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LACK OF NULL CONTROLLABILITY OF VISCOELASTIC FLOWS*

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Abstract. We consider controllability of linear viscoelastic flow with a localized control in the momentum equation. We show that, for Jeffreys fluids or for Maxwell fluids with more than one relaxation mode, exact null controllability does not hold. This contrasts with known results on approximate controllability.

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1. Introduction and main results

In this paper, we consider linear viscoelastic flows in a bounded domain Ω, a subset of $\mathbb{R}^d$, for $d = 2, 3$, with $\Gamma = \partial\Omega \in C^\infty$. The viscoelastic flows are governed by the equations of conservation of mass and momentum for the velocity u, pressure p and stress tensor τ:

$$\begin{aligned} &\nabla \cdot u = 0 \quad \text{in} \quad \Omega \times (0,T), \\ &\rho(\partial_t u + (u \cdot \nabla)u) = \nabla \cdot \tau - \nabla p \quad \text{in } \Omega \times (0,T), \end{aligned} \tag{1.1}$$

and a constitutive law relating the stress tensor to the motion. Maxwell's theory of linear viscoelasticity assumes that the stress is linked to the velocity by an ordinary differential equation

$$\partial_t \tau + \lambda\tau = \kappa \left(\nabla u + (\nabla u)^T\right), \tag{1.2}$$

where λ and κ are positive constants. The term $1/\lambda$ is the relaxation time. In Jeffreys models, the stress is taken to be a linear combination of a Maxwell stress term and a Newtonian stress term

$$\tau = \eta \left(\nabla u + (\nabla u)^T\right),$$

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for $\eta > 0$.

The linear viscoelastic flow of a single mode Jeffreys model (for $\eta \neq 0$) or Maxwell model (for $\eta = 0$) on $\Omega \times (0,T)$ governed by the velocity vector u and the elastic stress tensor τ, with a control f acting only in the velocity equation, is given by

$$\begin{aligned} &\rho \partial_t u \;=\; \eta \Delta u + \nabla \cdot \tau - \nabla p + f \quad \text{in } \Omega \times (0,T), \\ &\nabla \cdot u = 0 \quad \text{in } \Omega \times (0,T), \\ &\partial_t \tau + \lambda \tau = 2\kappa Du \quad \text{in } \Omega \times (0,T), \\ &u = 0 \quad \text{in } \partial\Omega \times (0,T), \\ &u(\cdot,0) = u_0, \quad \tau(\cdot,0) = \tau_0 \quad \text{in } \Omega, \end{aligned} \tag{1.3}$$

where ρ, λ and κ are positive constants and η is a nonnegative constant, and Du is the symmetrized gradient tensor defined by

$$Du := \frac{1}{2}\left(\nabla u + (\nabla u)^T\right).$$

By taking the stress to be a linear superposition of several stress terms, *i.e.*, $\tau = \sum_{i=1}^N \tau_i$, where τ_i satisfies

$$\partial_t(\tau_i) + \lambda_i \tau_i = \kappa_i \left(\nabla u + (\nabla u)^T\right), \quad i = 1, \dots, N, \tag{1.4}$$

we formulate multimode Jeffreys ($\eta > 0$) or Maxwell models ($\eta = 0$) as

$$\begin{aligned} &\rho \partial_t u \;=\; \eta \Delta u + \sum_{i=1}^N \nabla \cdot \tau_i - \nabla p + f \quad \text{in } \Omega \times (0,T), \\ &\nabla \cdot u = 0 \quad \text{in } \Omega \times (0,T), \\ &\partial_t(\tau_i) + \lambda_i \tau_i = 2\kappa_i Du \quad \text{in } \Omega \times (0,T), \quad i = 1, \dots, N, \\ &u = 0 \quad \text{in } \partial\Omega \times (0,T), \\ &u(\cdot,0) = u_0, \quad \tau_i(\cdot,0) = \tau_{i,0} \quad \text{in } \Omega, \quad i = 1, \dots, N, \end{aligned} \tag{1.5}$$

where ρ, λ_i and κ_i, for all $i = 1, \dots, N$, are positive constants and η is a nonnegative constant.

Our aim is to study if there exists any control f, able to bring (u, τ), the solution of (1.3), or $(u, (\tau_i)_{i=1}^N)$, the solution of (1.5), to rest at time $t = T$, for some $T > 0$, *i.e.*, if the system is null controllable at time T. In this paper, we see that using any control f localized in a fixed domain $\mathcal{O}$, any proper subset of Ω, the single mode and multimode Jeffreys systems (1.3) and (1.5), for $\eta > 0$, are not null controllable at any time $T > 0$. For $\eta = 0$, the multimode Maxwell system (1.5) is also not null controllable at any time $T > 0$ using a localized interior control acting only in velocity, and for the single mode Maxwell system (1.3), there exists a time T_0, such that the system is not null controllable at any time $0 < T < T_0$, using a localized interior control acting only on velocity.

First, we observe that due to the behaviour of the equation $(1.3)_3$ for the stress tensor τ, the null controllability of τ is possible only in a suitably defined subspace.

Remark 1.1. Note that, from $(1.3)_3$, we have

$$\tau(x,t) = e^{-\lambda t}\tau_0(x) + 2\kappa \int_0^t e^{-\lambda(t-s)} Du(x,s)\,\mathrm{d}s,$$

where (u, τ) is the solution of (1.3). The second term in the above representation constrains that the null controllability of τ can be obtained only for initial conditions of the form $\tau_0 = Dv_0$, for a divergence free vector field v_0 with homogeneous Dirichlet boundary conditions.

Before stating our main result, we define the spaces

$$V_n^0(\Omega) = \left\{ u \in (L^2(\Omega))^d \mid \text{div } u = 0, \text{ in } \Omega, \quad u \cdot n = 0, \text{ on } \partial\Omega \right\},$$

and

$$V^0(\Omega) = \left\{ u \in (L^2(\Omega))^d \mid \text{div } u = 0 \right\}, \quad V_0^s(\Omega) = \{ u \in (H^s(\Omega))^d \cap V^0(\Omega) \mid u = 0 \text{ on } \Gamma \},$$

for any $s \geq 1$. Here n denotes the normal to $\partial\Omega$.

We denote the space $L^2(\Omega; \mathcal{L}_s(\mathbb{R}^d))$, where $\mathcal{L}_s(\mathbb{R}^d)$ denotes the space of symmetric real $d \times d$ matrices, and the subspace of $L^2(\Omega; \mathcal{L}_s(\mathbb{R}^d))$

$$R_0 = \{ Dv \mid v \in V_0^1(\Omega) \}, \quad \text{and} \quad R_0^s = \{ Dv \mid v \in V_0^{s+1}(\Omega) \}, \quad \forall\, s \geq 1.$$

Note that v can be reconstructed from $\tau = 2Dv$ by solving the Stokes problem

$$\Delta v - \nabla p = \nabla \cdot \tau, \quad \nabla \cdot v = 0, \quad v|_{\partial\Omega} = 0,$$

so v is one order smoother than τ.

Our main result for single mode Jeffreys system is

Theorem 1.2. *Let $T > 0$ and $\mathcal{O}$ be a proper subset of Ω. There exists (u_0, τ_0), an initial condition of (1.3), in $V_0^s(\Omega) \times R_0^s$ for any $s \geq 1$, with compact support in Ω, such that the solution of system (1.3), for $\eta > 0$, with initial condition (u_0, τ_0) and with any control f localized in $\mathcal{O}$, acting only in the velocity equation, cannot reach zero at time T.*

To prove the above result, we first use a change of variable and show that the null controllability of (1.3) implies the null controllability of the new system (see Sect. 2). The new system is a coupling between a parabolic equation and an ordinary differential equation. The control localized in $\mathcal{O}$, a proper subset of Ω, is acting in the parabolic equation. Then we split the system into two systems – one is with the control but zero initial condition and the other system is without control but with the initial condition. Exploiting the regularity of the solution of the parabolic equation, we show that the solution of the first system with control is smooth outside $\mathcal{O}$ where the control acts, whereas for an appropriate choice of the initial conditions the solution of the second system cannot be smooth outside $\mathcal{O}$. This yields that the solutions of these two systems cannot cancel each other outside $\mathcal{O}$, and hence the original system cannot be null controllable using any localized control in the parabolic equation with support inside $\mathcal{O}$, a proper subset of Ω. The suitable initial condition having discontinuity around a neighbourhood of a point of $\Omega \backslash \overline{\mathcal{O}}$ can be chosen from the eigenspace of the linear operator associated with the new system taking advantage of the spectrum which has an accumulation point (see [3], Prop. 2.2).

A similar result holds for the Jeffreys system with several modes, *i.e.*,

Theorem 1.3. *Let $T > 0$ and $\mathcal{O}$ be a proper subset of Ω. There exists (u_0, τ_0), an initial condition of (1.5), in $V_0^s(\Omega) \times (R_0^s)^N$ for any $s \geq 1$, with compact support in Ω, such that the solution of system (1.5), for $\eta > 0$, with initial condition (u_0, τ_0) and with any control f localized in $\mathcal{O}$, acting only in the velocity equation, cannot reach zero at time T.*

For $\eta = 0$, the single mode linear Maxwell system is exactly controllable at sufficiently large enough time T, using a localized interior control acting only on velocity equation, under the restriction of the reachable set of the stress variable as outlined in the remark above [3]. For small time T, the system cannot be null controllable (due to the finite speed of propagation property the of Maxwell system), using a localized interior control acting only on velocity.

Theorem 1.4. *For any proper subset $\mathcal{O}$ of Ω, there exists a time $T_0 > 0$ and (u_0, τ_0), an initial condition of (1.3), in $V_0^s(\Omega) \times R_0^s$ for any $s \geq 1$, with compact support in Ω, such that the solution of system (1.3), for $\eta = 0$, with initial condition (u_0, τ_0) and with any control f localized in $\mathcal{O}$, acting only in the velocity equation, cannot reach zero at time T, for $0 < T < T_0$.*

However, for the multimode Maxwell model, the system is not null controllable at any time T using a localized interior control acting only on velocity. We have the following result.

Theorem 1.5. *Let $T > 0$ and $\mathcal{O}$ be a proper subset of Ω. There exists (u_0, τ_0), an initial condition of (1.5), in $V_0^s(\Omega) \times (R_0^s)^N$ for any $s \geq 1$, with compact support in Ω, such that the solution of system (1.5), for $\eta = 0$ and $N > 1$, with initial condition (u_0, τ_0) and with any control f localized in $\mathcal{O}$, acting only in the velocity equation, cannot reach zero at time T.*

Though Theorem 1.5 is similar to Theorem 1.3, the tools used to prove both theorems are completely different because of the parabolic behaviour of system (1.5) for $\eta > 0$, *i.e.*, regularity of its solution and infinite speed of propagation property, and hyperbolic behaviour of system (1.5) having no gain of regularity of its solution and finite speed of propagation property. To prove the nonnull controllability result for the Maxwell system, we exploit the finite speed of propagation property of the Maxwell system, *i.e.*, we choose an initial condition having a discontinuity outside the support of the control, and up to a certain time T_0, the domain of dependency of the solution in a neighbourhood of the discontinuity of the initial condition does not intersect the support of the control. Thus, the singularity of the initial condition persists in the solution up to time T_0. We prove this finite speed of propagation property using Holmgren's uniqueness theorem (see Thm. 4.10). Moreover, we show that there is a vertical characteristic of the system (see Lem. 4.9; this is where $N > 1$ is required), and we prove that for an appropriate initial condition, the singularity of the initial condition is on the vertical characteristic up to T_0 and then, using Hörmander's theorem (Thm. 4.6), we prove that the singularity propagates along the vertical characteristic for any time T and hence the solution cannot reach rest at any time $T > 0$.

We note that there are several different notions of controllability for PDE problems with memory. In early work on the subject, only the control of displacement (in the case of solids) and velocity is considered, see *e.g.* [10–14]; for hyperbolic PDEs with memory controllability is achieved by perturbation of the elastic case. Parabolic problems appear to be more subtle, since the incorporation of memory effects into Carleman inequalities is a nontrivial problem; we refer to [8, 24] for recent work showing both negative and positive results under certain situations. However, in a viscoelastic fluid the future evolution is not controlled to zero simply by controlling the velocity, and the focus of this paper is control of the full state of the system. Our negative result on exact null controllability is in this context, we make no claim about weaker notions of controllability such as control of only velocity. We note that an intermediate notion of "memory controllability" is used in [2, 15] where, in addition to the displacement and velocity, a memory integral is controlled to zero. This still falls short of controlling the future of the system. In this paper, we study full controllability of the state in the context of models where the relaxation function is a linear combination of exponentials. For a single exponential, such results are announced in [5], but without complete proofs. In [17], one-dimensional shear flows of multimode linear Maxwell and Jeffreys fluids are considered, with a distributed control localized on a subinterval. Exact controllability for single-mode Maxwell fluids and approximate controllability of multimode Maxwell and Jeffreys fluids are established. For Jeffreys or multimode Maxwell models, approximate controllability seems to be the best one can hope for, since observability estimates in any reasonable Sobolev norms cannot hold (see [21], Thm. 5). The result of the present paper confirms this.

A higher dimensional single mode Jeffreys fluid is considered in [4] and the approximate controllability result only for the velocity is obtained using a distributed control acting in the velocity equation. This result is not quite satisfactory because the controllability property of the stress is not proved. The case of a single mode Maxwell model with boundary or distributed control is considered in higher dimension in [1]. Under a geometric condition imposed on the controlled region along with an additional condition that the system is underdamped, an exact controllability result for both velocity and stress in a suitable space is established. In [3], a number of refinements of the results obtained in [1, 4] are obtained. The approximate controllability results for the single mode and several mode Jeffreys model for both velocity and stress tensor using an interior localized control acting only in the velocity equation are obtained in [3]. The exact controllability result for the single mode Maxwell model and the approximate controllability result for the several mode Maxwell model for both velocity and stress tensor are obtained at sufficiently large time using an interior, localized control acting only in velocity. In [16], these results are extended to Jeffreys and Maxwell models with infinitely many relaxation modes.

In view of Remark 1.1, control of the stress tensor of linear viscoelastic flows is possible in a suitable subspace, *i.e.* under the constraint that the stress tensor is the symmetric part of a gradient. This is a genuine obstacle to deriving the controllability result for the nonlinear viscoelastic system from the result obtained for the linear system, because the stress for the nonlinear system is not the symmetric part of a gradient. Even the characterization of the set of reachable states seems to be a very difficult problem on which only fragmentary results are known [18–20, 23].

In this paper, we establish that using an interior control in the velocity equation, which is localized in a fixed, proper subset of Ω, the system cannot be null controllable in any Sobolev space. In contrast, we anticipate that if the control has support in all of Ω or the support of the control is moving with respect to time as in [2, 15], the system can be null controllable. This problem is left for future research. Even for the simpler problem with control in all of Ω, the precise formulation of regularity conditions is likely to be subtle. For the one-dimensional case, an exact controllability result for the two-mode Maxwell model is shown in [17].

The paper is organized as follows. Section 2 is devoted to transforming the system (1.3) and (1.5), to more tractable systems using a change of variable and to show that if the transformed systems are not null controllable, (1.3) and (1.5) are also not so. For the well-posedness of the Maxwell and Jeffreys system, in either their original or transformed form, we refer to our earlier work on approximate controllability [16]. In Section 3, we prove that the transformed system for $\eta > 0$ is not null controllable and derive Theorem 1.2 from the result. Section 4 is devoted to the Maxwell system (1.5), for $\eta = 0$, and Theorem 1.5 is proved.

1.1. Notation

Throughout this paper, we shall use the notation of curl in both two and three dimensions. Let Ω be a bounded domain in $\mathbb{R}^d$, $d = 2, 3$. When $d = 2$, we define the *curl* operators as follows:

$$\begin{aligned} \operatorname{curl} u &= \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \qquad && \text{for any } u \in \mathcal{D}'(\Omega)^2 \\ \operatorname{curl} \varphi &= \left(\frac{\partial \varphi}{\partial x_2}, -\frac{\partial \varphi}{\partial x_1} \right) \qquad && \text{for any } \varphi \in \mathcal{D}'(\Omega). \end{aligned}$$

That is, the curl of a vector is a scalar, and the curl of a scalar is a vector. When $d = 3$ we define $\operatorname{curl} u = \nabla \times u$ for any $u \in \mathcal{D}'(\Omega)^3$. With the above notation, it is easy to see that,

$$\operatorname{curl} \operatorname{curl} u = -\Delta u + \nabla(\operatorname{div} u), \qquad d = 2, 3,$$

(see for instance [7], Chap. 1, Sect. 2.3).

2. Change of variables

As mentioned in Remark 1.1, we can expect controllability of (1.3) only if we assume that the viscoelastic stress τ is of the form $2Dv$, where v is a divergence free vector field vanishing on the boundary. We therefore assume that

$$\tau = 2Dv,$$

for some $v \in \mathbb{R}^d$ satisfying

$$\nabla \cdot v = 0 \text{ in } \Omega \text{ and } v = 0 \text{ on } \partial\Omega.$$

Then clearly $(1.3)_3$ is satisfied if

$$\begin{aligned} &\partial_t v + \lambda v = \kappa u, \quad \nabla \cdot v = 0 \quad \text{in } \Omega \times (0,T), \\ &v = 0 \quad \text{in } \partial\Omega \times (0,T), \quad v(\cdot,0) = v_0. \end{aligned}$$

Therefore we study the following controlled system for (u, v)

$$\begin{array}{l} \rho\partial_t u \ = \ \eta\Delta u + \Delta v - \nabla p + f \quad \text{in } \Omega \times (0,T), \\ \partial_t v + \lambda v = \kappa u \quad \text{in } \Omega \times (0,T), \\ \nabla \cdot u = 0, \quad \nabla \cdot v = 0 \quad \text{in } \Omega \times (0,T), \\ u = 0, \quad v = 0 \quad \text{in } \partial\Omega \times (0,T), \\ u(\cdot,0) = u_0, \quad v(\cdot,0) = v_0 \quad \text{in } \Omega. \end{array} \tag{2.1}$$

In this system, the equations $\nabla \cdot v = 0$ and $v = 0$ on $\partial\Omega$ should be viewed as constraints on the initial data; they automatically persist in time since $v_t + \lambda v = \kappa u$. To go back from (2.1) to (1.3), we simply set $\tau = 2Dv$ and apply D to the equation $v_t + \lambda v = \kappa u$.

In the following proposition, we prove the connection between null controllability of system (1.3) and system (2.1).

Proposition 2.1. *If the system (1.3) is null controllable in $V_n^0 \times R_0$ or $V_0^s \times R_0^s$ for any $s \geq 1$, by an interior control f at any time $T > 0$, the system (2.1) is null controllable in $V_n^0 \times V_0^1$ or $V_0^s \times V_0^{s+1}$ for any $s \geq 1$, at time T by the interior control f.*

Proof. Let (1.3) be null controllable in $V_0^s \times R_0^s$ at time T by an interior control f, *i.e.*,

$$(u, \tau)(x, T) = 0 \text{ for all } x \in \Omega.$$

Using the form of $\tau = 2Dv$, we get that $Dv(\cdot, T) = 0$. Since the kernel of D in the space V_0^1 is the zero vector, from it, we can derive that $v(\cdot, T) = 0$ in V_0^1. Hence, system (2.1) is null controllable in $V_n^0 \times V_0^1$ at time T. The result for $V_0^s \times R_0^s$, for any $s \geq 1$, follows in a similar way. □

In view of Proposition 2.1, to prove Theorem 1.2, we show the following result.

Theorem 2.2. *For any time $T > 0$ and any $s \geq 1$, there exists an initial condition of (2.1) for $\eta > 0$, in $V_0^s \times V_0^{s+1}$ such that the solution of (2.1) for $\eta > 0$ cannot be brought to zero at any time T, by any control f, localized in a fixed domain $\mathcal{O}$.*

Similarly, to study the controllability of multimode visco-elastic flows (1.5), we set for all $i = 1, \dots, N$, $\tau_i = 2Dv_i$, where v_i is also a divergence free vector field vanishing on the boundary and we consider system for $(u, (v_i)_{i=1}^N) \in \mathbb{R}^{d(N+1)}$

$$\begin{array}{l} \rho\partial_t u \ = \ \eta\Delta u + \displaystyle\sum_{i=1}^{N} \Delta v_i - \nabla p + f \quad \text{in } \Omega \times (0,T), \\ \partial_t v_i + \lambda_i v_i = \kappa_i u \quad i = 1, \dots, N, \quad \text{in } \Omega \times (0,T), \\ \nabla \cdot u = 0, \quad \nabla \cdot v_i = 0 \quad i = 1, \dots, N, \quad \text{in } \Omega \times (0,T), \\ u = 0, \quad v_i = 0 \quad i = 1, \dots, N, \quad \text{in } \partial\Omega \times (0,T), \\ u(\cdot,0) = u_0, \quad v_i(\cdot,0) = v_{i,0} \quad i = 1, \dots, N, \quad \text{in } \Omega. \end{array} \tag{2.2}$$

To prove Theorem 1.3, it is enough to prove the analogous version of Theorem 2.2 for the system (2.2) with $\eta > 0$. However, the argument to prove Theorem 1.3 for the Jeffreys model with several relaxation modes is completely analogous to that of the single mode case and hence, the proof of the theorem for the multimode case will be omitted.

Now for $\eta = 0$, to prove Theorem 1.5, we show the following result for (2.2).

Theorem 2.3. *Let $N > 1$. For any time $T > 0$ and any $s \geq 1$, there exists an initial condition of (2.2) for $\eta = 0$, in $V_0^s \times (V_0^{s+1})^N$ such that the solution of (2.2) for $\eta = 0$ cannot be brought to zero at any time T, by any control f, localized in a fixed domain $\mathcal{O}$, a proper subset of Ω.*

Note that to prove Theorem 1.4, it is enough to show the result analogous to the above theorem, for (2.1) with $\eta = 0$, at time T, where $0 < T < T_0$, for some T_0. The proof of the result can be derived from the first part of the arguments used in the proof of Theorem 2.3 and hence the details of Theorem 1.4 will be omitted.

3. Jeffreys system

In this section, we prove the result on lack of null controllability for the Jeffreys system (1.3) for $\eta > 0$. To do that, as mentioned in Section 2, we prove Theorem 2.2. Note that, the wellposedness of (2.1) on $V_n^0(\Omega) \times V_0^1(\Omega)$ can be derived showing that the linear operator associated to (2.1) denoted by $(A, D(A))$ as defined in (2.4) in [3] generates a C_0 semigroup on $V_n^0(\Omega) \times V_0^1(\Omega)$ (see [3], Rem. 2.1). As a byproducet of the arguments used in the next subsections to prove the lack of null controllability of (2.1), the wellposedness of the system is also obtained.

3.1. Formulation of the problem

To prove Theorem 2.2, we split the solution of (2.1) into a part resulting from the control, and another part resulting from the initial condition, as follows:

$$u = \widetilde{u} + \sigma, \quad p = \widetilde{p} + q, \quad v = \widetilde{v} + \phi,$$

where $(\widetilde{u}, \widetilde{p}, \widetilde{v})$ satisfies

$$\begin{aligned}
&\rho \widetilde{u}_t \ = \ \eta\Delta\widetilde{u} + \Delta\widetilde{v} - \nabla\widetilde{p} + f \quad \text{in } \Omega \times (0,T),\\
&\widetilde{v}_t + \lambda\widetilde{v} = \kappa\widetilde{u} \quad \text{in } \Omega \times (0,T),\\
&\nabla \cdot \widetilde{u} = 0, \quad \nabla \cdot \widetilde{v} = 0 \quad \text{in } \Omega \times (0,T),\\
&\widetilde{u} = 0, \quad \widetilde{v} = 0 \quad \text{in } \partial\Omega \times (0,T),\\
&\widetilde{u}(\cdot,0) = 0, \quad \widetilde{v}(\cdot,0) = 0 \quad \text{in } \Omega,
\end{aligned} \tag{3.1}$$

where the control f is localized in $\mathcal{O}$, a proper subset of Ω, and (σ, q, ϕ) satisfies

$$\begin{aligned}
&\rho\sigma_t \ = \ \eta\Delta\sigma + \Delta\phi - \nabla q \quad \text{in } \Omega \times (0,T),\\
&\phi_t + \lambda\phi = \kappa\sigma \quad \text{in } \Omega \times (0,T),\\
&\nabla \cdot \sigma = 0, \quad \nabla \cdot \phi = 0 \quad \text{in } \Omega \times (0,T),\\
&\sigma = 0, \quad \phi = 0 \quad \text{in } \partial\Omega \times (0,T),\\
&\sigma(\cdot,0) = u_0, \quad \phi(\cdot,0) = v_0 \quad \text{in } \Omega.
\end{aligned} \tag{3.2}$$

The first step to prove lack of null controllability of (2.1) is to show that $\widetilde{u}(\cdot,T)$ in (3.1) is C^∞ in a neighborhood of x_0 for any $x_0 \in \Omega\backslash\overline{\mathcal{O}}$ and for any $T > 0$. Then we show that for a suitable choice of (u_0, v_0), $\sigma(\cdot,T)$ in (3.2) is not C^∞ in any neighborhood of a x_0, for some $x_0 \in \Omega\backslash\overline{\mathcal{O}}$. Hence, $\widetilde{u} + \sigma$ cannot reach zero at any time T. More precisely, we prove the following two theorems:

Theorem 3.1. *Let $\mathcal{O}$ be a proper subset of Ω and $T>0$. Let us also assume that $f \in L^2(0,T;L^2(\Omega))$ with* $\mathrm{supp} f \subset \mathcal{O}$. *Then, for any $x_0 \in \Omega\backslash\overline{\mathcal{O}}$, there exists a $r_0>0$ such that $(\widetilde{u},\widetilde{p},\widetilde{v})$, the solution of (3.1), is $C^\infty(B_{r_0/2}(x_0)\times[0,T])$, where $B_{r_0}(x_0)$ is a ball with radius r_0 and the center at x_0.*

Theorem 3.2. *Let $\mathcal{O}$ be a proper subset of Ω, $x_0 \in \Omega \setminus \overline{\mathcal{O}}$, $s \geq 1$ and $T>0$. Then there exists $(u_0,v_0) \in V_0^s(\Omega)\times V_0^{s+1}(\Omega)$ such that, $\sigma(\cdot,T)$ is not C^∞ in any neighbourhood around x_0, where (σ,q,ϕ) is the solution of (3.2).*

Once we prove the above two theorems, the proof of Theorem 2.2 is obvious. Therefore, in the rest of this section we prove Theorem 3.1 and Theorem 3.2.

3.2. Study of the system (3.1)

In this subsection, our aim is to prove Theorem 3.1. For this, let us consider the following auxiliary system.

$$\begin{aligned}
&\rho a_t \;=\; \eta\Delta a + \Delta b + F \quad \text{in } \Omega\times(0,T),\\
&b_t + \lambda b = \kappa a \quad \text{in } \Omega\times(0,T),\\
&a = 0, \quad b = 0 \quad \text{in } \partial\Omega\times(0,T),\\
&a(\cdot,0) = 0, \quad b(\cdot,0) = 0 \quad \text{in } \Omega.
\end{aligned} \tag{3.3}$$

Let us set $Q_T = \Omega\times(0,T)$. For any $s \geq 0$, we introduce the following space

$$H^{s,s/2}(Q_T) = L^2(0,T;H^s(\Omega)) \cap H^{s/2}(0,T;L^2(\Omega)). \tag{3.4}$$

Theorem 3.3. *Let $m \in \mathbb{N}\cup\{0\}$. Let us assume that*

$$\frac{d^kF}{dt^k} \in L^2(0,T;H^{2m-2k}(\Omega)), \quad k = 0,1,\dots m,$$

satisfying

$$F(\cdot,0) = \cdots = \frac{d^{m-1}F}{dt^{m-1}}(\cdot,0) = 0, \tag{3.5}$$

if $m>0$. Then the system (3.3) admits a unique strong solution

$$a,b \in H^{2m+2,m+1}(Q_T).$$

Proof. We prove the theorem in the following steps:

Step 1. We consider the underlying base space as $L^2(\Omega)\times(H_0^1(\Omega)\cap H^2(\Omega))$ and the linearized operator associated to (3.3),

$$A_1(a,b)^T = \left(\frac{1}{\rho}\Delta(\eta a + b), \kappa a - \lambda b\right)^T, \quad D(A_1) = (H_0^1(\Omega)\cap H^2(\Omega))^2.$$

To prove that $(A_1, D(A_1))$ forms an analytic semigroup on $L^2(\Omega)\times(H_0^1(\Omega)\cap H^2(\Omega))$, we need the resolvent estimate

$$\|a\|_{L^2} + \|b\|_{H^2} \leq \frac{C}{|\mu|}(\|f\|_{L_2} + \|g\|_{H^2}),$$

where

$$f = \mu a - \frac{1}{\rho}\Delta(\eta a + b), \quad g = (\mu + \lambda)b - \kappa a,$$

for $\mu \in \mathbb{C}$ and $\Re\mu > 0$. The resolvent estimate is easily obtained by eliminating b to obtain the single equation

$$\mu a = f + \frac{1}{\rho(\mu+\lambda)}\Delta g + \frac{1}{\rho}\left(\eta + \frac{\kappa}{\mu+\lambda}\right)\Delta a.$$

Thus, for any given $F \in L^2(0,T;L^2(\Omega))$, using the regularity result for an analytic semigroup, we have

$$a \in H^{2,1}(Q_T), \quad b \in H^1(0,T;H^2(\Omega)).$$

Step 2. Let us now consider $m = 1$, *i.e.* we assume that $F \in H^{2,1}(Q_T)$ and $F(\cdot,0) = 0$. Setting, $\widetilde{a} = a_t$ and $\widetilde{b} = b_t$, we get that $(\widetilde{a},\widetilde{b})$ satisfies the following system

$$\begin{array}{l} \rho\widetilde{a}_t \;=\; \eta\Delta\widetilde{a} + \Delta\widetilde{b} + F_t \quad \text{in } \Omega\times(0,T), \\ \widetilde{b}_t + \lambda\widetilde{b} = \kappa\widetilde{a} \quad \text{in } \Omega\times(0,T), \\ \widetilde{a} = 0, \quad \widetilde{b} = 0 \quad \text{in } \partial\Omega\times(0,T), \\ \widetilde{a}(\cdot,0) = 0, \quad \widetilde{b}(\cdot,0) = 0 \quad \text{in } \Omega. \end{array} \tag{3.6}$$

Since $F_t \in L^2(0,T;L^2(\Omega))$ we obtain

$$\widetilde{a} \in H^{2,1}(Q_T), \quad \widetilde{b} \in H^1(0,T;H^2(\Omega)).$$

In particular

$$a \in H^1(0,T;H^2(\Omega)) \cap H^2(0,T;L^2(\Omega)), \quad b \in H^2(0,T;H^2(\Omega)).$$

Now we write for *a.e.* $0 \le t \le T$,

$$-\Delta(\eta a(\cdot,t) + b(\cdot,t)) = (F(\cdot,t) - \rho a_t(\cdot,t)) \text{ in } \Omega, \quad \eta a(\cdot,t) + b(\cdot,t) = 0 \text{ on } \partial\Omega.$$

Since F and a_t belong to $L^2(0,T;H^2(\Omega))$, by the standard elliptic regularity theory, we obtain $\eta a + b \in L^2(0,T;H^4(\Omega))$. Note that $(3.3)_2$ can be written as

$$b_t + (\lambda + \kappa/\eta)b = h \quad t \in [0,T], \qquad b(0) = 0,$$

where $h = \kappa(a + b/\eta)$. It is easy to see that $h = \kappa(a + b/\eta) \in H^{4,2}(Q_T)$. We have

$$b(x,t) = \int_0^t e^{-(\kappa/\eta+\lambda)(t-s)} h(x,s)\, ds. \tag{3.7}$$

It yields

$$\|b(\cdot,t)\|^2_{H^4(\Omega)} \le C\left(\int_0^t e^{-2(\lambda+\kappa/\eta)(t-s)}\, ds\right)\left(\int_0^t \|h(\cdot,s)\|^2_{H^4(\Omega)}\, ds\right) \le C\|h\|^2_{L^2(0,T;H^4(\Omega))},$$

for some generic positive constant C, depending on T. Therefore $b \in L^2(0,T;H^4(\Omega))$ and hence $a \in L^2(0,T;H^4(\Omega))$. The proof for larger m proceeds by the method of induction. □

As a corollary of the above theorem we obtain the following result

Corollary 3.4. *Let k be any odd integer. Let us assume that $F \in H^{k,k/2}(Q_T)$ satisfying*

$$F(\cdot,0) = \cdots = \frac{\mathrm{d}^j F}{\mathrm{d}t^j}(\cdot,0) = 0 \text{ for } j \leq (k-1)/2.$$

Then (3.3) has a unique solution $(a,b) \in (H^{2+k,1+k/2}(Q_T))^2$ satisfying the following estimate

$$\|a\|_{H^{2+k,1+k/2}(Q_T)} + \|b\|_{H^{2+k,1+k/2}(Q_T)} \leq C\|F\|_{H^{k,k/2}(Q_T)}. \tag{3.8}$$

Proof. The proof follows from interpolation. □

Now we are in a position to prove the regularity result for the system (3.1).

Proof of Theorem 3.1. We prove the theorem in the following three steps.

Step 1. Let $(A,D(A))$ be the linear operator associated to (3.1) on the base space $V_n^0(\Omega) \times V_0^2(\Omega)$, where $D(A) = (V_0^2(\Omega))^2$. In a similar fashion as for the operator A_1, arising in the proof of Theorem 3.3 above, we can show that $(A,D(A))$ generates an analytic semigroup on $V_n^0(\Omega) \times V_0^2(\Omega)$ and hence for any $f \in L^2(0,T;L^2(\mathcal{O}))$, $(\widetilde{u},\widetilde{p},\widetilde{v})$, the solution of (3.1), belongs to $H^{2,1}(Q_T) \times L^2(0,T;H_m^1(\Omega)) \times H^{2,1}(Q_T)$, where

$$H_m^1(\Omega) = \{p \in H^1(\Omega) \mid \int_\Omega p(x)\,\mathrm{d}x = 0\}.$$

Step 2. Now, let us take any point $x_0 \in \Omega\backslash\overline{\mathcal{O}}$ and choose a radius $r_0 > 0$ such that $B_{2r_0}(x_0) \subset \Omega\backslash\overline{\mathcal{O}}$.

Let $(\varepsilon_k)_{k\geq 0}$ be a sequence of positive numbers satisfying the following properties

$$\varepsilon_{k+1} < \varepsilon_k, \quad r_0 + \varepsilon_k < 2r_0 \text{ and } \lim_{k\to\infty} \varepsilon_k = 0.$$

Let $(\chi_k)_{k\geq 0}$ denotes a sequence of smooth cut-off function with the following properties

$$\begin{aligned}
&\text{supp } \chi_0 \subset B_{2r_0}(x_0) \text{ and } \chi_0 \equiv 1 \text{ in } B_{r_0+\varepsilon_0}(x_0),\\
&\text{supp } \chi_k \subset B_{r_0+\varepsilon_{k-1}}(x_0) \text{ and } \chi_k \equiv 1 \text{ in } B_{r_0+\varepsilon_k}(x_0) \text{ for } k \geq 1.
\end{aligned}$$

For all $k \geq 0$, we set

$$a^k = \chi_k \operatorname{curl} \widetilde{u}, \quad b^k = \chi_k \operatorname{curl} \widetilde{v}.$$

We also set

$$\begin{aligned}
&\Omega^0 = B_{2r_0}(x_0), \quad \Omega^k = B_{r_0+\varepsilon_{k-1}}(x_0) \text{ for } k \geq 1,\\
&Q_T^k = (0,T) \times \Omega^k \text{ for } k \geq 0.
\end{aligned}$$

It is easy to see that (a^k, b^k) satisfies the following system

$$\begin{array}{l} \rho a^k_t \;=\; \eta\Delta a^k + \Delta b^k + F^k \quad \text{in } (0,T)\times\Omega^k, \\ b^k_t + \lambda b^k = \kappa a^k \quad \text{in } (0,T)\times\Omega^k, \\ a^k = 0, \quad b^k = 0 \quad \text{in } (0,T)\times\partial\Omega^k, \\ a^k(\cdot,0) = 0, \quad b^k(\cdot,0) = 0 \quad \text{in } \Omega^k, \end{array} \tag{3.9}$$

where

$$F^k = -2\nabla\chi_k(\eta(\nabla \text{curl}\, \widetilde{u})^T + (\nabla \text{curl}\, \widetilde{v})^T) - \Delta\chi_k(\eta \text{curl}\, \widetilde{u} + \text{curl}\, \widetilde{v}).$$

We claim that $(a^k, b^k) \in (H^{k+2,(k+2)/2}(Q^k_T))^2$ for all $k \geq 0$. We already have $(\widetilde{u}, \widetilde{v}) \in (H^{2,1}(Q_T))^2$. This implies that $F^0 \in L^2(0,T;L^2(\Omega^0))$. Using Theorem 3.3 and Corollary 3.4, we obtain $(a^0, b^0) \in (H^{2,1}(Q^0_T))^2$. Let us assume that our claim is true for some $k > 0$. Then we have $(a^k, b^k) \in (H^{k+2,(k+2)/2}(Q^k_T))^2$. In particular, $(\text{curl}\, u, \text{curl}\, v) \in (H^{k+2,(k+2)/2}(Q^{k+1}_T))^2$. Thus $\nabla \text{curl}\, u \in H^{k+1,(k+1)/2}(Q^{k+1}_T)$ and $F^{k+1} \in H^{k+1,(k+1)/2}(Q^{k+1}_T)$. Therefore by Theorem 3.3 and Corollary 3.4, we get $(a^{k+1}, b^{k+1}) \in (H^{k+3,(k+3)/2}(Q^k_T))^2$ and the claim is proved.

From the definition of χ_k, we get

$$(\text{curl}\, u, \text{curl}\, v) \in (L^2(0,T;H^{k+2}(B_{r_0}(x_0))) \cap H^{k+1/2}(0,T;L^2(B_{r_0}(x_0))))^2 \text{ for all } k \geq 0.$$

Step 3. To derive the regularity of $(\widetilde{u}, \widetilde{v})$ at $B_{r_0}(x_0)$, we use the result on interior regularity of the elliptic equation for each $t \in (0,T)$,

$$\begin{aligned} -\Delta(\widetilde{u}(\cdot,t)) \; &= \text{curl}\,(\text{curl}\,(\widetilde{u}(\cdot,t)) \quad \text{in } B_{r_0}(x_0), \\ -\Delta(\widetilde{v}(\cdot,t) \; &= \text{curl}\,(\text{curl}\,(\widetilde{v}(\cdot,t))) \quad \text{in } B_{r_0}(x_0), \end{aligned}$$

where the regularity of the right hand side is obtained in the first part of the proof and it is $C^\infty(B_{r_0}(x_0) \times (0,T))$. □

3.3. Study of system (3.2)

In this subsection, we prove Theorem 3.2. To do that, we first consider the following system on $\mathbb{R}^d \times (0,T)$,

$$\begin{aligned} &\rho a_t \;=\; \eta\Delta a + \Delta b \quad \text{in } \mathbb{R}^d \times (0,T), \\ &b_t + \lambda b = \kappa a \quad \text{in } \mathbb{R}^d \times (0,T), \\ &a(\cdot,0) = \text{curl}\,(1_d(I-\Delta)^{-k}\delta_{x_0}), \quad b(\cdot,0) = b_0 \quad \text{in } \mathbb{R}^d, \quad \forall\, k \geq 2. \end{aligned} \tag{3.10}$$

Here 1_d is a unit vector in $\mathbb{R}^d$. We note that for $d = 2$, the curl of a vector field can be identified with a scalar (see the comment at the end of the introduction), so a and b are also scalars. Likewise, we shall identify the cross product ot two two-dimensional vectors with a scalar.

In the next theorem, we show that for a suitably given initial condition, the solution of (3.10) cannot be smooth on a neighbourhood of x_0 but it is smooth outside.

Theorem 3.5. *Let $T > 0$ and $x_0 \in \Omega$. Let us choose $a_0 = \text{curl}\,(1_d(I-\Delta)^{-k}\delta_{x_0})$, where $k \geq 2$. Then for a suitable b_0, chosen later, the solution of (3.10) is not C^∞ in $B_r(x_0) \times (0,T)$, where $B_r(x_0)$ is a ball around x_0 with radius r, for any $r > 0$.*

Further, the solution is C^∞ outside $B_r(x_0)$.

Proof. Using Fourier transform, we obtain the following ODE

$$\begin{aligned} \rho\hat{a}_t &= -|\xi|^2(\eta\hat{a}+\hat{b}) \quad \text{in } \mathbb{R}^d\times(0,T),\\ &\hat{b}_t+\lambda\hat{b}=\kappa\hat{a} \quad \text{in } \mathbb{R}^d\times(0,T),\\ &\hat{a}(\xi,0)=(i\xi\times 1_d)\tfrac{e^{-ix_0\cdot\xi}}{(1+|\xi|^2)^k}=\hat{a}_0(\xi), \quad \hat{b}(\xi,0)=\hat{b}_0(\xi) \quad \text{in } \mathbb{R}^d. \end{aligned}$$

Denoting the matrix in the above ODE by $\widehat{A}(\xi)$, the solution of the above system can be written as

$$(\hat{a},\hat{b})^T = e^{t\widehat{A}(\xi)}(\hat{a}_0,\hat{b}_0)^T, \tag{3.11}$$

We can see that the eigenvalues of $\widehat{A}(\xi)$ are

$$\begin{aligned} \mu^+(\xi) &= \frac{-(\lambda\rho+\eta|\xi|^2)-\sqrt{(\lambda\rho+\eta|\xi|^2)^2-4\rho(\lambda\eta+\kappa)|\xi|^2}}{2\rho}\\ \mu^-(\xi) &= \frac{-(\lambda\rho+\eta|\xi|^2)+\sqrt{(\lambda\rho+\eta|\xi|^2)^2-4\rho(\lambda\eta+\kappa)|\xi|^2}}{2\rho} \end{aligned}$$

and there exists $\xi_0>0$, such that for all $|\xi|\geq\xi_0$, $\mu^+(\xi)$ and $\mu^-(\xi)$ are all real and distinct. The eigenvalues satisfy

$$\lim_{|\xi|\mapsto\infty}\frac{-\mu^+(\xi)}{|\xi|^2}=\frac{\eta}{\rho}, \quad \lim_{|\xi|\mapsto\infty}-\mu^-(\xi)=\lambda+\frac{\kappa}{\eta}. \tag{3.12}$$

The eigenfunction of $\widehat{A}(\xi)$ corresponding to $\mu^-(\xi)$ is $(1,C_{\mu^-}(\xi))$, for $d=2$, and $(1_d,1_dC_{\mu^-}(\xi))$, for $d=3$, where

$$C_{\mu^-}(\xi)=\frac{\rho\mu^-(\xi)+\eta|\xi|^2}{-|\xi|^2}=\frac{\kappa}{\lambda+\mu^-(\xi)} \tag{3.13}$$

Our aim is to choose $\hat{b}_0(\xi)=\hat{a}_0(\xi)C_{\mu^-}(\xi)$ so that $(\hat{a}_0(\xi),\hat{b}_0(\xi))$ is an eigenfunction of $\hat{A}(\xi)$ corresponding to $\mu^-(\xi)$ and hence

$$(\hat{a},\hat{b})^T=e^{\mu^-(\xi)t}(\hat{a}_0(\xi),C_{\mu^-}(\xi)\hat{a}_0(\xi))^T.$$

Now, using the inverse Fourier transform, we get

$$a(x,t)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}(i\xi\times 1_d)\frac{e^{i(x-x_0).\xi}}{(1+|\xi|^2)^k}e^{\mu^-(\xi)t}\,d\xi,$$

and similarly, we get the expression for b.

To get an explicit representation of $a(\cdot,t)$, from the expression of $\mu^-(\xi)$, we note that

$$\mu^-(\xi)=-(\lambda+\kappa/\eta)+P((1+|\xi|^2)^{-1}),$$

and P can be represented as a regular Taylor series for large $|\xi|$. Thus $e^{\mu^-(\xi)t}$ can be written in the form

$$e^{\mu^-(\xi)t}=C_0(t)+\sum_{j=1}^{\infty}C_j(t)(1+|\xi|^2)^{-j},$$

where $C_j(\cdot)$ is $C^\infty([0,T])$ for all $j \in \mathbb{N}$, and hence after taking the inverse Fourier transform

$$a(\cdot,t) = C_0(t)(I-\Delta)^{-k}\operatorname{curl}(1_d\delta_{x_0}) + \sum_{j=1}^{\infty} C_j(t)(I-\Delta)^{-k-j}\operatorname{curl}(1_d\delta_{x_0}).$$

Note that each term is two orders more regular than its previous term. Thus, $a(\cdot,\cdot)$ is as smooth as $C_0(\cdot)(I-\Delta)^{-k}\operatorname{curl}(1_d\delta_{x_0})$ is.

It is clear that $(I-\Delta)^{-k}\operatorname{curl}(1_d\delta_{x_0})$ is not a smooth function in $\mathbb{R}^d$, specifically it is not smooth at the point x_0. It can be shown that it belongs to H^{2k-m-1} for all $m > d/2$, $d = 2,3$. Moreover, it follows from elliptic regularity that $a(\cdot,t)$ is C^∞ everywhere except the point x_0 for all $t \in [0,T]$. □

Remark 3.6. Note that in Theorem 3.5, we choose $a_0 = \operatorname{curl}(1_d(I-\Delta)^{-k}\delta_{x_0})$, for $k \geq 2$ and we construct b_0 by taking the inverse Fourier transform of $\hat{b}_0$, where

$$\hat{b}_0 = \hat{a}_0(\xi)C_{\mu^-}(\xi),$$

and $C_{\mu^-}(\xi)$ is given in (3.13). Thus, b_0 is also obtained as the curl of a vector field and in particular, we have

$$a_0 = \operatorname{curl} h_0, \quad b_0 = \operatorname{curl} g_0,$$

where $h_0 = 1_d(I-\Delta)^{-k}\delta_{x_0}$, and g_0 can be obtained by taking the inverse Fourier transform of $\hat{h}_0(\cdot)C_{\mu^-}(\cdot)$. Here $\hat{h}_0(\cdot)$ is the Fourier transform of $h_0(\cdot)$.

We also note that for $d = 3$,

$$\operatorname{div} a = 0, \quad \operatorname{div} b = 0,$$

where (a,b) is the solution of (3.10) with the initial condition (a_0,b_0), chosen above. This holds because on the whole space the divergence free condition is automatically satisfied as long as it holds for the initial conditions.

Next lemma gives how to reconstruct the initial condition (σ_0,ϕ_0) from (a_0,b_0), where (a_0,b_0) is chosen in Theorem 3.5.

Lemma 3.7. *Let $x_0 \in \Omega\backslash\overline{\mathcal{O}}$, $h_0 \in (H^s(\mathbb{R}^d))^d$, for $s \geq 1$ and $a_0 = \operatorname{curl} h_0$. Then there exists $\sigma_0 \in V_0^s(\Omega)$ such that*

$$\operatorname{curl}\sigma_0 = a_0 \text{ in a neighbourhood of} x_0. \tag{3.14}$$

Proof. Let $\mathbb{P}$ be the Leray projector from $(L^2(\Omega))^d$ onto $V_n^0(\Omega)$. Since $\mathbb{P}h_0$ is divergence free, there exists ψ_0, at least locally in some neighborhood of x_0, such that

$$\mathbb{P}h_0 = \operatorname{curl}\psi_0,$$

(see for instance Thm. 3.1 and Thm. 3.4 of [7]). We also have

$$\operatorname{curl} h_0 = \operatorname{curl}\mathbb{P}h_0,$$

since the difference $h_0 - \mathbb{P}h_0$ is a gradient. Let χ be a smooth cutoff function which vanishes near $\partial\Omega$ but equals 1 in $B_r(x_0)$, for some $r > 0$ such that $B_r(x_0) \subset \Omega\backslash\overline{\mathcal{O}}$. Setting

$$\sigma_0 = \operatorname{curl}(\chi\psi_0),$$

we have $\sigma_0 \in V_0^s(\Omega)$, and, on $B_r(x_0)$,

$$\operatorname{curl} \sigma_0 = \operatorname{curl} h_0.$$

□

We are now in a position to prove Theorem 3.2:

Proof of Theorem 3.2. We consider system (3.2). First, for any $s \geq 1$, we need to reconstruct the initial condition $(\sigma_0, \phi_0) \in V_0^s(\Omega) \times V_0^{s+1}(\Omega)$ from (a_0, b_0), where (a_0, b_0) is chosen in Theorem 3.5. Let x_0 be a point in $\Omega \backslash \overline{\mathcal{O}}$, and let (a_0, b_0), the initial data of (3.10) on $\mathbb{R}^d$ be as in Theorem 3.5. As mentioned in Remark 3.6, by our choice in Theorem 3.5, we have $a_0 = \operatorname{curl} h_0$, $b_0 = \operatorname{curl} g_0$. Note that, choosing k suitably large enough in h_0, we can make $(h_0, g_0) \in H^s(\Omega) \times H^{s+1}(\Omega)$. Using Lemma 3.7, there exists $(\sigma_0, \phi_0) \in V_0^s(\Omega) \times V_0^{s+1}(\Omega)$, such that on $B_r(x_0) \subset \Omega \backslash \overline{\mathcal{O}}$, for some $r > 0$,

$$\operatorname{curl} \sigma_0 = \operatorname{curl} (\chi \psi_0) = \operatorname{curl} h_0,$$

and, in an analogous fashion,

$$\operatorname{curl} \phi_0 = \operatorname{curl} (\chi \zeta_0) = \operatorname{curl} g_0.$$

We now consider the following system on $\Omega \times (0, T)$

$$\begin{aligned}
&\rho \sigma_t \; = \; \eta \Delta \sigma + \Delta \phi - \nabla q \quad \text{in } \Omega \times (0,T), \\
&\phi_t + \lambda \phi = \kappa \sigma \quad \text{in } \Omega \times (0,T), \\
&\nabla \cdot \sigma = 0, \quad \nabla \cdot \phi = 0 \quad \text{in } \Omega \times (0,T), \\
&\sigma(\cdot, t) = 0, \quad \phi(\cdot, t) = 0 \quad \text{on } \partial\Omega,\, t \in (0,T), \\
&\sigma(\cdot, 0) = \operatorname{curl} (\chi \psi_0), \quad \phi(\cdot, 0) = \operatorname{curl} (\chi \zeta_0) \quad \text{in } \Omega.
\end{aligned} \tag{3.15}$$

Now, note that as mentioned in Remark 3.6, for $d = 3$, (a, b), the solution of (3.10) for the initial condition (a_0, b_0), satisfies

$$\operatorname{div} a = \operatorname{div} b = 0, \quad \text{in} \quad \Omega.$$

Setting

$$\widetilde{\sigma}(x,t) = \operatorname{curl} \sigma(x,t) - a(x,t), \quad \widetilde{\phi}(x,t) = \operatorname{curl} \phi(x,t) - b(x,t), \qquad \text{for all } (x,t) \in \Omega \times (0,T).$$

we get that $(\widetilde{\sigma}, \widetilde{\phi})$ satisfies the following system:

$$\begin{aligned}
&\rho \widetilde{\sigma}_t \; = \; \eta \Delta \widetilde{\sigma} + \Delta \widetilde{\phi}, \quad \text{in } B_r(x_0) \times (0,T), \\
&\widetilde{\phi}_t + \lambda \widetilde{\phi} = \kappa \widetilde{\sigma}, \quad \text{in } B_r(x_0) \times (0,T), \\
&\nabla \cdot \widetilde{\sigma} = 0, \quad \nabla \cdot \widetilde{\phi} = 0 \quad \text{in } B_r(x_0) \times (0,T), \quad \text{for} \quad d = 3, \\
&\widetilde{\sigma}(\cdot, t) = a(\cdot, t), \quad \widetilde{\phi}(\cdot, t) = b(\cdot, t) \quad \text{on } \partial B_r(x_0),\, t \in (0,T), \\
&\widetilde{\sigma}(x, 0) = 0, \quad \widetilde{\phi}(x, 0) = 0 \quad \text{in } B_r(x_0).
\end{aligned} \tag{3.16}$$

Now, to conclude that $\widetilde{\sigma}$, and $\widetilde{\phi}$ are C^∞ in a neighborhood of the point x_0, we proceed as in Step 2 and Step 3 of the proof of Theorem 3.1 setting

$$\widetilde{a}^k = \chi_k \, \widetilde{\sigma}, \quad \widetilde{b}^k = \chi_k \, \widetilde{\phi}, \quad \forall \, k \geq 0,$$

where χ_k is a smooth cut-off function defined similarly to that introduced in Theorem 3.1. Note that, since we want to derive the interior regularity of the solution of (3.16) inside $B_r(x_0)$, the nonhomogeneous boundary data in (3.16) do not pose any difficulty to apply the arguments used in Steps 2 and 3 of the proof of Theorem 3.1 to (3.16). Thus, $\widetilde{\sigma}$ is C^∞ in a neighborhood of x_0. But, since $a(\cdot,\cdot)$ has a discontinuity at x_0, $\operatorname{curl}\sigma$ and hence σ cannot be C^∞ in any neighborhood of x_0. □

Thus, combining Theorems 3.1 and 3.2, Theorem 2.2 and Theorem 1.2 follow.

4. Maxwell system

In this section, we consider the following system on $\Omega\times(0,T)$ for a multi-mode Maxwell model with control f having support in $\mathcal{O}$, any proper open subset of Ω:

$$\begin{aligned}
&\rho u_t \;=\; \sum_{i=1}^N \nabla\cdot\tau_i - \nabla p + f \quad \text{in } \Omega\times(0,T),\\
&\nabla\cdot u = 0 \quad \text{in } \Omega\times(0,T),\\
&(\tau_i)_t + \lambda_i\tau_i = 2\kappa_i Du \quad \text{in } \Omega\times(0,T),\ i=1,\dots,N\\
&u = 0 \quad \text{in } \partial\Omega\times(0,T),\\
&u(\cdot,0) = u_0, \quad \tau_i(\cdot,0) = (\tau_0)_i, \quad i=1,\dots,N \quad \text{in } \Omega.
\end{aligned} \tag{4.1}$$

As described in Section 2, to obtain the lack of null controllability result for (4.1), it is enough to study the null controllability of the following system for $(u,p,(v_i)^N)\in\mathbb{R}^{d(N+1)}$:

$$\begin{aligned}
&\rho u_t \;=\; \sum_{i=1}^N \Delta v_i - \nabla p + f \quad \text{in } \Omega\times(0,T),\\
&(v_i)_t + \lambda_i v_i = \kappa_i u \quad \text{in } \Omega\times(0,T), \quad i=1,\dots,N,\\
&\nabla\cdot u = 0, \quad \nabla\cdot v_i = 0, \quad i=1,\dots,N, \quad \text{in } \Omega\times(0,T),\\
&u = 0, \quad v_i = 0, \quad i=1,\dots,N \quad \text{in } \partial\Omega\times(0,T),\\
&u(\cdot,0) = u_0, \quad v_i(\cdot,0) = (v_0)_i, \quad i=1,\dots,N \quad \text{in } \Omega,
\end{aligned} \tag{4.2}$$

where f has support in $\mathcal{O}$.

In the next lemma, we give the wellposedness of the above system in $V_n^0(\Omega)\times(V_0^1(\Omega))^N$ for any $f\in L^2(0,T;L^2(\mathcal{O}))$. The wellposedness of the system in $V_0^s(\Omega)\times(V^{s+1}(\Omega))^N$ can be derived similarly for suitable regular control f. To get the result on the wellposedness of the above system, as introduced in (2.5) of [3], we denote the Stokes operator $(A_0,D(A_0))$ on $V_n^0(\Omega)$ by

$$A_0 u = -\mathbb{P}u, \quad D(A_0) = H^2(\Omega)\times H_0^1(\Omega),$$

where $\mathbb{P}$ is the Leray projector, and from (5.4) of [3], we recall that the linear operator $(A,D(A))$ on $V_n^0(\Omega)\times(V_0^1(\Omega))^N$ associated to the above system is

$$\begin{aligned}
&D(A) = \{(u,(v_i)_{i=1}^N)\in(V_0^1(\Omega))^{N+1} \mid \textstyle\sum_{i=1}^N v_i\in D(A_0)\},\\
&A(u,(v_i)_{i=1}^N)^T = (A_0\textstyle\sum_{i=1}^N v_i, (\frac{\kappa_i}{\rho}u-\lambda_i v_i)_{i=1}^N)^T, \quad (u,(v_i)_{i=1}^N)\in D(A).
\end{aligned}$$

Lemma 4.1. *The operator associated to (4.2), $(A,D(A))$ generates a C_0-semigroup on $V_n^0(\Omega)\times(V_0^1(\Omega))^N$. For $f\in L^2(0,T;L^2(\Omega))$, (4.2) is wellposed in $V_n^0(\Omega)\times(V_0^1(\Omega))^N$.*

Proof. We consider $V_n^0(\Omega) \times (V_0^1(\Omega))^N$ with following weighted inner product:

$$\langle (u,(v_i)_{i=1}^N)^T, (\sigma,(\phi_i)_{i=1}^N)^T \rangle_{V_n^0(\Omega)\times(V_0^1(\Omega))^N} = \int_\Omega u\sigma \,\mathrm{d}x + \sum_{i=1}^N \frac{\rho}{\kappa_i}(\nabla v_i . \nabla \phi_i)\,\mathrm{d}x. \tag{4.3}$$

To prove $(A, D(A))$ generates a C_0-semigroup on $V_n^0(\Omega) \times (V_0^1(\Omega))^N$, we use the Lumer-Phillips theorem. By calculating with respect to the inner product (4.3), we get

$$\langle A(u,(v_i)_{i=1}^N)^T, (u,(v_i)_{i=1}^N)^T \rangle_{V_n^0(\Omega)\times(V_0^1(\Omega))^N} < 0, \quad \forall\, (u,(v_i)_{i=1}^N) \in D(A).$$

To show the maximality of $(A, D(A))$, we need to show that for a $\mu > 0$ and for any $(h,(g_i)_{i=1}^N) \in V_n^0(\Omega) \times (V_0^1(\Omega))^N$, there exists a unique $(u,(v_i)_{i=1}^N) \in D(A)$ such that following holds:

$$\begin{aligned} &\mu u + A_0 \textstyle\sum_{i=1}^N v_i = f, \\ &(\mu+\lambda_i)v_i - \tfrac{\kappa_i}{\rho} u = g_i, \quad \forall\, i = 1,\ldots,N. \end{aligned} \tag{4.4}$$

Now, reducing a single equation for u, we get

$$\mu u + \sum_{i=1}^N \frac{\kappa_i}{\rho(\mu+\lambda_i)} A_0 u = f - \frac{A_0 g_i}{\mu+\lambda_i} \in V^{-1}(\Omega), \tag{4.5}$$

and using the property of the Stokes operator $(A_0, V_0^1(\Omega))$ on $V^{-1}(\Omega)$, we get that there exists a unique $u \in V_0^1(\Omega)$ satisfying (4.5). Thus, from (4.4), we derive that for all $i = 1,\ldots,N$, $v_i \in V_0^1(\Omega)$ and $\sum_{i=1}^N v_i \in D(A_0)$. Hence the result is proved. □

Our aim of this section is to prove Theorem 2.3 and hence Theorem 1.5 by giving a choice of initial conditions for (4.2) with discontinuity in a neighborhood of a point in $\Omega \backslash \overline{\mathcal{O}}$ and showing that this discontinuity will propagate through the solution at any time T. Hence the solution cannot reach zero at any time T.

4.1. Formulation of the problem

Here, we first derive a more tractable system applying curl $(\cdot)$ to system (4.2) to eliminate the pressure term from the system.

We define the scalar $(\operatorname{curl} g)_1$ in the following way

$$(\operatorname{curl} g)_1 = \begin{cases} \operatorname{curl} g & \text{if } d = 2 \\ \text{first component of } \operatorname{curl} g & \text{if } d = 3. \end{cases} \tag{4.6}$$

Setting

$$a_0 = (\operatorname{curl} u_0)_1 \quad \text{and} \quad b_{i,0} = (\operatorname{curl} v_{i,0})_1, \quad i = 1,2,\cdots,N, \tag{4.7}$$

and

$$a = (\operatorname{curl} u)_1 \quad \text{and} \quad b_i = (\operatorname{curl} v_i)_1, \quad i = 1,2,\ldots,N, \tag{4.8}$$

where $(u,(v_i)_{i=1}^N)$ is the solution of (4.2) with initial condition $(u_0,(v_{i,0})_{i=1}^N)$ and control f having support in $\mathcal{O}$, a proper subset of Ω, we see that $(a,(b_i)_{i=1}^N)$ satisfies

$$\begin{aligned} \rho a_t &= \sum_{i=1}^N \Delta b_i + (\operatorname{curl} f)_1 \quad \text{in } \Omega\times(0,T),\\ &(b_i)_t + \lambda_i b_i = \kappa_i a \quad \text{in } \Omega\times(0,T), \quad i=1,\dots,N,\\ &a(\cdot,0)=a_0, \quad b_i(\cdot,0)=b_{i,0}, \quad i=1,\dots,N, \quad \text{in } \Omega. \end{aligned} \tag{4.9}$$

Further, if $(u,(v_i)_{i=1}^N)$, the solution of (4.2), satisfies $(u,(v_i)_{i=1}^N)(\cdot,T)=0$ in Ω for any $T>0$, then we have $(a,(b_i)_{i=1}^N)(\cdot,T)=0$ in Ω. Thus in the sequel, we shall prove the property of the propagation of singularities for system (4.9). Note that (4.2) is a well-posed system, and $(a,(b_i)_{i=1}^N)$, defined in (4.8), for $(u,(v_i)_{i=1}^N)$, the solution of (4.2), is a solution of (4.9). We never solve (4.9) directly in Ω; indeed we are missing a boundary condition for it which would be needed to formulate a well-posed problem. Therefore, when we say $(a,(b_i)_{i=1}^N)$ satisfies the system (4.9), it always means that $(a,(b_i)_{i=1}^N)$ is defined by (4.8) using the solution of (4.2).

For the rest of this section, we focus on the multimode case. For the case of single mode, see Remark 4.14.

Let us now briefly describe the steps to show a lack of null controllability result for (4.2). As mentioned earlier, our aim is to construct particular initial data $(u_0,(v_{i,0})_{i=1}^N)$ so that a, defined as in (4.8), has a singularity outside the support of control f. This is done in the following four steps:

Step 1. At first, we study the property of the propagation of singularities for a system similar to (4.9) in the whole space $\mathbb{R}^d$. More precisely, we consider the following system

$$\begin{aligned} \rho\widetilde{a}_t &= \textstyle\sum_{i=1}^N \Delta\widetilde{b}_i, \ \text{in } \mathbb{R}^d\times(0,T),\\ &(\widetilde{b}_i)_t + \lambda_i\widetilde{b}_i = \kappa_i\widetilde{a} \quad \text{in } \mathbb{R}^d\times(0,T), \quad i=1,\dots,N,\\ &\widetilde{a}(\cdot,0)=\widetilde{a}_0, \quad \widetilde{b}_i(\cdot,0)=\widetilde{b}_{i,0}, \quad i=1,\dots,N, \quad \text{in } \mathbb{R}^d, \end{aligned} \tag{4.10}$$

where $\widetilde{a}(x,t)\in\mathbb{R}$ and $\widetilde{b}_i(x,t)\in\mathbb{R}$. We will construct $(\widetilde{a}_0,(\widetilde{b}_{i,0})_{i=1}^N)$ having a singularity at $x_0\in\Omega\setminus\overline{\mathcal{O}}$, and show that $(\widetilde{a}(\cdot,t),(\widetilde{b}_i(\cdot,t))_{i=1}^N)$ has a singularity at x_0 for any $t>0$ (see Prop. 4.7).

Step 2. Next, we show that if $(a_0,(b_{i,0})_{i=1}^N)$ agrees with $(\widetilde{a}_0,(\widetilde{b}_{i,0})_{i=1}^N)$ in a small neighbourhood of x_0, then there is a region in $\Omega\times[0,T)$ containing $(x_0,0)$ such that $(a,(b_i)_{i=1}^N)$ coincides with $(\widetilde{a},(\widetilde{b}_i)_{i=1}^N)$ in that region (Thm. 4.10). This is proved using an extension of Holmgren's theorem (see [9], Thm. 8.6.8).

With the above uniqueness result, we then conclude that $a(\cdot,t)$, defined in (4.8), has a singularity at x_0 for $t\le T_0$ and for some $T_0>0$ (Cor. 4.11), provided $(u_0,(v_{i,0})_{i=1}^N)$ is such that $(a_0,(b_{i,0})_{i=1}^N)$ agrees with $(\widetilde{a}_0,(\widetilde{b}_{i,0})_{i=1}^N)$ in a small neighbourhood of x_0.

Step 3. At this step, we show that the singularity of a propagates for all $t>0$ (Thm. 4.12). This is proved using 'Hörmander theorem' (see [9], Thm. 8.3.3′) or a generalization of it which allows for multiple characteristics.

Step 4. In the final step, we show the existence of $(u_0,(v_{i,0})_{i=1}^N)\in V_0^s\times(V_0^{s+1})^N$ so that $(a_0,(b_{i,0})_{i=1}^N)$ agrees with $(\widetilde{a}_0,(\widetilde{b}_{i,0})_{i=1}^N)$ around a small neighbourhood of x_0 (Lem. 4.13).

4.2. Basic tools of microlocal analysis

In this subsection, we first recall some preliminaries and standard tools which are useful to study the propagation of the singularity of solutions of PDEs. We also recall Theorem 8.6.8 from [9] and Theorem 8.3.3′ from [9], which are the main ingredients of our proof.

Let us consider the expression $P(x,D)$ defined by

$$P(x,D) = \sum_{|\alpha|\leq m} C_\alpha(x) D^\alpha,$$

where

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_d^{\alpha_d}}, \quad \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d), \quad |\alpha| = \sum_{j=1}^{d} \alpha_j,$$

and $C_\alpha(\cdot)$ is the coefficient of D^α.

Definition 4.2 (Principal symbol)**.** The symbol of the expression $P(x,D)$ is defined by

$$P(x, i\xi) = \sum_{|\alpha|\leq m} C_\alpha(x)(i\xi)^\alpha, \quad \forall\, \xi \in \mathbb{R}^d,$$

where $\xi^\alpha = \xi_1^{\alpha_1}\xi_2^{\alpha_2}\cdots\xi_d^{\alpha_d}$. Moreover, the principal part of the symbol is

$$p_m(x, i\xi) = \sum_{|\alpha|= m} C_\alpha(x)(i\xi)^\alpha, \quad \forall\, \xi \in \mathbb{R}^d.$$

Definition 4.3 (Characteristic)**.** A characteristic surface of $P(x,D) = \sum_{|\alpha|\leq m} C_\alpha(x)D^\alpha$ at $x \in \mathbb{R}^d$ is a C^1 surface $S \subset \mathbb{R}^d$ with normal ξ at a point $x \in S$ satisfying $p_m(x, i\xi) = 0$.

We recall Theorem 8.6.8 from [9], which is an extension of Holmgren's uniqueness theorem.

Theorem 4.4. *Let X_1 and X_2 be two open convex sets in $\mathbb{R}^{d+1}$ such that $X_1 \subset X_2$. Let $P(D)$ be a differential operator with constant coefficients. Then the following are equivalent:*

1. *Every $u \in \mathcal{D}'(X_2)$ satisfying $P(D)u = 0$ in X_2 and vanishing in X_1 must also vanish in X_2.*
2. *Every hyperplane which is characteristic for $P(D)$ and intersects X_2 also intersects X_1.*

Next, we recall the definition of wave front set of any distribution $u \in \mathcal{D}'(X)$, where X is an open subset of $\mathbb{R}^d \times (0,T)$. For any $u \in \mathcal{D}'(\mathbb{R}^{d+1})$, $\Sigma(u)$ is a subset of $\mathbb{R}^{d+1}\backslash\{0\}$ containing $(\xi,\tau) \in \mathbb{R}^{d+1}\backslash\{0\}$, such that there is no conic neighbourhood V of (ξ,τ) in $\mathbb{R}^{d+1}$ satisfying

$$|\mathcal{F}(u)(\zeta,\theta)| \leq C_m(1 + |(\zeta,\theta)|)^{-m}, \quad \forall\,(\zeta,\theta) \in V,$$

for all $m \in \mathbb{N}$ and some positive constant C_m, where $\mathcal{F}$ is the Fourier transform in $\mathbb{R}^{d+1}$.

For any $u \in \mathcal{D}'(X)$, where X is any open set of $\mathbb{R}^d \times (0,T)$, we define for all $(x,t) \in X$,

$$\Sigma_{(x,t)}(u) = \bigcap_{\phi\in C_c^\infty(X),\phi(x,t)\neq 0} \Sigma(\phi u).$$

We recall Definition 8.12 from [9].

Definition 4.5. Let $u \in \mathcal{D}'(X)$, where X is any open set of $\mathbb{R}^d \times (0,T)$. Then the closed subset of $X \times (\mathbb{R}^d\backslash\{0\})$, defined by

$$WF(u) = \{((x,t),(\xi,\tau)) \in X \times (\mathbb{R}^d\backslash\{0\}) \mid (\xi,\tau) \in \Sigma_{(x,t)}(u)\}$$

is called the wave front set of u. The projection of $WF(u)$ onto X is the singular support of u.

We recall Theorem 8.3.3′, page 278 in [9].

Theorem 4.6. *Let $P(D)$ be of real principal type and principal symbol p_m. If $u \in \mathcal{D}'(\Omega \times (0,T))$, $P(D)u = f$ and $((x_0,t_0),(\xi,\tau)) \in WF(u)\backslash WF(f)$, then $p_m(i\xi, i\tau) = 0$ and $I \times \{(\xi,\tau)\} \subset WF(u)$, provided $I \subset \Omega \times (0,T)$ is a line segment containing (x_0,t_0) with direction $\nabla p_m(i\xi,i\tau)$ such that $I \times \{(\xi,\tau)\}$ does not meet $WF(f)$.*

We note that our definition of the principal symbol above differs from that of [9] by a power of i.

4.3. Lack of null controllability for multimode Maxwell system

In this subsection, we give the precise statements and proofs that we mentioned in Section 4.1. First we prove the following proposition:

Proposition 4.7. *Let $x_0 \in \Omega\backslash\overline{\mathcal{O}}$ and any $T > 0$. Let $\widetilde{a}_0 = \dfrac{\partial}{\partial x_2}(I-\Delta)^{-k}\delta_{x_0}$ (the derivative with respect to x_2 is included because $\widetilde{a}_0$ should be the first component of a curl), and $(\widetilde{b}_{i,0})_{i=1}^N$ be chosen suitably. Then, we have that $(x_0,t,\xi,0) \in WF(\widetilde{a})$, for all $t \in (0,T)$ and $\xi \in \mathbb{R}^d$ with $\xi_2 \neq 0$, where $(\widetilde{a},(\widetilde{b}_i)_{i=1}^N)$ is the solution of (4.10).*

Proof. We recall (4.10) in $\mathbb{R}^d \times (0,T)$,

$$\begin{aligned}
\rho\widetilde{a}_t \;&=\; \sum_{j=1}^N \Delta\widetilde{b}_j \quad \text{in } \mathbb{R}^d \times (0,T),\\
(\widetilde{b}_j)_t + \lambda_j\widetilde{b}_j &= \kappa_j\widetilde{a} \quad \text{in } \mathbb{R}^d \times (0,T), \quad j = 1,\dots,N,\\
\widetilde{a}(\cdot,0) = a_0, &\quad \widetilde{b}_j(\cdot,0) = b_{j,0}, \quad i = 1,\dots,N, \quad \text{in } \mathbb{R}^d.
\end{aligned}$$

Now using Fourier transform, we obtain the following Ode

$$\begin{aligned}
&\rho\hat{a}_t(\xi,t) = -|\xi|^2\sum_{j=1}^N \hat{b}_j(\xi,t) \quad \text{in } \mathbb{R}^d \times (0,T),\\
&(\hat{b}_j)_t(\xi,t) + \lambda_j\hat{b}_j(\xi,t) = \kappa_j\hat{a}(\xi,t), \quad j = 1,\dots,N \quad \text{in } \mathbb{R}^d \times (0,T),\\
&\hat{a}(\xi,0) = \hat{a}_0(\xi) = i\xi_2\tfrac{e^{-ix_0\cdot\xi}}{(1+|\xi|^2)^k}, \quad \hat{b}_j(\xi,0) = \hat{b}_{j,0}(\xi) \quad \text{in } \mathbb{R}^d.
\end{aligned}$$

Denoting the matrix in the above ODE by $\widehat{A}(\xi)$, the solution of the above system can be written as

$$(\hat{a},(\hat{b}_j)_{j=1}^N)^T(\xi,t) = e^{\widehat{A}(\xi)t}(\hat{a}_0,(\hat{b}_{j,0})_{j=1}^N)^T(\xi), \quad \forall\,\xi \in \mathbb{R}^d, \quad \forall\,t > 0. \tag{4.11}$$

We can show that $\{\mu_l(\xi)\}_{l=1}^{N+1}$, the eigenvalues of $\widehat{A}(\xi)$, for all $\xi \in \mathbb{R}^d$, satisfy

$$\mu(\xi) = -\frac{1}{\rho}|\xi|^2\sum_{j=1}^N \frac{\kappa_j}{\mu(\xi)+\lambda_j}, \quad \forall\,\xi \in \mathbb{R}^d. \tag{4.12}$$

Note that, for all $\xi \in \mathbb{R}^d$, there always is a root of (4.12), denoted by $\mu_1(\xi)$, between $-\lambda_2$ and $-\lambda_1$. Moreover, we can write the equation (4.12) in the form $F(\mu, 1/|\xi|^2) = 0$ with F analytic and use the implicit function theorem to show that $\mu_1(\xi)$ asymptotically satisfies

$$\mu_1(\xi) = \omega_0 + \frac{M_1}{|\xi|^2} + \frac{M_2}{|\xi|^4} + \cdots, \quad \text{as} \quad |\xi| \to \infty, \tag{4.13}$$

where ω_0 is a solution of $\sum_{j=1}^{N} \frac{\kappa_j}{\mu+\lambda_j} = 0$ and for all $k \in \mathbb{N}$, M_k is some constant.

The eigenfunction of $\widehat{A}(\xi)$ corresponding to $\mu_1(\xi)$ is $(1, C_j(\xi))_{j=1}^N)$, where

$$C_j(\xi) = \frac{\kappa_j}{\mu_1(\xi) + \lambda_j}, \quad \forall\, j = 1, \ldots, N. \tag{4.14}$$

We choose

$$\hat{b}_{j,0}(\xi) = i\xi_2 \frac{e^{-ix_0.\xi}}{(1+|\xi|^2)^k} C_j(\xi), \quad \forall \quad j = 1, \ldots, N,$$

so that $(\hat{a}_0(\xi), (\hat{b}_{j,0}(\xi))_{j=1}^N)$ is an eigenfunction of $\widehat{A}(\xi)$ corresponding to $\mu_1(\xi)$ and hence for all $t > 0$,

$$(\hat{a}, (\hat{b}_j)_{j=1}^N)^T(\xi, t) = e^{\mu_1(\xi)t} i\xi_2 \frac{e^{-ix_0\xi}}{(1+|\xi|^2)^k} (1, (C_j(\xi))_{j=1}^N)^T, \quad \forall\, \xi \in \mathbb{R}^d. \tag{4.15}$$

To prove $((x_0, t_0, \xi, 0) \in WF(\widetilde{a})$ for all $t_0 \in [0, T)$ and $\xi \in \mathbb{R}^d \backslash \{0\}$, it is enough to show that for any $\psi \in C_c^\infty(S)$ with $\psi(x_0, t_0) \neq 0$ and $\psi \geq 0$, the Fourier transform of $(\psi\widetilde{a})$ in R^{d+1} satisfies

$$|\mathcal{F}(\psi\widetilde{a})(\xi, 0)| > \frac{C|\xi_2|}{(1+|\xi|^2)^k}, \tag{4.16}$$

for some positive constant C, depending on ψ, and for $|\xi| \to \infty$ along any ray where $\xi_2 \neq 0$. The fact that it suffices to consider positive ψ follows from Lemma 8.1.1, page 253 in [9].

We have

$$\begin{aligned} \mathcal{F}(\psi\widetilde{a})(\xi, 0) &= \int_0^T \int_{\mathbb{R}^d} e^{-ix.\xi} \widetilde{a}(x,t)\psi(x,t)\, \mathrm{d}x\, \mathrm{d}t \\ &= \textstyle\int_0^T \psi(x_0,t) \int_{\mathbb{R}^d} e^{-ix.\xi} \widetilde{a}(x,t)\, \mathrm{d}x\, \mathrm{d}t + \int_0^T \int_{\mathbb{R}^d} e^{-ix.\xi} \widetilde{\psi}(x,t) \widetilde{a}(x,t)\, \mathrm{d}x\, \mathrm{d}t \\ &\geq \textstyle |\int_0^T \psi(x_0,t) \int_{\mathbb{R}^d} e^{-ix.\xi} \widetilde{a}(x,t)\, \mathrm{d}x\, \mathrm{d}t| - |\int_0^T \int_{\mathbb{R}^d} e^{-ix.\xi} \widetilde{\psi}(x,t) \widetilde{a}(x,t)\, \mathrm{d}x\, \mathrm{d}t|, \end{aligned} \tag{4.17}$$

where $\widetilde{\psi}(x,t) = \psi(x,t) - \psi(x_0,t)$. To estimate a lower bound of $\mathcal{F}(\psi\widetilde{a})(\xi, 0)$, the first term in the RHS of (4.17) can be estimated as follows:

$$|\int_0^T \psi(x_0,t) \int_{\mathbb{R}^d} e^{-ix.\xi} \widetilde{a}(x,t)\, \mathrm{d}x\, \mathrm{d}t| = \Big(\int_0^T \psi(x_0,t) \exp(\mu_1(\xi)t)\, \mathrm{d}t\Big) |\hat{a}_0(\xi)|.$$

Moreover, we have $|\hat{a}_0(\xi)| \geq C|\xi_2|/(1+|\xi|^2)^k$, and

$$\int_0^T \psi(x_0,t) \exp(\mu_1(\xi)t)\, \mathrm{d}t = \int_0^T \psi(x_0,t) \exp(\omega_0 t)\, \mathrm{d}t + O(1/|\xi|^2),$$

and the first term on the right hand side is positive. For $|\xi| \to \infty$ along any ray where $\xi_2 \neq 0$, we thus obtain a lower bound of the first term in the RHS of (4.17) which is of order $|\xi|^{1-2k}$, For the second term in (4.17), we find

$$\int_0^T \int_{\mathbb{R}^d} e^{-ix.\xi}(\psi(x,t)-\psi(x_0,t))\widetilde{a}(x,t)\,\mathrm{d}x\,\mathrm{d}t$$
$$= \frac{1}{(2\pi)^d}\Big(\int_0^T \int_{\mathbb{R}^d} \hat{a}(\xi-\zeta,t)\hat{\psi}(\zeta,t)\,\mathrm{d}\zeta\,\mathrm{d}t - \int_0^T \hat{a}(\xi,t)\int_{\mathbb{R}^d} \hat{\psi}(\zeta,t)e^{i\zeta.x_0}\,\mathrm{d}\zeta\,\mathrm{d}t\Big).$$

It is convenient to write

$$\hat{a}(\zeta,t) = e^{-ix_0\cdot\zeta}p(\zeta,t), \quad \hat{\psi}(\zeta,t) = e^{-ix_0\cdot\zeta}q(\zeta,t).$$

With this, the second term in (4.17) becomes

$$\frac{1}{(2\pi)^d}e^{-i\xi.x_0}\int_0^T \int_{\mathbb{R}^d}(p(\xi-\zeta,t)-p(\xi,t))q(\zeta,t)\,\mathrm{d}\zeta\,\mathrm{d}t. \tag{4.18}$$

For fixed t, we further investigate

$$\int_{\mathbb{R}^d}(p(\xi-\zeta,t)-p(\xi,t))q(\zeta,t)\,\mathrm{d}\zeta$$
$$= \int_{|\zeta|\le\epsilon|\xi|}(p(\xi-\zeta,t)-p(\xi,t))q(\zeta,t)\,\mathrm{d}\zeta + \int_{|\zeta|\ge\epsilon|\xi|}(p(\xi-\zeta,t)-p(\xi,t))q(\zeta,t)\,\mathrm{d}\zeta.$$

As $|\xi|\to\infty$ along any ray where $\xi_2\neq 0$, $|p(\xi,t)|$ is proportional to $|\xi|^{1-2k}$, and for $|\zeta|\le\epsilon|\xi|$ and ϵ sufficiently small, $|p(\xi-\zeta,t)-p(\xi,t)|$ is bounded by a constant times $\epsilon|\xi|^{1-2k}$. Since q is integrable, this bounds the first integral by a constant times $\epsilon|\xi|^{1-2k}$. On the other hand, $|q(\zeta)|$ is bounded by a constant time $|\zeta|^{-m}$ for any choice of m, which bounds the second integral by a constant time $(\epsilon|\xi|)^{d-m}$. ☐

Remark 4.8. Note that in Proposition 4.7, $\widetilde{a}_0\in H^{2k-m-1}$, where $m>d/2$ and $d=2,3$. For all $j=1,\dots,N$, we obtain $\widetilde{b}_{j,0}$ by taking the inverse Fourier transform of $\hat{b}_{j,0}$ and in the above proposition $\hat{b}_{j,0}$ is chosen as

$$\hat{b}_{j,0}(\xi) = i\xi_2\frac{e^{-ix_0.\xi}}{(1+|\xi|^2)^k}C_j(\xi), \quad \forall \quad j=1,\dots,N,$$

where $C_j(\xi)$ is defined in (4.14). Thus, it also follows that for all $j=1,\cdots,N$, $\widetilde{b}_{j,0}$ belongs to H^{2k-m-1} for $m>d/2$ and $d=2,3$ and $\widetilde{b}_{j,0}=\frac{\partial g_{j,0}}{\partial x_2}$, for some $g_{j,0}\in H^{2k-m}$.

We also note that for a given $s\ge 1$, choosing k large enough we can make $(\widetilde{a}_0,((\widetilde{b}_{j,0})_{j=1}^N)$ in $H^s\times H^{s+1}$.

Next we want to make a connection between $(a,(b_j)_{j=1}^N)$ and $(\widetilde{a},(\widetilde{b}_j)_{j=1}^N)$ where $(a,(b_j)_{j=1}^N)$ is defined in (4.8) and $(\widetilde{a},(\widetilde{b}_j)_{j=1}^N)$ is the solution of (4.10). As mentioned earlier, we want to use Theorem 4.4. To do it, we reduce the system satisfied by $(a,(b_j)_{j=1}^N)$ into a single equation for a. Applying $\prod_{j=1}^N(\partial_t+\lambda_j I)$ to $(4.9)_1$ and using $(4.9)_2$, we get

$$\begin{aligned}\rho\prod_{j=1}^N(\partial_t+\lambda_j I)a_t &= \sum_{i=1}^N\kappa_i\prod_{j\neq i}(\partial_t+\lambda_j I)\Delta a+\prod_{j=1}^N(\partial_t+\lambda_j I)(\operatorname{curl} f)_1 \quad \text{in } \Omega\times(0,T),\\ a(\cdot,0)=a_0,\quad \partial_t a(\cdot,0) &:= a_1 = \frac{1}{\rho}\Delta\sum_{i=1}^N b_{i,0}+(\operatorname{curl} f)_1(\cdot,0) \text{ in } \Omega,\end{aligned} \tag{4.19}$$

and the initial conditions for higher derivatives are calculated recursively from (4.9) in a similar fashion. Note that the terms involving $(\operatorname{curl} f)_1(\cdot,0)$ in initial conditions of (4.19) vanish in $\Omega\backslash\overline{\mathcal{O}}$ because f has support inside $\mathcal{O}$. Since for us, it is enough to consider (4.19) in a neighborhood of x_0, in $\Omega\backslash\overline{\mathcal{O}}$, we consider (4.19) on $\Omega\backslash\overline{\mathcal{O}}\times(0,T)$ and hence we do not need to assume any regularity for $(\operatorname{curl} f)_1$.

We introduce the linear differential operator defined by

$$P(D)a = \rho\prod_{j=1}^{N}(\partial_t+\lambda_j I)a_t \;-\; \sum_{i=1}^{N}\kappa_i\prod_{j\neq i}(\partial_t+\lambda_j I)\Delta a. \tag{4.20}$$

With the above notation, $(4.19)_1$ can be written as

$$P(D)a = F, \tag{4.21}$$

where

$$F = \prod_{j=1}^{N}(\partial_t+\lambda_j I)(\operatorname{curl} f)_1. \tag{4.22}$$

Note that in $(\Omega\setminus\overline{\mathcal{O}})\times(0,T)$, $P(D)a=0$ in the sense of distribution. From Definition 4.2, we get the symbol for $P(D)$, defined in (4.20), as

$$P(i\xi,i\tau) = \rho\prod_{j=1}^{N}(i\tau+\lambda_j)(i\tau) \;+\; \sum_{m=1}^{N}\kappa_m\prod_{j\neq m}(i\tau+\lambda_j)|\xi|^2, \tag{4.23}$$

and the principal part of the symbol is

$$p_{N+1}(i\xi,i\tau) = (i\tau)^{N-1}\left(-\rho\tau^2+\sum_{m=1}^{N}\kappa_m|\xi|^2\right). \tag{4.24}$$

Now we derive the equations for the characteristic surfaces for $P(D)$ (see Def. 4.3).

Lemma 4.9. *Let $P(D)$ be defined by (4.20). Then the characteristic surfaces for $P(D)$ are given by*

$$x\cdot\frac{\xi}{|\xi|}+\widetilde{\eta}t = C_1,\quad x\cdot\frac{\xi}{|\xi|}-\widetilde{\eta}t = C_2,\qquad x\cdot\xi = C_3,\quad \forall\, x\in\mathbb{R}^d,\, t\in\mathbb{R}, \tag{4.25}$$

for any $\xi\in\mathbb{R}^d$, where $\widetilde{\eta}=\sqrt{\sum_{j=1}^{N}\kappa_j/\rho}$.

We prove the following theorem

Theorem 4.10. *Let us consider the region in $\Omega\times(0,T)$,*

$$S = \{(x,t)\mid 0<t<T_0,\quad |x-x_0|<\widetilde{\eta}(T_0-t)\},$$

for $\widetilde{\eta}=\sqrt{\sum_{j=1}^{N}\kappa_j/\rho}$, a cone at the center (x_0,T_0), where $x_0\in\Omega\backslash\overline{\mathcal{O}}$ and T_0 is chosen such that

$$S_0 := \{x\in\Omega\mid |x-x_0|\le\widetilde{\eta}T_0\}\subset\Omega \text{ and } S_0\cap\mathcal{O} \text{ is empty.}$$

Let $(\widetilde{a}_0, (\widetilde{b}_{i,0})_{i=1}^N)$ be as in Proposition 4.7 and $(\widetilde{a}, (\widetilde{b}_i)_{i=1}^N)$ be the solution to the system (4.10) constructed in Proposition 4.7. Let us assume that $(u_0, (v_{i,0})_{i=1}^N) \in V_0^s \times (V_0^{s+1})^N$ be such that

$$a_0(x) = \widetilde{a}_0(x), \quad b_{i,0}(x) = \widetilde{b}_{i,0}(x),\ i = 1, 2, \dots, N,\ \ \textit{for all } x \in S_0, \tag{4.26}$$

where $(a_0, (b_{i,0})_{i=1}^N)$ is defined as in (4.7). Then

$$a(x, t) = \widetilde{a}(x, t) \textit{ for all } (x, t) \in S, \tag{4.27}$$

where $(a, (b_i)_{i=1}^N)$ is defined by (4.8).

Proof. We first note that $(\widetilde{a}, (\widetilde{b}_i)_{i=1}^N)$ is the solution of (4.10) if and only if $\widetilde{a}$ satisfies

$$\begin{aligned} &\rho \prod_{j=1}^N (\partial_t + \lambda_j I)\widetilde{a}_t \ = \ \sum_{i=1}^N \kappa_i \prod_{j\neq i} (\partial_t + \lambda_j I)\Delta\widetilde{a} \quad \text{in } \mathbb{R}^d \times (0, T), \\ &\widetilde{a}(\cdot, 0) = a_0, \quad \partial_t\widetilde{a}(\cdot, 0) = a_1 = \frac{1}{\rho}\Delta\sum_{i=1}^N b_{i,0}, \end{aligned} \tag{4.28}$$

and, recursively,

$$b_{i,m} = -\lambda_i b_{i,m-1} + \kappa_i a_{m-1}, \quad \partial_t^{m+1}\widetilde{a}(\cdot, 0) = a_{m+1} = \frac{1}{\rho}\Delta\sum_{i=1}^N b_{i,m}, \quad m = 1, \dots, N-1. \tag{4.29}$$

Similarly, if $(a, (b_i)_{i=1}^N)$ satisfies (4.8), 'a' satisfies (4.19).

Let us define

$$\begin{aligned} X_1 &= \{(x, t) \mid -T_0 < t < 0, \quad |x - x_0| < \widetilde{\eta}(t + T_0)\}, \\ X_2 &= \{(x, t) \mid -T_0 < t < T_0, \quad |x - x_0| < \widetilde{\eta}(T_0 - |t|)\}. \end{aligned} \tag{4.30}$$

It is easily checked that every characteristic plane which intersects X_2 also intersects X_1. We can define

$$\phi(x, t) = \begin{cases} \widetilde{a}(x, t) - a(x, t), & \text{if } (x, t) \in S, \\ 0, & \text{if } (x, t) \in X_1. \end{cases} \tag{4.31}$$

Since a and $\tilde{a}$ have identical initial conditions, it follows that ϕ is a solution of (4.19) in X_2. By Theorem 4.4, it follows that $a = \tilde{a}$ in S. □

A simple consequence of the above theorem and Proposition 4.7 is:

Corollary 4.11. *With the assumptions and notations in Theorem 4.10,*

$$(x_0, t, \xi, 0) \in WF(a), \text{ for all } t \in (0, T_0) \text{ and } \xi \in \mathbb{R}^d \text{ with } \xi_2 \neq 0.$$

Now we want to apply 4.6 to the pde (4.21). However, Theorem 4.6 applies to our problem only in the case $N = 2$. The reason is that, for $N > 2$, $\nabla p_{N+1}(i\xi, 0)$ is identically zero, due to the factor $(i\tau)^{N-1}$ appearing in p_{N+1}. However, a generalization to this case exists: it depends on the Levi condition. In our case this condition is met, due to the fact that no term in the symbol of P involves a power of ξ higher than the second, see (4.23). Instead of taking a line in the direction of ∇p_{N+1}, which is zero, we can then take a line in the t-direction. For a general statement of the theorem extending Theorem 4.6 to the case of multiple characteristics we refer to Theorem 5.3, Chapter 4 in [6].

Using Theorem 4.6 and its analogue for multiple characteristics, we derive the following result.

Theorem 4.12. *Let $\mathcal{O}$, the support of f, be a proper subset of Ω, $x_0 \in \Omega\backslash\overline{\mathcal{O}}$ and $T > 0$. Let us assume that $(x_0, t_0, \xi, 0) \in WF(a)$, for some $t_0 \in (0,T)$ and $\xi \in \mathbb{R}^d$ with $\xi_2 \neq 0$, where a satisfies (4.21). Let us consider the line segment I_T in $\Omega \times (0,T)$, defined by*

$$I_T = \{(x,t) \in \Omega \times (0,T) \mid x = x_0\}.$$

Then $(I_T \times (\xi, 0))$ is contained in $WF(a)$.

Proof. Let $x_0 \in \Omega\backslash\overline{\mathcal{O}}$ and T_0, as mentioned in Theorem 4.10. We also assume that $(x_0, t_0, \xi, 0) \in WF(a)$, for some $t_0 \in (0, T_0)$ and $x \in \mathbb{R}^d$ with $\xi_2 \neq 0$, where a satisfies (4.21). Note that for any $\xi \in \mathbb{R}^d$, and $\xi_2 \neq 0$, $((x_0,t_0),(\xi,0))$ does not belong to $WF(f)$, since (x_0, t_0) is outside of the support of f. Let us notice that for any $T > 0$, the line segment I_T, containing (x_0,t_0), never intersects $\mathcal{O} \times (0,T)$, the support of f and so, we can find a characteristic function χ_0 supported in a small neighbourhood of I_T and disjoint from $\mathcal{O} \times (0,T)$ such that $\chi_0 f = 0$. Thus, we can show that $(I_T \times (\xi,0)) \cap WF(f)$ is empty, for any $\xi \in \mathbb{R}^d$ and $\xi_2 \neq 0$. Now from Theorem 4.6 or the corresponding Theorem 5.3, Chapter 4 in [6], it follows that for any $T > 0$ and $\xi \in \mathbb{R}^d$, $(I_T \times (\xi, 0)) \subset WF(a)$, where a satisfies (4.21). □

Next we have the following lemma which helps us to reconstruct the initial data.

Lemma 4.13. *Let $x_0 \in \Omega$, $h_0 \in H^s(\mathbb{R}^d)$, $s \geq 1$ and $a_0 = \dfrac{\partial h_0}{\partial x_2}$. Then there exists $u_0 \in V_0^s(\Omega)$ such that*

$$(\operatorname{curl} u_0)_1 = a_0 \text{ in a neighbourhood of } x_0. \tag{4.32}$$

Proof. Let us define

$$H_0 = \begin{cases} (-h_0|_\Omega, 0) & \text{if } d = 2 \\ (0,0,h_0|_\Omega) & \text{if } d = 3 \end{cases}. \tag{4.33}$$

Then we get $H_0 \in (H^s(\Omega))^d$ for $s \geq 1$, and

$$(\operatorname{curl} H_0)_1(x) = a_0(x) \text{ for all } x \in \Omega.$$

Now, using Lemma 3.7, from this $H_0 \in (H^s(\Omega))^d$, we reconstruct $u_0 \in V_0^s(\Omega)$ satisfying (4.32). □

Now we are in a position to prove Theorem 2.3.

Proof of 2.3. We prove by contradiction. Let us assume that, for any initial conditions in $V_0^s \times (V_0^{s+1})^N$, there exists a control f with support in $\mathcal{O}$, a proper subset of Ω, such that the solution of system (4.2) satisfies

$$(u, (v_i)_{i=1}^N)(\cdot, T) = (0,0), \quad \text{in} \quad \Omega.$$

Next, we extend the system (4.2) on $\Omega \times (T, 2T)$ with the control $f = 0$ and the solution of the system is still zero on the time interval $[T, 2T)$. In particular,

$$a(\cdot,t) = b_i(\cdot,t) = 0, \quad \text{in} \quad \Omega, \quad \forall\, t \in [T,2T),\ i = 1,2,\ldots,N, \tag{4.34}$$

where $(a,(b_i)_{i=1}^N)$ is defined in (4.8). Our aim is to construct a suitable $(u_0,(v_{i,0})_{i=1}^N)$ in $V_0^s \times (V_0^{s+1})^N$ such that the above assertion fails.

To construct $(u_0, (v_{i,0})_{i=1}^N)$, let us set

$$h_0 = (I-\Delta)^{-k}\delta_{x_0} \text{ and } g_{i,0} = \mathcal{F}_\xi^{-1}\left(\frac{e^{-ix_0.\xi}}{(1+|\xi|^2)^k}C_i(\xi)\right), i = 1,2,\dots,N, \tag{4.35}$$

where $C_i(\xi)$ is defined in (4.14) and $\mathcal{F}^{-1}$ is the inverse Fourier transform. Then we see that

$$\widetilde{a}_0(\cdot) = \frac{\partial h_0(\cdot)}{\partial x_2} \text{ and } \widetilde{b}_{i,0}(\cdot) = \frac{\partial g_{i,0}(\cdot)}{\partial x_2}, i = 1,2,\dots,N, \tag{4.36}$$

where $(\widetilde{a}_0, (\widetilde{b}_{i,0})_{i=1}^N)$ is same as in Proposition 4.7. Next, we construct $(u_0, (v_{i,0})_{i=1}^N)$ from the above $(\widetilde{a}_0, (\widetilde{b}_{i,0})_{i=1}^N)$ using Lemma 4.13, such that $(u_0, (v_{i,0})_{i=1}^N) \in V_0^s \times (V_0^{s+1})^N$, for a suitable choice of k in (4.35) as mentioned in Remark 4.8 and

$$\begin{cases} a_0(x) = (\operatorname{curl} u_0)_1(x) = \widetilde{a}_0(x), \quad \forall\, x \in S_0, \\ b_{i,0}(x) = (\operatorname{curl} v_{i,0})_1(x) = \widetilde{b}_{i,0}(x),\ i = 1,2,\dots,N, \quad \forall\, x \in S_0, \end{cases} \tag{4.37}$$

where S_0 is defined as in Theorem 4.10.

With the above choice of $(u_0, (v_{i,0})_{i=1}^N)$ all the assumptions of Theorem 4.12 and Corollary 4.11 are fulfilled. Therefore, using Corollary 4.11 and Theorem 4.12, we conclude that

$$(x_0, 3T/2, \xi, 0) \in WF(a), \text{ for all } \xi \in \mathbb{R}^d \text{ with } \xi_2 \neq 0,$$

which is a contradiction to (4.34). Consequently, the system (4.2) is not null controllable. This completes the proof of Theorem 2.3. □

Remark 4.14. In the case of the single mode Maxwell system, the characteristic surfaces of the operator $P(D)$ defined in (4.20), for $N = 1$, are

$$x \cdot \frac{\xi}{|\xi|} + \widetilde{\eta}t = C_1, \quad x \cdot \frac{\xi}{|\xi|} - \widetilde{\eta}t = C_2, \quad \forall\, x \in \mathbb{R}^d,\, t \in \mathbb{R}, \tag{4.38}$$

for any $\xi \in \mathbb{R}^d$, where $\widetilde{\eta} = \sqrt{\sum_{j=1}^N \kappa_j/\rho}$. Therefore, results similar to Theorem 4.10 and Corollary 4.11 are also applicable to the analogous system for the single mode case, where the singularity of the solution propagates along the characteristics given by (4.38). Hence, for any proper subset $\mathcal{O}$ of Ω, there exists a T_0, such that the single mode Maxwell system cannot be null controllable at any time T, for $0 < T \leq T_0$, using a control localized in $\mathcal{O}$.

We note that (4.9) in both cases of single mode and multimode has the property of finite speed of propagation (this can be proved in a fashion similar to Theorem III.13 in [22]). Hence even approximate controllability is impossible until the domain of dependence of the controlled region has expanded to encompass all of Ω. This provides a (non-optimal) lower bound for T_0 in Theorem 1.4.

In contrast to the case of the multimode Maxwell system, there is no vertical characteristic of $P(D)$ in (x,t) plane and Theorem 4.6 is not applicable to the single mode case. In fact, for time T sufficiently large, the single mode Maxwell system is exactly controllable at time T using a localized control (see Thm. 4.1 and the next paragraph in [3]).

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