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PARAMETER ESTIMATION AND OUTPUT FEEDBACK STABILIZATION FOR THE LINEAR KORTEWEG-DE VRIES EQUATION WITH DISTURBED BOUNDARY MEASUREMENT☆

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Abstract. This paper is concerned with the parameter estimation and boundary feedback stabilization for the linear Korteweg-de Vries equation posed on a finite interval with the boundary observation at the right end and the non-collocated control at the left end. The boundary observation suffers from some unknown disturbance. An adaptive observer is designed and the adaptive laws of the parameters are obtained by the Lyapunov method. The resulted closed-loop system is proved to be well-posed and asymptotically stable in case that the length of the interval is not critical. Moreover, it is shown that the estimated parameter converges to the unknown parameter. As a by-product, a hidden regularity result is proved.

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1. Introduction

In this paper, we will consider the parameter estimation and boundary stabilization problem for the linear Korteweg-de Vries equation posed on $(0, \ell)$,

$$\begin{cases} u_t(x,t) + u_{xxx}(x,t) + u_x(x,t) = 0, & x \in (0,\ell),\ t \geq 0, \\ u(0,t) = \psi(t),\ u(\ell,t) = 0,\ u_x(\ell,t) = 0, & t \geq 0, \\ u(x,0) = u_0(x), & x \in (0,\ell), \\ y(t) = u_{xx}(\ell,t) + d, & t \geq 0, \end{cases} \tag{1.1}$$

where the real-valued function $u(x,t)$ is the system state. The function $\psi(t)$ represents the boundary input to be designed while the function $y(t)$ stands for the boundary measurement at the right end $x = \ell$ of the interval. The constant d appearing in the measurement is an unknown disturbance which is to be estimated. The system (1.1) is a non-collocated boundary control system with the boundary output at one end and the control at the other end.

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As it is well known, the tidy form of the Korteweg-de Vries (KdV for short) equation reads

$$u_t(x,t) + u_{xxx}(x,t) + u_x(x,t) + u(x,t)u_x(x,t) = 0. \tag{1.2}$$

It is a typical non-linear dispersive equation, which may serve as a mathematical model for the unidirectional propagation of small-amplitude long waves in non-linear dispersive systems. The KdV equation has been an object of prolific study in the past decades because of its mathematical properties and potential applications. From the viewpoint of mathematical control theory, the KdV equation has many interesting features. For instance, the phenomenon of critical lengths occurs when one considers the issue of controllability and stabilizability for the KdV equation on finite intervals. A set of critical lengths was first introduced by Rosier in [19] where the exact controllability in $L^2(0,\ell)$ of (1.2) with a Neumann boundary control on the right end was studied. The length $\ell > 0$ of the interval is said to be critical if the corresponding linearized equation around the trivial solution $u = 0$ posed on $(0,\ell)$ fails to be controllable. Concretely speaking, the following linear system

$$\begin{cases} u_t(x,t) + u_{xxx}(x,t) + u_x(x,t) = 0, \\ u(0,t) = 0, \ u(\ell,t) = 0, \ u_x(\ell,t) = h(t), \\ u(x,0) = u_0(x) \end{cases} \tag{1.3}$$

is exactly controllable if and only if ℓ does not belong to the set $\mathcal{N}$ [19],

$$\mathcal{N} = \left\{ 2\pi\sqrt{\frac{m^2+mn+n^2}{3}} \;\middle|\; m,n \in \mathbb{N}^* \right\}. \tag{1.4}$$

For $\ell \in \mathcal{N}$, the so-called critical length, the following observability inequality for the system (1.3) with $h(t) = 0$ fails [19],

$$\forall T > 0, \ \exists C_{\ell,T} > 0, \ \text{ s.t. } \ \|u_0\|_{L^2(0,\ell)} \le C_{\ell,T}\|u_x(0,\cdot)\|_{L^2(0,T)}, \ \forall u_0 \in L^2(0,\ell),$$

which leads to the uncontrollability of the system (1.3). It is believed that this is the influence of the first-order derivative operator ∂_x on the spectrum of the third-order derivative operator ∂_x^3 with the domain,

$$\mathfrak{D}\left(\partial_x^3\right) = \left\{ w \in H^3(0,\ell) \mid w(0) = w(\ell) = w'(\ell) = 0 \right\}.$$

Correspondingly, there exist initial data, such that the solutions of the linear system (1.3) with $h(t) = 0$ preserve their $L^2(0,\ell)-$ norms in case of $\ell \in \mathcal{N}$. Meanwhile, it has been proved in [18] that the linear system (1.3) without control (*i.e., $h(t) = 0$*) is exponentially stable in $L^2(0,\ell)$, *i.e.,*

$$\exists C > 0, \ \omega > 0, \quad \|u(\cdot,t)\|_{L^2(0,\ell)} \le Ce^{-\omega t}\|u_0\|_{L^2(0,\ell)}, \quad \forall u_0 \in L^2(0,\ell) \tag{1.5}$$

for $\ell \notin \mathcal{N}$. Of course, there is no more information about the decay rate. In order to stabilize the system with a desired decay rate, Cerpa and Crépeau considered a rapid exponential stabilization problem of the linear system (1.3) in [4], where $h(t)$ was designed as a feedback control forcing the solution of the resulted closed-loop system to decay exponentially with a prescribed rate. Required by the Gramian-based method adopted therein, the system should be controllable. So it was specified that the length ℓ is not a critical one. For the nonlinear case, the KdV equation (1.2) combined with the same boundary conditions as those in (1.3), the rapid exponential stabilization problem was investigated in [5], where the assumption $\ell \notin \mathcal{N}$ was still kept. By contrast, the situation where a Dirichlet boundary control acts on the left endpoint is different. In [3], the rapid exponential stabilization problem for (1.2) with homogeneous Dirichlet and Neumann boundary conditions on the right end was considered, where a left Dirichlet boundary feedback control law was designed by the backstepping method and the assumption $\ell \notin \mathcal{N}$ was not required any longer.

In last decades, there has been an abundance of results in the subject of the boundary stabilization of the KdV equation. See for example [3–5] aforementioned and [10–12, 16, 17, 20, 21, 24] to name but a few. In [3–5, 11, 12, 20, 24], some state feedback controllers were designed by the integral transformation method, the Gramian-based method or the control Lyapunov function method, respectively. Since it is hard to get the state of the system in most cases, it is more realistic to employ the partial measurement of the real state to design a feedback controller. In [10, 16, 17, 21], the output feedback control problems for the linear or nonlinear KdV equation have been investigated, where the boundary measurements at the end-point were used to design the boundary feedback laws. In [17], the output feedback control problem for a linear system similar to (1.1) was considered, where $y(t) = u_{xx}(\ell, t)$ was taken as the output. The output feedback controller presented therein can stabilize the system exponentially with any prescribed decay rate in $H^3(0, \ell)$. Inspired by these interesting works, we are now concerned with the output feedback control problem of the linear KdV equation where the boundary observation suffers from some unknown constant disturbance.

The objective of this paper is twofold. The first one is to construct a scheme to estimate the unknown constant d in the boundary measurement. The other is to present a feedback law stabilizing the whole system. Because there is some uncertainty in the system (1.1), we are going to design an adaptive observer and an observer-based feedback law. Since the closed-loop form of a non-collocated boundary control system is usually non-dissipative, it is difficult to analyze the stability by the traditional Lyapunov method. To achieve our goals, we will adopt the same strategy as those in [8, 9] where the parameter estimation and stabilization problems have been considered, respectively, for a wave equation or a one-dimensional Schrödinger equation with an unknown constant disturbance suffered from the boundary observation at one end and the control input at the other end. The main result of this paper is summarized in Theorem 5.1, which shows that the unknown disturbance d can be estimated and that the whole closed-loop system is asymptotically stable in $L^2(0, \ell)$ under our observer-based output feedback law. It should be noticed that the phenomenon of critical lengths is encountered here. Although we exert the Dirichlet boundary control on the left endpoint, it is still required that ℓ do not belong to the set $\mathcal{N}$. That is not the case in [3, 17] where the left Dirichlet boundary controllers were designed to stabilize the system exponentially with prescribed decay rates. As a by-product, the hidden regularity of the boundary term in the error system (2.11) is revealed when we employ the well-posedness theory of linear infinite-dimensional system to prove the well-posedness of the system (2.11). In fact, the hidden regularity can be proved rigorously as presented in the appendix.

In the rest of this paper, we will use both f_t and $\dot{f}$ to denote the derivative of a given function f with respect to the time variable t, while f_x or f' stands for the derivative with respect to the spatial variable x.

This paper is organized as follows. In Section 2, we present an adaptive observer. Based on the state feedback controller designed in [3], we construct an observer-based output feedback law. The adaptive laws of the parameters in the adaptive observer are obtained by the Lyapunov method. Section 3 is devoted to study the well-posedness and stability of the error system, which is divided into two subsections. The well-posedness and stability of the adaptive observer are analyzed in Section 4. We finish the analysis of the well-posedness and asymptotic stability of the whole closed-loop system in one strike in Section 5. The final section is an appendix which presents a proof of the hidden regularity proposed in Remark 4.2 in Section 4 .

2. Design of the boundary feedback law and the adaptive observer

In this section, we will present the structure of the boundary controller and an adaptive observer. In [3], the backstepping method was applied to design a boundary feedback controller for the system (1.1): consider the following target system,

$$\begin{cases} w_t(x,t) + w_{xxx}(x,t) + w_x(x,t) + \lambda w(x,t) = 0, \\ w(0,t) = 0,\ w(\ell,t) = 0,\ w_x(\ell,t) = 0, \\ w(x,0) = w_0(x), \end{cases} \tag{2.1}$$

where and in what follows λ is a positive constant. This system is exponentially stable with the decay rate λ,

$$\|w(\cdot,t)\|_{L^2(0,\ell)} \leq e^{-\lambda t}\|w_0\|_{L^2(0,\ell)}, \quad \forall t \geq 0. \tag{2.2}$$

Define the transformation $\Pi : L^2(0,\ell) \to L^2(0,\ell)$ as

$$w(x) = \Pi(u(x)) = u(x) - \int_x^\ell k(x,\eta)u(\eta)\mathrm{d}\eta, \tag{2.3}$$

with the kernel $k(\xi,\eta)$ satisfying

$$\begin{cases} k_{\xi\xi\xi}(\xi,\eta) + k_{\eta\eta\eta}(\xi,\eta) + k_\xi(\xi,\eta) + k_\eta(\xi,\eta) = -\lambda k(\xi,\eta), & (\xi,\eta) \in \mathcal{T}, \\ k(\xi,\ell) = 0,\ \ k(\xi,\xi) = 0, & \xi \in [0,\ell], \\ k_\xi(\xi,\xi) = \frac{\lambda}{3}(\ell - \xi), & \xi \in [0,\ell], \end{cases} \tag{2.4}$$

where $\mathcal{T}$ is a triangle domain in $\mathbb{R}^2$ defined as $\mathcal{T} = \{(\xi,\eta) \mid \xi \in [0,\ell],\ \ \eta \in [x,\ell]\}$. If the boundary control $\psi(t)$ in (1.1) takes the form

$$\psi(t) = \int_0^\ell k(0,\eta)u(\eta,t)d\eta, \tag{2.5}$$

Π maps the corresponding solution $u(x,t)$ of (1.1) into the solution $w(x,t)$ of the target system (2.1). Since Π is continuous and invertible, the closed-loop system (1.1) with the boundary feedback law (2.5) is also exponentially stable with the decay rate λ [3].

Now we focus on the design of an adaptive observer for the controlled system (1.1), which takes the following form,

$$\begin{cases} v_t(x,t) + v_{xxx}(x,t) + v_x(x,t) = 0, \\ v(0,t) = \psi(t),\ \ v(\ell,t) = -p(t)\left(y(t) - (v_{xx}(\ell,t) + q(t))\right),\ \ v_x(\ell,t) = 0, \\ \dot{p}(t) = h_1(t), \\ \dot{q}(t) = h_2(t). \end{cases} \tag{2.6}$$

The function $p(t)$ is the adaptive gain while $q(t)$ is an estimation of the unknown constant disturbance d. The functions $h_1(t)$ and $h_2(t)$ depict the adaptive laws of $p(t)$ and $q(t)$, respectively. This is an adaptive observer of Luenberger type which performs the reconstruction of the state $u(x,t)$ from the available output $y(t)$ with unknown disturbance and the input $\psi(t)$ of the original system (1.1).

First of all, we determine the adaptive laws of $p(t)$ and $q(t)$. For this purpose, define the state error $e(x,t)$ and the parameter error $\phi(t)$ as follows,

$$e(x,t) = u(x,t) - v(x,t), \quad \phi(t) = d - q(t).$$

It is direct from (1.1) and (2.6) to get the following error system,

$$\begin{cases} e_t(x,t)+e_{xxx}(x,t)+e_x(x,t)=0,\\ e(0,t)=0,\ \ e(\ell,t)=p(t)\left(e_{xx}(\ell,t)+\phi(t)\right),\ \ e_x(\ell,t)=0,\\ \dot{p}(t)=h_1(t),\\ \dot{\phi}(t)=-h_2(t). \end{cases} \tag{2.7}$$

Now consider the functional,

$$\mathcal{E}_e(t)=\frac{1}{2}\int_0^\ell |e(x,t)|^2\mathrm{d}x+\frac{|p(t)|^2}{2r_1}+\frac{|\phi(t)|^2}{2r_2},$$

where r_1 and r_2 are suitable positive constants. Differentiating $\mathcal{E}_e(t)$ with respect to the time variable t along the trajectory of (2.7) yields

$$\begin{aligned} \frac{d}{\mathrm{d}t}\mathcal{E}_e(t) &= -\frac{1}{2}e^2(\ell,t)-\frac{1}{2}e_x^2(0,t)-p(t)\left(e_{xx}(\ell,t)+\phi(t)\right)e_{xx}(\ell,t)+\frac{1}{r_1}p(t)\dot{p}(t)+\frac{1}{r_2}\phi(t)\dot{\phi}(t)\\ &= -\frac{1}{2}e^2(\ell,t)-\frac{1}{2}e_x^2(0,t)-p(t)\left(e_{xx}(\ell,t)+\phi(t)\right)^2+\frac{1}{r_1}p(t)\dot{p}(t)\\ &\quad +p(t)\phi(t)\left(e_{xx}(\ell,t)+\phi(t)\right)+\frac{1}{r_2}\phi(t)\dot{\phi}(t)\\ &= -\frac{1}{2}e^2(\ell,t)-\frac{1}{2}e_x^2(0,t)+p(t)\left(\frac{1}{r_1}h_1(t)-\left(e_{xx}(\ell,t)+\phi(t)\right)^2\right)\\ &\quad -\phi(t)\left(\frac{1}{r_2}h_2(t)-p(t)\left(e_{xx}(\ell,t)+\phi(t)\right)\right)\\ &= -\frac{1}{2}e^2(\ell,t)-\frac{1}{2}e_x^2(0,t)\le 0 \end{aligned}$$

in case that we take $h_1(t)$ and $h_2(t)$, respectively, as

$$\begin{cases} h_1(t)=r_1\left(e_{xx}(\ell,t)+\phi(t)\right)^2,\\ h_2(t)=r_2p(t)\left(e_{xx}(\ell,t)+\phi(t)\right). \end{cases} \tag{2.8}$$

Thus, $\|e(\cdot,t)\|_{L^2(0,\ell)}$ and $|\phi(t)|$ may be decreasing functions of the time variable t as desired. The adaptive laws $h_1(t)$ and $h_2(t)$ defined by (2.8) indicate that the gain $p(t)$ and the parameter estimation $q(t)$ are adjusted online according to both the state error $e(x,t)$ and the parameter error $\phi(t)$.

We will adopt the observer-based output feedback law for the system (1.1), which means that the observer state $v(x,t)$ takes the place of the real state $u(x,t)$ in the feedback law (2.5), *i.e.*,

$$\psi(t)=\int_0^\ell k(0,\eta)v(\eta,t)\mathrm{d}\eta. \tag{2.9}$$

Then the adaptive observer is actually designed as

$$\begin{cases} v_t(x,t) + v_{xxx}(x,t) + v_x(x,t) = 0, \\ v(0,t) = \int_0^\ell k(0,\eta)v(\eta,t)\mathrm{d}\eta, \quad v_x(\ell,t) = 0, \\ v(\ell,t) = -p(t)\left(y(t) - (v_{xx}(\ell,t) + q(t))\right), \\ \dot{p}(t) = r_1\left(y(t) - (v_{xx}(\ell,t) + q(t))\right)^2, \\ \dot{q}(t) = r_2 p(t)\left(y(t) - (v_{xx}(\ell,t) + q(t))\right), \\ v(x,0) = v_0(x), \quad p(0) = p_0 \geq 0, \quad q(0) = q_0. \end{cases} \tag{2.10}$$

The initial datum p_0 is designated to be positive so that the gain $p(t)$ keeps positive. Subsequently, the corresponding error system reads

$$\begin{cases} e_t(x,t) + e_{xxx}(x,t) + e_x(x,t) = 0, \\ e(0,t) = 0, \quad e(\ell,t) = p(t)\left(e_{xx}(\ell,t) + \phi(t)\right), \quad e_x(\ell,t) = 0, \\ \dot{p}(t) = r_1\left(e_{xx}(\ell,t) + \phi(t)\right)^2, \\ \dot{\phi}(t) = -r_2 p(t)\left(e_{xx}(\ell,t) + \phi(t)\right), \\ e(x,0) = e_0(x), \quad p(0) = p_0 \geq 0, \quad \phi(0) = \phi_0 := d - q_0. \end{cases} \tag{2.11}$$

The following questions arise naturally:

(i) Are the adaptive observer (2.10) and the error system (2.11) globally well-posed?
(ii) Does $e(x,t)$ decay to 0 in any sense as $t \to \infty$?

If the answers are affirmative as expected, we then conclude that $v(x,t)$ approximates the state $u(x,t)$ of the controlled system (1.1).

It is not easy to analyze directly the well-posedness and stability for the adaptive observer (2.10) because it is a non-linear PDE–ODE cascade system with the state $(v(x,t), p(t), q(t))$. Following the ideas presented in [3, 8, 9, 14], we make the following invertible transformation,

$$w(x,t) = \Pi(v(x,t)) = v(x,t) - \int_x^\ell k(x,\eta)v(\eta,t)\mathrm{d}\eta, \tag{2.12}$$

where the kernel $k(\cdot,\cdot)$ is given by (2.4). Then $w(x,t)$ is the solution to the following problem provided that $v(x,t)$ solves (2.10),

$$\begin{cases} w_t(x,t) + w_{xxx}(x,t) + w_x(x,t) + \lambda w(x,t) = -k_{\eta\eta}(x,\ell)p(t)\left(e_{xx}(\ell,t) + \phi(t)\right), \\ w(0,t) = 0, \quad w(\ell,t) = -p(t)\left(e_{xx}(\ell,t) + \phi(t)\right), \quad w_x(\ell,t) = 0, \\ \dot{p}(t) = r_1\left(e_{xx}(\ell,t) + \phi(t)\right)^2, \\ \dot{q}(t) = r_2 p(t)\left(e_{xx}(\ell,t) + \phi(t)\right), \\ w(x,0) = w_0(x), \; p(0) = p_0 \geq 0, \; q(0) = q_0, \end{cases} \tag{2.13}$$

Here $e(x,t)$ is determined by the problem (2.11). If it holds that $e(x,t) \equiv 0$ and $\phi(t) \equiv 0$, the transformed system (2.13) is reduced to the exponentially stable system (2.1). Roughly speaking, it is revealed through the transformation (2.12) that the $v-$subsystem of the adaptive observer (2.10) behaves in the same way as the system (2.1) does. On the other hand, the transformed system (2.13) shares the common (p,q)-subsystem with the error system (2.11). So we will investigate the well-posedness and stability of the following w-subsystem of

(2.13) instead of considering directly the adaptive observer (2.10),

$$\begin{cases} w_t(x,t) + w_{xxx}(x,t) + w_x(x,t) + \lambda w(x,t) = -k_{\eta\eta}(x,\ell)p(t)\left(e_{xx}(\ell,t) + \phi(t)\right), \\ w(0,t) = 0, \quad w(\ell,t) = -p(t)\left(e_{xx}(\ell,t) + \phi(t)\right), \quad w_x(\ell,t) = 0, \\ w(x,0) = w_0(x). \end{cases}$$

This will be postponed until the error system (2.11) is analyzed in the next section.

3. Well-posedness and stability analysis of the error system

3.1. Well-posedness of the error system (2.11)

Now we are concerned with the well-posedness of the error system (2.11). Let $\mathcal{H} = L^2(0,\ell) \times \mathbb{R} \times \mathbb{R}$ be the Hilbert space equipped with the following inner product,

$$\langle (f_1,p_1,\phi_1),(f_2,p_2,\phi_2)\rangle = \int_0^\ell f_1(x)f_2(x)\mathrm{d}x + \frac{p_1p_2}{2r_1} + \frac{\phi_1\phi_2}{r_2}$$

for any $(f_i,p_i,\phi_i) \in \mathcal{H}$, $i = 1,2$, whose norm will be denoted as $\|\cdot\|_{\mathcal{H}}$. We set $\mathcal{H}^+ = L^2(0,\ell) \times \mathbb{R}^+ \times \mathbb{R}$, which is a closed convex subset of $\mathcal{H}$. Define an operator $\mathcal{A} : \mathfrak{D}(\mathcal{A}) \subseteq \mathcal{H}^+ \to \mathcal{H}^+ \subseteq \mathcal{H}$ as follows:

$$\mathcal{A}(f,p,\phi) = \left(-f''' - f', r_1\left(f''(\ell) + \phi\right)^2, -r_2 p\left(f''(\ell) + \phi\right)\right) \tag{3.1}$$

with the domain

$$\mathfrak{D}(\mathcal{A}) = \{(f,p,\phi) \in H^3(0,\ell) \times \mathbb{R}^+ \times \mathbb{R} \mid f(0) = 0, f(\ell) = p(f''(\ell) + \phi), f'(\ell) = 0\}.$$

We can rewrite the error system (2.11) as the following non-linear evolution equation in the Hilbert space $\mathcal{H}$,

$$\begin{cases} \frac{d}{dt}X(t) = \mathcal{A}X(t), \\ X(0) = X_0, \end{cases} \tag{3.2}$$

where $X(\cdot) : \mathbb{R}^+ \to \mathcal{H}$ is the vector $X(t) = (e(\cdot,t), p(t), \phi(t))$ with $X_0 = (e_0, p_0, \phi_0)$. The main result of this subsection is presented as follows.

Theorem 3.1. *For any $X_0 \in \mathcal{H}^+$, the non-linear abstract evolution equation (3.2) admits a unique solution $X(\cdot) \in C(0,\infty;\mathcal{H}^+)$.*

Proof. It is enough to show that $\mathcal{A}$ generates a non-linear semigroup of contractions on $\mathcal{H}^+$. Firstly, the non-linear operator $\mathcal{A}$ is dissipative. In fact, for any $X_i = (f_i,p_i,\phi_i) \in \mathfrak{D}(\mathcal{A})$, $i = 1,2$, we have

$$\begin{aligned} \langle \mathcal{A}X_1 - \mathcal{A}X_2, X_1 - X_2\rangle = & \int_0^\ell \left[(f_2 - f_1)''' + (f_2 - f_1)'\right](f_1 - f_2)\,\mathrm{d}x \\ & + \frac{1}{2}\left[\left(f_1''(\ell) + \phi_1\right)^2 - \left(f_2''(\ell) + \phi_2\right)^2\right](p_1 - p_2) \\ & + \left[p_2\left(f_2''(\ell) + \phi_2\right) - p_1\left(f_1''(\ell) + \phi_1\right)\right](\phi_1 - \phi_2). \end{aligned}$$

Noting that

$$\begin{aligned}
&\int_0^\ell \left[(f_2-f_1)'''+(f_2-f_1)'\right](f_1-f_2)\,\mathrm{d}x\\
&\quad=-\frac{1}{2}\left(f_2'(0)-f_1'(0)\right)^2-\frac{1}{2}\left[p_1\left(f_1''(\ell)+\phi_1\right)-p_2\left(f_2''(\ell)+\phi_2\right)\right]^2\\
&\qquad-\left[(f_2''(\ell)+\phi_2)-(f_1''(\ell)+\phi_1)-(\phi_2-\phi_1)\right]\left[p_2\left(f_2''(\ell)+\phi_2\right)-p_1\left(f_1''(\ell)+\phi_1\right)\right],
\end{aligned}$$

we obtain after some elementary calculations that

$$\begin{aligned}
\langle \mathcal{A}X_1-\mathcal{A}X_2, X_1-X_2\rangle &= -\frac{1}{2}\left(f_2'(0)-f_1'(0)\right)^2-\frac{1}{2}\left[p_1\left(f_1''(\ell)+\phi_1\right)-p_2\left(f_2''(\ell)+\phi_2\right)\right]^2\\
&\quad+\frac{1}{2}\left[\left(f_1''(\ell)+\phi_1\right)^2-\left(f_2''(\ell)+\phi_2\right)^2\right](p_1-p_2)\\
&\quad-\left[(f_1''(\ell)+\phi_1)-(f_2''(\ell)+\phi_2)\right]\left[p_1\left(f_1''(\ell)+\phi_1\right)-p_2\left(f_2''(\ell)+\phi_2\right)\right]\\
&= -\frac{1}{2}\left(f_2'(0)-f_1'(0)\right)^2-\frac{1}{2}\left[p_1\left(f_1''(\ell)+\phi_1\right)-p_2\left(f_2''(\ell)+\phi_2\right)\right]^2\\
&\quad-\frac{1}{2}(p_1+p_2)\left[(f_1''(\ell)+\phi_1)-(f_2''(\ell)+\phi_2)\right]^2\\
&\leq 0.
\end{aligned}$$

Secondly, the operator $\lambda I-\mathcal{A}:\mathfrak{D}(\mathcal{A})\to\mathcal{H}^+$ is surjective for any $\lambda>0$, *i.e.*,

$$R(\lambda I-\mathcal{A})=\mathcal{H}^+. \tag{3.3}$$

Since the equation $(\lambda I-\mathcal{A})(f,p,\phi)=(\hat{f},\hat{p},\hat{\phi})$ is equivalent to the following system of equations for given $(\hat{f},\hat{p},\hat{\phi})\in\mathcal{H}^+$,

$$\begin{cases} f'''+f'+\lambda f=\hat{f},\\ \lambda p-r_1\left(f''(\ell)+\phi\right)^2=\hat{p},\\ \lambda\phi+r_2 p\left(f''(\ell)+\phi\right)=\hat{\phi}, \end{cases} \tag{3.4}$$

it is enough to prove the existence of solutions to (3.4). For the first step, we eliminate the unknown ϕ in (3.4) to obtain the following equations of f and p,

$$\begin{cases} f'''+f'+\lambda f=\hat{f},\\ f(0)=0,\quad f'(\ell)=0,\quad (\lambda+r_2p)f(\ell)-\lambda p f''(\ell)=p\hat{\phi} \end{cases} \tag{3.5}$$

and

$$\lambda r_2^2p^3+\left(2\lambda^2r_2-r_2^2\hat{p}\right)p^2+\left(\lambda^3-2\lambda r_2\hat{p}\right)p-r_1\left(\lambda f''(\ell)+\hat{\phi}\right)^2-\lambda^2\hat{p}=0. \tag{3.6}$$

It is seen that f and p are coupled. By fixing $p\in\mathbb{R}^+$, we investigate the existence of solutions to (3.5). The characteristic equation $\eta^3+\eta+\lambda=0$ admits a pair of conjugate complex roots and a real root by Cardan's method:

$$\eta_1=-\frac{\alpha+\beta}{2}+i\frac{\sqrt{3}(\alpha-\beta)}{2},\quad \eta_2=-\frac{\alpha+\beta}{2}-i\frac{\sqrt{3}(\alpha-\beta)}{2},\quad \eta_3=\alpha+\beta$$

with

$$\alpha = \sqrt[3]{\sqrt{\frac{\lambda^2}{4} + \frac{1}{27}} - \frac{\lambda}{2}}, \qquad \beta = -\sqrt[3]{\sqrt{\frac{\lambda^2}{4} + \frac{1}{27}} + \frac{\lambda}{2}}.$$

Then the real-valued general solution to the equation $f''' + f' + \lambda f = \hat{f}$ reads

$$f(x) = c_1 e^{\sigma_1 x} \cos(\sigma_2 x) + c_2 e^{\sigma_1 x} \sin(\sigma_2 x) + c_3 e^{-2\sigma_1 x} + f_1(x), \tag{3.7}$$

with $\sigma_1 = -\frac{\alpha+\beta}{2}$, $\sigma_2 = \frac{\sqrt{3}(\alpha-\beta)}{2}$, where c_1, c_2, c_3 are arbitrary real coefficients and $f_1(x)$ is the particular solution. Since the functions $e^{\sigma_1 x}\cos(\sigma_2 x)$, $e^{\sigma_1 x}\sin(\sigma_2 x)$ and $e^{-2\sigma_1 x}$ are linearly independent, the particular solution $f_1(x)$ can be obtained by the method of variation of parameters. The coefficients c_1, c_2 and c_3 are determined by solving the linear system

$$\underbrace{\begin{pmatrix} 1 & 0 & 1 \\ m_{21}(\lambda) & m_{22}(\lambda) & m_{23}(\lambda) \\ m_{31}(\lambda) & m_{32}(\lambda) & m_{33}(\lambda) \end{pmatrix}}_{\mathbf{M}} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -f_1(0) \\ -f_1'(\ell) \\ \lambda p f_1''(\ell) + p\hat{\phi} - (\lambda + r_2 p) f_1(\ell) \end{pmatrix}$$

with

$$\begin{aligned}
m_{21}(\lambda) &= e^{\sigma_1 \ell}\left(\sigma_1 \cos(\sigma_2 \ell) - \sigma_2 \sin(\sigma_2 \ell)\right), \\
m_{22}(\lambda) &= e^{\sigma_1 \ell}\left(\sigma_1 \sin(\sigma_2 \ell) + \sigma_2 \cos(\sigma_2 \ell)\right), \\
m_{23}(\lambda) &= -2\sigma_1 e^{-2\sigma_1 \ell}, \\
m_{31}(\lambda) &= e^{\sigma_1 \ell}\left[\left(\lambda + r_2 p - \lambda p \sigma_1^2 + \lambda p \sigma_2^2\right)\cos(\sigma_2 \ell) + 2\lambda p \sigma_1 \sigma_2 \sin(\sigma_2 \ell)\right], \\
m_{32}(\lambda) &= e^{\sigma_1 \ell}\left[\left(\lambda + r_2 p - \lambda p \sigma_1^2 + \lambda p \sigma_2^2\right)\sin(\sigma_2 \ell) - 2\lambda p \sigma_1 \sigma_2 \cos(\sigma_2 \ell)\right], \\
m_{33}(\lambda) &= e^{-2\sigma_1 \ell}\left(\lambda + r_2 p - 4\lambda p \sigma_1^2\right).
\end{aligned}$$

The determinant of $\mathbf{M}$ is obviously a continuous function of λ,

$$\begin{aligned}
\det(\mathbf{M})(\lambda) = {} & e^{-\sigma_1 \ell}\left[\left(3\lambda\sigma_1 + 3r_2\sigma_1 p - 6\lambda p \sigma_1^3 + 2\lambda p \sigma_1 \sigma_2^2\right)\sin(\sigma_2 \ell) - 4\lambda p \sigma_1^3 \sigma_2\right] \\
& + e^{-\sigma_1 \ell}\left(\lambda \sigma_2 + r_2 \sigma_2 p - 4\lambda p \sigma_1^2 \sigma_2\right)\cos(\sigma_2 \ell) \\
& - e^{2\sigma_1 \ell}\left(\lambda p \sigma_1^2 \sigma_2 + \lambda \sigma_2 + r_2 p \sigma_2 + \lambda p \sigma_2^3\right),
\end{aligned}$$

which is non-zero for $\lambda > 0$. Thus, we conclude that there is a unique function $f \in H^3(0,\ell)$ solving (3.5) for fixed $p \in \mathbb{R}^+$, which is denoted as f_p to emphasis its dependence on p. By trace theorem, we regard $f_p''(\ell) : \mathbb{R}^+ \to \mathbb{R}$ as a continuous function of p. Now substitute $f_p''(\ell)$ into the equation (3.6) and prove the existence of the solution $p \in \mathbb{R}^+$ to the corresponding (3.6), *i.e.*, the equation $\Upsilon(p) = 0$, with the continuous function $\Upsilon(p) : \mathbb{R}^+ \to \mathbb{R}$ defined as

$$\Upsilon(p) := \lambda r_2^2 p^3 + \left(2\lambda^2 r_2 - r_2^2 \hat{p}\right) p^2 + \left(\lambda^3 - 2\lambda r_2 \hat{p}\right) p - r_1 \left(\lambda f_p''(\ell) + \hat{\phi}\right)^2 - \lambda^2 \hat{p}.$$

Obviously, it holds that $\lim_{p \to 0^+} \Upsilon(p) \le -\lambda^2 \hat{p} < 0$ for $\lambda > 0$ and $\hat{p} > 0$. Moreover, it will be shown that

$$\lim_{p \to +\infty} \Upsilon(p) = +\infty. \tag{3.8}$$

The key point is to estimate $-r_1\left(\lambda f_p''(\ell)+\hat{\phi}\right)^2$ in terms of p. For the sake of simplicity, we set $\Lambda(p)=\lambda f_p''(\ell)+\hat{\phi}$, or equivalently,

$$f_p''(\ell)=\frac{1}{\lambda}\left(\Lambda(p)-\hat{\phi}\right). \tag{3.9}$$

Multiplying the first equation of (3.5) by f_p, integrating over $(0,\ell)$ and applying Hölder's inequality, we have

$$f_p(\ell)f_p''(\ell)\le\frac{1}{2\lambda}\|\hat{f}\|^2_{L^2(0,\ell)}, \tag{3.10}$$

where only the first two boundary conditions in (3.5) are involved. Taking the third boundary condition in (3.5) into account, we get

$$f_p(\ell)=\frac{p\Lambda(p)}{\lambda+r_2p},$$

which combines with (3.9) and (3.10) to yield

$$2p\left|\Lambda(p)\right|^2-2p\hat{\phi}\Lambda(p)-(\lambda+r_2p)\,\|\hat{f}\|^2_{L^2(0,\ell)}\le 0.$$

Then we have

$$\frac12\hat{\phi}-\frac12\sqrt{\hat{\phi}^2+2r_2M+\frac{2\lambda M}{p}}\le\Lambda(p)\le\frac12\hat{\phi}+\frac12\sqrt{\hat{\phi}^2+2r_2M+\frac{2\lambda M}{p}}$$

with the constant $M:=\|\hat{f}\|^2_{L^2(0,\ell)}$, which further implies

$$\frac12\hat{\phi}-\frac12\sqrt{\hat{\phi}^2+2r_2M}\le\lim_{p\to+\infty}\Lambda(p)\le\frac12\hat{\phi}+\frac12\sqrt{\hat{\phi}^2+2r_2M}. \tag{3.11}$$

By (3.11), it is easy to check that the term $-r_1\left(\lambda f_p''(\ell)+\hat{\phi}\right)^2$ in $\Upsilon(p)$ is bounded as $p\to+\infty$, and that (3.8) follows immediately. By the intermediate value theorem, the equation $\Upsilon(p)=0$ admits solutions in $\mathbb{R}^+$. Thus the claimed (3.3) is proved.

Finally, $\mathfrak{D}(\mathcal{A})$ is dense in the interior of $\mathcal{H}^+$. In fact, if $Z\in\mathcal{H}^+$ satisfies the equation

$$\langle X,Z\rangle=0,\quad\forall X\in\mathfrak{D}(\mathcal{A}), \tag{3.12}$$

there exists a $\tilde{X}\in\mathfrak{D}(\mathcal{A})$ solving the equation $(\lambda I-\mathcal{A})\,\tilde{X}=Z$ for arbitrary but fixed $\lambda>0$ due to (3.3). Noting that $\mathcal{A}(0)=0$ and that $\mathcal{A}$ is dissipative, we deduce from (3.12) that

$$\lambda\|\tilde{X}\|^2_{\mathcal{H}}=\langle\mathcal{A}\tilde{X},\tilde{X}\rangle\le 0,$$

which indicates $\tilde{X}=0$ and consequently $Z=0$.

By the non-linear semigroup theory [6, 15], we conclude that $\mathcal{A}$ generates a non-linear semigroup of contractions on $\mathcal{H}^+$. From now on, $\{S(t)\}_{t\ge0}$ will denote this non-linear semigroup of contractions associated with $\mathcal{A}$ on $\mathcal{H}^+$. Then for any $X_0\in\mathcal{H}^+$, (3.2) admits a unique solution $X(t)=S(t)X_0\in C(0,\infty;\mathcal{H}^+)$. This completes the proof. $\square$

3.2. Asymptotic stability of the error system (2.11)

The goal of this subsection is to investigate the asymptotic behavior of the error system as $t \to \infty$ *via* Lasalle's invariant principle [15].

Since the operator $\mathcal{A}$ is dissipative as shown in the proof of Theorem 3.1, $\lambda I - \mathcal{A}$ is injective for arbitrary but fixed $\lambda > 0$. Actually, if $(\lambda I - \mathcal{A}) X_1 = (\lambda I - \mathcal{A}) X_2$ for $X_1, X_2 \in \mathfrak{D}(\mathcal{A})$, or equivalently, $\lambda(X_1 - X_2) = \mathcal{A}X_1 - \mathcal{A}X_2$, we deduce that

$$\|X_1 - X_2\|_{\mathcal{H}}^2 = \frac{1}{\lambda}\langle \mathcal{A}X_1 - \mathcal{A}X_2, X_1 - X_2\rangle \leq 0.$$

It follows immediately that $X_1 = X_2$. Thus, $\lambda I - \mathcal{A}$ is bijective in view of (3.3). Let $(\lambda I - \mathcal{A})^{-1}$ be the inverse of $\lambda I - \mathcal{A}$ for $\lambda > 0$.

Lemma 3.2. *For any given $\lambda > 0$, the operator $(\lambda I - \mathcal{A})^{-1}$ is compact.*

Proof. Let $\left\{Y_n = \left(\hat{f}_n, \hat{p}_n, \hat{\phi}_n\right)\right\}_{n=1}^{\infty} \subseteq \mathcal{H}^+$ be a bounded sequence, *i.e.*,

$$\|Y_n\|_{\mathcal{H}} \leq C. \tag{3.13}$$

Here and in the sequel C denotes a generic positive constant which may vary from line to line. Set $X_n = (f_n, p_n, \phi_n) = (\lambda I - \mathcal{A})^{-1} Y_n$, $\forall n \in \mathbb{N}$. Noting that $\mathcal{A}(0) = 0$ and that $\mathcal{A}$ is dissipative, we have

$$\|X_n\|_{\mathcal{H}}^2 = \frac{1}{\lambda}\langle \mathcal{A}X_n, X_n\rangle + \frac{1}{\lambda}\langle Y_n, X_n\rangle \leq \frac{C}{\lambda}\|X_n\|_{\mathcal{H}},$$

which further implies that

$$\int_0^\ell |f_n|^2 \,\mathrm{d}x + \frac{1}{2r_1}|p_n|^2 + \frac{1}{r_2}|\phi_n|^2 \leq C, \quad \forall n \in \mathbb{N}. \tag{3.14}$$

On the other hand, it holds that

$$\begin{cases} f_n''' + f_n' + \lambda f_n = \hat{f}_n, \\ \lambda p_n - r_1 \left(f_n''(\ell) + \phi_n\right)^2 = \hat{p}_n, \\ \lambda \phi_n + r_2 p_n \left(f_n''(\ell) + \phi_n\right) = \hat{\phi}_n. \end{cases} \tag{3.15}$$

Multiplying the first equation of (3.15) by f_n''', f_n'' and integrating over $(0, \ell)$, respectively, we then obtain

$$\int_0^\ell |f_n'''|^2 \,\mathrm{d}x + \lambda p_n |f_n''(\ell)|^2 = \int_0^\ell \hat{f}_n f_n''' \mathrm{d}x + f_n''(0) f_n'(0) + \int_0^\ell |f_n''|^2 \,\mathrm{d}x - \lambda p_n \phi_n f_n''(\ell) - \frac{\lambda}{2}|f_n'(0)|^2, \tag{3.16}$$

and

$$\frac{1}{2}\left(|f_n''(0)|^2 + |f_n'(0)|^2\right) = \frac{1}{2}|f_n''(\ell)|^2 - \lambda \int_0^\ell |f_n'|^2 \,\mathrm{d}x - \int_0^\ell \hat{f}_n f_n'' \mathrm{d}x. \tag{3.17}$$

Combining (3.16) with (3.17) and applying Ehrling–Nirenberg–Gagliardo interpolation inequality, we get

$$\int_0^\ell |f_n'''|^2 \,\mathrm{d}x \leq C\left(\int_0^\ell \left|\hat{f}_n\right|^2 \mathrm{d}x + \int_0^\ell |f_n|^2 \,\mathrm{d}x + |f_n''(\ell)|^2 + \lambda^2 p_n^2 \phi_n^2\right).$$

Taking advantage of (3.13), (3.14) and the second equation of (3.15), we then conclude that

$$\int_0^\ell |f_n'''|^2 \,\mathrm{d}x \leq C. \tag{3.18}$$

(3.18) together with (3.14) implies that $\{f_n\}_{n=1}^\infty$ is a bounded sequence in $H^3(0,\ell)$. By Sobolev embedding theorems and (3.14), we deduce that there is a subsequence $\{X_{n_k}\}_{k=1}^\infty \subseteq \{X_n\}_{n=1}^\infty$ and some $X_0 \in \mathcal{H}$ satisfying $\lim\limits_{k\to\infty} \|X_{n_k} - X_0\|_{\mathcal{H}} = 0$. This completes the proof. ☐

Remark 3.3. For $X_0 \in \mathcal{H}^+$, the orbit of (3.2) through X_0 is defined as

$$\gamma(X_0) = \bigcup_{t\in\mathbb{R}^+} S(t)X_0.$$

By Lemma 3.2, we conclude that $\gamma(X_0)$ is precompact for any $X_0 \in \mathcal{H}^+$ [7, 15].

The following is the main result of this subsection.

Theorem 3.4. *Under the assumption of $\ell \notin \mathcal{N}$ with $\mathcal{N}$ defined by (1.4), the error system (2.11) is asymptotically stable, i.e.,*

$$\lim_{t\to\infty} \|e(\cdot,t)\|_{L^2(0,\ell)} = 0 \tag{3.19}$$

for any given $(e_0, p_0, \phi_0) \in \mathcal{H}^+$. Moreover, it holds that

$$p(\cdot) \in L^\infty(0,\infty), \quad \lim_{t\to\infty} q(t) = d. \tag{3.20}$$

Proof. We will apply Lasalle's invariant principle [15] to prove this theorem. First of all, we set

$$\mathcal{S} := \{X_0 = (e_0, p_0, \phi_0) \in \mathcal{H}^+ \mid \dot{V}_e(S(t)X_0) = 0\},$$

where $V_e\,(S(t)X_0)$ is a functional defined on $\mathcal{H}^+$,

$$V_e(S(t)X_0) = \frac{1}{2}\int_0^\ell |e(x,t)|^2 \mathrm{d}x + \frac{1}{2r_1}\,|p(t) - \zeta|^2 + \frac{1}{2r_2}|\phi(t)|^2$$

with $S(t)X_0 = (e(\cdot,t), p(t), \phi(t))$ and a given constant $\zeta > 0$. Since $\mathfrak{D}(\mathcal{A})$ is dense in $\mathcal{H}^+$, we take $X_0 \in \mathfrak{D}(\mathcal{A})$ hereinafter without loss of generality.

For any given $X_0 = (e_0, p_0, \phi_0) \in \mathfrak{D}(\mathcal{A})$, the error system (2.11) admits a strong solution $X(t) = S(t)X_0 = (e(\cdot,t), p(t), \phi(t)) \in C^1(0,\infty;\mathcal{H}^+) \cap C(0,\infty;\mathfrak{D}(\mathcal{A}))$. Differentiating $V_e(S(t)X_0)$ with respect to t along the trajectory of (2.11) yields

$$\dot{V}_e(S(t)X_0) = -\frac{1}{2}\,|e_x(0,t)|^2 - \frac{1}{2}\,|e(\ell,t)|^2 - \zeta\,|e_{xx}(\ell,t) + \phi(t)|^2 \leq 0. \tag{3.21}$$

By Lasalle's invariant principle and Remark 3.3, we know that all solutions of (2.11) go to the maximal invariant subset of $\mathcal{S}$.

If $\dot{V}_e(S(t)X_0) = 0$, it follows from (3.21) that

$$e_x(0,t) = 0, \quad e(\ell,t) = 0, \quad e_{xx}(\ell,t) + \phi(t) = 0. \tag{3.22}$$

By (2.11), the solution $X(t) = S(t)X_0$ satisfies

$$\begin{cases} e_t(x,t) + e_{xxx}(x,t) + e_x(x,t) = 0, \\ e(0,t) = e_x(0,t) = 0, \;\; e(\ell,t) = e_x(\ell,t) = 0, \\ e(x,0) = e_0(x), \end{cases} \tag{3.23}$$

and

$$\begin{cases} \dot{p}(t) = 0, \quad \dot{\phi}(t) = 0, \\ p(0) = p_0 \geq 0, \;\; \phi(0) = \phi_0. \end{cases} \tag{3.24}$$

Referring to Lemma 3.4 of [19], we get from (3.23) that $e_0 = 0$ in $L^2(0,\ell)$ for any $\ell \notin \mathcal{N}$. Combining (3.22), (3.23) and (3.24), we further deduce that $\phi_0 = 0$. Thus, the set $\mathcal{S}$ consists of such points $(0, p_0, 0)$ with $p_0 \geq 0$. This completes the proof. □

Remark 3.5. It seems a little strange that the assumption $\ell \notin \mathcal{N}$ is imposed on the length of the interval. As pointed out in [19], there are non-trivial steady solutions to the problem (3.23) in case of $\ell \in \mathcal{N}$. For example, the time-independent function $e(x,t) = \cos x - 1$ solves (3.23) with $\ell = 2\pi \in \mathcal{N}$.

4. Well-posedness and stability analysis of the adaptive observer

We now go back to analyze the well-posedness and the asymptotic stability of the adaptive observer (2.10). As announced at the end of Section 2, we will focus on the following system instead,

$$\begin{cases} w_t(x,t) + w_{xxx}(x,t) + w_x(x,t) + \lambda w(x,t) = -k_{\eta\eta}(x,\ell)p(t)\left(e_{xx}(\ell,t) + \phi(t)\right), \\ w(0,t) = 0, \;\; w(\ell,t) = -p(t)\left(e_{xx}(\ell,t) + \phi(t)\right), \;\; w_x(\ell,t) = 0, \\ w(x,0) = w_0(x), \end{cases} \tag{4.1}$$

i.e., the $w-$subsystem of the transformed system (2.13), since the (p,q)-subsystem has been analyzed in the error system (2.11). By Theorem 3.1, the functions $e(x,t)$, $p(t)$ and $\phi(t)$ are well-defined functions for any given $(e_0, p_0, q_0) \in \mathcal{H}^+$.

We will apply the well-posedness theory of linear infinite-dimensional systems [22, 23] to prove that the system (4.1) is well-posed in $L^2(0,\ell)$ with the inner product denoted as $\langle \cdot, \cdot \rangle_2$. Define the operator $\mathbf{A} : \mathfrak{D}(\mathbf{A}) \subseteq L^2(0,\ell) \to L^2(0,\ell)$ as

$$\mathbf{A}f = -f''' - f' - \lambda f, \qquad \mathfrak{D}(\mathbf{A}) = \{f \in H^3(0,\ell) \cap H_0^1(0,\ell) \mid f'(\ell) = 0\}.$$

It is easy to get the adjoint $\mathbf{A}^*$ of $\mathbf{A}$,

$$\mathbf{A}^* g = g''' + g' - \lambda g, \qquad \mathfrak{D}(\mathbf{A}^*) = \{g \in H^3(0,\ell) \cap H_0^1(0,\ell) \mid g'(0) = 0\}.$$

Since $\mathbf{A}$ and $\mathbf{A}^*$ are both dissipative operators, $\mathbf{A}$ generates a C_0-semigroup of contractions $\left\{e^{\mathbf{A}t}\right\}_{t\geq 0}$ which is also exponentially stable as shown by (2.2) [3]. Let $[\mathfrak{D}(\mathbf{A}^*)]'$ be the dual space of $\mathfrak{D}(\mathbf{A}^*)$ with respect to the pivot sapce $L^2(0,\ell)$ and let $\tilde{\mathbf{A}} : L^2(0,\ell) \to [\mathfrak{D}(\mathbf{A}^*)]'$ be the continuous extension of $\mathbf{A}$ defined by

$$\langle \tilde{\mathbf{A}} f, g \rangle_{[\mathfrak{D}(\mathbf{A}^*)]', \mathfrak{D}(\mathbf{A}^*)} = \langle f, \mathbf{A}^* g \rangle_2, \quad \forall f \in L^2(0,\ell), \; \forall g \in \mathfrak{D}(\mathbf{A}^*).$$

It is apparent that $\tilde{\mathbf{A}}f = \mathbf{A}f$ for any $f \in \mathfrak{D}(\mathbf{A})$. Correspondingly, $\left\{e^{\mathbf{A}t}\right\}_{t\geq 0}$ can also be extended from $L^2(0,\ell)$ onto $[\mathfrak{D}(\mathbf{A}^*)]'$. With a slight abuse of notations, we still denote the extensions as $\mathbf{A}$ and $\left\{e^{\mathbf{A}t}\right\}_{t\geq 0}$, respectively in the sequel. Let $U = \mathbb{R}$ be the control space and let $\mathbf{B} : U \to [\mathfrak{D}(\mathbf{A}^*)]'$ be the control operator defined by the identity

$$\langle \mathbf{B}r, g\rangle_{[\mathfrak{D}(\mathbf{A}^*)]',\mathfrak{D}(\mathbf{A}^*)} = r\left(\int_0^\ell k_{\eta\eta}(x,\ell)g(x)\mathrm{d}x - g''(\ell)\right), \quad \forall g \in \mathfrak{D}(\mathbf{A}^*).$$

Obviously, $\mathbf{B} \in \mathcal{L}\left(U, [\mathfrak{D}(\mathbf{A}^*)]'\right)$. Moreover, it is easy to check that

$$\mathbf{B}^* g = \int_0^\ell k_{\eta\eta}(x,\ell)g(x)\mathrm{d}x - g''(\ell), \quad \forall g \in \mathfrak{D}(\mathbf{A}^*).$$

The function $z(t) = -p(t)(e_{xx}(\ell,t) + \phi(t))$ is viewed as an input. By (3.21), one can deduce that

$$\zeta \int_0^t |e_{xx}(\ell,t) + \phi(t)|^2 \mathrm{d}t \leq V_e(X_0), \quad \forall t \geq 0, \tag{4.2}$$

which together with (3.20) implies that $z \in L^2(0,\infty)$. Then the system (4.1) can be translated into an abstract form in $[\mathfrak{D}(\mathbf{A}^*)]'$,

$$\begin{cases} \dfrac{d}{\mathrm{d}t} w(t) = \mathbf{A}w(t) + \mathbf{B}z(t), \quad t > 0, \\ w(0) = w_0. \end{cases} \tag{4.3}$$

Theorem 4.1. *For any initial data $w_0 \in L^2(0,\ell)$ and $(e_0, p_0, q_0) \in \mathcal{H}^+$, there exists a unique solution $w \in C(0,\infty; L^2(0,\ell))$ to the system (4.3) (or equivalently the system (4.1)), which can be written as*

$$w(t) = e^{\mathbf{A}t} w_0 + \int_0^t e^{\mathbf{A}(t-s)} \mathbf{B} z(s) \mathrm{d}s. \tag{4.4}$$

Proof. It is enough to show that the control operator $\mathbf{B}$ is admissible, which is equivalent to prove, that for some $T > 0$,

$$\int_0^T \left|\mathbf{B}^* e^{\mathbf{A}^* t} \hat{w}_0\right|^2 \mathrm{d}t \leq C\|\hat{w}_0\|^2_{\mathfrak{D}(\mathbf{A}^*)} \tag{4.5}$$

holds true for any $\hat{w}_0 \in \mathfrak{D}(\mathbf{A}^*)$ with a positive constant C independent of $\hat{w}_0$ [22, 23]. Here and in what follows, $\|\cdot\|_{\mathfrak{D}(\mathbf{A}^*)}$ denotes the graph norm on $\mathfrak{D}(\mathbf{A}^*)$. For arbitrary but fixed $\hat{w}_0 \in \mathfrak{D}(\mathbf{A}^*)$, consider the observation problem for the dual system of (4.3),

$$\begin{cases} \frac{d}{\mathrm{d}t} \hat{w}(t) = \mathbf{A}^* \hat{w}(t), \quad \hat{w}(0) = \hat{w}_0, \\ \hat{y}(t) = \mathbf{B}^* \hat{w}(t), \end{cases}$$

which can be written as

$$\begin{cases} \hat{w}_t(x,t) - \hat{w}_{xxx}(x,t) - \hat{w}_x(x,t) + \lambda \hat{w}(x,t) = 0, \\ \hat{w}(0,t) = \hat{w}_x(0,t) = \hat{w}(\ell,t) = 0, \\ \hat{w}(x,0) = \hat{w}_0(x), \\ \hat{y}(t) = \int_0^\ell k_{\eta\eta}(x,\ell)\hat{w}(x,t)\mathrm{d}x - \hat{w}_{xx}(\ell,t). \end{cases} \tag{4.6}$$

Then the inequality (4.5) is equivalent to the following one

$$\int_0^T |\hat{y}(t)|^2 \mathrm{d}t \leq C\|\hat{w}_0\|^2_{\mathfrak{D}(\mathbf{A}^*)}. \tag{4.7}$$

To prove (4.7), we multiply the first equation of (4.6) by $x\hat{w}_{xx}(x,t)$ and integrate by parts with respect to the spatial variable x over $(0,\ell)$ to obtain

$$\begin{aligned} &\frac{\ell}{2}\left(|\hat{w}_{xx}(\ell,t)|^2 + |\hat{w}_x(\ell,t)|^2\right) + \lambda \int_0^\ell x\,|\hat{w}(x,t)|^2\,\mathrm{d}x \\ &\quad = \frac{1}{2}\int_0^\ell \left(|\hat{w}_{xx}(x,t)|^2 + |\hat{w}_x(x,t)|^2\right)\mathrm{d}x + \int_0^\ell x\hat{w}_{xx}(x,t)\hat{w}_t(x,t)\mathrm{d}x, \end{aligned}$$

which, together with Ehrling–Nirenberg–Gagliardo interpolation inequality, indicates

$$\begin{aligned} |\hat{w}_{xx}(\ell,t)|^2 &\leq \left(1+\frac{1}{\ell}\right)\int_0^\ell \hat{w}_{xx}(x,t)^2\mathrm{d}x + \frac{1}{\ell}\int_0^\ell \hat{w}_x(x,t)^2\mathrm{d}x + \int_0^\ell |\hat{w}_t(x,t)|^2\mathrm{d}x \\ &\leq C\int_0^\ell \left(|\hat{w}(x,t)|^2 + |\hat{w}_{xxx}(x,t)|^2\right)\mathrm{d}x + \int_0^\ell |\hat{w}_t(x,t)|^2\mathrm{d}x \end{aligned} \tag{4.8}$$

with a positive constant C depending on ℓ. Set the constat $\kappa = \int_0^\ell |k_{\eta\eta}(x,\ell)|^2\,\mathrm{d}x$. It follows from (4.8) that

$$\begin{aligned} |\hat{y}(t)|^2 &\leq 2\kappa\int_0^\ell |\hat{w}(x,t)|^2\mathrm{d}x + 2|\hat{w}_{xx}(\ell,t)|^2 \\ &\leq 2(\kappa+C)\int_0^\ell |\hat{w}(x,t)|^2\mathrm{d}x + 2C\int_0^\ell |\hat{w}_{xxx}(x,t)|^2\mathrm{d}x + 2\int_0^\ell |\hat{w}_t(x,t)|^2\mathrm{d}x. \end{aligned} \tag{4.9}$$

Multiplying the first equation of (4.6) by $\hat{w}_{xxx}(x,t)$ and applying the Cauchy–Schwarz inequality, one then get

$$\int_0^\ell |\hat{w}_{xxx}(x,t)|^2\mathrm{d}x \leq 3\left(\int_0^\ell |\hat{w}_t(x,t)|^2\mathrm{d}x + \lambda^2\int_0^\ell |\hat{w}(x,t)|^2\mathrm{d}x\right) + 3\int_0^\ell |\hat{w}_x(x,t)|^2\mathrm{d}x. \tag{4.10}$$

Multiplying the first equation of (4.6) by $(\ell-x)\hat{w}(x,t)$ and integrating by parts $(0,\ell)$, we have

$$\frac{d}{\mathrm{d}t}\int_0^\ell (\ell-x)|\hat{w}(x,t)|^2\mathrm{d}x + 3\int_0^\ell |\hat{w}_x(x,t)|^2\mathrm{d}x - \int_0^\ell |\hat{w}(x,t)|^2\mathrm{d}x + 2\lambda\int_0^\ell (\ell-x)|\hat{w}(x,t)|^2\mathrm{d}x = 0.$$

Integrate with respect to the time variable t over $(0,T)$ to obtain

$$\int_0^\ell (\ell-x)|\hat{w}(x,T)|^2\mathrm{d}x + \int_0^T\int_0^\ell |\hat{w}_x(x,t)|^2\mathrm{d}x\mathrm{d}t + 2\lambda\int_0^T\int_0^\ell (\ell-x)|\hat{w}(x,t)|^2\mathrm{d}x\mathrm{d}t$$
$$= \int_0^\ell (\ell-x)|\hat{w}(x,0)|^2\mathrm{d}x + \int_0^T\int_0^\ell |\hat{w}(x,t)|^2\mathrm{d}x\mathrm{d}t,$$

which implies immediately that

$$\int_0^T\int_0^\ell |\hat{w}_x(x,t)|^2\mathrm{d}x\mathrm{d}t \le \ell\int_0^\ell |\hat{w}(x,0)|^2\mathrm{d}x + \int_0^T\int_0^\ell |\hat{w}(x,t)|^2\mathrm{d}x\mathrm{d}t, \tag{4.11}$$

Combining (4.9)–(4.11), one can deduce that

$$\int_0^T |\hat{y}(t)|^2\mathrm{d}t \le C\int_0^\ell |\hat{w}(x,0)|^2\mathrm{d}x + C\int_0^T\int_0^\ell |\hat{w}(x,t)|^2\mathrm{d}x\mathrm{d}t + C\int_0^T\int_0^\ell |\hat{w}_t(x,t)|^2\mathrm{d}x\mathrm{d}t \tag{4.12}$$

holds true for some positive constant C depending on κ and ℓ. Let $E_{\hat{w}}(t) = \frac{1}{2}\int_0^\ell |\hat{w}(x,t)|^2\mathrm{d}x$. Differentiating $E_{\hat{w}}(t)$ with respect to the time variable t yields

$$\frac{d}{\mathrm{d}t}E_{\hat{w}}(t) = -\frac{1}{2}\left|\hat{w}_x(\ell,t)\right|^2 - 2\lambda E_{\hat{w}}(t) \le -2\lambda E_{\hat{w}}(t),$$

which implies that

$$E_{\hat{w}}(t) \le e^{-2\lambda t}E_{\hat{w}}(0), \quad \forall t \ge 0. \tag{4.13}$$

Set $\hat{v}(x,t) = \hat{w}_t(x,t)$. It is easy to check that $\hat{v}(x,t)$ satisfies

$$\begin{cases} \hat{v}_t(x,t) - \hat{v}_{xxx}(x,t) - \hat{v}_x(x,t) + \lambda\hat{v}(x,t) = 0 \\ \hat{v}(0,t) = \hat{v}_x(0,t) = \hat{v}(\ell,t) = 0. \end{cases}$$

Similar arguments yield that $E_{\hat{v}}(t) \le e^{-2\lambda t}E_{\hat{v}}(0)$, equivalently,

$$\int_0^\ell |\hat{w}_t(x,t)|^2\mathrm{d}x \le e^{-2\lambda t}\int_0^\ell |\hat{w}_t(x,0)|^2\mathrm{d}x. \tag{4.14}$$

By (4.12)–(4.14), we have

$$\int_0^T |\hat{y}(t)|^2\mathrm{d}t \le C\left(\int_0^\ell |\hat{w}(x,0)|^2\mathrm{d}x + \int_0^\ell |\hat{w}_{xxx}(x,0)|^2\mathrm{d}x\right) \le C\|\hat{w}_0\|^2_{H^3(0,\ell)}, \tag{4.15}$$

which yields (4.7) immediately. This completes the proof. □

Remark 4.2. If the function $p(t)\left(e_{xx}(\ell,t) + \phi(t)\right)$ is treated as an external signal, the system (4.1) is a non-homogeneous initial-boundary-value problem of the linear KdV equation as far as its well-posedness is concerned. A common approach is to render its boundary conditions homogeneous. In fact, the function

$$\tilde{w}(x,t) = w(x,t) - \frac{1}{\ell^2}(x^2 - 2\ell x)p(t)\left(e_{xx}(\ell,t) + \phi(t)\right)$$

solves an inhomogeneous linear equation with homogeneous boundary conditions if $w(x,t)$ is the solution to (4.1). But it is required that the functions $e(x,t)$, $p(t)$ and $\phi(t)$ possess much higher regularities, which is not guaranteed by Theorem 3.1. An alternative approach is the boundary integral operator method developed by Bona, Sun and Zhang in [1, 2]. By this method, the sharp regularity of boundary traces can be clearly revealed. For example, it can be proved by this method that $w(\ell,t)$ will belong to $H^{\frac{1}{3}}(0,T)$ for any $T>0$ if the initial datum w_0 is taken from $L^2(0,\ell)$ for the system (4.1) [1, 11]. Unfortunately, the boundary function $-p(t)\left(e_{xx}(\ell,t)+\phi(t)\right)$ of (4.1) lies in $L^2(0,T)$ according to (3.20) and (4.2). Here we employ the well-posedness theory of linear infinite-dimensional systems [22, 23] to prove its well-posedness in $L^2(0,\ell)$. The main reason is that the function $p(t)\left(e_{xx}(\ell,t)+\phi(t)\right)$ is in fact not a pure external signal. By (2.10)–(2.12), $w(x,t)$ is coupled with $e(x,t)$, $p(t)$ and $\phi(t)$. Moreover, the hidden regularity of the boundary function is revealed by this method, which means that $p(t)\left(e_{xx}(\ell,t)+\phi(t)\right)$ may be in $H^{\frac{1}{3}}(0,T)$. This could be explained that $p(t)$ and $\phi(t)$ compensate the lost regularity of $e_{xx}(\ell,t)$. In the Appendix, we will prove the hidden regularity of $p(t)\left(e_{xx}(\ell,t)+\phi(t)\right)$ for the error system (2.11).

The inequality (4.7) for the linear system (4.6) is a direct consequence of the so-called sharp Kato-smoothing property since it holds true that

$$\|\hat{w}_{xx}(\ell,\cdot)\|_{H^{\frac{2}{3}}(0,T)} \le C\|\hat{w}_0\|_{H^3(0,\ell)}.$$

We refer the readers to [1, 2, 13] for the details with some minor modifications.

Besides its well-posedness, we should further analyze the asymptotic behavior of the solution to the system (4.1). The well-posedness theory of linear infinite-dimensional systems also facilitates us to study the stability of (4.1), which is presented as follows.

Theorem 4.3. *The system (4.1) is asymptotically stable in the following sense: for any $w_0 \in L^2(0,\ell)$ and $(e_0,p_0,q_0)\in\mathcal{H}^+$, the corresponding solution $w(x,t)$ of the system (4.1) satisfies*

$$\lim_{t\to\infty}\|w(\cdot,t)\|_{L^2(0,\ell)} = 0. \tag{4.16}$$

Proof. By (4.4), we obtain that for arbitrary $t_0>0$,

$$w(t) = e^{\mathbf{A}t}w_0 + e^{\mathbf{A}(t-t_0)}\int_0^{t_0} e^{\mathbf{A}(t_0-s)}\mathbf{B}z(s)\mathrm{d}s + \int_{t_0}^t e^{\mathbf{A}(t-s)}\mathbf{B}z(s)\mathrm{d}s,\ \forall t\ge 0.$$

As shown by (2.2), it holds that

$$\|e^{\mathbf{A}t}w_0\|_{L^2(0,\ell)} \le e^{-\lambda t}\|w_0\|_{L^2(0,\ell)}, \quad \forall t\ge 0. \tag{4.17}$$

On the other hand, the admissibility of the operator $\mathbf{B}$ implies that for any $t>0$,

$$\int_0^t e^{\mathbf{A}(t-s)}\mathbf{B}\cdot ds \in \mathcal{L}\left(L^2(0,t), L^2(0,\ell)\right). \tag{4.18}$$

Thus, there exists a positive constant C independent of z so that

$$\left\|\int_0^{t_0} e^{\mathbf{A}(t_0-s)}\mathbf{B}z(s)\mathrm{d}s\right\|_{L^2(0,\ell)} \le C\|z\|_{L^2(0,t_0)},$$

which together with (2.2) yields

$$\left\| e^{\mathbf{A}(t-t_0)} \int_0^{t_0} e^{\mathbf{A}(t_0-s)} \mathbf{B} z(s) \mathrm{d}s \right\|_{L^2(0,\ell)} \leq C e^{-\lambda(t-t_0)} \|z\|_{L^2(0,t_0)}, \quad \forall t \geq t_0. \tag{4.19}$$

Let $g \underset{\tau}{\diamondsuit} h$ be the $\tau-$concatenation of g and h [23], *i.e.*,

$$\left(g \underset{\tau}{\diamondsuit} h \right)(t) = \begin{cases} g(t), & 0 \leq t < \tau; \\ h(t-\tau), & t \geq \tau. \end{cases}$$

By (4.18) and the fact that $\|g\|_{L^2(0,t)} = \|0 \underset{\tau-t}{\diamondsuit} g\|_{L^2(0,\tau)}$, we have

$$\left\| \int_{t_0}^t e^{\mathbf{A}(t-s)} \mathbf{B} z(s) \mathrm{d}s \right\|_{L^2(0,\ell)} \leq \left\| \int_0^t e^{\mathbf{A}(t-s)} \mathbf{B} \left(0 \underset{t_0}{\diamondsuit} z \right)(s) \mathrm{d}s \right\|_{L^2(0,\ell)} \leq C\|z\|_{L^2(t_0,t)} \leq C\|z\|_{L^2(t_0,\infty)}. \tag{4.20}$$

Combining (4.17)–(4.20) yields

$$\|w(\cdot,t)\|_{L^2(0,\ell)} \leq C \left(e^{-\lambda t} \|w_0\|_{L^2(0,\ell)} + e^{-\lambda(t-t_0)} \|z\|_{L^2(0,t_0)} + \|z\|_{L^2(t_0,\infty)} \right), \tag{4.21}$$

where C is a generic positive constant independent of z. For any given $\varepsilon > 0$, we choose $t_0 > 0$ large enough so that

$$\|z\|_{L^2(t_0,\infty)} \leq \frac{\varepsilon}{C}$$

holds true due to $z \in L^2(0,\infty)$. By (4.21), one then obtains that

$$\varlimsup_{t\to\infty} \|w(\cdot,t)\|_{L^2(0,\ell)} \leq \varepsilon,$$

and (4.16) follows immediately. This completes the proof. □

5. Well-posedness and asymptotic stability of the closed-loop system

We are now in a position to complete our analysis of the closed-loop system under the feedback law (2.9). In fact, the whole closed-loop system reads

$$\begin{cases} u_t(x,t) + u_{xxx}(x,t) + u_x(x,t) = 0, \\ u(0,t) = v(0,t),\ u(\ell,t) = 0,\ u_x(\ell,t) = 0, \\ v_t(x,t) + v_{xxx}(x,t) + v_x(x,t) = 0, \\ v(0,t) = \int_0^\ell k(0,\eta) v(\eta,t) \mathrm{d}\eta,\ \ v_x(\ell,t) = 0, \\ v(\ell,t) = -p(t) \left(u_{xx}(\ell,t) + d - (v_{xx}(\ell,t) + q(t)) \right), \\ \dot{p}(t) = r_1 \left(u_{xx}(\ell,t) + d - (v_{xx}(\ell,t) + q(t)) \right)^2, \\ \dot{q}(t) = r_2 p(t) \left(u_{xx}(\ell,t) + d - (v_{xx}(\ell,t) + q(t)) \right), \\ u(x,0) = u_0(x),\ \ v(x,0) = v_0(x), \\ p(0) = p_0 \geq 0,\ \ q(0) = q_0. \end{cases} \tag{5.1}$$

Let $\mathbb{X} = \{(u,v,p,q) \in L^2(0,\ell) \times L^2(0,\ell) \times \mathbb{R} \times \mathbb{R} \mid u(0) = v(0)\}$ be the underlying space. Consider the transformation

$$\begin{pmatrix} e(x,t) \\ w(x,t) \\ p(t) \\ q(t) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & \Pi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u(x,t) \\ v(x,t) \\ p(t) \\ q(t) \end{pmatrix} \tag{5.2}$$

with Π given by (2.12), which is obviously bounded and invertible. Then (e,w,p,q) solves the system (2.11)–(2.13) if (u,v,p,q) is the solution to (5.1). Owing to the stability of the error system (2.11) (see Thm. 3.4) and the transferred system (4.1) corresponding to (2.13) (see Thm. 4.3), we claim the well-posedness and stability of the system (5.1) as below.

Theorem 5.1. *For any $(u_0,v_0,p_0,q_0) \in \mathbb{X}$ with $p_0 \geq 0$, the system (5.1) admits a unique solution $(u,v,p,q) \in C(0,\infty;\mathbb{X})$. Under the assumption of $\ell \notin \mathcal{N}$ with $\mathcal{N}$ defined by (1.4), the closed-loop system (5.1) is asymptotically stable in the following sense,*

$$\lim_{t\to\infty} \left(\|u(\cdot,t)\|_{L^2(0,\ell)} + \|v(\cdot,t)\|_{L^2(0,\ell)}\right) = 0. \tag{5.3}$$

Moreover, the unknown constant disturbance d in the output is estimated in the following way,

$$\lim_{t\to\infty} q(t) = d. \tag{5.4}$$

Proof. For any $(u_0,v_0,p_0,q_0) \in \mathbb{X}$ with $p_0 \geq 0$, let (e_0,w_0,p_0,q_0) be the corresponding quadruples determined by (5.2). It is obvious that $(e_0,p_0,d-q_0) \in \mathcal{H}^+$ and $w_0 \in L^2(0,\ell)$. By Theorems 3.1 and 4.1, there exist $(e,p,\phi) \in C(0,\infty;\mathcal{H}^+)$ and $w \in C(0,\infty;L^2(0,\ell))$ solving (2.11) and (4.1), respectively. Since the transformation defined by (5.2) is invertible, there exists a unique $(u,v,p,q) \in \mathbb{X}$ solving (5.1). The asymptotic stability (5.3) and the convergence (5.4) are direct consequences of Theorems 3.4 and 4.3 because the transformation defined by (5.2) is bounded and invertible. This completes the proof. $\square$

Appendix A. Proof of the hidden regularity of the boundary term in (4.1)

Proposition A.1. *For any given $T > 0$ and $e_0 \in L^2(0,\ell)$, it holds that*

$$p(t)\left(e_{xx}(\ell,t) + \phi(t)\right) \in H^{\frac{1}{3}}(0,T)$$

for the error system (2.11).

Proof. Let $T > 0$ be arbitrary but fixed. For any $e_0 \in L^2(0,\ell)$, $g_1 \in H^{\frac{1}{3}}(0,T)$, $g_2 \in L^2(0,T)$ and $g_3 \in H^{-\frac{1}{3}}(0,T)$, it is well known that the initial-boundary-value problem

$$\begin{cases} e_t(x,t) + e_{xxx}(x,t) + e_x(x,t) = 0, & x \in (0,\ell), \quad t \in (0,T), \\ e(0,t) = g_1(t), \quad e_x(\ell,t) = g_2(t), & e_{xx}(\ell,t) = g_3(t), \\ e(x,0) = e_0(x). \end{cases} \tag{A.1}$$

admits a unique solution [11, 13]

$$e \in \mathcal{Y}_T := C([0,T];L^2(0,\ell)) \cap L^2(0,T;H^1(0,\ell)) \cap L^\infty_x(0,\ell;H^{\frac{1}{3}}(0,T)) \tag{A.2}$$

satisfying

$$\begin{aligned}\|e\|_{\mathcal{Y}_T} &:= \sup_{0\le t\le T}\|e(\cdot,t)\|_{L^2(0,\ell)} + \|e\|_{L^2(0,T;H^1(0,\ell))} + \sup_{0\le x\le \ell}\|e(x,\cdot)\|_{H^{\frac{1}{3}}(0,T)} \\ &\le C\left(\|e_0\|_{L^2(0,\ell)} + \|g_1\|_{H^{\frac{1}{3}}(0,T)} + \|g_2\|_{L^2(0,T)} + \|g_3\|_{H^{-\frac{1}{3}}(0,T)}\right) \end{aligned} \tag{A.3}$$

for some positive constant C.

We now turn to the following initial-boundary-value problem

$$\begin{cases} e_t(x,t) + e_{xxx}(x,t) + e_x(x,t) = 0, \quad x\in(0,\ell), \quad t\ge 0,\\ e(0,t) = g_1(t), \quad e_x(\ell,t) = g_2(t), \quad e_{xx}(\ell,t) - \alpha(t)e(\ell,t) = g_3(t),\\ e(x,0) = e_0(x), \end{cases} \tag{A.4}$$

where $\alpha(\cdot)\in L^\infty(0,\infty)$ is a given function. Let $\eta>0$ and $0<\theta\le\max\{1,T\}$ be constants to be determined and set

$$B_{\theta,\eta} = \{z\in\mathcal{Y}_\theta \mid \|z\|_{\mathcal{Y}_\theta}\le\eta\},$$

where $\mathcal{Y}_\theta$ and $\|\cdot\|_{\mathcal{Y}_\theta}$ are defined in the same way as in (A.2) and (A.3), respectively. Let $e_0\in L^2(0,\ell)$, $g_1\in H^{\frac{1}{3}}(0,T)$, $g_2\in L^2(0,T)$ and $g_3\in H^{-\frac{1}{3}}(0,T)$ in (A.4) be arbitrary but fixed. Define a map Γ on $B_{\theta,\eta}$ by $e=\Gamma(z)$ with e being the unique solution of the following initial-boundary-value problem

$$\begin{cases} e_t(x,t) + e_{xxx}(x,t) + e_x(x,t) = 0, \quad x\in(0,\ell), \quad t\ge 0,\\ e(0,t) = g_1(t), \quad e_x(\ell,t) = g_2(t), \quad e_{xx}(\ell,t) = \alpha(t)z(\ell,t) + g_3(t),\\ e(x,0) = e_0(x) \end{cases} \tag{A.5}$$

corresponding to a given $z\in B_{\theta,\eta}$. By (A.3) and the imbedding theorem, we have

$$\begin{aligned}\|\mathit{\Gamma}(z)\|_{\mathcal{Y}_\theta} &\le C\left(\|e_0\|_{L^2(0,\ell)} + \|g_1\|_{H^{\frac{1}{3}}(0,T)} + \|g_2\|_{L^2(0,T)} + \|g_3\|_{H^{-\frac{1}{3}}(0,T)}\right) + C\|\alpha(\cdot)z(\ell,\cdot)\|_{H^{-\frac{1}{3}}(0,\theta)}\\ &\le C\left(\|e_0\|_{L^2(0,\ell)} + \|g_1\|_{H^{\frac{1}{3}}(0,T)} + \|g_2\|_{L^2(0,T)} + \|g_3\|_{H^{-\frac{1}{3}}(0,T)}\right) + C_1\|\alpha\|_\infty\|z(\ell,\cdot)\|_{L^{\frac{18}{11}}(0,\theta)}\\ &\le C\left(\|e_0\|_{L^2(0,\ell)} + \|g_1\|_{H^{\frac{1}{3}}(0,T)} + \|g_2\|_{L^2(0,T)} + \|g_3\|_{H^{-\frac{1}{3}}(0,T)}\right) + C_2\theta^{\frac{4}{9}}\|\alpha\|_\infty\|z(\ell,\cdot)\|_{L^6(0,\theta)}\\ &\le C\left(\|e_0\|_{L^2(0,\ell)} + \|g_1\|_{H^{\frac{1}{3}}(0,T)} + \|g_2\|_{L^2(0,T)} + \|g_3\|_{H^{-\frac{1}{3}}(0,T)}\right) + C_3\theta^{\frac{4}{9}}\|\alpha\|_\infty\|z(\ell,\cdot)\|_{H^{\frac{1}{3}}(0,\theta)}.\end{aligned}$$

If we choose such η and θ that

$$\eta = 2C\left(\|e_0\|_{L^2(0,\ell)} + \|g_1\|_{H^{\frac{1}{3}}(0,T)} + \|g_2\|_{L^2(0,T)} + \|g_3\|_{H^{-\frac{1}{3}}(0,T)}\right)$$

and

$$C_3\|\alpha\|_\infty\theta^{\frac{4}{9}} \le \frac{1}{2},$$

we have

$$\|\mathit{\Gamma}(z)\|_{\mathcal{Y}_\theta}\le\eta, \quad \forall z\in B_{\theta,\eta}.$$

Moreover, it holds that

$$\|\Gamma(z_1) - \Gamma(z_2)\|_{\mathcal{Y}_\theta} \leq \frac{1}{2}\|z_1 - z_2\|_{\mathcal{Y}_\theta}$$

for any $z_1, z_2 \in \mathcal{Y}_\theta$. Thus, Γ is a contraction mapping on $\mathcal{Y}_\theta$ whose fixed point $e \in \mathcal{Y}_\theta$ is the unique solution of the initial-boundary-value problem (A.4) on the time interval $(0, \theta)$. By a standard argument, the time interval $(0, \theta)$ can be extended to $(0, T)$ since θ depends only on $\|\alpha\|_\infty$. In conclusion, (A.4) admits a unique solution

$$e \in C([0,T]; L^2(0,\ell)) \cap L^2(0,T; H^1(0,\ell)), \quad e(x,\cdot) \in H^{\frac{1}{3}}(0,T), \ \forall x \in (0,\ell). \tag{A.6}$$

Noting that the $e-$subsystem of the error system (2.11) can be rewritten in the form of (A.4), *i.e.*,

$$\begin{cases} e_t(x,t) + e_{xxx}(x,t) + e_x(x,t) = 0, \\ e(0,t) = 0, \ \ e_x(\ell,t) = 0, \ \ e_{xx}(\ell,t) - \frac{1}{p(t)}e(\ell,t) = -\phi(t), \\ e(x,0) = e_0(x) \end{cases}$$

on account of $p(\cdot), \phi(\cdot) \in L^\infty(0,\infty)$. Thus, it holds that $e(\ell,\cdot) \in H^{\frac{1}{3}}(0,T)$ by (A.6), or equivalently

$$p(t)\left(e_{xx}(\ell,t) + \phi(t)\right) \in H^{\frac{1}{3}}(0,T),$$

which is the desired conclusion. □

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