ESAIM: COCV 25 (2019) 73
https://doi.org/10.1051/cocv/2018071

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

MULTIPLE POSITIVE BOUND STATES FOR CRITICAL SCHRÖDINGER-POISSON SYSTEMS

Giovanna Cerami[1] and Riccardo Molle[2,*]

Abstract. Using variational methods we prove some results about existence and multiplicity of positive bound states of to the following Schrödinger-Poisson system:

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi(x)u = u^5 \\ -\Delta\phi = K(x)u^2 \qquad x \in \mathbb{R}^3. \end{cases} \qquad (SP)$$

We remark that (SP) exhibits a "double" lack of compactness because of the unboundedness of $\mathbb{R}^3$ and the critical growth of the nonlinear term and that in our assumptions ground state solutions of (SP) do not exist.

Mathematics Subject Classification. 35J20, 35J60.

Received March 12, 2018. Accepted December 8, 2018.

1. Introduction

This paper deals with the question of finding solutions to systems of the type

$$\begin{cases} ih\frac{\partial\xi}{\partial t} = -\frac{h^2}{2m}\Delta\xi + (V(x)+E)\xi + K(x)\phi(x)\xi - f(x,|\xi|) \\ -\Delta\phi = K(x)\xi^2 \qquad x \in \mathbb{R}^3 \end{cases}$$

where h is the Planck constant, i is the imaginary unit, m is a positive constant, E is a real number, $\xi : \mathbb{R}^3 \times [0,T] \to \mathbf{C}$.

Such equations have strongly attracted the researchers attention because their deep physical meaning: we just mention they appear in Semiconductor Theory and Quantum Mechanics models (see f.i. the celebrated papers [4, 7, 9, 21, 22] and the book [25]).

Keywords and phrases: Schrödinger-Poisson system, lack of compactness, bound states, variational methods.

[1] Dipartimento di Matematica, Politecnico di Bari, Via Amendola 126/B, 70126 Bari, Italy.
[2] Dipartimento di Matematica, Università di Roma "Tor Vergata", Via della Ricerca Scientifica no 1, 00133 Roma, Italy.

* Corresponding author: molle@mat.uniroma2.it

Our interest is focused in the search of standing waves, that is solutions $\xi(x,t) = e^{-\frac{iEt}{h}}u$ where u is a real function. In this case one is led to study the existence of functions u satisfying

$$\begin{cases} -\frac{h^2}{2m}\Delta u + V(x)u + K(x)\phi(x)u = f(x,|u|) \\ -\Delta\phi = K(x)u^2 \qquad x \in \mathbb{R}^3. \end{cases} \qquad (sp)$$

Such a system is usually known as Schrödinger-Poisson system because first equation, which is a nonlinear stationary Schrödinger equation, is coupled with a Poisson equation. This model has been introduced in [3] to describe electrostatic situations in which the interaction between an electrostatic field and solitary waves has to be considered; the nonlinear term f simulates the interaction between many particles, while, by the effect of the Poisson equation, the potential is determined by the charge of the wave function itself.

More precisely we are interested in the existence of positive, physically meaningful solutions of (sp) when $f(x,u) = u^p$ and $p = 5$ is the *critical exponent* with respect to the Sobolev embedding, moreover we look for solutions when $\frac{h^2}{2m}$ is a constant, which without loss of generality can be assumed equal to 1. Therefore, the problem we address becomes

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi(x)u = u^5 \\ -\Delta\phi = K(x)u^2 \qquad x \in \mathbb{R}^3. \end{cases} \qquad (SP)$$

It is well known that, in studying Schrödinger-Poisson systems, even in subcritical cases, one has to face many difficulties: some come from the coupling and appear in the potential, some originate from the lack of compactness due to the invariance of $\mathbb{R}^3$ under translations. In addition, in the critical case the invariance by dilations of $\mathbb{R}^3$ has to be considered and make the things even harder to handle.

During last fifteen years system (sp) has been widely investigated, mainly considering nonlinearities having subcritical growth: first results have been obtained for equations with coefficients $V(x)$ and $K(x)$ constants or radially symmetric; then, an increasing number of papers has been devoted to cases in which no symmetry assumptions are requested.

Describing the interesting and various contributions given to the study of the subcritical case in a exhaustive way, without forgetting something, would be an hard task, so we prefer to refer readers interested in a rich bibliography to the papers [1, 5] and to the book [6].

Less contributions has received the analysis of (sp) in the critical case. Between those appeared in the latest years and treating equations with non constant coefficients we refer the readers to [10, 19, 20, 23, 24, 29, 30] and references therein. However, we remark that the researchers attention has been mainly devoted to the question of existence and multiplicity of *semi-classical* solutions (*i.e.* the case in which, in (sp), $\frac{h^2}{2m} \to 0$, as in [19, 20, 24]). Furthermore, we also stress that all the researches we are aware, concerning the search either of semi-classical either of classical solution (as in [10, 23, 29, 30]), are carried out under assumptions which allow to work in frameworks that ensure the existence of ground state solutions.

Here we consider situations that must be faced by more refined tools. We ask the potentials $V(x)$ and $K(x)$ satisfy:

$$\begin{cases} \lim_{|x|\to+\infty} V(x) = V_\infty \geq 0 & (i) \\ V(x) \geq V_\infty \quad \forall x \in \mathbb{R}^3 & (ii) \\ (V - V_\infty) \in L^{3/2}(\mathbb{R}^3) & (iii) \end{cases} \qquad (H_V)$$

and

$$\begin{cases} \lim_{|x|\to+\infty} K(x) = 0 & (i) \\ K(x) \geq 0 \quad \forall x \in \mathbb{R}^3, \ K \not\equiv 0 & (ii) \\ K \in L^2(\mathbb{R}^3). & (iii) \end{cases} \tag{H_K}$$

Indeed, as shown in Section 2, Proposition 2.8, under the above assumptions the existence of positive solutions cannot be obtained by minimization methods and ground state solutions do not exist. Similar topological situations related to (SP) have been considered in the subcritical case in [12, 15]. Here the critical growth of the nonlinear term makes more difficult the question.

The results we obtain are stated in the following theorems, where S denotes the best Sobolev constant.

First theorem is concerned with potentials vanishing at infinity:

Theorem 1.1. *Let $V_\infty = 0$. Let (H_V), (H_K), and*

$$\left(1 + \frac{|V|_{L^{3/2}}}{S} + \frac{|K|^2_{L^2}}{S^{3/2}}\right)^3 \left(1 + \frac{3}{4\,S}\,|V|_{L^{3/2}}\right) < 2 \tag{1.1}$$

be satisfied.

Then (SP) has at least a positive solution.

Second theorem provides existence and multiplicity of positive solutions when $V_\infty \neq 0$, namely:

Theorem 1.2. *Let $V_\infty > 0$. Let (H_V) and (H_K) be satisfied.*

Then a real number $\bar{V} > 0$ exists such that if $V_\infty \in (0, \bar{V})$, then (SP) has at least a positive solution.

Moreover, if in addition to (H_V) and (H_K),

$$\left(1 + \frac{|V - V_\infty|_{L^{3/2}}}{S} + \frac{|K|^2_{L^2}}{S^{3/2}}\right)^3 \left(1 + \frac{3}{4\,S}\,|V - V_\infty|_{L^{3/2}}\right) < 2 \tag{1.2}$$

holds, $\bar{V} > 0$ can be found so that, when $V_\infty \in (0, \bar{V})$, (SP) has at least two distinct positive solutions.

It is worth observing that Theorem 1.1 generalizes to Schrödinger-Poisson systems a well known result proven in [2] for nonlinear Schrödinger equations. Indeed, in [2] the non-existence of ground state solutions and the existence of a positive not least energy solution to

$$\begin{cases} -\Delta u + V(x)u = |u|^{\frac{N+2}{N-2}} & x \in \mathbb{R}^N \\ u(x) \to 0 \quad \text{as} \quad x \to \infty \end{cases} \tag{SE}$$

has been shown assuming $V_\infty = 0$, $V(x) \not\equiv 0$, $V(x) \in L^{N/2}$, and, in addition, a restriction on $|V|_{L^{3/2}}$ quite analogous to the bound $|V|_{L^{3/2}} < (2^4 - 1)S$ one can deduce from (1.1) setting $K = 0$.

On the other hand, as far as we know, no results concerning solutions of (SE) when $V_\infty > 0$ are available, while a Pohozaev type inequality shows that when V is a positive constant (SE) has no solutions. Therefore, it seems interesting to remark that from Theorem 1.2 it follows, as corollary, a non trivial existence and multiplicity theorem for (SE). We state it explicitly, assuming $N = 3$ because Schrödinger-Poisson systems are here considered in $\mathbb{R}^3$, nevertheless, reading the paper and the proof of Theorem 1.2, it is not difficult to understand that everything also holds for any dimension $N \geq 3$.

Theorem 1.3. *Let $N = 3, V_\infty > 0$. Let (H_V) be satisfied and $|V - V_\infty|_{L^{3/2}} \neq 0$.*

Then a real number $\bar{V} > 0$ exists such that if $V_\infty \in (0, \bar{V})$, then (SE) has at least a positive solution. Moreover, if in addition to (H_V)

$$|V - V_\infty|_{L^{3/2}} < (2^4 - 1)S$$

holds, $\bar{V} > 0$ can be found so that, when $V_\infty \in (0, \bar{V})$, (SE) has at least two distinct positive solutions.

Of course some of our arguments, mainly those related to the lack of compactness question, take advantage of some ideas introduced in [2]. However we strongly point out that in the present paper we face different situations, we need new delicate estimates concerning the nonlocal term, the variational framework in which we work must be different, and we use here very refined and more subtle tools to control translations and concentrations of Palais-Smale sequences.

The paper is organized as follows: in Section 2 the variational framework is introduced, some useful facts are stated, the compactness question is studied and the nonexistence of ground state solutions is proved, Section 3 contains some basic, deep estimates, and in Section 4 the proof of Theorems 1.1 and 1.2 is performed.

2. Variational framework, compactness study, nonexistence result

Hereafter we use the following notation

– $\|\cdot\|$ denotes the norm in H, that is

$$\|u\| = \left(\int_{\mathbb{R}^3}(|\nabla u|^2 + V_\infty u^2)\mathrm{d}x\right)^{1/2} \qquad \text{when } H = H^1(\mathbb{R}^3)$$
$$\|u\| = \left(\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\right)^{1/2} \qquad \text{when } H = \mathcal{D}^{1,2}(\mathbb{R}^3).$$

– $|u|_q$, $1 \le q \le +\infty$ denotes the norm in the Lebesgue space $L^q(\mathbb{R}^3)$, while the norm of u in $L^q(\Omega)$, $\Omega \subset \mathbb{R}^3$, is denoted by $|u|_{q,\Omega}$.
– $B_\rho(y)$, $\forall y \in \mathbb{R}^3$, denotes the open ball of radius ρ centered at y, $(\cdot|\cdot)_{\mathbb{R}^3}$ denotes the scalar product in $\mathbb{R}^3$, and for any measurable set $\mathcal{O} \subset \mathbb{R}^3$, $|\mathcal{O}|$ denotes its Lebeasgue measure.
– S is the best Sobolev constant, that is

$$S = \inf_{u \in H^1(\mathbb{R}^3)} \frac{\|u\|^2}{|u|_6^2} = \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^3)} \frac{\|u\|^2}{|u|_6^2}. \tag{2.1}$$

Throughout the paper we set

$$W(x) := V(x) - V_\infty$$

moreover we assume V and K satisfy (H_V) and (H_K) respectively.

It is well known (see f.i. [12, 27]) that (SP) can be transformed in a nonlinear Schrödinger equation with a non local term. Indeed, the Poisson equation can be solved by using the Lax-Milgram theorem, thus for all $u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ a unique $\phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$, satisfying

$$-\Delta\phi = K(x)u^2, \tag{2.2}$$

is obtained. Then, inserting ϕ_u into the first equation of (SP) one gets

$$-\Delta u + V(x)u + K(x)\phi_u(x)u = u^5. \qquad (SP')$$

(SP') is variational and its solutions are the critical points of the functional

$$I(u)=\frac{1}{2}\left(\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x+\int_{\mathbb{R}^3}V(x)u^2\mathrm{d}x\right)+\frac{1}{4}\int_{\mathbb{R}^3}K(x)\phi_u(x)u^2\mathrm{d}x-\frac{1}{6}\int_{\mathbb{R}^3}u^6\mathrm{d}x$$

which is defined in the space H where H is either $H^1(\mathbb{R}^3)$ or $\mathcal{D}^{1,2}(\mathbb{R}^3)$ according to whether $V_\infty>0$ or $V_\infty=0$.

Let $\Phi: H\to\mathcal{D}^{1,2}(\mathbb{R}^3)$ be the operator defined by

$$\Phi(u)=\phi_u.$$

Next two propositions collect some properties of Φ.

Proposition 2.1.

(1) Φ *is continuous;*
(2) Φ *maps bounded sets into bounded sets;*
(3) $\Phi(tu)=t^2\Phi(u)$ *for all* $t\in\mathbb{R}$*;*
(4) the following representation formula holds

$$\Phi(u)=\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{K(y)}{|x-y|}\,u^2(y)\,dy=\frac{1}{4\pi}\frac{1}{|x|}*K\,u^2. \tag{2.3}$$

The proof of *(1)* and *(2)* can be found f.i. in [27] or [15] while properties *(3)* and *(4)* are straight consequence of the fact that $\Phi(u)$ solves (2.2).

Proposition 2.2. *Let* $(u_n)_n$, $u_n\in\mathcal{D}^{1,2}(\mathbb{R}^3)$ *be such that*

$$u_n\rightharpoonup 0\qquad in\ \ \mathcal{D}^{1,2}(\mathbb{R}^3). \tag{2.4}$$

Then, up to subsequences

(a) $\Phi(u_n)\to 0$ *in* $\mathcal{D}^{1,2}(\mathbb{R}^3)$
(b) $\int_{\mathbb{R}^3}K(x)\phi_{u_n}u_n^2\mathrm{d}x\longrightarrow 0$
(c) $\int_{\mathbb{R}^3}K(x)\phi_{u_n}u_n\varphi\,\mathrm{d}x\to 0\quad\forall\varphi\in\mathcal{D}^{1,2}(\mathbb{R}^3)$.

The proof of Proposition 2.2 can be carried out exactly as that of Proposition 2.2 of [12] once stated the following regularity result, which is also useful in the study of the compactness question.

Lemma 2.3. *Let* $(u_n)_n$ *be as in Proposition 2.2. Then, up to subsequences,*

$$u_n\longrightarrow 0\qquad in\ L^p_{\mathrm{loc}}(\mathbb{R}^3)\quad\forall p\in[2,6). \tag{2.5}$$

Proof. By the Hardy inequality

$$\int_{\mathbb{R}^3}u_n^2(1+|x|)^{-2}\mathrm{d}x\le c\int_{\mathbb{R}^3}|\nabla u_n|^2\mathrm{d}x\le C.$$

Let $\mathcal{A}\subset\subset\mathbb{R}^3$ be arbitrarily chosen and let $r>0$ be such that $\mathcal{A}\subset B_r(0)$, thus

$$\int_{\mathcal{A}}u_n^2\mathrm{d}x\le(1+r)^2\int_{B_r(0)}\frac{u_n^2}{(1+|x|)^2}\,\mathrm{d}x\le\hat{C}$$

which, together with (2.4) gives $(u_n)_n$ bounded in $H^1(\mathcal{A})$. Hence, up to a subsequence, $(u_n)_n$ converges strongly to 0 in $L^2(\mathcal{A})$ by the Sobolev embedding and (2.4). Then, the claim follows by interpolation and Sobolev embedding, because for all $p \in [2,6)$

$$|u_n|_{p,\mathcal{A}} \leq |u_n|_{2,\mathcal{A}}^{\alpha} |u_n|_{6,\mathcal{A}}^{1-\alpha}$$

where $\frac{\alpha}{2} + \frac{1-\alpha}{6} = \frac{1}{p}$. □

It is not difficult to verify that the functional I is bounded neither from below, nor from above. So, it is suitable to consider I restricted to the Nehari natural constraint:

$$\mathcal{N} := \{u \in H \setminus \{0\} \ : \ I'(u)[u] = 0\}$$

and remark that we can write $I_{|\mathcal{N}}$ as

$$\begin{aligned} I_{|\mathcal{N}}(u) &= \frac{1}{4}\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)\mathrm{d}x + \frac{1}{12}\int_{\mathbb{R}^3} u^6 \mathrm{d}x \\ &= \frac{1}{3}\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)\mathrm{d}x + \frac{1}{12}\int_{\mathbb{R}^3} K(x)\phi_u(x)u^2 \mathrm{d}x \end{aligned} \tag{2.6}$$

from which one at once deduces that I is bounded from below on $\mathcal{N}$. Furthermore, for all $u \in H \setminus \{0\}$, there exists a unique $t_u \in \mathbb{R}^+ \setminus \{0\}$ such that $t_u u \in \mathcal{N}$. Indeed, t_u satisfies

$$0 = I'(tu)[tu] = t^2 \left(\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \int_{\mathbb{R}^3} V(x)u^2 \mathrm{d}x \right) + t^4 \int_{\mathbb{R}^3} K(x)\phi_u(x)u^2 \mathrm{d}x - t^6 \int_{\mathbb{R}^3} u^6 \mathrm{d}x \tag{2.7}$$

which is easily seen to have a unique positive solution.

The function $t_u u \in \mathcal{N}$ is called the projection of u on $\mathcal{N}$ and we also point out that

$$I(t_u u) = \max_{t>0} I(tu).$$

Actually, more precise information is available on $\mathcal{N}$ and $I_{|\mathcal{N}}$ and it can be summarized in the following lemma, whose proof can be found in [15]

Lemma 2.4.

(1) $\mathcal{N}$ is a C^1 regular manifold diffeomorphic to the sphere of H;
(2) I is bounded from below on $\mathcal{N}$ by a positive constant;
(3) u is a free critical point of I if and only if u is a critical point of I constrained on $\mathcal{N}$.

For what follows it is also useful to introduce the "problem at infinity" related to (SP)

$$\begin{cases} -\Delta u = u^5 & x \in \mathbb{R}^3 \\ u \in \mathcal{D}^{1,2}(\mathbb{R}^3). \end{cases} \tag{P_∞}$$

Combining the results of [17, 18, 28] the following statement can be obtained.

Proposition 2.5. *Any positive solution of (P_∞) must be of the form*

$$\Psi_{\sigma,y}(x) := \frac{1}{\sigma^{1/2}}\,\Psi\left(\frac{x-y}{\sigma}\right) = \frac{[3\sigma^2]^{1/4}}{[\sigma^2 + |x-y|^2]^{1/2}} \tag{2.8}$$

where

$$\Psi(x) = 3^{1/4}\,\frac{1}{(1+|x|^2)^{1/2}} = \frac{\Psi^*(x)}{|\Psi^*|_6}$$

and

$$\Psi^*(x) = \frac{1}{(1+|x|^2)^{1/2}}$$

is the unique minimizer for S, up to translations and scaling.

Throughout the paper we denote by

$$I_\infty : H \to \mathbb{R}$$

the functional whose critical points are solutions of (P_∞), that is

$$I_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x - \frac{1}{6}\int_{\mathbb{R}^3} u^6 \mathrm{d}x$$

and by

$$\mathcal{N}_\infty = \{u \in H \setminus \{0\} :\ I'_\infty(u)[u] = 0\}.$$

A straight computation shows

$$I_\infty(\Psi_{\sigma,y}) = \min_{\mathcal{N}_\infty} I_\infty(u) = \frac{1}{3}\,S^{3/2}.$$

Remark 2.6. It is worth pointing out that the existence of infinitely many non-radial changing sign solutions to (P_∞) has been proved by Ding [16]; however, for any such solution $u \in \mathcal{N}_\infty$, $u = u^+ - u^-$, $u^+ \neq 0 \neq u^-$, the estimate $I_\infty(u) \geq \frac{2}{3}\,S^{3/2}$ can easily be shown (see f.i. [14]).

For all $u \in H \setminus \{0\}$, there exists unique $\tau_u \in \mathbb{R}^+ \setminus \{0\}$ such that $\tau_u u \in \mathcal{N}_\infty$, we call $\tau_u u$ projection of u on $\mathcal{N}_\infty$.

Lemma 2.7. *Let $u \in H \setminus \{0\}$ and let $\tau_u u$ and $t_u u$ be the projections of u on $\mathcal{N}_\infty$ and $\mathcal{N}$ respectively. Then*

$$\tau_u \leq t_u. \tag{2.9}$$

Proof. By definition we have

$$\tau_u^4 = \frac{\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x}{|u|_6^6} = \frac{t_u^4 |u|_6^6 - t_u^2 \int_{\mathbb{R}^3} K(x)\phi_u u^2 \mathrm{d}x - \int_{\mathbb{R}^3} V(x) u^2}{|u|_6^6} \leq t_u^4.$$

□

Proposition 2.8. *Set*

$$\inf\{I(u)\ :\ u \in \mathcal{N}\} =: m. \tag{2.10}$$

Then

$$m = \frac{1}{3} S^{3/2} \tag{2.11}$$

and the minimization problem (2.10) has no solution.

Proof. Let $u \in \mathcal{N}$ be arbitrarily chosen and let $\tau_u u$ be its projection on $\mathcal{N}_\infty$,

$$\begin{aligned} I(u) \geq I(\tau_u u) &= I_\infty(\tau_u u) + \frac{1}{2}\int_{\mathbb{R}^3} W(x)(\tau_u u)^2 \mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_{\tau_u u}(x)(\tau_u u)^2 \mathrm{d}x \\ &\geq I_\infty(\tau_u u) \geq \frac{1}{3} S^{3/2} \end{aligned}$$

from which

$$m \geq \frac{1}{3} S^{3/2}$$

follows. To show that the equality holds, let us consider the sequence

$$\tilde{\Psi}_n(x) = \chi(|x|)\Psi_{\frac{1}{n},0}(x) = \chi(|x|)\,\frac{3^{1/4}\cdot\sqrt{1/n}}{((1/n)^2 + |x|^2)^{1/2}}$$

where $\chi \in \mathcal{C}_0^\infty([0,+\infty))$ is a nonnegative real function such that $\chi(s) = 1$ if $s \in [0, 1/2]$, and $\chi(s) = 0$ if $s \geq 1$. Well known computations (see f.i. [8]) give

$$I_\infty(\tilde{\Psi}_n(x)) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\tilde{\Psi}_n(x)|^2 \mathrm{d}x - \frac{1}{6}\int_{\mathbb{R}^3} \tilde{\Psi}_n^6(x)\,\mathrm{d}x = \frac{1}{3}S^{3/2} + O(1/n), \tag{2.12}$$

$$|\tilde{\Psi}_n(x)|_2^2 = O(1/n), \tag{2.13}$$

$$\int_{\mathbb{R}^3} |\nabla\tilde{\Psi}_n(x)|^2 \mathrm{d}x - \int_{\mathbb{R}^3} \tilde{\Psi}_n^6(x)\,\mathrm{d}x = O(1/n). \tag{2.14}$$

Thus, $\tau_{\tilde{\Psi}_n} = 1 + O(1/n)$. Now, using *(2)* of Proposition 2.1, we get

$$\begin{aligned} \int_{\mathbb{R}^3} K(x)\phi_{\tilde{\Psi}_n}(x)\tilde{\Psi}_n^2(x)\,\mathrm{d}x &= \int_{B_1(0)} K(x)\phi_{\tilde{\Psi}_n}(x)\tilde{\Psi}_n^2(x)\mathrm{d}x \\ &= \int_{B_1(0)\setminus B_{1/\sqrt{n}}(0)} K(x)\phi_{\tilde{\Psi}_n}(x)\tilde{\Psi}_n^2(x)\,\mathrm{d}x + \int_{B_{1/\sqrt{n}}(0)} K(x)\phi_{\tilde{\Psi}_n}(x)\tilde{\Psi}_n^2(x)\,\mathrm{d}x \\ &\leq |K|_2|\phi_{\tilde{\Psi}_n}|_6|\tilde{\Psi}_n|^2_{6,B_1(0)\setminus B_{1/\sqrt{n}}(0)} + |K|_{2,B_{1/\sqrt{n}}(0)}|\phi_{\tilde{\Psi}_n}|_6|\tilde{\Psi}_n|_6^2 \\ &\leq c_1|\tilde{\Psi}_n|^2_{6,B_1(0)\setminus B_{1/\sqrt{n}}(0)} + c_2|K|_{2,B_{1/\sqrt{n}}(0)} \end{aligned}$$

with $c_1, c_2 > 0$ not depending on n. Thus, considering that

$$|\tilde{\Psi}_n|_{6,B_1(0)\setminus B_{1/\sqrt{n}}(0)} = o(1), \qquad |K|_{2,B_{1/\sqrt{n}}(0)} = o(1)$$

we deduce

$$\int_{\mathbb{R}^3} K(x)\phi_{\tilde{\Psi}_n}(x)\tilde{\Psi}_n^2(x)\,\mathrm{d}x = o(1). \tag{2.15}$$

Furthermore, for all $\rho > 0$

$$\int_{\mathbb{R}^3} W(x)\tilde{\Psi}_n^2(x)\,\mathrm{d}x = \int_{B_\rho(0)} W(x)\tilde{\Psi}_n^2(x)\,\mathrm{d}x + \int_{\mathbb{R}^3\setminus B_\rho(0)} W(x)\tilde{\Psi}_n^2(x)\,\mathrm{d}x$$
$$\leq |\tilde{\Psi}_n|_6^2 |W|_{3/2,B_\rho(0)} + |W|_{3/2}|\tilde{\Psi}_n|^2_{6,\mathbb{R}^3\setminus B_\rho(0)}$$

and, in view of

$$\lim_{n\to\infty} |\tilde{\Psi}_n|_{6,\mathbb{R}^3\setminus B_\rho(0)} = 0, \qquad \lim_{n\to\infty} |\tilde{\Psi}_n|_6 = \mathrm{const}$$

we obtain for all $\rho > 0$

$$\int_{\mathbb{R}^3} W(x)\tilde{\Psi}_n^2(x)\,\mathrm{d}x \leq \mathrm{const}\, |W|_{3/2,B_\rho(0)}.$$

Thus, from $\lim_{\rho\to 0} |W|_{3/2,B_\rho(0)} = 0$ we get

$$\int_{\mathbb{R}^3} W(x)\tilde{\Psi}_n^2(x)\,\mathrm{d}x = o(1). \tag{2.16}$$

Therefore, from (2.12), (2.13), (2.14), (2.15) and (2.16)

$$t_{\tilde{\Psi}_n} = 1 + o(1) \tag{2.17}$$

follows.

Finally, setting $\hat{\Psi}_n(x) = t_{\tilde{\Psi}_n}\tilde{\Psi}_n(x)$ we conclude that (2.11) holds because $\hat{\Psi}_n \in \mathcal{N}$ and, by (2.12), (2.13), (2.14), (2.15), (2.16), (2.17)

$$\lim_{n\to\infty} I(\hat{\Psi}_n) = \frac{1}{2}t^2_{\tilde{\Psi}_n}\left[\int_{\mathbb{R}^3}(|\nabla\tilde{\Psi}_n(x)|^2 + V(x)\tilde{\Psi}_n^2(x))\mathrm{d}x\right]$$
$$+\frac{1}{4}t^4_{\tilde{\Psi}_n}\int_{\mathbb{R}^3} K(x)\phi_{\tilde{\Psi}_n}\tilde{\Psi}_n^2(x)\,\mathrm{d}x - \frac{1}{6}t^6_{\tilde{\Psi}_n}\int_{\mathbb{R}^3}\tilde{\Psi}_n^6(x)\,\mathrm{d}x$$
$$= \frac{1}{3}\,S^{3/2}.$$

To show that $m = \frac{1}{3}\,S^{3/2}$ is not achieved we argue by contradiction and we assume $\bar{u} \in \mathcal{N}$ exists so that $I(\bar{u}) = \frac{1}{3}\,S^{3/2}$. Then, by using (2.6), $(H_k)(ii)$, $(H_V)(ii)$ we have

$$\frac{1}{3}\,S^{3/2} = I(\bar{u}) = \frac{1}{3}\int_{\mathbb{R}^3}(|\nabla\bar{u}|^2 + V(x)\bar{u}^2)\mathrm{d}x + \frac{1}{12}\int_{\mathbb{R}^3} K(x)\phi_{\bar{u}}(x)\bar{u}^2\mathrm{d}x$$
$$\geq \frac{1}{3}\int_{\mathbb{R}^3}|\nabla\bar{u}|^2\mathrm{d}x \geq \frac{1}{3}\int_{\mathbb{R}^3}|\nabla\tau_{\bar{u}}\bar{u}|^2\mathrm{d}x \geq \frac{1}{3}\,S^{3/2} \tag{2.18}$$

from which

$$\int_{\mathbb{R}^3} K(x)\phi_{\bar{u}}\bar{u}^2\mathrm{d}x = 0, \qquad \int_{\mathbb{R}^3} V(x)\bar{u}^2\mathrm{d}x = 0 \tag{2.19}$$

follow, that, if $V_\infty \neq 0$ gives at once a contradiction.

When $V_\infty = 0$, we observe that (2.19), (2.18), (2.9) imply $\tau_{\bar u} = 1$, so

$$\bar u(x) = \Psi_{\sigma,y}(x) > 0 \qquad \forall x \in \mathbb{R}^3$$

for some $\sigma \in \mathbb{R}$, $y \in \mathbb{R}^3$. Then, by $(H_k)(ii)$

$$\int_{\mathbb{R}^3} K(x)\phi_{\bar u}\bar u^2(x)\,\mathrm{d}x > 0$$

has to be true, contradicting (2.19). □

Problem (SP) is affected by a lack of compactness due to the unboundedness of $\mathbb{R}^3$ and to the critical exponent. Next proposition gives a picture of the compactness situation describing the Palais-Smale sequences behaviour.

Proposition 2.9. *Let $(u_n)_n$ be a PS-sequence of $I_{|\mathcal{N}}$, i.e.*

$$u_n \in \mathcal{N}, \qquad I(u_n) \longrightarrow c$$

$$\nabla I_{|\mathcal{N}}(u_n) \longrightarrow 0.$$

Then there exist a number $k \in \mathbb{N}$, k sequences of points $(y_n^j)_n$, $y_n^j \in \mathbb{R}^3$, $1 \le j \le k$, k sequences of positive numbers $(\sigma_n^j)_n$, $1 \le j \le k$, $(k+1)$ sequences of functions $(u_n^j)_n$, $u_n^j \in \mathcal{D}^{1,2}(\mathbb{R}^3)$, $0 \le j \le k$, such that, replacing $(u_n)_n$, if necessary, by a subsequence still denoted by $(u_n)_n$

$$(i)\;\; u_n(x) = u_n^0(x) + \sum_{j=1}^k \frac{1}{(\sigma_n^j)^{1/2}}\, u_n^j\left(\frac{x - y_n^j}{\sigma_n^j}\right)$$
$$(ii)\;\; u_n^j \longrightarrow u^j \;\; \text{strongly in } \mathcal{D}^{1,2}(\mathbb{R}^3),\;\; 0 \le j \le k$$

where u^0 is a solution of (SP'), u^j are solutions of (P_∞), and

$$(iii)\quad \text{if } y_n^j \to \bar y_j \quad \text{then} \quad \begin{cases} V_\infty \neq 0 \;\Rightarrow\; \sigma_n^j \to 0 \\ V_\infty = 0 \;\Rightarrow\; \text{either } \sigma_n^j \to 0 \;\text{ or } \sigma_n^j \to \infty \end{cases}$$
$$\text{if } |y_n^j| \to \infty \quad \text{then} \quad \begin{cases} V_\infty \neq 0 \;\Rightarrow\; \sigma_n^j \to 0 \\ V_\infty = 0 \;\Rightarrow\; \text{either } \sigma_n^j \to \bar\sigma^j \in \mathbb{R}^+ \;\text{ or } \sigma_n^j \to \infty. \end{cases}$$

Moreover as $n \to \infty$

$$\|u_n\|^2 = \sum_{j=0}^k \|u^j\|^2 + o(1)$$

$$I(u_n) \longrightarrow I(u^0) + \sum_{j=1}^k I_\infty(u^j).$$

The proof of Proposition 2.9 can be carried out quite analogously to the proof of Theorem 2.5 in [2] taking advantage of Proposition 2.2, Lemma 2.3 and *(3)* of Lemma 2.4.

Corollary 2.10. *Assume $(u_n)_n$ satisfies the assumptions of Proposition 2.9 with $c \in \left(\frac{1}{3}\, S^{3/2}, \frac{2}{3}\, S^{3/2}\right)$, then $(u_n)_n$ is relatively compact.*

Proof. It suffices to apply Proposition 2.9 considering that any nontrivial solution u of (SP'), by Proposition 2.8, verifies $I(u) > \frac{1}{3}\, S^{3/2}$, that any positive solution of (P_∞) has energy $\frac{1}{3}\, S^{3/2}$, and for any changing sign solution v of (P_∞) the relation $I(v) \geq \frac{2}{3}\, S^{3/2}$ holds. □

3. Basic estimates

Let us introduce a barycenter type map $\beta : H \setminus \{0\} \to \mathbb{R}^3$:

$$\beta(u) = \frac{1}{|u|_6^6} \int_{\mathbb{R}^3} \frac{x}{1+|x|}\, u^6(x)\, \mathrm{d}x$$

and a kind of inertial momentum $\gamma : H \setminus \{0\} \to \mathbb{R}$ to estimate the concentration of a function u around its barycenter:

$$\gamma(u) = \frac{1}{|u|_6^6} \int_{\mathbb{R}^3} \left| \frac{x}{1+|x|} - \beta(u) \right| u^6(x)\, \mathrm{d}x.$$

See [2, 11, 13, 26] for similar definitions of barycenter and of a parameter measuring the spread of a function around its barycenter.

The maps β and γ are continuous and verify

$$\beta(tu) = \beta(u), \quad \gamma(tu) = \gamma(u) \qquad \forall t \in \mathbb{R} \quad \forall u \in H \setminus \{0\}. \tag{3.1}$$

Proposition 3.1. *The inequality*

$$\inf \{I(u) \ : \ u \in \mathcal{N},\ \beta(u) = 0,\ \gamma(u) = 1/2\} > \frac{1}{3}\, S^{3/2} \tag{3.2}$$

holds true.

Proof. By (2.11), clearly

$$\inf \{I(u) \ : \ u \in \mathcal{N},\ \beta(u) = 0,\ \gamma(u) = 1/2\} \geq \frac{1}{3}\, S^{3/2}. \tag{3.3}$$

To prove (3.2) we argue by contradiction and suppose the equality holds true in (3.3). Thus a sequence $(u_n)_n$ exists such that

$$\begin{cases} u_n \in \mathcal{N},\ \beta(u_n) = 0,\ \gamma(u_n) = 1/2 & (a) \\ \lim\limits_{n\to\infty} I(u_n) = \frac{1}{3}\, S^{3/2}. & (b) \end{cases} \tag{3.4}$$

Therefore, since $K(x) \geq 0$, $V(x) \geq 0$, $\phi_{u_n} \geq 0$, by using (2.6) and (2.9), we can write

$$\begin{aligned} \frac{1}{3} S^{3/2} &= \lim_{n\to\infty} \left[\frac{1}{3} \int_{\mathbb{R}^3} (|\nabla u_n|^2 + V(x) u_n^2) \mathrm{d}x + \frac{1}{12} \int_{\mathbb{R}^3} K(x) \phi_{u_n} u_n^2 \right] \\ &\geq \lim_{n\to\infty} \frac{1}{3} \int_{\mathbb{R}^3} |\nabla u_n|^2 \mathrm{d}x \geq \lim_{n\to\infty} \frac{1}{3} \tau_{u_n}^2 \int_{\mathbb{R}^3} |\nabla u_n|^2 \mathrm{d}x \\ &\geq \frac{1}{3} S^{3/2}, \end{aligned} \tag{3.5}$$

from which we infer

$$\lim_{n\to\infty} \frac{1}{3} \int_{\mathbb{R}^3} |\nabla u_n|^2 \mathrm{d}x = \frac{1}{3} S^{3/2}$$

and

$$\lim_{n\to\infty} \tau_{u_n} = 1. \tag{3.6}$$

By the uniqueness of the family of ground state positive solutions of (P_∞) stated in Proposition 2.5, and by Proposition 2.9 we deduce

$$\tau_{u_n} u_n(x) = \Psi_{\sigma_n, y_n}(x) + \varepsilon_n(x)$$

where $\sigma_n \in \mathbb{R}$, $\sigma_n > 0$, $y_n \in \mathbb{R}^3$, $\varepsilon_n \in \mathcal{D}^{1,2}(\mathbb{R}^3)$, $\varepsilon_n \to 0$ strongly in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ and $L^6(\mathbb{R}^3)$. Furthermore, by (3.6)

$$u_n(x) = \Psi_{\sigma_n, y_n}(x) + \tilde{\varepsilon}_n(x) \tag{3.7}$$

with $\tilde{\varepsilon}_n \to 0$ strongly in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ and $L^6(\mathbb{R}^3)$.

We claim that, up to subsequences,

$$(a) \quad \lim_{n\to\infty} \sigma_n = \bar{\sigma} > 0 \qquad (b) \lim_{n\to\infty} y_n = \bar{y} \in \mathbb{R}^3. \tag{3.8}$$

Indeed, once the claim is shown true, the proof can be quickly concluded: it is enough to observe that in this case

$$\Psi_{\sigma_n, y_n} \to \Psi_{\bar{\sigma}, \bar{y}} \text{ strongly in } \mathcal{D}^{1,2}(\mathbb{R}^3) \text{ and } L^6(\mathbb{R}^3) \tag{3.9}$$

so (3.4)(b), (3.7), and (3.9), together with $(H_K)(ii)$, $\Psi_{\bar{\sigma},\bar{y}} > 0$, $\phi_{\Psi_{\bar{\sigma},\bar{y}}} > 0$ give

$$\begin{aligned} S^{3/2} &= \lim_{n\to\infty} \left[\int_{\mathbb{R}^3} (|\nabla u_n|^2 + V(x) u_n^2) \mathrm{d}x + \frac{1}{4} \int_{\mathbb{R}^3} K(x) \phi_{u_n} u_n^2 \mathrm{d}x \right] \\ &= \int_{\mathbb{R}^3} (|\nabla \Psi_{\bar{\sigma},\bar{y}}|^2 + V(x) \Psi_{\bar{\sigma},\bar{y}}^2) \mathrm{d}x + \frac{1}{4} \int_{\mathbb{R}^3} K(x) \phi_{\Psi_{\bar{\sigma},\bar{y}}} \Psi_{\bar{\sigma},\bar{y}}^2 \mathrm{d}x \\ &> \int_{\mathbb{R}^3} |\nabla \Psi_{\bar{\sigma},\bar{y}}|^2 = S^{3/2} \end{aligned} \tag{3.10}$$

that is impossible.

Let us now prove the claim. To show (3.8)(a), we start establishing that $(\sigma_n)_n$ is bounded. Assume $(\sigma_n)_n$ unbounded, then, passing eventually to a subsequence, $\sigma_n \longrightarrow \infty$ occurs, thus for all $\rho > 0$

$$\lim_{n\to\infty} \int_{B_\rho(0)} u_n^6 \mathrm{d}x = \lim_{n\to\infty} \int_{B_\rho(0)} \Psi^6_{\sigma_n,y_n}(x)\,\mathrm{d}x = 0. \tag{3.11}$$

Hence, from (3.4)(a) we get

$$0 = \beta(u_n) = \beta(\Psi_{\sigma_n,y_n}) + o(1), \tag{3.12}$$

and, taking into account *(2)* of Lemma 2.4 we obtain for all $\rho > 0$

$$\begin{aligned}
\gamma(u_n) &= \frac{1}{|u_n|_6^6} \int_{\mathbb{R}^3} \frac{|x|}{1+|x|}\, u_n^6(x)\,\mathrm{d}x \\
&= \frac{1}{|u_n|_6^6} \left[\int_{B_\rho(0)} \frac{|x|}{1+|x|}\, u_n^6(x)\,\mathrm{d}x + \int_{\mathbb{R}^3\setminus B_\rho(0)} \frac{|x|}{1+|x|}\, u_n^6(x)\,\mathrm{d}x \right] \\
&= \frac{1}{|u_n|^6_{6,\mathbb{R}^3\setminus B_\rho(0)} + o(1)} \left[\int_{\mathbb{R}^3\setminus B_\rho(0)} \frac{|x|}{1+|x|}\, u_n^6(x)\,\mathrm{d}x + o(1) \right] \\
&\geq \frac{\rho}{1+\rho} + o(1),
\end{aligned}$$

so $\liminf\limits_{n\to\infty} \gamma(u_n) \geq \frac{\rho}{1+\rho}$, $\forall \rho > 0$, which implies

$$\lim_{n\to\infty} \gamma(u_n) = 1$$

contradicting (3.4)(a).

Thus, up to a subsequence, $\sigma_n \longrightarrow \bar\sigma \in \mathbb{R}^+$. If $\bar\sigma = 0$ would occur, then for all $\rho > 0$

$$\lim_{n\to\infty} \int_{\mathbb{R}^3\setminus B_\rho(y_n)} u_n^6(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_{\mathbb{R}^3\setminus B_\rho(y_n)} (\Psi_{\sigma_n,y_n}(x))^6 \mathrm{d}x = 0$$

and

$$0 < c < |u_n|_6^6 = |u_n|^6_{6,B_\rho(y_n)} + o(1),$$

from which, for all $\rho > 0$

$$\begin{aligned}
\frac{|y_n|}{1+|y_n|} &= \left| \frac{y_n}{1+|y_n|} - \beta(u_n) \right| = \frac{1}{|u_n|_6^6} \left| \int_{\mathbb{R}^3} \left(\frac{y_n}{1+|y_n|} - \frac{x}{1+|x|} \right) u_n^6(x)\,\mathrm{d}x \right| \\
&\leq \frac{1}{|u_n|_6^6} \left[\int_{B_\rho(y_n)} \left| \frac{y_n}{1+|y_n|} - \frac{x}{1+|x|} \right| u_n^6(x)\,\mathrm{d}x + \int_{\mathbb{R}^3\setminus B_\rho(y_n)} \left| \frac{y_n}{1+|y_n|} - \frac{x}{1+|x|} \right| u_n^6(x)\,\mathrm{d}x \right] \\
&\leq \frac{1}{|u_n|^6_{6,B_\rho(y_n)} + o(1)} \left[\rho |u_n|^6_{6,B_\rho(y_n)} + o(1) \right] \\
&\leq \rho + o(1)
\end{aligned}$$

which implies $|y_n| \to 0$ as $n \to \infty$. Therefore

$$0 \le \gamma(u_n) = \frac{1}{|u_n|_6^6} \int_{\mathbb{R}^3} \left| \frac{x}{1+|x|} - \frac{y_n}{1+|y_n|} \right| u_n^6(x)\, \mathrm{d}x + o(1) \le \rho + o(1) \qquad \forall \rho > 0.$$

So, we obtain $\lim\limits_{n\to\infty} \gamma(u_n) = 0$ against (3.4)(a), therefore (3.8)(a) is proved.

Let us now show that $(|y_n|)_n$ is bounded and, then, convergent up to subsequences. By contradiction we suppose that a subsequence, still denoted by $(y_n)_n$, exists for which $\lim\limits_{n\to\infty} |y_n| = \infty$. Then for all $\varepsilon > 0$, and all $R > 0$, $\bar{n} \in \mathbb{N}$ can be found so that $\forall n > \bar{n}$

$$|x - y_n| < R \;\Rightarrow\; \left| \frac{x}{1+|x|} - \frac{y_n}{1+|y_n|} \right| < \varepsilon.$$

Moreover, for all $\varepsilon > 0$, $\bar{\rho} > 0$ depending only on ε exists such that $\forall R > \bar{\rho}$

$$\int_{\mathbb{R}^3 \setminus B_R(y_n)} \Psi^6_{\bar{\sigma}, y_n}(x)\, \mathrm{d}x = \int_{\mathbb{R}^3 \setminus B_R(0)} \Psi^6_{\bar{\sigma}, 0}(x)\, \mathrm{d}x < \varepsilon. \tag{3.13}$$

Now, let us choose arbitrarily $\varepsilon > 0$ and fix $R > 0$ so that (3.13) holds true; for large n we get

$$\begin{aligned}
\left| \beta(u_n) - \frac{y_n}{1+|y_n|} \right| &\le \frac{1}{|u_n|_6^6} \int_{\mathbb{R}^3} \left| \frac{x}{1+|x|} - \frac{y_n}{1+|y_n|} \right| u_n^6(x)\, \mathrm{d}x \\
&\le \frac{1}{|\Psi_{\bar{\sigma}, y_n}|_6^6 + o(1)} \left[\int_{B_R(y_n)} \left| \frac{x}{1+|x|} - \frac{y_n}{1+|y_n|} \right| \Psi^6_{\bar{\sigma}, y_n}(x)\, \mathrm{d}x \right. \\
&\quad \left. + \int_{\mathbb{R}^3 \setminus B_R(y_n)} \left| \frac{x}{1+|x|} - \frac{y_n}{1+|y_n|} \right| \Psi^6_{\bar{\sigma}, y_n}(x)\, \mathrm{d}x \right] + o(1) \\
&\le \hat{c}\varepsilon
\end{aligned}$$

with $\hat{c} > 0$ independent of y_n and R. Thus $|\beta(u_n)| \longrightarrow 1$ as $n \to \infty$, giving a contradiction with (3.4)(a) and completing the proof of the claim and of the proposition. □

Proposition 3.2. *Let assume $V_\infty > 0$. Set*

$$\mu = \inf\{ I(u) \;:\; u \in \mathcal{N},\ \beta(u) = 0,\ \gamma(u) \ge 1/2 \}.$$

Then

$$\mu > \frac{1}{3} S^{3/2}. \tag{3.14}$$

Proof. We follow an analogous argument to that of Proposition 3.1. We start observing that by (2.11)

$$\mu \ge \frac{1}{3} S^{3/2} \tag{3.15}$$

and if the equality in (3.15) holds a sequence $(u_n)_n$ exists so that

$$\begin{cases} u_n \in \mathcal{N},\ \beta(u_n) = 0,\ \gamma(u_n) \geq 1/2 & (a) \\ \lim\limits_{n\to\infty} I(u_n) = \frac{1}{3}\, S^{3/2}. & (b) \end{cases} \tag{3.16}$$

Then, the same computations made in Proposition 3.1 allow to assert that

$$u_n(x) = \Psi_{\sigma_n, y_n}(x) + \varepsilon_n(x)$$

where $\sigma_n \in \mathbb{R}$, $\sigma_n > 0$, $y_n \in \mathbb{R}^3$, $\varepsilon_n \to 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$.

The sequence $(\sigma_n)_n$ is bounded. Indeed, otherwise, up to subsequences, $\sigma_n \to \infty$ and, by (3.16), $(H_V)(ii)$, $(H_K)(ii)$, (2.6),

$$\begin{aligned} \frac{1}{3}\, S^{3/2} &= \lim_{n\to\infty} I(u_n) \geq \liminf_{n\to\infty} \left(\frac{1}{3} \int_{\mathbb{R}^3} |\nabla u_n|^2 \mathrm{d}x + V_\infty \int_{B_{\sigma_n}(y_n)} u_n^2 \mathrm{d}x \right) \\ &\geq \lim_{n\to\infty} \frac{1}{3} \int_{\mathbb{R}^3} |\nabla \Psi_{\sigma_n, y_n}|^2 \mathrm{d}x + \lim_{n\to\infty} V_\infty \sigma_n^2 \left[\int_{B_1(0)} \Psi_{1,0}^2(x)\, \mathrm{d}x + o(1) \right] \\ &= +\infty. \end{aligned}$$

Hence, passing eventually to a subsequence, the relation $\lim\limits_{n\to\infty} \sigma_n = \bar{\sigma}$ holds. Working again as in Proposition (3.2), $\bar{\sigma} > 0$ is shown and, furthermore, $(y_n)_n$ bounded is proved, so that $y_n \to \bar{y}$, up to a subsequence. Thus, we deduce

$$\Psi_{\sigma_n, y_n} \longrightarrow \Psi_{\bar{\sigma}, \bar{y}} \quad \text{strongly in } \mathcal{D}^{1,2}(\mathbb{R}^3) \text{ and } L^2_{\mathrm{loc}}(\mathbb{R}^3)$$

and the impossible relation

$$\begin{aligned} \frac{1}{3}\, S^{3/2} &= \lim_{n\to\infty} I(u_n) \geq \frac{1}{3} \left[\int_{\mathbb{R}^3} |\nabla \Psi_{\bar{\sigma}, \bar{y}}|^2 \mathrm{d}x + V_\infty \int_{B_{\bar{\sigma}}(\bar{y})} \Psi_{\bar{\sigma}, \bar{y}}^2 \mathrm{d}x \right] \\ &> \frac{1}{3} \int_{\mathbb{R}^3} |\nabla \Psi_{\bar{\sigma}, \bar{y}}|^2 \mathrm{d}x = \frac{1}{3}\, S^{3/2} \end{aligned}$$

which brings to conclude the equality in (3.15) cannot occur. □

Remark 3.3. Let us remark that if $V_\infty = 0$ then $\mu = \frac{1}{3}\, S^{3/2}$.

In what follows we use the notation

$$\mathcal{B}_{V_\infty} := \inf\{ I(u) \ :\ u \in \mathcal{N},\ \beta(u) = 0,\ \gamma(u) = 1/2 \}, \qquad \text{if} \quad V_\infty > 0,$$

and

$$\mathcal{B}_0 := \inf\{ I(u) \ :\ u \in \mathcal{N},\ \beta(u) = 0,\ \gamma(u) = 1/2 \}, \qquad \text{if} \quad V_\infty = 0.$$

Obviously, by Proposition 3.1,

$$\mathcal{B}_0 > \frac{1}{3}\, S^{3/2} \qquad \text{and} \qquad \mathcal{B}_{V_\infty} > \frac{1}{3}\, S^{3/2}.$$

Remark 3.4. Let us observe that

$$\mathcal{B}_0 \le \mathcal{B}_{V_\infty} \qquad \forall V_\infty \in \mathbb{R}^+ \setminus \{0\}. \tag{3.17}$$

Indeed, for all $u \in H^1(\mathbb{R}^3)$ such that $\beta(u) = 0$, $\gamma(u) = 1/2$ and

$$\int_{\mathbb{R}^3} (|\nabla u|^2 + W(x)u^2)\mathrm{d}x + \int_{\mathbb{R}^3} K(x)\phi_u u^2 \mathrm{d}x - \int_{\mathbb{R}^3} u^6 \mathrm{d}x = 0$$

let us consider $t_u u$ such that

$$t_u^2 \int_{\mathbb{R}^3} [|\nabla u|^2 + V(x)u^2]\mathrm{d}x + t_u^4 \int_{\mathbb{R}^3} K(x)\phi_u u^2 \mathrm{d}x - t_u^6 \int_{\mathbb{R}^3} u^6 \mathrm{d}x = 0,$$

then $\beta(t_u u) = \beta(u) = 0$, $\gamma(t_u u) = \gamma(u) = 1/2$ and

$$\begin{aligned}
&\frac{1}{2} t_u^2 \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)\mathrm{d}x + \frac{1}{4} t_u^4 \int_{\mathbb{R}^3} K(x)\phi_u u^2 \mathrm{d}x - \frac{1}{6} t_u^6 \int_{\mathbb{R}^3} u^6 \mathrm{d}x \\
&\ge \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + (V_\infty + W(x))u^2)\,\mathrm{d}x + \frac{1}{4} \int_{\mathbb{R}^3} K(x)\phi_u u^2 \mathrm{d}x - \frac{1}{6} \int_{\mathbb{R}^3} u^6 \mathrm{d}x \\
&\ge \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + W(x)u^2)\mathrm{d}x + \frac{1}{4} \int_{\mathbb{R}^3} K(x)\phi_u u^2 \mathrm{d}x - \frac{1}{6} \int_{\mathbb{R}^3} u^6 \mathrm{d}x
\end{aligned}$$

from which considering also $\mathcal{D}^{1,2}(\mathbb{R}^3) \supset H^1(\mathbb{R}^3)$ (3.17) follows.

Let us now fix a number $\bar{c}$ such that

$$\frac{1}{3}\, S^{3/2} < \bar{c} < \min\left(\frac{\mathcal{B}_0 + \frac{1}{3}\, S^{3/2}}{2}, \frac{1}{2}\, S^{3/2} \right). \tag{3.18}$$

Moreover, if either (1.1) or (1.2) holds, then we also require

$$\bar{c} < \frac{2}{3}\, S^{3/2} \left(1 + \frac{|W|_{3/2}}{S} + \frac{|K|_2^2}{S^{3/2}} \right)^{-3} \left(1 + \frac{3}{4\,S}\, |W|_{3/2} \right)^{-1} \tag{3.19}$$

and we observe that $W(x) = V(x)$ when $V_\infty = 0$.

Clearly $\bar{c} \in \left(\frac{1}{3}\, S^{3/2}, \frac{2}{3}\, S^{3/2}\right)$.

We denote by $\omega(x)$ a function having the following properties:

$$\begin{cases}
(i) & \omega \in \mathcal{C}_0^\infty(B_1(0)), \\
(ii) & \omega(x) \ge 0 \qquad \forall x \in B_1(0) \\
(iii) & \omega \in \mathcal{N}_\infty \text{ and } I_\infty(\omega) = \Sigma \in \left(\frac{1}{3}\, S^{3/2}, \bar{c}\right) \\
(iv) & \omega(x) = \omega(|x|) \text{ and } |x_1| \le |x_2| \ \Rightarrow\ \omega(x_1) \ge \omega(x_2).
\end{cases} \tag{3.20}$$

Moreover, for every $\sigma > 0$ and $y \in \mathbb{R}^3$ we set

$$\omega_{\sigma,y}(x) = \begin{cases} \sigma^{-1/2}\omega\left(\frac{x-y}{\sigma}\right) & x \in B_\sigma(y) \\ 0 & x \notin B_\sigma(y). \end{cases}$$

Remark that

$$|\omega|_6 = |\omega|_{6,B_1(0)} = |\omega_{\sigma,y}|_{6,B_\sigma(y)} = |\omega_{\sigma,y}|_6. \tag{3.21}$$

Lemma 3.5. *The following relations hold*

$$\begin{array}{ll} (a) & \lim_{\sigma\to 0} \sup\left\{\int_{\mathbb{R}^3} W(x)\,\omega^2_{\sigma,y}(x)\,\mathrm{d}x \ : \ y \in \mathbb{R}^3\right\} = 0 \\ (b) & \lim_{\sigma\to \infty} \sup\left\{\int_{\mathbb{R}^3} W(x)\,\omega^2_{\sigma,y}(x)\,\mathrm{d}x \ : \ y \in \mathbb{R}^3\right\} = 0 \\ (c) & \lim_{r\to \infty} \sup\left\{\int_{\mathbb{R}^3} W(x)\,\omega^2_{\sigma,y}(x)\,\mathrm{d}x \ : \ y \in \mathbb{R}^3,\ |y| = r,\ \sigma > 0\right\} = 0. \end{array} \tag{3.22}$$

Proof. Let $y \in \mathbb{R}^3$ be arbitrarily chosen. Then for all $\sigma > 0$

$$\int_{\mathbb{R}^3} W(x)\,\omega^2_{\sigma,y}(x)\,\mathrm{d}x = \int_{B_\sigma(y)} W(x)\,\omega^2_{\sigma,y}(x)\,\mathrm{d}x \le |W|_{3/2,B_\sigma(y)}|\omega|^2_{6,B_1(0)} \le c|W|_{3/2,B_\sigma(y)}$$

with c not depending on σ. So

$$\sup_{y\in\mathbb{R}^3} \int_{\mathbb{R}^3} W(x)\omega^2_{\sigma,y}(x)\,\mathrm{d}x \le c\sup\{|W|_{3/2,B_\sigma(y)} \ : \ y \in \mathbb{R}^3\}$$

and, since $\lim_{\sigma\to 0} |W|_{3/2,B_\sigma(y)} = 0$, uniformly in $y \in \mathbb{R}^3$, (3.22)(a) follows.

To prove (3.22)(b), again let us choose arbitrarily $y \in \mathbb{R}^3$. Then $\forall\rho > 0$, $\forall\sigma > 0$

$$\begin{aligned} \int_{\mathbb{R}^3} W(x)\,\omega^2_{\sigma,y}\mathrm{d}x &= \int_{B_\rho(0)} W(x)\,\omega^2_{\sigma,y}\mathrm{d}x + \int_{\mathbb{R}^3\setminus B_\rho(0)} W(x)\,\omega^2_{\sigma,y}\mathrm{d}x \\ &\le |W|_{3/2,B_\rho(0)}|\omega_{\sigma,y}|^2_{6,B_\rho(0)} + |W|_{3/2,\mathbb{R}^3\setminus B_\rho(0)}|\omega_{\sigma,y}|^2_{6,\mathbb{R}^3\setminus B_\rho(0)} \\ &\le |W|_{3/2,B_\rho(0)} \sup_{y\in\mathbb{R}^3} |\omega_{\sigma,y}|^2_{6,B_\rho(0)} + c|W|_{3/2,\mathbb{R}^3\setminus B_\rho(0)} \end{aligned}$$

with c not depending on σ and ρ.

Now, $\lim_{\sigma\to\infty} |\omega_{\sigma,y}|_{6,B_\rho(0)} = 0$, uniformly with respect to $y \in \mathbb{R}^3$, so we get $\forall\rho > 0$

$$\lim_{\sigma\to\infty} \sup_{y\in\mathbb{R}^3} \int_{\mathbb{R}^3} W(x)\,\omega^2_{\sigma,y}(x)\,\mathrm{d}x \le c|W|_{3/2,\mathbb{R}^3\setminus B_\rho(0)}$$

and, letting $\rho \to +\infty$, (3.22)(b) follows.

To verify (3.22)(c), we argue by contradiction, so we assume the existence of a sequence $(y_n)_n$, $y_n \in \mathbb{R}^3$, and a sequence $(\sigma_n)_n$, $\sigma_n \in \mathbb{R}^+ \setminus \{0\}$, such that

$$|y_n| \longrightarrow \infty \tag{3.23}$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} W(x)\,\omega^2_{\sigma_n,y_n}(x)\,\mathrm{d}x > 0. \tag{3.24}$$

In view of (3.22)(a)–(b), passing eventually to a subsequence, we can suppose $\lim\limits_{n\to\infty}\sigma_n=\bar\sigma>0$ which, together (3.23) and $(H_v)\,(iii)$, implies

$$\lim_{n\to\infty}|W|_{3/2,B_{\sigma_n(y_n)}}=0.$$

Therefore we deduce the relation

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} W(x)\,\omega^2_{\sigma_n,y_n}(x)\,\mathrm{d}x \le \lim_{n\to\infty}\left[|W|_{3/2,B_{\sigma_n}(y_n)}\cdot|\omega_{\sigma_n,y_n}|^2_{6,B_{\sigma_n}(y_n)}\right]=0$$

contradicting (3.24). □

Lemma 3.6. *The following relations hold*

$$\begin{array}{ll} (a) & \lim\limits_{\sigma\to 0}\sup\left\{\int_{\mathbb{R}^3} K(x)\,\phi_{\omega_{\sigma,y}}(x)\,\omega^2_{\sigma,y}(x)\,\mathrm{d}x \ :\ y\in\mathbb{R}^3\right\}=0\\ (b) & \lim\limits_{\sigma\to\infty}\sup\left\{\int_{\mathbb{R}^3} K(x)\,\phi_{\omega_{\sigma,y}}(x)\,\omega^2_{\sigma,y}(x)\,\mathrm{d}x \ :\ y\in\mathbb{R}^3\right\}=0\\ (c) & \lim\limits_{r\to\infty}\sup\left\{\int_{\mathbb{R}^3} K(x)\,\phi_{\omega_{\sigma,y}}(x)\,\omega^2_{\sigma,y}(x)\,\mathrm{d}x \ :\ y\in\mathbb{R}^3,\ |y|=r,\ \sigma>0\right\}=0. \end{array} \tag{3.25}$$

Proof. We start remarking that, by *(2)* of Proposition 2.1, $\{\phi_{\omega_{\sigma,y}}\ :\ \sigma>0,\ y\in\mathbb{R}^3\}$ is a bounded set in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ and $L^6(\mathbb{R}^3)$. Let $y\in\mathbb{R}^3$ arbitrarily chosen, then for all $\sigma>0$

$$\begin{aligned}\int_{\mathbb{R}^3} K(x)\phi_{\omega_{\sigma,y}}(x)\,\omega^2_{\sigma,y}(x)\,\mathrm{d}x &= \int_{B_\sigma(y)} K(x)\phi_{\omega_{\sigma,y}}(x)\,\omega^2_{\sigma,y}(x)\,\mathrm{d}x\\ &\le |K|_{2,B_\sigma(y)}|\phi_{\omega_{\sigma,y}}|_6|\omega_{\sigma,y}|^2_{6,B_\sigma(y)}\\ &\le c|K|_{2,B_\sigma(y)}\end{aligned}$$

$c>0$ independent of y and σ. Hence

$$\sup_{y\in\mathbb{R}^3}\int_{\mathbb{R}^3} K(x)\phi_{\omega_{\sigma,y}}(x)\,\omega^2_{\sigma,y}(x)\,\mathrm{d}x \le c\sup_{y\in\mathbb{R}^3}|K|_{2,B_\sigma(y)}$$

which gives (3.25)(a) because

$$\lim_{\sigma\to 0}|K|_{2,B_\sigma(y)}=0 \qquad \text{uniformly in } y\in\mathbb{R}^3.$$

To verify (3.25)(b), let us fix arbitrarily $y\in\mathbb{R}^3$. Then $\forall\rho>0\ \forall\sigma>0$

$$\begin{aligned}&\int_{\mathbb{R}^3} K(x)\phi_{\omega_{\sigma,y}}(x)\,\omega^2_{\sigma,y}(x)\,\mathrm{d}x\\ &\qquad = \int_{\mathbb{R}^3\setminus B_\rho(0)} K(x)\phi_{\omega_{\sigma,y}}(x)\,\omega^2_{\sigma,y}(x)\,\mathrm{d}x + \int_{B_\rho(0)} K(x)\phi_{\omega_{\sigma,y}}(x)\,\omega^2_{\sigma,y}(x)\,\mathrm{d}x\end{aligned}$$

$$\le |K|_{2,\mathbb{R}^3\setminus B_\rho(0)}|\phi_{\omega_{\sigma,y}}|_6|\omega_{\sigma,y}|^2_{6,\mathbb{R}^3\setminus B_\rho(0)} + |K|_{2,B_\rho(0)}|\phi_{\omega_{\sigma,y}}|_6|\omega_{\sigma,y}|^2_{6,B_\rho(0)}$$
$$\le \bar{c}_1|K|_{2,\mathbb{R}^3\setminus B_\rho(0)}| + \bar{c}_2|K|_{2,B_\rho(0)}|\sup_{y\in\mathbb{R}^3}|\omega_{\sigma,y}|^2_{6,B_\rho(0)}$$

with $\bar{c}_1, \bar{c}_2 \in \mathbb{R}^+\setminus\{0\}$ depending neither on y nor on σ. Thus, considering $|\omega_{\sigma,y}|_{6,B_\rho(0)} \to 0$ as $\sigma\to\infty$, uniformly with respect to $y\in\mathbb{R}^3$, we get

$$\lim_{\sigma\to\infty}\sup\left\{\int_{\mathbb{R}^3} K(x)\phi_{\omega_{\sigma,y}}(x)\,\omega^2_{\sigma,y}(x)\,\mathrm{d}x \;:\; y\in\mathbb{R}^3\right\} \le \bar{c}_1|K|_{2,\mathbb{R}^3\setminus B_\rho(0)}$$

and, letting $\rho\to\infty$, we obtain (3.25)(b). To show (3.25)(c), working by contradiction, we assume it false, so that $(y_n)_n$, $y_n\in\mathbb{R}^3$ and $(\sigma_n)_n$, $\sigma_n\in\mathbb{R}^3\setminus\{0\}$, exist for which

$$(a)\quad |y_n|\to\infty, \qquad (b)\quad \lim_{n\to\infty}\int_{\mathbb{R}^3} K(x)\phi_{\omega_{\sigma_n,y_n}}(x)\,\omega^2_{\sigma_n,y_n}(x)\,\mathrm{d}x > 0. \tag{3.26}$$

By (3.25)(a)–(b), up to a subsequence, we have $\lim_{n\to\infty}\sigma_n = \bar{\sigma}\in(0,\infty)$, and using (3.26)$(a)$ we deduce

$$\lim_{n\to\infty}|K|_{2,B_{\sigma_n}(y_n)} = 0.$$

Thus

$$\int_{\mathbb{R}^3} K(x)\phi_{\omega_{\sigma_n,y_n}}(x)\omega^2_{\sigma_n,y_n}(x)\,\mathrm{d}x = \int_{B_{\sigma_n}(y_n)} K(x)\phi_{\omega_{\sigma_n,y_n}}(x)\omega^2_{\sigma_n,y_n}(x)\,\mathrm{d}x$$
$$\le |K|_{2,B_{\sigma_n}(y_n)}|\phi_{\omega_{\sigma_n,y_n}}|_6|\omega_{\sigma_n,y_n}|^2 \le \tilde{c}|K|_{2,B_{\sigma_n}(y_n)}$$

$\tilde{c}$ not depending on y_n nor on σ_n. Therefore,

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} K(x)\phi_{\omega_{\sigma_n,y_n}}(x)\omega^2_{\sigma_n,y_n}(x)\,\mathrm{d}x \le \tilde{c}\lim_{n\to\infty}|K|_{2,B_{\sigma_n}(y_n)} = 0$$

follows and contradicts (3.26)(b). □

Corollary 3.7. *Set* $t_{\sigma,y} = t_{\omega_{\sigma,y}}$. *Then*

$$\begin{aligned} &(a) \qquad \lim_{\sigma\to 0}\sup\left\{|t_{\sigma,y}-1| \;:\; y\in\mathbb{R}^3\right\} = 0 \\ &(b) \qquad \lim_{\sigma\to\infty}\sup\left\{|t_{\sigma,y}-1| \;:\; y\in\mathbb{R}^3\right\} = 0 \\ &(c) \qquad \lim_{r\to\infty}\sup\left\{|t_{\sigma,y}-1| \;:\; y\in\mathbb{R}^3,\ |y|=r,\ \sigma>0\right\} = 0. \end{aligned} \tag{3.27}$$

Proof. It is enough to observe that, since $\omega_{\sigma,y}\in\mathcal{N}_\infty$, the equalities

$$1 = \frac{\|\omega_{\sigma,y}\|^2}{|\omega_{\sigma,y}|_6^6} = \frac{t^4_{\sigma,y}|\omega_{\sigma,y}|_6^6 - t^2_{\sigma,y}\int_{\mathbb{R}^3} K\phi_{\omega_{\sigma,y}}\omega^2_{\sigma,y} - \int_{\mathbb{R}^3} W\omega^2_{\sigma,y}}{|\omega_{\sigma,y}|_6^6}$$

hold true, so by Lemmas 3.5 and 3.6, relations (3.27) follow at once. □

Lemma 3.8. *The following relations hold*

$$\begin{aligned} &(a) \qquad \lim_{\sigma\to 0}\sup\left\{\gamma(\omega_{\sigma,y})\ :\ y\in\mathbb{R}^3\right\}=0\\ &(b) \qquad \lim_{\sigma\to\infty}\inf\left\{\gamma(\omega_{\sigma,y})\ :\ y\in\mathbb{R}^3,\ |y|\le r\right\}=1 \quad \forall r>0\\ &(c) \qquad (\beta(\omega_{\sigma,y})|y)_{\mathbb{R}^3}>0 \quad \forall y\in\mathbb{R}^3,\ \forall\sigma>0. \end{aligned} \tag{3.28}$$

Proof. For all $y\in\mathbb{R}^3$ and for all $\sigma>0$

$$\begin{aligned} 0\le\gamma(\omega_{\sigma,y}) &= \frac{1}{|\omega_{\sigma,y}|^6_{6,B_\sigma(y)}}\int_{B_\sigma(y)}\left|\frac{x}{1+|x|}-\beta(\omega_{\sigma,y})\right|\omega^6_{\sigma,y}\,\mathrm{d}x\\ &\le \frac{1}{|\omega_{\sigma,y}|^6_{6,B_\sigma(y)}}\int_{B_\sigma(y)}\left|\frac{x}{1+|x|}-\frac{y}{1+|y|}\right|\omega^6_{\sigma,y}\,\mathrm{d}x+\left|\frac{y}{1+|y|}-\beta(\omega_{\sigma,y})\right|\\ &\le \frac{1}{|\omega_{\sigma,y}|^6_{6,B_\sigma(y)}}\left[\int_{B_\sigma(y)}|x-y|\,\omega^6_{\sigma,y}\mathrm{d}x+\int_{B_\sigma(y)}\left|\frac{x}{1+|x|}-\frac{y}{1+|y|}\right|\omega^6_{\sigma,y}\,\mathrm{d}x\right]\\ &\le 2\sigma \end{aligned}$$

from which

$$0\le\sup\{\gamma(\omega_{\sigma,y})\ :\ y\in\mathbb{R}^3\}\le 2\sigma$$

follows and, then, (3.28)(a).

In order to prove (3.28)(b), let us first show that for all $y\in\mathbb{R}^3$

$$\lim_{\sigma\to\infty}\beta(\omega_{\sigma,y})=0. \tag{3.29}$$

Indeed, by symmetry $\beta(\omega_{\sigma,0})=0$, $\forall\sigma>0$ considering (3.21), we deduce

$$\begin{aligned} |\beta(\omega_{\sigma,y})| &= \frac{1}{|\omega_{\sigma,y}|^6_6}\left|\int_{\mathbb{R}^3}\frac{x}{1+|x|}\,\omega^6_{\sigma,y}\mathrm{d}x\right|\\ &= \frac{1}{|\omega_{\sigma,0}|^6_6}\left|\int_{\mathbb{R}^3}\frac{x}{1+|x|}(\omega^6_{\sigma,y}-\omega^6_{\sigma,0})\mathrm{d}x\right|\\ &\le \frac{1}{|\omega_{\sigma,0}|^6_6}\int_{\mathbb{R}^3}\frac{|x|}{1+|x|}|\omega^6_{\sigma,y}-\omega^6_{\sigma,0}|\mathrm{d}x\\ &\le c\int_{\mathbb{R}^3}|\omega^6_{1,y/\sigma}-\omega^6_{1,0}|\mathrm{d}x\longrightarrow 0 \qquad \text{as } \sigma\to\infty. \end{aligned}$$

Now, let us choose arbitrarily $r>0$ and $y\in\mathbb{R}^3$ such that $|y|\le r$. For all $\sigma>0$, we have

$$\gamma(\omega_{\sigma,y})=\frac{1}{|\omega_{\sigma,y}|^6_6}\int_{\mathbb{R}^3}\left|\frac{x}{1+|x|}-\beta(\omega_{\sigma,y})\right|\omega^6_{\sigma,y}\,\mathrm{d}x\le 1+|\beta(\omega_{\sigma,y})|$$

from which, by (3.29)

$$\limsup_{\sigma\to\infty}\inf\{\gamma(\omega_{\sigma,y})\ :\ y\in\mathbb{R}^3,\ |y|\le r\}\le 1$$

follows. Hence, to obtain (3.28)(b) we just need to show

$$\liminf_{\sigma\to\infty} \inf\{\gamma(\omega_{\sigma,y}) \ : \ y \in \mathbb{R}^3,\ |y| \le r\} \ge 1. \tag{3.30}$$

To do this let us take the sequences $(y_n)_n$, $y_n \in \mathbb{R}^3$, $|y_n| \le r$, and $(\sigma_n)_n$, $\sigma_n \in \mathbb{R}^+$, $\sigma_n \to \infty$. Then, considering (3.29) and that $|\omega_{\sigma,y}|_{6,B_\rho(0)} \to 0$, as $\sigma \to \infty$, we deduce for all $\rho > 0$

$$\begin{aligned}
\gamma(\omega_{\sigma_n,y_n}) &= \frac{1}{|\omega_{\sigma_n,y_n}|_6^6} \int_{\mathbb{R}^3} \left| \frac{x}{1+|x|} - \beta(\omega_{\sigma_n,y_n}) \right| \omega_{\sigma_n,y_n}^6 \mathrm{d}x \\
&\ge \frac{1}{|\omega_{\sigma_n,y_n}|_{6,\mathbb{R}^3\setminus B_\rho(0)}^6 + o(1)} \int_{\mathbb{R}^3\setminus B_\rho(0)} \frac{|x|}{1+|x|}\, \omega_{\sigma_n,y_n}^6 \mathrm{d}x - o(1) \\
&\ge \frac{\rho}{1+\rho} - o(1)
\end{aligned}$$

which, letting $\rho \to \infty$, gives $\lim_{n\to\infty} \gamma(\omega_{\sigma_n,y_n}) = 1$ and proves (3.30).

Lastly, let us remark that (3.28)(c) is immediate if $0 \notin B_\sigma(y)$. If $0 \in B_\sigma(y)$, to prove (3.28)(c) we just need to consider that $\forall \bar{x} \in B_\sigma(y)$ such that $(\bar{x}|y)_{\mathbb{R}^3} > 0$ the point $-\bar{x}$ verifies $\omega_{\sigma,y}(-\bar{x}) < \omega_{\sigma,y}(\bar{x})$. □

4. Proof of Theorems

In this section we use the notation I_0 to denote the functional I when $V_\infty = 0$, while when we write I we intend that $V_\infty \neq 0$. Namely, $\forall u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$

$$I_0(u) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + W(x)u^2)\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u u^2 \mathrm{d}x - \frac{1}{6}\int_{\mathbb{R}^3} u^6 \mathrm{d}x$$

and, $\forall u \in H^1(\mathbb{R}^3)$

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + (V_\infty + W(x))u^2)\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u u^2 \mathrm{d}x - \frac{1}{6}\int_{\mathbb{R}^3} u^6 \mathrm{d}x$$

with $V_\infty \neq 0$.

According to this we have

$$\mathcal{N}_0 := \{u \in \mathcal{D}^{1,2}(\mathbb{R}^3) \setminus \{0\} \ : \ I_0'(u)[u] = 0\}$$

$$\mathcal{N} := \{u \in H^1(\mathbb{R}^3) \setminus \{0\} \ : \ I'(u)[u] = 0\}$$

and we denote by

$$\hat{\omega}_{\sigma,y} := t_{\sigma,y,0}\,\omega_{\sigma,y} = t^0_{\omega_{\sigma,y}}\omega_{\sigma,y}$$

$$\tilde{\omega}_{\sigma,y} := t_{\sigma,y}\,\omega_{\sigma,y} = t_{\omega_{\sigma,y}}\omega_{\sigma,y}$$

the projections of $\omega_{\sigma,y}$ respectively on $\mathcal{N}_0$ and on $\mathcal{N}$.

Lemma 4.1. *There exist real numbers $\bar{r} > 0$, $\sigma_1, \sigma_2 :\ 0 < \sigma_1 < \frac{1}{2} < \sigma_2$ such that*

$$\gamma(\hat{\omega}_{\sigma_1,y}) < 1/2,\ \gamma(\hat{\omega}_{\sigma_2,y}) > 1/2, \quad \forall y \in \mathbb{R}^3 \tag{4.1}$$

and

$$\sup\{I_0(\hat{\omega}_{\sigma,y})\ :\ (\sigma, y) \in \partial\mathcal{H}\} < \bar{c} \tag{4.2}$$

where $\bar{c}$ is defined in (3.18) and

$$\mathcal{H} = \{(\sigma, y) \in \mathbb{R}^+ \times \mathbb{R}^3\ :\ \sigma \in [\sigma_1, \sigma_2],\ |y| < \bar{r}\}. \tag{4.3}$$

Proof. Since

$$I_0(\hat{\omega}_{\sigma,y}) = \frac{1}{2}t^2_{\sigma,y,0}\int_{\mathbb{R}^3}|\nabla\omega_{\sigma,y}|^2 - \frac{1}{6}t^6_{\sigma,y,0}\int_{\mathbb{R}^3}\omega^6_{\sigma,y}\mathrm{d}x + \frac{1}{2}t^2_{\sigma,y,0}\int_{\mathbb{R}^3}W(x)\omega^2_{\sigma,y}\mathrm{d}x + \frac{1}{4}t^4_{\sigma,y,0}\int_{\mathbb{R}^3}K(x)\phi_{\omega_{\sigma,y}}\omega^2_{\sigma,y}\mathrm{d}x \tag{4.4}$$

the existence of $\sigma_1 \in (0, 1/2)$ such that $\gamma(\hat{\omega}_{\sigma,y}) < 1/2$ and $I_0(\hat{\omega}_{\sigma,y}) < \bar{c}$ holds true when $\sigma = \sigma_1$ for all $y \in \mathbb{R}^3$ is a consequence of (3.1), (3.22)(a), (3.25)(a), (3.27)(a), (3.28)(a), and (3.20). Furthermore (3.22)(c), (3.25)(c), (3.27)(c) and (3.20) allow us to choose $\bar{r} > 0$ such that, if $|y| = \bar{r}$, $I_0(\hat{\omega}_{\sigma,y}) < \bar{c}$ is satisfied for all $\sigma > 0$. Once $\bar{r}$ is fixed, we use again (3.1), (3.20) plus (3.22)(b), (3.25)(b), (3.27)(b), (3.28)(b) in (4.4) and we find $\sigma_2 > 1/2$ for which

$$\gamma(\hat{\omega}_{\sigma_2,y}) > 1/2 \quad \forall y \in \mathbb{R}^3,\ |y| \leq \bar{r}$$

and $I_0(\hat{\omega}_{\sigma,y}) < \bar{c}$ is verified when $\sigma = \sigma_2$ and $|y| \leq \bar{r}$.

□

Lemma 4.2. *Let $\sigma_1, \sigma_2, \bar{r}, \mathcal{H}$ as in Lemma 4.1. Then, $(\tilde{\sigma}, \tilde{y}) \in \partial\mathcal{H}$ and $(\bar{\sigma}, \bar{y}) \in \overset{\circ}{\mathcal{H}}$ exist so that*

$$\beta(\omega_{\tilde{\sigma},\tilde{y}}) = 0 \qquad \gamma(\omega_{\tilde{\sigma},\tilde{y}}) \geq 1/2 \tag{4.5}$$

$$\beta(\omega_{\bar{\sigma},\bar{y}}) = 0 \qquad \gamma(\omega_{\bar{\sigma},\bar{y}}) = 1/2. \tag{4.6}$$

Proof. The existence of $(\tilde{\sigma}, \tilde{y})$ verifying (4.5) is an easy consequence of Lemma 4.1, indeed it is enough to choose $(\tilde{\sigma}, \tilde{y}) = (\sigma_2, 0)$.

Set $\forall(\sigma, y) \in \mathcal{H}$

$$\theta(\sigma, y) = (\gamma(\omega_{\sigma,y}), \beta(\omega_{\sigma,y}))$$

and for all $(\sigma, y) \in \mathcal{H}$ and $s \in (0, 1)$

$$\mathcal{G}(\sigma, y, s) = (1 - s)(\sigma, y) + s\theta(\sigma, y). \tag{4.7}$$

In order to prove (4.6) it is enough to show

$$d(\theta, \overset{\circ}{\mathcal{H}}, (1/2, 0)) = 1, \tag{4.8}$$

and, since $d(\mathrm{Id}, \overset{\circ}{\mathcal{H}}, (1/2,0)) = 1$, (4.8) straightly follows by the topological degree homotopy invariance, if for all $(\sigma, y) \in \partial\mathcal{H}$, for all $s \in [0,1]$ $\mathcal{G}(\sigma, y, s) \neq (1/2, 0)$.

Therefore, in view of (3.1), to prove (4.5) and (4.6) we just need to verify

$$((1-s)\sigma + s\gamma(\omega_{\sigma,y}), (1-s)y + s\beta(\omega_{\sigma,y})) \neq (1/2, 0) \quad \forall (y,\sigma) \in \partial\mathcal{H},\ \forall s \in [0,1].$$

Now $\partial\mathcal{H} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$ with

$$\mathcal{F}_1 = \{(\sigma, y) \in \partial\mathcal{H} \ : \ |y| \leq \bar{r},\ \sigma = \sigma_1\}$$

$$\mathcal{F}_2 = \{(\sigma, y) \in \partial\mathcal{H} \ : \ |y| \leq \bar{r},\ \sigma = \sigma_2\}$$

$$\mathcal{F}_3 = \{(\sigma, y) \in \partial\mathcal{H} \ : \ |y| = \bar{r},\ \sigma \in [\sigma_1, \sigma_2]\}.$$

By (4.1) we get

$$\forall (\sigma, y) \in \mathcal{F}_1 \ : \ (1-s)\sigma_1 + s\gamma(\omega_{\sigma_1,y}) < 1/2$$

$$\forall (\sigma, y) \in \mathcal{F}_2 \ : \ (1-s)\sigma_2 + s\gamma(\omega_{\sigma_2,y}) > 1/2.$$

If $(\sigma, y) \in \mathcal{F}_3$, using (3.28)$(c)$ we get

$$((1-s)y + s\beta(\omega_{\sigma,y})|y) = (1-s)|y|^2 + s(\beta(\omega_{\sigma,y})|y) > 0$$

hence $(1-s)y + s\beta(\omega_{\sigma,y}) \neq 0$.

□

Lemma 4.3. *Let $\sigma_1, \sigma_2, \bar{r}, \mathcal{H}$ as in Lemma 4.1. Assume that (1.1) holds, then*

$$L = \sup\{I_0(\hat{\omega}_{\sigma,y}) \ : \ (\sigma, y) \in \mathcal{H}\} < \frac{2}{3} S^{3/2}. \tag{4.9}$$

Proof. Taking into account that $\omega_{\sigma,y} \in \mathcal{N}_\infty$ by definition, $t_{\sigma,y,0} \geq 1$ by (2.9), and using (3.21) we have for all $(\sigma, y) \in \mathcal{H}$

$$\begin{aligned} I_0(\hat{\omega}_{\sigma,y}) &= \frac{1}{4}\int_{\mathbb{R}^3} [|\nabla \hat{\omega}_{\sigma,y}|^2 + W(x)\hat{\omega}_{\sigma,y}^2]\mathrm{d}x + \frac{1}{12}\int_{\mathbb{R}^3} \hat{\omega}_{\sigma,y}^6 \mathrm{d}x \\ &= \frac{1}{4}\int_{\mathbb{R}^3} t_{\sigma,y,0}^2 [|\nabla \omega_{\sigma,y}|^2 + W(x)\omega_{\sigma,y}^2]\mathrm{d}x + \frac{1}{12} t_{\sigma,y,0}^6 \int_{\mathbb{R}^3} \omega_{\sigma,y}^6 \mathrm{d}x \\ &\leq t_{\sigma,y,0}^6 \left[\frac{1}{3}\|\omega\|^2 + \frac{1}{4}|W|_{3/2}|\omega|_6^2\right] \\ &\leq t_{\sigma,y,0}^6 \left[\Sigma + \frac{1}{4\,S}|W|_{3/2}\|\omega\|^2\right] \\ &\leq t_{\sigma,y,0}^6 \left(1 + \frac{3}{4\,S}|W|_{3/2}\right)\Sigma. \end{aligned} \tag{4.10}$$

On the other hand, $\hat{\omega}_{\sigma,y} \in \mathcal{N}_0$ implies

$$t^2_{\sigma,y,0}\int_{\mathbb{R}^3}[|\nabla\omega_{\sigma,y}|^2+W(x)\omega^2_{\sigma,y}]\mathrm{d}x+t^4_{\sigma,y,0}\int_{\mathbb{R}^3}K(x)\phi_{\omega_{\sigma,y}}\omega^2_{\sigma,y}\mathrm{d}x-t^6_{\sigma,y,0}\int_{\mathbb{R}^3}\omega^6_{\sigma,y}\mathrm{d}x=0$$

hence, using (2.9)

$$\left(\|\omega_{\sigma,y}\|^2+\int_{\mathbb{R}^3}W(x)\omega^2_{\sigma,y}\mathrm{d}x+\int_{\mathbb{R}^3}K(x)\phi_{\omega_{\sigma,y}}\omega^2_{\sigma,y}\mathrm{d}x\right)-t^2_{\sigma,y,0}|\omega_{\sigma,y}|^6_6\geq 0.$$

Moreover, since $|\phi_{\omega_{\sigma,y}}|_6\leq \frac{1}{S}\,|K|_2|\omega_{\sigma,y}|^2_6$,

$$\begin{aligned} t^2_{\sigma,y,0} &\leq 1+\frac{|W|_{3/2}}{|\omega|^4_6}+\frac{|K|_2|\phi_{\omega_{\sigma,y}}|_6}{|\omega|^4_6} \\ &\leq 1+\frac{|W|_{3/2}}{(3\Sigma)^{2/3}}+\frac{1}{S}\,\frac{|K|^2_2}{|\omega|^2_6} \\ &\leq 1+\frac{|W|_{3/2}}{S}+\frac{|K|^2_2}{S^{3/2}}. \end{aligned} \tag{4.11}$$

So, by (1.1), (4.10), (4.11) we get

$$I_0(\hat{\omega}_{\sigma,y})\leq\left(1+\frac{|W|_{3/2}}{S}+\frac{|K|^2_2}{S^{3/2}}\right)^3\left(1+\frac{3}{4\,S}|W|_{3/2}\right)\Sigma<\frac{2}{3}\,S^{3/2}.$$

□

In what follows we use the notation

$$I^c_0=\{u\in\mathcal{N}_0\ :\ I_0(u)\leq c\}.$$

Proof of Theorem 1.1. Collecting the results of Proposition 3.1, Lemma 4.2 and Lemma 4.3, setting $\hat{\omega}_{\bar{\sigma},\bar{y}}=t_{\bar{\sigma},\bar{y},0}\omega_{\bar{\sigma},\bar{y}}$, where $\omega_{\bar{\sigma},\bar{y}}\in\overset{\circ}{\mathcal{H}}$ is defined in Lemma 4.2, we obtain

$$\frac{1}{3}\,S^{3/2}<\bar{c}<\mathcal{B}_0\leq I_0(\hat{\omega}_{\bar{\sigma},\bar{y}})\leq L<\frac{2}{3}\,S^{3/2}.$$

Let us show that a critical level of I_0 constrained on $\mathcal{N}_0$ exists in the interval $\left(\frac{1}{3}\,S^{3/2},\frac{2}{3}\,S^{3/2}\right)$. We argue by contradiction and we assume there are not critical levels in $\left(\frac{1}{3}\,S^{3/2},\frac{2}{3}\,S^{3/2}\right)$. So, since I_0 constrained on $\mathcal{N}_0$ satisfies the Palais-Smale condition in the energy range $\left(\frac{1}{3}\,S^{3/2},\frac{2}{3}\,S^{3/2}\right)$, by using standard deformation arguments we find a number $\delta>0$ such that $\mathcal{B}_0-\delta>\bar{c}$, $L+\delta<\frac{2}{3}\,S^{3/2}$ and a continuous function

$$\eta: I^{L+\delta}_0\longrightarrow I^{\mathcal{B}_0-\delta}_0$$

such that

$$\eta(u)=u\qquad\forall u\in I^{\mathcal{B}_0-\delta}_0.$$

Then, we remark that

$$\forall(\sigma, y) \in \mathcal{H} \qquad I_0(\eta(\hat{\omega}_{\sigma,y})) \leq \mathcal{B}_0 - \delta$$

so

$$\Theta(\sigma, y) := (\gamma(\eta(\hat{\omega}_{\sigma,y})), \beta(\eta(\hat{\omega}_{\sigma,y}))) \neq (1/2, 0). \tag{4.12}$$

On the other hand, by Lemma 4.1 $\forall(\sigma, y) \in \partial\mathcal{H}$

$$I_0(\hat{\omega}_{\sigma,y}) < \bar{c} < \mathcal{B}_0 - \delta \quad \Rightarrow \quad \eta(\hat{\omega}_{\sigma,y}) = \hat{\omega}_{\sigma,y}$$

from which

$$\Theta(\sigma, y) = \theta(\sigma, y) = (\gamma(\hat{\omega}_{\sigma,y}), \beta(\hat{\omega}_{\sigma,y})) \quad \forall(\sigma, y) \in \partial\mathcal{H}.$$

Therefore, by the homotopy invariance of topological degree, we deduce

$$1 = d(\theta, \mathring{\mathcal{H}}, (1/2, 0)) = d(\Theta, \mathring{\mathcal{H}}, (1/2, 0))$$

that implies the existence of $(\hat{\sigma}, \hat{y}) \in \mathcal{H}$ for which

$$\Theta(\hat{\sigma}, \hat{y}) = (1/2, 0)$$

contradicting (4.12).

To conclude the proof we only must show that solutions corresponding to critical levels lying in the interval $\left(\frac{1}{3} S^{3/2}, \frac{2}{3} S^{3/2}\right)$ cannot change sign. Indeed, assume $u = u^+ - u^-$ is a solution with $u^+ \neq 0$, $u^- \neq 0$. Then, by Proposition 2.8 and taking into account that $\phi_{u^+} \leq \phi_u$, we get

$$\begin{aligned} \frac{1}{3} S^{3/2} &\leq I(t_{u^+} u^+) \\ &= \frac{1}{3} t_{u^+}^2 \int_{\mathbb{R}^3} (|\nabla u^+|^2 + W(x)(u^+)^2)\, \mathrm{d}x + \frac{1}{12} t_{u^+}^4 \int_{\mathbb{R}^3} K(x)\phi_{u^+}(x)(u^+)^2 \mathrm{d}x \\ &\leq \frac{1}{3} t_{u^+}^2 \int_{\mathbb{R}^3} (|\nabla u^+|^2 + W(x)(u^+)^2)\, \mathrm{d}x + \frac{1}{12} t_{u^+}^4 \int_{\mathbb{R}^3} K(x)\phi_u(x)(u^+)^2 \mathrm{d}x. \end{aligned} \tag{4.13}$$

We claim that $t_{u^+} \leq 1$. Once the claim is proved, by (4.13) we obtain

$$\frac{1}{3} S^{3/2} \leq \frac{1}{3} \int_{\{u>0\}} (|\nabla u|^2 + W(x)\, u^2)\, \mathrm{d}x + \frac{1}{12} \int_{\{u>0\}} K(x)\phi_u(x)\, u^2 \mathrm{d}x. \tag{4.14}$$

Likewise,

$$\frac{1}{3} S^{3/2} \leq \frac{1}{3} \int_{\{u<0\}} (|\nabla u|^2 + W(x)\, u^2)\, \mathrm{d}x + \frac{1}{12} \int_{\{u<0\}} K(x)\phi_u(x)\, u^2 \mathrm{d}x,$$

so that $I(u) \geq \frac{2}{3} S^{3/2}$ follows.

Let us prove that $t_{u^+} \leq 1$. By definition of projection,

$$t_{u^+}^2 \int_{\mathbb{R}^3} (|\nabla u^+|^2 + W(x)(u^+)^2)\, \mathrm{d}x + t_{u^+}^4 \int_{\mathbb{R}^3} K(x)\phi_{u^+}(x)(u^+)^2 \mathrm{d}x = t_{u^+}^6 |u^+|_6^6, \tag{4.15}$$

and, since u is a critical point,

$$|u^+|_6^6 = \int_{\mathbb{R}^3} (|\nabla u^+|^2 + W(x)(u^+)^2)\, \mathrm{d}x + \int_{\mathbb{R}^3} K(x)\phi_u(x)(u^+)^2 \mathrm{d}x. \tag{4.16}$$

So t_{u^+} is the positive solution of $\varphi(t) = 0$, where

$$\varphi(t) := t^4 |u^+|_6^6 - t^2 \int_{\mathbb{R}^3} K(x)\phi_{u^+}(x)(u^+)^2 \mathrm{d}x + \int_{\mathbb{R}^3} K(x)\phi_u(x)(u^+)^2 \mathrm{d}x - |u^+|_6^6.$$

A straight computation shows that $\varphi'(t) > 0$ $\forall t \geq 1$, because, by (4.16),

$$\begin{aligned} \frac{1}{|u^+|_6^6} \int_{\mathbb{R}^3} K(x)\phi_{u^+}(x)(u^+)^2 \mathrm{d}x &\leq \frac{1}{|u^+|_6^6} \int_{\mathbb{R}^3} K(x)\phi_u(x)(u^+)^2 \mathrm{d}x \\ &= 1 - \frac{1}{|u^+|_6^6} \int_{\mathbb{R}^3} (|\nabla u^+|^2 + W(x)(u^+)^2)\, \mathrm{d}x < 1. \end{aligned}$$

So the claim follows from

$$\varphi(1) = \int_{\mathbb{R}^3} K(x)[\phi_u(x) - \phi_{u^+}(x)](u^+)^2 \mathrm{d}x \geq 0.$$

□

Lemma 4.4. *Let $\bar{r}, \sigma_1, \sigma_2$ and $\mathcal{H}$ as in Lemma 4.1. Then there exists a number $\bar{V} > 0$ such that for all $V_\infty \in (0, \bar{V})$*

$$\gamma(\tilde{\omega}_{\sigma_1,y}) < 1/2, \quad \gamma(\tilde{\omega}_{\sigma_2,y}) > 1/2, \qquad \forall y \in \mathbb{R}^3,\ |y| < \bar{r}, \tag{4.17}$$

$$\tilde{l} := \sup\{I(\tilde{\omega}_{\sigma,y}) \ :\ (\sigma, y) \in \partial\mathcal{H}\} < \bar{c}. \tag{4.18}$$

Furthermore, if (1.2) holds true, then $\bar{V}$ can be found so that, in addition to (4.17) and (4.18),

$$\tilde{s} := \sup\{I(\tilde{\omega}_{\sigma,y}) \ :\ (\sigma, y) \in \mathcal{H}\} < \frac{2}{3} S^{3/2} \tag{4.19}$$

is satisfied.

Proof. By (3.1)

$$\gamma(\tilde{\omega}_{\sigma,y}) = \gamma(\hat{\omega}_{\sigma,y}) = \gamma(\omega_{\sigma,y}) \qquad \forall(\sigma, y) \in (0, +\infty) \times \mathbb{R}^3.$$

Hence relations (4.17) straightly follow from (4.1). Moreover

$$1 = \frac{\|\omega_{\sigma,y}\|_{\mathcal{D}^{1,2}}^2}{|\omega_{\sigma,y}|_6^6} = t_{\sigma,y}^4 - \frac{\int_{\mathbb{R}^3} (V_\infty + W(x))\omega_{\sigma,y}^2 - t_{\sigma,y}^2 \int_{\mathbb{R}^3} K(x)\phi_{\omega_{\sigma,y}} \omega_{\sigma,y}^2}{|\omega_{\sigma,y}|_6^6}$$

and

$$\int_{\mathbb{R}^3} V_\infty \omega_{\sigma,y}^2(x)\,\mathrm{d}x = V_\infty \sigma^2 \int_{B_1(0)} \omega^2(x)\,\mathrm{d}x \tag{4.20}$$

imply

$$\lim_{V_\infty\to 0} \sup_{(\sigma,y)\in\mathcal{H}} |t_{\sigma,y} - t_{0,\sigma,y}| = 0. \tag{4.21}$$

Then, if V_∞ is suitably small (4.18) and (4.19) are consequence of (4.20), (4.21), (4.2) and (4.9).

□

Proof of Theorem 1.2. In what follows $\bar{V}$ denotes the number whose existence is stated in Lemma 4.4 and we assume $V_\infty \in (0,\bar{V})$.

To prove the theorem we intend to show that a critical level exists in the interval $\left(\frac{1}{3}\,S^{3/2},\bar{c}\right)$ and that if (1.2) holds another critical level exists in $\left(\bar{c},\frac{2}{3}\,S^{3/2}\right)$.

By using (3.14), (4.5) together with (3.1), (4.18), and (3.18) we deduce

$$\frac{1}{3}\,S^{3/2} < \mu \le I(\tilde{\omega}_{\tilde{\sigma},\tilde{y}}) \le \tilde{l} < \bar{c} < \mathcal{B}_0. \tag{4.22}$$

Arguing by contradiction, we assume there are no critical levels in $\left(\frac{1}{3}\,S^{3/2},\bar{c}\right)$. Since the Palais-Smale compactness condition holds in that energy range, we can find a positive number $\delta_1 > 0$ such that

$$\mu - \delta_1 > \frac{1}{3}\,S^{3/2} \qquad \tilde{l} + \delta_1 < \bar{c}$$

and a continuous function

$$\eta : [0,1] \times I^{\tilde{l}+\delta_1} \longrightarrow I^{\tilde{l}+\delta_1}$$

such that

$$\begin{aligned} \eta(0,u) &= u \\ \eta(s,u) &= u && \forall u \in I^{\mu-\delta_1},\ \forall s\in[0,1] \\ I\circ\eta\,(s,u) &\le I(u) && \forall s\in[0,1] && (4.23) \\ \eta(1, I^{\tilde{l}+\delta_1}) &\subset I^{\mu-\delta_1}. && && (4.24) \end{aligned}$$

Therefore, definition of $\tilde{l}$ and (4.24) give

$$(\sigma,y)\in\partial\mathcal{H} \ \Rightarrow\ I(\tilde{\omega}_{\sigma,y}) \le \tilde{l} \ \Rightarrow\ I(\eta(1,\tilde{\omega}_{\sigma,y})) \le \mu - \delta_1. \tag{4.25}$$

Let us consider $\forall s\in[0,1]$, $\forall(\sigma,y)\in\mathcal{H}$

$$\Gamma(\sigma,y,s) \ = \ \begin{cases} \mathcal{G}(\sigma,y,2s) & s\in[0,1/2] \\ (\gamma\circ\eta\,(2s-1,\tilde{\omega}_{\sigma,y}), \beta\circ\eta\,(2s-1,\tilde{\omega}_{\sigma,y})) & s\in[1/2,1] \end{cases}$$

where $\mathcal{G}$ is the map defined in (4.7). As already shown in Lemma 4.2,

$$\forall s\in[0,1/2],\ \forall(\sigma,y)\in\partial\mathcal{H},\ \Gamma(\sigma,y,s)\neq(1/2,0). \tag{4.26}$$

Furthermore, by using (4.22) and (4.23) we deduce $\forall s \in [1/2, 1]$ $\forall (\sigma, y) \in \mathcal{H}$

$$I(\eta(2s-1, \tilde{\omega}_{\sigma,y})) \leq I(\tilde{\omega}_{\sigma,y}) \leq \tilde{l} < \bar{c} < \mathcal{B}_0 \leq \mathcal{B}_{V_\infty}$$

which gives

$$\forall s \in [1/2, 1],\ \forall (\sigma, y) \in \partial\mathcal{H},\ \Gamma(\sigma, y, s) \neq (1/2, 0). \tag{4.27}$$

Then $(\check{\sigma}, \check{y}) \in \partial\mathcal{H}$ must exist such that

$$\beta \circ \eta\,(1, \tilde{\omega}_{\check{\sigma},\check{y}}) = 0, \quad \gamma \circ \eta\,(1, \tilde{\omega}_{\check{\sigma},\check{y}}) \geq 1/2. \tag{4.28}$$

Indeed, let us observe that $\partial\mathcal{H}$ is homotopic to a sphere in $\mathbb{R}^4$ and that $(1/2, 0) \in \overset{\circ}{\mathcal{H}}$, so the line $(1/2, +\infty) \times \{0\}$ crosses $\partial\mathcal{H}$. Thus, being $\partial\mathcal{H}$ homotopic to $\Gamma(\partial\mathcal{H}, 1)$, (4.28) could be false only if $\Gamma(\sigma, y, 1) = (1/2, 0)$ for some $s \in [0, 1]$ and $(\sigma, y) \in \partial\mathcal{H}$ and this relation is impossible by (4.26) and (4.27). Then

$$I(\eta(1, \tilde{\omega}_{\check{\sigma},\check{y}})) \geq \mu$$

which contradicts (4.25).

Now, let us suppose that (1.2) holds. By using (3.19), (3.17), (4.6) together with (3.1), and (4.19) we deduce

$$\bar{c} < \mathcal{B}_0 \leq \mathcal{B}_{V_\infty} \leq I(\tilde{\omega}_{\bar{\sigma},\bar{y}}) \leq \tilde{s} < \frac{2}{3}\, S^{3/2}. \tag{4.29}$$

Repeating the argument of the proof of Theorem 1.1, the existence of a critical level lying in the energy interval $\left(\bar{c}, \frac{2}{3}\, S^{3/2}\right)$ follows. The proof is completed showing, exactly as for Theorem 1.1, that critical points u such that $I(u) < \frac{2}{3}\, S^{3/2}$ are functions that cannot change sign.

□

Proof of Theorem 1.3. We first remark that Proposition 2.8 holds true even if $K(x) \equiv 0$, because $V_\infty \not\equiv 0$.

To obtain the proof of Theorem 1.3 one has just to follow the argument displayed in Section 4 to prove Theorem 1.2. However, it must be pointed out that some basic estimates, appearing in (4.22) and (4.29) and proven assuming $K(x) \not\equiv 0$ and $V(x) - V_\infty \geq 0$ still hold when $K(x) \equiv 0$ if $V(x) - V_\infty \geq 0$, $V(x) - V_\infty \not\equiv 0$. Actually, looking at Proposition 3.1, one easily sees that this statement and its proof are still true, even if $K(x) \equiv 0$, when $V_\infty > 0$, because the inequality (3.10) holds and the argument by contradiction works. Thus, the numbers $\mathcal{B}_{V_\infty}$ are well defined. On the other hand, in order to be able to define the number $\bar{c}$ in (3.18) independently of $\mathcal{B}_{V_\infty}$ (so that (4.22) and (4.29) are verified), a bound from below to $\mathcal{B}_{V_\infty}$ as $V_\infty \to 0$ is needed. This bound is given by $\mathcal{B}_0$ and the relation $\mathcal{B}_0 > \frac{1}{3}\, S^{3/2}$ holds true when $K(x) \equiv 0$ if $|W|_{L^{3/2}} = |V - V_\infty|_{L^{3/2}} \neq 0$. Indeed looking again at Proposition 3.1 one sees that the statement and the proof works, even if $K(x) \equiv 0$, when $|W|_{L^{3/2}} \neq 0$ and $V_\infty = 0$ because in the inequality (3.10) one gets

$$\int_{\mathbb{R}^3} V(x)\Psi^2_{\bar{\sigma},\bar{y}}(x)\mathrm{d}x = \int_{\mathbb{R}^3} W(x)\Psi^2_{\bar{\sigma},\bar{y}}(x)\mathrm{d}x \neq 0.$$

□

Acknowledgements. The authors have been supported by the "Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA)" of the *Istituto Nazionale di Alta Matematica (INdAM)* – Project: Sistemi differenziali ellittici nonlineari derivanti dallo studio di fenomeni elettromagnetici. The second author has been supported also by the research project: "Consolidate the Foundations: Nonlinear Differential Problems and their Applications", of the University of Rome "Tor Vergata".

References

[1] A. Ambrosetti, On Schrödinger-Poisson systems. *Milan J. Math.* **10** (2008) 391–404.

[2] V. Benci and G. Cerami, Existence of positive solutions of the equation $-\Delta u + a(x)u = u^{(N+2)/(N-2)}$ in $\mathbb{R}^N$. *J. Funct. Anal.* **88** (1990) 90–117.

[3] V. Benci and D. Fortunato, An eigenvalue problem for the Schrödinger-Maxwell equations. *Top. Methods Nonlin. Anal.* **11** (1998) 283–293.

[4] V. Benci and D. Fortunato, Solitary waves of the nonlinear Klein–Gordon equation coupled with Maxwell equations. *Rev. Math. Phys.* **14** (2002) 409–420.

[5] V. Benci and D. Fortunato, Solitons in Schrödinger-Maxwell equations. *J. Fixed Point Theory Appl.* **15** (2014) 101–132.

[6] V. Benci and D. Fortunato, Variational methods in nonlinear field equations. *Springer Monographs in Math.* Springer Int. Publ. Switzerland (2014).

[7] R. Benguria, H. Brézis and E. Lieb, The Thomas-Fermi-Von Weizsäcker theory of atoms and molecules. *Commun. Math. Phys.* **79** (1981) 167–180.

[8] H. Brézis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. *Commun. Pure Appl. Math.* **36** (1983) 437–477.

[9] I. Catto and P.L. Lions, Binding of atoms and stability of molecules in Hartree and Thomas-Fermi type theories. I. *Commun Partial Differ. Equ.* **17** (1992) 1051–1110.

[10] J. Chen, E. Rocha and L. Huang, Positive and sign-changing solutions of a Schrödinger-Poisson system involving a critical nonlinearity. *J. Math. Anal. Appl.* **408** (2013) 55–69.

[11] G. Cerami and R. Molle, Multiple positive solutions for nonautonomous quasicritical elliptic problems in unbounded domains. *Adv. Nonlin. Stud.* **6** (2006) 233–254.

[12] G. Cerami and R. Molle, Positive bound state solutions for some Schrödinger-Poisson systems. *Nonlinearity* **29** (2016) 3103–3119.

[13] G. Cerami and D. Passaseo, High energy positive solutions for mixed and Neumann elliptic problems with critical nonlinearity. *J. Anal. Math.* **71** (1997) 1–39.

[14] G. Cerami, S. Solimini and M. Struwe, Some existence results for superlinear elliptic boundary value problems involving critical exponents. *J. Funct. Anal.* **69** (1986) 289–306.

[15] G. Cerami and G. Vaira, Positive solutions for some non-autonomous Schrödinger-Poisson systems. *J. Differ. Equ.* **248** (2010) 521–543.

[16] W-Y. Ding, On a conformally invariant elliptic equation on $\mathbb{R}^n$. *Commun. Math. Phys.* **107** (1986) 331–335.

[17] B. Gidas, Symmetry properties and isolated singularities of positive solutions of nonlinear elliptic equations, in *Proc of Nonlinear Partial Differential Equations In Engineering and Applied Science*, edited by R.L Sternberg, A.J Kalinowski, J.S Papadakis. Dekker, New York (1979).

[18] B. Gidas, W.M. Ni and L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^n$. *Math. Anal. Appl. A: Adv. Math. Suppl. Stud.* **7**A (1981) 369–402.

[19] Y. He, L. Lu and W. Shuai, Concentrating ground state solutions of Schrödinger-Poisson equations in $\mathbb{R}^3$ involving critical Sobolev exponents. *Commun. Pure Appl. Anal.* **15** (2016) 103–125.

[20] X. He and W. Zou, Existence and concentration of ground states for Schrödinger-Poisson equations with critical growth. *J. Math. Phys.* **53** (2012) 023702.

[21] P.L. Lions, Solutions of Hartree-Fock equations for Coulomb systems. *Commun. Math. Phys.* **109** (1987) 33–97.

[22] P.L. Lions and B. Simon, The Thomas Fermi theory of atoms, molecules and solids. *Adv. Math.* **23** (1977) 22–116.

[23] Z. Liu and S. Guo, On ground state solutions for the Schrödinger-Poisson equations with critical growth. *J. Math. Anal. Appl.* **412** (2014) 435–448.

[24] Z. Liu, S. Guo and Y. Fang, Multiple semi classical states for Schrödinger-Poisson equations with critical exponential growth. *J. Math. Phys.* **56** (2015) 041505.

[25] P.A. Markowich, C.A. Ringhofer and C. Schmeiser, Semiconductor Equations. Springer Velag, Vienna (1990).

[26] D. Passaseo, Some sufficient conditions for the existence of positive solutions to the equation $-\Delta u + a(x)u = u^{2^*-1}$ in bounded domains. *Ann. Inst. Henri Poincaré Anal. Non Linéaire* **13** (1996) 185–227.

[27] D. Ruiz, The Schrödinger-Poisson equation under the effect of a nonlinearlocal term. *J. Funct. Anal.* **237** (2006) 655–674.

[28] G. Talenti, Best constant in Sobolev inequality. *Ann. Mat. Pura Appl.* **110** (1976) 353–372.

[29] J. Zhang, On ground state and nodal solutions of Schrödinger-Poisson equations with critical growth. *J. Math. Anal. Appl.* **428** (2015) 387–404.

[30] J. Zhang, Ground state and multiple solutions for Schrödinger-Poisson equations with critical nonlinearity. *J. Math. Anal. Appl.* **440** (2016) 466–482.