TURNPIKE PROPERTIES OF OPTIMAL RELAXED CONTROL PROBLEMS*

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Abstract. In this paper, three kinds of turnpike properties for optimal relaxed control problems are considered. Under some convexity and controllability assumptions, we obtain the uniform boundedness of the optimal pairs and the adjoint functions. On the basis, we prove the integral turnpike property, the mean square turnpike property and the exponential turnpike property, respectively.

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1. INTRODUCTION

The term "turnpike" represents a kind of highway. For variational problems and optimal control problems, "turnpike property" describes the relationship between the optimal pairs and their corresponding stationary optimal pairs. For instance, if a system describes that a person wants to travel from Place A to Place B (Place B is very far from Place A) by car as soon as possible, the turnpike property tells us that the straight line path between Place A and Place B may not be the best choice. One may suggest that the person drives from Place A to a turnpike (the steady state) nearby, and travels along the turnpike until he approaches Place B. Then, the person leaves the turnpike and reaches Place B.

The research of turnpike property comes from economic fields. The idea can be traced back to von Neumann's paper [11] in 1945. Then, Dorfman, *et al.* named the property metaphorically in [4]. Afterwards, numerous relevant studies were established (see, for example [3, 5, 6, 9, 10, 12–15, 17] and the references therein). Regarding the optimal control theory, if an optimal control problem has the turnpike property, we get an approximate description of the optimal pairs, which will help us economize the calculation costs (see [15]). Thus, the study of turnpike properties corresponding to the optimal control problems is meaningful. However, such results are very limited.

Recently, several researchers discussed the turnpike properties of finite-dimensional optimal control problems and provided exponential rate of the stability. Specifically, Porretta and Zuazua focused on a linear system [12],

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which can be described as follows:

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = -Ax(t) + Bu(t), & t \in [0, T], \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{n \times n}$. The optimal control problem (P_T) is to find $(x^T(\cdot), u^T(\cdot)) \in \mathscr{P}_{ad}$, such that

$$\begin{aligned} J^T(x^T(\cdot), u^T(\cdot)) &= \inf_{(x(\cdot), u(\cdot)) \in \mathscr{P}_{ad}} J^T(x(\cdot), u(\cdot)) \\ &= \inf_{(x(\cdot), u(\cdot)) \in \mathscr{P}_{ad}} \int_0^T \frac{1}{2} \big(|u(t)|^2 + |Cx(t) - z|^2 \big) \, \mathrm{d}t, \end{aligned}$$

where z is a given vector in $\mathbb{R}^n$, and

$$\mathscr{P}_{ad} = \left\{ (x(\cdot), u(\cdot)) \in \mathrm{C}^0([0, T]; \mathbb{R}^n) \times \mathrm{L}^2([0, T]; \mathbb{R}^m) \; \middle| \; (1.1) \text{ holds on } [0, T] \right\}.$$

The optimization problem (Q) is

$$\inf_{\substack{(x, u) \in \mathbb{R}^n \times \mathbb{R}^m \\ -Ax + Bu = 0}} \frac{1}{2} \big(|u|^2 + |Cx - z|^2 \big),$$

whose solution is $(\bar{x}, \bar{u})$. Suppose (A, B) is controllable, and (A, C) is observable, which implies the uniqueness of Problem (Q). Then, they proved that there exist positive constants λ and K, independent of $T > 0$, such that

$$|u^T(t) - \bar{u}| + |x^T(t) - \bar{x}| \le K(e^{-\lambda t} + e^{-\lambda(T-t)}), \quad \forall \, t \in [0, T]. \tag{1.2}$$

Further, Trélat and Zuazua established similar results for the nonlinear cases in [15]. They considered

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = f(x(t), u(t)), & t \in [0, T], \\ R(x(0), x(T)) = 0. \end{cases} \tag{1.3}$$

The optimal control problem (P_T) is to find $(x^T(\cdot), u^T(\cdot)) \in \mathscr{P}_{ad}$, such that

$$\begin{aligned} J^T(x^T(\cdot), u^T(\cdot)) &= \inf_{(x(\cdot), u(\cdot)) \in \mathscr{P}_{ad}} J^T(x(\cdot), u(\cdot)) \\ &= \inf_{(x(\cdot), u(\cdot)) \in \mathscr{P}_{ad}} \int_0^T f^0(x(t), u(t)) \, \mathrm{d}t, \end{aligned}$$

where

$$\mathscr{P}_{ad} = \left\{ (x(\cdot), u(\cdot)) \in \mathrm{C}^0([0, T]; \mathbb{R}^n) \times \mathrm{L}^\infty([0, T]; \mathbb{R}^m) \; \middle| \; (1.3) \text{ holds on } [0, T] \right\}.$$

The optimization problem (Q) is

$$\inf_{\substack{(x,u)\in\mathbb{R}^n\times\mathbb{R}^m \\ f(x,u)=0}} f^0(x,u),$$

whose solution is denoted as $(\bar{x}, \bar{u})$. They assumed that $R(\cdot),\ f(\cdot),\ f^0(\cdot) \in C^2$ and linearized Problem (P_T) by Taylor expansion at $(\bar{x}, \bar{u})$. By estimating the linearized part and the remainder terms respectively, they got the same turnpike properties as stated in (1.2). It is worth noting that the authors assumed a controllability condition, which can not ensure the uniqueness of $(\bar{x}, \bar{u})$. Meanwhile, it is more complicated to prove that the constant K is independent of T. So they need some extra conditions on the initial value and the terminal value of the adjoint equation, which is not easy to handle.

Observing the results mentioned above, we regard Problem (Q) as an optimization problem which requires the solution to satisfy some stationary conditions of (1.1)/(1.3). In [12, 15], the stationary condition is $-Ax + Bu = 0/f(x,u) = 0$. Such condition seems a little strict. It is worth discussing whether and to which extent the condition can be relaxed. Meanwhile, the existing results also require the existence of Problem (P_T) and Problem (Q), whose necessity needs to be discussed.

For this reason, we will consider a relaxed optimal control problem and study the corresponding turnpike properties. For given $T > 0$ and $x_0 \in \mathbb{R}^n$, consider the following system

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = Ax(t) + \displaystyle\int_{\mathbb{R}^m} Bu\,\sigma(t)(\mathrm{d}u), & t \in [0,T], \\ x(0) = x_0, \end{cases}$$

and the corresponding cost functional

$$J_T(\sigma(\cdot); x_0) = \int_0^T \mathrm{d}t \int_{\mathbb{R}^m} (f(x(t)) + g(u))\,\sigma(t)(\mathrm{d}u), \quad \forall\, \sigma(\cdot) \in \mathcal{R}_T(\mathbb{R}^m), \tag{1.4}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $f(\cdot) : \mathbb{R}^n \to \mathbb{R}$, $g(\cdot) : \mathbb{R}^m \to \mathbb{R}$, and

$$\mathcal{M}_+^1(\mathbb{R}^m) = \left\{ \theta \,\middle|\, \theta \ \text{is a probability measure on } \mathbb{R}^m \text{ with a compact support set} \right\},$$

$$\mathcal{R}_T(\mathbb{R}^m) = \left\{ \sigma(\cdot) \,\middle|\, \sigma(t) \in \mathcal{M}_+^1(\mathbb{R}^m),\ \exists\, U \subset\subset \mathbb{R}^m,\ \text{s.t. } \operatorname{supp}\sigma(t) \subseteq U,\ \text{a.e. } t \in [0,T], \right.$$

$$\left. \text{and } t \mapsto \int_{\mathbb{R}^m} l(u)\,\sigma(t)(\mathrm{d}u) \text{ is measurable, } \forall\, l(\cdot) \in C^0(\mathbb{R}^m) \right\}.$$

The optimal relaxed control problem is

Problem $RP(x_0, T)$: Find $\bar{\sigma}_T(\cdot) \in \mathcal{R}_T(\mathbb{R}^m)$, such that

$$J_T(\bar{\sigma}_T(\cdot); x_0) = \inf_{\sigma(\cdot)\in\mathcal{R}_T(\mathbb{R}^m)} J_T(\sigma(\cdot); x_0).$$

The optimal pair is $(\bar{x}_T(\cdot), \bar{\sigma}_T(\cdot))$. We say that $\bar{x}_T(\cdot)$ is an optimal trajectory, and $\bar{\sigma}_T(\cdot)$ is an optimal relaxed control.

Correspondingly, denote

$$\mathcal{M}_S(\mathbb{R}^n, \mathbb{R}^m) = \left\{ (x, \theta) \in \mathbb{R}^n \times \mathcal{M}_+^1(\mathbb{R}^m) \ \Big| \ Ax + \int_{\mathbb{R}^m} Bu \, \theta(\mathrm{d}u) = 0 \right\},$$

$$\mathscr{M}_S(\mathbb{R}^n, \mathbb{R}^m) = \left\{ (\mu, \theta) \in \mathcal{M}_+^1(\mathbb{R}^n) \times \mathcal{M}_+^1(\mathbb{R}^m) \ \Big| \ \int_{\mathbb{R}^n} Ax \, \mu(\mathrm{d}x) + \int_{\mathbb{R}^m} Bu \, \theta(\mathrm{d}u) = 0 \right\}. \tag{1.5}$$

The relaxed optimization problem is

Problem $RQ(\mathbb{R}^n, \mathbb{R}^m)$: Find $(x^*, \theta^*) \in \mathcal{M}_S(\mathbb{R}^n, \mathbb{R}^m)$, such that

$$f(x^*) + \int_{\mathbb{R}^m} g(u) \, \theta^*(\mathrm{d}u) = \inf_{(x, \theta) \in \mathcal{M}_S(\mathbb{R}^n, \mathbb{R}^m)} \left(f(x) + \int_{\mathbb{R}^m} g(u) \, \theta(\mathrm{d}u) \right).$$

Denote P_* be the solution set of Problem $RQ(\mathbb{R}^n, \mathbb{R}^m)$, and S_* be the projection of P_* on $\mathbb{R}^n$. For any $(x^*, \theta^*) \in P_*$, we call x^* an optimal equilibrium state, and call θ^* an optimal equilibrium measure.

We will prove the existence of Problem $RP(x_0, T)$ and Problem $RQ(\mathbb{R}^n, \mathbb{R}^m)$ in the next two sections.

There are different versions of turnpike properties in the literature. We aim to analyze three kinds of turnpike properties, which are progressively enhanced. For convenience, we state the definitions as follows (see [2, 12, 17], etc.).

Definition 1.1 (Integral Turnpike Property)**.** We say Problem $RP(x_0, T)$ has integral turnpike property, if it holds that

$$\frac{1}{T} \int_0^T \bar{x}_T(t) \, \mathrm{d}t \to x^*, \quad \text{as } T \to +\infty. \tag{1.6}$$

Definition 1.2 (Mean Square Turnpike Property)**.** We say Problem $RP(x_0, T)$ has mean square turnpike property, if it holds that

$$\frac{1}{T} \int_0^T |\bar{x}_T(t) - x^*|^2 \, \mathrm{d}t \to 0, \quad \text{as } T \to +\infty. \tag{1.7}$$

Definition 1.3 (Exponential Turnpike Property)**.** We say Problem $RP(x_0, T)$ has exponential turnpike property, if it holds that

$$|\bar{x}_T(t) - x^*| \leq C_*(e^{-\lambda_* t} + e^{-\lambda_*(T-t)}), \quad \forall \, t \in [0, T], \ T > T_* \tag{1.8}$$

for some $T_* > 0$, $C_* > 0$ and $\lambda_* > 0$, independent of $T > 0$.

Remark 1.4. In Sections 4–6, we will prove that the convergence in (1.6), (1.7) and the estimation in (1.8) are uniform for $x_0 \in X$, where $X \subseteq \mathbb{R}^n$ is bounded. Moreover, we will see that the adjoint function $\bar{\psi}_T(\cdot)$ has the same turnpike properties.

Remark 1.5. The above three kinds of turnpike properties are progressively enhanced. Precisely, it holds that

$$\text{Integral Turnpike} \quad \overset{\Longleftarrow}{\nRightarrow} \quad \text{Mean Square Turnpike} \quad \overset{\Longleftarrow}{\nRightarrow} \quad \text{Exponential Turnpike.}$$

In order to illustrate this, an example will be given in Section 7.

The rest of the paper is organized as follows: In Sections 2 and 3, we prove the existence results and necessary conditions of Problems $RQ(\mathbb{R}^n, \mathbb{R}^m)$ and $RP(x_0, T)$, respectively. In Sections 4–6, we present the integral

turnpike property, the mean square turnpike property and the exponential turnpike property, respectively. Subsequently, we take some examples to illustrate our results in Section 7. Finally, we state further remarks in the last section.

2. Relaxed optimization problem

In this section, we will study the existence and necessary conditions of Problem $RQ(\mathbb{R}^n, \mathbb{R}^m)$.

Firstly, we introduce auxiliary problems as follows:

$$RQ(X,U): \quad \inf_{(x,\theta)\in\mathcal{M}_S(X,U)} \Big(f(x) + \int_U g(u)\,\theta(\mathrm{d}u) \Big),$$

and

$$\mathscr{RQ}(X,U): \quad \inf_{(\mu,\theta)\in\mathscr{M}_S(X,U)} \Big(\int_X f(x)\,\mu(\mathrm{d}x) + \int_U g(u)\,\theta(\mathrm{d}u) \Big),$$

where $X \subseteq \mathbb{R}^n$, $U \subseteq \mathbb{R}^m$, $\mathcal{M}_S(X,U)$ and $\mathscr{M}_S(X,U)$ are defined in (1.5) by replacing $\mathbb{R}^n$ and $\mathbb{R}^m$ with X and U.

Meanwhile, for two measures θ_1 and θ_2 on U, define

$$\|\theta_1 - \theta_2\| = |\theta_1 - \theta_2|(U)$$

$$= \sup\left\{ \sum_{j=1}^k |\theta_1(A_j) - \theta_2(A_j)| \ \Big|\ \bigcup_{j=1}^k A_j = U,\ A_1,\ A_2,\ \cdots,\ A_k \text{ are pairwise disjoint sets} \right\}.$$

Then, we establish the existence and necessary conditions of Problem $RQ(X,U)$.

Lemma 2.1. *Let $X \subseteq \mathbb{R}^n$ be convex and compact, and $U \subseteq \mathbb{R}^m$ be compact. Moreover, let $f(\cdot) : X \to \mathbb{R}$ be continuous and convex, and $g(\cdot) : U \to \mathbb{R}$ be continuous. Then,*

1. *Problem $RQ(X,U)$ has at least one solution;*
2. *if (x^*, θ^*) is a solution of Problem $RQ(X,U)$, there exists a nontrivial pair $(\psi_0^*, \psi^*) \in \mathbb{R} \times \mathbb{R}^n$, such that*

$$\psi_0^* \geq 0, \tag{2.1}$$

$$\langle \psi^*, w \rangle = 0, \quad \text{if } A^\top w = 0,\ B^\top w = 0,\ w \in \mathbb{R}^n, \tag{2.2}$$

$$\psi_0^* f(x^*) + \langle \psi^*, Ax^* \rangle \leq \psi_0^* f(x) + \langle \psi^*, Ax \rangle, \quad \forall\, x \in X, \tag{2.3}$$

$$\psi_0^* \int_U g(u)\,\theta^*(\mathrm{d}u) + \Big\langle \psi^*, \int_U Bu\,\theta^*(\mathrm{d}u) \Big\rangle$$
$$\leq \psi_0^* \int_U g(u)\,\theta(\mathrm{d}u) + \Big\langle \psi^*, \int_U Bu\,\theta(\mathrm{d}u) \Big\rangle, \quad \forall\, \theta \in \mathcal{M}_+^1(U), \tag{2.4}$$

and inequality (2.4) *can be rewritten equivalently as*

$$\text{supp}\, \theta^* \subseteq \left\{ u \in U \;\Big|\; \psi_0^* g(u) + \langle \psi^*, Bu \rangle = \min_{v \in U} \left(\psi_0^* g(v) + \langle \psi^*, Bv \rangle \right) \right\}. \tag{2.5}$$

The proof of the above lemma is quite standard, we give it in Appendix A.

Theorem 2.2. *Let* $f(\cdot) : \mathbb{R}^n \to \mathbb{R}$ *be continuously differentiable and convex, and* $g(\cdot) : \mathbb{R}^m \to \mathbb{R}$ *be continuously differentiable. Moreover, let* $f(\cdot)$ *and* $g(\cdot)$ *satisfy*

$$\lim_{x \to \infty} \frac{f(x)}{|x|} = +\infty, \quad \lim_{u \to \infty} \frac{g(u)}{|u|} = +\infty, \tag{2.6}$$

and $\mathcal{M}_S(\mathbb{R}^n, \mathbb{R}^m)$ *be nonempty. Then,*

1. *Problem* $RQ(\mathbb{R}^n, \mathbb{R}^m)$ *has at least one solution;*
2. *if* (x^*, θ^*) *is a solution of Problem* $RQ(\mathbb{R}^n, \mathbb{R}^m)$*, there exists an adjoint vector* $\psi^* \in \mathbb{R}^n$*, such that*

$$\langle \psi^*, w \rangle = 0, \quad \text{if } A^\top w = 0, \; B^\top w = 0, \; w \in \mathbb{R}^n, \tag{2.7}$$

$$f(x^*) + \langle \psi^*, Ax^* \rangle \leq f(x) + \langle \psi^*, Ax \rangle, \quad \forall\, x \in \mathbb{R}^n, \tag{2.8}$$

$$\int_{\mathbb{R}^m} g(u)\, \theta^*(\mathrm{d}u) + \left\langle \psi^*, \int_{\mathbb{R}^m} Bu\, \theta^*(\mathrm{d}u) \right\rangle$$
$$\leq \int_{\mathbb{R}^m} g(u)\, \theta(\mathrm{d}u) + \left\langle \psi^*, \int_{\mathbb{R}^m} Bu\, \theta(\mathrm{d}u) \right\rangle, \quad \forall\, \theta \in \mathcal{M}_+^1(\mathbb{R}^m), \tag{2.9}$$

and inequality (2.9) *can be rewritten equivalently as*

$$\text{supp}\, \theta^* \subseteq \left\{ u \in \mathbb{R}^m \;\Big|\; g(u) + \langle \psi^*, Bu \rangle = \min_{v \in \mathbb{R}^m} \left(g(v) + \langle \psi^*, Bv \rangle \right) \right\}. \tag{2.10}$$

Proof. 1. It is noticeable that $\mathcal{M}_+^1(\mathbb{R}^n)$ is not sequentially compact.

For any pair $(x_0, \theta_0) \in \mathcal{M}_S(\mathbb{R}^n, \mathbb{R}^m)$, there exists an $M > 0$ satisfying[1]

$$|x_0| \leq M, \quad \text{supp}\, \theta_0 \subseteq B_M^m(0).$$

Then, for any $k \geq M$, $\mathcal{M}_S\left(\overline{B_k^n(0)}, \overline{B_k^m(0)} \right)$ is nonempty. From Lemma 2.1, there is a pair $(x_k^*, \theta_k^*) \in \mathcal{M}_S\left(\overline{B_k^n(0)}, \overline{B_k^m(0)} \right)$, which is a solution of Problem $RQ\left(\overline{B_k^n(0)}, \overline{B_k^m(0)} \right)$. Moreover, there is a nontrivial pair $(\psi_{0,k}^*, \psi_k^*) \in \mathbb{R} \times \mathbb{R}^n$, such that

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[1]The set $B_M^m(0)$ represents an m-dimensional open ball centered at 0 with radius M.

$$\psi_{0,k}^* \geq 0,$$

$$\langle \psi_k^*, w \rangle = 0, \quad \text{if } A^\top w = 0, \; B^\top w = 0, \; w \in \mathbb{R}^n, \tag{2.11}$$

$$\psi_{0,k}^* f(x_k^*) + \langle \psi_k^*, A x_k^* \rangle \leq \psi_{0,k}^* f(x) + \langle \psi_k^*, Ax \rangle, \quad \forall \, x \in \overline{B_k^n(0)}, \tag{2.12}$$

$$\operatorname{supp} \theta_k^* \subseteq V_k \triangleq \left\{ u \in \overline{B_k^m(0)} \; \middle| \; \psi_{0,k}^* g(u) + \langle \psi_k^*, Bu \rangle = \min_{|v| \leq k} \left(\psi_{0,k}^* g(v) + \langle \psi_k^*, Bv \rangle \right) \right\}. \tag{2.13}$$

Since $\lim_{u \to \infty} g(u) = +\infty$ and $g(\cdot)$ is continuous, it holds that $g(\cdot)$ is bounded below. Combining with $\lim_{x \to \infty} f(x) = +\infty$, we realize that $\{x_k^*\}$ is bounded, which means that

$$x_k^* \in W \subseteq B_k^n(0), \quad \forall \, k \gg 1,$$

for some compact set $W \subseteq \mathbb{R}^n$, independent of k. Then (2.12) implies that

$$\psi_{0,k}^* \nabla f(x_k^*) + A^\top \psi_k^* = 0, \quad \forall \, k \gg 1. \tag{2.14}$$

Similarly, we have that

$$\psi_{0,k}^* \nabla g(u) + B^\top \psi_k^* = 0, \quad \forall \, u \in B_k^m(0) \cap V_k, \; k = 1, 2, \ldots. \tag{2.15}$$

We claim that $\psi_{0,k}^* \neq 0$ for any $k \gg 1$. Otherwise, if $B_k^m(0) \cap V_k \neq \varnothing$, then

$$B^\top \psi_k^* = 0.$$

Combining the above with (2.11) and (2.14), we deduce that $\langle \psi_k^*, \psi_k^* \rangle = 0$, which contradicts to the triviality of $(\psi_{0,k}^*, \psi_k^*)$. On the other hand, if $B_k^m(0) \cap V_k = \varnothing$, then $\operatorname{supp} \theta_k^* \subseteq \partial B_k^m(0)$,

$$f(x_k^*) + \int_{\overline{B_k^m(0)}} g(u) \, \theta_k^*(\mathrm{d}u) \geq \min_{x \in \mathbb{R}^n} f(x) + \min_{|u|=k} g(u) \to +\infty, \quad \text{as } k \to +\infty,$$

and

$$f(x_k^*) + \int_{\overline{B_k^m(0)}} g(u) \, \theta_k^*(\mathrm{d}u) \leq f(x_0) + \int_{\mathbb{R}^n} g(u) \, \theta_0(\mathrm{d}u), \quad \forall \, k \geq M, \tag{2.16}$$

which is a contradiction.

Based on the analysis above, we have that when $k \gg 1$, $\psi_{0,k}^* \neq 0$, that is $\psi_{0,k}^* > 0$. Thus, (2.14) and (2.15) can be rewritten as

$$\nabla f(x_k^*) + A^\top \frac{\psi_k^*}{\psi_{0,k}^*} = 0, \quad \forall \, k \gg 1,$$

$$\nabla g(u) + B^\top \frac{\psi_k^*}{\psi_{0,k}^*} = 0, \quad \forall \, u \in B_k^m(0) \cap V_k, \; k \gg 1.$$

Moreover, since $g(\cdot) \in \mathrm{C}^1(\mathbb{R}^n; \mathbb{R})$, we claim that $\left\{ \frac{B^\top \psi_k^*}{\psi_{0,k}^*} \right\}$ is bounded. Otherwise, there is a subsequence (without relabeling it) $M_k \to +\infty$ as $k \to +\infty$, such that

$$M_k < k, \quad \forall\, k = 1, 2, \ldots,$$

$$\operatorname{supp} \theta_k^* \subseteq \overline{B_k^m(0)} \setminus B_{M_k}^m(0), \quad \forall\, k = 1, 2, \ldots,$$

which implies that

$$f(x_k^*) + \int_{\overline{B_k^m(0)}} g(u)\, \theta_k^*(\mathrm{d}u) \geq \min_{x \in \mathbb{R}^n} f(x) + \min_{|u| = M_k} g(u) \to +\infty, \quad \text{as } k \to +\infty.$$

This contradicts to (2.16). Thus, $\left\{ \frac{B^\top \psi_k^*}{\psi_{0,k}^*} \right\}$ is bounded.

Combining with (2.6) and (2.13), we deduce that there is a compact set V, independent of k, such that $V_k \subseteq V$. Then, we can regard θ_k^* as a probability measure defined on V for any $k = 1, 2, \ldots$. It follows that (x_k^*, θ_k^*) converges to $(x^*, \theta^*) \in \mathbb{R}^n \times \mathcal{M}_+^1(\mathbb{R}^m)$ as $k \to +\infty$, at least along a subsequence. Thus, we get the existence of solution to Problem $RQ(\mathbb{R}^n, \mathbb{R}^m)$.

2 By repeating the same strategies in Lemma 2.1 (2), we can get the necessary conditions of Problem $RQ(\mathbb{R}^n, \mathbb{R}^m)$. □

Remark 2.3. For any $(x^*, \theta^*) \in P_*$, the corresponding adjoint vector ψ^*, which satisfies (2.7)–(2.10), is unique. To see this, we assume that $\hat{\psi}$ is another vector satisfying (2.7)–(2.10). That is

$$\left\langle \hat{\psi}, w \right\rangle = 0, \quad \text{if } A^\top w = 0,\ B^\top w = 0,\ w \in \mathbb{R}^n, \tag{2.17}$$

$$f(x^*) + \left\langle \hat{\psi}, Ax^* \right\rangle \leq f(x) + \left\langle \hat{\psi}, Ax \right\rangle, \quad \forall\, x \in \mathbb{R}^n, \tag{2.18}$$

$$\operatorname{supp} \theta^* \subseteq \left\{ u \in \mathbb{R}^m \ \middle|\ g(u) + \left\langle \hat{\psi}, Bu \right\rangle = \min_{v \in \mathbb{R}^m} \left(g(v) + \left\langle \hat{\psi}, Bv \right\rangle \right) \right\}. \tag{2.19}$$

Then, by (2.8) and (2.18), we have

$$\nabla f(x^*) + A^\top \psi^* = \nabla f(x^*) + A^\top \hat{\psi} = 0.$$

Similarly, by (2.10) and (2.19), we get

$$\nabla g(u) + B^\top \psi^* = \nabla g(u) + B^\top \hat{\psi} = 0, \quad \forall\, u \in \operatorname{supp} \theta^*.$$

Noticing that $\operatorname{supp} \theta^*$ is nonempty, we deduce that

$$A^\top (\psi^* - \hat{\psi}) = 0, \quad B^\top (\psi^* - \hat{\psi}) = 0.$$

Therefore, it follows from (2.7) and (2.17) that

$$\left\langle \psi^*, \psi^* - \hat{\psi} \right\rangle = 0, \quad \left\langle \hat{\psi}, \psi^* - \hat{\psi} \right\rangle = 0,$$

which implies $\hat{\psi} = \psi^*$.

Remark 2.4. If we remove the convexity of $f(\cdot)$, then (2.7), (2.9) and (2.10) still hold. Equation (2.8) becomes

$$\nabla f(x^*) + A^\top \psi^* = 0. \tag{2.20}$$

Remark 2.5. If the convexity of $f(\cdot)$ is reduced to the "single point convexity" of $f(\cdot)$ at x^*, that is (when $f(\cdot)$ is continuous) if

$$\sum_{j=1}^k \alpha_j x_j = x^*, \quad \sum_{j=1}^k \alpha_j = 1, \quad \alpha_1, \alpha_2, \ldots, \alpha_k \geq 0, \quad x_1, x_2, \ldots, x_k \in \mathbb{R}^n,$$

then

$$f(x^*) \leq \sum_{j=1}^k \alpha_j f(x_j).$$

Thus, the conclusions in Theorem 2.2 are sill tenable.

Remark 2.6. If $f(\cdot)$ is convex, then P_* and S_* are convex sets. If $f(\cdot)$ is strictly convex, then S_* is a singleton.

Remark 2.7. If S_* is a singleton, then the adjoint vector ψ^* satisfying (2.7)–(2.10) is unique. In other words, ψ^* is independent of the equilibrium measure. To see this, suppose there are two adjoint vectors ψ^* and $\hat{\psi}$ with respect to $(x^*, \theta^*) \in P_*$ and $(x^*, \hat{\theta}) \in P_*$. Let $\tilde{\theta} = \frac{\hat{\theta} + \theta^*}{2}$. It is obvious that $(x^*, \tilde{\theta}) \in P_*$. Correspondingly, we have an adjoint vector $\tilde{\psi}$ satisfying (2.7)–(2.10). And for any $u \in \operatorname{supp} \theta^*$ and $v \in \operatorname{supp} \hat{\theta}$, it holds that u, $v \in \operatorname{supp} \tilde{\theta}$, which implies that

$$B^\top \psi^* = -\nabla g(u) = B^\top \tilde{\psi} = -\nabla g(v) = B^\top \hat{\psi}.$$

Further, we have that

$$A^\top \psi^* = -\nabla f(x^*) = A^\top \hat{\psi}.$$

Thus, $\hat{\psi} = \psi^*$ holds.

Remark 2.8. Assume that $g(\cdot) \in \mathrm{C}^1(\mathbb{R}^m; \mathbb{R})$ and $\displaystyle\lim_{u \to \infty} \frac{g(u)}{|u|} = +\infty$. Then by setting $A = 0, B = 0$ and $f(x) = |x|^2$, we can see that there exists a $\theta^* \in \mathcal{M}_+^1(\mathbb{R}^m)$ such that

$$\int_{\mathbb{R}^m} g(u)\, \theta^*(\mathrm{d}u) = \inf_{\theta \in \mathcal{M}_+^1(\mathbb{R}^m)} \int_{\mathbb{R}^m} g(u)\, \theta(\mathrm{d}u).$$

3. Optimal relaxed control problems

Now, we will discuss the existence and necessary conditions of Problem $RP(x_0, T)$. Like what we have done in Section 2, we define auxiliary optimal relaxed control problems as follows.

Problem $RP_U(x_0, T)$: Find $\bar{\sigma}(\cdot) \in \mathcal{R}_T(U)$, such that

$$J_T(\bar{\sigma}(\cdot); x_0) = \inf_{\bar{\sigma}(\cdot) \in \mathcal{R}_T(U)} J_T(\bar{\sigma}(\cdot); x_0),$$

where $J_T(\cdot; \cdot)$ is defined by (1.4).

Lemma 3.1. *Let $U \subseteq \mathbb{R}^m$ be compact, $f(\cdot) : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable, and $g(\cdot) : U \to \mathbb{R}$ be continuous. Then, Problem $RP_U(x_0, T)$ has at least one solution. Moreover, if $(\bar{x}(\cdot), \bar{\sigma}(\cdot))$ is an optimal pair of Problem $RP_U(x_0, T)$, then it holds that*

$$\operatorname{supp} \bar{\sigma}(t) \subseteq \left\{ u \in U \ \Big| \ g(u) + \langle \bar{\psi}(t), Bu \rangle = \min_{v \in U} \Big(g(v) + \langle \bar{\psi}(t), Bv \rangle \Big) \right\}, \quad \text{a.e. } t \in [0, T],$$

where $\bar{\psi}(\cdot)$ satisfies

$$\begin{cases} \dfrac{\mathrm{d}\bar{\psi}(t)}{\mathrm{d}t} = -A^\top \bar{\psi}(t) - \nabla f(\bar{x}(t)), \quad t \in [0, T], \\ \bar{\psi}(T) = 0. \end{cases}$$

Remark 3.2. We omit the proof of Lemma 3.1 since it is quite standard. One can see [7, 8] for reference.

For Problem $RP(x_0, T)$, we have the following result.

Theorem 3.3. *Assume that $f(\cdot) : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable, $g(\cdot) : \mathbb{R}^m \to \mathbb{R}$ is continuous and (2.6) holds. Then, Problem $RP(x_0, T)$ has at least one solution. Moreover, if $(\bar{x}_T(\cdot), \bar{\sigma}_T(\cdot))$ is an optimal pair of Problem $RP(x_0, T)$, it holds that*

$$\operatorname{supp} \bar{\sigma}_T(t) \subseteq \left\{ u \in \mathbb{R}^m \ \Big| \ g(u) + \langle \bar{\psi}_T(t), Bu \rangle = \min_{v \in \mathbb{R}^m} \Big(g(v) + \langle \bar{\psi}_T(t), Bv \rangle \Big) \right\}, \quad \text{a.e. } t \in [0, T],$$

where $\bar{\psi}_T(\cdot)$ satisfies

$$\begin{cases} \dfrac{\mathrm{d}\bar{\psi}_T(t)}{\mathrm{d}t} = -A^\top \bar{\psi}_T(t) - \nabla f(\bar{x}_T(t)), \quad t \in [0, T], \\ \bar{\psi}_T(T) = 0. \end{cases}$$

Proof. The necessary conditions of Problem $RP(x_0, T)$ are easy to derive. We focus on the existence of a solution for Problem $RP(x_0, T)$.

From Lemma 3.1, for any $k \geq 1$, Problem $RP_{\overline{B_k^m(0)}}(x_0, T)$ has a solution $(\bar{x}_k(\cdot), \bar{\sigma}_k(\cdot))$, satisfying

$$\operatorname{supp} \bar{\sigma}_k(t) \subseteq \left\{ u \in \overline{B_k^m(0)} \ \Big| \ g(u) + \langle \bar{\psi}_k(t), Bu \rangle = \min_{v \in \overline{B_k^m(0)}} \Big(g(v) + \langle \bar{\psi}_k(t), Bv \rangle \Big) \right\}, \quad \text{a.e. } t \in [0, T], \qquad (3.1)$$

where $\bar{\psi}_k(\cdot)$ satisfies

$$\begin{cases} \dfrac{\mathrm{d}\bar{\psi}_k(t)}{\mathrm{d}t} = -A^\top \bar{\psi}_k(t) - \nabla f(\bar{x}_k(t)), \quad t \in [0, T], \\ \bar{\psi}_k(T) = 0. \end{cases} \qquad (3.2)$$

Since $(\bar{x}_k(\cdot), \bar{\sigma}_k(\cdot))$ is an optimal pair of Problem $RP_{\overline{B_k^m(0)}}(x_0, T)$ and $f(\cdot)$ is bounded below, we have an $\widehat{M} > 0$, such that

$$\left| \int_0^T \mathrm{d}t \int_{\overline{B_k^m(0)}} g(u)\, \bar{\sigma}_k(t)(\mathrm{d}u) \right| \leq \widehat{M}, \quad \forall\, k = 1, 2, \ldots.$$

Combining the above with (2.6), we have an $\widetilde{M} > 0$, such that

$$\left| \int_0^T \mathrm{d}t \int_{\overline{B_k^m(0)}} |u|\, \bar{\sigma}_k(t)(\mathrm{d}u) \right| \leq \widetilde{M}, \quad \forall\, k = 1, 2, \ldots.$$

Then, $\{\bar{x}_k(\cdot)\}_{k=1}^{+\infty}$ is uniformly bounded on $[0, T]$, which implies that $\{\nabla f(\bar{x}_k(t))\}_{k=1}^{+\infty}$ is uniformly bounded on $[0, T]$. Combining the above with (3.2), we deduce that $\{\bar{\psi}_k(\cdot)\}_{k=1}^{+\infty}$ is uniformly bounded on $[0, T]$. It follows from (2.6) and (3.1) that there exists a compact set $V \subseteq \mathbb{R}^m$, independent of k, such that

$$\mathrm{supp}\, \bar{\sigma}_k(t) \subseteq V, \quad \text{a.e. } t \in [0, T].$$

It follows that $\{\bar{x}_k(\cdot)\}_{k=1}^{+\infty}$ is equi-continuous. Therefore, at least along a subsequence, $\{\bar{x}_k(\cdot)\}_{k=1}^{+\infty}$ converges uniformly to $\bar{x}_T(\cdot)$ on $[0, T]$ as $k \to +\infty$, and $\{\bar{\sigma}_k(\cdot)\}_{k=1}^{+\infty}$ converges to $\bar{\sigma}_T(\cdot)$ as $k \to +\infty$.

Furthermore, for any $k \gg 1$, it is easy to verify that $(\bar{x}_T(\cdot), \bar{\sigma}_T(\cdot))$ is an optimal pair of Problem $RP_{\overline{B_k^m(0)}}(x_0, T)$. Thus, we know that Problem $RP(x_0, T)$ admits at least one solution. $\qquad\square$

Remark 3.4. From the proof above, we see that if x_0 takes value in a bounded set $X \subseteq \mathbb{R}^n$, the corresponding optimal pair $(\bar{x}_T(\cdot), \bar{\sigma}_T(\cdot))$ and the adjoint function $\bar{\psi}_T(\cdot)$ are uniformly bounded with respect to $x_0 \in X$.

It is worth noticing that we haven't proved the uniform boundedness of $(\bar{x}_T(\cdot), \bar{\sigma}_T(\cdot))$ and $\bar{\psi}_T(\cdot)$ with respect to $T > 0$ by now.

4. INTEGRAL TURNPIKE PROPERTY

In this section, we will discuss the integral turnpike property of Problem $RP(x_0, T)$. We need the following assumptions.

(A1) The pair (A, B) is controllable.

(A2) The functions $f(\cdot): \mathbb{R}^n \to \mathbb{R}$ and $g(\cdot): \mathbb{R}^m \to \mathbb{R}$ are continuously differentiable, which satisfy

$$\lim_{x \to \infty} \frac{f(x)}{|x|} = +\infty, \quad \lim_{u \to \infty} \frac{g(u)}{|u|} = +\infty. \tag{4.1}$$

(A3) The function $f(\cdot)$ is convex.

Before considering the integral turnpike property, we review the existence and the necessary conditions of Problem $RP(x_0, T)$ and Problem $RQ(\mathbb{R}^n, \mathbb{R}^m)$, obtained in Sections 2 and 3. Assume that (A2) holds. If $(\bar{x}_T(\cdot), \bar{\sigma}_T(\cdot))$ is an optimal pair of Problem $RP(x_0, T)$, then

$$\begin{cases} \dfrac{\mathrm{d}\bar{x}_T(t)}{\mathrm{d}t} = A\bar{x}_T(t) + B \displaystyle\int_{\mathbb{R}^m} u\, \bar{\sigma}_T(t)(\mathrm{d}u), & t \in [0, T], \\[2mm] \dfrac{\mathrm{d}\bar{\psi}_T(t)}{\mathrm{d}t} = -A^\top \bar{\psi}_T(t) - \nabla f(\bar{x}_T(t)), & t \in [0, T], \\[2mm] \bar{x}(0) = x_0,\ \bar{\psi}_T(T) = 0, \end{cases}$$

and

$$\operatorname{supp} \bar{\sigma}_T(t) \subseteq \left\{ u \in \mathbb{R}^m \ \Big| \ g(u) + \langle \bar{\psi}_T(t), Bu \rangle = \min_{v \in \mathbb{R}^m} \left(g(v) + \langle \bar{\psi}_T(t), Bv \rangle \right) \right\},$$

$$\subseteq \left\{ u \in \mathbb{R}^m \ \Big| \ \nabla g(u) + B^\top \bar{\psi}_T(t) = 0 \right\}, \quad \text{a.e. } t \in [0,T]. \tag{4.2}$$

On the other hand, by Theorem 2.2, there exists a solution (x^*, θ^*) of Problem $RQ(\mathbb{R}^n, \mathbb{R}^m)$ and an adjoint vector ψ^* satisfying (2.7)–(2.10). We have

$$f(x^*) + \int_{\mathbb{R}^m} g(u)\,\theta^*(\mathrm{d}u) + \left\langle \psi^*, Ax^* + \int_{\mathbb{R}^m} Bu\,\theta^*(\mathrm{d}u) \right\rangle$$

$$\leq f(x) + \int_{\mathbb{R}^m} g(u)\,\theta(\mathrm{d}u) + \left\langle \psi^*, Ax + \int_{\mathbb{R}^m} Bu\,\theta(\mathrm{d}u) \right\rangle, \quad \forall\, x \in \mathbb{R}^n,\ \theta \in \mathcal{M}_+^1(\mathbb{R}^m), \tag{4.3}$$

and

$$\begin{cases} \dfrac{\mathrm{d}(\bar{x}_T(t) - x^*)}{\mathrm{d}t} = A(\bar{x}_T(t) - x^*) + B \displaystyle\int_{\mathbb{R}^m} u\,(\bar{\sigma}_T(t) - \theta^*)(\mathrm{d}u), & t \in [0,T], \\[2ex] \dfrac{\mathrm{d}(\bar{\psi}_T(t) - \psi^*)}{\mathrm{d}t} = -A^\top(\bar{\psi}_T(t) - \psi^*) - \left(\nabla f(\bar{x}_T(t)) - \nabla f(x^*) \right), & t \in [0,T], \\[2ex] \bar{x}_T(0) - x^* = x_0 - x^*, \quad \bar{\psi}_T(T) - \psi^* = -\psi^*. \end{cases} \tag{4.4}$$

Denote

$$I^* = f(x^*) + \int_{\mathbb{R}^m} g(u)\,\theta^*(\mathrm{d}u)$$

$$= f(x^*) + \int_{\mathbb{R}^m} g(u)\,\theta^*(\mathrm{d}u) + \left\langle \psi^*, Ax^* + \int_{\mathbb{R}^m} Bu\,\theta^*(\mathrm{d}u) \right\rangle.$$

And we assume X be a bounded subset in $\mathbb{R}^n$ in the following part of the paper.

Theorem 4.1. *Let (A1), (A2) and (A3) hold. Let $(\bar{x}_T(\cdot), \bar{\sigma}_T(\cdot))$ be an optimal pair of Problem $RP(x_0, T)$, and $\bar{\psi}_T(\cdot)$ be the corresponding adjoint function. Then, $\{\bar{x}_T(\cdot)\}_{T>0}$ and $\{\bar{\psi}_T(\cdot)\}_{T>0}$ are uniformly bounded with respect to $T > 0$ and $x_0 \in X$. Moreover, there is a compact set $V \subseteq \mathbb{R}^n$, such that $\operatorname{supp}\bar{\sigma}_T(\cdot) \subseteq V$ for any $T > 0$ and $x_0 \in X$.*

Proof.
Step 1. $\{\bar{x}_T(T)\}_{T>0}$ is uniformly bounded with respect to $T > 0$ and $x_0 \in X$.

Take an $(x^*, \theta^*) \in P_*$. The controllability of the pair (A, B) ensures that the matrix

$$\int_0^1 e^{(1-s)A} BB^\top e^{-sA^\top}\,\mathrm{d}s$$

is invertible. Let

$$\tilde{u}(t) = B^\top e^{-tA} \left(\int_0^1 e^{(1-s)A} BB^\top e^{-sA^\top}\,\mathrm{d}s \right)^{-1} (x^* - e^A x_0), \quad \forall\, t \in [0,1].$$

Then,

$$x(1; x_0, \tilde{u}(\cdot)) = x^*.$$

Let

$$\tilde{\sigma}(t) = \begin{cases} \delta_{\tilde{u}(t)}, & t \in [0,1], \\ \theta^*, & t \in (1, +\infty). \end{cases}$$

We deduce that

$$x(t; x_0, \tilde{\sigma}(\cdot)) = x^*, \quad \forall \, t \geq 1.$$

When $T \geq 1$, it holds that

$$\begin{aligned} J_T(\bar{\sigma}_T(\cdot); x_0) - TI^* &\leq J_T(\tilde{\sigma}(\cdot); x_0) - TI^* \\ &= \int_0^T \left(f(x(t; x_0, \tilde{\sigma}(\cdot)) + \int_{\mathbb{R}^m} g(u) \, \tilde{\sigma}(t)(\mathrm{d}u) - I^* \right) \mathrm{d}t \\ &= \int_0^1 \left(f(x(t; x_0, \tilde{u}(\cdot)) + g(\tilde{u}(t)) - I^* \right) \mathrm{d}t \\ &\triangleq M_0(X), \quad \forall \, T \geq 1, \end{aligned} \tag{4.5}$$

where $M_0(X)$ is a constant, independent of $x_0 \in X$ and $T \geq 1$. Meanwhile, from (4.1) and the continuity of $f(\cdot)$ and $g(\cdot)$, it is easy to see that there is an $M > 0$ satisfying

$$f(x) + g(u) - I^* \geq (\|A\| \, |\psi^*| + \|B\| \, |\psi^*| + 1)(|x| + |u|) - M, \quad \forall \, (x, u) \in \mathbb{R}^n \times \mathbb{R}^m. \tag{4.6}$$

We can suppose $X \supseteq \overline{B_{2M}^n(0)} \supseteq S_*$ without loss of generality. It follows that

$$\begin{aligned} |\bar{x}_T(T) - x_0| &= \left| \int_0^T A\bar{x}_T(t) \, \mathrm{d}t + \int_0^T \mathrm{d}t \int_{\mathbb{R}^m} Bu \, \bar{\sigma}_T(t)(\mathrm{d}u) \right| \\ &\leq (\|A\| + \|B\|) \int_0^T \mathrm{d}t \int_{\mathbb{R}^m} (|\bar{x}_T(t)| + |u|) \, \bar{\sigma}_T(t)(\mathrm{d}u) \\ &\leq (\|A\| + \|B\|) \int_0^T \mathrm{d}t \int_{\mathbb{R}^m} (f(\bar{x}_T(t)) + g(u) - I^* + M) \, \bar{\sigma}_T(t)(\mathrm{d}u) \\ &= (\|A\| + \|B\|) \Big(J_T(\bar{\sigma}_T(\cdot); x_0) - TI^* + MT \Big) \\ &\leq (\|A\| + \|B\|)(M_0(X) + MT), \quad \forall \, T \geq 1, \, x_0 \in X. \end{aligned}$$

When $T \in (0,1)$, we have similar results by adopting the similar strategies.

Then, for any $T > 0$, define

$$R_T = \bigcup_{\substack{x_0 \in X \\ 0 \leq S \leq T}} \left\{ \bar{x}_S(S) \mid \bar{x}_S(\cdot) \text{ is an optimal trajectory of Problem } RP(x_0, S) \right\}.$$

It is easy to see that R_T is bounded and monotonically increasing.

For $T \geq T^\vee \triangleq \frac{M_0(X)}{M} + 3$, $x_0 \in X$ and the corresponding optimal trajectory $\bar{x}_T(\cdot)$, we claim that the Lebesgue measure of $\{t \in [0, T] \mid |\bar{x}_T(t)| \leq 2M\}$ is bigger than 1. Otherwise, (4.6) implies that

$$J_T(\bar{\sigma}_T(\cdot); x_0) - TI^* \geq -M + (T-1)M > M_0(X),$$

which contradicts to (4.5).

Then, there is a $t_0 \in [1, T]$, such that $\bar{x}_T(t_0) \in \overline{B_{2M}^n(0)} \subseteq X$. By Bellman's optimality principle, $\bar{x}_T(\cdot)$ is also an optimal trajectory of Problem $RP(\bar{x}_T(t_0), T - t_0)$. Therefore

$$\bar{x}_T(T) \in R_{T-t_0} \subseteq R_{T-1}, \quad \forall\, T \geq T^\vee + 1$$

and

$$R_T \subseteq R_{T-1}, \quad \forall\, T \geq T^\vee + 1.$$

Combining the above with the monotonicity of R_T, we deduce that

$$R_T \subseteq R_{T^\vee}, \quad \forall\, T > 0,$$

which means $\{\bar{x}_T(T)\}_{T>0}$ is uniformly bounded with respect to $T > 0$ and $x_0 \in X$.

Step 2. $\{\bar{x}_T(\cdot)\}_{T>0}$ is uniformly bounded with respect to $T > 0$ and $x_0 \in X$.
    We have

$$\int_0^T \left\langle \psi^*, A\bar{x}_T(t) + \int_{\mathbb{R}^m} Bu\, \bar{\sigma}_T(t)(\mathrm{d}u) \right\rangle \mathrm{d}t$$
$$= \int_0^T \left\langle \psi^*, \frac{\mathrm{d}\bar{x}_T(t)}{\mathrm{d}t} \right\rangle \mathrm{d}t = \langle \psi^*, \bar{x}_T(T) - x_0 \rangle.$$

Then, there is an $M_1(X) > 0$, independent of $x_0 \in X$ and $T > 0$, such that

$$\int_0^T \left(f(\bar{x}_T(t)) + \int_{\mathbb{R}^m} g(u)\, \bar{\sigma}_T(t)(\mathrm{d}u) + \left\langle \psi^*, A\bar{x}_T(t) + \int_{\mathbb{R}^m} Bu\, \bar{\sigma}_T(t)(\mathrm{d}u) \right\rangle - I^* \right) \mathrm{d}t$$
$$\leq M_0(X) + \langle \psi^*, \bar{x}_T(T) - x_0 \rangle \leq M_1(X), \quad \forall\, T > 0,\ x_0 \in X. \tag{4.7}$$

For convenience, we only discuss the case $T \geq 1$ in the following part. The case $T < 1$ can be treated similarly. Choose $S \geq 0$ satisfying $[S, S+1] \subseteq [0, T]$. By (4.3), we have

$$\int_S^{S+1} \left(f(\bar{x}_T(t) + \int_{\mathbb{R}^m} g(u)\, \bar{\sigma}_T(t)(\mathrm{d}u) + \left\langle \psi^*, A\bar{x}_T(t) + \int_{\mathbb{R}^m} Bu\, \bar{\sigma}_T(t)(\mathrm{d}u) \right\rangle - I^* \right) \mathrm{d}t$$
$$\leq \int_0^T \left(f(\bar{x}_T(t) + \int_{\mathbb{R}^m} g(u)\, \bar{\sigma}_T(t)(\mathrm{d}u) + \left\langle \psi^*, A\bar{x}_T(t) + \int_{\mathbb{R}^m} Bu\, \bar{\sigma}_T(t)(\mathrm{d}u) \right\rangle - I^* \right) \mathrm{d}t$$
$$\leq M_1(X). \tag{4.8}$$

Combining (4.8) with (4.6), we deduce

$$\int_S^{S+1} \left(|\bar{x}_T(t)| + \int_{\mathbb{R}^m} |u|\, \bar{\sigma}_T(t)(\mathrm{d}u) \right) \mathrm{d}t \leq M_1(X) + M. \tag{4.9}$$

Then, there exists a $\xi \in [S, S+1]$, such that $|\bar{x}_T(\xi)| \leq M_1(X) + M$. Thus,

$$|\bar{x}_T(t)| = \left| e^{A(t-\xi)} \bar{x}_T(\xi) + \int_\xi^t \mathrm{d}s \int_{\mathbb{R}^m} e^{A(t-s)} Bu\, \bar{\sigma}_T(s)(\mathrm{d}u) \right|$$

$$\leq e^{\|A\|} |\bar{x}_T(\xi)| + e^{\|A\|} \|B\| \int_S^{S+1} \mathrm{d}s \int_{\mathbb{R}^m} |u|\, \bar{\sigma}_T(s)(\mathrm{d}u)$$

$$\leq e^{\|A\|} (1 + \|B\|)(M_1(X) + M)$$

$$\triangleq M_2(X), \quad \forall\, t \in [S, S+1],\ x_0 \in X,$$

which yields

$$|\bar{x}_T(t)| \leq M_2(X), \quad \forall\, t \in [0, T],\ x_0 \in X. \tag{4.10}$$

Step 3. $\{\bar{\psi}_T(\cdot)\}_{T>0}$ is uniformly bounded with respect to $T > 0$ and $x_0 \in X$.

Since (A, B) is controllable, we know that $(A^\top, B^\top)$ is observable. It implies the observability inequality

$$|\bar{\psi}_T(S)|^2 \leq c \left[\int_S^{S+1} |B^\top \bar{\psi}(t)|^2\, \mathrm{d}t + \int_S^{S+1} |\nabla f(\bar{x}_T(t))|^2\, \mathrm{d}t \right],$$

for some $c > 0$, independent of $T > 0$ and $S \in [0, T-1]$. Denote

$$E_S = \left\{ t \in [S, S+1] \ \middle|\ \mathrm{supp}\, \bar{\sigma}_T(t) \subseteq \mathbb{R}^m \setminus B_{M_3(X)}^m \right\},$$

where $M_3(X) = (2c\|B^\top\|^2 + 1)(M_1(X) + M)$. The inequality (4.9) implies that

$$M_3(X)|E_S| \leq M_1(X) + M.$$

Then, we have

$$|E_S| \leq \frac{1}{2c\|B^\top\|^2 + 1}.$$

Moreover, denote

$$F_S = \left\{ t \in [S, S+1] \ \middle|\ |B^\top \bar{\psi}_T(t)| \geq \max_{|u| \leq M_3(X)} |\nabla g(u)| + 1 \right\}.$$

It holds that $F_S \subseteq E_S$ from (4.2). Then, we have

$$|F_S| \leq \frac{1}{2c\|B^\top\|^2 + 1}.$$

We suppose that $|\bar{\psi}_T(\cdot)|$ takes the maximal value m_T on $[0, T]$ at the point S_T. If $S_T \in [T-1, T]$, we have that

$$m_T = |\bar{\psi}_T(S_T)|$$

$$= \left| \int_T^{S_T} e^{-A^\top(S_T - t)} \nabla f(\bar{x}_T(t))\, \mathrm{d}t \right|$$

$$\leq e^{\|A^\top\|} \int_{T-1}^T |\nabla f(\bar{x}_T(t))|\, \mathrm{d}t$$

$$\leq e^{\|A^\top\|} \max_{|x| \leq M_2(X)} |\nabla f(x)|.$$

If $S_T \in [0, T-1)$, then

$$
\begin{aligned}
m_T^2 &\le c\Big[\int_{S_T}^{S_T+1} |B^\top \bar\psi_T(t)|^2 \, \mathrm{d}t + \int_{S_T}^{S_T+1} |\nabla f(\bar x_T(t))|^2 \, \mathrm{d}t\Big] \\
&\le c\int_{[S_T, S_T+1]\setminus F_{S_T}} |B^\top \bar\psi_T(t)|^2 \, \mathrm{d}t + c\int_{F_{S_T}} |B^\top \bar\psi_T(t)|^2 \, \mathrm{d}t + c \max_{|x|\le M_2(X)} |\nabla f(x)|^2 \\
&\le c\Big(\max_{|u|\le M_3(X)} |\nabla g(u)| + 1\Big)^2 + \frac{c\|B^\top\|^2 m_T^2}{2c\|B^\top\|^2 + 1} + c \max_{|x|\le M_2(X)} |\nabla f(x)|^2 \\
&\le c\Big(\max_{|u|\le M_3(X)} |\nabla g(u)| + 1\Big)^2 + \frac{m_T^2}{2} + c \max_{|x|\le M_2(X)} |\nabla f(x)|^2.
\end{aligned}
$$

Thus, we deduce that

$$
m_T^2 \le 2c\Big(\max_{|u|\le M_3(X)} |\nabla g(u)| + 1\Big)^2 + 2c \max_{|x|\le M_2(X)} |\nabla f(x)|^2.
$$

This implies that $\bar\psi_T(\cdot)$ is uniformly bounded.

Step 4. The uniform boundedness of $\operatorname{supp} \bar\sigma_T(\cdot)$ can be obtained by the uniform boundedness of $\bar\psi_T(\cdot)$, (4.1) and (4.2). $\qquad\square$

By reviewing Remark 2.6, we know that if $f(\cdot)$ is strictly convex, then the optimal equilibrium state x^* is unique. In the following part of the paper, we need the following assumption.

(A3 $'$) The function $f(\cdot)$ is strictly convex.

We give the following integral turnpike property.

Theorem 4.2. *Let (A1), (A2) and (A3 $'$) hold. Then, for any $\varepsilon > 0$, there exists a $T_\varepsilon > 0$, such that*

$$
\begin{aligned}
&\Big|\frac{1}{T}\int_0^T \bar x_T(t)\,\mathrm{d}t - x^*\Big| + \Big|\frac{1}{T}\int_0^T \mathrm{d}t \int_{\mathbb{R}^m} Bu\,\bar\sigma_T(t)(\mathrm{d}u) - \int_{\mathbb{R}^m} Bu\,\theta^*(\mathrm{d}u)\Big| \\
&+ \Big|\frac{1}{T}\int_0^T \mathrm{d}t \int_{\mathbb{R}^m} g(u)\,\bar\sigma_T(t)(\mathrm{d}u) - \int_{\mathbb{R}^m} g(u)\,\theta^*(\mathrm{d}u)\Big| < \varepsilon, \quad \forall\, T \ge T_\varepsilon,
\end{aligned} \tag{4.11}
$$

where $(\bar x_T(\cdot), \bar\sigma_T(\cdot))$ is an optimal pair of Problem $RP(x_0, T)$, $(x^*, \theta^*) \in P_*$ and x_0 is any point in X.

Proof. If the conclusion does not hold, we have an $\varepsilon_0 > 0$, a sequence $\{T_k\}_{k=1}^{+\infty} \subseteq \mathbb{R}$, a sequence $\{x_k\}_{k=1}^{+\infty} \subseteq X$ and an optimal pair $(\bar x_{T_k}(\cdot), \bar\sigma_{T_k}(\cdot))$ of Problem $RP(x_k, T_k)$ $(k = 1, 2, \ldots)$, such that

$$
\lim_{k\to+\infty} T_k = +\infty,
$$

and

$$
\begin{aligned}
&\Big|\frac{1}{T_k}\int_0^{T_k} \bar x_{T_k}(t)\,\mathrm{d}t - x^*\Big| + \Big|\frac{1}{T_k}\int_0^{T_k} \mathrm{d}t \int_{\mathbb{R}^m} Bu\,\bar\sigma_{T_k}(t)(du) - \int_{\mathbb{R}^m} Bu\,\theta^*(du)\Big| \\
&+ \Big|\frac{1}{T_k}\int_0^{T_k} \mathrm{d}t \int_{\mathbb{R}^m} g(u)\,\bar\sigma_{T_k}(t)(du) - \int_{\mathbb{R}^m} g(u)\,\theta^*(du)\Big| \ge \varepsilon_0, \quad \forall\, k = 1, 2, \ldots.
\end{aligned} \tag{4.12}
$$

By Theorem 4.1, $\{\bar{x}_{T_k}(\cdot)\}_{k=1}^{+\infty}$ and $\{\operatorname{supp} \bar{\sigma}_{T_k}(\cdot)\}_{k=1}^{+\infty}$ are uniformly bounded, which implies two convergent subsequences, respectively. Without relabeling the subsequence, we suppose that

$$\frac{1}{T_k}\int_0^{T_k} \bar{x}_{T_k}(t)\,\mathrm{d}t \to \widehat{x}, \quad \frac{1}{T_k}\int_0^{T_k} \bar{\sigma}_{T_k}(t)\,\mathrm{d}t \to \hat{\theta}, \quad \text{as } k \to +\infty.$$

Since $f(\cdot)$ is convex and (4.5) holds, we have

$$f\Big(\frac{1}{T_k}\int_0^{T_k}\bar{x}_{T_k}(t)\,\mathrm{d}t\Big) + \int_{\mathbb{R}^m} g(u)\Big(\frac{1}{T_k}\int_0^{T_k}\bar{\sigma}_{T_k}(t)\,\mathrm{d}t\Big)(\mathrm{d}u)$$
$$\leq \frac{1}{T_k}\int_0^{T_k} f(\bar{x}_{T_k}(t))\,\mathrm{d}t + \frac{1}{T_k}\int_0^{T_k}\mathrm{d}t\int_{\mathbb{R}^m} g(u)\,\bar{\sigma}_{T_k}(t)(\mathrm{d}u)$$
$$\leq \frac{M_0(X)}{T_k} + I^*, \quad \forall\, k = 1, 2, \dots.$$

Let $k \to +\infty$, it follows that

$$f(\widehat{x}) + \int_{\mathbb{R}^m} g(u)\,\hat{\theta}(\mathrm{d}u) \leq I^*. \tag{4.13}$$

Meanwhile,

$$\frac{1}{T_k}\int_0^{T_k} A\bar{x}_{T_k}(t)\,\mathrm{d}t + \frac{1}{T_k}\int_0^{T_k}\mathrm{d}t\int_{\mathbb{R}^m} Bu\,\bar{\sigma}_{T_k}(t)(\mathrm{d}u) = \frac{\bar{x}_{T_k}(T_k) - x_k}{T_k}.$$

Since $\bar{x}_{T_k}(T_k)$ is bounded, we have that

$$A\widehat{x} + \int_{\mathbb{R}^m} Bu\,\hat{\theta}(du) = 0$$

by taking $k \to +\infty$. Then, we deduce that $(\widehat{x}, \hat{\theta}) \in P_*$. As we mentioned in Remark 2.6, S_* is a singleton. Thus, $\widehat{x} = x^*$ holds. Moreover,

$$\int_{\mathbb{R}^m} Bu\,\hat{\theta}(\mathrm{d}u) = -Ax^* = \int_{\mathbb{R}^m} Bu\,\theta^*(\mathrm{d}u).$$

And the optimality of (x^*, θ^*) makes (4.13) become an equality. Therefore,

$$\int_{\mathbb{R}^m} g(u)\,\hat{\theta}(\mathrm{d}u) = \int_{\mathbb{R}^m} g(u)\,\theta^*(\mathrm{d}u).$$

By taking $k \to +\infty$ in (4.12), we obtain a contradiction.

We get the proof. $\qquad\square$

5. Mean square turnpike property

In this section, we will state the mean square turnpike property. For this reason, we need an extra assumption.

(A4) The function $f(\cdot)$ is twice continuously differentiable in a deleted neighborhood of x^*, and the Hessian matrix f_{xx} is positive definite in the neighborhood.

Remark 5.1. The assumption (A4) holds in many cases. For instance, if

$$f(x) = \langle Cx, x \rangle \quad \text{or} \quad f(x) = e^{|x|},$$

where C is a positive definite matrix, the Hessian matrix f_{xx} is positive definite for any $x \in \mathbb{R}^n$, which implies (A4) holds. Moreover, if

$$f(x) = |x|^k, \quad \forall \, k = 2, 3, \dots,$$

the assumption (A4) holds, although $f_{xx}(0)$ does not exist or $f_{xx}(0) = 0$.

Theorem 5.2. *Let (A1), (A2), (A3′) and (A4) hold. Then, for any $\varepsilon > 0$, we have a $T_\varepsilon > 0$, such that*

$$\frac{1}{T} \int_0^T |\bar{x}_T(t) - x^*|^2 \, \mathrm{d}t < \varepsilon, \quad \forall \, T \geq T_\varepsilon,$$

where $(\bar{x}_T(\cdot), \bar{\sigma}_T(\cdot))$ is an optimal pair of Problem $RP(x_0, T)$, $x^* \in S_*$ and x_0 is an arbitrary point in X.

Proof. By (A4), there is a $\delta_0 > 0$, such that $f(\cdot)$ is twice continuously differentiable in $B_{\delta_0}^n(x^*) \setminus \{x^*\}$, and

$$f_{xx}(x) > 0, \quad \forall \, x \in B_{\delta_0}^n(x^*) \setminus \{x^*\}.$$

Noticing that $f(\cdot)$ is strictly convex, it is easy to see that for any $0 < \varepsilon < \min(\delta_0, \frac{1}{4})$, we can find an $\alpha_\varepsilon > 0$, such that

$$
f(x) - f(x^*) - \langle \nabla f(x^*), x - x^* \rangle
$$
$$
\geq \begin{cases} \alpha_\varepsilon \min \left(|x - x^*|^2, \delta_0^2 \right), & \text{if } x \in \mathbb{R}^n \setminus B_\varepsilon^n(x^*), \\ 0, & \text{if } x \in B_\varepsilon^n(x^*). \end{cases}
$$

Combining the above with Theorem 4.1 (see (4.10)), we have an $M_2(X) > 0$, such that

$$|\bar{x}_T(t)| \leq M_2(X), \quad \forall \, t \in [0, T], \; x_0 \in X.$$

Then,

$$
f(\bar{x}_T(t)) - f(x^*) - \langle \nabla f(x^*), \bar{x}_T(t) - x^* \rangle
$$
$$
\geq \begin{cases} \beta_\varepsilon \, |\bar{x}_T(t) - x^*|^2, & \text{if } \bar{x}_T(t) \in \mathbb{R}^n \setminus B_\varepsilon^n(x^*), \\ 0, & \text{if } \bar{x}_T(t) \in B_\varepsilon^n(x^*), \end{cases}
$$

where

$$\beta_\varepsilon = \min \left(\alpha_\varepsilon, \frac{\alpha_\varepsilon \delta_0^2}{(M_2(X) + |x^*|)^2} \right) > 0.$$

By Theorem 4.2, there exists a $\tilde{T}_\varepsilon > 0$, such that

$$\left| \frac{1}{T} \int_0^T \mathrm{d}t \int_{\mathbb{R}^m} Bu \, \bar{\sigma}_T(t)(\mathrm{d}u) - \int_{\mathbb{R}^m} Bu \, \theta^*(\mathrm{d}u) \right| \leq \frac{\varepsilon \beta_\varepsilon}{4 |\psi^*|}, \quad \forall \, T \geq \tilde{T}_\varepsilon,$$

and

$$\left| \frac{1}{T} \int_0^T \mathrm{d}t \int_{\mathbb{R}^m} g(u)\, \bar{\sigma}_T(t)(\mathrm{d}u) - \int_{\mathbb{R}^m} g(u)\, \theta^*(\mathrm{d}u) \right| \le \frac{\varepsilon \beta_\varepsilon}{4}, \quad \forall\, T \ge \tilde{T}_\varepsilon.$$

Further, since (2.20) and (4.7) hold, we have

$$\frac{1}{T} \int_0^T |\bar{x}_T(t) - x^*|^2 \, \mathrm{d}t$$

$$\le \frac{1}{\beta_\varepsilon T} \int_0^T \big(f(\bar{x}_T(t)) - f(x^*) - \langle \nabla f(x^*), \bar{x}_T(t) - x^* \rangle \big) \, \mathrm{d}t + \varepsilon^2$$

$$= \frac{1}{\beta_\varepsilon T} \int_0^T \big(f(\bar{x}_T(t)) - f(x^*) + \langle \psi^*, A(\bar{x}_T(t) - x^*) \rangle \big) \, \mathrm{d}t + \varepsilon^2$$

$$= \frac{1}{\beta_\varepsilon T} \int_0^T \left(f(\bar{x}_T(t)) + \int_{\mathbb{R}^m} g(u)\, \bar{\sigma}_T(t)(\mathrm{d}u) + \left\langle \psi^*, A\bar{x}_T(t) + \int_{\mathbb{R}^m} Bu\, \bar{\sigma}_T(t)(\mathrm{d}u) \right\rangle \right) \mathrm{d}t$$

$$- \frac{I^*}{\beta_\varepsilon} + \frac{1}{\beta_\varepsilon T} \int_0^T \mathrm{d}t \int_{\mathbb{R}^m} \big(g(u) + \langle \psi^*, Bu \rangle \big) \, (\theta^* - \bar{\sigma}_T(t))(\mathrm{d}u) + \varepsilon^2$$

$$\le \frac{M_1(X)}{\beta_\varepsilon T} + \frac{1}{\beta_\varepsilon} \left| \frac{1}{T} \int_0^T \mathrm{d}t \int_{\mathbb{R}^m} g(u)\, \bar{\sigma}_T(t)(\mathrm{d}u) - \int_{\mathbb{R}^m} g(u)\, \theta^*(\mathrm{d}u) \right|$$

$$+ \frac{|\psi^*|}{\beta_\varepsilon} \left| \frac{1}{T} \int_0^T \mathrm{d}t \int_{\mathbb{R}^m} Bu\, \bar{\sigma}_T(t)(\mathrm{d}u) - \int_{\mathbb{R}^m} Bu\, \theta^*(\mathrm{d}u) \right| + \varepsilon^2$$

$$\le \frac{M_1(X)}{\beta_\varepsilon T} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4}, \quad \forall\, T > \tilde{T}_\varepsilon.$$

Denote

$$T_\varepsilon = \max \left(\tilde{T}_\varepsilon, \frac{4M_1(X)}{\varepsilon \beta_\varepsilon} \right).$$

We complete the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 5.3. The above theorem tells us that $\frac{1}{T} \int_0^T |\bar{x}_T(t) - x^*|^2 \, \mathrm{d}t \to 0$ as $T \to +\infty$, which is uniformly for any $x_0 \in X$.

Thus, by the uniform boundedness of $\bar{x}_T(\cdot)$ with respect to $x_0 \in X$ and $T > 0$, the above theorem implies that for any $p \in [1, +\infty)$, $\frac{1}{T} \int_0^T |\bar{x}_T(t) - x^*|^p \, \mathrm{d}t \to 0$ as $T \to +\infty$, which is uniformly for any $x_0 \in X$.

Moreover, for any $F(\cdot) \in \mathrm{C}^0(\mathbb{R}; \mathbb{R})$ and $p \in [1, +\infty)$, it holds that $\frac{1}{T} \int_0^T |F(\bar{x}_T(t)) - F(x^*)|^p \, \mathrm{d}t \to 0$ as $T \to +\infty$ uniformly for $x_0 \in X$ under the assumptions in the above theorem.

Actually, denote M be an upper bound of $|\bar{x}_T(\cdot)|$ for any $T > 0$ and $x_0 \in X$. And denote $\omega_F(\cdot)$ be the continuous modulus of $F(\cdot)$ defined on $\overline{B_M^n(0)}$, that is

$$\omega_F(r) \triangleq \sup_{\substack{|\xi| \le M, |\eta| \le M \\ |\xi - \eta| \le r}} |F(\xi) - F(\eta)|, \quad \forall\, r \ge 0.$$

Then,

$$\int_0^T \left| F(\bar{x}_T(t)) - F(x^*) \right| \mathrm{d}t$$

$$= \int_{|\bar{x}_T(t) - x^*| > \delta} \left| F(\bar{x}_T(t)) - F(x^*) \right| \mathrm{d}t + \int_{|\bar{x}_T(t) - x^*| \leq \delta} \left| F(\bar{x}_T(t)) - F(x^*) \right| \mathrm{d}t$$

$$\leq \frac{2}{\delta^2} \sup_{|x| \leq M} |F(x)| \int_0^T |\bar{x}_T(t) - x^*|^2 \mathrm{d}t + T \omega_F(\delta), \quad \forall \, \delta > 0.$$

It follows that $\frac{1}{T} \int_0^T |F(\bar{x}_T(t)) - F(x^*)|^p \, \mathrm{d}t \to 0$ as $T \to +\infty$ uniformly for any $x_0 \in X$.

Corollary 5.4. *Let the assumptions in Theorem 5.2 hold. Then, for any $\varepsilon > 0$, there is a $T_\varepsilon > 0$, such that*

$$\left| \frac{1}{T} \int_0^T \bar{\psi}_T(t) \, \mathrm{d}t - \psi^* \right| < \varepsilon, \quad \forall \, T \geq T_\varepsilon, \tag{5.1}$$

where $\bar{\psi}_T(\cdot)$ is the adjoint function of Problem $RP(x_0, T)$, ψ^* is the adjoint vector of Problem $RQ(\mathbb{R}^n, \mathbb{R}^m)$, and x_0 is an arbitrary point in X.

Proof. If the assertion does not hold, we can find an $\varepsilon_0 > 0$, a sequence $\{T_k\}_{k=1}^{+\infty} \subseteq \mathbb{R}$, a sequence $\{x_k\}_{k=1}^{+\infty} \subseteq X$ and an optimal pair $(\bar{x}_{T_k}(\cdot), \bar{\sigma}_{T_k}(\cdot))$ of Problem $RP(x_k, T_k)$ $(k = 1, 2, \ldots)$, such that

$$\lim_{k \to +\infty} T_k = +\infty,$$

and

$$\left| \frac{1}{T_k} \int_0^{T_k} \bar{\psi}_{T_k}(t) \, \mathrm{d}t - \psi^* \right| \geq \varepsilon_0, \quad \forall \, k = 1, 2, \ldots, \tag{5.2}$$

where $\bar{\psi}_{T_k}(\cdot)$ is the corresponding adjoint function of Problem $RP(x_k, T_k)$. By Theorem 4.1, $\{\bar{\psi}_{T_k}(\cdot)\}_{k=1}^{+\infty}$ and $\{\operatorname{supp} \bar{\sigma}_{T_k}(\cdot)\}_{k=1}^{+\infty}$ are uniformly bounded. Thus, there is a $\hat{\psi} \in \mathbb{R}^n$ and a $\hat{\theta} \in \mathcal{M}_+^1(\mathbb{R}^m)$ satisfying

$$\frac{1}{T_k} \int_0^{T_k} \bar{\psi}_{T_k}(t) \, \mathrm{d}t \to \hat{\psi}, \quad \frac{1}{T_k} \int_0^{T_k} \bar{\sigma}_{T_k}(t) \, \mathrm{d}t \to \hat{\theta}, \quad \text{as } k \to +\infty.$$

Meanwhile, Theorem 5.2 implies that

$$\frac{1}{T_k} \int_0^{T_k} |\bar{x}_{T_k}(t) - x^*|^2 \, \mathrm{d}t \to 0, \quad \text{as } k \to +\infty. \tag{5.3}$$

Combining (5.3) with $f(\cdot) \in C^1(\mathbb{R}^n; \mathbb{R})$ and Remark 5.3, we find out that

$$\frac{1}{T_k} \int_0^{T_k} \left| \nabla f(\bar{x}_{T_k}(t)) - \nabla f(x^*) \right| \mathrm{d}t \to 0, \quad \text{as } k \to +\infty.$$

Then,

$$\left| \frac{1}{T_k} \int_0^{T_k} \nabla f(\bar{x}_{T_k}(t)) \, \mathrm{d}t - \nabla f(x^*) \right| \to 0, \quad \text{as } k \to +\infty.$$

By observing the structure of the adjoint equation (4.4), we deduce that

$$\frac{\bar{\psi}_{T_k}(0)}{T_k} = A^\top \Big(\frac{1}{T_k} \int_0^{T_k} \bar{\psi}_{T_k}(t)\,\mathrm{d}t - \psi^* \Big) + \frac{1}{T_k} \int_0^{T_k} \nabla f(\bar{x}_{T_k}(t))\,\mathrm{d}t - \nabla f(x^*), \quad \forall\, k = 1, 2, \dots.$$

Let $k \to +\infty$, we get

$$A^\top (\hat{\psi} - \psi^*) = 0.$$

On the other hand, (4.2) implies that

$$B^\top \hat{\psi} + \int_{\mathbb{R}^m} \nabla g(u)\, \hat{\theta}(\mathrm{d}u) = 0.$$

Like the proof of Theorem 4.2, we can prove that $(x^*, \hat{\theta}) \in P_*$. Meanwhile, Remark 2.7 shows that $\hat{\psi} = \psi^*$. By taking $k \to +\infty$ in (5.2), it holds that $0 \geq \varepsilon_0$, which is a contradiction.

We get the proof. □

It seems that the mean square turnpike property of $\bar{\psi}_T(\cdot)$ is another difficulty in turnpike theory of optimal relaxed control problems. Under the assumptions in Theorem 5.2, although we have already proved the integral turnpike property of $\bar{\psi}_T(\cdot)$, it is still not enough to establish the mean square turnpike property of $\bar{\psi}_T(\cdot)$. Fortunately, we have the following results.

Corollary 5.5. *Under the assumptions in Theorem 5.2, for any $\varepsilon > 0$, there is a $T_\varepsilon > 0$, such that*

$$\frac{1}{T} \int_0^T \Big| \int_{\mathbb{R}^m} \big(g(u) + \langle \psi^*, Bu \rangle \big)\, (\bar{\sigma}_T(t) - \theta^*)(\mathrm{d}u) \Big|\, \mathrm{d}t < \varepsilon, \quad \forall\, T \geq T_\varepsilon, \tag{5.4}$$

$$\frac{1}{T} \Big| \int_0^T \Big(\int_{\mathbb{R}^m} \langle \bar{\psi}_T(t), Bu \rangle\, \bar{\sigma}_T(t)(\mathrm{d}u) - \int_{\mathbb{R}^m} \langle \psi^*, Bu \rangle\, \theta^*(\mathrm{d}u) \Big)\, \mathrm{d}t \Big| < \varepsilon, \quad \forall\, T \geq T_\varepsilon, \tag{5.5}$$

and

$$\frac{1}{T} \int_0^T \Big| \Big\langle \bar{\psi}_T(t) - \psi^*, \int_{\mathbb{R}^m} Bu\, (\bar{\sigma}_T(t) - \theta^*)(\mathrm{d}u) \Big\rangle \Big|\, \mathrm{d}t < \varepsilon, \quad \forall\, T \geq T_\varepsilon, \tag{5.6}$$

where $(\bar{x}_T(\cdot), \bar{\sigma}_T(\cdot))$ is an optimal pair of Problem $RP(x_0, T)$, $\bar{\psi}_T(\cdot)$ is the corresponding adjoint function, x_0 is an arbitrary point in X, $(x^*, \theta^*) \in P_*$ and ψ^* is the adjoint vector.

Proof. Based on (4.7), we have

$$\frac{1}{T} \int_0^T \Big| f(\bar{x}_T(t)) + \int_{\mathbb{R}^m} g(u)\, \bar{\sigma}_T(t)(\mathrm{d}u) + \Big\langle \psi^*, A\bar{x}_T(t) + \int_{\mathbb{R}^m} Bu\, \bar{\sigma}_T(t)(\mathrm{d}u) \Big\rangle - I^* \Big|\, \mathrm{d}t$$
$$\leq \frac{M_1(X)}{T}, \quad \forall\, T > 0,\ x_0 \in X.$$

Then,

$$\frac{1}{T} \int_0^T \Big| \int_{\mathbb{R}^m} \big(g(u) + \langle \psi^*, Bu \rangle \big) (\bar{\sigma}_T(t) - \theta^*)(\mathrm{d}u) \Big| \, \mathrm{d}t$$

$$\leq \frac{1}{T} \int_0^T \big| f(\bar{x}_T(t)) - f(x^*) \big| \, \mathrm{d}t + \frac{1}{T} \int_0^T \big| \langle \psi^*, A(\bar{x}_T(t) - x^*) \rangle \big| \, \mathrm{d}t$$

$$+ \frac{M_1(X)}{T}, \quad \forall \, T > 0, \ x_0 \in X.$$

Then, by Remark 5.3, we get (5.4).

Meanwhile, it holds that

$$- \langle \psi^*, \bar{x}_T(T) - x^* \rangle - \langle \bar{\psi}_T(0) - \psi^*, x_0 - x^* \rangle$$

$$= \langle \bar{\psi}_T(T) - \psi^*, \bar{x}_T(T) - x^* \rangle - \langle \bar{\psi}_T(0) - \psi^*, x_0 - x^* \rangle$$

$$= \int_0^T \mathrm{d}t \int_{\mathbb{R}^m} \langle \bar{\psi}_T(t) - \psi^*, Bu \rangle \, (\bar{\sigma}_T(t) - \theta^*)(\mathrm{d}u)$$

$$- \int_0^T \langle \bar{x}_T(t) - x^*, \nabla f(\bar{x}_T(t)) - \nabla f(x^*) \rangle \, \mathrm{d}t$$

$$= \int_0^T \Big(\int_{\mathbb{R}^m} \langle \bar{\psi}_T(t), Bu \rangle \, \bar{\sigma}_T(t)(\mathrm{d}u) - \int_{\mathbb{R}^m} \langle \psi^*, Bu \rangle \, \theta^*(\mathrm{d}u) \Big) \mathrm{d}t$$

$$- \int_0^T \Big(\int_{\mathbb{R}^m} \langle \bar{\psi}_T(t) - \psi^*, Bu \rangle \, \theta^*(\mathrm{d}u) + \int_{\mathbb{R}^m} \langle \psi^*, Bu \rangle \, (\bar{\sigma}_T(t) - \theta^*)(\mathrm{d}u) \Big) \mathrm{d}t$$

$$- \int_0^T \langle \bar{x}_T(t) - x^*, \nabla f(\bar{x}_T(t)) - \nabla f(x^*) \rangle \, \mathrm{d}t,$$

which implies (5.5).

Furthermore, it follows from (4.2) that

$$\frac{1}{T} \int_0^T \Big| \Big\langle \bar{\psi}_T(t) - \psi^*, \int_{\mathbb{R}^m} Bu \, (\bar{\sigma}_T(t) - \theta^*)(\mathrm{d}u) \Big\rangle \Big| \, \mathrm{d}t$$

$$\leq \frac{1}{T} \int_0^T \Big| \int_{\mathbb{R}^m} \big(g(u) + \langle \psi^*, Bu \rangle \big) (\bar{\sigma}_T(t) - \theta^*)(\mathrm{d}u) \Big| \, \mathrm{d}t$$

$$+ \frac{1}{T} \int_0^T \Big| \int_{\mathbb{R}^m} \big(g(u) + \langle \bar{\psi}_T(t), Bu \rangle \big) (\bar{\sigma}_T(t) - \theta^*)(\mathrm{d}u) \Big| \, \mathrm{d}t$$

$$= \frac{1}{T} \int_0^T \Big| \int_{\mathbb{R}^m} \big(g(u) + \langle \psi^*, Bu \rangle \big) (\bar{\sigma}_T(t) - \theta^*)(\mathrm{d}u) \Big| \, \mathrm{d}t$$

$$+ \frac{1}{T} \int_0^T \mathrm{d}t \int_{\mathbb{R}^m} \big(g(u) + \langle \bar{\psi}_T(t), Bu \rangle \big) (\theta^* - \bar{\sigma}_T(t))(\mathrm{d}u)$$

$$\leq \frac{1}{T} \int_0^T \Big| \int_{\mathbb{R}^m} \big(g(u) + \langle \psi^*, Bu \rangle \big) (\bar{\sigma}_T(t) - \theta^*)(\mathrm{d}u) \Big| \, \mathrm{d}t$$

$$+ \int_{\mathbb{R}^m} g(u) \, \theta^*(\mathrm{d}u) - \frac{1}{T} \int_0^T \mathrm{d}t \int_{\mathbb{R}^m} g(u) \, \bar{\sigma}_T(t)(\mathrm{d}u)$$

$$+ \Big| \frac{1}{T} \int_0^T \Big(\int_{\mathbb{R}^m} \langle \bar{\psi}_T(t), Bu \rangle \, \bar{\sigma}_T(t)(\mathrm{d}u) - \int_{\mathbb{R}^m} \langle \psi^*, Bu \rangle \, \theta^*(\mathrm{d}u) \Big) \mathrm{d}t \Big|$$

$$+ \frac{1}{T} \int_0^T \mathrm{d}t \int_U \langle \bar{\psi}_T(t) - \psi^*, Bu \rangle \, \theta^*(\mathrm{d}u).$$

Combining the above with (4.11), (5.1), (5.4) and (5.5), we get (5.6). $\qquad\square$

Theorem 5.6. *Let E be an $n \times n$ symmetric matrix. Assume all the assumptions in Theorem 5.2 hold. Then, for any $\varepsilon > 0$, there is a $T_\varepsilon > 0$, such that*

$$\left| \frac{1}{T} \int_0^T \left\langle (EA^\top + AE)(\bar{\psi}_T(t) - \psi^*), \bar{\psi}_T(t) - \psi^* \right\rangle \mathrm{d}t \right| < \varepsilon, \quad \forall\, T \geq T_\varepsilon, \tag{5.7}$$

where $\bar{\psi}_T(\cdot)$ is the adjoint function of Problem $RP(x_0, T)$, corresponding to the optimal pair $(\bar{x}_T(\cdot), \bar{\sigma}_T(\cdot))$, ψ^* is the adjoint vector corresponding to $(x^*, \theta^*) \in P_*$, and x_0 is an arbitrary point in X.

Proof. It is easy to see that

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\langle E(\bar{\psi}_T(t) - \psi^*), \bar{\psi}_T(t) - \psi^* \right\rangle$$
$$= -2 \left\langle E(\bar{\psi}_T(t) - \psi^*), A^\top(\bar{\psi}_T(t) - \psi^*) + \nabla f(\bar{x}_T(t)) - \nabla f(x^*) \right\rangle$$
$$= - \left\langle (EA^\top + AE)(\bar{\psi}_T(t) - \psi^*), \bar{\psi}_T(t) - \psi^* \right\rangle$$
$$\quad - 2 \left\langle E(\bar{\psi}_T(t) - \psi^*), \nabla f(\bar{x}_T(t)) - \nabla f(x^*) \right\rangle.$$

Then,

$$\left| \frac{1}{T} \int_0^T \left\langle (EA^\top + AE)(\bar{\psi}_T(t) - \psi^*), \bar{\psi}_T(t) - \psi^* \right\rangle \mathrm{d}t \right|$$
$$\leq \frac{1}{T} \left| \left\langle E(\bar{\psi}_T(T) - \psi^*), \bar{\psi}_T(T) - \psi^* \right\rangle - \left\langle E(\bar{\psi}_T(0) - \psi^*), \bar{\psi}_T(0) - \psi^* \right\rangle \right|$$
$$\quad + 2 \max_{s \in [0,T]} |E(\bar{\psi}_T(s) - \psi^*)| \frac{1}{T} \int_0^T |\nabla f(\bar{x}_T(t)) - \nabla f(x^*)| \, \mathrm{d}t.$$

Then, by the uniform boundedness of $\bar{\psi}_T(\cdot)$ and the mean square turnpike property of $\bar{x}_T(\cdot)$, we obtain (5.7) (see Rem. 5.3). $\qquad\square$

If A is stable, that is all the eigenvalues of A have negative real parts, then we have a positive definite matrix $E \in \mathbb{R}^{n \times n}$ satisfying the algebraic Lyapunov equation (see [1])

$$EA^\top + AE + I = 0, \tag{5.8}$$

where I denotes the $n \times n$ unit matrix.

It is easy to verify that if A is invertible, there is a symmetric matrix $E \in \mathbb{R}^{n \times n}$, which satisfies (5.8). Then, we can prove the mean square turnpike property of the adjoint function by adopting Theorem 5.6. We state the result precisely as follows.

Theorem 5.7. *Assume all assumptions in Theorem 5.2 hold, and A be invertible. Then, for any $\varepsilon > 0$, there is a $T_\varepsilon > 0$, such that*

$$\frac{1}{T} \int_0^T |\bar{\psi}_T(t) - \psi^*|^2 \, \mathrm{d}t < \varepsilon, \quad \forall\, T \geq T_\varepsilon, \tag{5.9}$$

where $\bar{\psi}_T(\cdot)$ is the adjoint function of Problem $RP(x_0, T)$, corresponding to the optimal pair $(\bar{x}_T(\cdot), \bar{\sigma}_T(\cdot))$, ψ^* is the adjoint vector corresponding to $(x^*, \theta^*) \in P_*$, and x_0 is an arbitrary point in X.

Remark 5.8. In Theorem 5.7, A is invertible, which seems a little strict. We suspect that it is a kind of compensation for lack of convexity of $g(\cdot)$. However, for some specific $g(\cdot)$, we can obtain the uniform convergence of $\frac{1}{T} \int_0^T |B^\top(\bar{\psi}_T(t) - \psi^*)|^2 \, \mathrm{d}t$ with respect to $x_0 \in X$ by (5.4)–(5.6) in Corollary 5.5. Then, we can obtain (5.9).

Actually, since (A, B) is controllable, (A, B) is stable, that means we have an $m \times n$ matrix K, such that $A + BK$ is stable. Then, there is an $n \times n$ symmetric matrix E, such that

$$E(A + BK)^\top + (A + BK)E + I = 0.$$

It follows that

$$
\begin{aligned}
&\frac{1}{T} \int_0^T |\bar{\psi}_T(t) - \psi^*|^2 \, \mathrm{d}t \\
&= -\frac{1}{T} \int_0^T \left\langle (E(A + BK)^\top + (A + BK)E)(\bar{\psi}_T(t) - \psi^*), \bar{\psi}_T(t) - \psi^* \right\rangle \mathrm{d}t \\
&\leq \frac{1}{T} \left| \int_0^T \left\langle (EA^\top + AE)(\bar{\psi}_T(t) - \psi^*), \bar{\psi}_T(t) - \psi^* \right\rangle \mathrm{d}t \right| \\
&\quad + \frac{1}{T} \left| \int_0^T \left\langle (EK^\top B^\top + BKE)(\bar{\psi}_T(t) - \psi^*), \bar{\psi}_T(t) - \psi^* \right\rangle \mathrm{d}t \right| \\
&\leq \frac{1}{T} \left| \int_0^T \left\langle (EA^\top + AE)(\bar{\psi}_T(t) - \psi^*), \bar{\psi}_T(t) - \psi^* \right\rangle \mathrm{d}t \right| \\
&\quad + 2\|KE\| \max_{s \in [0,T]} |\bar{\psi}_T(s) - \psi^*| \frac{1}{T} \int_0^T \left| B^\top(\bar{\psi}_T(t) - \psi^*) \right| \mathrm{d}t.
\end{aligned}
$$

Thus, we have (5.9).

By checking the above proof carefully, we have the stronger version of mean square turnpike property, stated as follows.

Theorem 5.9. *Assume all assumptions in Theorem 5.7 hold. Then, for any $\varepsilon > 0$, there is a $T_\varepsilon > 0$, such that when $T \geq T_\varepsilon$, it holds that*

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \left(|\bar{x}_T(t) - x^*|^2 + |\bar{\psi}_T(t) - \psi^*|^2 \right) \mathrm{d}t < \varepsilon, \quad \forall \, 0 \leq t_1 < t_2 \leq T, \ t_2 - t_1 \geq T_\varepsilon,$$

where $(\bar{x}_T(\cdot), \bar{\sigma}_T(\cdot))$ is the optimal pair of Problem $RP(x_0, T)$, $\bar{\psi}_T(\cdot)$ is the corresponding adjoint function, and x_0 is an arbitrary point in X. Moreover, x^* is the optimal equilibrium state of Problem $RQ(\mathbb{R}^n, \mathbb{R}^m)$ and ψ^* is the corresponding adjoint vector.

Corollary 5.10. *Assume all assumptions in Theorem 5.7 hold. Then, for any $\varepsilon > 0$, we can find a $T_\varepsilon > 0$. If $T \geq T_\varepsilon$, there are $\eta_1, \eta_2 \in (0, T_\varepsilon)$, such that*

$$|\bar{x}_T(\eta_1) - x^*| + |\bar{\psi}_T(\eta_1) - \psi^*| < \varepsilon,$$

$$|\bar{x}_T(T - \eta_2) - x^*| + |\bar{\psi}_T(T - \eta_2) - \psi^*| < \varepsilon,$$

where $(\bar{x}_T(\cdot), \bar{\sigma}_T(\cdot))$ is an optimal pair of Problem $RP(x_0, T)$, $\bar{\psi}_T(\cdot)$ is the corresponding function, and x_0 is an arbitrary point in X. Meanwhile, x^* is the optimal equilibrium state of Problem $RQ(\mathbb{R}^n, \mathbb{R}^m)$ and ψ^* is the corresponding adjoint vector.

Proof. By Theorem 5.9, for any $\varepsilon > 0$, there exists a $T_\varepsilon > 0$, such that

$$\frac{1}{T_\varepsilon} \int_0^{T_\varepsilon} \left(|\bar{x}_T(t) - x^*| + |\bar{\psi}_T(t) - \psi^*| \right) \mathrm{d}t < \varepsilon, \quad \forall \, T \geq T_\varepsilon,$$

$$\frac{1}{T_\varepsilon} \int_{T-T_\varepsilon}^{T} \left(|\bar{x}_T(t) - x^*| + |\bar{\psi}_T(t) - \psi^*| \right) \mathrm{d}t < \varepsilon, \quad \forall \, T \geq T_\varepsilon.$$

Then, we can get the conclusion immediately. $\qquad\square$

6. Exponential turnpike property

In this section, we will study the exponential turnpike property. Before we prove the main results, we state some notations and preliminaries.

Let

$$V(x) = f(x) + \langle \psi^*, Ax \rangle + \inf_{\theta \in \mathcal{M}_+^1(\mathbb{R}^m)} \int_{\mathbb{R}^m} \left(g(u) + \langle \psi^*, Bu \rangle \right) \theta(\mathrm{d}u).$$

Remark 6.1. If (A2) and (A3′) hold, $V(\cdot)$ is continuous and

$$V(x) - I^* > 0, \quad \forall \, x \neq x^*.$$

Moreover, if $f(\cdot)$ is twice continuously differentiable, it is easy to see that

$$\begin{aligned}
V(x) &= f(x) + \langle \psi^*, Ax \rangle + \inf_{\theta \in \mathcal{M}_+^1(\mathbb{R}^m)} \int_{\mathbb{R}^m} \left(g(u) + \langle \psi^*, Bu \rangle \right) \theta(\mathrm{d}u) \\
&= f(x) + \langle \psi^*, Ax \rangle + \int_{\mathbb{R}^m} \left(g(u) + \langle \psi^*, Bu \rangle \right) \theta^*(\mathrm{d}u) \\
&= I^* + f(x) - f(x^*) + \langle \psi^*, A(x - x^*) \rangle \\
&= I^* + \int_0^1 (1-s) \langle f_{xx}(x^* + s(x-x^*))(x-x^*), x-x^* \rangle \, \mathrm{d}s.
\end{aligned} \tag{6.1}$$

Thus, we have

$$V(x) - I^* \leq C|x - x^*|^2, \quad \forall \, |x - x^*| \leq 1$$

for some constant $C > 0$.

Remark 6.2. If (A2) and (A3′) hold, by Remark 6.1 and (4.1), we know that $V(\cdot) - I^*$ has a positive minimum m_k in $\mathbb{R}^n \setminus B_{\frac{1}{2k}}(x^*)$ for any $k \geq 1$, that is

$$m_k = \min \left\{ V(x) - I^* \ \middle| \ x \in \mathbb{R}^n \setminus B_{\frac{1}{2k}}(x^*) \right\} > 0, \quad \forall \, k = 1, 2, \ldots.$$

For convenience, we denote

$$f^*(x) = f(x) + \langle \psi^*, Ax \rangle, \quad g^*(u) = g(u) + \langle \psi^*, Bu \rangle, \quad \forall \, x \in \mathbb{R}^n, \ u \in \mathbb{R}^m. \tag{6.2}$$

Then, we have the following result.

Lemma 6.3. *Let (A2) and (A3′) hold. Then, for any $k \geq 1$, there is a $c_k > 0$, such that for any $t_2 > t_1 > 0$ and $\sigma(\cdot) \in \mathcal{R}_{t_2}(\mathbb{R}^m)$, if*

$$x(t_1; \sigma(\cdot)) \in \overline{B^n_{\frac{1}{2k}}(x^*)}, \quad x(t_2; \sigma(\cdot)) \notin \overline{B^n_{\frac{1}{k}}(x^*)},$$

then

$$\int_{t_1}^{t_2} \left(f^*(x(t; \sigma(\cdot))) + \int_{\mathbb{R}^m} g^*(u)\,\sigma(t)(\mathrm{d}u) - I^* \right) \mathrm{d}t \geq c_k.$$

Proof. Similar to (4.6), by (4.1) and the continuity of $f(\cdot)$ and $g(\cdot)$, we have an $M^* > 0$ satisfying

$$f^*(x) + g^*(u) - I^* \geq |x| + |u| - M^*, \quad \forall\, (x, u) \in \mathbb{R}^n \times \mathbb{R}^m.$$

Since $x(\cdot; \sigma(\cdot))$ is continuous, we can find $t_1 \leq s_1 < s_2 \leq t_2$, such that

$$|x(s_1; \sigma(\cdot)) - x^*| = \frac{1}{2k}, \quad |x(s_2; \sigma(\cdot)) - x^*| = \frac{1}{k},$$

and

$$x(t; \sigma(\cdot)) \in \overline{B^n_{\frac{1}{k}}(x^*)} \setminus B^n_{\frac{1}{2k}}(x^*), \quad \forall\, t \in [s_1, s_2].$$

There are two possible cases. If $s_2 - s_1 \geq \frac{1}{8M^*k(\|A\|+\|B\|)}$, it holds that

$$\int_{t_1}^{t_2} \left(f^*(x(t; \sigma(\cdot))) + \int_{\mathbb{R}^m} g^*(u)\,\sigma(t)(\mathrm{d}u) - I^* \right) \mathrm{d}t$$

$$\geq \int_{s_1}^{s_2} \left(f^*(x(t; \sigma(\cdot))) + \int_{\mathbb{R}^m} g^*(u)\,\sigma(t)(\mathrm{d}u) - I^* \right) \mathrm{d}t$$

$$\geq m_k(s_2 - s_1) \geq \frac{m_k}{8M^*k(\|A\| + \|B\|)}.$$

If $s_2 - s_1 < \frac{1}{8M^*k(\|A\|+\|B\|)}$, then

$$\frac{1}{2k} \leq |x(s_2; \sigma(\cdot)) - x(s_1; \sigma(\cdot))|$$

$$= \left| \int_{s_1}^{s_2} \left(Ax(t; \sigma(\cdot)) + \int_{\mathbb{R}^m} Bu\,\sigma(t)(\mathrm{d}u) \right) \mathrm{d}t \right|$$

$$\leq (\|A\| + \|B\|) \int_{s_1}^{s_2} \left(|x(t; \sigma(\cdot))| + \int_{\mathbb{R}^m} |u|\,\sigma(t)(\mathrm{d}u) \right) \mathrm{d}t$$

$$= (\|A\| + \|B\|) \int_{s_1}^{s_2} \left(|x(t; \sigma(\cdot))| + \int_{|u| < M^* - |x(t;\sigma(\cdot))|} |u|\,\sigma(t)(\mathrm{d}u) \right) \mathrm{d}t$$

$$+ (\|A\| + \|B\|) \int_{s_1}^{s_2} \left(|x(t; \sigma(\cdot))| + \int_{|u| \geq M^* - |x(t;\sigma(\cdot))|} |u|\,\sigma(t)(\mathrm{d}u) \right) \mathrm{d}t$$

$$\leq 2M^*(\|A\| + \|B\|)(s_2 - s_1)$$

$$+ 2(\|A\| + \|B\|) \int_{s_1}^{s_2} \left(|x(t; \sigma(\cdot))| + \int_{|u| \geq M^* - |x(t;\sigma(\cdot))|} |u|\,\sigma(t)(\mathrm{d}u) - M^* \right) \mathrm{d}t$$

$$\leq 2M^*(\|A\| + \|B\|)(s_2 - s_1)$$
$$+ 2(\|A\| + \|B\|) \int_{s_1}^{s_2} \left(f^*(x(t; \sigma(\cdot))) + \int_{|u| \geq M^* - |x(t; \sigma(\cdot))|} g^*(u)\, \sigma(t)(\mathrm{d}u) - I^* \right) \mathrm{d}t$$
$$\leq \frac{1}{4k} + 2(\|A\| + \|B\|) \int_{s_1}^{s_2} \left(f^*(x(t; \sigma(\cdot))) + \int_{\mathbb{R}^m} g^*(u)\, \sigma(t)(\mathrm{d}u) - I^* \right) \mathrm{d}t.$$

It follows that

$$\int_{t_1}^{t_2} \left(f^*(x(t; \sigma(\cdot))) + \int_{\mathbb{R}^m} g^*(u)\, \sigma(t)(\mathrm{d}u) - I^* \right) \mathrm{d}t$$
$$\geq \int_{s_1}^{s_2} \left(f^*(x(t; \sigma(\cdot))) + \int_{\mathbb{R}^m} g^*(u)\, \sigma(t)(\mathrm{d}u) - I^* \right) \mathrm{d}t$$
$$\geq \frac{1}{8k(\|A\| + \|B\|)}.$$

Define $c_k = \frac{\min\{m_k, M^*\}}{8kM^*(\|A\| + \|B\|)}$. We complete the proof. $\qquad\square$

In order to establish the exponential turnpike property, we set the following assumptions.
(A5) The matrix A is invertible.
(A6) There is a $C_0 > 0$, such that

$$V(x) - I^* \geq C_0 |x - x^*|^2, \quad \forall\, |x - x^*| \leq 1.$$

Remark 6.4. By (6.1), we see that the assumption (A6) holds if $f_{xx} \geq 2C_0 I$.

Now, we state the exponential turnpike property of Problem $RP(x_0, T)$.

Theorem 6.5. *Assume that $f(\cdot) : \mathbb{R}^n \to \mathbb{R}$ and $g(\cdot) : \mathbb{R}^m \to \mathbb{R}$ are twice continuously differentiable, and (4.1) holds. Moreover, assume (A1), (A3 '), (A4), (A5) and (A6) hold. Then, there exist constants $C_* > 0$ and $\lambda_* > 0$, such that*

$$|\bar{x}_T(t) - x^*| + |\bar{\psi}_T(t) - \psi^*| \leq C_*(e^{-\lambda_* t} + e^{-\lambda_*(T-t)}), \quad \forall\, t \in [0, T], \ T > 0,$$

where $(\bar{x}_T(\cdot), \bar{\sigma}_T(\cdot))$ is an optimal pair of Problem $RP(x_0, T)$, $\bar{\psi}_T(\cdot)$ is the corresponding function, and x_0 is an arbitrary point in X. Meanwhile, x^* is the optimal equilibrium state of Problem $RQ(\mathbb{R}^n, \mathbb{R}^m)$ and ψ^* is the corresponding adjoint vector.

Proof.
Step 1. We suppose $T > 0$ is big enough without loss of generality.

By Corollary 5.10, we can find a $T_1 > 0$, independent of $x_0 \in X$ and $T > 0$, and $a_1 \in [0, T_1]$, $b_1 \in [T - T_1, T]$, which may depend on $T > 0$ and $x_0 \in X$, such that

$$|\bar{x}_T(a_1) - x^*| < \frac{1}{2}, \quad |\bar{x}_T(b_1) - x^*| < \frac{1}{2}.$$

We claim that there is an $S > 2$, independent of $x_0 \in X$ and $T > 0$, and two sequences $\{a_j\}_{j=1}^k$ and $\{b_j\}_{j=1}^k$, which may depend on $T > 0$ and $x_0 \in X$, such that

$$0 < a_1 < a_2 < a_3 < \ldots a_k < b_k < \cdots < b_3 < b_2 < b_1 < T, \tag{6.3}$$

$$a_{j+1} - a_j \leq S, \quad b_j - b_{j+1} \leq S, \quad \forall\, j = 1, 2, \ldots, k-1, \tag{6.4}$$

$$b_k - a_k \le 2S, \tag{6.5}$$

and

$$|\bar{x}_T(a_j) - x^*| \le \frac{1}{2^j}, \quad |\bar{x}_T(b_j) - x^*| < \frac{1}{2^j}, \quad \forall\, j = 1, 2, \dots k. \tag{6.6}$$

We adopt mathematical induction to prove the assertion. Suppose we already get a_j and b_j. If we restrict Problem $RP(x_0, T)$ on $[a_j, b_j]$ with state constraints $x(a_j) = \bar{x}_T(a_j)$ and $x(b_j) = \bar{x}_T(b_j)$, the relaxed control $\bar{\sigma}_T(\cdot)\big|_{[a_j, b_j]}$ is the minimizer of the functional

$$J_j(\sigma(\cdot)) \triangleq \int_{a_j}^{b_j} \left(f(x(t)) + \int_{\mathbb{R}^m} g(u)\, \sigma(t)(\mathrm{d}u) \right) \mathrm{d}t,$$

where

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = Ax(t) + \int_{\mathbb{R}^m} Bu\, \sigma(t)(\mathrm{d}u), \quad t \in [a_j, b_j]. \tag{6.7}$$

Let

$$u_j(t) = B^\top e^{-(t-a_j)A^\top} \left(\int_0^1 e^{-sA} BB^\top e^{-sA^\top} \, \mathrm{d}s \right)^{-1} (x^* - \bar{x}_T(a_j)), \quad t \in [a_j, a_j + 1],$$

$$v_j(t) = B^\top e^{-(t-b_j)A^\top} \left(\int_{-1}^0 e^{-sA} BB^\top e^{-sA^\top} \, \mathrm{d}s \right)^{-1} (\bar{x}_T(b_j) - x^*), \quad t \in [b_j - 1, b_j],$$

and

$$\sigma_j(t) = \begin{cases} \tau_{u_j(t)}\theta^*, & t \in [a_j, a_j + 1], \\ \theta^*, & t \in (a_j + 1, b_j - 1), \\ \tau_{v_j(t)}\theta^*, & t \in [b_j - 1, b_j], \end{cases}$$

where $\tau_{u_j(t)}\theta^*$ represents a probability measure which is defined as

$$\int_{\mathbb{R}^m} h(u)\, \tau_{u_j(t)}\theta^*(\mathrm{d}u) = \int_{\mathbb{R}^m} h(u_j(t) + u)\, \theta^*(\mathrm{d}u), \quad \forall\, h(\cdot) \in \mathrm{C}^0(\mathbb{R}^m),$$

and $\tau_{v_j(t)}\theta^*$ can be defined similarly. Let $x_j(\cdot)$ be the solution of (6.7) with initial data $x_j(a_j) = \bar{x}_T(a_j)$ and relaxed control $\sigma_j(\cdot)$. Obviously, $x_j(b_j) = \bar{x}_T(b_j)$. Then, it follows that

$$\int_{a_j}^{b_j} \left(f(\bar{x}_T(t)) + \int_{\mathbb{R}^m} g(u)\, \bar{\sigma}_T(t)(\mathrm{d}u) + \left\langle \psi^*, A\bar{x}_T(t) + \int_{\mathbb{R}^m} Bu\, \bar{\sigma}_T(t)(\mathrm{d}u) \right\rangle - I^* \right) \mathrm{d}t$$

$$\le \int_{a_j}^{b_j} \left(f(x_j(t)) + \int_{\mathbb{R}^m} g(u)\, \sigma_j(t)(\mathrm{d}u) + \left\langle \psi^*, A\bar{x}_T(t) + \int_{\mathbb{R}^m} Bu\, \bar{\sigma}_T(t)(\mathrm{d}u) \right\rangle - I^* \right) \mathrm{d}t$$

$$= \int_{a_j}^{b_j} \left(f(x_j(t)) + \int_{\mathbb{R}^m} g(u)\, \sigma_j(t)(\mathrm{d}u) - I^* \right) \mathrm{d}t + \langle \psi^*, x_j(b_j) - x_j(a_j) \rangle$$

---

[2]The inequality (6.5) is to determine k.

$$\begin{aligned}
&= \int_{a_j}^{b_j} \left(f(x_j(t)) + \int_{\mathbb{R}^m} g(u)\,\sigma_j(t)(\mathrm{d}u) + \left\langle \psi^*, Ax_j(t) + \int_{\mathbb{R}^m} Bu\,\sigma_j(t)(\mathrm{d}u) \right\rangle - I^* \right) \mathrm{d}t \\
&\leq C_1 \frac{|\bar{x}_T(b_j) - x^*|^2 + |\bar{x}_T(a_j) - x^*|^2}{2} \leq \frac{C_1}{2^{2j}},
\end{aligned}$$

where C_1 is a constant independent of $x_0 \in X$ and $T > 0$. Thus,

$$\begin{aligned}
&\int_{a_j}^{b_j} \left(V(\bar{x}_T(t)) - I^* \right) \mathrm{d}t \\
&\leq \int_{a_j}^{b_j} \left(f(\bar{x}_T(t)) + \int_{\mathbb{R}^m} g(u)\,\bar{\sigma}_T(t)(\mathrm{d}u) + \left\langle \psi^*, A\bar{x}_T(t) + \int_{\mathbb{R}^m} Bu\,\bar{\sigma}_T(t)(\mathrm{d}u) \right\rangle - I^* \right) \mathrm{d}t \\
&\leq C_1 \frac{|\bar{x}_T(b_j) - x^*|^2 + |\bar{x}_T(a_j) - x^*|^2}{2} \leq \frac{C_1}{2^{2j}}.
\end{aligned}$$

Let $S = \max\left\{ \frac{4C_1}{C_0}, 1 \right\}$. Suppose $b_j - a_j > 2S$, otherwise we have already proved the assertion. Since $V(\bar{x}_T(t)) - I^*$ is nonnegative and (A6) holds, we can get an $a_{j+1} \in [a_j, a_j + S]$ and a $b_{j+1} \in [b_j - S, b_j]$ satisfying

$$|\bar{x}_T(a_{j+1}) - x^*| \leq \frac{1}{2^{j+1}}, \quad |\bar{x}_T(b_{j+1}) - x^*| < \frac{1}{2^{j+1}}.$$

Then, we deduce $\{a_j\}_{j=1}^k$ and $\{b_j\}_{j=1}^k$ satisfying (6.3)–(6.6).

Furthermore, by (A6) and Lemma 6.3, if the maximum value δ_j of $|\bar{x}_T(\cdot) - x^*|$ on $[a_j, b_j]$ is bigger than or equal to $\frac{2}{2^j}$, it is tenable that

$$\int_{a_j}^{b_j} \left(f^*(\bar{x}_T(t)) + \int_{\mathbb{R}^m} g^*(u)\,\bar{\sigma}_T(t)(\mathrm{d}u) - I^* \right) \mathrm{d}t \geq C_2 \delta_j^3,$$

for some $C_2 > 0$, independent of $x_0 \in X$ and $T > 0$, where $f^*(\cdot)$ and $g^*(\cdot)$ are defined in (6.2). Thus,

$$\delta_j = \max_{t \in [a_j, b_j]} |\bar{x}_T(t) - x^*| \leq \left(\frac{C_1}{C_2} \right)^{\frac{1}{3}} \frac{1}{(\sqrt[3]{4})^j},$$

which implies that

$$|\bar{x}_T(t) - x^*| \leq C_3 (e^{-\lambda_1 t} + e^{-\lambda_1 (T-t)}), \quad \forall\, t \in [0, T], \tag{6.8}$$

where $C_3 > 0$ and $\lambda_1 > 0$ are constants independent of $x_0 \in X$ and $T > 0$.

Step 2. Now, we prove the exponential turnpike property of $\bar{\psi}_T(\cdot)$. By (A5), we can find an $n \times n$ matrix P, such that

$$-PA^\top P^{-1} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

where A_1 is an $n_1 \times n_1$ matrix, and A_2 is an $n_2 \times n_2$ matrix $(n_1 + n_2 = n)$. Also, all the eigenvalues of A_1 have negative real parts, and the case of A_2 is the opposite. Denote

$$P(\bar{\psi}_T(t) - \psi^*) = \begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix}, \quad \forall\, t \in [0, T],$$

where $\psi_1(\cdot)$ and $\psi_2(\cdot)$ take values in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, respectively. Then, the adjoint equation becomes

$$\begin{cases} \dfrac{\mathrm{d}\psi_1(t)}{\mathrm{d}t} = A_1\psi_1(t) + \xi_1(t), & t \in [0, T], \\ \dfrac{\mathrm{d}\psi_2(t)}{\mathrm{d}t} = A_2\psi_2(t) + \xi_2(t), & t \in [0, T], \end{cases}$$

where $\xi_1(\cdot)$ and $\xi_2(\cdot)$ take values in complex field, and

$$|\xi_1(t)| + |\xi_2(t)| \le C_4(e^{-\lambda_1 t} + e^{-\lambda_1(T-t)}), \quad \forall\, t \in [0, T]$$

for some constant $C_4 > 0$, independent of $x_0 \in X$ and $T > 0$. Meanwhile, we have

$$\psi_1(t) = e^{A_1 t}\psi_1(0) + \int_0^t e^{A_1(t-s)}\xi_1(s)\,\mathrm{d}s,$$

and

$$\psi_2(t) = e^{A_2(t-T)}\psi_2(T) + \int_T^t e^{A_2(t-s)}\xi_2(s)\,\mathrm{d}s,$$

which implies

$$|\bar{\psi}_T(t) - \psi^*| \le C_5(e^{-\lambda_2 t} + e^{-\lambda_2(T-t)}), \quad \forall\, t \in [0, T], \tag{6.9}$$

where $C_5 > 0$ and $\lambda_2 > 0$ are independent of $x_0 \in X$ and $T > 0$.

We complete the proof by (6.8) and (6.9). $\qquad\square$

7. Examples

Example 7.1. Consider a 1-d system with fixed initial data

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = \int_{\mathbb{R}} u\,\sigma(t)(\mathrm{d}u), & t \in [0, T], \\ x(0) = 0, \end{cases}$$

and a cost functional

$$J_T(\sigma(\cdot); 0) = \int_0^T \mathrm{d}t \int_{\mathbb{R}} (x^2(t) + u^4 - 8u^2)\,\sigma(t)(\mathrm{d}u).$$

Problem $RP(0, T)$: Find $\bar{\sigma}_T(\cdot) \in \mathcal{R}_T(\mathbb{R})$, such that

$$J_T(\bar{\sigma}_T(\cdot); 0) = \inf_{\sigma(\cdot) \in \mathcal{R}_T(\mathbb{R})} J_T(\sigma(\cdot); 0).$$

First of all, we study the corresponding non-relaxed problem. Precisely, we consider the system

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = u(t), & t \in [0, T], \\ x(0) = 0, \end{cases} \tag{7.1}$$

and the cost functional

$$J_T(u(\cdot); 0) = \int_0^T \left(x^2(t) + u^4(t) - 8u^2(t) \right) \mathrm{d}t.$$

Problem $P(0, T)$: Find $\bar{u}_T(\cdot) \in \mathcal{U}_T \triangleq \{ u(\cdot) : [0, T] \to \mathbb{R} \mid u(\cdot) \text{ is measurable} \}$, such that

$$J_T(\bar{u}_T(\cdot); 0) = \inf_{u(\cdot) \in \mathcal{U}_T} J_T(u(\cdot); 0).$$

We claim that Problem $P(0, T)$ has no optimal solution. To see this, for any $k \in \mathbb{N}$, denote

$$u_k(t) = \begin{cases} 2, & t \in \displaystyle\bigcup_{j=0}^{k-1} \left[\dfrac{jT}{k}, \dfrac{jT}{k} + \dfrac{T}{2k} \right), \\ -2, & t \in \displaystyle\bigcup_{j=0}^{k-1} \left[\dfrac{jT}{k} + \dfrac{T}{2k}, \dfrac{(j+1)T}{k} \right), \end{cases}$$

and $x_k(\cdot)$ be the solution of (7.1) with control $u_k(\cdot)$. It is easy to check that

$$|x_k(t)| \leq \frac{T}{k}, \quad \forall\, t \in [0, T].$$

Then, we deduce that

$$J_T(u_k(\cdot); 0) = \int_0^T \left(x_k^2(t) + u_k^4(t) - 8u_k^2(t) \right) \mathrm{d}t \leq \frac{T^3}{k^2} - 16T. \tag{7.2}$$

Meanwhile, it holds that

$$J_T(u(\cdot); 0) \geq -16T, \quad \forall\, u(\cdot) \in \mathcal{U}_T. \tag{7.3}$$

Equations (7.2) and (7.3) ensure that

$$\inf_{u(\cdot) \in \mathcal{U}_T} J_T(u(\cdot); 0) = \lim_{k \to +\infty} J_T(u_k(\cdot); 0) = -16T.$$

However, in order to achieve the infimum, we deduce that

$$\bar{x}_T(t) = 0, \quad |\bar{u}_T(t)| = 2, \quad \text{a.e. } t \in [0, T],$$

which contradicts to the system (7.1).

Now, we turn to consider the optimal relaxed control problem $RP(0, T)$. The corresponding optimization problem can be defined as follows.

Problem $RQ(\mathbb{R}, \mathbb{R})$: Find $(x^*, \theta^*) \in \mathcal{M}_S(\mathbb{R}, \mathbb{R})$, such that

$$(x^*)^2 + \int_{\mathbb{R}} (u^4 - 8u^2)\, \theta^*(\mathrm{d}u) = \inf_{(x,\theta) \in \mathcal{M}_S(\mathbb{R},\mathbb{R})} \left(x^2 + \int_{\mathbb{R}} (u^4 - 8u^2)\, \theta(\mathrm{d}u) \right).$$

The optimal solution of Problem $RQ(\mathbb{R}, \mathbb{R})$ is

$$x^* = 0, \quad \theta^* = \frac{1}{2}\delta_2 + \frac{1}{2}\delta_{-2},$$

where δ represents the Dirac measure. Then, we can check that (A1), (A2), (A3′) and (A4) hold. By Theorems 4.2 and 5.2, we have that Problem $RP(0, T)$ satisfies the integral turnpike property and the mean square turnpike property. Moreover, since the example is very simple, the optimal solution $(\bar{x}_T(\cdot), \bar{\sigma}_T(\cdot))$ of Problem $RP(0, T)$ can be calculated explicitly, which is

$$\bar{x}_T(t) = 0, \quad \bar{\sigma}_T(t) = \frac{1}{2}\delta_2 + \frac{1}{2}\delta_{-2}, \quad \text{a.e. } t \in [0, T].$$

We realize that the solutions of Problem $RP(0, T)$ and $RQ(\mathbb{R}, \mathbb{R})$ are same. It implies that the exponential turnpike property is satisfied, although (A5) does not hold, which illustrates that Theorem 6.5 is a sufficient result of exponential turnpike property.

Remark 7.2. Although Example 7.1 is very simple, it can not be covered in the existing literature (see [12, 15], etc.) since Example 7.1 does not admit a classical optimal solution. There are more general examples.

Example 7.3. Consider a 1-d system

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = x(t) + \displaystyle\int_{\mathbb{R}} u\, \sigma(t)(\mathrm{d}u), \quad t \in [0, T], \\ x(0) = x_0, \end{cases}$$

and a cost functional

$$J_T(\sigma(\cdot); x_0) = \int_0^T \mathrm{d}t \int_{\mathbb{R}} (x^2(t) + u^4 - u^2)\, \sigma(t)(\mathrm{d}u).$$

Problem $RP(x_0, T)$: Find $\bar{\sigma}_T(\cdot) \in \mathcal{R}_T(\mathbb{R})$, such that

$$J_T(\bar{\sigma}_T(\cdot); x_0) = \inf_{\sigma(\cdot) \in \mathcal{R}_T(\mathbb{R})} J_T(\sigma(\cdot); x_0).$$

In this example, all the assumptions in the paper are satisfied. Thus, the three kinds of turnpike properties are satisfied.

Then, we take an example to illustrate that the integral turnpike property need not imply the mean square turnpike property. For convenience, we consider a non-relaxed case.

Example 7.4. Consider a 1-d system

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = u(t), \quad t \in [0, T], \\ x(0) = 1, \end{cases}$$

and a cost functional

$$J_T(u(\cdot); 1) = \int_0^T (f(x(t)) + g(u(t))) \, \mathrm{d}t,$$

where

$$f(x) = \begin{cases} -x - 1, & x \in (-\infty, -1], \\ 0, & x \in (-1, 1], \\ x - 1, & x \in (1, +\infty), \end{cases} \quad \text{and} \quad g(u) = \begin{cases} -u - 2, & u \in (-\infty, -1], \\ u, & u \in (-1, 0], \\ -u, & u \in (0, 1], \\ u - 2, & u \in (1, +\infty). \end{cases}$$

Problem $P(1, T)$: Find a $\bar{u}_T(\cdot) \in \mathcal{U}_T$, such that

$$J_T(\bar{u}_T(\cdot); 1) = \inf_{u(\cdot) \in \mathcal{U}_T} J_T(u(\cdot); 1).$$

Suppose $T \gg 1$ and denote $T_0 = [\frac{T}{2}]$, where $[\frac{T}{2}]$ represents the integer part of $\frac{T}{2}$. One of the optimal solution $(\bar{x}_T(\cdot), \bar{u}_T(\cdot))$ can be stated as

$$\bar{x}_T(t) = \sum_{k=1}^{T_0} (-1)^{k-1}(2k - 1 - t)\chi_{[2k-2,2k)}(t) + (-1)^{T_0}(2T_0 + 1 - t)\chi_{[2T_0, T]}(t), \quad \forall \, t \in [0, T],$$

$$\bar{u}_T(t) = \sum_{k=1}^{T_0} (-1)^k \chi_{[2k-2,2k)}(t) + (-1)^{T_0+1}\chi_{[2T_0, T]}(t), \quad \forall \, t \in [0, T],$$

where $\chi_A(\cdot)$ is the characteristic function.

Then, the corresponding optimization problem is

Problem $Q(\mathbb{R}, \mathbb{R})$: Find $(x^*, u^*) \in \mathbb{R} \times \mathbb{R}$, such that

$$f(x^*) + g(u^*) = \inf_{\substack{(x,u) \in \mathbb{R} \times \mathbb{R} \\ u = 0}} (f(x) + g(u)).$$

One of the optimal solution of Problem $Q(\mathbb{R}, \mathbb{R})$ is

$$x^* = 0, \quad u^* = 0.$$

Based on the above analysis, it can be verified that

$$\frac{1}{T}\int_0^T \bar{x}_T(t) \, \mathrm{d}t \to x^*, \quad \frac{1}{T}\int_0^T |\bar{x}_T(t) - x^*|^2 \, \mathrm{d}t \nrightarrow 0, \quad \text{as } T \to +\infty,$$

which means that the integral turnpike property holds, while the mean square turnpike property fails.

8. CONCLUDING REMARKS

Without the convexity of $g(\cdot)$ in the cost functional, it is interesting to consider relaxed problems. In this paper, we have established turnpike properties of optimal relaxed control problems. Among them, there are integral turnpike property, mean square turnpike property and exponential turnpike property.

Since the optimal relaxed control problems, considered in this paper, generalize the LQ problems without the cross-terms of the trajectory and the control, more assumptions are needed to derive the corresponding turnpike properties than those in the existing literature. We set some assumptions on $f(\cdot)$, $g(\cdot)$ and (A, B), which are easily verifiable. However, there are many challenging problems that need to be studied. For relaxed problems, optimal solutions exist under relatively weak conditions. Therefore, we think it will be interesting to consider turnpike properties for optimal relaxed control problems when $f(\cdot)$ is not convex ((A3) fails) or not convex enough ((A4) or (A6) fails), and A is degenerate ((A5) fails). Moreover, there are different kinds of turnpike properties in the existing literature besides what we have mentioned above. It is interesting to prove different turnpike properties under different assumptions and discuss the relation between them.

In addition, the turnpike properties of optimal control problems with state and control constraints have attracted some researchers' attention (see [14] for instance). It will be very challenging to study the turnpike properties of optimal relaxed control problems with state and control constraints.

Appendix A. Proof of Lemma 2.1

Proof. 1. Since $\mathcal{M}_S(X, U)$ is nonempty, $\mathscr{M}_S(X, U)$ is nonempty. Take a minimizing sequence $\{(\mu_k, \theta_k)\}_{k=1}^{+\infty} \subseteq \mathscr{M}_S(X, U)$ of Problem $\mathscr{RQ}(X, U)$, which implies that

$$\lim_{k \to +\infty} \left(\int_X f(x) \, \mu_k(\mathrm{d}x) + \int_U g(u) \, \theta_k(\mathrm{d}u) \right) = \inf_{(\mu, \theta) \in \mathscr{M}_S(X, U)} \left(\int_X f(x) \, \mu(\mathrm{d}x) + \int_U g(u) \, \theta(\mathrm{d}u) \right).$$

From the sequential compactness of $\mathcal{M}_+^1(X)$ and $\mathcal{M}_+^1(U)$, we have a $\mu^* \in \mathcal{M}_+^1(X)$ and a $\theta^* \in \mathcal{M}_+^1(U)$ satisfying

$$\mu_k \to \mu^* \quad \text{and} \quad \theta_k \to \theta^*, \quad \text{as } k \to +\infty.$$

Obviously, $(\mu^*, \theta^*) \in \mathscr{M}_S(X, U)$. Combining with the continuity of $f(\cdot)$ and $g(\cdot)$, it holds that

$$\int_X f(x) \, \mu^*(\mathrm{d}x) + \int_U g(u) \, \theta^*(\mathrm{d}u) = \inf_{(\mu, \theta) \in \mathscr{M}_S(X, U)} \left(\int_X f(x) \, \mu(\mathrm{d}x) + \int_U g(u) \, \theta(\mathrm{d}u) \right).$$

Then, (μ^*, θ^*) is a solution of Problem $\mathscr{RQ}(X, U)$.

Meanwhile, let $x^* = \int_X x \, \mu^*(\mathrm{d}x)$. Since X is convex and compact, it holds that $x^* \in X$. Also, the convexity of $f(\cdot)$ ensures that

$$f(x^*) \leq \int_X f(x) \, \mu^*(\mathrm{d}x).$$

Thus, (δ_{x^*}, θ^*) is also a solution of Problem $\mathscr{RQ}(X, U)$, which implies that (x^*, θ^*) is a solution of Problem $RQ(X, U)$.

2. Let (x^*, θ^*) be a solution of Problem $RQ(X, U)$. Based on the analysis in (1), we know that (δ_{x^*}, θ^*) is also a solution of Problem $\mathscr{RQ}(X, U)$.

For convenience, we suppose

$$f(x^*) + \int_U g(u) \, \theta^*(\mathrm{d}u) = 0.$$

For any $\varepsilon > 0$ and $(\mu, \theta) \in \mathcal{M}_+^1(X) \times \mathcal{M}_+^1(U)$, denote

$$I_\varepsilon(\mu, \theta) = \sqrt{\left[\left(\int_X f(x) \, \mu(\mathrm{d}x) + \int_U g(u) \, \theta(\mathrm{d}u) + \varepsilon \right)^+ \right]^2 + \left[\int_X Ax \, \mu(\mathrm{d}x) + \int_U Bu \, \theta(\mathrm{d}u) \right]^2}.$$

Then, it holds that

$$I_\varepsilon(\mu, \theta) > 0, \quad \forall\, (\mu, \theta) \in \mathcal{M}^1_+(X) \times \mathcal{M}^1_+(U),$$

$$I_\varepsilon(\delta_{x^*}, \theta^*) = \varepsilon \leq \inf_{(\mu,\theta)\in\mathcal{M}^1_+(X)\times\mathcal{M}^1_+(U)} I_\varepsilon(\mu, \theta) + \varepsilon.$$

By Ekeland's variational principle, there exists a pair $(\mu^\varepsilon, \theta^\varepsilon) \in \mathcal{M}^1_+(X) \times \mathcal{M}^1_+(U)$, such that

$$\|\mu^\varepsilon - \delta_{x^*}\| + \|\theta^\varepsilon - \theta^*\| \leq \sqrt{\varepsilon},$$

$$I_\varepsilon(\mu^\varepsilon, \theta^\varepsilon) \leq I_\varepsilon(\mu, \theta) + \sqrt{\varepsilon}(\|\mu - \mu^\varepsilon\| + \|\theta - \theta^\varepsilon\|), \quad \forall\, (\mu, \theta) \in \mathcal{M}^1_+(X) \times \mathcal{M}^1_+(U).$$

For any $\mu \in \mathcal{M}^1_+(X)$, $\theta \in \mathcal{M}^1_+(U)$ and $\alpha \in (0, 1)$, we have

$$\begin{aligned}
I_\varepsilon(\mu^\varepsilon, \theta^\varepsilon) &\leq I_\varepsilon(\mu^\varepsilon + \alpha(\mu - \mu^\varepsilon), \theta^\varepsilon + \alpha(\theta - \theta^\varepsilon)) + \alpha\sqrt{\varepsilon}(\|\mu - \mu^\varepsilon\| + \|\theta - \theta^\varepsilon\|) \\
&\leq I_\varepsilon(\mu^\varepsilon + \alpha(\mu - \mu^\varepsilon), \theta^\varepsilon + \alpha(\theta - \theta^\varepsilon)) + 4\alpha\sqrt{\varepsilon}.
\end{aligned}$$

It follows that

$$\begin{aligned}
-4\sqrt{\varepsilon} &\leq \lim_{\alpha\to 0^+} \frac{I_\varepsilon(\mu^\varepsilon + \alpha(\mu - \mu^\varepsilon), \theta^\varepsilon + \alpha(\theta - \theta^\varepsilon)) - I_\varepsilon(\mu^\varepsilon, \theta^\varepsilon)}{\alpha} \\
&= \psi_0^\varepsilon \Big(\int_X f(x)\,(\mu - \mu^\varepsilon)(\mathrm{d}x) + \int_U g(u)\,(\theta - \theta^\varepsilon)(\mathrm{d}u) \Big) \\
&\quad + \Big\langle \psi^\varepsilon, \int_X Ax\,(\mu - \mu^\varepsilon)(\mathrm{d}x) + \int_U Bu\,(\theta - \theta^\varepsilon)(\mathrm{d}u) \Big\rangle,
\end{aligned} \tag{A.1}$$

where

$$\psi_0^\varepsilon = \frac{\Big(\int_X f(x)\,\mu^\varepsilon(\mathrm{d}x) + \int_U g(u)\,\theta^\varepsilon(\mathrm{d}u) + \varepsilon \Big)^+}{I_\varepsilon(\mu^\varepsilon, \theta^\varepsilon)}, \quad \psi^\varepsilon = \frac{\int_X Ax\,\mu^\varepsilon(\mathrm{d}x) + \int_U Bu\,\theta^\varepsilon(\mathrm{d}u)}{I_\varepsilon(\mu^\varepsilon, \theta^\varepsilon)}.$$

Then,

$$\psi_0^\varepsilon \geq 0, \tag{A.2}$$

$$\langle \psi^\varepsilon, w \rangle = 0, \quad \text{if } A^\top w = 0,\ B^\top w = 0,\ w \in \mathbb{R}^n, \tag{A.3}$$

$$|\psi_0^\varepsilon|^2 + |\psi^\varepsilon|^2 = 1.$$

Thus, we have a subsequence of $\{(\psi_0^\varepsilon, \psi^\varepsilon)\}_{\varepsilon>0}$ (we do not relabel the sequence) and a (ψ_0^*, ψ^*), satisfying

$$(\psi_0^\varepsilon, \psi^\varepsilon) \to (\psi_0^*, \psi^*), \quad \text{as } \varepsilon \to 0^+,$$

$$|\psi_0^*|^2 + |\psi^*|^2 = 1. \tag{A.4}$$

Combining with (A.1), we have

$$
\begin{aligned}
\psi_0^* \Big(&\int_X f(x)\,(\mu - \delta_{x^*})(\mathrm{d}x) + \int_U g(u)\,(\theta - \theta^*)(\mathrm{d}u) \Big) \\
&+ \Big\langle \psi^*, \int_X Ax\,(\mu - \delta_{x^*})(\mathrm{d}x) + \int_U Bu\,(\theta - \theta^*)(\mathrm{d}u) \Big\rangle \geq 0, \\
&\qquad \forall\, \mu \in \mathcal{M}_+^1(X),\ \theta \in \mathcal{M}_+^1(U),
\end{aligned}
\tag{A.5}
$$

To conclude, (A.2) and (A.3) ensure that (2.1) and (2.2) are tenable. Equation (A.4) is the nontrivial condition of (ψ_0^*, ψ^*). Also, (A.5) implies (2.3) and (2.4)/(2.5) hold. □

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