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ERGODIC CONTROL OF INFINITE-DIMENSIONAL STOCHASTIC DIFFERENTIAL EQUATIONS WITH DEGENERATE NOISE

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Abstract. This paper is devoted to the study of the asymptotic behaviour of the value functions of both finite and infinite horizon stochastic control problems and to the investigation of their relationship with suitable stochastic ergodic control problems. Our methodology is based only on probabilistic techniques, as, for instance, the so-called randomisation of the control method, thus avoiding completely analytical tools from the theory of viscosity solutions. We are then able to treat with the case where the state process takes values in a general (possibly infinite-dimensional) real separable Hilbert space, and the diffusion coefficient is allowed to be degenerate.

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1. Introduction

In this paper, we study the asymptotic behaviour of the value functions both for finite horizon stochastic control problems (as the horizon diverges) and for discounted infinite horizon control problems (as the discount vanishes) and investigate their relationship with suitable stochastic ergodic control problems. We refer to such limits as *ergodic limits*. The main novelty of this work is that we deal with ergodic limits for control problems in which the state process is allowed to take values in a general (possibly infinite-dimensional) real separable Hilbert space and the diffusion coefficient is allowed to be degenerate.

On the one hand, the infinite-dimensional framework imposes the use of purely probabilistic techniques essentially based on backward stochastic differential equations (BSDEs; see, the introduction of [12]). On the other hand, the degeneracy of the noise prevents the use of standard BSDE techniques as they are implemented, for similar problems, in [9]. Indeed, see again [12], the identification between solutions of BSDEs and value functions of stochastic optimal control problems can be easily obtained, by a Girsanov argument, as far as the so-called *structure condition*, imposing large enough image of diffusion operator, holds. Here, we wish to avoid such a requirement.

Keywords and phrases: Ergodic control, infinite-dimensional SDEs, BSDEs, randomisation of the control method.

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To our knowledge, the only paper that deals, by means of BSDEs, with ergodic limits in the degenerate case is [5], where authors use the same tool of randomised control problems and related *constrained* BSDEs that we eventually employ here. Note, however, that in [5], the state process lives in a finite-dimensional Euclidean space, and probabilistic methods are combined with PDE techniques, relying on powerful tools from the theory of viscosity solutions. Here, as already mentioned, we have to completely avoid these arguments. As a matter of fact, viscosity solutions require, in the infinite-dimensional case, additional artificial assumptions that we cannot impose here (see the theory of B-continuous viscosity solutions for second-order PDEs, [8, 20]). On the other hand, to separate difficulties, we consider, as in [9] but differently from [5], only additive and uncontrolled noise. This, in particular, considerably simplifies the proof of estimate (4.3).

Let us now give a more precise idea of the results obtained in the paper. Consider the following infinite and finite horizon stochastic control problems:

$$v^{\beta}(x) \;=\; \inf_{u\in\mathcal{U}} \mathbb{E}\bigg[\int_0^{\infty} e^{-\beta t}\,\ell(X_s^{x,u},u_s)\,\mathrm{d}s\bigg]$$

and

$$v^{T}(x) \;=\; \inf_{u\in\mathcal{U}} \mathbb{E}\bigg[\int_0^{T} \ell(X_s^{x,u},u_s)\,\mathrm{d}s + \phi(X_T^{x,u})\bigg],$$

where the discount coefficient β can be any positive real number, as well as the time horizon T, while the controlled state process $X^{x,u}$ takes values in some real separable Hilbert space H and is a (mild) solution to the time-homogenous stochastic differential equation

$$\mathrm{d}X_t \;=\; AX_t\mathrm{d}t + F(X_t,u_t)\mathrm{d}t + G\mathrm{d}W_t^1, \qquad X_0 \;=\; x.$$

Here, W^1 is a cylindrical Wiener process and A is a possibly unbounded linear operator on H. We assume that both A and F are dissipative. The control process u is progressively measurable and takes values in some real separable Hilbert space U (actually, U can be taken more general; see Rem. 2.3). Note that the diffusion coefficient G is only assumed to be a bounded linear operator, so that it can be degenerate. Our aim is to study the asymptotic behaviour of

$$v^{\beta}(x)-v^{\beta}(0), \qquad\qquad \beta\, v^{\beta}(0), \qquad\qquad \frac{v^T(x)}{T}, \tag{1.1}$$

as $\beta\to 0$ and $T\to+\infty$. In order to do it, we find nonlinear Feynman–Kac representations for both v^{β} and v^T in terms of suitable BSDEs (which can be seen as the probabilistic counterparts of the Hamilton–Jacobi–Bellman equations). Since G can be degenerate, we adopt the recently introduced the so-called randomisation method (see, *e.g.* [2, 10, 17]), which was also implemented in [5]. Here we use it in a rather different way, as we will explain below. The idea of the randomisation (of the control) method is to introduce a new control problem (called the randomised infinite/finite horizon stochastic control problem), where we replace the family of control processes u by a particular class of processes (depending on the control parameter α), here denoted by $\mathfrak{I}^{a,\alpha}$, which is, roughly speaking, dense in $\mathcal{U}$. More precisely, focusing for simplicity only on the infinite horizon case, we define the value function of the randomised infinite horizon stochastic control problem as follows:

$$v^{\beta,\mathcal{R}}(x,a) \;=\; \inf_{\alpha\in\mathcal{A}} \mathbb{E}\bigg[\int_0^{\infty} e^{-\beta t}\,\ell(\mathfrak{X}_t^{x,a,\alpha},\mathfrak{I}_t^{a,\alpha})\,\mathrm{d}t\bigg],$$

where $\mathcal{A}$ is the set of progressively measurable and uniformly bounded processes taking values in U, while the state process is the pair $(X^{x,a,\alpha}, \mathfrak{I}^{a,\alpha})$ satisfying

$$\begin{cases} \mathrm{d}\mathfrak{X}_t = A\mathfrak{X}_t \mathrm{d}t + F(\mathfrak{X}_t, \mathfrak{I}_t)\mathrm{d}t + \ G\mathrm{d}W_t^1, & \mathfrak{X}_0 \ = \ x, \\ \mathrm{d}\mathfrak{I}_t = R\alpha_t \mathrm{d}t + R\mathrm{d}W_t^2, & \mathfrak{I}_0 \ = \ a \end{cases}$$

with $R\colon U \to U$ being a trace class injective linear operator with dense image, while W^2 is a cylindrical Wiener process independent of W^1. We prove by a density argument (Prop. 3.3) that $v^\beta(x) = v^{\beta,\mathcal{R}}(x,a)$, for every $(x,a) \in H \times U$, so, in particular, $v^{\beta,\mathcal{R}}$ does not depend on its second argument a. Note that we randomise the control by means of an independent cylindrical Wiener process W^2, instead of using an independent Poisson random measure on $\mathbb{R}_+ \times U$ (as it is usually the case in the literature on the randomisation method). Taking a Poisson random measure has the advantage that U can be any Borel space, while here we have to impose some restrictions on U (see Rem. 2.3). However, randomising the control by means of a cylindrical Wiener process is simpler and more natural in an infinite-dimensional setting, as many important fundamental results on SDEs and BSDEs can only be found for the case where the driving noise is of Wiener-type. Moreover, with this choice, the results presented here can receive more attention in the infinite-dimensional literature. Furthermore, the Wiener-type randomisation has not been enough investigated in the literature, since it was implemented only in [3], where the proof of the fundamental equality $v^\beta(x) = v^{\beta,\mathcal{R}}(x,a)$ was based on PDE techniques (in particular, viscosity solutions' arguments) adapted from [17], instead of using purely probabilistic arguments, as it was done in [2, 10] for the Poisson-type randomisation. So, in particular, this is the first time that the equality $v^\beta(x) = v^{\beta,\mathcal{R}}(x,a)$ is proved in purely probabilistic terms for the Wiener-type randomisation.

Once we know that $v^\beta(x) = v^{\beta,\mathcal{R}}(x,a)$, it is fairly standard in the framework of the randomisation approach, to derive a nonlinear Feynman–Kac formula for v^β (see [17]). As a matter of fact, note that, for each positive integer n, the control problem with value function

$$v^{\beta,\mathcal{R},n}(x,a) \ = \ \inf_{\alpha\in\mathcal{A}:|\alpha|\leq n} \mathbb{E}\bigg[\int_0^\infty e^{-\beta t}\,\ell(X_t^{x,a,\alpha}, I_t^{a,\alpha})\,\mathrm{d}t\bigg]$$

is a dominated problem. Therefore, by standard BSDE techniques, $v^{\beta,\mathcal{R},n}$ admits a nonlinear Feynman–Kac representation in terms of some BSDEs depending on the parameter n. Passing to the limit as n goes to infinity, we find, as in [5], a non-standard BSDE for $v^\beta(x) = v^{\beta,\mathcal{R}}(x,a) = \lim_n v^{\beta,\mathcal{R},n}(x,a)$ (see Prop. 4.1 and 4.4). As a matter of fact, such a BSDE, which we shall call 'constrained', involves a reflection term and is characterised by its maximality (see [17]). We eventually exploit the probabilistic representation of v^β (and similarly of v^T) to study the limits in (1.1). In particular, in Section 5, we prove that, up to a subsequence, the limit of $v^\beta(x) - v^\beta(0)$ (resp. $\beta v^\beta(0)$) exists and is given by a function $\hat{v}$ (resp. constant λ). Moreover, we prove that $\hat{v}$ and λ are related to a suitable constrained ergodic BSDE, again of the non-standard, constrained type (see Thm. 5.3). This is the most technical result of the paper. In the previous literature (see [5]), PDE techniques are indeed of great help at this level. In the present context, we have to prove in a direct way that the candidate solution to the ergodic constrained BSDE enjoys the required maximality property. To do that we exploit the extra regularity of the trajectories of the state equation implied by Assumption (A.2) (see estimate (3.3) in Prop. 3.1).

Concerning the long-time asymptotics of $v^T(x)/T$, we show in Theorem 6.1 that this quantity converges to the same constant λ. We end Section 5 proving that, under suitable assumptions, λ coincides with the value function of an ergodic control problem. The latter result is again proved using only probabilistic techniques, while in [5] the proof is based on PDE arguments (see Rem. 6.6 for more details on this point).

The rest of the paper is organised as follows. In Section 2, we first introduce the notation used throughout the paper, then we formulate both the infinite and finite horizon stochastic optimal control problems on a generic probabilistic setting; afterwards, we formulate both control problems on a specific probabilistic, product space, setting and we prove (Prop. 2.5) that, even if the probabilistic setting has changed, value functions are still the same. Section 3 is devoted to the formulation of the randomised control problems; we prove (Prop. 3.3) that the value functions of the randomised problems coincide with the value functions of the original control problems. In Section 4, we find nonlinear Feynman–Kac representation formulae for the value functions in terms of constrained BSDEs. In Section 5, we introduce an ergodic BSDE and study the asymptotic behaviour of the infinite horizon problem. Finally, in Section 6, we introduce an ergodic control problem and study the long-time asymptotics of the finite horizon problem.

2. Infinite/finite horizon optimal control problems

In this section, we introduce both infinite and finite horizon stochastic optimal control problem, first on a generic probability space and then on an enlarged probability space in product form (the latter will then be used throughout the paper). First, we fix some notation.

2.1. General notation

Let $\mathfrak{X}i$, H and U be real separable Hilbert spaces. In the sequel, we use the notation $|\cdot|_{\mathfrak{X}}i$, $|\cdot|_H$ and $|\cdot|_U$ to denote the norms on $\mathfrak{X}i$, H and U, respectively; if no confusion arises, we simply write $|\cdot|$. We use similar notation for the scalar products. We denote the dual spaces of $\mathfrak{X}i$, H and U by $\mathfrak{X}i^*$, H^* and U^*, respectively. We also denote by $L(H,H)$ the space of bounded linear operators from H to H, endowed with the operator norm. Moreover, we denote by $L_2(\mathfrak{X}i, H)$ the space of Hilbert–Schmidt operators from $\mathfrak{X}i$ to H. Finally, we denote by $\mathcal{B}(\Lambda)$, the Borel σ-algebra of any topological space Λ.

Given a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ together with a filtration $(\mathcal{F}_t)_{t\geq 0}$ (satisfying the usual conditions of $\mathbb{P}$-completeness and right-continuity) and an arbitrary real separable Hilbert space V, we define the following classes of processes for fixed $0 \leq t \leq T$ and $p \geq 1$:

- $L^p_{\mathcal{P}}(\Omega \times [t,T]; V)$ denotes the set of equivalence classes of $(\mathcal{F}_s)$-predictable processes $Y \in L^p(\Omega \times [t,T]; V)$ such that the following norm is finite:

$$|Y|_p \;=\; \left(\mathbb{E}\int_t^T |Y_s|^p \,\mathrm{d}s\right)^{1/p}.$$

- $L^{p,\mathrm{loc}}_{\mathcal{P}}(\Omega \times [0,+\infty[; V)$ denotes the set of processes defined on $\mathbb{R}^+$, the restriction to an arbitrary time interval $[0,T]$ of which belongs to $L^p_{\mathcal{P}}(\Omega \times [0,T]; V)$.
- $L^p_{\mathcal{P}}(\Omega; C([t,T]; V))$ denotes the set of $(\mathcal{F}_s)$-predictable processes Y on $[t,T]$ with continuous paths in V, such that the norm

$$\|Y\|_p \;=\; \big(\mathbb{E}\sup_{s\in[t,T]} |Y_s|^p\big)^{1/p}$$

  is finite. The elements of $L^p_{\mathcal{P}}(\Omega; C([t,T]; V))$ are identified up to indistinguishability.
- $L^{p,\mathrm{loc}}_{\mathcal{P}}(\Omega; C[0,+\infty[; V)$ denotes the set of processes defined on $\mathbb{R}^+$, the restriction to an arbitrary time interval $[0,T]$ of which belongs to $L^p_{\mathcal{P}}(\Omega; C([0,T]; V))$.
- $\mathcal{K}^2_{\mathcal{P}}(0,T)$ denotes the set of real-valued $(\mathcal{F}_s)$-adapted non-decreasing continuous processes K on $[0,T]$ such that $\mathbb{E}|K_T|^2 < \infty$ and $K_0 = 0$.
- $\mathcal{K}^{2,\mathrm{loc}}_{\mathcal{P}}$ denotes the set of processes defined on $\mathbb{R}^+$, the restriction to an arbitrary time interval $[0,T]$ of which belongs to $\mathcal{K}^2_{\mathcal{P}}(0,T)$.

2.2. Formulation of the control problems

We formulate here both the discounted, infinite horizon, control problem and the finite horizon one the asymptotic behaviour of which is the main focus of this paper. The notation chosen here may seem a bit artificial but this is done in order to keep the notation simple in the product space and randomised setting (see Sects. 2 and 3) where the technical arguments are developed.

We fix a complete probability space $(\bar{\Omega}^1, \bar{\mathcal{F}}^1, \bar{\mathbb{P}}^1)$ on which a cylindrical Wiener process $\bar{W}^1 = (\bar{W}^1_t)_{t\geq 0}$ with values in $\mathfrak{X}i$ is defined. By $(\bar{\mathcal{F}}^1_t)_{t\geq 0}$, or simply $(\bar{\mathcal{F}}^1_t)$, we denote the natural filtration of $\bar{W}^1$, augmented with the family $\bar{\mathcal{N}}^1$ of $\bar{\mathbb{P}}^1$-null sets of $\bar{\mathcal{F}}^1$. Obviously, the filtration $(\bar{\mathcal{F}}^1_t)$ satisfies the usual conditions of right-continuity and $\bar{\mathbb{P}}^1$-completeness.

In this section, the notion of measurability and progressive measurability will always refer to the filtration $\bar{\mathcal{F}}^1$.

Let $\bar{\mathcal{U}}$ be the family of $(\bar{\mathcal{F}}^1_t)$-progressively measurable processes taking values in U (see Rem. 2.3 for the case where the space of control actions U is not necessarily a Hilbert space).

State process. Given $x \in H$ and $\bar{u} \in \bar{\mathcal{U}}$, we consider the controlled stochastic differential equation

$$\mathrm{d}\bar{X}_t \;=\; A\bar{X}_t \mathrm{d}t + F(\bar{X}_t, \bar{u}_t)\mathrm{d}t + G\mathrm{d}\bar{W}^1_t, \qquad \bar{X}_0 \;=\; x. \tag{2.1}$$

On the coefficients A, F and G, we impose the following assumptions.

(A.1) $A\colon \mathcal{D}(A) \subset H \to H$ is a linear, possibly unbounded operator generating an analytic semigroup $\{e^{tA}\}_{t\geq 0}$. We assume that A is dissipative, *i.e.* $\langle Ax, x\rangle \leq 0$, for all $x \in \mathcal{D}(A)$.

(A.2) $G\colon \mathfrak{X}i \to H$ is a bounded linear operator. Moreover, there exist positive constants M_A and $\gamma \in [0, \frac{1}{2}[$ such that

$$\left|e^{sA}G\right|_{L_2(\mathfrak{X}i,H)} \;\leq\; \frac{M_A}{s^{\gamma}} \qquad \text{for all } s \in (0,1).$$

(A.3) Fixed $\delta > 0$, we denote by $\mathcal{D}((\delta I - A)^{\rho})$, the domain of the fractional power of the operator $\delta I - A$ (see [18]). We assume that there exists a $\rho \in (0, \frac{1}{2} - \gamma)$ such that the domain of the fractional power $\mathcal{D}((\delta I - A)^{\rho})$ is compactly embedded in H.

(A.4) $F\colon H \times U \to H$ is continuous and there exists $C_F > 0$ such that

$$|F(x,a)| \leq C_F(1 + |x|)$$

for all $x \in H$ and $a \in U$.
Moreover, there exists $L_F > 0$ such that

$$|F(x,a) - F(x',a)|_H \;\leq\; L_F|x - x'|_H$$

for all $x, x' \in H$ and $a \in U$.

(A.5) F is assumed to be strongly dissipative: there exists $\mu > 0$ such that

$$\langle F(x,a) - F(x',a), x - x'\rangle_H \;\leq\; -\mu|x - x'|^2_H$$

for all $x, x' \in H$ and $a \in U$.

Remark 2.1. Assumption (A.5) can be balanced with the dissipativity of A replacing A by $A + \lambda I$ and F by $F - \lambda I$ ($\lambda \in \mathbb{R}$). In particular, we can always think that both A and F are strongly dissipative.

Proposition 2.2. *Assume (A.1)–(A.5). Then, for any $x \in H$ and $\bar{u} \in \bar{\mathcal{U}}$, there exists a unique (up to indistinguishability) process $\bar{X}^{x,\bar{u}} = (\bar{X}^{x,\bar{u}}_t)_{t\geq 0}$ that belongs to $L^{p,\mathrm{loc}}_{\mathcal{P}}(\bar{\Omega}^1; C([0,+\infty[; H))$ for all $p \geq 1$ and is a mild solution of (2.1), it that is:*

$$\bar{X}^{x,\bar{u}}_t \;=\; e^{tA}x + \int_0^t e^{(t-s)A}F(\bar{X}^{x,\bar{u}}_s, \bar{u}_s)\,\mathrm{d}s + \int_0^t e^{(t-s)A}G\,\mathrm{d}\bar{W}^1_s \quad \textit{for all } t \geq 0,\ \bar{\mathbb{P}}^1\textit{-a.s.}$$

Moreover, the following estimates hold:

- *For every $T > 0$ and $p \geq 1$, there exists a positive constant $\kappa_{p,T}$, independent of $x \in H$ and $\bar{u} \in \bar{\mathcal{U}}$, such that*

$$\bar{\mathbb{E}}^1\bigg[\sup_{t\in[0,T]}|\bar{X}_t^{x,\bar{u}}|^p\bigg] \;\leq\; \kappa_{p,T}(1+|x|^p).$$

- *There exists a positive constant κ, independent of $x \in H$, $\bar{u} \in \bar{\mathcal{U}}$, $t \geq 0$, such that*

$$\bar{\mathbb{E}}^1|\bar{X}_t^{x,\bar{u}}| \;\leq\; \kappa(1+|x|).$$

Proof. This is a standard result (see [9], Prop. 3.6) for the proof in a general Banach space context. Note that the presence of the control process $\bar{u}$ does not cause any additional difficulty since Assumptions (A.4) and (A.5) hold uniformly with respect to the control variable $a \in U$. □

Finally, we fix a running cost $\ell\colon H \times U \to \mathbb{R}$ and we impose the following assumption:

(A.6) ℓ is continuous and bounded, moreover there exists $L_\ell > 0$ such that

$$|\ell(x,a) - \ell(x',a)| \;\leq\; L_\ell|x-x'|_H$$

for all $x, x' \in H$ and $a \in U$.

Infinite horizon control problem. Given a positive discount $\beta > 0$, the cost corresponding to control $\bar{u} \in \mathcal{U}$, and the initial condition x is defined as

$$\bar{J}^\beta(x,\bar{u}) \;:=\; \bar{\mathbb{E}}^1\bigg[\int_0^\infty e^{-\beta t}\,\ell(\bar{X}_t^{x,\bar{u}},\bar{u}_t)\,\mathrm{d}t\bigg],$$

where $\bar{\mathbb{E}}^1$ denotes the expectation with respect to $\bar{\mathbb{P}}^1$. Moreover, the value function is given by

$$\bar{v}^\beta(x) \;:=\; \inf_{\bar{u}\in\bar{\mathcal{U}}}\bar{J}^\beta(x,\bar{u}) \quad \text{for every } x \in H.$$

Finite horizon control problem. Fix a function $\phi : H \to \mathbb{R}$ satisfying:

(A.7) ϕ is continuous and there exists $C_\phi > 0$ such that

$$|\phi(x)| \;\leq\; C_\phi(1+|x|_H) \qquad \text{for all } x \in H.$$

The cost with (finite) horizon $T > 0$ and discount $\beta \geq 0$ relative to the control $\bar{u}$ and initial condition x is defined as

$$\bar{J}^{\beta,T}(x,\bar{u}) \;:=\; \bar{\mathbb{E}}^1\bigg[\int_0^T e^{-\beta s}\ell(\bar{X}_s^{x,\bar{u}},\bar{u}_s)\,\mathrm{d}s + e^{-\beta T}\phi(\bar{X}_T^{x,\bar{u}})\bigg].$$

Finally, the value function is given by

$$\bar{v}^{\beta,T}(x) \;:=\; \inf_{\bar{u}\in\bar{\mathcal{U}}}\bar{J}^{\beta,T}(x,\bar{u}) \quad \text{for every } x \in H.$$

Remark 2.3. The request on the space of control actions U to be an Hilbert space can be relaxed. As a matter of fact, suppose that the space of control actions is a certain set $\tilde{U}$, so that in the formulation of the stochastic optimal control problem, drift and running cost are defined on $H \times \tilde{U}$:

$$\tilde{F}\colon H \times \tilde{U} \to H, \quad \tilde{\ell}\colon H \times \tilde{U} \to \mathbb{R}.$$

Suppose that $\tilde{U}$ has the following property: there exists a continuous surjection $\varphi\colon U \to \tilde{U}$ for some real separable Hilbert space U. This holds true, for instance, if $\tilde{U}$ is a compact, connected, locally connected subset of $\mathbb{R}^n$ for some positive integer n (in this case, the existence of a continuous surjection $\varphi\colon U \to \tilde{U}$, with $U = \mathbb{R}$, follows from the Hahn–Mazurkiewicz theorem (see Thm. 6.8 in [19])). Then, we define $F\colon H \times U \to H$ and $\ell\colon H \times U \to \mathbb{R}$ as

$$F(x,a) \ := \ \tilde{F}(x,\varphi(a)), \quad \ell(x,a) \ := \ \tilde{\ell}(x,\varphi(a))$$

for every $(x,a) \in H \times U$. Note that, if $\tilde{F}$ (resp. $\tilde{\ell}$) satisfies Assumptions (A.4) and (A.5) (resp. (A.6)), then F (resp. ℓ) still satisfies the same assumptions. Replacing $\tilde{U}$, $\tilde{F}$ and $\tilde{\ell}$ by U, F and ℓ, respectively, we find a stochastic control problem of the form studied in this work, which has the same value function of the original control problem.

2.3. Formulation of the control problems on a product space

For a technical reason imposed by the randomisation method (see Sect. 3), we have to reformulate our control problems in a product probability space. The main point of this section (see Prop. 2.5) will be to show that this new setting does not affect the value function.

Let $\bar{W}^2 = (\bar{W}^2_t)_{t\geq 0}$ be a cylindrical Wiener process with values in U, defined on a complete probability space $(\bar{\Omega}^2, \bar{\mathcal{F}}^2, \bar{\mathbb{P}}^2)$. We define $(\Omega, \mathcal{F}, \mathbb{P})$, W^1 and W^2 as follows: $\Omega := \bar{\Omega}^1 \times \bar{\Omega}^2$, $\mathcal{F}$ the $\bar{\mathbb{P}}^1 \otimes \bar{\mathbb{P}}^2$-completion of $\bar{\mathcal{F}}^1 \otimes \bar{\mathcal{F}}^2$, $\mathbb{P}$ the extension of $\bar{\mathbb{P}}^1 \otimes \bar{\mathbb{P}}^2$ to $\mathcal{F}$, $W^1(\omega^1,\omega^2) := \bar{W}^1(\omega^1)$, $W^2(\omega^1,\omega^2) := \bar{W}^2(\omega^2)$, for every $(\omega^1,\omega^2) \in \Omega$.

By $(\mathcal{F}_t)_{t\geq 0}$ we denote the natural filtration of (W^1, W^2), augmented with the family $\mathcal{N}$ of $\mathbb{P}$-null sets of $\mathcal{F}$. Clearly, $(\mathcal{F}_t)$ satisfies the usual conditions of right-continuity and $\mathbb{P}$-completeness. Finally, we denote by $\mathcal{U}$ the family of $(\mathcal{F}_t)$-progressively measurable processes with values in U. In this section, measurability will always be referred to such a filtration.

As before, given $x \in H$ and $u \in \mathcal{U}$, we consider the controlled stochastic differential equation

$$\mathrm{d}X_t \ = \ AX_t\mathrm{d}t + F(X_t,u_t)\mathrm{d}t + G\mathrm{d}W^1_t, \quad X_0 \ = \ x. \tag{2.2}$$

Exactly as for equation (2.1), we have the following result.

Proposition 2.4. *Assume (A.1)–(A.5). Then, for any $x \in H$ and $u \in \mathcal{U}$, there exists a unique (up to indistinguishability) process $X^{x,u} = (X^{x,u}_t)_{t\geq 0}$ that belongs to $L^{p,\mathrm{loc}}_{\mathcal{P}}(\Omega; C([0,+\infty[;H))$ for all $p \geq 1$ and is a mild solution of (2.2), that is:*

$$X^{x,u}_t \ = \ e^{tA}x + \int_0^t e^{(t-s)A}F(X^{x,u}_s,u_s)\,\mathrm{d}s + \int_0^t e^{(t-s)A}G\,\mathrm{d}W^1_s \quad \textit{for all } t \geq 0,\ \mathbb{P}\textit{-a.s.}$$

Moreover, the following estimates hold:

- *For every $T > 0$ and $p \geq 1$, there exists a positive constant $\kappa_{p,T}$, independent of $x \in H$ and $u \in \mathcal{U}$, such that*

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |X^{x,u}_t|^p\Big] \ \leq \ \kappa_{p,T}(1 + |x|^p). \tag{2.3}$$

- *There exists a positive constant κ, independent of $x \in H$, $u \in \mathcal{U}$, $t \geq 0$, such that*

$$\mathbb{E}|X_t^{x,u}| \;\leq\; \kappa(1+|x|). \tag{2.4}$$

Again, for every $\beta > 0$, and for any $x \in H$, $u \in \mathcal{U}$, we define the *infinite horizon* cost

$$J^\beta(x,u) \;:=\; \mathbb{E}\bigg[\int_0^\infty e^{-\beta t}\,\ell(X_t^{x,u},u_t)\,\mathrm{d}t\bigg]$$

and the corresponding value function

$$v^\beta(x) \;:=\; \inf_{u\in\mathcal{U}} J^\beta(x,u) \quad \text{for every } x\in H.$$

On the other hand, for every $T>0$, $\beta \geq 0$, and for any $x \in H$, $u \in \mathcal{U}$, we define the *finite horizon* cost:

$$J^{\beta,T}(x,u) \;:=\; \mathbb{E}\bigg[\int_0^T e^{-\beta s}\ell(X_s^{x,u},u_s)\,\mathrm{d}s + e^{-\beta T}\phi(X_T^{x,u})\bigg]$$

and the corresponding value function

$$v^{\beta,T}(x) \;:=\; \inf_{u\in\mathcal{U}} J^{\beta,T}(x,u).$$

In Proposition 2.5, we give a detailed argument showing that, as expected, the value function of both infinite and finite horizon problems is not affected by the product space formulation. For the definition of $\bar v^\beta$ and $\bar v^T$, see Section 2.2.

Proposition 2.5. *Suppose that Assumptions (A.1)–(A.7) hold. Then:*

(i) *For all $\beta > 0$, $\bar v^\beta(x) = v^\beta(x)$, for every $x \in H$.*
(ii) *For all $T > 0$ and $\beta \geq 0$, $\bar v^{\beta,T}(x) = v^{\beta,T}(x)$, for every $x\in H$.*

Proof. We prove only the first statement, since the proof of (ii) can be done proceeding along the same lines.

Fix $\beta > 0$ and $x \in H$. We begin noting that the inequality $\bar v^\beta(x) \geq v^\beta(x)$ is immediate. As a matter of fact, given $\bar u \in \bar{\mathcal{U}}$ (thus, $\bar u$ is a process defined on $[0,\infty[\times\bar\Omega^1$) let $u_t(\omega^1,\omega^2) := \bar u_t(\omega^1)$ for every $(\omega^1,\omega^2)\in\Omega$. Then, $u\in\mathcal{U}$ and $\bar J^\beta(x,\bar u) = J^\beta(x,u) \geq v^\beta(x)$. Taking the infimum over $\bar u \in \bar{\mathcal{U}}$, we conclude that $\bar v^\beta(x)\geq v^\beta(x)$.

We now prove the other inequality. To this end, we recall that $(\bar{\mathcal{F}}_t^1)$ is the natural filtration on $(\bar\Omega^1,\bar{\mathcal{F}}^1,\bar{\mathbb{P}}^1)$ of $\bar W^1$, augmented with the family $\bar{\mathcal{N}}^1$ of $\bar{\mathbb{P}}^1$-null sets. In a similar way, we define $(\bar{\mathcal{F}}_t^2)$. Now, let $(\tilde{\mathcal{F}}_t)$ be the filtration defined as $\tilde{\mathcal{F}}_t := \bar{\mathcal{F}}_t^1\otimes\bar{\mathcal{F}}_t^2$, for every $t\geq 0$. Observe that $(\tilde{\mathcal{F}}_t)$ is right-continuous (as it can be shown proceeding as in the proof of Thm. 1 in [13]), but not necessarily $\mathbb{P}$-complete. We also note that $(\mathcal{F}_t)$ is the augmentation of $(\tilde{\mathcal{F}}_t)$.

Now, fix $u\in\mathcal{U}$. Since u is $(\mathcal{F}_t)$-progressively measurable, by Lemma B.21 in [1] or Theorem 3.7 in [4], we deduce that there exists an $(\mathcal{F}_t)$-predictable process $\hat u$ with values in U such that $u = \hat u$, $d\mathbb{P}\otimes\mathrm{d}t$-a.e., so, in particular, $J^\beta(x,u) = J^\beta(x,\hat u)$. By Lemma 2.17b in [16], it follows that there exists an $(\tilde{\mathcal{F}}_t)$-predictable process $\tilde u$ with values in U which is indistinguishable from $\hat u$, so that $J^\beta(x,\hat u) = J^\beta(x,\tilde u)$. In addition, since $\tilde u$ is $(\tilde{\mathcal{F}}_t)$-progressively measurable, for every $\omega^2\in\bar\Omega^2$ we have that the process $\tilde u^{\omega^2}$ on $(\bar\Omega^1,\bar{\mathcal{F}}^1,\bar{\mathbb{P}}^1)$, given by $\tilde u_t^{\omega^2}(\omega^1) := \tilde u_t(\omega^1,\omega^2)$, is $(\bar{\mathcal{F}}_t^1)$-progressively measurable. In other words, $\tilde u^{\omega^2}\in\bar{\mathcal{U}}$ for every $\omega^2\in\bar\Omega^2$.

Consider now, for every $\omega^2\in\bar\Omega^2$, the process $(\bar X^{x,\tilde u^{\omega^2}})_{t\geq 0}$ solving the following controlled equation:

$$\bar X_t^{x,\tilde u^{\omega^2}} \;=\; e^{tA}x + \int_0^t e^{(t-s)A}F\big(\bar X_s^{x,\tilde u^{\omega^2}},\tilde u_s^{\omega^2}\big)\,\mathrm{d}s + \int_0^t e^{(t-s)A}G\,\mathrm{d}\bar W_s^1 \quad \text{for all } t\geq 0,\ \bar{\mathbb{P}}^1\text{-a.s.}$$

On the other hand, we recall that the process $(X^{x,\tilde{u}})_{t\geq 0}$ solves the controlled equation

$$X_t^{x,\tilde{u}} \;=\; e^{tA}x + \int_0^t e^{(t-s)A}F(X_s^{x,\tilde{u}}, \tilde{u}_s)\,\mathrm{d}s + \int_0^t e^{(t-s)A}G\,\mathrm{d}W_s^1 \quad \text{for all } t \geq 0,\ \mathbb{P}\text{-a.s.}$$

So, in particular, there exists a $\mathbb{P}$-null set $N \subset \Omega$ such that the above equality holds for all $t \geq 0$ and for every $\omega \notin N$. Therefore, there exists a $\bar{\mathbb{P}}^2$-null set $\bar{N}^2 \in \bar{\mathcal{F}}^2$ such that, for every $\omega^2 \notin \bar{N}^2$,

$$X_t^{x,\tilde{u}}(\cdot,\omega^2) \;=\; e^{tA}x + \int_0^t e^{(t-s)A}F(X_s^{x,\tilde{u}}(\cdot,\omega^2), \tilde{u}_s(\cdot,\omega^2))\,\mathrm{d}s + \int_0^t e^{(t-s)A}G\,\mathrm{d}W_s^1 \quad \text{for all } t \geq 0,\ \bar{\mathbb{P}}^1\text{-a.s.},$$

which can be rewritten in terms of $\tilde{u}^{\omega^2}$ as

$$X_t^{x,\tilde{u}}(\cdot,\omega^2) \;=\; e^{tA}x + \int_0^t e^{(t-s)A}F(X_s^{x,\tilde{u}}(\cdot,\omega^2), \tilde{u}_s^{\omega^2})\,\mathrm{d}s + \int_0^t e^{(t-s)A}G\,\mathrm{d}W_s^1, \quad \text{for all } t \geq 0,\ \bar{\mathbb{P}}^1\text{-a.s..}$$

Then, we see that, for every $\omega^2 \notin \bar{N}^2$, the two processes $(\bar{X}^{x,\tilde{u}^{\omega^2}})_{t\geq 0}$ and $(X^{x,\tilde{u}}(\cdot,\omega^2))_{t\geq 0}$ solve the same equation. By pathwise uniqueness, it follows that, for every $\omega^2 \notin \bar{N}^2$, $(\bar{X}^{x,\tilde{u}^{\omega^2}})_{t\geq 0}$ and $(X^{x,\tilde{u}}(\cdot,\omega^2))_{t\geq 0}$ are indistinguishable. An application of Fubini's Theorem yields

$$J^\beta(x,\tilde{u}) \;=\; \int_{\bar{\Omega}^2} \bar{\mathbb{E}}^1\Big[\int_0^\infty e^{-\beta t}\ell\big(\bar{X}_t^{x,\tilde{u}^{\omega^2}}, \tilde{u}_t^{\omega^2}\big)\,\mathrm{d}t\Big]\,\bar{\mathbb{P}}^2(\mathrm{d}\omega^2) \;=\; \bar{\mathbb{E}}^2\big[\bar{J}^\beta\big(x,\tilde{u}^{\omega^2}\big)\big] \;\geq\; \bar{v}^\beta(x).$$

Recalling that $J^\beta(x,u) = J^\beta(x,\tilde{u})$, the claim follows taking the infimum over all $u \in \mathcal{U}$. □

3. Randomised optimal control problems

In this section, we formulate the randomised versions (see [17]) of both the infinite and the finite horizon stochastic optimal control problems introduced in Section 2.

We consider the same probabilistic setting as in Section 2.3. In particular, we adopt the same notation: $(\Omega,\mathcal{F},\mathbb{P})$, W^1, W^2, $(\mathcal{F}_t)$ and $\mathcal{U}$. Progressive measurability of processes will always be intended with respect to the filtration $(\mathcal{F}_t)$.

By $\mathcal{A}_n$, we denote the family of progressively measurable processes α with values in U such that $|\alpha| \leq n$, $\mathbb{P}\otimes \mathrm{d}t$-almost surely. Moreover, $\mathcal{A} := \bigcup_{n\in\mathbb{N}} \mathcal{A}_n$ is the set of progressively measurable and essentially bounded processes with values in U.

State process. Given $(x,a) \in H \times U$ and $\alpha \in \mathcal{A}$, we consider the system of controlled stochastic differential equations:

$$\begin{cases} \mathrm{d}\mathfrak{X}_t &= A\mathfrak{X}_t\mathrm{d}t + F(\mathfrak{X}_t,\mathfrak{I}_t)\mathrm{d}t + G\mathrm{d}W_t^1, \quad \mathfrak{X}_0 \;=\; x, \\ \mathrm{d}\mathfrak{I}_t &= R\alpha_t\mathrm{d}t + R\mathrm{d}W_t^2, \qquad\qquad\qquad\quad\;\; \mathfrak{I}_0 \;=\; a. \end{cases} \tag{3.1}$$

On F and G, we impose the same assumptions as in Section 2, while on R we impose the following:

(A.8) $R\colon U \to U$ is a trace class injective linear operator with dense image.

Proposition 3.1. *Assume (A.1)–(A.5) and (A.8). Then, for any $(x,a) \in H\times U$ and $\alpha \in \mathcal{A}$, there exists a unique (up to indistinguishability) pair of processes $\mathfrak{X}^{x,a,\alpha} = (\mathfrak{X}_t^{x,a,\alpha})_{t\geq 0}$ and $\mathfrak{I}^{a,\alpha} = (\mathfrak{I}_t^{a,\alpha})_{t\geq 0}$ (the process $\mathfrak{I}^{a,\alpha}$ is independent of x) such that:*

- $\mathfrak{I}^{a,\alpha}$ *is given by*

$$\mathfrak{I}_t^{a,\alpha} = a + \int_0^t R\alpha_s \,\mathrm{d}s + RW_t^2 \quad \textit{for all } t \geq 0,\ \mathbb{P}\text{-a.s.}, \tag{3.2}$$

thus satisfies the second equation in (3.1) *and belongs to* $L_{\mathcal{P}}^{p,\mathrm{loc}}(\Omega; C([0,+\infty[; U))$ *for all* $p \geq 1$*;*

- $\mathfrak{X}^{x,a,\alpha}$ *belongs to* $L_{\mathcal{P}}^{p,\mathrm{loc}}(\Omega; C([0,+\infty[; H))$ *for all* $p \geq 1$ *and is a mild solution of the first equation in* (3.1)*, that is:*

$$\mathfrak{X}_t^{x,a,\alpha} = e^{tA}x + \int_0^t e^{(t-s)A} F(\mathfrak{X}_s^{x,a,\alpha}, \mathfrak{I}_s^{a,\alpha}) \,\mathrm{d}s + \int_0^t e^{(t-s)A} G \,\mathrm{d}W_s^1 \quad \textit{for all } t \geq 0,\ \mathbb{P}\text{-a.s.}$$

Moreover, for every $0 < \rho < \frac{1}{2} - \gamma$ *(with* γ *as in Asm. (A.3)), and for any* $T > 0$*, the following estimate holds:*

$$\sup_{t\in[0,T]} t^\rho \,\mathbb{E}||\mathfrak{X}_t^{x,a,\alpha}||_{\mathcal{D}((\delta I - A)^\rho)} \ \leq\ c \tag{3.3}$$

for some positive constant c*, depending only on* T*,* x*,* ρ*, and on the constants introduced in Assumptions (A.1)–(A.5), but independent of* a *and* α*.*

Proof. This is quite a classical result (see [6]) and the randomisation framework has nothing special here (we just formulate the result in the case in which we need it). For the sake of completeness, we report the proof of estimate (3.3). We have

$$\begin{aligned}
\mathbb{E}\|\mathfrak{X}_t\|_{\mathcal{D}(A^\rho)} &\leq \mathbb{E}\|e^{tA}x\|_{\mathcal{D}(A^\rho)} + \mathbb{E}\left\|\int_0^t e^{(t-s)A}F(\mathfrak{X}_s,\mathfrak{I}_s)\,\mathrm{d}s\right\|_{\mathcal{D}((\delta I-A)^\rho)} + \mathbb{E}\left\|\int_0^t e^{(t-s)A}G\,\mathrm{d}W_s^1\right\|_{\mathcal{D}((\delta I-A)^\rho)} \\
&\leq \frac{|x|_H}{t^\rho} + k\int_0^t (t-s)^{-\rho}\mathbb{E}|F(\mathfrak{X}_s,\mathfrak{I}_s)|_H\,\mathrm{d}t + \left(\mathbb{E}\left\|\int_0^t e^{(r-s)A}G\mathrm{d}W_s^1\right\|^2_{\mathcal{D}((\delta I-A)^\rho)}\right)^{\frac{1}{2}} \\
&\leq \frac{|x|_H}{t^\rho} + kC_F t^{1-\rho}(1+\mathbb{E}\sup_{t\in[0,T]}|\mathfrak{X}_t|_H) + L\Big(\mathbb{E}\int_0^t (t-r)^{-2\rho-2\gamma}\,\mathrm{d}r\Big)^{1/2} \\
&\leq \frac{|x|_H}{t^\rho} + kCt^{1-\rho} + Lt^{\frac{1}{2}-\rho-\gamma}.
\end{aligned}$$

Thus, for every $\rho > 0$ such that $\rho + \gamma < \frac{1}{2}$, we deduce estimate (3.3). □

Remark 3.2. Note that $\mathfrak{X}^{x,a,\alpha} = X^{x,\mathfrak{I}^{a,\alpha}}$, with $X^{x,u}$ defined in Proposition 2.4. Indeed, for every $a \in U$ and $\alpha \in \mathcal{A}$, the process $\mathfrak{I}^{a,\alpha}$ belongs to $\mathcal{U}$. Thus, the analogue of estimates (2.3) and (2.4) hold for $\mathfrak{X}^{x,a,\alpha}$ (uniformly with respect to a and α).

Once more we define, in this new setting, the finite and infinite horizon costs as well as the corresponding value functions.

Infinite horizon control problem. For every $\beta > 0$, and for any $(x,a) \in H \times U$, $\alpha \in \mathcal{A}$, the infinite horizon cost functional is

$$J^{\beta,\mathcal{R}}(x,a,\alpha) \ :=\ \mathbb{E}\Big[\int_0^\infty e^{-\beta s}\,\ell(\mathfrak{X}_s^{x,a,\alpha},\mathfrak{I}_s^{a,\alpha})\,\mathrm{d}t\Big],$$

with the corresponding value function

$$v^{\beta,\mathcal{R}}(x,a) \ := \ \inf_{\alpha\in\mathcal{A}} J^{\beta,\mathcal{R}}(x,a,\alpha).$$

Finite horizon control problem. For every $T>0$, and for any $(x,a)\in H\times U$, $\alpha\in\mathcal{A}$, the finite horizon cost is

$$J^{\beta,T,\mathcal{R}}(x,a,\alpha) \ := \ \mathbb{E}\bigg[\int_0^T e^{-\beta s}\ell(\mathfrak{X}_s^{x,a,\alpha},\mathfrak{I}_s^{a,\alpha})\,\mathrm{d}s + e^{-\beta T}\phi(\mathfrak{X}_T^{x,a,\alpha})\bigg],$$

with the corresponding value function

$$v^{\beta,T,\mathcal{R}}(x,a) \ := \ \inf_{\alpha\in\mathcal{A}} J^{\beta,T,\mathcal{R}}(x,a,\alpha).$$

Next statement entitles us to study the (asymptotic) behaviour of $v^{\beta,\mathcal{R}}$ and $v^{\beta,T,\mathcal{R}}$ instead of $\bar v^{\beta}$ and $\bar v^{\beta,T}$ (or of v^{β} and $v^{\beta,T}$). Moreover, it implies that $v^{\beta,\mathcal{R}}$ and $v^{\beta,T,\mathcal{R}}$ do not depend on their last argument.

Proposition 3.3. *Suppose that Assumptions (A.1)–(A.8) hold. Then, we have (recalling Prop. 2.5):*

(i) *For every $\beta>0$, $\bar v^{\beta}(x)=v^{\beta}(x)=v^{\beta,\mathcal{R}}(x,a)$, for all $(x,a)\in H\times U$. In particular, the function $v^{\beta,\mathcal{R}}$ is independent of its second argument.*
(ii) *For every $T>0$ and $\beta\geq 0$, $\bar v^{\beta,T}(x)=v^{\beta,T}(x)=v^{\beta,T,\mathcal{R}}(x,a)$ for all $(x,a)\in H\times U$. In particular, the function $v^{\beta,T,\mathcal{R}}$ is independent of its second argument.*

Proof. We only report the proof of the first statement, as item (ii) can be proved in an analogous way. We split the proof of (i) into some steps.
Step 1. We are going to prove that, for every fixed $T>0$, $x\in H$ and $u\in\mathcal{U}$,

$$\text{If } u^n \xrightarrow{\mathbb{P}} u \ \text{ in } \Omega\times[0,T], \quad \text{then } X^{x,u^n}\to X^{x,u} \text{ in } L^2_{\mathcal{P}}(\Omega\times[0,T];H). \tag{3.4}$$

We set $\bar X^n = X^{x,u^n}-X^{x,u}$, thus

$$\bar X^n_t = \int_0^t e^{(t-s)A}[F(X^{x,u^n}_s,u^n_s)-F(X^{x,u}_s,u_s)]\,\mathrm{d}s \quad \text{for all } t\in[0,T],\ \mathbb{P}\text{-a.s.}$$

and there is a positive constant C (that depends only on the Lipschitz constant of F and on T) such that

$$|\bar X^n_t|^2 \leq C\left[\int_0^t |\bar X^n_s|^2\,\mathrm{d}s + \int_0^t |F(X^{x,u}_s,u^n_s)-F(X^{x,u}_s,u_s)|^2\,\mathrm{d}s\right].$$

Thus,

$$\mathbb{E}\sup_{t\in[0,r]}|\bar X^n_t|^2 \leq C\int_0^r \mathbb{E}\sup_{\sigma\in[0,s]}|\bar X^n_\sigma|^2\,\mathrm{d}s + C\,\mathbb{E}\int_0^T |F(X^{x,u}_s,u^n_s)-F(X^{x,u}_s,u_s)|^2\,\mathrm{d}s.$$

Hence, by Gronwall's lemma

$$\mathbb{E}\sup_{t\in[0,r]}|\bar X^n_t|^2 \leq Ce^{CT}\,\mathbb{E}\int_0^T |F(X^{x,u}_s,u^n_s)-F(X^{x,u}_s,u_s)|^2\,\mathrm{d}s.$$

Note that by (A.4), we have

$$|F(X_s^{x,u}, u_s^n) - F(X_s^{x,u}, u_s)| \leq 2C_F(1 + |X_s^{x,u}|).$$

Henceforth, thanks to Proposition 2.4, we can apply Lebesgue's dominated convergence Theorem to derive (3.4). Observe that, actually, we have proved the following:

$$\lim_{n\to+\infty} \mathbb{E} \sup_{t\in[0,r]} |X_t^{x,u^n} - X_t^{x,u}|^2 \to 0.$$

Step 2. Fix $x \in H$, $u \in \mathcal{U}$. We show that

$$\text{If, } \forall T > 0,\ u^n \xrightarrow{\mathbb{P}\otimes \mathrm{d}t} u \ \text{ in } \Omega \times [0,T] \quad \text{then } \lim_{n\to\infty} J^\beta(x, u^n) = J^\beta(x, u). \tag{3.5}$$

Thanks to the presence of the discount term $e^{-s\beta}$ and the boundedness of ℓ, see (A.6), it is enough to prove that

$$\lim_{n\to\infty} \mathbb{E} \int_0^T \ell(X_s^{x,u^n}, u_s^n)\, \mathrm{d}s = \mathbb{E} \int_0^T \ell(X_s^{x,u}, u_s)\, \mathrm{d}s.$$

Indeed, for every $T > 0$,

$$|J^\beta(x, u^n) - J^\beta(x, u)| \leq \mathbb{E} \int_0^T \left|\ell(X_s^{x,u^n}, u_s^n) - \ell(X_s^{x,u}, u_s)\right| \mathrm{d}s + 2M_\ell \int_T^{+\infty} e^{-\beta s}\, \mathrm{d}s.$$

From Step 1 and the boundedness of ℓ, we obtain

$$\lim_{n\to\infty} \mathbb{E} \int_0^T \ell(X_s^{x,u^n}, u_s^n)\, \mathrm{d}s = \mathbb{E} \int_0^T \ell(X_s^{x,u}, u_s)\, \mathrm{d}s.$$

By the previous considerations, we deduce the validity of (3.5).
Step 3. Let $\mathcal{U}_{\mathrm{bdd}}$ denote the subset of $\mathcal{U}$ of all uniformly bounded processes. By the previous step, we deduce the following equality:

$$\inf_{u\in\mathcal{U}} J^\beta(x, u) \ = \ \inf_{u\in\mathcal{U}_{\mathrm{bdd}}} J^\beta(x, u) \quad \text{for every } x \in H. \tag{3.6}$$

As a matter of fact, given $u \in \mathcal{U}$, it is enough to apply (3.5) to the sequence $u^n = I_{B(0,n)}(|u|)u$, $n \in \mathbb{N}$.
Step 4. Fix $a \in U$. Given $u \in \mathcal{U}_{\mathrm{bdd}}$, we claim that there exists a sequence $(\alpha^n)_n \subset \mathcal{A}$ such that (see (3.2) for the definition of $\mathfrak{I}^{a,\alpha^n}$)

$$\mathfrak{I}^{a,\alpha^n} \to u \qquad \text{in } L^2([0,T] \times \Omega; U) \quad \forall T > 0.$$

The above follows if we prove that the affine set

$$\{\mathfrak{I}^{a,\alpha} \colon \alpha \in \mathcal{A}\}$$

is dense in $L^2_{\mathcal{P}}([0,T] \times \Omega; U)$. Since $\mathfrak{I}^{a,\alpha} := \hat{\mathfrak{I}}^\alpha + a + RW^2$ with $\hat{\mathfrak{I}}_t^\alpha := \int_0^t R\,\alpha_s\, \mathrm{d}s$, it is enough to prove that the linear subspace $\{\hat{\mathfrak{I}}^\alpha \colon \alpha \in \mathcal{A}\}$ is dense in $L^2_{\mathcal{P}}([0,T] \times \Omega; U)$.

Now, consider the linear space generated by the set of functions

$$\{\eta I_{[t_0,T)} : t_0 \in [0,T),\ \eta : \Omega \to U\ \mathcal{F}_{t_0}\text{-meas. and bounded}\}.$$

As is well known, such linear space is dense in $L^2([0,T] \times \Omega; U)$ (note that it coincides with the linear space generated by the set of functions of the form: $\eta I_{[t_0,t_1)}$, with $0 \leq t_0 < t_1 \leq T$, η $\mathcal{F}_{t_0}$-measurable and bounded).

Thus, it is enough to prove that, for all $t_0 \in [0,T)$ and every $\mathcal{F}_{t_0}$-measurable and bounded η, there exists a sequence $(\alpha^n)_n \subset \mathcal{A}$, such that:

$$\mathbb{E}\int_0^T |\eta I_{[t_0,T)}(t) - \hat{\mathfrak{I}}_t^{\alpha^n}|^2 \mathrm{d}t \to 0.$$

Now, let $(U_m)_m$ be a sequence of finite-dimensional subspaces of U such that $U_m \subset U_{m+1}$ and $\bigcup_{m=1}^\infty U_m = U$ and define $E_m = RU_m$. Clearly, $E_m \subset E_{m+1}$, $\bigcup_{m=1}^\infty E_m$ is dense in U and R is invertible from U_m to E_m (thus bounded with bounded inverse).

We can always assume that η takes values in E_m for a suitable m, possibly approximating η by its projection on E_m.

Now take $\alpha_s^n = nI_{[t_0,t_0+\frac{1}{n}]}(s)R^{-1}\eta$, $n \in \mathbb{N}$, so that for every n we have that $\alpha^n \in \mathcal{A}$. Moreover $\hat{\mathfrak{I}}^{\alpha^n} = n(1-(t-t_0))\eta I_{[t_0,t_0+1/n[} + \eta I_{[t_0+1/n,T]}$ and consequently

$$\mathbb{E}\int_0^T |\eta I_{[t_0,T)}(t) - \hat{\mathfrak{I}}_t^{\alpha^n}|^2 \mathrm{d}t = \frac{1}{n}\mathbb{E}|\eta|^2.$$

Conclusion. Recalling from Remark 3.2 that $J^{\beta,\mathcal{R}}(x,a,\alpha) = J^\beta(x, \mathfrak{I}^{a,\alpha})$, we immediately see that $v^\beta(x) \leq v^{\beta,\mathcal{R}}(x,a)$.

On the other hand, given $u \in \mathcal{U}_{\mathrm{bdd}}$, by Step 4 and a standard diagonal argument it is possible to construct a sequence $(\alpha^n)_n \in \mathcal{A}$ such that $\forall T > 0$, $\mathfrak{I}^{a,\alpha^n} \xrightarrow{\mathbb{P}\otimes \mathrm{d}t} u$ in $\Omega \times [0,T]$. Thus, by Step 2,

$$v^{\beta,\mathcal{R}}(x,a) \leq \lim_{n\to\infty} J^{\beta,\mathcal{R}}(x,a,\alpha^n) = \lim_{n\to\infty} J^\beta(x, \mathfrak{I}^{a,\alpha^n}) = J^\beta(x,u).$$

Hence, using also (3.6), we see that the reverse inequality $v^\beta(x) \geq v^{\beta,\mathcal{R}}(x,a)$ holds as well. □

4. BSDE representation of the value functions

In this section, we obtain a nonlinear Feynman–Kac formula for the value functions of both the infinite and the finite horizon stochastic optimal control problems. This representation will be the essential tool to study the asymptotic behaviour of the value functions.

4.1. Elliptic BSDE: infinite horizon optimal control problem

We denote by $\mathfrak{X}^{x,a}$ and $\mathfrak{I}^a$ the solution to (3.1) corresponding to $\alpha \equiv 0$. For any $\beta > 0$ and $n \in \mathbb{N}$, we consider the following standard BSDE with infinite terminal time and generator being Lipschitz with respect to the martingale variable (Z,Γ) and strongly dissipative with respect to Y:

$$\begin{aligned} Y_t^{x,a,\beta,n} = Y_T^{x,a,\beta,n} - \beta \int_t^T Y_s^{x,a,\beta,n}\,\mathrm{d}s + \int_t^T \ell(\mathfrak{X}_s^{x,a}, \mathfrak{I}_s^a)\,\mathrm{d}s - n\int_t^T |\Gamma_s^{x,a,\beta,n}|\,\mathrm{d}s \\ - \int_t^T Z_s^{x,a,\beta,n}\,\mathrm{d}W_s^1 - \int_t^T \Gamma_s^{x,a,\beta,n}\,\mathrm{d}W_s^2, \qquad 0\le t\le T<+\infty. \end{aligned} \tag{4.1}$$

Proposition 4.1. *Suppose that Assumptions (A.1)–(A.8) hold. Then*

(i) For every $(x,a)\in H\times U$, there exists a unique solution $(Y^{x,a,\beta,n},\Gamma^{x,a,\beta,n},Z^{x,a,\beta,n})$ of the BSDE (4.1) with $Y^{x,a,\beta,n}$ bounded, continuous and progressively measurable, $Z^{x,a,\beta,n}$ belonging to $L^{2,\mathrm{loc}}_{\mathcal{P}}(\Omega\times[0,+\infty[;\mathfrak{X}i^*)$ and $\Gamma^{x,a,\beta,n}$ belonging to $L^{2,\mathrm{loc}}_{\mathcal{P}}(\Omega\times[0,+\infty[;U^*)$.

(ii) The following bounds hold (uniformly with respect to n):

$$|Y_t^{x,a,\beta,n}| \le \frac{M_\ell}{\beta}, \tag{4.2}$$

$$|Z_t^{x,a,\beta,n}| \le \frac{L_\ell|G|}{\mu}, \tag{4.3}$$

$$\mathbb{E}\int_0^{+\infty} e^{-2\beta s}|\Gamma_s^{x,a,\beta,n}|^2\,\mathrm{d}s<\infty. \tag{4.4}$$

(iii) For every $(x,a)\in H\times U$, if we define the value function for the infinite horizon problem, in the randomised framework and with bounded set of controls, namely

$$v^{\beta,n,\mathcal{R}}(x,a) := \inf_{\alpha\in\mathcal{A}_n} J^{\beta,\mathcal{R}}(x,a,\alpha), \tag{4.5}$$

then, for all $t\ge 0$:

$$Y_t^{x,a,\beta,n} = v^{\beta,n,\mathcal{R}}(\mathfrak{X}_t^{x,a},\mathfrak{I}_t^a), \qquad \mathbb{P}\text{- a.s.} \tag{4.6}$$

In particular,

$$Y_0^{x,a,\beta,n} \;=\; v^{\beta,n,\mathcal{R}}(x,a) \;=\; \inf_{\alpha\in\mathcal{A}_n} J^{\beta,\mathcal{R}}(x,a,\alpha). \tag{4.7}$$

Proof. Equation (4.1) fulfils the standard assumptions in [15], Lemma 2.1, thus we already know that (i), as well as estimates (4.2) and (4.4) in (ii), hold true.

It remains to prove the uniform estimate (4.3). To do that we need to introduce finite horizon approximations of (4.1) and then to smooth up its coefficients.

Denote by $(Y^{x,M},\Gamma^{x,M},Z^{x,M})$ the solution to the finite horizon BSDE on $[0,M]$, with $M\in\mathbb{N}$:

$$\begin{aligned} Y_t^{x,M} = &-\beta\int_t^M Y_s^{x,M}\,\mathrm{d}s + \int_t^M \ell(\mathfrak{X}_s^{x,a},\mathfrak{I}_s^a)\,\mathrm{d}s - n\int_t^M |\Gamma_s^{x,M}|\,\mathrm{d}s \\ &- \int_t^M Z_s^{x,M}\,\mathrm{d}W_s^1 - \int_t^T \Gamma_s^{x,M}\,\mathrm{d}W_s^2, \forall t\in[0,M], \end{aligned}$$

where we have omitted some parameters in the notation to keep it readable. We have (see again [15] Lem. 2.1):

$$\mathbb{E}\int_0^T |Z_t^{x,M} - Z_t^{x,a,\beta,n}|^2 \mathrm{d}t \to 0 \quad \text{as } M \to \infty.$$

Thus, it is enough to prove (4.3) for $Z^{x,M}$. To this end, we construct sequences $(F_k(\cdot))_{k\in\mathbb{N}}$ and $(\ell_k(\cdot,\cdot))_{k\in\mathbb{N}}$ of Gateaux differentiable functions converging pointwise to F and ℓ such that Assumptions (A.4), (A.5) and (A.6) hold with the same constants (see [11] for such a construction).

We introduce the notation $\rho_k(u) = \sqrt{|u|^2 + k^{-1}}$. Consider the regularised forward–backward system parametrised with final time M (we omit parameters in the equations to simplify notation):

$$\begin{cases} \mathrm{d}\mathfrak{X}_t = A\mathfrak{X}_t \mathrm{d}t + F_k(\mathfrak{X}_t, \mathfrak{I}_t)\mathrm{d}t + G\mathrm{d}W_t^1, & \mathfrak{X}_0 = x,\ t \in [0, M], \\ \mathrm{d}\mathfrak{I}_t = R\mathrm{d}W_t^2, & \mathfrak{I}_0 = a,\ t \in [0, M], \\ \mathrm{d}Y_t = -\beta Y_t \mathrm{d}t - \ell_k(\mathfrak{X}_t, \mathfrak{I}_t)\mathrm{d}t - n\rho_k(\Gamma_t)\mathrm{d}t - Z_t \mathrm{d}W_t^1 - \Gamma_t \mathrm{d}W_t^2, & Y_M = 0. \end{cases} \tag{4.8}$$

Denote by $(\mathfrak{X}^{x,k}, \mathfrak{I}, Y^{x,k,M}, Z^{x,k,M}, \Gamma^{x,k,M})$ the solution to the above forward–backward system (for readability sake we do not report parameter a). Proceeding as in [11], it is easy to verify by parameter depending contraction theorem that, as $k \to \infty$,

$$\mathbb{E}\int_0^M |Z_t^{x,k,M} - Z_t^{x,M}|^2 \mathrm{d}t \to 0.$$

Thus, once more, it is enough to prove (4.3) for $Z^{x,k,M}$.

Set $\nu^M(\tau, x, a) = Y_0^{x,k,M-\tau}$. Then, recalling that all the coefficients in (4.8) are differentiable we have the following identifications (see [12]):

$$Y_t^{x,k,M} = \nu^M(t, \mathfrak{X}_t^{x,k}, \mathfrak{I}_t), \qquad Z_t^{x,k,M} = \nabla_x \nu^M(t, \mathfrak{X}_t^{x,k}, \mathfrak{I}_t)\, G.$$

Thus, to obtain (4.3), it is enough to prove that, for all $M > 0$,

$$|\nabla_x Y_\tau^{x,k,M}| \leq \frac{L_\ell}{\mu + \beta}.$$

Differentiating (4.8) with respect to x in the direction ξ, we obtain

$$\begin{cases} \mathrm{d}\nabla_x^\xi \mathfrak{X}_t = A\nabla_x^\xi \mathfrak{X}_t \mathrm{d}t + \nabla_x F_k(\mathfrak{X}_t, \mathfrak{I}_t)\nabla_x^\xi \mathfrak{X}_t \mathrm{d}t, \\ \nabla_x^\xi \mathfrak{X}_\tau = \xi, \\ -\mathrm{d}\nabla_x^\xi Y_t = -\beta \nabla_x^\xi Y_t \mathrm{d}t - \nabla_x \ell_k(\mathfrak{X}_t, \mathfrak{I}_t)\nabla_x^\xi \mathfrak{X}_t \mathrm{d}t - n\nabla_x^\xi \rho_k(U_t)\nabla_x^\xi U_t \mathrm{d}t - \nabla_x^\xi Z_t \mathrm{d}W_t^1 - \nabla_x^\xi \Gamma_t \mathrm{d}W_t^2, \\ \nabla_x^\xi Y_\ell = 0. \end{cases}$$

Exploiting the dissipativity of A and F_k (namely (A.1) and (A.4)) (see [15]), we obtain $|\nabla_x^\xi \mathfrak{X}_t| \leq e^{-\mu(t-\tau)}|\xi|$ and, again by a standard Girsanov argument, we deduce that, with respect to a suitable probability $\tilde{\mathbb{P}}$,

$$|\nabla_x^\xi Y_\tau| = \left| \tilde{\mathbb{E}} \int_\tau^\ell e^{-\beta(t-\tau)} \nabla_x \ell_k(\mathfrak{X}_t, \mathfrak{I}_t)\nabla_x^\xi \mathfrak{X}_t \mathrm{d}t \right| \leq \frac{L_\ell}{\mu + \beta}.$$

Thus, estimate (4.3) holds.

Let us now prove (4.7). For any $\alpha \in \mathcal{A}_n$ and every $T > 0$, by Girsanov's theorem there exists a probability measure $\mathbb{P}^\alpha$ on $(\Omega, \mathcal{F}_T)$, equivalent to $\mathbb{P}$, under which $(W_t^1)_{t\in[0,T]}$ and $(W_t^2 - \int_0^t \alpha_s ds)_{t\in[0,T]}$ are independent Wiener processes. So, in particular, the pair $(\mathfrak{X}_t^{x,a,\alpha}, \mathfrak{I}_t^a)_{t\in[0,T]}$ under $\mathbb{P}$ has the same law as the pair $(\mathfrak{X}_t^{x,a}, \mathfrak{I}_t^a)_{t\in[0,T]}$ under $\mathbb{P}^\alpha$. Therefore, by equation (4.1), we obtain (denoting by $\mathbb{E}^\alpha$ the expectation with respect to $\mathbb{P}^\alpha$):

$$
\begin{aligned}
Y_0^{x,a,\beta,n} &= \mathbb{E}^\alpha\Big[e^{-\beta T}\, Y_T^{x,a,\beta,n} + \int_0^T e^{-\beta s}\,\ell(\mathfrak{X}_s^{x,a}, \mathfrak{I}_s^a)\,\mathrm{d}s + \int_0^T e^{-\beta s}\big(\Gamma_s^{x,a,\beta,n}\alpha_s - n|\Gamma_s^{x,a,\beta,n}|\big)\mathrm{d}s\Big] \\
&\le e^{-\beta T}\,\frac{M_\ell}{\beta} + \mathbb{E}^\alpha\Big[\int_0^T e^{-\beta s}\,\ell(\mathfrak{X}_s^{x,a}, \mathfrak{I}_s^a)\,\mathrm{d}s\Big] = e^{-\beta T}\,\frac{M_\ell}{\beta} + \mathbb{E}\Big[\int_0^T e^{-\beta s}\,\ell(\mathfrak{X}_s^{x,a,\alpha}, \mathfrak{I}_s^{a,\alpha})\,\mathrm{d}s\Big],
\end{aligned}
$$

where the inequality follows from estimate (4.2) and $|\alpha| \le n$. Then, letting $T \to +\infty$, we find

$$
Y_0^{x,a,\beta,n} \le J^{\beta,\mathcal{R}}(x,a,\alpha).
$$

It remains to prove the reverse inequality. To this end, take $\alpha_s^* = n\Gamma_s^{x,a,\beta,n}|\Gamma_s^{x,a,\beta,n}|^{-1}1_{\{\Gamma_s^{x,a,\beta,n}\neq 0\}}$. Then, from the same calculations as above, we obtain

$$
\begin{aligned}
Y_0^{x,a,\beta,n} = \mathbb{E}^{\alpha^*}\Big[e^{-\beta T} Y_T^{x,a,\beta,n} + \int_0^T e^{-\beta s}\,\ell(\mathfrak{X}_s^{x,a}, \mathfrak{I}_s^a)\mathrm{d}s\Big] &\ge e^{-\beta T}\,\frac{M_\ell}{\beta} + \mathbb{E}^{\alpha^*}\Big[\int_0^T e^{-\beta s}\,\ell(\mathfrak{X}_s^{x,a}, \mathfrak{I}_s^a)\mathrm{d}s\Big] \\
&= e^{-\beta T}\,\frac{M_\ell}{\beta} + \mathbb{E}\Big[\int_0^T e^{-\beta s}\,\ell(\mathfrak{X}_s^{x,a,\alpha^*}, \mathfrak{I}_s^{a,\alpha^*})\mathrm{d}s\Big].
\end{aligned}
$$

Letting $T \to +\infty$, we get $Y_0^{x,a,\beta,n} \ge J^{\beta,\mathcal{R}}(x,a,\alpha^*)$, which concludes the proof of (4.7).
Finally, concerning (4.6), we note that by the uniqueness of equations (3.1) and (4.1), it follows that for every $t \in [0,T]$, we have $Y_s^{x,a,\beta,n} = Y_s^{\mathfrak{X}_t^{x,a},\mathfrak{I}_t^a,\beta,n}$, for all $s \in [t,T]$, $\mathbb{P}$-a.s., and the same property holds for $Z^{x,a,\beta,n}$ and $\Gamma^{x,a,\beta,n}$. From this, together with equality (4.7), we see that (4.6) holds. □

Taking into account the definition of $v^{\beta,n,\mathcal{R}}(x,a)$ and identification (4.5), Proposition 2.5 and 4.1 yield the following.

Proposition 4.2. *If Assumptions (A.1)–(A.8) are verified, then the following properties hold for every $x, x' \in H$, $a \in A$:*

$$
|v^{\beta,n,\mathcal{R}}(x,a)| \le \frac{M_\ell}{\beta}, \quad |v^{\beta,n,\mathcal{R}}(x,a) - v^{\beta,n,\mathcal{R}}(x',a)| \le L_\ell\,|x - x'|, \tag{4.9}
$$

$$
v^{\beta,n,\mathcal{R}}(x,a) \downarrow \inf_{\alpha\in\mathcal{A}} J^{\beta,\mathcal{R}}(x,a,\alpha) = v^{\beta,\mathcal{R}}(x) = v^\beta(x), \tag{4.10}
$$

$$
|v^\beta(x)| \le \frac{M_\ell}{\beta}, \quad |v^\beta(x) - v^\beta(x')| \le L_\ell\,|x - x'|. \tag{4.11}
$$

Our aim is to characterise $v^{\beta,\mathcal{R}}$ in terms of the maximal solution to the following non-standard 'constrained' infinite horizon BSDE involving a reflection term:

$$
Y_t = Y_T - \beta\int_t^T Y_s\,\mathrm{d}s + \int_t^T \ell(\mathfrak{X}_s^{x,a}, \mathfrak{I}_s^a)\,\mathrm{d}s + K_t - K_T - \int_t^T Z_s\,\mathrm{d}W_s^1, \quad \forall 0 \le t \le T, \ \ \mathbb{P}\text{-a.s.} \tag{4.12}
$$

Definition 4.3. A solution to (4.12) is a triple (Y,Z,K) such that Y is a progressively measurable bounded process with continuous trajectories, $Z \in L_{\mathcal{P}}^{2,\mathrm{loc}}(\Omega \times [0,+\infty[;\mathfrak{X}i^*)$, $K \in \mathcal{K}^{2,\mathrm{loc}}$ and (4.12) holds.

Proposition 4.4. *Assume (A.1)–(A.8). Then, for every $\beta > 0$ and $(x,a) \in H \times U$, there exists a triple $(Y^{x,a,\beta}, Z^{x,a,\beta}, K^{x,a,\beta})$ of progressively measurable processes such that:*

1. *$Y^{x,a,\beta}$ is the non-increasing limit of $(Y^{x,a,\beta,n})_n$ (see (4.1)); moreover, the following holds, for all $t \geq 0$,*

$$Y_t^{x,a,\beta} = v^\beta(\mathfrak{X}_t^{x,a}), \quad |Y_t^{x,a,\beta}| \leq \frac{M_\ell}{\beta}, \qquad \mathbb{P}\text{-}a.s. \tag{4.13}$$

2. *The following estimate holds, for all $t \geq 0$:*

$$|Z_t^{x,a,\beta}| \leq \frac{L_\ell}{\mu}, \quad \mathbb{P}\text{-}a.s., \tag{4.14}$$

and, for all $T > 0$:

$$Z^{x,a,\beta,n} \rightharpoonup Z^{x,a,\beta} \quad in\ L^2_{\mathcal{P}}(\Omega \times [0,T]; \mathfrak{X}i^*).$$

3. *The following convergences take place for all $T > 0$:*

$$\Gamma^{x,a,\beta,n} \rightharpoonup 0 \quad in\ L^2_{\mathcal{P}}(\Omega \times [0,T]; U^*),$$

$$n \int_0^T |\Gamma_s^{x,a,\beta,n}| \, \mathrm{d}s \rightharpoonup K_T^{x,a,\beta}\, in\ L^2(\Omega, \mathcal{F}_t, \mathbb{R}).$$

4. *$(Y^{x,a,\beta}, Z^{x,a,\beta}, K^{x,a,\beta})$ is the maximal solution of equation (4.12) in the sense that if there exists another solution $(\bar{Y}, \bar{Z}, \bar{K})$, then $Y_t^{x,a,\beta} \geq \bar{Y}_t^{x,a,\beta}$ for all $t \geq 0$, $\mathbb{P}$-a.s.*

Proof. The proof is essentially identical to the proof of Prop. 3.3 in [5] and we omit it. □

4.2. Results on the finite horizon case

We report here, for further use, the finite horizon analogue of the results stated in Prop. 4.4.

Definition 4.5. For every $\beta \geq 0$, $T > 0$ and $(x,a) \in H \times U$, a solution to the finite horizon BSDE on $[0,T]$:

$$Y_t = \phi(\mathfrak{X}_T^{x,a}) - \beta \int_t^T Y_s \, \mathrm{d}s + \int_t^T \ell(\mathfrak{X}_s^{x,a}, \mathfrak{I}_s^a) \, \mathrm{d}r - K_T + K_t - \int_t^T Z_s \, \mathrm{d}W_s^1 \tag{4.15}$$

is a triple (Y,Z,K) in $L^2_{\mathcal{P}}(\Omega; C([0,T];\mathbb{R})) \times L^2_{\mathcal{P}}(\Omega \times [0,T]; \mathfrak{X}i^*) \times \mathcal{K}^2(0,T)$, satisfying (4.15) for all $t \in [0,T]$, $\mathbb{P}$-a.s.

Now, for every $n \in \mathbb{N}$, $\beta \geq 0$, $T > 0$, $(x,a) \in H \times U$, let $(Y^{x,a,\beta,T,n}, Z^{x,a,\beta,T,n}, \Gamma^{x,a,\beta,T,n}) \in L^2_{\mathcal{P}}(\Omega; C([0,T];\mathbb{R})) \times L^2_{\mathcal{P}}(\Omega \times [0,T]; \mathfrak{X}i^*) \times L^2_{\mathcal{P}}(\Omega \times [0,T]; U^*)$ be the solution of the standard BSDE on $[0,T]$ (see [12]):

$$Y_t = \phi(\mathfrak{X}_T^{x,a}) - \beta \int_t^T Y_s \, \mathrm{d}s + \int_t^T \ell(\mathfrak{X}_s^{x,a}, \mathfrak{I}_s^a) \, \mathrm{d}s - n \int_t^T |\Gamma_s| \, \mathrm{d}s - \int_t^T \Gamma_s \, \mathrm{d}W_s^2 - \int_t^T Z_s \, \mathrm{d}W_s^1. \tag{4.16}$$

Then, similar to Proposition 4.4, we have the following result concerning the finite horizon case.

Proposition 4.6. *Assume (A.1)–(A.8). Then, for every $\beta \geq 0$, $T > 0$ and $(x,a) \in H \times U$, there exists $(Y^{x,a,\beta,T}, Z^{x,a,\beta,T}, K^{x,a,\beta,T})$ such that:*

1. *$(Y^{x,a,\beta,T}, Z^{x,a,\beta,T}, K^{x,a,\beta,T})$ is a solution to (4.15).*
2. *$Y^{x,a,\beta,T}$ is the non-increasing limit of $(Y^{x,a,\beta,T,n})_n$, and the following representation holds*

$$Y_t^{x,a,\beta,T} = v^{\beta,T-t}(\mathfrak{X}_t^{x,a}).$$

3. *$Z^{x,a,\beta,T,n}$ converges towards $Z^{x,a,\beta,T}$ weakly in $L^2_{\mathcal{P}}(\Omega \times [0,T]; \mathfrak{X}i^*)$.*
4. *$\Gamma^{x,a,\beta,T,n}$ converges towards 0 weakly in $L^2_{\mathcal{P}}(\Omega \times [t,T]; U^*)$.*
5. *For all $t \in [0,T]$, $n \int_0^t |\Gamma_r^{x,a,\beta,T,n}| \, \mathrm{d}r$ converges towards $K_t^{x,a,\beta,T}$ weakly in $L^2(\Omega, \mathcal{F}_t, \mathbb{R})$.*
6. *$Y^{x,a,\beta}$ is the maximal solution of equation (4.15) in the sense that given any other solution (Y', Z', K') of (4.15) then $Y'_t \leq Y_t$, for all $t \in [0,T]$, $\mathbb{P}$-a.s.*

Remark 4.7. Concerning the representation at point 2 of Proposition 4.6, this is a well-known result for the solution to the standard BSDE (4.16) and it is enough to pass to the limit as $n \to +\infty$.

Standard estimates on the value function $v^{\beta,T}$ imply that there exists a constant $c_{\beta,T}$, non-decreasing with respect to T, such that

$$|Y_t^{x,a,\beta,T}| \leq c_{\beta,T}(1 + |\mathfrak{X}_t^{x,a}|).$$

To proceed with our arguments, we need to prove that the maximal solution $(Y^{x,a,\beta}, Z^{x,a,\beta}, K^{x,a,\beta})$ of the infinite horizon BSDE (4.12) (see Prop. 4.4), when restricted to an arbitrary compact subset $[0,T]$, coincides with the maximal solution of equation (4.15) with final condition ϕ replaced by v^β itself, namely of the BSDE:

$$Y_t = v^\beta(\mathfrak{X}_T^{x,a}) - \beta \int_t^T Y_s \, \mathrm{d}s + \int_t^T \ell(\mathfrak{X}_s^{x,a}, \mathfrak{I}_s^a) \, \mathrm{d}s - K_T + K_t - \int_t^T Z_s \, \mathrm{d}W_s^1. \tag{4.17}$$

In the existing literature, this result is a straightforward consequence of the characterisation of v^β as the unique viscosity solution of an elliptic HJB equation (see [5]). Here, where such tools are not available, we give a direct proof that avoids the use of PDE techniques.

Proposition 4.8. *Assume (A.1)–(A.8) and fix $T > 0$. Then $(Y^{x,a,\beta}, Z^{x,a,\beta}, K^{x,a,\beta})$, restricted to $[0,T]$, is the maximal solution of the finite horizon BSDE (4.17).*

Proof. Checking that the restriction of $(Y^{x,a,\beta}, Z^{x,a,\beta}, K^{x,a,\beta})$ to $[0,T]$ is a solution of equation (4.17) is straightforward. It remains to prove its maximality.

To this purpose let $(Y^{x,a,\beta,n}, Z^{x,a,\beta,n}, \Gamma^{x,a,\beta,n})$ be the solution of equation (4.1). Moreover, let (Y^T, Z^T, K^T) be a generic solution of the constrained equation (4.17). Applying Itô's formula to $e^{-\beta t}\,(Y_t^{x,a,\beta,n} - Y_t^T)$, we obtain

$$\begin{aligned} \mathrm{d}[e^{-\beta t}\,(Y_t^{x,a,\beta,n}-Y_t^T)] = &- e^{-\beta t}(n|\Gamma^{x,a,\beta,n}|)\,\mathrm{d}t + e^{-\beta t}\mathrm{d}K_t^T - e^{-\beta t}(Z_t^{x,a,\beta,n}-Z_t^T)\,\mathrm{d}W_t^1 \\ &- e^{-\beta t}\Gamma^{x,a,\beta,n}\,\mathrm{d}W_t^2. \end{aligned}$$

Let $\psi^{\beta,n}$ be defined as follows:

$$\psi_s^{\beta,n} \;:=\; \begin{cases} -n\frac{\Gamma^{x,a,\beta,n}}{|\Gamma^{x,a,\beta,n}|}, & \Gamma^{x,a,\beta,n}\neq 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Since $\psi^{\beta,n}$ is bounded, its exponential local martingale is indeed a $\mathbb{P}$-martingale:

$$L_t^{\beta,n} \;:=\; \mathcal{E}\bigg(\int_0^t \psi_s^{\beta,n}\,\mathrm{d}s\bigg).$$

Then, we introduce the new probability $\mathbb{P}^{\beta,n}$ under which $(\tilde W_t^1,\tilde W_t^2) = (-\int_0^t \psi_s^{\beta,n}\,\mathrm{d}s + W_t^1, W_t^2)$ is a cylindrical Wiener process (with respect to $(\mathcal{F})$). As a consequence, taking also into account that K^T is non-decreasing, we find:

$$Y_t^{x,a,\beta,n} - Y_t^T \geq \mathbb{E}^{\beta,n}\left[\left(e^{-\beta T}(v^{\beta,n,\mathcal{R}}(\mathfrak{X}_T^{x,a},\mathfrak{I}_T^a) - v^\beta(\mathfrak{X}_T^{x,a}))\right)\Big|\mathcal{F}_t\right]$$

and the claim follows by (4.10). □

5. Asymptotic behaviour of the infinite horizon problem and the ergodic BSDE

We are now finally able to introduce the objective of our analysis. For every $(x,a)\in H\times U$, we introduce the following infinite horizon constrained BSDE (ergodic constrained BSDE):

$$Y_t \;=\; Y_T + \int_t^T (\ell(\mathfrak{X}_s^{x,a},\mathfrak{I}_s^a)-\lambda)\,\mathrm{d}s - K_T + K_t - \int_t^T Z_s\mathrm{d}W_s^1, \qquad 0\le t\le T<+\infty, \tag{5.1}$$

where the real number λ is part of the unknowns, so, in particular, we shall denote by (Y,Z,K,λ) a solution to equation (5.1). Such a kind of equation has been already studied in the infinite-dimensional setting in [7, 9] under structure condition, while the randomised case has been addressed in [5], in the finite-dimensional case.

5.1. Asymptotic behaviour of the infinite horizon problem

As in Theorem 3.8 of [7], we have the following result, which immediately follows from (4.11).

Lemma 5.1. *Assume (A.1)–(A.8) and set $\hat v^\beta(x) := v^\beta(x) - v^\beta(0)$. Then there exists a sequence $\beta_k \searrow 0$ such that:*

$$\lim_{k\to\infty} \hat v^{\beta_k}(x) \;=\; \hat v(x), \qquad \forall x\in H, \tag{5.2}$$

$$\lim_{k\to\infty} \beta_k\hat v^{\beta_k}(0) \;=\; \lambda, \tag{5.3}$$

for a suitable function $\hat v$ and a suitable real number λ.

Remark 5.2. By (4.11) and the definition of $\hat{v}$, we have

$$|\hat{v}^\beta(x)| \leq C|x|, \quad |\hat{v}(x)| \leq C|x|, \tag{5.4}$$

where C does not depend on β.

Using the pair $\hat{v}$ and λ, we can easily construct a solution to equation (5.1) (see [7] and [9] for the same argument in the 'structure condition' case).

Theorem 5.3. *Assume (A.1)–(A.8). Let $\hat{v}$, λ and $(\beta_k)_k$ as in Lemma 5.1. Fix $(x,a) \in H \times U$ and let $\hat{Y}_t^{x,a} = \hat{v}(\mathfrak{X}_t^{x,a})$.*

There exist $\hat{Z}^{x,a} \in L^{2,\mathrm{loc}}_{\mathcal{P}}(\Omega \times [0,+\infty[; \mathfrak{X}i^*)$, $\hat{K}^{x,a} \in \mathcal{K}^{2,\mathrm{loc}}$ and a subsequence $\{\beta_{k_h}\}$ such that, for all $T > 0$,

$$\begin{aligned} Z^{x,a,\beta_{k_h}} &\rightharpoonup \hat{Z}^{x,a} \ \text{in } L^2_{\mathcal{P}}(\Omega \times [0,T]; \mathfrak{X}i^*), \\ K_T^{x,a,\beta_{k_h}} &\rightharpoonup \hat{K}_T^{x,a} \ \text{in } L^2_{\mathcal{P}}(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}), \end{aligned}$$

Moreover, $\hat{Y}^{x,a} \in L^{2,\mathrm{loc}}_{\mathcal{P}}(\Omega; C([0,+\infty[;\mathbb{R}))$.
Finally, (5.1) is verified by $(\hat{Y}^{x,a}, \hat{Z}^{x,a}, \hat{K}^{x,a}, \lambda)$, $\mathbb{P}$-a.s. for all t and T with $0 \leq t \leq T$.

Proof. To simplify notation, we denote $(Y^{x,a,\beta}, Z^{x,a,\beta}, K^{x,a,\beta})$ simply by $(Y^\beta, Z^\beta, K^\beta)$.
Set $\hat{Y}^\beta = \hat{v}^\beta(\mathfrak{X}_t^{x,a}) = v^\beta(\mathfrak{X}_t^{x,a}) - v^\beta(0)$. By trivial computations, we find

$$\hat{Y}_t^\beta \ = \ \hat{v}^\beta(\mathfrak{X}_T^{x,a}) - \beta \int_t^T \hat{Y}_s^\beta \, \mathrm{d}s - \beta v^\beta(0)(T-t) + \int_t^T \ell(\mathfrak{X}_s^{x,a}, \mathfrak{I}_s^a) \, \mathrm{d}s - K_T^\beta + K_t^\beta - \int_t^T Z_s^\beta \, \mathrm{d}W_s^1.$$

From (4.13) and (5.2), we have

$$\begin{aligned} \lim_{h\to\infty} \hat{Y}_t^{\beta_{k_h}} &= \hat{Y}_t^{x,a}, \quad \forall t \in [0,T], \quad \mathbb{P}\text{-a.s.} \\ \lim_{h\to\infty} \beta_{k_h} \hat{Y}_t^{\beta_{k_h}} &= 0, \quad \forall t \in [0,T], \quad \mathbb{P}\text{-a.s.} \\ \lim_{h\to\infty} \beta_{k_h} v_{k_h}^\beta(0) &= \lambda. \end{aligned}$$

Thanks to estimate (4.14), we see that, for all $T > 0$, the sequence $\{Z^{\beta_k}_{|[0,T]}\}_k$ is bounded in the Hilbert space $L^2_{\mathcal{P}}(\Omega \times [0,T]; \mathfrak{X}i^*)$. It follows that there exists a subsequence $\{Z^{\beta_{k_h}}_{|[0,T]}\}_h$ and a stochastic process $\hat{Z}^{x,a,T}$ in $L^2_{\mathcal{P}}(\Omega \times [0,T]; \mathfrak{X}i^*)$ such that, for all $T > 0$, $\{Z^{\beta_{k_h}}_{|[0,T]}\}_h$ weakly converges to $\hat{Z}^{x,a,T}$ in $L^2_{\mathcal{P}}(\Omega \times [0,T]; \mathfrak{X}i^*)$:

$$Z^{\beta_{k_h}}_{|[0,T]} \rightharpoonup \hat{Z}^{x,a,T} \quad \text{in } L^2_{\mathcal{P}}(\Omega \times [0,T]; \mathfrak{X}i^*).$$

We also see that there exists $\hat{Z}^{x,a}$ in $L^{2,\mathrm{loc}}_{\mathcal{P}}(\Omega \times [0,+\infty[; \mathfrak{X}i^*)$ such that $\hat{Z}^{x,a}_{|[0,T]} = \hat{Z}^{x,a,T}$, for all $T > 0$. Thus all terms, apart from $K_T^\beta - K_t^\beta$, weakly converge. Choosing $t = 0$, we deduce that, for all $T \geq 0$, K_T^β, as well, converges weakly in $L^2(\mathcal{F}_T)$ to some $\hat{K}_T^{x,a}$. Therefore, we deduce that we can pass to the limit as $h \to \infty$ to get that $(\hat{Y}, \hat{Z}, \hat{K}, \lambda)$ is a solution to (5.1). □

Remark 5.4. Note that, thanks to (4.9), we have

$$|\hat{Y}_t| \leq L_\ell |\mathfrak{X}_t^{x,a}| \quad \forall t \in [0,T].$$

We wish to prove that the above solution is maximal (in a suitable sense).

The notion of maximality is the same as in [5] but the proof is completely different since, once more, we cannot use PDE interpretation.

The following is, as a matter of fact, the main technical result of the paper.

To state it, fixed an horizon $T > 0$, letting λ and $\hat{v}$ be the ones defined in (5.2) and (5.3), we consider the following BSDE on $[0, T]$

$$Y_t \; = \; \hat{v}(\mathfrak{X}_T^{x,a}) + \int_t^T (\ell(\mathfrak{X}_s^{x,a}, \mathfrak{I}_s^a) - \lambda)\, \mathrm{d}s - K_T + K_t - \int_t^T Z_s \mathrm{d}W_s^1, \qquad t \in [0,T]. \tag{5.5}$$

Note that the above equation is indeed (4.15) with $\beta = 0$, final condition ϕ replaced by $\hat{v}$ and generator ℓ replaced by $\ell - \lambda$ (see Def. 4.5 for the definition of solution).

Theorem 5.5. *Assume (A.1)–(A.8). The quadruple $(\hat{Y}^{x,a}, \hat{Z}^{x,a}, \hat{K}^{x,a}, \lambda)$ is the maximal solution of equation (5.1) in the following sense: for all fixed $T \geq 0$, given any triple (Y, Z, K) defined on $[0,T]$ solution to (5.5), then $\hat{Y}_t^{x,a} \geq Y_t$ for all $t \in [0,T]$. In particular, we have*

$$\hat{v}(x) \; = \; \inf_{u \in \mathcal{U}} \hat{J}^T(x,u), \quad \forall x \in H,\, T > 0, \tag{5.6}$$

where

$$\hat{J}^T(x,u) \; := \; \mathbb{E}\left[\int_0^T (\ell(X_s^{x,u}, u_s) - \lambda)\, \mathrm{d}s + \hat{v}(X_T^{x,u})\right].$$

Proof. First of all, we know from Proposition 4.6 that equation (5.5) admits a maximal solution that can be rewritten as $Y_t = v^{0,T-t}(\mathfrak{X}_t^{x,a})$ (for $v^{0,T-t}$, we refer to the notation in Proposition 4.6 with ℓ replaced by $\ell - \lambda$, we recall that 0 indicates the absence of discount, *that is* $\beta = 0$). Moreover, by Remark 4.7, the following holds:

$$|v^{0,T-t}(\mathfrak{X}_t^{x,a}))| \leq c(1 + |\mathfrak{X}_t^{x,a}|) \quad \forall t \in [0,T], \tag{5.7}$$

for some positive constant $c > 0$, depending only T and on the constants introduced in Assumption (A).

To prove that $v^{0,T-t}(\mathfrak{X}_t^{x,a})) \leq \hat{v}(\mathfrak{X}_t^{x,a}) \forall t \in [0,T]$, we come back to the penalised, finite horizon BSDE making a step 'backward'. As a matter of fact, we introduce the classical (finite horizon) BSDE in $[0,T]$:

$$\begin{aligned} \check{Y}_t^{\beta,n} \; &= \; v^\beta(\mathfrak{X}_T^{x,a}) - \beta \int_t^T \check{Y}_s^{\beta,n}\, \mathrm{d}s + \int_t^T \ell(\mathfrak{X}_s^{x,a}, \mathfrak{I}_s^a)\, \mathrm{d}s - n \int_t^T |\check{\Gamma}_s^{\beta,n}|\, \mathrm{d}s \\ &\quad - \int_t^T \check{Z}_s^{\beta,n}\, \mathrm{d}W_s^1 - \int_t^T \check{\Gamma}_s^{\beta,n}\, \mathrm{d}W_s^2, \qquad 0 \leq t \leq T. \end{aligned}$$

See Proposition 4.2 for the definition on v^β. By Proposition 4.4, point *4*, we have that, for all $t \in [0,T]$,

$$\check{Y}_t^{\beta,n} \searrow \check{Y}_t^\beta \qquad \text{as } n \to \infty,$$

where $\check{Y}_t^\beta$ is the first component of the maximal solution to the following constrained BSDE on $[0,T]$:

$$Y_t \; := \; v^\beta(\mathfrak{X}_T^{x,a}) - \beta \int_t^T Y_s\, \mathrm{d}s + \int_t^T \ell(\mathfrak{X}_s^{x,a}, \mathfrak{I}_s^a)\, \mathrm{d}s - K_T + K_t - \int_t^T Z_s\, \mathrm{d}W_s^1, \quad 0 \leq t \leq T.$$

Note, however, that by Proposition 4.8, the maximal solution of the above equation is indeed $(Y^\beta, Z^\beta, K^\beta)$ (recall that we denote by $(Y^\beta, Z^\beta, K^\beta)$ the triple $(Y^{x,a,\beta}, Z^{x,a,\beta}, K^{x,a,\beta})$ founded in Prop. 4.4). So, in particular, $\check{Y}_t^{\beta,n} \searrow Y_t^\beta$ as $n \to \infty$.
Thus, setting $\hat{\check{Y}}_t^{\beta,n} = \check{Y}_t^{\beta,n} - v^\beta(0)$, we find, for all $t \in [0,T]$: $\hat{\check{Y}}_t^{\beta,n} \searrow Y_t^\beta - v^\beta(0) = \hat{Y}_t^\beta$ as $n \to \infty$.
Moreover, for all $t \leq T$, it holds, $\mathbb{P}$-a.s.,

$$\begin{aligned}\hat{\check{Y}}_t^{\beta,n} \;&=\; \hat{v}^\beta(\mathfrak{X}_T^{x,a}) - \beta \int_t^T \hat{\check{Y}}_s^{\beta,n}\,\mathrm{d}s - \beta \int_t^T v^\beta(0)\,\mathrm{d}s + \int_t^T \ell(\mathfrak{X}_s^{x,a}, \mathfrak{I}_s^a)\,\mathrm{d}s - n\int_t^T |\hat{\check{\Gamma}}_s^{\beta,n}|\,\mathrm{d}s \\ &\quad - \int_t^T \hat{\check{Z}}_s^{\beta,n}\,\mathrm{d}W_s^1 - \int_t^T \hat{\check{\Gamma}}_s^{\beta,n}\,\mathrm{d}W_s^2.\end{aligned}$$

Then $(\hat{\check{Y}}^{\beta,n}, \hat{\check{Z}}^{\beta,n}, \hat{\check{\Gamma}}^{\beta,n})$ becomes the candidate triple to be compared with a generic solution (Y, Z, K) of equation (5.1) such that $Y_T = \hat{v}(\mathfrak{X}_T^{x,a})$ (recall that $\hat{Y}_t^{\beta_k} \to \hat{Y}_t^\beta$ as $k \to \infty$). Note that, setting $Y_t^\sharp = e^{-\beta t} Y_t$, $Y_t^{\sharp,\beta,n} = e^{-\beta t}\hat{\check{Y}}_t^{\beta,n}$, $Z_t^\sharp = e^{-\beta t} Z_t$, $Z_t^{\sharp,\beta,n} = e^{-\beta t}\hat{\check{Z}}_t^{\beta,n}$, $\Gamma_t^{\sharp,\beta,n} = e^{-\beta t}\hat{\check{\Gamma}}_t^{\beta,n}$, we find, $\mathbb{P}$-a.s. for all $t \in [0,T]$:

$$\begin{aligned}Y_t^{\sharp,\beta,n} - Y_t^\sharp \;&= e^{-\beta T}\hat{v}^\beta(\mathfrak{X}_T^{x,a}) - e^{-\beta T}\hat{v}(\mathfrak{X}_T^{x,a}) - \int_t^T e^{-\beta s}[\beta v^\beta(0) - \lambda]\,\mathrm{d}s + \beta\int_t^T Y_s^\sharp\,\mathrm{d}s \\ &\quad - n\int_t^T |\Gamma_s^{\sharp,\beta,n}|\,\mathrm{d}s + \int_t^T e^{-\beta s}\mathrm{d}\,K_s + \int_t^T Z_s^{\sharp,\beta,n}\,\mathrm{d}W_s^1 - \int_t^T \Gamma_s^{\sharp,\beta,n}\,\mathrm{d}W_s^2.\end{aligned}$$

By Girsanov's theorem, there exists an equivalent probability $\tilde{\mathbb{P}}^{\beta,n}$ and a $\tilde{\mathbb{P}}^{\beta,n}$-Wiener process $\tilde{W}^2$ such that W^1 is still a Wiener process under $\tilde{\mathbb{P}}^{\beta,n}$ and the above equation can be rewritten as:

$$\begin{aligned}Y_t^{\sharp,\beta,n} - Y_t^\sharp \;&= e^{-\beta T}\hat{v}^\beta(\mathfrak{X}_T^{x,a}) - e^{-\beta T}\hat{v}(\mathfrak{X}_T^{x,a}) - \int_t^T e^{-\beta s}[\beta v^\beta(0) - \lambda]\,\mathrm{d}s + \beta\int_t^T Y_s^\sharp\,\mathrm{d}s \\ &\quad + \int_t^T e^{-\beta s}\mathrm{d}\,K_s + \int_t^T Z_s^{\sharp,\beta,n}\,\mathrm{d}W_s^1 - \int_t^T \Gamma_s^{\sharp,\beta,n}\,\mathrm{d}W_s^2.\end{aligned}$$

Hence, taking the expectation with respect to $\tilde{\mathbb{P}}^{\beta,n}$, we obtain

$$\begin{aligned}Y_0^{\sharp,\beta,n} - Y_0^\sharp &\geq (\lambda - \beta v^\beta(0))\frac{e^{-\beta T} - 1}{\beta} + \beta\tilde{\mathbb{E}}^{\beta,n}\int_t^T Y_s^\sharp\,\mathrm{d}s + e^{-\beta T}\tilde{\mathbb{E}}^{\beta,n}[\hat{v}^\beta(\mathfrak{X}_T^{x,a}) - \hat{v}(\mathfrak{X}_T^{x,a})] \\ &= I_1^\beta + I_2^{\beta,n} + I_3^{\beta,n}.\end{aligned}$$

We immediately have that:

$$\lim_{\beta\to\infty} I_1^\beta = 0.$$

Regarding the other two terms, we note that the law of $\mathfrak{X}^{x,a}$, under $\tilde{\mathbb{P}}^{\beta,n}$, is the law, under $\mathbb{P}$, of the solution $\mathfrak{X}^{x,a,\beta,n}$ of

$$\begin{cases} \mathrm{d}\mathfrak{X}_t = A\mathfrak{X}_t\mathrm{d}t + F(\mathfrak{X}_t, \mathfrak{I}_t^{\beta,n})\mathrm{d}t + G\mathrm{d}W_t^1, & \mathfrak{X}_0 \;=\; x, \\ \mathrm{d}\mathfrak{I}_t^{\beta,n} \;=\; R\gamma_t^{\beta,n}\mathrm{d}t + R\mathrm{d}W_t^2, & \mathfrak{I}_0 \;=\; a, \end{cases}$$

where

$$\gamma_s^{\beta,n} = \begin{cases} -n\frac{\Gamma_s^{\sharp,\beta,n}}{|\Gamma_s^{\sharp,\beta,n}|}, & \Gamma_s^{\sharp,\beta,n} \neq 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Comparing with (3.1), we also note that $\mathfrak{X}_t^{x,a,\beta,n} = \mathfrak{X}_t^{x,a,\gamma^{\beta,n}}$.
By Remark 3.2 and Proposition 2.2, we have that, for any $p \geq 1$,

$$\mathbb{E} \sup_{t\in[0,T]} |\mathfrak{X}_t^{x,a,\beta,n}|_H^p = \tilde{\mathbb{E}}^{\beta,n} \sup_{t\in[0,T]} |\mathfrak{X}_t^{x,a}|_H^p \leq C, \tag{5.8}$$

where the positive constant C may depend on x, T, p, but neither on n nor on β, a, R, $\gamma^{\beta,n}$ (since the dissipativity on F is uniform with respect to the control variable). Moreover, by (5.7), we get, for every $t \in [0,T]$,

$$|Y_t^\sharp| \leq Ce^{-\beta t}(1 + |\mathfrak{X}_t^{x,a,\beta,n}|).$$

This together with (5.8) yields:

$$\tilde{\mathbb{E}}^{\beta,n}\beta\int_0^T |Y_t^\sharp|\,\mathrm{d}t \to 0 \quad \text{as } \beta \to 0,$$

it that is,

$$\lim_{\beta\to 0} I_2^{\beta,n} = 0.$$

It remains to study the term $I_3^{\beta,n}$. We note that

$$\tilde{\mathbb{E}}^{\beta,n}[\hat{v}^\beta(\mathfrak{X}_T^{x,a}) - \hat{v}(\mathfrak{X}_T^{x,a})] = \mathbb{E}[\hat{v}^\beta(\mathfrak{X}_T^{x,a,\beta,n}) - \hat{v}(\mathfrak{X}_T^{x,a,\beta,n})].$$

Thanks to (4.9), we deduce that $(\hat{v}^\beta)_\beta$ is equicontinuous and equibounded on H. Taking into account (3.3), we obtain

$$\mathbb{E}||\mathfrak{X}_T^{x,a,\beta,n}||_{D((\delta I - A)^\rho)} \leq C,$$

where again C depends on T and on the quantities introduced in the assumptions, but neither on n nor on β. Thus, for every $R > 0$, set B_R the centred ball of radius R in $D(\delta I - A)^\rho$ and denote by B_R^c its complementary set. Then, we have

$$|\mathbb{E}[\hat{v}^\beta(\mathfrak{X}_T^{x,a,\beta,n}) - \hat{v}(\mathfrak{X}_T^{x,a,\beta,n})]| \leq \sup_{y\in B_R} |\hat{v}^\beta(y) - \hat{v}(y)| + \mathbb{E}[|\hat{v}^\beta(\mathfrak{X}_T^{x,a,\beta,n}) - \hat{v}(\mathfrak{X}_T^{x,a,\beta,n})| I_{B_R^c}(\mathfrak{X}_T^{x,a,\beta,n})].$$

By Ascoli–Arzelá Theorem, $\hat{v}^\beta$ converges uniformly on compact subsets of H as β tends to 0, thus recalling (A.3):

$$\lim_{\beta\to 0} \sup_{y\in B_R} |\hat{v}^\beta(y) - \hat{v}(y)| = 0.$$

On the other hand, (5.4) yields:

$$\begin{aligned}
&\mathbb{E}[|\hat{v}^\beta(\mathfrak{X}_T^{x,a,\beta,n}) - \hat{v}(\mathfrak{X}_T^{x,a,\beta,n})| I_{B_R^c}(X_T^{x,a,\beta,n})] \\
&\leq (\mathbb{E}[|\hat{v}^\beta(\mathfrak{X}_T^{x,a,\beta,n}) - \hat{v}(\mathfrak{X}_T^{x,a,\beta,n})|^2])^{1/2} (\mathbb{P}(||\mathfrak{X}_T^{x,a,\beta,n}||_{D(\delta I - A)^\rho} > R))^{1/2} \\
&\leq C(\mathbb{E}(|\mathfrak{X}_T^{x,a,\beta,n}|^2))^{1/2} (\mathbb{E}||\mathfrak{X}_T^{x,a,\beta,n}||_{\mathcal{D}(\delta I - A)^\rho})^{1/2} R^{-1/2} \leq C R^{-1/2},
\end{aligned}$$

for some positive constant C independent of n and β. Therefore, for every $\varepsilon > 0$, we can find R large enough and then β small enough such that

$$|I_3^\beta| < \varepsilon.$$

This concludes the proof. □

6. Long-time asymptotics of the finite horizon problem and the ergodic control problem

Once the existence of a maximal solution $(\hat{Y}^{x,a}, \hat{Z}^{x,a}, \hat{K}^{x,a}, \lambda)$ to the constrained ergodic BSDE (5.1) has been proved together with the Markovian representation of $\hat{Y}_t$ in terms of $\hat{v}(\mathfrak{X}^{x,a})$, this final section is devoted to the application of such results to the study of the asymptotic expansion of the value function of finite horizon problems as $T \to +\infty$ as well as to the study of the value function of an ergodic optimal control problem.

We note that the key point for the results below is that we have been able to identify through the constrained ergodic BSDE a function $\hat{v}$ (independent of T) and a constant λ that, for all $T > 0$, gives the value function of a control problem with horizon T running cost $\ell - \lambda$ and final cost $\hat{v}$ itself.

Theorem 6.1. *Assume (A.1)–(A.5) and (A.7). Fix $T > 0$ and choose any function ϕ that satisfies (A.7). Let $v^T := v^{0,T}$ be defined as in Proposition 2.4 with $\beta = 0$. For every $x \in H$, it holds:*

$$|v^T(x) - \hat{v}(x) - \lambda T| \leq C(1 + |x|)$$

for some positive constant C. In particular,

$$\lim_{T \to +\infty} \frac{v^T(x)}{T} \;=\; \lambda. \tag{6.1}$$

Remark 6.2. Note that by Theorem 6.1, the component λ of the maximal solution $(\hat{Y}^{x,a}, \hat{Z}^{x,a}, \hat{K}^{x,a}, \lambda)$ to equation (5.1) is unique in the following sense: let $(\tilde{Y}^{x,a}, \tilde{Z}^{x,a}, \tilde{K}^{x,a}, \tilde{\lambda})$ be another maximal solution to equation (5.1) such that there exists a function $\tilde{v}\colon H \to \mathbb{R}$, satisfying Assumption (A.7), with $\tilde{Y}_t^{x,a} = \tilde{v}(\mathfrak{X}_t^{x,a})$, $t \geq 0$. Then, since Theorem 6.1 works for both $\hat{v}$ and $\tilde{v}$, we conclude that $\lambda = \tilde{\lambda}$.

Remark 6.3. Note that in Theorem 6.1, we only deal with limit (6.1), while we do not investigate the rate of convergence as done in [14]. A result of this kind seems, at the moment, quite complex to achieve since we cannot adapt easily their approach due to the lack of regularity of our state equation.

Proof. By equality (5.6) and Proposition 2.4 (with $\phi = \hat{v}$ and ℓ replaced by $\ell - \lambda$), we obtain

$$|v^T(x) - \hat{v}(x) - \lambda T| \leq \sup_{u \in \mathcal{U}} |J^T(x,u) - \hat{J}^T(x,u) - \lambda T| \leq \sup_{u \in \mathcal{U}} \big(\mathbb{E}|\phi(X_T^{x,u})| + \mathbb{E}|\hat{v}(X_T^{x,u})|\big).$$

By 5.4 and (A.7), we deduce

$$|\hat{v}(X_T^{x,u})| + |\phi(X_T^{x,u})| \leq (1 + |X_T^{x,u}|)$$

and the claim follows by (2.3). □

Proposition 6.4. *Assume (A.1)–(A.7) and let $x \in H$.*

(i) *The real number λ in* (5.3) *satisfies the following inequality:*

$$\lambda \ \leq \ \inf_{u\in\mathcal{U}} \liminf_{T\to+\infty} \frac{1}{T}\mathbb{E}\bigg[\int_0^T \ell(X_t^{x,u}, u_t)\,\mathrm{d}t\bigg].$$

(ii) *Assume in addition that there exists $\hat{u} \in \mathcal{U}$ such that the stochastic process*

$$\bigg(\hat{v}(X_t^{x,\hat{u}}) + \int_0^t \ell(X_s^{x,\hat{u}}, \hat{u}_s)\,\mathrm{d}s - \lambda\, t\bigg)_{t\geq 0}$$

is a local martingale. Then

$$\lambda \ = \ \lim_{T\to+\infty} \frac{1}{T}\mathbb{E}\bigg[\int_0^T \ell(X_t^{x,\hat{u}}, \hat{u}_t)\,\mathrm{d}t\bigg] \ \inf_{u\in\mathcal{U}} \liminf_{T\to+\infty} \frac{1}{T}\mathbb{E}\bigg[\int_0^T \ell(X_t^{x,u}, u_t)\,\mathrm{d}t\bigg].$$

Remark 6.5. We observe that the functional

$$\hat{J}(x,u) := \liminf_{T\to+\infty} \frac{1}{T}\mathbb{E}\bigg[\int_0^T \ell(X_t^{x,u}, u_t)\,\mathrm{d}t\bigg]$$

is a natural definition of *ergodic cost*. Thus, the result of Proposition 6.4 can be rephrased saying that λ is the optimal cost for the ergodic cost $\hat{J}(x,\cdot)$ (independently on x) and $\hat{u}$ is the optimal control).

Remark 6.6. Note that a similar result to Proposition 5.1 was proved in the finite-dimensional context in [5], Remark 5.6, based on PDE, rather than probabilistic, techniques. More precisely, in [5], the authors assume the existence of a (Lipschitz) feedback control which is, in a suitable sense, optimal for the ergodic PDE, while here in item (ii) above we assume the existence of a control $\hat{u} \in \mathcal{U}$ (not necessarily in feedback form) which is optimal according to the martingale principle of optimality.

Proof. We split the proof into two steps.
Proof of (i). Recall from Theorem 6.1 that

$$\lambda \ = \ \lim_{T\to+\infty} \frac{v^T(x)}{T}.$$

By its definition $v^T(x) \leq \mathbb{E}[\int_0^T \ell(X_t^{x,u}, u_t)\,\mathrm{d}t + \phi(X_T^{x,u})]$ for any $u \in \mathcal{U}$, we obtain

$$\lambda \ \leq \ \liminf_{T\to+\infty} \frac{1}{T}\mathbb{E}\bigg[\int_0^T \ell(X_t^{x,u}, u_t)\,\mathrm{d}t + \phi(X_T^{x,u})\bigg], \qquad \forall u \in \mathcal{U}.$$

By Assumption (A.7) and (2.3), we deduce that $\liminf_{T\to+\infty} \mathbb{E}[\phi(X_T^{x,u})]/T = 0$. Consequently

$$\lambda \;\le\; \inf_{u\in\mathcal{U}} \liminf_{T\to+\infty} \frac{1}{T}\mathbb{E}\bigg[\int_0^T \ell(X_t^{x,u}, u_t)\,\mathrm{d}t\bigg].$$

Proof of (ii). By point (i), it is enough to prove the inequality

$$\lambda \;\ge\; \inf_{u\in\mathcal{U}} \liminf_{T\to+\infty} \frac{1}{T}\mathbb{E}\bigg[\int_0^T \ell(X_t^{x,u}, u_t)\,\mathrm{d}t\bigg].$$

By the local martingale property of $(\hat v(X_t^{x,\hat u}) + \int_0^t \ell(X_s^{x,\hat u}, \hat u_s)\,\mathrm{d}s - \lambda\, t)_{t\ge 0}$, it follows, through a straightforward localisation argument, that, for every $T>0$,

$$\hat v(x) \;=\; \mathbb{E}\bigg[\hat v(X_T^{x,\hat u}) + \int_0^T \ell(X_t^{x,\hat u}, \hat u_t)\,\mathrm{d}t - \lambda\, T\bigg],$$

which can be rewritten as

$$\lambda \;=\; \mathbb{E}\bigg[\frac{1}{T}\int_0^T \ell(X_t^{x,\hat u}, \hat u_t)\,\mathrm{d}t + \frac{\hat v(X_T^{x,\hat u}) - \hat v(x)}{T}\bigg].$$

By (5.4) and (2.4), we have

$$\lim_{T\to+\infty} \mathbb{E}\bigg[\frac{\hat v(X_T^{x,\hat u}) - \hat v(x)}{T}\bigg] \;=\; 0.$$

In conclusion, we obtain

$$\lambda \;=\; \lim_{T\to+\infty} \frac{1}{T}\mathbb{E}\bigg[\int_0^T \ell(X_t^{x,\hat u}, \hat u_t)\,\mathrm{d}t\bigg] \;\ge\; \inf_{u\in\mathcal{U}} \liminf_{T\to+\infty} \frac{1}{T}\mathbb{E}\bigg[\int_0^T \ell(X_t^{x,u}, u_t)\,\mathrm{d}t\bigg].$$

□

Remark 6.7. The above results have been stated referring to the product space formulation of the problem (see Sect. 2.3). Thanks to the equality of the value functions (see Prop. 2.5), identical results can be proved with the same arguments in the original formulation of the control problem (see Sect. 2.2).

Example 6.8. We consider an ergodic control problem for a stochastic heat equation as in [7]. The difference here is that we can handle the degenerate noise without the *structure condition*; in particular, in the equation below, the diffusion coefficient σ need not to stay away from 0. On the other hand, we need to ask strong dissipativity for the drift f. As a matter of fact, we consider:

$$\begin{cases} \mathrm{d}_t X^u(t,\xi) = [\frac{\partial}{\partial\xi^2} X^u(t,\xi) + f(\xi, X^u(t,\xi), u(t,\xi))]\,\mathrm{d}t + \sigma(\xi)\dot W(t,\xi)\,\mathrm{d}t,\ t\ge 0,\ \ \xi\in[0,1], \\ X^u(t,0) = X^u(t,1) = 0, \\ X^u(0,\xi) = x_0(\xi), \end{cases}$$

where W is the space-time white noise on $[0,+\infty)\times[0,1]$. An admissible control u is a predictable process $u:\Omega\times[0,+\infty)\times[0,1]\to\mathbb{R}$, such that $\int_0^1 u^2(\omega,t,\xi)\,\mathrm{d}\xi<\infty$, $\forall t\in[0,+\infty)$, $\mathbb{P}$-a.s. The cost functional is

$$J(x_0,u)=\liminf_{T\to+\infty}\frac{1}{T}\,\mathbb{E}\int_0^T\int_0^1 \ell(t,X^u(t,\xi),u(t,\xi))\,\mathrm{d}\xi\,\mathrm{d}s.$$

The abstract formulation in $H=L^2(0,1)$ and $U=L^2(0,1)$ follows as in ([11], Sect. 5). We note that the realisation of the second-order derivative with Dirichlet boundary conditions fulfils Assumptions (A.1) and (A.3) (see [18]).

Then, Theorem 6.1 and Proposition 6.4 apply provided that one asks, for instance,

1. $f:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ is a continuous function such that

$$|f(\xi,x,u)|\le M_f(1+|x|+|u|),$$
$$|f(\xi,x,u)-f(\xi,x',u)|\le L_f|x-x'|$$

for suitable positive constants M_f, L_f, for almost all $\xi\in[0,1]$ and every $x,x',u\in\mathbb{R}$. Moreover, we assume $f(\xi,\cdot,u)\in C^1(\mathbb{R})$ for almost all $\xi\in[0,1]$ and every $u\in\mathbb{R}$ to be such that, for some $\mu>0$,

$$\frac{\partial}{\partial x}f(\xi,x,u)\le-\mu,$$

for almost all $\xi\in[0,1]$ and every $x,u\in\mathbb{R}$.
2. $\sigma:[0,1]\to\mathbb{R}$ is a measurable and bounded function.
3. $\ell:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ is a continuous and bounded function such that

$$|\ell(\xi,x,u)-\ell(\xi,x',u)|\le L_\ell|x-x'|,$$

for a suitable positive constant L_ℓ, for almost all $\xi\in[0,1]$ and every $x,x',u\in\mathbb{R}$.

Example 6.9. We can also handle an SPDE, in a bounded open regular domain $D\subset\mathbb{R}^2$ driven by coloured (in space) noise. We note that in this case the structure condition would be a very artificial request. Namely we consider

$$\begin{cases}\mathrm{d}_tX^u(t,\xi)=[\mathcal{A}X^u(t,\xi)+f(\xi,X^u(t,\xi),u(t,\xi))]\,\mathrm{d}t+\dfrac{\partial\mathcal{W}^G}{\partial t}(t,\xi)\,\mathrm{d}t, t\ge0,\xi\in D,\\ X^u(t,0)=X^u(t,1)=0,\\ X^u(0,\xi)=x_0(\xi),\end{cases}$$

where

1. $\mathcal{A}$ is a second-order operator in divergence form $\mathcal{A}=\frac{\partial}{\partial\xi_h}\left(\sum_{h,k=1}^2 a_{h,k}(\xi)\frac{\partial}{\partial\xi_k}\right)$, the matrix $(a_{h,k})_{h,k}$ is non-negative and symmetric, with regular coefficients; moreover, the uniform elliptic condition

$$\sup_{\xi\in\bar{D}}\sum_{h,k=1}^2 a_{h,k}(\xi)\lambda_h\lambda_k\ge\mu|\lambda|^2,\qquad \lambda\in\mathbb{R}^2$$

is fulfilled for a positive constant μ.
Under such an assumption the realisation of $\mathcal{A}$ in $H=L^2(D)$ is a self-adjoint strongly dissipative operator A that generates an analytic semigroup with dense domain in H, so that conditions (A.1) and (A.3) are fulfilled.

Moreover, it is diagonalised by a complete orthonormal basis of functions e_k such that

$$Ae_k = \lambda_k e_k, \quad \text{with } \lambda_k \sim k, \quad k = 1, 2, \dots$$

2. $f : D \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function such that

$$|f(\xi, x, u)| \le M_f(1 + |x| + |u|),$$
$$|f(\xi, x, u) - f(\xi, x', u)| \le L_f|x - x'|$$

for suitable positive constants M_f, L_f, for almost all $\xi \in D$ and every $x, x', u \in \mathbb{R}$. Moreover, we assume $f(\xi, \cdot, u) \in C^1(\mathbb{R})$ for almost all $\xi \in D$ and every $u \in \mathbb{R}$; in addition, for some $\nu > 0$,

$$\frac{\partial}{\partial x} f(\xi, x, u) \le -\nu,$$

for almost all $\xi \in D$ and every $x, u \in \mathbb{R}$.

3. $\ell : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous and bounded function such that

$$|\ell(\xi, x, u) - \ell(\xi, x', u)| \le L_\ell|x - x'|,$$

for a suitable positive constant L_ℓ, for almost all $\xi \in [0, 1]$ and every $x, x', u \in \mathbb{R}$.

4. An admissible control u is a predictable process $u : \Omega \times [0, +\infty) \times D \to \mathbb{R}$, such that

$$\int_D u^2(\omega, t, \xi)\, \mathrm{d}\xi < \infty, \quad \forall t \in [0, +\infty), \mathbb{P}\text{-a.s.}$$

5. $\frac{\partial \mathcal{W}^G}{\partial t}(t, \xi)$ stands for a gaussian noise that is assumed to be white in time and coloured in space. More precisely, the infinite-dimensional reformulation of the equation is driven by an H-valued Wiener process W^G defined using the sum:

$$W^G(t, \xi) = \sum_{k=1}^{\infty} Ge_k(\xi)\beta_k(t), \quad t \ge 0, \quad \xi \in D,$$

where $\{\beta_k\}$ is a sequence of mutually independent standard Brownian motions and $G : H \to H$ is given by $G := (-A)^{-\eta}$ for a fixed $\eta > \frac{1}{4}$, in order to ensure that condition (A.2) holds true. The operator G is clearly not invertible.

Under the above assumptions, Theorem 6.1 and Proposition 6.4 apply.

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