 & \text{(Current)} \\ \mu h^3 \left(\frac{1}{24} A_1^1 - \frac{1}{24} (A_3^0)_x - \frac{h^2}{320} (A_3^2)_x \right) = 0 & \text{(Second current moment)} \end{array} \right.$$

$$\begin{cases} \frac{h^3}{12} (-\alpha w_{xx} + \gamma \phi^2) = 0 & \text{(Bending moment)} \\ -\rho \ddot{w}_x + \alpha w_{xxx} - \gamma (\phi^2)_x = 0 & \text{(Shear)} \\ \varepsilon_1 h \left(\dot{A}_1^0 + \frac{h^2}{24} \dot{A}_1^2 + (\phi^0)_x + \frac{h^2}{24} (\phi^2)_x \right) = 0 & \text{(Charge)} \\ \varepsilon_1 h^3 \left(\frac{1}{12} \dot{A}_1^0 + \frac{h^2}{160} \dot{A}_1^2 + \frac{1}{12} (\phi^0)_x + \frac{h^2}{160} (\phi^2)_x \right) = 0 & \text{(Second charge moment)} \\ \frac{\mu h^3}{12} (A_1^2 - (A_3^1)_x) = 0 & \text{(First current moment)} \end{cases} \quad (3.3)$$

It will be assumed, as is common in practice, that there is no free body charge or current, *i.e.* $\sigma_b \equiv i_b \equiv 0$. This implies that the surface continuity condition for each actuation is

$$\frac{di_s}{dx} = 0, \quad \dot{\sigma}_s = 0. \quad (3.4)$$

Then the bending equations (3.2) are entirely uncontrolled and also decoupled from the stretching equations (3.5). The focus of this work is stabilizability of the piezo-electric beam. Clearly, only the bending component can be controlled. Therefore, from this point on, only the stretching equations (3.5) are considered with the corresponding boundary conditions at $x = 0, L$.

The last equation in (3.1) can be simplified by using the previous one to obtain

$$\begin{cases} \rho h \ddot{v} - \alpha h v_{xx} - \gamma h \left((\phi^1)_x + (\dot{A}_3^0)_x + \frac{h^2}{24} (\dot{A}_3^2)_x \right) = 0 \\ -\frac{\varepsilon_1 h^3}{12} \left((\phi^1)_{xx} + (\dot{A}_1^1)_x \right) + \varepsilon_3 h \left(\dot{A}_3^0 + \frac{h^2}{24} \dot{A}_3^2 + \phi^1 \right) - \gamma h v_x = \sigma_s \\ \frac{\varepsilon_1 h^3}{12} \ddot{A}_1^1 + \frac{\varepsilon_1 h^3}{12} (\dot{\phi}^1)_x - \mu h \left((A_3^0)_x + \frac{h^2}{24} (A_3^2)_x - A_1^1 \right) = i_s^1 \\ \varepsilon_3 h \left(\ddot{A}_3^0 + \frac{h^2}{24} \ddot{A}_3^2 + \dot{\phi}^1 \right) - \mu h \left((A_3^0)_{xx} + \frac{h^2}{24} (A_3^2)_{xx} - (A_1^1)_x \right) - \gamma h \dot{v}_x = 0 \\ \frac{\varepsilon_3 h^3}{24} \left(\ddot{A}_3^0 + \frac{h^2}{24} \ddot{A}_3^2 + \dot{\phi}^1 \right) - \frac{\mu h^3}{24} \left((A_3^0)_{xx} + \frac{h^2}{24} (A_3^2)_{xx} - (A_1^1)_x \right) - \frac{\gamma h^3}{24} \dot{v}_x + \left(\frac{\varepsilon_3 h^5}{720} \ddot{A}_3^2 - \frac{\mu h^5}{720} (A_3^2)_{xx} \right) = 0 \end{cases} \quad (3.5)$$

The current and second current moment boundary conditions in (3.3) can be simplified as

$$A_1^1 - (A_3^0)_x, (A_3^2)_x = 0 \quad \text{at } x = 0, L. \quad (3.6)$$

This leads to

$$\begin{cases} \alpha v_x + \gamma h \left(\phi^1 + \dot{A}_3^0 + \frac{h}{24} \dot{A}_3^2 \right) = 0 & \text{(Lateral force)} \\ \dot{A}_1^1 + (\phi^1)_x = 0 & \text{(First charge moment)} \\ A_1^1 - (A_3^0)_x = (A_3^2)_x = 0, \quad \text{at } x = 0, L. \end{cases} \quad (3.7)$$

The magnetic potential vector A and the electric potential ϕ are not uniquely defined by (2.8). In fact, the Lagrangian \mathbf{L} (2.6) is invariant under a large class of transformations.

Theorem 3.1. *For any scalar C^1 function $\chi = \chi(x, z, t)$, the Lagrangian \mathbf{L} is invariant under the transformation*

$$\begin{aligned} A &\mapsto \tilde{A} := A + \nabla \chi \\ \phi &\mapsto \tilde{\phi} := \phi - \dot{\chi}. \end{aligned} \quad (3.8)$$

Proof. By (3.8), \tilde{A} and $\tilde{\phi}$ satisfy

$$\begin{aligned}\tilde{B} &= \nabla \times \tilde{A} = \nabla \times A + \nabla \times \nabla \chi = \nabla \times A = B, \\ \tilde{E} &= -\dot{\tilde{A}} - \nabla \tilde{\phi} = -\dot{A} - \nabla \dot{\chi} - \nabla \phi + \nabla \dot{\chi} = -\dot{A} - \nabla \phi = E.\end{aligned}$$

This implies that $\mathbf{E} - \mathbf{P}$ and \mathbf{B} defined by (2.20) and (2.21) respectively are invariant under the transformation. Since \mathbf{K} in (2.19) is independent of A and ϕ , we need to check if \mathbf{W} defined by (2.22) is invariant under (3.8).

Choose the arbitrary scalar function χ to be quadratic-through thickness $\chi = \chi^0 + z\chi^1 + \frac{z^2}{2}\chi^2$ to be consistent with the structure of $\tilde{\varphi}$ and \tilde{A} in (2.16) and (2.17), respectively. According to the principle of virtual work, the solution trajectory is the trajectory which renders stationary the Lagrangian under all kinematically admissible displacements. Let $\mathbf{y} = \{v, w, \phi^0, \phi^1, \phi^2, A_1^0, A_1^1, A_1^2, A_3^0, A_3^1, A_3^2\}$ denote test function on $[0, L] \times (0, T)$ for which

$$\delta \mathbf{y}, \delta \dot{\mathbf{y}}, |_{t=0, T} = 0.$$

Since we assume these conditions for \mathbf{y} , for compatibility, we assume them for the χ function as well. Using (3.8),

$$\begin{aligned}\tilde{A}_1^0 &= A_1^0 + (\chi^0)_x, & A_1^1 &= A_1^1 + (\chi^1)_x, & A_1^2 &= A_1^2 + (\chi^2)_x, & \tilde{A}_3^0 &= A_3^0 + \chi^1, & \tilde{A}_3^2 &= A_3^2, \\ \tilde{\phi}^0 &= \phi^0 - \dot{\chi}^0, & \tilde{\phi}^1 &= \phi^1 - \dot{\chi}^1, & \tilde{\phi}^2 &= \phi^2 - \dot{\chi}^2,\end{aligned}$$

and therefore

$$\begin{aligned}\int_0^T \tilde{\mathcal{W}} dt &= \int_0^T \int_0^L \left(-\sigma_b \left(\tilde{\phi}^0 + \frac{h^2}{24} \tilde{\phi}^2 \right) - \sigma_s \tilde{\phi}^1 + i_b^1 \left(\tilde{A}_1^0 + \frac{h^2}{24} \tilde{A}_1^2 \right) + i_s^1 \tilde{A}_1^1 \right) dx dt \\ &= \int_0^T \mathcal{W} dt + \int_0^T \int_0^L \left(\sigma_b \left(\chi^0 + \frac{h^2}{24} \chi^2 \right) + \sigma_s \dot{\chi}^1 + \left(i_b^1 \left((\chi^0)_x + \frac{h^2}{24} (\chi^2)_x \right) + i_s^1 (\chi^1)_x \right) \right) dx dt. \\ &= \int_0^T \mathcal{W} dt + \int_0^T \int_0^L \left[\left(\frac{di_s^1}{dx} \right) \chi^1 - \frac{di_b^1}{dx} \left(\chi^0 + \frac{h^2}{24} \chi^2 \right) \right] dx dt \\ &\quad + \int_0^T \int_0^L \left(\dot{\sigma}_b \left(\chi^0 + \frac{h^2}{24} \chi^2 \right) + \dot{\sigma}_s \chi^1 \right) dx dt + h \left[\int_0^T \left(i_s^1 \chi^1 + i_b^1 \left(\chi^0 + \frac{h^2}{24} \chi^2 \right) \right) dt \right]_0^L \\ &\quad + \left[\int_0^L \left(\sigma_b \left(\chi^0 + \frac{h^2}{24} \chi^2 \right) + \sigma_s \chi^1 \right) dt \right]_0^T \\ &= \int_0^T \mathcal{W} dt,\end{aligned}$$

using $i_s^1 = i_b^1 = 0$ at the insulated edges of electrodes. Hence, the Lagrangian \mathbf{L} is invariant under the transformation (3.8). \square

Since \mathbf{L} is invariant under transformations of type (3.8), the electric and magnetic potentials are not uniquely determined by (3.5) and (3.7). An additional condition can be added to remove the ambiguity. The additional condition is generally known as a *gauge* and it is generally chosen to simplify the equations. It is common to choose the gauge to decouple the electrical potential equation from the equations of the magnetic potential. The Coulomb gauge, a commonly used gauge in standard electromagnetic theory, is

$$\nabla \cdot A = 0 \quad \text{in } \Omega, \quad A \cdot n = 0 \quad \text{on } \partial\Omega. \quad (3.9)$$

With this additional condition, the Maxwell equations (2.8) written in terms of the potentials

$$\begin{aligned} -\nabla^2 \phi - \frac{\partial(\nabla \cdot A)}{\partial t} &= 0 \\ \frac{\partial^2 A}{\partial t^2} - \nabla^2 A &= -\nabla \frac{\partial \phi}{\partial t} - \nabla(\nabla \cdot A) \end{aligned} \quad (3.10)$$

are decoupled (for instance see [7], p. 80) and (3.10) becomes

$$-\nabla^2 \phi = 0.$$

In this work, it is assumed that A and ϕ are quadratic in the thickness variable z . So in (2.20)–(2.22) integration by parts is with respect to the x variable, but not z . Because of this, (3.9) does not decouple the equations. Examining equation (3.5b) a closely related, but more appropriate condition is

$$-\frac{\varepsilon_1 h^2}{12} (A_1^1)_x + \varepsilon_3 \left(A_3^0 + \frac{h^2}{24} A_3^2 \right) = 0. \quad (3.11)$$

The boundary conditions

$$(A_1^1)(0) = (A_1^1)(L) = 0 \quad (3.12)$$

are analogous to that in (3.9) and decouple the boundary conditions (3.6).

The gauge condition (3.11) uniquely determines ϕ and A . Let A, \tilde{A} be potentials related by a transformation of the form (3.8) that satisfy (3.11) and (3.12). Then

$$\begin{aligned} 0 &= -\frac{\varepsilon_1 h^2}{12} (\tilde{A}_1^1)_x + \varepsilon_3 \left(\tilde{A}_3^0 + \frac{h^2}{24} \tilde{A}_3^2 \right) \\ &= -\frac{\varepsilon_1 h^2}{12} ((A_1^1)_x + (\chi^1)_{xx}) + \varepsilon_3 \left(A_3^0 + \chi^1 + \frac{h^2}{24} A_3^2 \right) \end{aligned} \quad (3.13)$$

$$= -\frac{\varepsilon_1 h^2}{12} (\chi^1)_{xx} + \varepsilon_3 \chi^1. \quad (3.14)$$

Also, (3.12) implies the boundary conditions

$$(\chi^1)_x(0) = (\chi^1)_x(L) = 0.$$

Since (3.14) with these boundary conditions has only the trivial solution $\chi^1 \equiv 0$ it follows that the additional conditions (3.11,3.12) uniquely define the potentials ϕ and A in (3.8). The existence and uniqueness of the solutions of the system (3.5) with (3.11), (3.12) will be analyzed in detail in Section 4.

The linear-through thickness assumption for the magnetic variables is more than enough since the variables A_1^2 and A_3^2 would not have any contribution in the analysis of this paper. Therefore, we assume that $A_3^2 \equiv 0$ for simplicity. For convenience of notation, define $\xi = \frac{\varepsilon_1 h^2}{12\varepsilon_3}, \theta := A_1^1, \eta := A_3^0$. The gauge condition (3.11) and boundary conditions (3.12) are

$$-\xi \theta_x + \eta = 0, \quad \theta(0) = \theta(L) = 0. \quad (3.15)$$

Using (3.15) in equations (3.5) and the boundary conditions (3.7) and (3.6) yields

$$\begin{cases} \rho\ddot{v} - \alpha v_{xx} - \gamma((\phi^1)_x + \dot{\eta}_x) = 0 \\ -\xi\varepsilon_3(\phi^1)_{xx} + \varepsilon_3\phi^1 - \gamma v_x = \frac{\sigma_s(t)}{h} \\ \xi\varepsilon_3\ddot{\theta} + \mu(\theta - \eta_x) + \xi\varepsilon_3(\dot{\phi}^1)_x = \frac{i_s^1(t)}{h} \\ \varepsilon_3\dot{\eta} + \mu(\theta - \eta_x)_x + \varepsilon_3\dot{\phi}^1 - \gamma\dot{v}_x = 0, \end{cases} \quad (x, t) \in (0, L) \times \mathbb{R}^+. \quad (3.16)$$

The boundary conditions (3.6) become

$$(\phi^1)_x = 0, \quad \theta = 0, \quad \eta_x = 0, \quad \alpha v_x + \gamma(\phi^1 + \dot{\eta}) = 0, \quad x = 0, L. \quad (3.17)$$

Remark 3.2. Distributed mechanical damping such as viscous damping $k\dot{v}$ or Kelvin-Voigt damping $k\ddot{v}_x$ with $k > 0$, can be considered in the model (3.16). However, the goal of the paper is to see the effect of the active electrical controller on the control of the piezoelectric beam model. Additional mechanical control could also be included, but is not considered here.

4. WELL-POSEDNESS

It is important to note that the controller $\sigma_s(t)$ or $i_s(t)$ are constant along the electroded region $(0, L)$ of the piezo-electric beam. Therefore, for the following argument, we consider the controllers together with the characteristic function of the interval $(0, L)$, which we denote by $\mathbb{1}_{(0,L)}(x)$, to emphasize their space dependency of the electroded region.

First consider the elliptic equation in (3.16). Define $D_x^2\phi = \phi_{xx}$ with domain

$$\text{Dom}(D_x^2) = \{\phi \in H^2(0, L), \quad \phi_x(0) = \phi_x(L) = 0\},$$

The equation

$$-\xi D_x^2\phi + \phi = z. \quad (4.1)$$

has a unique solution for ϕ for any $z \in \mathcal{L}_2(0, L)$. Define the operator

$$P := (-\xi D_x^2 + I)^{-1}. \quad (4.2)$$

It is well-known that P is a compact non-negative operator on $\mathcal{L}_2(0, L)$.

In the case of current actuation the solution to

$$-\xi\phi_{xx}^1 + \phi^1 = \frac{\gamma}{\varepsilon_3}v_x, \quad (\phi^1)_x(0) = (\phi^1)_x(L) = 0. \quad (4.3)$$

is $\phi^1 = \frac{\gamma}{\varepsilon_3} P v_x$. However, since the ϕ^1 -equation differs in the case of charge equation, *i.e.*

$$-\xi\phi_{xx}^1 + \phi^1 = \frac{\gamma}{\varepsilon_3}v_x + \frac{1}{\varepsilon_3 h}\sigma_s(t) \cdot \mathbb{1}_{(0,L)}(x), \quad (\phi^1)_x(0) = (\phi^1)_x(L) = 0. \quad (4.4)$$

the solution is $\phi^1 = P\left(\frac{\gamma}{\varepsilon_3}v_x + \frac{1}{\varepsilon_3 h}\sigma_s(t) \cdot \mathbb{1}_{(0,L)}(x)\right)$ where $P(\sigma_s(t)) = \sigma_s(t) + K$ and K is a constant. Once ϕ^1 is substituted into the other equations,

$$\dot{\phi}^1 = \begin{cases} \frac{\gamma}{\varepsilon_3} P\dot{v}_x, & \sigma_s(t) \equiv 0, i_s^1(t) \neq 0, \\ \frac{\gamma}{\varepsilon_3} P\dot{v}_x, & \sigma_s(t) \neq 0, i_s^1(t) \equiv 0. \end{cases} \quad (4.5)$$

since $\dot{\sigma} = 0$ by (3.4) and $\dot{K} = 0$, and

$$\phi_x^1 = \begin{cases} \frac{\gamma}{\varepsilon_3} (Pv_x)_x, & \sigma_s(t) \equiv 0, i_s^1(t) \neq 0, \\ \frac{\gamma}{\varepsilon_3} (Pv_x)_x + \frac{\sigma_s(t)}{\varepsilon_3 h} \mathbb{1}'_{(0,L)}(x), & \sigma_s(t) \neq 0, i_s^1(t) \equiv 0. \end{cases} \quad (4.6)$$

where $K_x = 0$. Note that $\mathbb{1}'_{(0,L)}(x) = (\mathbb{1}_{(0,L)}(x))_x = \delta(x) - \delta(x - L)$ where δ is the Dirac delta distribution.

The stretching equations (3.16) are rewritten as

$$\begin{cases} \rho\ddot{v} - \alpha v_{xx} - \frac{\gamma^2}{\varepsilon_3} (Pv_x)_x - \gamma\dot{\eta}_x = \frac{\gamma\sigma_s(t) \cdot \mathbb{1}'_{(0,L)}(x)}{\varepsilon_3 h} \\ \xi\varepsilon_3\ddot{\theta} + \mu(\theta - \eta_x) + \xi\gamma(P\dot{v}_x)_x = \frac{i_s^1(t)}{h} \\ \varepsilon_3\ddot{\eta} + \mu(\theta - \eta_x)_x - \gamma(\dot{v}_x - (P\dot{v}_x)) = 0 \end{cases} \quad (4.7)$$

The boundary conditions (3.17) are, at $x = 0, L$,

$$\alpha v_x + \gamma \left(\frac{\gamma}{\varepsilon_3} Pv_x + \dot{\eta} \right) = 0, \quad \text{with } \theta = 0, \eta_x = 0, \quad x = 0, L. \quad (4.8)$$

Note that the electrical coupling to the v -equation makes the beam much stiffer. In fact, if we assume that the constant $\varepsilon_1 \rightarrow 0$ in (3.16) and therefore $\xi \rightarrow 0$, which is a common assumption in other approaches, the term $\xi\varepsilon_3\phi_{xx}^1 \rightarrow 0$ in (3.16) since $\xi \rightarrow 0$. This implies that $P \rightarrow I$, *i.e.* $Pv_x \rightarrow v_x$. Therefore the first equation in (4.7) reads

$$\rho\ddot{v} - \left(\alpha + \frac{\gamma^2}{\varepsilon_3} \right) v_{xx} - \gamma\dot{\eta}_x = \frac{\gamma\sigma_s(t) \cdot \mathbb{1}'_{(0,L)}(x)}{\varepsilon_3 h}.$$

This implies that the stretching equation has a bigger stiffness coefficient $\alpha + \frac{\gamma^2}{\varepsilon_3}$.

The natural energy associated with (4.7) is

$$\mathbb{E}(t) = \frac{1}{2} \int_0^L \left\{ \rho|\dot{v}|^2 + \alpha|v_x|^2 + \frac{\gamma^2}{\varepsilon_3} (Pv_x)_x^2 + \xi\varepsilon_3|\dot{\theta}|^2 + \varepsilon_3\xi^2|\dot{\theta}_x|^2 + \mu|\theta - \eta_x|^2 \right\} dx.$$

This motivates defining the space

$$\mathbb{H} = (\mathcal{L}_2(0, L))^2 \times H_0^1(0, L) \times \mathcal{L}_2(0, L) \quad (4.9)$$

with the bilinear form

$$\langle \mathbf{y}, \mathbf{z} \rangle_{\mathbb{H}} = \int_0^L \left\{ \rho y_1 \bar{z}_1 + \alpha y_2 \bar{z}_2 + \frac{\gamma^2}{\varepsilon_3} (Py_2) \bar{z}_2 + \xi\varepsilon_3 y_3 \bar{z}_3 + \xi^2 \varepsilon_3 (y_3)_x (\bar{z}_3)_x + \mu y_4 \bar{z}_4 \right\} dx. \quad (4.10)$$

The following result is straightforward.

Lemma 4.1. *The form (4.10) defines an inner product on the linear space \mathbb{H} . Moreover, \mathbb{H} is complete.*

Define the state variable,

$$\mathbf{y} = \begin{bmatrix} \dot{v} \\ v_x \\ \dot{\theta} \\ \theta - \eta_x \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix},$$

the system (4.7) can be written

$$\dot{\mathbf{y}} = \mathcal{A}\mathbf{y} - \frac{1}{\xi\varepsilon_3 h} \mathcal{B}i_s^1(t), \quad \mathbf{y}(x, 0) = \mathbf{y}^0 = (v^1, v_x^0, \theta^1, \theta^0 - \eta_x^0) \quad (4.11)$$

where

$$\mathcal{A} = \begin{bmatrix} 0 & \frac{\alpha}{\rho} D_x + \frac{\gamma^2}{\varepsilon_3 \rho} D_x P & \frac{\xi\gamma}{\rho} D_x^2 & 0 \\ D_x & 0 & 0 & 0 \\ -\frac{\gamma}{\varepsilon_3} D_x P D_x & 0 & 0 & \frac{-\mu}{\xi\varepsilon_3} I \\ 0 & 0 & I - \xi D_x^2 & 0 \end{bmatrix}, \quad B\psi = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\xi\varepsilon_3 h} \\ 0 \end{bmatrix}, \quad (4.12)$$

$$\text{Dom}(\mathcal{A}) = (H^1(0, L))^2 \times (H^2(0, L) \cap H_0^1(0, L)) \times H_0^1(0, L) \quad (4.13)$$

$$\cap \left\{ \mathbf{y} \in \mathbf{H} : \left(\alpha I + \frac{\gamma^2}{\varepsilon_3} P \right) y_2 + \gamma \xi (y_3)_x = 0 \text{ at } x = 0, L \right\}, \quad (4.14)$$

$$\text{Dom}(\mathcal{A}) = \left\{ y \in (H^1(0, L))^2 \times (H_0^1(0, L) \cap H^2(0, L)) \times H_0^1(0, L); \left(\alpha I + \frac{\gamma^2}{\varepsilon_3} P \right) y_2 + \gamma \xi (y_3)_x = 0 \text{ at } x = 0, L \right\}.$$

For any constant C , $(C, 0, 0, 0)^T$ is the only solution of the eigenvalue problem $\mathcal{A}\mathbf{y} = \lambda\mathbf{y}$ for the eigenvalue $\lambda = 0$. Let \tilde{X} represents the space of such solutions. Define the factor spaces $W_2 := \text{Dom}(\mathcal{A})/\tilde{X}$ and $W_1 := H/\tilde{X}$.

It will be shown that \mathcal{A} generates a strongly continuous semigroup on a Hilbert space with norm corresponding to the physical energy. This implies well-posedness of the uncontrolled system. Since \mathcal{B} will be a bounded operator into the state space this will also imply well-posedness of the controlled system.

Lemma 4.2. *Let $\text{Dom}(D_x^2) = \{w \in H^2(0, L) : w_x(0) = w_x(L) = 0\}$. The operator $\frac{1}{\xi}(P - I)$ is continuous, self-adjoint and non-positive on $\mathcal{L}_2(0, L)$. Moreover, for all $w \in \text{Dom}(P)$,*

$$J = D_x^2 P = D_x^2 (I - \xi D_x^2)^{-1} w.$$

Proof. Define $J = \frac{1}{\xi}(P - I)$. Continuity and self-adjointness easily follow from the definition of J . We first prove that J is a non-positive operator. Let $u \in \mathcal{L}_2(0, L)$. Then $(I - \xi D_x^2)^{-1} u = s$ implies that $s \in \text{Dom}(D_x^2)$ and $s - \xi s_{xx} = u$

$$\langle Ju, u \rangle_{\mathcal{L}_2(0, L)} = \left\langle \frac{1}{\xi} (P - I)u, u \right\rangle_{\mathcal{L}_2(0, L)} = \frac{1}{\xi} \langle s - s + \xi s_{xx}, s - \xi s_{xx} \rangle_{\mathcal{L}_2(0, L)} - \|s_x\|_{\mathcal{L}_2(0, L)}^2 - \xi \|s_{xx}\|_{\mathcal{L}_2(0, L)}^2.$$

Let $Jw = \frac{1}{\xi}(P - I)w$ and $v := Pw$. Then $v - \xi v_{xx} = w$. By a simple rearrangement of the terms

$$Jw = \frac{1}{\xi}(v - w) = \frac{1}{\xi}(v - v + \xi v_{xx}) = v_{xx} = D_x^2 Pw.$$

□

Lemma 4.3. *The operator $\mathcal{A} : W_2 \rightarrow W_1$ is densely defined in W_1 .*

Proof. Let $\mathbf{y} \in \text{Dom}(\mathcal{A})$. Then

$$\mathcal{A}\mathbf{y} = \begin{pmatrix} \frac{\alpha}{\rho}(y_2)_x + \frac{\gamma^2}{\varepsilon_3\rho}(Py_2)_x + \frac{\gamma\xi}{\rho}(y_3)_{xx} \\ (y_1)_x \\ -\frac{\gamma}{\varepsilon_3}(P(y_1)_x)_x - \frac{\mu}{\xi\varepsilon_3}y_4 \\ y_3 - \xi(y_3)_{xx} \end{pmatrix}.$$

Since $\mathbf{y} \in W_1$, $y_1 \in H^1(0, L)$, $y_2 \in H^1(0, L)$, $y_3 \in H^2(0, L) \cap H_0^1(0, L)$ and $y_4 \in H_0^1(0, L)$, $(y_1)_x \in \mathcal{L}_2(0, L)$, and by the definition of the operator in (4.2) $(P(y_1)_x)_x \in H_0^1(0, L)$. Therefore $(P(y_1)_x)_x + y_4 \in H_0^1(0, L)$. Hence, $\mathcal{A}\mathbf{y} \in W_1$. \square

Theorem 4.4. *The operator $\mathcal{A} : W_2 \subset W_1 \rightarrow W_1$ satisfies $\mathcal{A}^* = -\mathcal{A}$ on W_1 , and \mathcal{A} defined by (4.12) is the generator of a unitary semigroup $\{e^{\mathcal{A}t}\}_{t \geq 0}$ on W_1 .*

Proof. Let $\mathbf{y}, \mathbf{z} \in W_2$. Then we have

$$\begin{aligned} \langle \mathcal{A}\mathbf{y}, \mathbf{z} \rangle_{W_1} &= \left\langle \begin{pmatrix} \frac{\alpha}{\rho}(y_2)_x + \frac{\gamma^2}{\varepsilon_3\rho}(Py_2)_x + \frac{\gamma\xi}{\rho}(y_3)_{xx} \\ (y_1)_x \\ -\frac{\gamma}{\varepsilon_3}(P(y_1)_x)_x - \frac{\mu}{\xi\varepsilon_3}y_4 \\ y_3 - \xi(y_3)_{xx} \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \right\rangle_{W_1} \\ &= \int_0^L \left\{ \left(\frac{\alpha}{\rho}(y_2)_x + \frac{\gamma^2}{\varepsilon_3\rho}(Py_2)_x + \frac{\gamma}{\rho}(y_3)_{xx} \right) \bar{z}_1 + \alpha(y_1)_x \bar{z}_2 + \frac{\gamma^2}{\varepsilon_3}(P(y_1)_x) \bar{z}_2 \right. \\ &\quad \left. - (\gamma\xi(P(y_1)_x)_x + \mu y_4) \bar{z}_3 - [\gamma\xi^2(P(y_1)_x)_x + \mu\xi y_4]_x (\bar{z}_3)_x + \mu(y_3 - (y_3)_{xx}) \bar{z}_4 \right\} dx \\ &= \int_0^L \left\{ \left(\frac{\alpha}{\rho}(y_2)_x + \frac{\gamma^2}{\varepsilon_3\rho}(Py_2)_x + \frac{\gamma}{\rho}(y_3)_{xx} \right) \bar{z}_1 + \alpha(y_1)_x \bar{z}_2 + \frac{\gamma^2}{\varepsilon_3}(P(y_1)_x) \bar{z}_2 \right. \\ &\quad \left. - (\gamma\xi(P(y_1)_x)_x + \mu y_4) \bar{z}_3 - \gamma\xi[P(y_1)_x - (y_1)_x] (\bar{z}_3)_x + \mu\xi(y_4)_x (\bar{z}_3)_x + \mu(y_3 - (y_3)_{xx}) \bar{z}_4 \right\} dx \end{aligned}$$

By using boundary conditions (4.13), Lemma 4.2, and integration by parts,

$$\begin{aligned} \langle \mathcal{A}\mathbf{y}, \mathbf{z} \rangle_{W_1} &= \left[\left(\alpha y_2 + \frac{\gamma^2}{\varepsilon_3}(Py_2) + \gamma(y_3)_x \right) \bar{z}_1 + \left(\alpha \bar{z}_2 + \frac{\gamma^2}{\varepsilon_3}(P\bar{z}_2) + \gamma(\bar{z}_3)_x \right) y_1 + \gamma\xi y_1 (\bar{z}_3)_x \right]_{x=0}^{x=L} \\ &\quad - [\mu y_4 (\bar{z}_3)_x - \gamma\xi P(y_1)_x \bar{z}_3]_{x=0}^{x=L} + \int_0^L \left\{ - \left(\alpha y_2 + \frac{\gamma^2}{\varepsilon_3}(Py_2) + \gamma(y_3)_x \right) (\bar{z}_1)_x - \mu y_4 \bar{z}_3 + \mu y_3 \bar{z}_4 \right. \\ &\quad \left. - \left(\alpha (\bar{z}_2)_x + \frac{\gamma^2}{\varepsilon_3}(P\bar{z}_2)_x + \gamma(\bar{z}_3)_{xx} \right) y_1 + \mu y_4 (\bar{z}_3)_{xx} - \mu(y_3)_{xx} \bar{z}_4 \right\} dx, \end{aligned}$$

$$\begin{aligned} \langle \mathbf{y}, -\mathcal{A}\mathbf{z} \rangle_{W_1} &= \int_0^L - \left\{ \left(\frac{\alpha}{\rho}(\bar{z}_2)_x + \frac{\gamma^2}{\varepsilon_3\rho}(P\bar{z}_2)_x + \frac{\gamma}{\rho}(\bar{z}_3)_{xx} \right) y_1 + \alpha(\bar{z}_1)_x y_2 \right. \\ &\quad \left. + \frac{\gamma^2}{\varepsilon_3}(P(\bar{z}_1)_x) y_2 - (\gamma\xi(P(\bar{z}_1)_x)_x + \mu \bar{z}_4) y_3 - [\gamma\xi^2(P(\bar{z}_1)_x)_x + \mu\xi \bar{z}_4]_x (y_3)_x + \mu(\bar{z}_3 - (\bar{z}_3)_{xx}) y_4 \right\} dx \\ &= \int_0^L - \left\{ \left(\frac{\alpha}{\rho}(\bar{z}_2)_x + \frac{\gamma^2}{\varepsilon_3\rho}(P\bar{z}_2)_x + \frac{\gamma}{\rho}(\bar{z}_3)_{xx} \right) y_1 + \alpha(\bar{z}_1)_x y_2 + \frac{\gamma^2}{\varepsilon_3}(P(\bar{z}_1)_x) y_2 \right. \\ &\quad \left. - (\gamma\xi(P(\bar{z}_1)_x)_x + \mu \bar{z}_4) y_3 - \gamma\xi(P(\bar{z}_1)_x - (\bar{z}_1)_x) (y_3)_x - \mu\xi(\bar{z}_4)_x (y_3)_x + \mu(\bar{z}_3 - (\bar{z}_3)_{xx}) y_4 \right\} dx \end{aligned}$$

$$= \int_0^L \left\{ - \left(\alpha y_2 + \frac{\gamma^2}{\varepsilon_3} (P y_2) + \gamma (y_3)_x \right) (\bar{z}_1)_x - \mu y_4 \bar{z}_3 + \mu y_3 \bar{z}_4 \right. \\ \left. - \left(\alpha (\bar{z}_2)_x + \frac{\gamma^2}{\varepsilon_3} (P \bar{z}_2)_x + \gamma (\bar{z}_3)_{xx} \right) y_1 + \mu y_4 (\bar{z}_3)_{xx} - \mu (y_3)_{xx} \bar{z}_4 \right\} dx,$$

This implies that \mathcal{A} is skew-symmetric. To prove that \mathcal{A} is skew-adjoint on W_1 , *i.e.* $\mathcal{A}^* = -\mathcal{A}$ on W_1 , with the same domains, it is required to show that for any $\mathbf{g} \in W_1$ there is $\mathbf{y} \in W_2$ so that $\mathcal{A}\mathbf{y} = \mathbf{g}$. This is equivalent to solving the system of equations for $\mathbf{y} \in W_2$. Simplifying the equations leads to

$$\begin{aligned} \frac{\alpha}{\rho} (y_2)_x + \frac{\gamma^2}{\varepsilon_3 \rho} (P y_2)_x + \frac{\gamma \xi}{\rho} (y_3)_{xx} &= g_1 \in \mathcal{L}_2(0, L) \\ (y_1)_x &= g_2 \in \mathcal{L}_2(0, L) \\ -\frac{\gamma}{\varepsilon_3} (P (y_1)_x)_x - \frac{\mu}{\xi \varepsilon_3} y_4 &= g_3 \in H_0^1(0, L) \\ y_3 - \xi (y_3)_{xx} &= g_4 \in \mathcal{L}_2(0, L) \end{aligned} \quad (4.15)$$

where g_1 satisfies $\int_0^L g_1(x) dx = 0$ due to the space \tilde{X} . Integration and using the boundary conditions, yields that $\mathbf{y} \in W_2$ and

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} \int g_2(x) dx \\ (\alpha I + \frac{\gamma^2}{\varepsilon_3} P)^{-1} \left[-\gamma \xi (\tilde{P} g_4)_x + \rho \int_0^x g_1(x) dx \right] \\ \tilde{P} g_4 \\ \frac{1}{\mu} [-\gamma \xi (P g_2)_x - \xi \varepsilon_3 g_3] \end{pmatrix}$$

where $\tilde{P} = (I - \xi D_x^2)^{-1}$ is the self-adjoint operator defined on $\mathcal{L}_2(0, L)$ with $\text{Dom}(D_x^2) = H^2(0, L) \cap H_0^1(0, L)$. This proves that $\mathbf{y} \in \text{Dom}(\mathcal{A})$, and therefore \mathcal{A} is skew-adjoint. By taking $\mathbf{z} = \mathbf{y}$, $\text{Re} \langle \mathcal{A}\mathbf{y}, \mathbf{y} \rangle_{W_1} = \text{Re} \langle \mathcal{A}^*\mathbf{y}, \mathbf{y} \rangle_{W_1} = 0$.

It follows then from the Lumer-Phillips Theorem [29], that \mathcal{A} generates a dissipative semigroup on H . Since \mathcal{A} is skew-adjoint, the semigroup is unitary. \square

The fact that \mathcal{A} generates a unitary semigroup means that the norm and hence the energy $E(t)$ is conserved along solution trajectories of (4.12) if there is no control term; that is in the absence of control

$$\|\mathbf{y}(t)\| = \|\mathbf{y}_0\|.$$

Consider the current actuation case: $i_s(t) \neq 0, \sigma_s(t) \equiv 0$. Since \mathcal{B} is a bounded operator into the state-space, it now follows that the control system is well-posed.

Theorem 4.5. *Let $T > 0$, and $i_s(t) \in L^2(0, T)$. For any $\mathbf{y}_0 \in W_1$, $\mathbf{y} \in C[[0, T]; W_1]$, and there exists a positive constants $c(T)$ such that (4.11) satisfies*

$$\|\mathbf{y}(T)\|_{W_1}^2 \leq c(T) \left\{ \|\mathbf{y}_0\|_{W_1}^2 + \|i_s\|_{L^2(0, T)}^2 \right\}. \quad (4.16)$$

Theorem 4.6. *The spectrum $\sigma(\mathcal{A})$ of \mathcal{A} contains only isolated eigenvalues.*

Proof. Since W_2 is densely defined and compact in W_1 , $0 \in \rho(\mathcal{A})$, we have $(\lambda I - \mathcal{A})^{-1}$ is compact at $\lambda = 0$, thus compact for all $\lambda \in \rho(\mathcal{A})$. Hence the spectrum of \mathcal{A} consists of isolated eigenvalues. \square

5. STABILIZATION

The control operator \mathcal{B} is bounded in W_1 and rank 1. Thus, it is compact. Since the semigroup \mathcal{A} is unitary, the system (4.11) cannot be exponentially stabilized; see [14] or the textbook [6]. The question is then, is it possible to asymptotically stabilize the system? The adjoint operator $\mathcal{B}^* : W_1 \rightarrow \mathcal{U}$ is

$$\mathcal{B}^*\psi = \frac{1}{h} \int_0^L \psi_3(x) dx.$$

The \mathcal{B}^* -feedback refers to the time derivative of the “envelope” magnetic flux in the x -direction. This is called “Faraday emf” and it is the induced voltage accumulated at the electrodes of beam. There are sensors to measure this quantity; see for instance [37]. If all the eigenvalues are stabilizable, that is $\mathcal{B}^*\phi \neq 0$ for all eigenfunctions, then

$$i_s(t) = -k\mathcal{B}^*\mathbf{y}(t)$$

provides asymptotic stabilization for any gain $k > 0$.

In this section it is shown that for some materials there are unstabilizable eigenvalues and that for these systems, the system is not stabilizable by any feedback. A characterization of eigenvalues and eigenfunctions for some material properties is first calculated.

Theorem 5.1. *For real τ , define*

$$\begin{aligned} A &= \alpha\mu\xi\varepsilon_3 \\ B &= -(\alpha\varepsilon_3 + \gamma^2)(\mu - \varepsilon_3\xi\tau^2) + \xi\varepsilon_3\mu\rho\tau^2 \\ C &= -\rho\tau^2\varepsilon_3(\mu - \varepsilon_3\xi\tau^2), \end{aligned} \tag{5.1}$$

and $a = \frac{1}{\sqrt{\xi}}$,

$$a_3 = \sqrt{\frac{B + \sqrt{B^2 - 4AC}}{2A}}, \quad a_4 = \sqrt{\frac{B - \sqrt{B^2 - 4AC}}{2A}}, \tag{5.2}$$

$$b_3 = -\frac{\alpha\mu}{\gamma\tau(\mu - \varepsilon_3\xi\tau^2)} \left(-\frac{\tau^2\rho}{\alpha} + \frac{a^2}{a^2 + a_3^2} \frac{\gamma^2}{\alpha\varepsilon_3\mu} (\mu - \varepsilon_3\xi\tau^2) a_3^2 + a_3^2 \right), \tag{5.3}$$

$$b_4 = -\frac{\alpha\mu}{\gamma\tau(\mu - \varepsilon_3\xi\tau^2)} \left(-\frac{\tau^2\rho}{\alpha} + \frac{a^2}{a^2 + a_4^2} \frac{\gamma^2}{\alpha\varepsilon_3\mu} (\mu - \varepsilon_3\xi\tau^2) a_4^2 + a_4^2 \right) \tag{5.4}$$

$$b_{33} = \frac{ia^2a_3}{a^2 + a_3^2}, \quad b_{44} = \frac{ia^2a_4}{a^2 + a_4^2} \tag{5.5}$$

Assume that for some integers m, n , with $m \neq n$,

$$a_3 = \frac{2n\pi}{L} \quad \text{and} \quad a_4 = \frac{2m\pi}{L}, \tag{5.6}$$

and $\mu - \varepsilon_3\xi\tau^2 < 0$. Then \mathcal{A} has eigenvalues $i\tau_{\mp}^{(j)}$, $j = 3, 4$ satisfying

$$\tau_{\mp}^{(j)} = \frac{\sqrt{\frac{\mu\rho + \xi a_j^2(\alpha\varepsilon_3 + \gamma^2 + \mu\rho) \mp \sqrt{(\mu\rho + \xi a_j^2(\alpha\varepsilon_3 + \gamma^2 + \mu\rho))^2 - 4a_j^2\mu\xi\rho(\varepsilon_3(\alpha\xi a_j^2 + \alpha) + \gamma^2)}}{\xi\rho\varepsilon_3}}{\sqrt{2}}. \tag{5.7}$$

The corresponding eigenfunctions of \mathcal{A} are

$$\mathbf{y} = \begin{pmatrix} \imath\tau(C(a_3) \sin(a_4x) + C(a_4) \sin(a_3x)) \\ a_4C(a_4) \cos(a_4x) + a_3C(a_3) \cos(a_3x) \\ \imath\tau(b_4C(a_3) \sin(a_4x) + b_3C(a_4) \sin(a_3x)) \\ b_4\left(\frac{a_4^2}{a^2} + 1\right)C(a_3) \sin(a_4x) + b_3\left(\frac{a_3^2}{a^2} + 1\right)C(a_4) \sin(a_3x) \end{pmatrix} \quad (5.8)$$

where

$$C(a_j) = (-1)^j (-b_{jj}\gamma^2 + \imath a_j \varepsilon_3 (b_j \gamma \xi \tau + \imath \alpha)), \quad j = 3, 4. \quad (5.9)$$

Proof. Since all the eigenvalues are imaginary, setting $\lambda = \imath\tau$ for some real τ , the eigenvalue problem $\mathcal{A}\mathbf{y} = \lambda\mathbf{y}$ can be written

$$\begin{cases} \alpha(y_2)_x + \frac{\gamma^2}{\varepsilon_3}(Py_2)_x + \gamma\xi(y_3)_{xx} = \imath\rho\tau \\ (y_1)_x = \imath\tau y_2 \\ -\gamma\xi(P(y_1)_x)_x - \mu y_4 = \imath\tau\xi\varepsilon_3 y_3 \\ y_3 - \xi(y_3)_{xx} = \imath\tau y_4 \end{cases} \quad (5.10)$$

with the boundary conditions

$$\alpha y_2 + \frac{\gamma^2}{\varepsilon_3} P y_2 + \gamma \xi (y_3)_x, \quad y_3, \quad y_4 = 0, \quad x = 0, L.$$

Defining $r := P(y_1)_x$, i.e. $r - \xi r_{xx} = (y_1)_x$, (5.10) reduces to

$$\begin{cases} \alpha(y_1)_{xx} + \tau^2 \rho y_1 + \frac{\gamma^2}{\varepsilon_3} r_x + \imath\tau \gamma \xi (y_3)_{xx} = 0 \\ -\gamma \xi \imath\tau r_x - \mu (y_3 - \xi (y_3)_{xx}) + \tau^2 \xi \varepsilon_3 y_3 = 0 \\ r - \xi r_{xx} = (y_1)_x \end{cases} \quad (5.11)$$

with the boundary conditions

$$\alpha(y_1)_x + \frac{\gamma^2}{\varepsilon_3} r + \imath\tau \gamma \xi (y_3)_x, \quad r_x, \quad y_3 = 0, \quad x = 0, L. \quad (5.12)$$

Solving (5.11) for r_{xx} , $(y_1)_{xx}$ and $(y_3)_{xx}$ yields

$$\begin{cases} (y_1)_{xx} = -\frac{\tau^2 \rho}{\alpha} y_1 + \left(\frac{\gamma^2 \tau^2 \xi}{\alpha \mu} - \frac{\gamma^2}{\alpha \varepsilon_3}\right) r_x + \left(\frac{\imath\tau^3 \xi \gamma \varepsilon_3}{\alpha \mu} - \frac{\imath\tau \gamma}{\alpha}\right) y_3. \\ (y_3)_{xx} = \frac{1}{\mu} \left(\gamma \imath\tau r_x + \frac{\mu}{\xi} y_3 - \tau^2 \varepsilon_3 y_3\right) \\ r_{xx} = \frac{1}{\xi} (r - (y_1)_x) \end{cases} \quad (5.13)$$

TABLE 2. Signs of B and C determines the type of eigenvalues of (5.17).

	$\tau^2 > \frac{\mu}{\varepsilon_3 \xi}$	$\frac{\mu}{\varepsilon_3 \xi} \frac{\alpha \varepsilon_3 + \gamma^2}{\alpha \varepsilon_3 + \gamma^2 + \mu \rho} < \tau^2 < \frac{\mu}{\varepsilon_3 \xi}$	$\tau^2 < \frac{\mu}{\varepsilon_3 \xi} \frac{\alpha \varepsilon_3 + \gamma^2}{\alpha \varepsilon_3 + \gamma^2 + \mu \rho}$
Sign of B	+	+	-
Sign of C	+	-	-

Define $Z = [y_1 \ (y_1)_x \ y_3 \ (y_3)_x \ r \ r_x]^T$. The system (5.13) is rewritten as

$$\frac{dZ}{dx} = \mathcal{D}Z = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{\tau^2 \rho}{\alpha} & 0 & \frac{-\iota \tau \gamma}{\alpha} + \frac{\iota \tau^3 \gamma \xi \varepsilon_3}{\alpha \mu} & 0 & 0 & -\frac{\gamma^2}{\alpha \varepsilon_3} + \frac{\gamma^2 \tau^2 \xi}{\alpha \mu} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\xi} - \frac{\tau^2 \varepsilon_3}{\mu} & 0 & 0 & \frac{\gamma \tau \iota}{\mu} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -\frac{1}{\xi} & 0 & 0 & \frac{1}{\xi} & 0 \end{pmatrix} Z. \quad (5.14)$$

with the boundary conditions

$$\alpha z_2 + \frac{\gamma^2}{\varepsilon_3} z_5 + \iota \tau \gamma \xi z_4, \quad z_3, \quad z_6, \quad x = 0, L. \quad (5.15)$$

The solution to (5.14) is

$$Z = e^{\mathcal{D}x} K \quad (5.16)$$

where $K = [k_1 \ k_2 \ k_3 \ k_4 \ k_5 \ k_6]^T$ is a vector of arbitrary coefficients. The characteristic equation $\text{Det}(\mathcal{D} - \lambda I) = 0$ yields

$$\left(\xi \tilde{\lambda}^2 - 1 \right) \left[\alpha \mu \xi \varepsilon_3 \tilde{\lambda}^4 + (-\alpha \varepsilon_3 + \gamma^2)(\mu - \varepsilon_3 \xi \tau^2) + \xi \varepsilon_3 \mu \rho \tau^2 \right] \tilde{\lambda}^2 - \varepsilon_3 \rho \tau^2 (\mu - \xi \varepsilon_3 \tau^2) = 0. \quad (5.17)$$

This equation has two real roots $\tilde{\lambda}_1 := a$ and $\tilde{\lambda}_2 = -a$, and four other roots depending on the discriminant of the fourth order polynomial

$$\begin{aligned} \Delta &= B^2 - 4AC = [(\alpha \varepsilon_3 + \gamma^2)(\mu - \varepsilon_3 \xi \tau^2) - \xi \varepsilon_3 \mu \rho \tau^2]^2 + 4\alpha \mu \xi \rho \varepsilon_3^2 \tau^2 (\mu - \xi \tau^2 \varepsilon_3) \\ &= [\alpha \varepsilon_3 (\mu - \varepsilon_3 \xi \tau^2) + \varepsilon_3 \xi \mu \rho \tau^2]^2 + \gamma^4 (\mu - \varepsilon_3 \xi \tau^2)^2 + 2\gamma^2 [\alpha \varepsilon_3 (\mu - \varepsilon_3 \xi \tau^2) - \varepsilon_3 \xi \mu \rho \tau^2] (\mu - \varepsilon_3 \xi \tau^2). \end{aligned} \quad (5.18)$$

The type of eigenvalues of (5.17) depends on the sign of B and $B^2 - 4AC$. There are mainly three cases as in Table 2:

Consider $\tau^2 > \frac{\mu}{\varepsilon_3 \xi}$. Since $A, B, C, \Delta > 0$ and $\frac{-B \mp \sqrt{B^2 - 4AC}}{2A} < 0$,

$$\mp \sqrt{\frac{-B \mp \sqrt{B^2 - 4AC}}{2A}} = \mp \iota \sqrt{\frac{B \pm \sqrt{B^2 - 4AC}}{2A}}.$$

The other four complex conjugate roots of the characteristic equation (5.17) are

$$\tilde{\lambda}_3 = \iota a_3, \quad \tilde{\lambda}_4 = -\iota a_3, \quad \tilde{\lambda}_5 = \iota a_4, \quad \tilde{\lambda}_6 = -\iota a_4 \quad (5.19)$$

where $a_3 \neq a_4$, and a_3, a_4 are defined by (5.2).

Since $a_3 \neq a_4$, in (5.5), $b_3 - b_4 \neq 0, b_{33} - b_{44} \neq 0$. The solution of (5.14) is $Z = Pe^{Jx}P^{-1}K$ where, defining $b = \frac{-i\tau\varepsilon_3 a}{\gamma}$,

$$P = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & ia_3 & -ia_3 & ia_4 & -ia_4 \\ 1 & 1 & b_3 & b_3 & b_4 & b_4 \\ a & -a & ia_3b_3 & -ia_3b_3 & ia_4b_4 & -ia_4b_4 \\ b & -b & b_{33} & -b_{33} & b_{44} & -b_{44} \\ ab & ab & ia_3b_{33} & ia_3b_{33} & ia_4b_{44} & ia_4b_{44} \end{pmatrix}, \quad e^{Jx} = \begin{pmatrix} e^{xa} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-xa} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{ia_3x} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-ia_3x} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{ia_4x} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-ia_4x} \end{pmatrix}.$$

The boundary conditions $z_3(0) = z_6(0) = 0$, imply that $k_3 = k_6 = 0$. Considering the other boundary condition at $x = 0$: $\alpha z_2(0) + \frac{\gamma^2}{\varepsilon_3} z_5(0) + i\tau\gamma\xi z_4(0) = 0$, leads to $k_2 = -\frac{i\gamma\xi\tau k_4}{\alpha} - \frac{\gamma^2 k_5}{\alpha\varepsilon_3}$. The other three boundary conditions at $x = L$ together with $a_3 = \frac{2n\pi}{L}$ and $a_4 = \frac{2m\pi}{L}$, defines a system of equations for the unknowns $\{k_1, k_4, k_5\}$. Using Mathematica symbolic calculation, the coefficient matrix is row reduced to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{a_4 b_4 b_{33} \gamma^2 + a_3 (-b_3 b_{44} \gamma^2 - i\alpha \varepsilon_3 a_4 (b_3 - b_4))}{\varepsilon_3 a_4 (\alpha + i\gamma \xi \tau b_4) b_{33} - \varepsilon_3 a_3 (\alpha + i\gamma \xi \tau b_3) b_{44}} \\ 0 & 0 & 0 \end{pmatrix}.$$

This immediately implies that $k_1 = 0$. Since k_5 can be chosen to be a free variable, for simplicity take $k_5 = 1$. The unknown coefficients are determined by

$$k_2 = \frac{i\gamma (a_3 (a_4 (b_4 - b_3) \xi \tau \varepsilon_3 - b_{44} \gamma) + a_4 b_{33} \gamma)}{a_4 b_{33} \varepsilon_3 (b_4 \gamma \xi \tau - i\alpha) + ia_3 b_{44} \varepsilon_3 (\alpha + ib_3 \gamma \xi \tau)}, \quad k_4 = \frac{(-a_4 b_4 b_{33} \gamma^2 + a_3 (b_3 b_{44} \gamma^2 + i\alpha a_4 (b_3 - b_4) \varepsilon_3))}{a_4 b_{33} \varepsilon_3 (\alpha + ib_4 \gamma \xi \tau) - a_3 b_{44} \varepsilon_3 (\alpha + ib_3 \gamma \xi \tau)}$$

so that the nonzero solution $\mathbf{z} \neq \mathbf{0}$ of (5.14) is

$$\mathbf{z} = \begin{pmatrix} C(a_3) \sin(a_4 x) + C(a_4) \sin(a_3 x) \\ a_4 C(a_4) \cos(a_4 x) + a_3 C(a_3) \cos(a_3 x) \\ b_4 C(a_3) \sin(a_4 x) + b_3 C(a_4) \sin(a_3 x) \\ a_4 b_4 C(a_3) \cos(a_4 x) + a_3 b_3 C(a_4) \cos(a_3 x) \\ -ib_{33} C(a_4) \cos(a_3 x) - ib_{44} C(a_3) \cos(a_4 x) \\ ia_4 b_{44} C(a_3) + ia_3 b_{33} C(a_4) \sin(a_3 x) \end{pmatrix}$$

where $\tau, b_3, b_{33}, b_4, b_{44}$ are determined by (5.5), $C(a_3), C(a_4) \neq 0$ are defined in (5.9).

Thus, the nonzero solution $\mathbf{y} \neq \mathbf{0}$ of the eigenvalue problem (5.10) is (5.8). □

For some material parameters, the form of a_3 and a_4 in (5.6) is attained for some $m \neq n$ with suitably large (5.7); see the experimental values in Table 3 and the calculated values in Table 4.

Theorem 5.2. *If there are positive integers $m, n, m \neq n$ so that a_3 and a_4 in (5.2) satisfy $a_3 = \frac{2n\pi}{L}, a_4 = \frac{2m\pi}{L}$ then (4.11) is not asymptotically stabilizable by any bounded state feedback.*

Proof. Theorem 5.1 provides a characterization of the eigenfunctions \mathbf{y} for systems with parameters where $a_3 = \frac{2n\pi}{L}, a_4 = \frac{2m\pi}{L}$. For such \mathbf{y} ,

$$\int_0^L y_3(x) dx = 0$$

TABLE 3. Piezoceramic PZT beam physical parameters and experimental values [38].

ρ	Density	7600 kg/m ³
γ	Electromechanical coefficients	10 ⁻³ C/m ²
α_1	Stiffness constant	121 × 10 ⁹ N/m ²
ε_1	permittivity constant	0.25 × 10 ⁻¹² F/m
ε_3	permittivity constant	0.25 × 10 ⁻¹² F/m
ξ	$\frac{\varepsilon_1 h^2}{12\varepsilon_3}$	8.3 × 10 ⁻¹⁰ $\frac{1}{\text{m}^2}$
μ	Magnetic impermeability	1.2 × 10 ⁻⁶ H/m
h	Thickness of the beam	10 ⁻⁴ m
L	Length of the beam	1 m

TABLE 4. Eigenvalues $\{\lambda_3, \lambda_4\}$ of \mathcal{A} for the material parameters in Table 3. The numbers are rounded to the nearest thousandth, and $\tau_+^{(j)} - \sqrt{\frac{\mu}{\xi\varepsilon_3}} > 0$ for every $j = 3, 4$ where $\sqrt{\frac{\mu}{\xi\varepsilon_3}} = 7.589 \times 10^7$.

n	m	$\lambda_3 = \iota\tau_+^{(3)}$	$\lambda_4 = \iota\tau_+^{(4)}$	$a_3 = \frac{n\pi}{L}$	$a_4 = \frac{m\pi}{L}$
2	4000	7.589 × 10 ⁷ ι	1.003 × 10 ⁸ ι	12.567	34, 641.016
5	30	7.589 × 10 ⁷ ι	7.590 × 10 ⁷ ι	31.416	188.496
1	2	7.589 × 10 ⁷ ι	7.589 × 10 ⁷ ι	6.283	12.567

and so $\mathcal{B}^*\mathbf{y} = 0$. This means that for any scalar k , $\mathcal{A} + k\mathcal{B}\mathcal{B}^*$ has imaginary eigenvalues and so does not generate an asymptotically stable semigroup. Thus the state-feedback $\mathcal{B}^*\mathbf{y}$ is not able to stabilize the system (4.11) and also $\{\mathcal{A}, \mathcal{B}^*\}$ is not approximately observable and $\{\mathcal{A}, \mathcal{B}\}$ is not approximately controllable. Recall that \mathcal{A} is a unitary operator on \mathcal{H} , its spectrum $\{\iota\tau_{\mp}^{(3)}, \mp\iota\tau_{\mp}^{(4)}\}$ consists of isolated eigenvalues and \mathcal{B} is a bounded operator. By Benchimol's Theorem [1], $\{\mathcal{A}, \mathcal{B}\}$ is not asymptotically stabilizable. \square

Remark 5.3. In the case of charge-actuation the control operator \mathcal{B} is an unbounded operator with its adjoint $\mathcal{B}^*\psi = \frac{\gamma}{\varepsilon_3 h}(\psi_1(0) - \psi_1(L))$. The \mathcal{B}^* measurement is now mechanical (the difference between tip velocities). Eigenfunctions $\mathbf{y} \neq 0$ with

$$(y_1(L) - y_1(0))^2 = 0$$

and so $\mathcal{B}^*\mathbf{y} \equiv 0$ can be constructed by following the same argument as above for current control. It follows that these systems are not stabilizable. The details are omitted due to space limitations.

6. CONCLUSIONS

In this paper a model for current actuation of a piezo-electric beam with fully dynamic electro-magnetic effects was derived using Hamilton's Principle. An Euler-Bernoulli model was used for the mechanical model. If the Mindlin-Timoshenko small displacement assumptions are used instead, the bending equations (3.2) change substantially. However, stretching equations in (3.5) remain the same. Since the control only affects the stretching equations, the choice of beam model does not affect stabilizability.

With dynamic magnetic effects, the adjoint \mathcal{B}^* feedback in both the voltage- and current-controlled cases is electrical. This is quite different from electrostatic models for piezoelectric beams where this feedback is mechanical [22].

For current actuation with magnetic effects, the control operator is bounded and rank 1. Thus, it is not possible to exponentially stabilize the piezo-electric beam; see [14] or the textbook [6]. It was shown here that

in many cases the system is not even asymptotically stabilizable. The question of whether the system fails to be asymptotically stabilizable under conditions more general than those in Theorem 5.2 is an open research problem.

The analysis in the paper shows that current actuation is quite different from voltage actuation. The control operator in that case, whether or not magnetic effects are included, is unbounded on the state space. When magnetic effects are included, the voltage-actuated piezo-electric beam is exactly observable and exponentially stabilizable if the material parameters satisfy number-theoretical conditions. The system is asymptotically stabilizable under a wider set of parameter values [22].

Stabilizability is different for electro-static models than for the models with electro-magnetic dynamics, as shown in this paper and in [22]. For a voltage-controlled system, an elliptic-type differential equation is obtained for charge, and once this equation is solved and back substituted to the mechanical equations, the system reduces to a simple wave equation with the voltage control $V(t)$ acting at the free end of the beam. This model is well-known to be exponentially stabilizable with B^* feedback, see for example [4, 25]. Hamilton's principle cannot be used to derive a current-controlled system with electrostatic (or quasi-static assumptions). Such a model can be obtained by adding a circuit equation for the capacitance $\dot{V} = \frac{1}{C_p} i$ to the voltage controlled model. The control operator is bounded, so, as for the model with magnetic effects, the system is not exponentially stabilizable. The B^* feedback involves voltage. The same analysis used for the voltage-controlled case in [22] can be used to show that the system is asymptotically stabilizable only for certain parameter values.

Charge actuation is mathematically very similar to voltage actuation for both electrostatic and quasi-static assumptions. This is because $\theta, \eta, \dot{\theta}, \dot{\eta} \ll \phi^1$. Without the terms $\theta, \eta, \dot{\theta}, \dot{\eta}$ in (4.7) and (4.8), the model (4.7) and (4.8) for the clamped-free case becomes

$$\begin{cases} \rho \ddot{v} - \alpha v_{xx} - \frac{\gamma^2}{\epsilon_3} (Pv_x)_x = \frac{\gamma \sigma_s(t)}{\epsilon_3 h} \delta(x-L) & \text{in } (0, L) \times \mathbb{R}^+ \\ v(0) = 0, \quad \alpha v_x(L) + \frac{\gamma^2}{\epsilon_3} Pv_x(L) = 0, \quad t \in \mathbb{R}^+ \\ v(x, 0) = v_0(x), \quad \dot{v}(x, 0) = v_1(x) & \text{in } (0, L) \end{cases} \quad (6.1)$$

As for voltage control of the electrostatic model, B^* feedback control $\sigma_s(t) = -kB^* \mathbf{y} = -k\dot{v}(L, t)$ where $k > 0$, leads to an exponentially stable system; details can be found in [26].

For all of voltage, current and charge- control, magnetic effects have a significant effect on stabilizability and other systems properties. Issues with stabilizability are discussed above. The analysis also shows that unlike quasi-static models, the adjoint operator B^* is electrical, not mechanical. Although the magnetic coupling μ may very small, the systems theoretic properties of piezo-electric beams are qualitatively different for models with dynamic magnetics than for electrostatic models. The model obtained in this paper can be adapted for other smart materials where the magnetic energy is as significant as the electrical and mechanical energies.

If damping is included in the model, then the system can be shown to be stable. However, the lack of stabilizability in the undamped model suggests that control does not affect certain modes. Determining which subspaces are unstabilizable is an open question. Another topic for investigation is to demonstrate theoretically that current control reduces hysteresis as compared to voltage control.

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