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REGULARITY AND STABILITY OF COUPLED PLATE EQUATIONS WITH INDIRECT STRUCTURAL OR KELVIN-VOIGT DAMPING

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Abstract. In this paper, the regularity and stability of the semigroup associated with a system of coupled plate equations is considered. Indirect structural or Kelvin-Voigt damping is imposed, *i.e.*, only one equation is directly damped by one of these two damping. By the frequency domain method, we show that the associated semigroup of the system with indirect structural damping is analytic and exponentially stable. However, with the much stronger indirect Kelvin-Voigt damping, we prove that, by the asymptotic spectral analysis, the semigroup is even not differentiable. The exponential stability is still maintained. Finally, some numerical simulations of eigenvalues of the corresponding one-dimensional systems are also given.

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1. Introduction

It is known that the semigroup associated with the following thermoelastic plate equations of type I or type III thermoelasticity are exponentially stable and analytic (see [6, 15–17]).

$$\begin{cases} u_{tt}(x,t) = -\Delta^2 u(x,t) - \gamma\Delta\theta(x,t), \\ \theta_t(x,t) = \gamma\Delta u_t(x,t) + k(\Delta - I)\theta(x,t); \end{cases} \tag{1.1}$$

and

$$\begin{cases} u_{tt}(x,t) = -\Delta^2 u(x,t) - \gamma\Delta\eta_t(x,t), \\ \eta_{tt}(x,t) = k_1(\Delta - I)\eta(x,t) + \gamma\Delta u_t(x,t) + k_2(\Delta - I)\eta_t(x,t). \end{cases} \tag{1.2}$$

These two models can be regarded as a conservative plate equation indirectly damped by the heat dissipation from the heat equations through the coupling term [22]. Note that when $\gamma = 0$, the semigroup associated with the decoupled heat equations is analytic and exponentially stable. In this paper, we are interested in whether the regularity and stability properties can be kept for a conservative plate equation indirectly damped by

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another dissipative plate equation through the coupling term. The damping mechanism considered here are the structural or the Kelvin-Voigt damping.

More precisely, we shall consider the following system of linear coupled plate equations,

$$\begin{cases} u_{tt}(x,t) = -\Delta^2 u - \gamma\Delta w_t, \quad x \in \Omega,\ t > 0, \\ w_{tt}(x,t) = -\Delta^2 w + d_{st}\Delta w_t - d_{kv}\Delta^2 w_t + \gamma\Delta u_t, \quad x \in \Omega,\ t > 0, \\ u|_{\partial\Omega} = \frac{\partial u}{\partial\nu}|_{\partial\Omega} = 0,\ \ t > 0, \\ w|_{\partial\Omega} = \frac{\partial w}{\partial\nu}|_{\partial\Omega} = 0,\ \ t > 0, \\ u(x,0) = u^0(x),\ u_t(x,0) = u^1(x),\ w(x,0) = w^0(x),\ w_t(x,0) = w^1(x),\ x \in \Omega,\ t > 0, \end{cases} \tag{1.3}$$

where $u(x,t)$ and $w(x,t)$ are the displacement of the system at time t in the domain $\Omega \in \mathbb{R}^N$ with smooth boundary $\partial\Omega$, $\gamma \neq 0$ is the coupling coefficient. Only one of the damping coefficients $d_{st} \geq 0$ and $d_{kv} \geq 0$ is positive.

Systems of coupled elastic equations with indirect damping have been investigated extensively in the literature. It is impossible to give a complete review. We just refer to [1–4, 7, 9, 11, 18, 23] and the references therein. Most of the works are on the stability of the system with various coupling, damping locations, and damping types.

Note that when $\gamma = 0$, the system decouples into a conservative plate equation and a damped plate equation. It is known that the semigroup associated with the damped plate equation with either the structural damping or the Kelvin-Voigt damping is analytic and exponentially stable (see [5, 13, 14]). This is similar to the thermoelastic plate equation. When $\gamma \neq 0$, we would like to know how strongly the conservative plate is indirectly damped, and what are the regularity and stability properties of the coupled system.

The natural energy of this system is defined by

$$E(t) = \frac{1}{2}\int_\Omega |\Delta u|^2 + |u_t|^2 + |\Delta w|^2 + |w_t|^2)\mathrm{d}x. \tag{1.4}$$

By a straight forward calculation, we obtain

$$E'(t) = -\int_\Omega d_{st}|\nabla w_t|^2\mathrm{d}x - \int_\Omega d_{kv}|\Delta w_t|^2\mathrm{d}x \leq 0, \tag{1.5}$$

which implies that this system is dissipative.

We first show that the semigroup associated with the system with indirect structural damping is analytic and exponentially stable by the frequency domain method. Thus, it is reasonable to expect that under Kelvin-Voigt damping, the semigroup should be still analytic, since the Kelvin-Voigt damping is much stronger than the structural one (see [5]). However, by a detailed spectral analysis, we find that a branch of eigenvalues of the system with indirect Kelvin-Voigt damping has a vertical asymptote $\mathcal{R}e\lambda = -\frac{\gamma^2}{2d_{kv}}$. This implies that the associated semigroup can't be analytic. In fact, it even lacks differentiability because of the distribution of the spectrum. On the other hand, we are able to show that the exponential stability is maintained. Similar results are obtained for the system with both indirect structural and Kelvin-Voigt damping.

We would like to point out that the regularity and stability results obtained in this paper still hold for the system with some other type of boundary conditions, such as the simply supported boundary condition.

This paper is organized as follows. In Section 2, the system is set up as a first order evolution equation in an appropriate Hilbert space, and its well-posedness is proved. Section 3 is devoted to proving the analyticity and exponential stability for the system with indirect structural damping. In Section 4, we show that the system with indirect Kelvin-Voigt damping lacks differentiability but it is still exponentially stable. Numerical simulations of eigenvalues of the corresponding one-dimensional systems are presented in Section 5.

2. Semigroup setting

This section is devoted to giving the well-posedness of system (1.3) in the frame of semigroup. Let us first reformulate system (1.3) in an appropriate Hilbert state space setting.

The state space is chosen to be the following Hilbert space

$$\mathcal{H} = H_0^2(\Omega) \times L^2(\Omega) \times H_0^2(\Omega) \times L^2(\Omega),$$

equipped with an inner product

$$\langle Z_1, Z_2 \rangle_{\mathcal{H}} = \int_\Omega (v_1 \overline{v_2} + \Delta u_1 \Delta \overline{u_2} + z_1 \overline{z_2} + \Delta w_1 \Delta \overline{w_2}) \mathrm{d}x,$$

for $Z_j = (u_j, v_j, w_j, z_j) \in \mathcal{H},\ j = 1, 2$. We define the system operator $\mathcal{A}$ in $\mathcal{H}$ as follows:

$$\mathcal{A} \begin{bmatrix} u \\ v \\ w \\ z \end{bmatrix} = \begin{pmatrix} v \\ -\Delta^2 u - \gamma \Delta z \\ z \\ -\Delta^2 w + d_{st} \Delta z - d_{kv} \Delta^2 z + \gamma \Delta v \end{pmatrix}$$

with domain

$$\mathcal{D}(\mathcal{A}) = \left\{ (u, v, w, z) \in \mathcal{H} \;\middle|\; \begin{array}{l} v \in H_0^2(\Omega),\ z \in H_0^2(\Omega) \\ \Delta u \in H^2(\Omega) \\ (\Delta w + d_{kv} \Delta z) \in H^2(\Omega) \\ \dfrac{\partial u}{\partial \nu}|_{\partial\Omega} = \dfrac{\partial w}{\partial \nu}|_{\partial\Omega} = 0 \end{array} \right\}.$$

Then, system (1.3) can be rewritten as an evolution equation in $\mathcal{H}$:

$$\begin{cases} \dfrac{\mathrm{d}U(t)}{\mathrm{d}t} = \mathcal{A}U(t), \quad t > 0, \\ U(0) = U_0 \end{cases} \tag{2.1}$$

where $U(t) = (u(\cdot, t), v(\cdot, t), w(\cdot, t), z(\cdot, t))^T$ and $U_0 = (u^0, u^1, w^0, w^1)^T \in \mathcal{H}$.

Using the classic semigroup theory (see [20]), we get the following result on the well-posedness of system (2.1).

Theorem 2.1. *Let $\mathcal{A}$ and $\mathcal{H}$ be defined as before. Then*

(1) $\mathcal{A}$ is dissipative in $\mathcal{H}$ and satisfies

$$\langle \mathcal{A}U, U \rangle_{\mathcal{H}} = -d_{st} \|\nabla z\|^2 - d_{kv} \|\Delta z\|^2 \leq 0. \tag{2.2}$$

(2) $\mathcal{A}$ generates a C_0 semigroup $S(t)$ of contractions on $\mathcal{H}$.

Moreover, it is easy to verify the following result.

Lemma 2.2. *Let $\mathcal{A}$ and $\mathcal{H}$ be defined as before. Then $i\mathbb{R} \in \rho(\mathcal{A})$.*

3. System with indirect structural damping

In this section, we consider the case when $d_{st} > 0,\ d_{kv} = 0$ in system (1.3), that is, the system with indirect structural damping. We need the following well known result (see [8, 19, 20]).

Lemma 3.1. *Let $A : \mathcal{D}(\mathcal{A}) \subseteq \mathcal{H} \to \mathcal{H}$ generates a C_0-semigroup $e^{\mathcal{A}t}$ on $\mathcal{H}$ such that*

$$\|e^{\mathcal{A}t}\| \leq M, \quad \forall t \geq 0,$$

for some $M \geq 1$ and

$$i\beta \in \rho(\mathcal{A}), \quad \forall \beta \in \mathbb{R}.$$

Then the semigroup $e^{\mathcal{A}t}$ is analytic if and only if for some $a \in \mathbb{R}$ and $b, C > 0$ such that

$$\rho(\mathcal{A}) \supseteq \sum(a,b) := \{\lambda \in \mathbb{C} | \mathcal{R}e\lambda > a - b|\mathcal{I}m\lambda|\}, \tag{3.1}$$

and

$$\|(i\lambda - \mathcal{A})^{-1}\| \leq \frac{C}{1+|\lambda|}, \quad \lambda \in \sum(a,b).$$

This is equivalent to

$$\limsup_{\beta \in \mathbb{R}, |\beta| \to \infty} |\beta| \|(i\beta I - \mathcal{A})^{-1}\| < \infty.$$

Our main result in this section is

Theorem 3.2. *When $d_{st} > 0$, $d_{kv} = 0$, the semigroup $e^{\mathcal{A}t}$ associated with system (1.3) is analytic on $\mathcal{H}$. As a consequence, it is exponentially stable.*

Proof. By Lemma 3.1, it is sufficient to check the following estimate for the resolvent operator along the imaginary axis.

$$\limsup_{\beta_n \to \infty} \beta_n \|(i\beta_n I - \mathcal{A})^{-1}\| < \infty. \tag{3.2}$$

If (3.2) is not true, by the Banach-Steinhaus theorem, there exist a sequence $\beta_n \in \mathbb{R}$ with $|\beta_n| \to \infty$ and a sequence $W_n = (u_n, v_n, w_n, z_n)^T \in \mathcal{D}(\mathcal{A})$ with $\|W_n\|_{\mathcal{H}} = 1$ such that

$$\lim_{n \to \infty} \beta_n^{-1} \|(i\beta_n I - \mathcal{A}) W_n\|_{\mathcal{H}} = 0,$$

i.e.,

$$iu_n - \beta_n^{-1} v_n = f_{1n} \to 0, \quad \text{in } H^2(\Omega), \tag{3.3}$$
$$iv_n + \beta_n^{-1} \Delta^2 u_n + \beta_n^{-1} \gamma \Delta z_n = f_{2n} \to 0, \quad \text{in } L^2(\Omega), \tag{3.4}$$
$$iw_n - \beta_n^{-1} z_n = f_{3n} \to 0, \quad \text{in } H^2(\Omega), \tag{3.5}$$
$$iz_n + \beta_n^{-1} \Delta^2 w_n - \beta_n^{-1} d_{st} \Delta z_n - \beta_n^{-1} \gamma \Delta v_n = f_{4n} \to 0, \quad \text{in } L^2(\Omega). \tag{3.6}$$

Since $\mathcal{A}$ is dissipative, we obtain that

$$\mathcal{R}e\langle \beta_n^{-1}(i\beta_n - \mathcal{A}) W_n, W_n \rangle_{\mathcal{H}} = d_{st} \|\beta_n^{-\frac{1}{2}} \nabla z_n\|^2 \to 0. \tag{3.7}$$

We are going to derive that $\|W_n\|_{\mathcal{H}} \to 0$ which is a contradiction to $\|W_n\|_{\mathcal{H}} = 1$. By (3.3) and (3.5), it is obvious that $\|\beta_n^{-1}\Delta v_n\|$, $\|\beta_n^{-1}\Delta z_n\|$ are bounded. Thus, in reference of (3.4) and (3.6), we obtain that $\|\beta_n^{-1}\Delta^2 u_n\|$, $\beta_n^{-1}\|\Delta^2 w_n\|$ are also bounded. Moreover, by interpolation,

$$\|\beta_n^{-\frac{1}{2}}\nabla v_n\| \le \|v_n\|^{\frac{1}{2}}\|\beta_n^{-1}\Delta v_n\|^{\frac{1}{2}}, \tag{3.8}$$

which implies that $\|\beta_n^{-\frac{1}{2}}\nabla v_n\|$ is also bounded.

Taking the inner product of (3.6) with z_n yields

$$\langle iz_n, z_n\rangle + \langle \beta_n^{-1}\Delta^2 w_n, z_n\rangle - d_{st}\langle \beta_n^{-1}\Delta z_n, z_n\rangle - \gamma\langle \beta_n^{-1}\Delta v_n, z_n\rangle \to 0. \tag{3.9}$$

Note that by Gagliardo-Nirenberg interpolation inequality,

$$\|\beta_n^{-\frac{1}{2}}\nabla(\Delta w_n)\| \le \|\Delta w_n\|^{\frac{1}{2}}\|\beta_n^{-1}\Delta^2 w_n\|^{\frac{1}{2}} + \beta_n^{-1/2}\|\Delta w_n\|, \tag{3.10}$$

which together with the boundedness of $\|\Delta w_n\|$, implies that $\|\beta_n^{-\frac{1}{2}}\nabla(\Delta w_n)\|$ is bounded. Thus, by (3.7), the second term in (3.9) satisfies

$$|\langle \beta_n^{-1}\Delta^2 w_n, z_n\rangle| = |\langle \beta_n^{-\frac{1}{2}}\nabla(\Delta w_n), \beta_n^{-\frac{1}{2}}\nabla z_n\rangle| \le \|\beta_n^{-\frac{1}{2}}\nabla(\Delta w_n)\|\,\|\beta_n^{-\frac{1}{2}}\nabla z_n\| \to 0. \tag{3.11}$$

On the other hand,

$$\begin{aligned}\langle \beta_n^{-1}\Delta^2 w_n, z_n\rangle &= \langle \Delta w_n, \beta_n^{-1}\Delta z_n\rangle \\ &= -i\|\Delta w_n\|^2 - \langle \Delta w_n, \Delta f_{3n}\rangle.\end{aligned} \tag{3.12}$$

Since $\|\Delta f_{3n}\| \to 0$, combining (3.11) and (3.12), we get

$$\|\Delta w_n\| \to 0. \tag{3.13}$$

Hence, due to (3.10), we also get

$$\|\beta_n^{-\frac{1}{2}}\nabla(\Delta w_n)\| \to 0. \tag{3.14}$$

It is easy to see that the third term and fourth term in (3.9) all converge to 0. In fact, by (3.7) and (3.8),

$$\langle \beta_n^{-1}\Delta z_n, z_n\rangle = -\|\beta_n^{-\frac{1}{2}}\nabla z_n\|^2 \to 0, \tag{3.15}$$

and

$$|\langle \beta_n^{-1}\Delta v_n, z_n\rangle| = |\langle \beta_n^{-\frac{1}{2}}\nabla v_n, \beta_n^{-\frac{1}{2}}\nabla z_n\rangle| \le \|\beta_n^{-\frac{1}{2}}\nabla v_n\|\,\|\beta_n^{-\frac{1}{2}}\nabla z_n\| \to 0. \tag{3.16}$$

Therefore, the first term in (3.9) converges to 0, *i.e.*,

$$\|z_n\| \to 0. \tag{3.17}$$

Next, we take the inner product of (3.6) with v_n to get

$$\langle iz_n, v_n\rangle + \langle \beta_n^{-1}\Delta^2 w_n, v_n\rangle - d_{st}\langle \beta_n^{-1}\Delta z_n, v_n\rangle - \gamma\langle \beta_n^{-1}\Delta v_n, v_n\rangle \to 0. \tag{3.18}$$

It is easy to see that the first term $\langle iz_n, v_n\rangle$ converges to 0 due to (3.17). By (3.14) and the boundedness of $\|\beta_n^{-\frac{1}{2}}\nabla v_n\|$, we get

$$\langle\beta_n^{-1}\Delta^2 w_n, v_n\rangle = -\langle\beta_n^{-\frac{1}{2}}\nabla(\Delta w_n), \beta_n^{-\frac{1}{2}}\nabla v_n\rangle \to 0,$$

and

$$\langle\beta_n^{-1}\Delta z_n, v_n\rangle = -\langle\beta_n^{-\frac{1}{2}}\nabla z_n, \beta_n^{-\frac{1}{2}}\nabla v_n\rangle \to 0.$$

Hence, by (3.18), we get

$$\langle\beta_n^{-1}\Delta v_n, v_n\rangle = -\|\beta_n^{-\frac{1}{2}}\nabla v_n\|^2 \to 0. \tag{3.19}$$

On the other hand, the inner product of (3.4) with v_n leads to

$$i\|v_n\|^2 + \langle\beta_n^{-1}\Delta^2 u_n, v_n\rangle + \gamma\langle\beta_n^{-1}\Delta z_n, v_n\rangle \to 0. \tag{3.20}$$

Similar to the discussion in (3.10)–(3.13), by (3.19), we have

$$\langle\beta_n^{-1}\Delta^2 u_n, v_n\rangle = -\langle\beta_n^{-\frac{1}{2}}\nabla(\Delta u_n), \beta_n^{-\frac{1}{2}}\nabla v_n\rangle \to 0, \tag{3.21}$$

and

$$\begin{aligned}\langle\beta_n^{-1}\Delta^2 u_n, v_n\rangle &= \langle\beta_n^{-1}\Delta u_n, \Delta v_n\rangle\\ &= \langle\Delta u_n, i\Delta u_n - \Delta f_{1n}\rangle.\end{aligned} \tag{3.22}$$

Since $\|\Delta f_{1n}\| \to 0$, (3.21) and (3.22) imply that

$$\|\Delta u_n\| \to 0. \tag{3.23}$$

It is easy to see that the third term in (3.20) also converges to 0. In fact,

$$\langle\beta_n^{-1}\Delta z_n, v_n\rangle = -\langle\beta_n^{-\frac{1}{2}}\nabla z_n, \beta_n^{-\frac{1}{2}}\nabla v_n\rangle \to 0 \tag{3.24}$$

because of (3.7) and (3.19). Applying (3.21) and (3.24) to (3.20) yields

$$\|v_n\| \to 0. \tag{3.25}$$

Therefore, by (3.13), (3.17), (3.23) and (3.25), we arrive at the promised contradiction $\|W_n\|_{\mathcal{H}} \to 0$. The analyticity of this system follows.

It follows from Lemma 2.2 and the analyticity of $e^{\mathcal{A}t}$ that the semigroup is exponentially stable. The proof is complete. □

4. System with indirect Kelvin-Voigt damping

This section is devoted to discussing the stability and regularity of system (1.3) with $d_{st} = 0$, $d_{kv} > 0$, *i.e.*, the system with indirect Kelvin-Voigt damping. We will show that the system lacks analyticity, and in fact is not even differentiable. However, it still maintains the exponential stability. We have the following result.

Theorem 4.1. *Let $\mathcal{A}$ and $\mathcal{H}$ be defined as before. When $d_{st}=0$, $d_{kv}>0$, the semigroup $e^{\mathcal{A}t}$ associated with system (1.3) is not differentiable.*

Proof. We will prove the conclusion by carrying out a detailed spectral analysis for the system operator $\mathcal{A}$.

Let us consider the eigenvalue problem

$$(\lambda I-\mathcal{A})(u,\lambda u,w,\lambda w)^T=0, \tag{4.1}$$

i.e.,

$$\begin{cases} \lambda^2 u=-\Delta^2 u-\lambda\gamma\Delta w,\\ \lambda^2 w=-\Delta^2 w-\lambda d_{kv}\Delta^2 w+\lambda\gamma\Delta u,\\ u|_{\partial\Omega}=\frac{\partial u}{\partial\nu}|_{\partial\Omega}=0,\\ w|_{\partial\Omega}=\frac{\partial w}{\partial\nu}|_{\partial\Omega}=0. \end{cases} \tag{4.2}$$

Let μ_k and $\Phi_k(x)$ be the eigenvalue and its eigenfunction of the Laplace operator $-\Delta$ with Dirichlet boundary condition. μ_k goes to $+\infty$ as $k\to+\infty$. Set

$$\begin{cases} u_k=C\Phi_k(x),\ x\in\Omega,\\ w_k=D\Phi_k(x),\ x\in\Omega. \end{cases} \tag{4.3}$$

Substituting the above into the first two equations in (4.2), we get

$$\begin{pmatrix} \lambda_k^2+\mu_k^2 & -\lambda_k\gamma\mu_k\\ \lambda_k\gamma\mu_k & \lambda_k^2+\mu_k^2+\lambda_k d_{kv}\mu_k^2 \end{pmatrix}\begin{pmatrix} C\\ D\end{pmatrix}\Phi_k(x)=0. \tag{4.4}$$

Thus, λ_k is an eigenvalue of $\mathcal{A}$, if

$$\zeta(\lambda_k)=\det\begin{pmatrix} \lambda_k^2+\mu_k^2 & -\lambda_k\gamma\mu_k\\ \lambda_k\gamma\mu_k & \lambda_k^2+\mu_k^2+\lambda_k d_{kv}\mu_k^2 \end{pmatrix}=0. \tag{4.5}$$

A direct calculation yields

$$\begin{aligned} \zeta(\lambda_k)&=(\lambda_k^2+\mu_k^2)(\lambda_k^2+\mu_k^2+\lambda_k d_{kv}\mu_k^2)+\lambda_k^2\gamma^2\mu_k^2\\ &=(\lambda_k^2+\mu_k^2)^2+\lambda_k d_{kv}\mu_k^2(\lambda_k^2+\mu_k^2)+\lambda_k^2\gamma^2\mu_k^2\\ &=(\lambda_k^2+\mu_k^2+\frac{\lambda_k d_{kv}\mu_k^2}{2})^2-(\frac{\lambda_k d_{kv}\mu_k^2}{2})^2+\lambda_k^2\gamma^2\mu_k^2\\ &=(\lambda_k^2+\mu_k^2+\frac{\lambda_k d_{kv}\mu_k^2}{2})^2-\frac{\lambda_k^2 d_{kv}^2\mu_k^4}{4}[1-\frac{4\gamma^2}{d_{kv}^2\mu_k^2}]\\ &=\left(\lambda_k^2+\mu_k^2+\frac{\lambda_k d_{kv}\mu_k^2}{2}+\frac{\lambda_k d_{kv}\mu_k^2}{2}\sqrt{1-\frac{4\gamma^2}{d_{kv}^2\mu_k^2}}\right)\\ &\qquad\times\left(\lambda_k^2+\mu_k^2+\frac{\lambda_k d_{kv}\mu_k^2}{2}-\frac{\lambda_k d_{kv}\mu_k^2}{2}\sqrt{1-\frac{4\gamma^2}{d_{kv}^2\mu_k^2}}\right)\\ &=0. \end{aligned} \tag{4.6}$$

Hence, at least one of the following two equations hold.

$$\lambda_k^2+\mu_k^2+\frac{\lambda_k d_{kv}\mu_k^2}{2}+\frac{\lambda_k d_{kv}\mu_k^2}{2}\sqrt{1-\frac{4\gamma^2}{d_{kv}^2\mu_k^2}}=0, \tag{4.7}$$

$$\lambda_k^2+\mu_k^2+\frac{\lambda_k d_{kv}\mu_k^2}{2}-\frac{\lambda_k d_{kv}\mu_k^2}{2}\sqrt{1-\frac{4\gamma^2}{d_{kv}^2\mu_k^2}}=0. \tag{4.8}$$

Let us calculate λ_k by (4.7) and (4.8), respectively.

(1) Calculate λ_k from (4.7).
Note that $\frac{1}{\mu_k}\to 0$ as $k\to+\infty$. Thus,

$$\sqrt{1-\frac{4\gamma^2}{d_{kv}^2\mu_k^2}}=1-\frac{1}{2}\frac{4\gamma^2}{d_{kv}^2\mu_k^2}-\frac{1}{8}\frac{16\gamma^4}{d_{kv}^4\mu_k^4}+O(\frac{1}{\mu_k^6}).$$

Hence, we have

$$\lambda_k^2+\mu_k^2+\frac{\lambda_k d_{kv}\mu_k^2}{2}+\frac{\lambda_k d_{kv}\mu_k^2}{2}(1-\frac{1}{2}\frac{4\gamma^2}{d_{kv}^2\mu_k^2}-\frac{1}{8}\frac{16\gamma^4}{d_{kv}^4\mu_k^4}+O(\frac{1}{\mu_k^6}))=0,$$

which is further simplified into

$$\lambda_k^2+\lambda_k(d_{kv}\mu_k^2-\frac{\gamma^2}{d_{kv}}-\frac{\gamma^4}{d_{kv}^3\mu_k^2}+O(\frac{1}{\mu_k^4}))+\mu_k^2=0.$$

A direct calculation yields

$$\begin{aligned}\lambda_k&=\frac{-(d_{kv}\mu_k^2-\frac{\gamma^2}{d_{kv}}-\frac{\gamma^4}{d_{kv}^3\mu_k^2}+O(\frac{1}{\mu_k^4}))\pm\sqrt{(d_{kv}\mu_k^2-\frac{\gamma^2}{d_{kv}}-\frac{\gamma^4}{d_{kv}^3\mu_k^2}+O(\frac{1}{\mu_k^4}))^2-4\mu_k^2}}{2}\\&=\frac{-d_{kv}\mu_k^2+\frac{\gamma^2}{d_{kv}}+\frac{\gamma^4}{d_{kv}^3\mu_k^2}+O(\frac{1}{\mu_k^4})\pm\mu_k^2\sqrt{(d_{kv}-\frac{\gamma^2}{d_{kv}\mu_k^2}+O(\frac{1}{\mu_k^4}))^2-4/\mu_k^2}}{2}\\&=\frac{-d_{kv}\mu_k^2+\frac{\gamma^2}{d_{kv}}+\frac{\gamma^4}{d_{kv}^3\mu_k^2}+O(\frac{1}{\mu_k^4})\pm\mu_k^2\sqrt{d_{kv}^2-\frac{4+2\gamma^2}{\mu_k^2}+O(\frac{1}{\mu_k^4})}}{2}\\&=\frac{-d_{kv}\mu_k^2+\frac{\gamma^2}{d_{kv}}+\frac{\gamma^4}{d_{kv}^3\mu_k^2}+O(\frac{1}{\mu_k^4})\pm\mu_k^2 d_{kv}(1-\frac{1}{2}(\frac{4+2\gamma^2}{d_{kv}^2\mu_k^2})+O(\frac{1}{\mu_k^4}))}{2}\\&=\begin{cases}-\frac{1}{d_{kv}}+O(\frac{1}{\mu_k^2}),\\-d_{kv}\mu_k^2+\frac{\gamma^2+1}{d_{kv}}+O(\frac{1}{\mu_k^2}).\end{cases}\end{aligned} \tag{4.9}$$

Thus, as $k\to+\infty$, $-\frac{1}{d_{kv}}$ is the cluster point of the eigenvalues of $\mathcal{A}$. The negative real axis is also one of the asymptotes of the eigenvalues.

(2) Calculate λ_k from (4.8).
Similar to the above discussion, by (4.8), we have

$$\lambda_k^2+\mu_k^2+\frac{\lambda_k d_{kv}\mu_k^2}{2}-\frac{\lambda_k d_{kv}\mu_k^2}{2}(1-\frac{1}{2}\frac{4\gamma^2}{d_{kv}^2\mu_k^2}-\frac{1}{8}\frac{16\gamma^4}{d_{kv}^4\mu_k^4}+O(\frac{1}{\mu_k^6}))=0.$$

Hence,

$$\lambda_k^2+\lambda_k(\frac{\gamma^2}{d_{kv}}+\frac{\gamma^4}{d_{kv}^3\mu_k^2}+O(\frac{1}{\mu_k^4}))+\mu_k^2=0.$$

A direct calculation yields

$$\begin{aligned}\lambda_k &= \frac{-(\frac{\gamma^2}{d_{kv}} + \frac{\gamma^4}{d_{kv}^3\mu_k^2} + O(\frac{1}{\mu_k^4})) \pm \sqrt{(\frac{\gamma^2}{d_{kv}} + \frac{\gamma^4}{d_{kv}^3\mu_k^2} + O(\frac{1}{\mu_k^4}))^2 - 4\mu_k^2}}{2} \\ &= -\frac{\frac{\gamma^2}{d_{kv}} + \frac{\gamma^4}{d_{kv}^3\mu_k^2} + O(\frac{1}{\mu_k^4})}{2} \pm i\mu_k\sqrt{1 - \frac{1}{4\mu_k^2}(\frac{\gamma^2}{d_{kv}} + \frac{\gamma^4}{d_{kv}^3\mu_k^2} + O(\frac{1}{\mu_k^4}))^2} \\ &= -\frac{\gamma^2}{2d_{kv}} + O(\frac{1}{\mu_k^2}) \pm i(\mu_k + O(\frac{1}{\mu_k})). \end{aligned} \tag{4.10}$$

Thus, we have found a vertical asymptote $\mathcal{R}e\lambda = -\frac{\gamma^2}{2d_{kv}}$ of the spectrum $\sigma(\mathcal{A})$. By the characteristic properties on the spectrum for differentiability (see [20]), we conclude that the system lacks differentiability. The proof is complete.

□

Now, let us further consider the stability of system (1.3) for this case. We have the following result on the stability of the system.

Theorem 4.2. *When $d_{st} = 0$, $d_{kv} > 0$, the semigroup $e^{\mathcal{A}t}$ associated with the system (1.3) is exponentially stable.*

Proof. By the necessary and sufficient condition for exponential stability of semigroup. It is sufficient to check the following two conditions (see [10, 12, 21]).

1) $i\sigma \in \rho(\mathcal{A})$, $\forall \sigma \in \mathbb{R}$;
2) $\limsup_{\beta_n \to \infty} \|(i\beta_n - \mathcal{A})^{-1}\| < \infty$.

The condition 1) has been obtained in Lemma 2.2. Thus we only need to check the condition 2). Similar to the proof of Theorem 3.2, if condition 2) is not true, there exist a sequence $\beta_n \in \mathbb{R}$ with $|\beta_n| \to \infty$ and a sequence $W_n = (u_n, v_n, w_n, z_n)^T \in \mathcal{D}(\mathcal{A})$ with $\|W_n\|_{\mathcal{H}} = 1$ such that

$$\lim_{n\to\infty} \|(i\beta_n I - \mathcal{A})W_n\|_{\mathcal{H}} = 0,$$

i.e.,

$$i\beta_n u_n - v_n = f_{1n} \to 0, \quad \text{in } H^2(\Omega), \tag{4.11}$$
$$i\beta_n v_n + \Delta^2 u_n + \gamma\Delta z_n = f_{2n} \to 0, \quad \text{in } L^2(\Omega), \tag{4.12}$$
$$\beta_n w_n - z_n = f_{3n} \to 0, \quad \text{in } H^2(\Omega), \tag{4.13}$$
$$i\beta_n z_n + \Delta^2 w_n + d_{kv}\Delta^2 z_n - \gamma\Delta v_n = f_{4n} \to 0, \quad \text{in } L^2(\Omega). \tag{4.14}$$

We obtain the following estimate since $\mathcal{A}$ is dissipative.

$$\mathcal{R}e\langle(i\beta_n - \mathcal{A})W_n, W_n\rangle_{\mathcal{H}} = d_{kv}\|\Delta z_n\|^2 \to 0. \tag{4.15}$$

Thus, by the Poincaré inequality,

$$\|z_n\| \to 0. \tag{4.16}$$

By (4.13), we also have

$$\|\beta_n \Delta w_n\| \to 0 \Rightarrow \|\Delta w_n\| \to 0. \tag{4.17}$$

Dividing (4.14) by β_n and taking the inner product with v_n leads to

$$\langle iz_n, v_n\rangle + \langle \beta_n^{-1}\Delta^2 w_n, v_n\rangle + d_{kv}\langle \beta_n^{-1}\Delta^2 z_n, v_n\rangle - r\langle \beta_n^{-1}\Delta v_n, v_n\rangle \to 0. \tag{4.18}$$

Since $\|\beta_n^{-1}\Delta v_n\|$ is bounded which can be seen from (4.11) and the boundedness of $\|\Delta u_n\|$, we obtain that the second and third term in (4.18) converge to 0. Moreover, it is clear that the first term in (4.18) also converges to 0. Hence,

$$\langle \beta_n^{-1}\Delta v_n, v_n\rangle = \|\beta_n^{-\frac{1}{2}}\nabla v_n\|^2 \to 0. \tag{4.19}$$

Next, dividing (4.12) by β_n and taking the inner product with v_n yields

$$i\|v_n\|^2 + \langle \beta_n^{-1}\Delta^2 u_n, v_n\rangle + r\langle \beta_n^{-1}\Delta z_n, v_n\rangle \to 0. \tag{4.20}$$

Note that $\|\beta_n^{-1}\Delta^2 u_n\|$ is bounded due to (4.12). Therefore, $\|\beta_n^{-\frac{1}{2}}\nabla(\Delta u_n)\|$ is also bounded by interpolation

$$\|\beta_n^{-\frac{1}{2}}\nabla(\Delta u_n)\| \le \|\Delta u_n\|^{\frac{1}{2}}\|\beta_n^{-1}\Delta^2 u_n\|^{\frac{1}{2}}.$$

This, together with (4.19), further leads to

$$\langle \beta_n^{-1}\Delta^2 u_n, v_n\rangle = -\langle \beta_n^{-\frac{1}{2}}\nabla(\Delta u_n), \beta_n^{-\frac{1}{2}}\nabla v_n\rangle \to 0, \tag{4.21}$$

and

$$\begin{aligned}\langle \beta_n^{-1}\Delta^2 u_n, v_n\rangle &= \langle \beta_n^{-1}\Delta u_n, \Delta v_n\rangle \\ &= \langle \Delta u_n, i\Delta u_n - \Delta f_{1n}\rangle.\end{aligned} \tag{4.22}$$

Since $\|\Delta f_{1n}\| \to 0$, it follows from (4.21) and (4.22) that

$$\|\Delta u_n\| \to 0. \tag{4.23}$$

Finally, from (4.20) we obtain

$$\|v_n\| \to 0. \tag{4.24}$$

Therefore, by (4.16), (4.17), (4.23) and (4.24), we arrive at the promised contradiction $\|W_n\|_{\mathcal{H}} \to 0$.

□

Remark 4.3. We further discuss the regularity and stability of the system, when the parameters d_{st} and d_{kv} are all positive, that is, both of indirect structural and Kelvin-Voigt damping exist in the system. Similar to the calculation of the spectrum in the proof of Theorem 4.1, we obtain that there is a branch of the spectrum given as follows.

$$\lambda_k = -\frac{\gamma^2}{2d_{kv}} - \frac{\gamma^2 d_{st}}{2d_{kv}^2\mu_k} + O\left(\frac{1}{\mu_k^2}\right) \pm i\left(\mu_k + O\left(\frac{1}{\mu_k}\right)\right).$$

Thus, there exists a vertical asymptote $\mathcal{R}e\lambda = -\frac{\gamma^2}{2d_{kv}}$ of the spectrum of this system, which is the same as the one for the system with only indirect Kelvin-Voigt damping. This implies that the system with $d_{st} > 0,\ d_{kv} > 0$ is still not differentiable.

However, by the similar method as in the proof of Theorem 4.2, we can further get the exponential stability of the system with $d_{st} > 0,\ d_{kv} > 0$.

Remark 4.4. It is plausible to consider the damping in the form of $\Delta^{\alpha} w_t, 0 \le \alpha \le 2$ so that the structural and Kelvin-Voigt damping considered in this paper are two special cases. In fact, a much more general system is under investigation which also considers the order of the coupling terms, damping terms, and the elastic operators of the two equations.

5. Numerical simulation on the spectrum

In this section, we give some numerical simulation on the distribution of the spectrum of the system operator $\mathcal{A}$.

For simplicity, we consider the one dimensional case. Set $\Omega = (0, 1)$ and $\gamma = 3$. Thus, the eigenvalue problem of system (1.3) is as follows

$$\begin{cases} \lambda^2 u = -u_{xxxx} - 3\lambda w_{xx}, \quad x \in (0,1), \\ \lambda^2 w = -w_{xxxx} + d_{st}\lambda w_{xx} - d_{kv}\lambda w_{xxxx} + 3\lambda u_{xx}, \quad x \in (0,1), \\ u(0) = u_x(0) = w(0) = w_x(0) = 0, \\ u(1) = u_x(1) = w(1) = w_x(1) = 0. \end{cases} \tag{5.1}$$

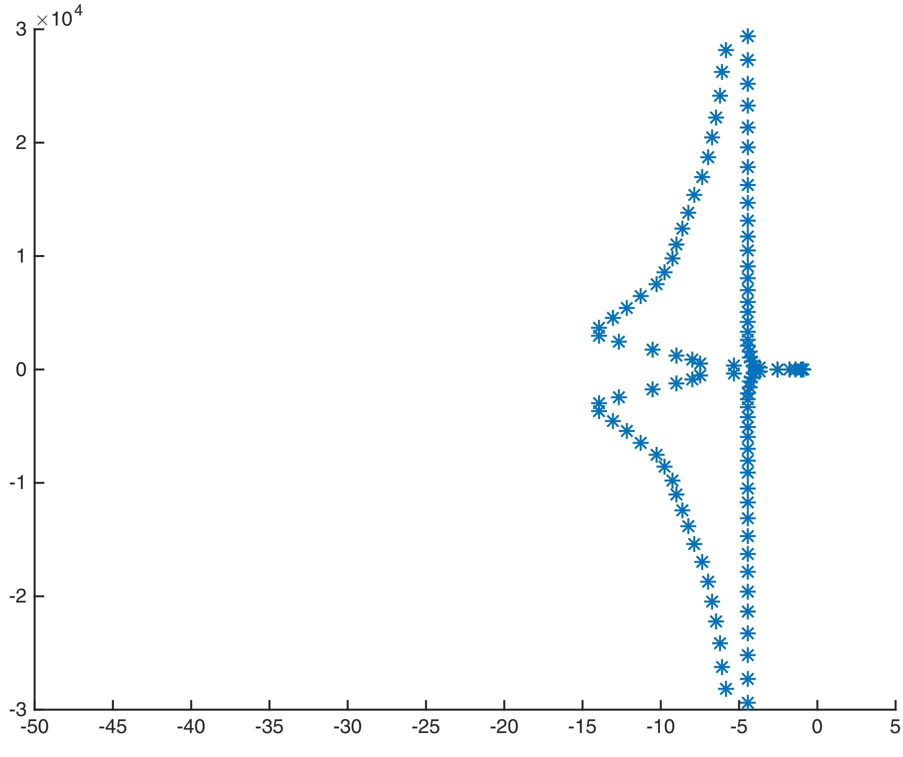

FIGURE 1. Spectrum: $d_{st} = 1, d_{kv} = 0$.

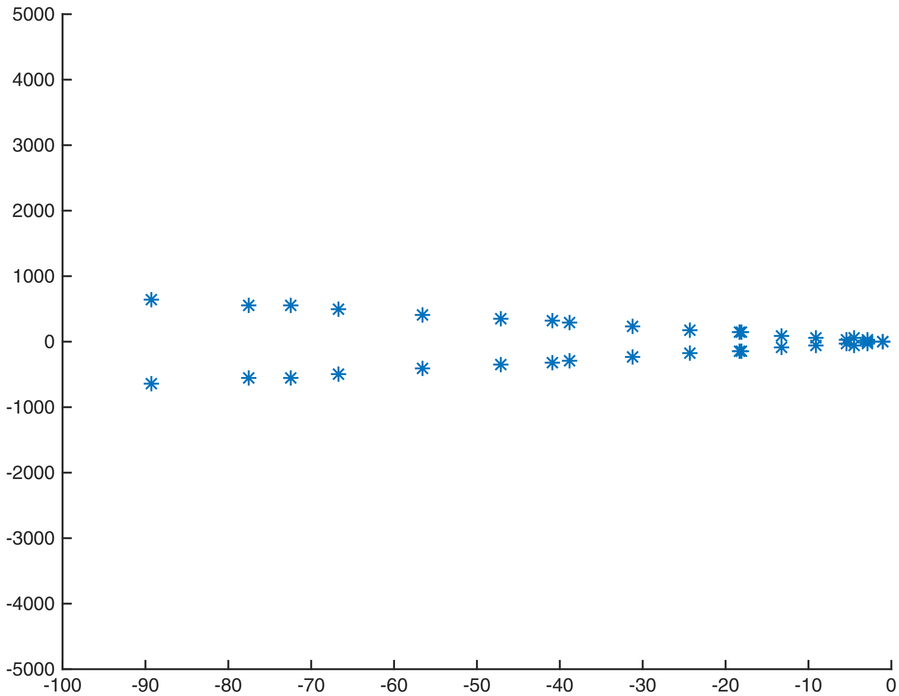

FIGURE 2. Spectrum: $d_{st} = 0, d_{kv} = 1$.

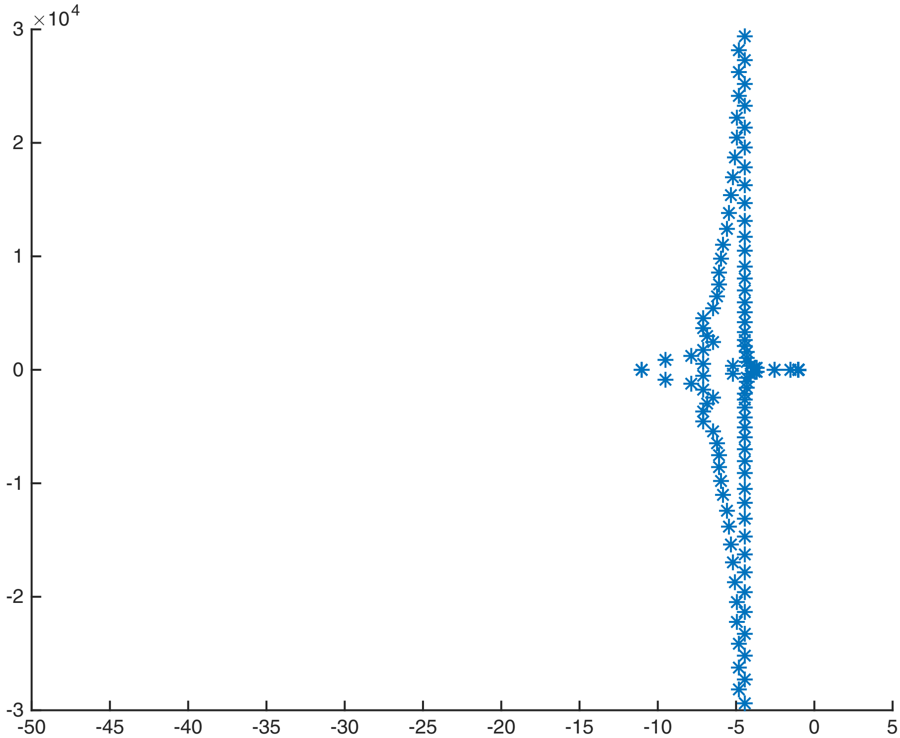

FIGURE 3. Spectrum: $d_{st} = 1, d_{kv} = 1$.

We employed the Chebyshev spectral method in space (spatial grid size: $N = 600$) in a Matlab environment (see [24]) so as to discretize PDEs (5.1) into ODEs, and based on which we calculate its eigenvalues by the following three cases, respectively (Figs. 1–3).

Case 1. $d_{st} = 1, d_{kv} = 0$.
Case 2. $d_{st} = 0, d_{kv} = 1$.
Case 3. $d_{st} = 1, d_{kv} = 1$.

From Figure 1, we see that the distribution of the spectrum for case 1 is consistent with the one for the analyticity (see Lem. 3.1). Moreover, the spectra are all located on the left hand side of the complex plane, which implies the exponential stability of the system for this case.

From Figures 2 and 3, it is easy to see that there is a vertical asymptote of the spectrum for Case 2 and 3, that is, $\mathcal{R}e\lambda = -4.5$. It implies the non-differentiability of the system for these two cases. However, all spectra are also located on the left hand side of the complex plane and the asymptote is away from the imaginary axis. Hence, the exponential stability holds for these two cases, respectively.

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