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A VARIATIONAL APPROACH TO NONLINEAR STOCHASTIC DIFFERENTIAL EQUATIONS WITH LINEAR MULTIPLICATIVE NOISE

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Abstract. One introduces a new concept of generalized solution for nonlinear infinite dimensional stochastic differential equations of subgradient type driven by linear multiplicative Wiener processes. This is defined as solution of a stochastic convex optimization problem derived from the Brezis-Ekeland variational principle. Under specific conditions on nonlinearity, one proves the existence and uniqueness of a variational solution which is also a strong solution in some significant situations. Applications to the existence of stochastic total variational flow and to stochastic parabolic equations with mild nonlinearity are given.

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1. Introduction

Consider the infinite-dimensional stochastic differential equation

$$\begin{aligned} &\mathrm{d}X(t) + A(t)X(t)\mathrm{d}t + \lambda X(t)\mathrm{d}t = X(t)\mathrm{d}W(t),\ t \in [0,T],\\ &X(0) = x, \end{aligned} \tag{1.1}$$

where $X : [0,T] \to V$ and $A : [0,T] \times V \to V^*$. The hypotheses below will be assumed in the following.

(i) V is a real separable Banach space with dual space V^* and H is a real separable Hilbert space such that $V \subset H$, with continuous and dense embedding.
Moreover, $C^2(\overline{\mathcal{O}}) \subset H$, where $\mathcal{O}$ is a bounded and open subset of $\mathbb{R}^d$, $1 \le d < \infty$, with smooth boundary $\partial\mathcal{O}$.

(ii) W is a Wiener cylindrical H-valued process of the form

$$W(t) = \sum_{j=1}^{\infty} \mu_j e_j \beta_j(t), \tag{1.2}$$

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where $\{\beta_j\}_j$ is an independent system of Brownian motions in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the normal filtration $(\mathcal{F}_t)_{t\geq 0}$, $\mu_j \in \mathbb{R}$ and $\{e_j\}_{j=1}^{\infty} \subset C^2(\overline{\mathcal{O}})$ is an orthonormal basis in H. Moreover, one assumes that

$$\sum_{j=1}^{\infty} \mu_j^2 \gamma_j^2 |e_j|_\infty^2 < \infty,\ |ye_j|_H \leq \gamma_j |e_j|_\infty |y|_H, \quad \forall y \in H, \gamma_j \geq 1, j = 1, ..., \tag{1.3}$$

and that e^W is a multiplier in V, H and V^*. (Here, $|\cdot|_H$ is the norm of H and $|\cdot|_\infty$ the norm of $L^\infty(\mathcal{O})$.) We set

$$\widetilde{\mu}(x) = \frac{1}{2} \sum_{j=1}^{\infty} |xe_j|_H^2 \mu_j^2,\ \forall x \in H. \tag{1.4}$$

(iii) $A(t) = \partial\varphi(t),\ \forall t \in [0, T]$, where $\varphi = \varphi(t, y) : [0, T] \times V \to \mathbb{R}^+ = [0, +\infty)$ is continuous, convex in y for each $t \in [0, T]$ and

$$\varphi(t, y) \leq C_N,\ \forall t \in [0, T],\ \|y\|_V \leq N,\ \forall N > 0, \tag{1.5}$$

$$\liminf_{y_n \overset{H}{\to} y,\ y_n \in V} \varphi(t, y_n) \geq \varphi(t, y),\ \forall (t, y) \in [0, T] \times V, \tag{1.6}$$

$$\varphi(t, y) \geq \alpha_1 \|y\|_V - \alpha_2 |y|_H - \alpha_3,\ \forall y \in V, \tag{1.7}$$

where $\alpha_1 > 0,\ \alpha_2, \alpha_3 \in \mathbb{R}$, and $\|\cdot\|_V$ is the norm of V.

Here, for each $t \in [0, T]$, $\partial\varphi(t) : V \to 2^{V^*}$ is the subdifferential of $\varphi(t)$, that is,

$$\partial\varphi(t, y) = \{z \in V^*;\ \varphi(t, y) \leq \varphi(t, u) + {}_{V^*}\langle z, y - u\rangle_V\ ,\ \forall u \in V\}.$$

We denote by ${}_{V^*}\langle\cdot,\cdot\rangle_V$ the duality pairing between V and V^* and by the same symbol we denote the scalar product $\langle\cdot,\cdot\rangle$ on $H \times H$.

Assumption (1.6) means that, for each $t \in [0, T]$, the function $\widetilde{\varphi}(t, y) : H \to \overline{\mathbb{R}} = (-\infty, +\infty]$ defined by

$$\widetilde{\varphi}(t, y) = \begin{cases} \varphi(t, y) & \text{if } y \in V, \\ +\infty & \text{if } y \overline{\in} V, \end{cases} \tag{1.8}$$

is lower-semicontinuous in H.

We note that, by hypothesis (i), V and H are spaces of functions on $\overline{\mathcal{O}}$ (for instance, $L^p(\mathcal{O})$ or Sobolev space) while $A(t)$ is a nonlinear elliptic operator on $\mathcal{O}$ with appropriate boundary conditions. (Of course, hypotheses (i)–(ii) can be formulated under a more general functional setting.)

The exact significance of the right-hand side of (1.1) is $X\,dW = \sigma(X)d\widetilde{W}$, where $\sigma : H \to \mathcal{L}_2(H)$ (the space of Hilbert–Schmidt operators on H) is defined by

$$\sigma(y)(z) = \sum_{j=1}^{\infty} \mu_j y \langle z, e_j\rangle e_j,\ \forall z \in H,$$

and $\widetilde{W}$ is the cylindrical Wiener process on H formally written as

$$\widetilde{W}(t)=\sum_{j=1}^{\infty}\beta_j(t)e_j$$

(see [14], p. 89).

Definition 1.1. The stochastic process $X^*:[0,T]\to V$ is said to be a variational solution to equation (1.1) if it is $(\mathcal{F}_t)_{t\ge 0}$-adapted, pathwise weakly continuous,

$$\begin{aligned} X^*\in L^2(\Omega;L^2(0,T;H))\cap L^1((0,T)\times\Omega;V),\\ \varphi(t,X^*)\in L^1((0,T)\times\Omega), \end{aligned} \tag{1.9}$$

and there is an $(\mathcal{F}_t)_{t\ge 0}$-adapted stochastic process $Z^*:[0,T]\to V^*$ such that

$$\varphi(\cdot,Z^*)\in L^1((0,T)\times\Omega), \tag{1.10}$$

$$\mathrm{d}X^*+\lambda X^*\mathrm{d}t+Z^*\mathrm{d}t=X^*\mathrm{d}W,\ t\in(0,T), \tag{1.11}$$
$$X^*(0)=x,$$

$$\begin{aligned}\mathbb{E}\int_0^T(\varphi(t,X^*(t))+\varphi^*(t,Z^*(t))+\lambda|X^*(t)|_H^2-\widetilde{\mu}(X^*(t)))\mathrm{d}t\\ +\frac{1}{2}\,\mathbb{E}|X^*(T)|_H^2\le\mathbb{E}\int_0^T(\varphi(t,X(t))+\varphi^*(t,Z(t))\\ +\lambda|X(t)|_H^2-\widetilde{\mu}(X(t)))\mathrm{d}t+\frac{1}{2}\,|X(T)|_H^2,\end{aligned} \tag{1.12}$$

for all $(\mathcal{F}_t)_{t\ge 0}$-adapted processes $(X,Z):[0,T]\to V\times V^*$ which satisfy the stochastic differential equation

$$\begin{aligned}&\mathrm{d}X+\lambda X\,\mathrm{d}t+Z\,\mathrm{d}t=X\,\mathrm{d}W,\ t\in(0,T),\\ &X(0)=x.\end{aligned} \tag{1.13}$$

Here, $\widetilde{\mu}$ is defined by (1.4) and $\varphi^*:[0,T]\times V^*\to\overline{\mathbb{R}}=]-\infty,+\infty]$ is the Legendre, conjugate of φ, that is (see, *e.g.*, [8]),

$$\varphi^*(t,z)=\sup\left\{{}_{V^*}\langle z,y\rangle_V-\varphi(t,y);\ y\in V\right\}. \tag{1.14}$$

Equations (1.11) and (1.13) are taken in sense of Itô's (see, *e.g.*, [14]). Since, by (1.5), it follows that

$$\lim_{\|z\|_{V^*}\to\infty}\frac{\varphi^*(t,z)}{\|z\|_{V^*}}=+\infty,$$

condition (1.10) imply that $Z^*\in L^1((0,T)\times\Omega;V^*)$ and so equation (1.11) is well defined.)

We note that (1.10)–(1.13) can be viewed a stochastic optimal control problem with the state equation (1.13) (Z is the stochastic controller).

The stochastic process $X^*:[0,T]\to V$ is said to be a strong solution to (1.1) if it is $(\mathcal{F}_t)_{t\ge 0}$-adapted, pathwise weakly continuous and

$$X^*(t)=x-\int_0^t(Z^*(s)+\lambda X^*(s))\mathrm{d}s+\int_0^t X^*(s)\mathrm{d}W(s),\ \forall t\in[0,T],$$

where X^*, Z^* satisfy (1.9) and

$$Z^*(t) \in \partial\varphi(t, X^*(t)), \text{ a.e. } t \in (0,T) \times \Omega, \tag{1.15}$$

If X^* is a strong solution to (1.1), then it is a variational solution to (1.1). Indeed, applying the Itô formula in equation (1.13) and taking into account (1.4), we obtain that

$$\begin{aligned} &\mathbb{E}\int_0^T {}_{V^*}\langle Z(t), X(t)\rangle_V \,\mathrm{d}t \\ &\quad = \frac{1}{2}\,(\mathbb{E}(X(T)|_H^2 - |x|_H^2) - \mathbb{E}\int_0^T (\lambda|X(t)|_H^2 - \widetilde{\mu}(X(t))))\mathrm{d}t, \end{aligned} \tag{1.16}$$

for all solutions (X, Z) to (1.13). Then, by the conjugacy formulae (see, *e.g.*, [8], p. 70),

$$\begin{aligned} \varphi(t, X^*(t)) + \varphi^*(t, Z^*(t) &= {}_{V^*}\langle Z^*(t), X^*(t)\rangle_V \\ \varphi(t, X) + \varphi^*(t, Z) &\geq {}_{V^*}\langle Z, X\rangle_V, \ \forall (X, Z) \in V \times V^*, \end{aligned} \tag{1.17}$$

we see that X^* is variational solution to (1.1), that is,

$$\begin{aligned} X^* = \arg\min\Big\{&\mathbb{E}\int_0^T (\varphi(t, X(t)) + \varphi^*(t, Z(t)) + \lambda|X(t)|_H^2 - \widetilde{\mu}(X(t)))\mathrm{d}t \\ &+ \frac{1}{2}\,\mathbb{E}|X(T)|_H^2 \text{ for all } (X, Z) \text{ subject to (1.13)}\Big\} \end{aligned} \tag{1.18}$$

and

$$\begin{aligned} &\mathbb{E}\int_0^T \big(\varphi(t, X^*(t)) + \varphi^*(t, Z^*(t)) + \lambda|X^*(t)|_H^2 - \widetilde{\mu}(X^*(t))\big)\,\mathrm{d}t \\ &\qquad - \frac{1}{2}\,(\mathbb{E}|X^*(T)|_H^2 - |x|_H^2)\mathrm{d}t = 0. \end{aligned} \tag{1.19}$$

It should be emphasized, however, that a variational solution X^* in sense of Definition 1.1 is not a strong solution to (1.1) until (1.16) and (1.19) hold for X^*, Z^*.

In Theorem 2.1, we shall prove the existence of a unique variational solution which, under additional assumption on φ, is also strong solution. In general, however, the notion of variational solution is a more general one.

As regards Definition 1.1, it should be mentioned that a related concept of the generalized solution to equation (1.1) was given in [9] in the special case $A(t)X = -\mathrm{div}\left(\frac{\nabla X}{|\nabla X|}\right)$, $H = L^2(\mathcal{O})$, that is, for total variation stochastic flows. More will be said about this in Section 3.1.

The main advantage of this variational formulation of equation (1.1) is that it reduces the stochastic equation to a convex minimization problem in the Hilbert space $L^2(\Omega \times [0,T]; H)$.

The idea to represent the nonlinear subgradient Cauchy problems in Hilbert spaces as a convex minimization problems goes back to Brezis and Ekeland and was developed in [17, 20] to more general nonlinear evolution problems. In [11], this approach was extended to a more general class of equations (1.1) with $A(t) = \partial\varphi(t)$. For earlier results, we mention [4, 6, 10] (see, also, [12]).

Notations. Given a Banach space $\mathcal{X}$ and $(a, b) \in \mathbb{R}$, we denote by $C([a, b]; \mathcal{X})$ the space of continuous $\mathcal{X}$-valued function s on $[a, b]$ and by $L^p(a, b; \mathcal{X})$, $1 \leq p \leq \infty$, the space of L^p-Bochner integrable functions on (a, b). The spaces $L^p((a, b) \times \Omega; \mathcal{X})$ are similarly defined. By $W^{1,p}([a, b]; \mathcal{X})$ we denote the space of all absolutely continuous and a.e. differentiable functions on $[a, b]$ such that $\frac{du}{dt} \in L^p(a, b; \mathcal{X})$. $L^p(\mathcal{O})$ is the space of Lebesgue p-integrable functions on $\mathcal{O} \subset \mathbb{R}^d$. The Euclidean norm of $\mathbb{R}^d$ will be denoted by $|\cdot|_d$. We refer to [7, 14] for basic results and notations pertaining infinite dimensional stochastic differential equations and spaces of infinite dimensional stochastic processes.

2. The main result

We set

$$\lambda_0 = \frac{1}{2}\sum_{j=1}^{\infty} \mu_j^2 \gamma_j^2 |e_j|_\infty^2.$$

Theorem 2.1. *Assume that hypotheses* (i)–(iii) *hold and, in addition, one of the following two conditions hold.*

(k) *Either V is reflexive or V^* is separable.*
(kk) $\varphi(t,y) = \alpha\|y\|_V,\ \forall y \in V$, *where $\alpha > 0$.*

Then, for $\lambda > \lambda_0$, there is a unique variational solution X^* to (1.1) *which satisfies*

$$X^* \in L^1((0,T)\times\Omega;V),\ \varphi(t,X^*) \in L^1((0,T)\times\Omega), \tag{2.1}$$

$$e^{-W}X^* \in W^{1,1}([0,T];V^*),\ \mathbb{P}\text{-}a.s. \tag{2.2}$$

Moreover, if the function $(t,\omega) \to {}_{V^*}\langle Z^*(t,\omega), X^*(t,\omega)\rangle_V$ is in $L^1((0,T)\times\Omega)$, then X^* is a strong solution to (1.1)*, that is,* (1.15) *holds.*

Since $\varphi(t,X^*), \varphi^*(t,Z^*) \in L^1((0,T)\times\Omega)$, while by (1.17) we have

$${}_{V^*}\langle Z^*(t), X^*(t)\rangle_V \le \varphi(t,X^*(t)) + \varphi^*(t,Z^*(t)),\ \mathbb{P}\times \mathrm{d}t \text{ a.e.}$$

and

$$-{}_{V^*}\langle Z^*(t), X^*(t)\rangle_V \le \varphi(t,-X^*(t)) + \varphi^*(t,Z^*(t)),\ \mathbb{P}\times \mathrm{d}t \text{ a.e.},$$

it follows that ${}_{V^*}\langle Z^*(t), X^*(t)\rangle_V \in L^1((0,T)\times\Omega)$ if one assumes further that

$$\varphi(t,-x) \le C_1\varphi(t,x) + C_2,\ \forall (t,x) \in [0,T]\times V,\ \|x\|_V \ge C_3, \tag{2.3}$$

for some $C_1, C_2, C_3 > 0$.

Condition (2.3) can be viewed as a *far away quasi-symmetry property* for the function φ and can be checked in most situations of interest.

By Theorem 2.1, we have

Corollary 2.2. *Under hypotheses of Theorem* 2.1 *and* (2.3)*, there is a unique variational solution X^* to* (1.1) *which satisfies* (2.1)–(2.2) *and is strong solution to* (1.1)*.*

Remark 2.3. Condition $\lambda > \lambda_0$ can be dispensed with by the substitution $X = e^{\lambda t}Y$ which transforms equation (1.1) into

$$\begin{aligned} &dY + \widetilde{A}(t)Y\,\mathrm{d}t + \lambda Y\,\mathrm{d}t = Y\,\mathrm{d}W,\ t\in[0,T],\\ &Y(0) = x, \end{aligned} \tag{2.4}$$

where $\widetilde{A}(t)Y = e^{-\lambda t}A(t)(e^{\lambda t}Y)$. We note that $\widetilde{A}(t)Y = \partial\phi(t,Y),\ \phi(t,Y) \equiv e^{-2\lambda t}\varphi(t, e^{\lambda t}Y)$ and so hypothesis (iii) as well as (iii)′ remain invariant under this substitution and equations (1.1) and (2.4) are equivalent.

Remark 2.4. In Definition 1.1 and Theorem 2.1, one might ask instead of (1.18) that

$$\begin{aligned} X^*_{[0,t]} = \arg\min\Big\{&\mathbb{E}\int_0^t (\varphi(s,X(s)) + \varphi^*(x,Z(s)) + \lambda|X(s)|^2_H - \widetilde{\mu}(X(s)) \\ &+ \frac{1}{2}\,\mathbb{E}|X(t)|^2_H)\mathrm{d}s; \mathrm{d}X + \lambda X\,\mathrm{d}s + Z\,\mathrm{d}s = X\,\mathrm{d}W,\ s\in(0,t),\ X(0)=x\Big\}, \\ &\forall t\in(0,T). \end{aligned} \tag{2.5}$$

In fact, the existence and uniqueness of solution (X^*, Z^*) to (2.5) on each interval $(0,t)$ implies that $X^*_{[0,t]}(s) = X^*_{[0,\tilde{t}]}(s)$, for all $0 \le s \le t \le \tilde{t} \le T$, and so equations (1.18) and (2.5) are equivalent.

In [18], equations of form (1.1) are studied under the main assumptions that $A(t) : V \to V^*$ is monotone (not necessarily of subgradient type) and

$$\|A(t)u\|_{V^*} \le C(\|u\|^p_V + 1),\ \forall u \in V, \tag{2.6}$$

$$_{V^*}(A(t)u,u)_V \ge \alpha_1\|u\|^{p-1}_V + \alpha_2|u|^2_H,\ \forall u \in U, \tag{2.7}$$

where $1 < p < \infty$ and $\alpha_1 > 0$, $\alpha_2 \in R$. (See, also, [10] for an operatorial approach to equation (1.1) under assumptions (2.6)–(2.7).)

The main novelty here is that in Theorem 2.1 the space V is, in general, nonreflexive, and in (2.6)–(2.7) $1 \le p < \infty$. As seen below, this nontrivial extension of results of [18] obtained by variational arguments are applicable to a new class of parabolic stochastic equations, so far not covered by a standard maximal monotone theory. In fact, by the operatorial approach developed in [10] under assumptions (2.6)–(2.7), equation (1.1) can be rewritten as $\mathcal{B}y + \mathcal{A}y = f$, where $\mathcal{B}, \mathcal{A}$ are maximal monotone coercive operators on $\mathcal{V}\times\mathcal{V}^*$, $D(\mathcal{A}) = \mathcal{V}$ and $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}^*$. Here, $\mathcal{V}$ is a reflexive Banach space of $(\mathcal{F}_t)_{t\ge0}$-adapted, V-valued processes on $[0,T]$, continuously and densely embedded in a Hilbert space $\mathcal{H}$ and $\mathcal{V}^*$ is its dual. Hence, $\mathcal{B}+\mathcal{A}$ is surjective on $\mathcal{V}^*$ by the general theory of maximal monotone operators in reflexive Banach spaces (see, *e.g.*, [5]). In the case considered here, that is, under hypotheses (i)–(iii), the space $\mathcal{V}$ is not reflexive, $p = 1$, and so the above surjective result is not applicable and a new approach was necessary.

To put this work in perspective, we note that, contrary to what happens in the deterministic case, we do not have so far a general existence theory for strong or weak (martingale) solutions to equation (1.1) for $A(t) \equiv A : H \to H$, a time-independent unbounded maximal monotone operator in the Hilbert space H. The motivation is that, as seen later on, equation (1.1) reduces by the rescaling transformation $X = e^W y$ to a random differential equation with a time-dependent equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} + e^{-W(t)}A(e^{W(t)}y) + \mu y = 0,\ t\in(0,T),\ \mathbb{P}\text{-a.s.},$$

and since the function $t \to e^{-W(t)}A(e^{W(t)}y)$ has not a bounded variation, the general existence theory for the Cauchy problem is not applicable (see, *e.g.*, [5], p. 177). Under these circumstances, the variational approach proposed here seems to be a convenient and apparently the unique alternative to the existence of the stochastic PDE, though it does not provide in all situations a strong solution. (As a matter of fact, a similar approach can be developed to equation (1.1) with additive Gaussian noise.)

One of the main advantages of this approach is that it reduces the stochastic equation to a convex minimization problem in a suitable space of stochastic processes.

3. Examples

We shall give below two significant examples to which Theorem 2.1 applies neatly. In all these examples, the basic space V is either the space of functions with bounded variations or a Sobolev Orlicz space on a domain $\mathcal{O} \subset \mathbb{R}^d$ while the pivot space H in $L^2(\mathcal{O})$. Of course, Theorem 2.1 applies to finite dimensional stochastic differential equations and in other cases as well.

3.1. The stochastic total flow

Consider here the equation

$$\begin{aligned} &\mathrm{d}X - \operatorname{div}\left(\frac{\nabla X}{|\nabla X|_d}\right)\mathrm{d}t = X\,\mathrm{d}W && \text{in } (0,T)\times\mathcal{O},\\ &X(0)=x(\xi), && \xi\in\mathcal{O},\\ &X(t,\xi)=0, && \forall (t,\xi)\in(0,T)\times\partial\mathcal{O}, \end{aligned} \tag{3.1}$$

in the space $BV(\mathcal{O})$ of functions with bounded variation in the open and bounded domain $\mathcal{O} \subset \mathbb{R}^d$. We recall (see, *e.g.*, [2]) that

$$BV(\mathcal{O}) = \{u \in L^1(\mathcal{O}); \nabla u \in M(\mathcal{O}, \mathbb{R}^d)\},$$

where $M(\mathcal{O}, \mathbb{R}^d)$ is the space of all the d-valued Borel measures, that is, the dual of $C_0^\infty(\mathcal{O}, \mathbb{R}^d)$. (Here, $\nabla u = Du$ is the gradient of u in sense of Schwartz distributions.) Here W is a Wiener process of the form (1.2), which satisfies (1.3) on $H = L^2(\mathcal{O})$.

For the sake of simplicity, we take here $d = 2$. Then $BV(\mathcal{O}) \subset L^2(\mathcal{O})$.

Denote by $\|Du\|$ the total variation of $u \in BV(\mathcal{O})$, that is,

$$\|Du\| = \sup\left\{\int_{\mathcal{O}} u \operatorname{div}\psi\, d\xi,\ \psi \in C_0^\infty(\mathbb{R}^d),\ |\psi|_\infty \le 1\right\}.$$

Define the function $\varphi : L^2(\mathcal{O}) \to \overline{\mathbb{R}} =]-\infty, +\infty]$,

$$\varphi(u) = \begin{cases} \|Du\| & \text{if } u \in BV_0(\mathcal{O}),\\ +\infty & \text{if } L^2(\mathcal{O}) \setminus BV_0(\mathcal{O}), \end{cases} \tag{3.2}$$

where $BV_0(\mathcal{O}) = \{u \in BV(\mathcal{O});\ \gamma_0(u) = 0\}$. Here, $\gamma_0(u)$ is the trace of $u \in BV(\mathcal{O})$ on $\partial\mathcal{O}$ (see, *e.g.*, [2, 3]).

The norm of $BV_0(\mathcal{O})$ is given by

$$\|u\|_{BV_0(\mathcal{O})} = \|Du\|.$$

We can rewrite equation (3.1) as (see [3, 9, 14])

$$\begin{aligned} &\mathrm{d}X + \partial\varphi(X)\mathrm{d}t + \lambda\, X\,\mathrm{d}t = X\,\mathrm{d}W \text{ in } (0,T),\\ &X(0) = x, \end{aligned} \tag{3.3}$$

where $\partial\varphi : H = L^2(\mathcal{O}) \to 2^H$ is the subdifferential of the function φ. We shall apply here Theorem 2.1, where $V = BV_0(\mathcal{O})$, $H = L^2(\mathcal{O})$ and $\varphi : V \to \mathbb{R}$ is defined as above. In this case, condition (k) of Theorem 2.1 is not satisfied, but (kk) holds. So, by Theorem 2.1, equation (3.3) has a unique variational solution

$$\begin{aligned} X^* &\in L^2((0,T)\times\Omega\times\mathcal{O})\cap L^1((0,T)\times\Omega;BV_0(\mathcal{O})) \\ e^{-W}X^* &\in W^{1,1}([0,T];V^*),\ \mathbb{P}\text{-a.s.}, \end{aligned} \tag{3.4}$$

which, by Corollary 2.2, is also a strong solution to (3.3) because condition (2.3) is obviously satisfied.

According to Definition 1.1, this means that there is a unique stochastic process

$$(X^*,Z^*)\in L^2((0,T)\times\Omega;L^2(\mathcal{O}))\cap L^1((0,T)\times\Omega;BV_0(\mathcal{O}))\times L^1((0,T)\times\Omega;V^*)$$

such that

$$\begin{aligned} &\mathbb{E}\int_0^T(\varphi(X^*(t))+\varphi^*(Z^*(t))+\lambda|X^*(t)|_H^2-\widetilde{\mu}(X^*(t))\mathrm{d}t+\frac{1}{2}\,\mathbb{E}|X^*(T)|_H^2 \\ &\quad\le\mathbb{E}\int_0^T(\varphi(X(t))+\varphi^*(Z(t))+\lambda|X(t)|_H^2-\widetilde{\mu}(X(t))\mathrm{d}t+\frac{1}{2}\,\mathbb{E}|X(T)|_H^2, \end{aligned} \tag{3.5}$$

for all the stochastic $(\mathcal{F}_t)_{t\ge0}$-adapted processes $(X,Z)\in L^1((0,T)\times\Omega;V)\times L^1((0,T)\times\Omega;V)$ satisfying the equation

$$\begin{aligned} &\mathrm{d}X+\lambda X\,\mathrm{d}t+Z\,\mathrm{d}t=X\,\mathrm{d}W\ \text{ in } (0,T), \\ &X(0)=x. \end{aligned} \tag{3.6}$$

Moreover, X^* is a strong solution to (3.3).

Remark 3.1. This result can be compared most closely with that obtained in [9]. It should be said also that the subdifferential $\partial\varphi$ is quite hard to describe explicitly (see [3]) and so the variational formulation (3.5) of the solution X^* seems, however, to be a convenient way to represent it.

Remark 3.2. The stochastic equation for the total variation flow with Neumann's boundary condition

$$\begin{aligned} &\mathrm{d}X-\mathrm{div}\left(\frac{\nabla X}{|\nabla X|_d}\right)\mathrm{d}t=X\,\mathrm{d}W && \text{in } (0,T)\times\mathcal{O}, \\ &\frac{\partial X}{\partial n}=0 && \text{on } (0,T)\times\partial\mathcal{O}, \\ &X(0)=x && \text{in } \mathcal{O}, \end{aligned}$$

can be treated similarly if we take $\varphi: BV(\mathcal{O})\to\mathbb{R}$ defined by (see [3])

$$\varphi(u)=\|Du\|.$$

The details are omitted.

3.2. Quasilinear parabolic stochastic equations with nonpolynomial mild growth nonlinearity

Consider the equation

$$\begin{aligned} &\mathrm{d}X - \mathrm{div}(a(\nabla X))\mathrm{d}t = X\,\mathrm{d}W && \text{in } (0,T)\times\mathcal{O},\\ &a(\nabla X)\cdot n = 0 && \text{on } (0,T)\times\partial\mathcal{O},\\ &X(0,\xi) = x(\xi), && \xi\in\mathcal{O}, \end{aligned} \tag{3.7}$$

on a bounded and open smooth domain $\mathcal{O}\subset\mathbb{R}^d$ with normal at boundary n. Here, $a:\mathbb{R}^d\to\mathbb{R}^d$ is a continuous monotone function of the form $a=\partial j$, where $j:\mathbb{R}^d\to\mathbb{R}^+$, is a convex continuous function which satisfies the conditions

$$\alpha_1\rho(|r|_d)\le j(r)\le\alpha_2\rho(|r|_d)+\alpha_3,\ \forall r\in\mathbb{R}^d, \tag{3.8}$$

where $\alpha_1,\alpha_2>0$ and $\rho:[0.\infty)\to[0,\infty)$ is a continuous convex and increasing function such that

$$\lim_{s\to 0^+}\frac{\rho(s)}{s}=0,\quad \lim_{s\to+\infty}\frac{\rho(s)}{s}=+\infty, \tag{3.9}$$

$$s\to\frac{\rho(s)}{s}\ \text{is strictly increasing} \tag{3.10}$$

$$\rho(2s)\le k_1\rho(s),\ \rho^*(s)\le k_2\rho^*(s),\ \forall s\in\mathbb{R}^+, \tag{3.11}$$

for some $k_i>0$, $i=1,2$. Here ρ^* is the Legendre conjugate of ρ, that is,

$$\rho^*(s)=\sup\{sv-\rho(v);\ v\in\mathbb{R}\}.$$

Equation (3.7) can be treated in the framework of Theorem 2.1 if we take $H=L^2(\mathcal{O})$, $V=W^1(L_\rho(\mathcal{O}))$ and

$$\varphi(u)=\int_{\mathcal{O}} j(\nabla u)\mathrm{d}\xi,\ \forall u\in V.$$

If $L_\rho(\mathcal{O})$ is the Orlicz space corresponding to the N-function ρ, then $W^1(L_\rho(\mathcal{O}))$ is the Orlicz-Sobolev space

$$\{u\in L_\rho(\mathcal{O});\ D_j u\in L_\rho(\mathcal{O}),\ j=1,...,d\}.$$

(See [1], p. 246.) We note that under assumptions (3.9)–(3.11) the space $V=W^1(L_p(\mathcal{O}))$ is reflexive and so condition (k) holds. Then, by Theorem 2.1 it follows the existence and uniqueness of a variational solution X^* which is a strong solution if j satisfies (2.5) (see [1], p. 247).

Two examples of this type are

$$\begin{aligned} a(r) &\equiv \alpha\,\mathrm{sgn}\,r\log(1+|r|_d),\ \forall r\in\mathbb{R}^d,\\ a(r) &= \alpha\,\mathrm{sgn}\,r\exp(|r|_d),\ \forall r\in\mathbb{R}^d. \end{aligned}$$

$\alpha>0$, $\operatorname{sgn} r=\frac{r}{|r|_d}$ for $r\neq 0$, $\operatorname{sgn} 0=B(0;1)$, which lead to the stochastic equations

$$\begin{aligned}&\mathrm{d}X-\alpha\,\mathrm{div}\left(\log(1+|\nabla X|_d)\frac{\nabla X}{|\nabla X|_d}\right)=X\,\mathrm{d}W\ \text{ in } (0,T)\times\mathcal{O},\\ &X=0 \text{ on } (0,T)\times\partial\mathcal{O},\quad X(0)=x \text{ in } \mathcal{O},\end{aligned}$$

and, respectively,

$$\begin{aligned}&\mathrm{d}X-\alpha\,\mathrm{div}\left(\exp\left(|\nabla X|_d\right)\frac{\nabla X}{|\nabla X|_d}\right)=X\,\mathrm{d}W,\\ &X=0 \text{ on } (0,T)\times\partial\mathcal{O},\quad X(0)=x \text{ in } \mathcal{O}.\end{aligned}$$

Note also that equation (3.7) with Dirichlet boundary conditions can be treated in a similar way.

The existence for equations of this type cannot be derived by the general results in [19] and, as far as we know, was not treated so far in literature.

4. Proof of Theorem 2.1

By the transformation

$$X=e^W y$$

we reduce, *via* Itô's formula, equation (1.1) to the random differential equation

$$\begin{aligned}&\frac{\mathrm{d}y}{\mathrm{d}t}+e^{-W}A(t,e^W y)+(\lambda+\mu)y=0,\ t\in(0,T),\\ &y(0)=x,\end{aligned}\tag{4.1}$$

where $\mu(\xi)=\frac{1}{2}\sum_{j=1}^{\infty}\mu_j^2e_j^2(\xi),\ \xi\in\overline{\mathcal{O}}$. (By (1.3), we see that $\mu\in L^\infty(\mathcal{O})$.) Of course, the equivalence of (1.1), (4.1) is true for strong solutions X and $(\mathcal{F}_t)_{t\geq 0}$-adapted smooth solutions y, only. (See [9], [10]) for rigorous proof.)

Then, in terms of the function $y^*=e^{-W}X^*$, the stochastic optimization problem (1.18) reduces to the random optimal control problem

$$\begin{aligned}y^*{=}\arg\min\Bigg\{&\mathbb{E}\int_0^T(\varphi(t,e^{W(t)}y(t))+\varphi^*(t,z(t))\\ &+\lambda|e^{W(t)}y(t)|_H^2-\mu(e^Wy(t))\mathrm{d}t+\frac{1}{2}\left(\mathbb{E}|e^{W(T)}y(T)|_H^2-||x|_H^2\right);\\ &\frac{\mathrm{d}y}{\mathrm{d}t}+e^{-W}z+(\lambda+\mu)y=0,\ y(0)=x;\\ &y\in W^{1,1}([0,T];V^*),\ \mathbb{P}\text{-a.s.}\\ &y\in L^2((0,T)\times\Omega;H),\ \varphi(t,e^Wy)\in L^1((0,T)\times\Omega),\\ &z\in L^1((0,T)\times\Omega;V^*),\ (y,z),\ (\mathcal{F}_t)_{t\geq 0}-\mathit{adapted}\Bigg\}.\end{aligned}\tag{4.2}$$

Into this form, for $\lambda>\lambda_0$, problem (4.2) (as well as (1.18)) is a convex optimal control problem in the space $L^1((0,T)\times\Omega;V)$ with z as controller input. More precisely, by (1.3)-(1.4) we see that, for $\lambda>\lambda_0$, there is $\nu>0$

such that

$$\lambda|y|_H^2 - \widetilde{\mu}(y) \geq \nu|y|_H^2,\ \forall y \in H,$$

and, since the function $y \to \lambda|t|_H^2 - \widetilde{\mu}(y)$ is quadratic, this implies that the integrand in (4.2) is strictly convex.

Denote by $I = I(y,z)$ the convex function arising in the right-hand side of (4.2) and consider a sequence $\{(y_n, z_n)\}$ of $(\mathcal{F}_t)_{t\geq 0}$-adapted processes such that, $\mathbb{P}$-a.s.,

$$\inf I \leq I(y_n, z_n) \leq \inf I + \frac{1}{n}, \tag{4.3}$$

$$\begin{aligned} &\frac{dy_n}{\mathrm{d}t} + e^{-W} z_n + (\lambda+\mu) y_n = 0,\ \text{a.e. } t \in (0,T), \\ &y_n(0) = x. \end{aligned} \tag{4.4}$$

Assume first that condition (k) holds, that is, either V is reflexive or V^* is separable. By hypothesis (iii) part (1.5), it follows that (see, *e.g.*, [8], p. 76)

$$\lim_{\|v\|_{V^*}\to+\infty} \frac{\varphi^*(t,v)}{\|v\|_{V^*}} = +\infty \ \text{ uniformly on } [0,T]. \tag{4.5}$$

By (4.3), we also have

$$\mathbb{E}\int_0^T \varphi^*(t,z_n)\mathrm{d}t \leq C,\ \forall n \in \mathbb{N}, \tag{4.6}$$

which, along with (4.5), the latter implies that, for each $\varepsilon > 0$,

$$\int_G \|z_n\|_{V^*}\mathrm{d}t\, d\mathbb{P} \leq \varepsilon,\ \forall n \in \mathbb{N},$$

if $\int_G dt\, d\mathbb{P} \leq \delta(\varepsilon)$, where G is any $\mathrm{d}t \times d\mathbb{P}$ measurable set $(0,T)\times\Omega$. Since the space V^* and its dual have the Radon-Nikodym property (as reflexive or separable dual spaces), we infer by the Dunford-Pettis theorem in $L^1((0,T)\times\Omega; V^*)$ (see [13], Thm. 1) that the sequence $\{z_n\}$ is weakly compact in $L^1((0,T)\times\Omega;\, V^*)$. Hence, there is $z^* \in L^1((0,T)\times\Omega; V^*)$ such that, on a subsequence again denoted $\{z_n\}$, we have

$$z_n \to z^* \ \text{ weakly in } L^1((0,T)\times\Omega; V^*) \tag{4.7}$$

and, therefore,

$$\liminf_{n\to\infty} \mathbb{E}\int_0^T \varphi^*(t,z_n)\mathrm{d}t \geq \mathbb{E}\int_0^T \varphi^*(t,z^*)\mathrm{d}t, \tag{4.8}$$

because the function $z \to \mathbb{E}\int_0^T \varphi^*(t,z(t))\mathrm{d}t$ is convex and lower-semicontinuous on $L^1((0,T)\times\Omega; V^*)$.

Note also that, by (4.3) and hypothesis (iii), it follows that on a subsequence, again denoted $\{n\}$, we have

$$e^W y_n \to e^W y^* \quad \text{weakly in } L^2((0,T)\times\Omega; H) \tag{4.9}$$

and so, since e^{-W} is a multiplier in $L^2((0,T)\times\Omega;H)$, by (4.4) we have

$$\begin{aligned}\frac{dy_n}{\mathrm{d}t} \to \frac{\mathrm{d}y^*}{\mathrm{d}t} &= -(e^{-W}z^* + (\lambda+\mu)y^*)\\ &\text{weakly in } L^1((0,T)\times\Omega;V^*) + L^2((0,T)\times\Omega;H).\end{aligned} \tag{4.10}$$

(Here $\frac{d}{\mathrm{d}t}\,y^*$ is the distributional derivative of $y^*\in W^{1,1}([0,T];V^*)$.)

By (4.7) and (4.9), it follows also that the process (y^*,z^*) is $(\mathcal{F}_t)_{t\ge 0}$-adapted.

Now, taking into account that, by virtue of (1.6)–(1.7), (4.9) and of weak lower-semicontinuity in H of the function $\widetilde{\varphi}$ (see (1.9)), we have

$$\begin{aligned}\liminf_{n\to\infty}\mathbb{E}\int_0^T \varphi(t,e^W y_n)\mathrm{d}t \;&\ge \mathbb{E}\int_0^T \varphi(t,e^W y^*)\mathrm{d}t\\ &\ge \mathbb{E}\int_0^T (\alpha_1\|e^W y\|_V - \alpha_2|e^W y|_H)\mathrm{d}t,\end{aligned}$$

we get by (4.3), (4.7), (4.10) that

$$\begin{aligned}\mathbb{E}\int_0^T (\varphi(t,e^W y^*) + \varphi^*(t,z^*) + \lambda|e^W y|_H^2 - \widetilde{\mu}(e^W y^*))\mathrm{d}t\\ +\frac{1}{2}\,(\mathbb{E}|e^{W(T)}y^*(T)|_H^2 - |x|_H^2) = \inf I.\end{aligned}$$

Hence (4.2) holds and so $X^* = e^W y^*$ is a variational solution to (1.1).

Assume now that condition (kk) in Theorem 2.1 holds. Then, it follows that

$$\varphi^*(t,v) = \begin{cases} 0 & \text{if } \|v\|_{V^*}\le\alpha,\\ +\infty & \text{if } \|v\|_{V^*}>\alpha.\end{cases}$$

Then, if $\{z_n\}$ is chosen as above, by (4.6) we have

$$\|z_n(t,\omega)\|_{V^*}\le\alpha,\quad \text{a.e. } (t,\omega)\in(0,T)\times\Omega, \tag{4.11}$$

and so

$$\varphi^*(t,z_n(t,\omega)) = \alpha,\ \text{ a.e. } (t,\omega)\in(0,T)\times\Omega. \tag{4.12}$$

We set

$$\mathcal{X}_T = \left\{\varphi\in L^2((0,T)\times\Omega;V),\frac{\mathrm{d}\varphi}{\mathrm{d}t}\in L^2((0,T)\times\Omega;H),\varphi(T)=0\right\}$$

and note that by (4.4) it follows that

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}\int_0^T\left(\frac{\mathrm{d}y_n}{\mathrm{d}t},\varphi(t)\right)\mathrm{d}t&=-\mathbb{E}\int_0^T\left(y(t),\frac{\mathrm{d}\varphi}{\mathrm{d}t}(t)\right)\mathrm{d}t\\&+\mathbb{E}(x,\varphi(0))=-\lim_{n\to\infty}\mathbb{E}\int_0^T(e^{-W}z_n,\varphi)\mathrm{d}t\\&-\mathbb{E}\int_0^T((\lambda+\mu)y,\varphi)\mathrm{d}t,\ \forall\varphi\in\mathcal{X}_T.\end{aligned}$$

This implies that, for $n\to\infty$,

$$e^{-W}z_n\to\eta \text{ weakly in } \mathcal{X}_T^*, \tag{4.13}$$

where

$$\begin{aligned}&\frac{\mathrm{d}y}{\mathrm{d}t}+\eta+(\lambda+\mu)y=0,\ t\in(0,T),\\&y(0)=x.\end{aligned} \tag{4.14}$$

(Here, $\mathcal{X}_T^*$ is the dual of the space $\mathcal{X}^T$.)

On the other hand, by (4.11), it follows that $\|z^*\|_{V^*}\le\alpha$ on $(0,T)\times\Omega$ and so

$$\varphi^*(t,z^*(t,\omega))=0,\ \text{a.e. } (t,\omega)\in(0,T)\times\Omega.$$

Then, comparing with (4.11), we see that (4.8) holds in this case too.

The uniqueness is immediate because, as seen earlier, for $\lambda\ge\lambda_0$ the functional $I=I(y,z)$ is strictly convex in y.

We have shown, therefore, that problem (4.2) and, consequently, (1.18) has a unique solution (X^*,Z^*).

It remains to prove that, if ${}_{V^*}\langle Z^*,X^*\rangle_V\in L^1((0,T))$ (or, equivalently, ${}_{V^*}\left\langle z^*,e^Wy^*\right\rangle_V\in L^1((0,T)\times\Omega)$), then X^* is strong solution to (1.1), that is (1.15) holds. To this end, we shall prove first that

$$\begin{aligned}\mathbb{E}\int_0^T(\varphi(t,e^Wy^*)\ &+\varphi^*(t,z^*)+\lambda|e^Wy^*|_H^2-\widetilde{\mu}(e^Wy^*))\mathrm{d}t\\&+\frac{1}{2}\,\mathbb{E}|e^{W(T)}y^*(T)|_H^2-\frac{1}{2}\,|x|_H^2=0.\end{aligned} \tag{4.15}$$

To prove (4.15), we approximate (4.2) by the family of smooth optimization problems

Minimize

$$\left\{\mathbb{E}\int_0^T(\varphi_\varepsilon(t,e^Wy)+(\varphi_\varepsilon)^*(t,z)+\lambda|e^Wy|_H^2-\widetilde{\mu}(e^Wy))\mathrm{d}t+\frac{1}{2}\,\mathbb{E}|e^{W(T)}y(T)|_H^2\right\} \tag{4.16}$$

subject to

$$\begin{aligned}&\frac{\mathrm{d}y}{\mathrm{d}t}+e^{-W}z+(\lambda+\mu)y=0,\ \text{a.e. in } (0,T),\\&y(0)=x.\end{aligned} \tag{4.17}$$

Here, φ_ε is the Yosida-Moreau regularization of φ, that is (see, *e.g.*, [8], p. 97)

$$\varphi_\varepsilon(t,u)=\inf\left\{\frac{1}{2\varepsilon}\,|u-v|_H^2+\varphi(t,v);\ v\in H\right\}.$$

Taking into account that φ_ε is continuous on $[0,T]\times H$ (and Lipschitz in u), it follows that

$$\lim_{|v|_H\to\infty}\frac{\varphi_\varepsilon^*(t,v)}{|v|_H}=+\infty \text{ uniformly on } [0,T]$$

and so, arguing as above, it follows that problem (4.16) has a unique solution $(y_\varepsilon,z_\varepsilon)\in L^2((0,T)\times\Omega;H)\times L^1((0,T)\times\Omega;H)$.

Moreover, since $A_\varepsilon(t)=\partial\varphi_\varepsilon(t)$ is Lipschitz on H, it follows that equation (4.1) has a unique solution $\widetilde{y}_\varepsilon$ and so, as easily follows by (4.17) and by the equation $\widetilde{z}_\varepsilon=\partial\varphi_\varepsilon(t,e^{W_\varepsilon}y_\varepsilon)$, we have

$$\begin{aligned}&\varphi_\varepsilon(t,e^W\widetilde{y}_\varepsilon)+(\varphi_\varepsilon)^*(t,\widetilde{z}_\varepsilon)-{}_{V^*}\langle\widetilde{z}_\varepsilon,\widetilde{y}_\varepsilon\rangle_V=0,\\ &\frac{\mathrm{d}\widetilde{y}_\varepsilon}{\mathrm{d}t}+e^{-W}\widetilde{z}_\varepsilon+(\lambda+\mu)\widetilde{y}_\varepsilon=0,\ \text{a.e. in } (0,T),\end{aligned}\tag{4.18}$$

where $\widetilde{z}_\varepsilon=A(t,e^{W_\varepsilon}\widetilde{y}_\varepsilon)$. By Itô's formula, we have

$$\begin{aligned}\mathrm{d}(e^W\widetilde{y}_\varepsilon)+\widetilde{z}_\varepsilon\mathrm{d}t+\lambda e^W\widetilde{y}_\varepsilon\mathrm{d}t&=e^W\widetilde{y}_\varepsilon\mathrm{d}W\\ \frac{1}{2}\,\mathrm{d}|e^W\widetilde{y}_\varepsilon|_H^2={}_{V^*}\big\langle\widetilde{z}_\varepsilon,e^W\widetilde{y}_\varepsilon\big\rangle_V&+\big\langle e^W\widetilde{y}_\varepsilon,e^W\widetilde{y}_\varepsilon\mathrm{d}W\big\rangle\\ -\lambda|e^W\widetilde{y}_\varepsilon|_H^2&+\widetilde{\mu}(e^W\widetilde{y}_\varepsilon)\mathrm{d}t,\end{aligned}\tag{4.19}$$

and by (4.18) this yields $\widetilde{y}_\varepsilon=y_\varepsilon,\ \widetilde{z}_\varepsilon=z_\varepsilon$ and

$$\begin{aligned}\mathbb{E}\int_0^T(\varphi_\varepsilon(t,e^Wy_\varepsilon)+(\varphi_\varepsilon)^*(t,z_\varepsilon)+\lambda|e^Wy_\varepsilon|_H^2-\widetilde{\mu}(e^Wy_\varepsilon))\mathrm{d}t\\ +\frac{1}{2}\,|e^{W(T)}y_\varepsilon(T)|_H^2-\frac{1}{2}\,|x|_H^2=0,\quad\forall\varepsilon>0.\end{aligned}\tag{4.20}$$

Taking into account that

$$\begin{aligned}\varphi_\varepsilon(t,e^Wy_\varepsilon)&=\varphi(t,u_\varepsilon)+\frac{1}{2\varepsilon}\,|u_\varepsilon-e^Wy_\varepsilon|_H^2\\ u_\varepsilon&=(I+\varepsilon\partial\varphi)^{-1}(e^Wy_\varepsilon)\\ (\varphi_\varepsilon)^*(t,z_\varepsilon)&=\sup\{{}_{V^*}(z_\varepsilon,v)_V-\varphi_\varepsilon(t,v);\ v\in V\}\\ &\ge\sup\{{}_{V^*}(z_\varepsilon,v)_V-\varphi(t,v);\ v\in V\}\\ &=\varphi^*(t,z_\varepsilon),\end{aligned}\tag{4.21}$$

we get by (4.20) that

$$\begin{aligned}\mathbb{E}\int_0^T(\varphi(t,u_\varepsilon)+\varphi^*(t,z_\varepsilon)+\frac{1}{2\varepsilon}\,|u_\varepsilon-e^Wy_\varepsilon|_H^2-\widetilde{\mu}(e^Wy_\varepsilon)+\lambda|e^Wy_\varepsilon|_H^2)\mathrm{d}t\\ +\frac{1}{2}\,|e^{W(T)}y_\varepsilon(T)|_H^2\le 0.\end{aligned}\tag{4.22}$$

On the other hand, by (4.20)–(4.21) it follows, as above, under assumption (k) *via* the Dunford–Pettis weak compactness theorem that, on a subsequence $\{\varepsilon\}\to 0$, we have

$$\begin{aligned} y_\varepsilon &\to \widetilde{y}^* \quad \text{weakly in } L^2((0,T)\times\Omega;H)\\ z_\varepsilon &\to \widetilde{z}^* \quad \text{weakly in } L^1((0,T)\times\Omega;V^*)\\ u_\varepsilon &\to \widetilde{u}^* \quad \text{weakly in } L^2((0,T)\times\Omega;H) \end{aligned} \tag{4.23}$$

and so, by the weak lower-semicontinuity of convex integrands, it follows by (4.22) that $I(\widetilde{y}^*,z^*)=\inf I$, and so $(\widetilde{y}^*,\widetilde{z}^*)=(y^*,z^*)$ is the solution to the minimization problem (4.2). Moreover, letting ε tend to zero in (4.20), by (4.23) and the weak-lower semicontinuity of convex integrands, it follows that

$$\begin{aligned} \mathbb{E}\int_0^T |\varphi(t,e^W\widetilde{y}^*)+\varphi^*(t,\widetilde{z}^*)-\widetilde{\mu}(e^W\widetilde{y}^*)+\lambda|e^W\widetilde{y}^*|_H^2)\mathrm{d}t&\\ +\frac12\,|e^{W(T)}\widetilde{y}^*(T)|_H^2-\frac12\,|x|_H^2\le 0.& \end{aligned} \tag{4.24}$$

As noted earlier, the same inequality can be obtained under assumption (k).

We also have that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\,\widetilde{y}^*+e^{-W}\widetilde{z}^*+(\lambda+\mu)\widetilde{y}^*=0,\quad \text{a.e. in } (0,T),\\ &\widetilde{y}^*(0)=x, \end{aligned} \tag{4.25}$$

and by (4.23) it follows also that the processes $\widetilde{y}^*,\widetilde{z}^*$ are $(\mathcal{F}_t)_{t\ge0}$-adapted.

Taking into account that ${}_{V^*}\left\langle\widetilde{z}^*,e^W\widetilde{y}^*\right\rangle_V\in L^1((0,T)\times\Omega)$, arguing as in (4.19), it follows by (4.25) *via* Itô's formula that

$$\begin{aligned} \frac12\,\mathrm{d}|e^W\widetilde{y}^*|_H^2 = -\,{}_{V^*}\left\langle\widetilde{z}^*,e^W\widetilde{y}^*\right\rangle_V\mathrm{d}t-\lambda|e^W\widetilde{y}^*|_H^2dt&\\ +\widetilde{\mu}(e^W\widetilde{y}^*)\mathrm{d}t+\left\langle e^W\widetilde{y}^*,e^W\widetilde{y}^*\,\mathrm{d}W\right\rangle.& \end{aligned}$$

This yields

$$\begin{aligned} \mathbb{E}\int_0^T(\lambda|e^W\widetilde{y}^*|_H^2-\widetilde{\mu}(e^W\widetilde{y}^*))\mathrm{d}t+\frac12\,\mathbb{E}|e^{W(T)}\widetilde{y}^*|_H^2-\frac12\,|x|_H^2&\\ =-\mathbb{E}\int_0^T {}_{V^*}\left\langle\widetilde{z}^*,e^W\widetilde{y}^*\right\rangle_V\mathrm{d}t,& \end{aligned}$$

and so, by (4.23) we have

$$\mathbb{E}\int_0^T \varphi(t,e^W\widetilde{y}^*)+\varphi^*(t,\widetilde{z}^*)-{}_{V^*}\left\langle z^*,e^W\widetilde{y}^*\right\rangle_V)\mathrm{d}t\le 0$$

and, therefore,

$$\varphi(t,e^Wy^*)+\varphi^*(t,z^*)-{}_{V^*}\left\langle(z^*,e^Wy^*\right\rangle_V=0,\ \mathrm{d}t\times\mathbb{P},\ \text{a.e.}$$

Finally, $X^*(t) = e^W y^*$ and $Z^* = z^*$ satisfy

$$\varphi(t, X^*(t)) + \varphi^*(t, Z^*(t)) - {}_{V^*}\langle Z^*(t), X^*(t)\rangle_V = 0, \ \mathrm{d}t \times \mathbb{P}, \ \text{a.e.}$$

and so (1.15) follows. Hence, X^* is a strong solution to (1.1), as claimed.

This concludes the proof of Theorem 2.1. □

Remark 4.1. As explicitly follows by the previous proof, Theorem 2.1 and Corollary 2.2 remain true for the operator $A : [0, T] \times V \times \Omega \to V^*$ of the form $A(t, \omega) \equiv \partial\varphi(t, \omega)$, which is progressively measurable, that is, for every t, A restricted to $(0, t) \times V \times \Omega$ is $\mathcal{B}([0, t]) \otimes \mathcal{B}(V) \otimes \mathcal{F}_t$ measurable.

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