ESAIM: COCV 25 (2019) 67
https://doi.org/10.1051/cocv/2018054

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

EXPLICIT EXPONENTIAL STABILIZATION OF NONAUTONOMOUS LINEAR PARABOLIC-LIKE SYSTEMS BY A FINITE NUMBER OF INTERNAL ACTUATORS

Karl Kunisch[1] and Sérgio S. Rodrigues[2,*]

Abstract. An explicit feedback controller is proposed for stabilization of linear parabolic equations, with a time-dependent reaction–convection operator. The range of the feedback controller is finite-dimensional, and is typically modeled by indicator functions of small subdomains. Its dimension depends polynomially on a suitable norm of the reaction–convection operator. A sufficient condition for stabilizability is given, which involves the asymptotic behavior of the eigenvalues of the (time-independent) diffusion operator, the norm of the reaction–convection operator, and the norm of the nonorthogonal projection onto the controller's range along a suitable infinite-dimensional (higher-modes) eigenspace. To construct the explicit feedback, the essential step consists in computing the nonorthogonal projection. Numerical simulations are presented, in 1D and 2D, showing the practicability of the controller and its response to measurement errors, where the actuators are indicator functions of suitable small subsets.

Mathematics Subject Classification. 93D15, 93B52, 93C05, 93C20.

Received November 29, 2017. Accepted September 24, 2018.

1. Introduction

We consider a nonautonomous evolutionary system in the abstract form

$$\dot{y}(t) + Ay(t) + A_{\mathrm{rc}}(t)y(t) - \sum_{i=1}^{M} u_i(t)\Psi_i = 0, \quad y(0) = y_0, \tag{1.1}$$

where y is the state, y_0 and Ψ_i, $i \in \{1, 2, \ldots, M\}$, are given in a Hilbert space H, and $u(t) = (u_1, \ldots, u_M)(t)$ is a control function at our disposal, taking values in $\mathbb{R}^M$.

Under general conditions on the linear operator A, the nonautonomous operator A_{rc}, and on the family $\{\Psi_i \mid i \in \{1, 2, \ldots, M\}\}$, we present an explicit stabilizing feedback controller for system (1.1). The operators A and A_{rc} are diffusion and reaction–convection-like operators. The functions Ψ_i represent the actuators. We stress that, at the abstract level, these functions may be any elements in H. In applications to concrete parabolic

Keywords and phrases: Exponential stabilization, nonautonomous parabolic systems, finite-dimensional controller, explicit feedback.

[1] Institute for Mathematics and Scientific Computing, Karl-Franzens-Universität, Heinrichstr. 36, 8010 Graz, Austria.
[2] Johann Radon Institute for Computational and Applied Mathematics, ÖAW, Altenberger Straße 69, 4040 Linz, Austria.

[*] Corresponding author: sergio.rodrigues@ricam.oeaw.ac.at

equations, we are particularly interested in the case where these functions are indicator functions of small subdomains.

To introduce the concepts of this paper, let us describe a consequence of the main result. For this purpose we suppose that A is a positive densely defined self-adjoint operator in H, and with compact inverse. We introduce $\mathcal{U}$ as the linear span of our actuators $\mathcal{U} \coloneqq \mathrm{span}\{\Psi_i \mid i \in \{1, 2, \ldots, M\}\}$.

Let $0 < \alpha_i$, $i \in \mathbb{N}_0$, be the increasing sequence of (repeated) eigenvalues of A and let E_M be the span of the eigenfunctions of the operator A associated with its first M eigenvalues. We will assume that our actuators satisfy $H = \mathcal{U} \oplus E_M^\perp$, which allow us to define the (nonorthogonal) projection $P_{\mathcal{U}}^{E_M^\perp} : H \to \mathcal{U}$ onto $\mathcal{U}$ along $E_M^\perp$. Furthermore, we will have that $A \in \mathcal{L}(V, V')$, maps V into V', for a suitable Hilbert space $V \hookrightarrow H$, with dual V'.

A consequence of the main result of this paper is: if

$$H = \mathcal{U} \oplus E_M^\perp, \tag{1.2a}$$

$$\overline{\mu}_M \coloneqq \alpha_{M+1} - \left(6 + 4 \left|P_{\mathcal{U}}^{E_M^\perp}\right|_{\mathcal{L}(H)}^2\right) |A_{\mathrm{rc}}|_{L^\infty((0,+\infty),\mathcal{L}(H,V'))}^2 > 0, \tag{1.2b}$$

then for any given constant $\lambda > 0$ a feedback stabilizing control is given by

$$y \mapsto \mathcal{K}(t)y \coloneqq P_{\mathcal{U}}^{E_M^\perp} \left(Ay + A_{\mathrm{rc}}(t)y - \lambda y\right). \tag{1.3}$$

More precisely, the system

$$\dot{y}(t) + Ay(t) + A_{\mathrm{rc}}(t)y(t) - \mathcal{K}(t)y(t) = 0, \quad y(0) = y_0, \tag{1.4}$$

is exponentially stable: there exist suitable constants $\mu > 0$ and $D \geq 1$ such that

$$|y(t)|_H^2 \leq D\mathrm{e}^{-\mu(t-s)} |y(s)|_H^2, \quad \text{for all} \quad t \geq s \geq 0.$$

Observe that (1.4) is exactly (1.1), with u given by

$$\sum_{i=1}^{M} u_i \Psi_i = P_{\mathcal{U}}^{E_M^\perp}(Ay + A_{\mathrm{rc}}y - \lambda y). \tag{1.5}$$

Concerning condition (1.2a) we shall see that it is satisfied in a practically relevant case with Ψ_i as indicator functions of subdomains. Moreover (1.2a) implies that the family of actuators $\{\Psi_i \mid i \in \{1, 2, \ldots, M\}\}$ is linearly independent. Thus, the control u as in (1.5) is uniquely defined, and we may write $u = [\Psi]^{-1} P_{\mathcal{U}}^{E_M^\perp}(Ay + A_{\mathrm{rc}}y - \lambda y)$, where $[\Psi] \colon \mathbb{R}^M \to \mathcal{U}$, with $u \mapsto \sum\limits_{i=1}^{M} u_i \Psi_i$, is a bijection.

The main ideas in the procedure are the following:

1. Condition (1.2a) will allow us to set up a one-to-one correspondence between the *orthogonal* projection $q = P_{E_M} y$ of the solution and the control $\sum_{i=1}^{M} u_i(t)\Psi_i(x)$. In particular we will see that, with the chosen control as in (1.5) we have $\dot{q} = -\lambda q$, $q(0) = P_{E_M} y_0$ which is exponentially stable.
2. Next we look at (1.4) as a system where q is seen as a perturbation: with $y = Q + q$, we can see that $Q = (1 - P_{E_M})y$ solves the system

$$\dot{Q} + AQ + P_{E_M^\perp}^{\mathcal{U}} A_{\mathrm{rc}} Q = -P_{E_M^\perp}^{\mathcal{U}}(Aq + A_{\mathrm{rc}}q) - \lambda P_{E_M^\perp} P_{\mathcal{U}}^{E_M^\perp} q, \quad Q(0) = (1 - P_{E_M})y_0, \tag{1.6}$$

with the complementary projections $P^{\mathcal{U}}_{E_M^\perp} \coloneqq 1 - P^{E_M^\perp}_{\mathcal{U}}$ and $P_{E_M^\perp} \coloneqq 1 - P_{E_M}$. We will see that condition (1.2b) is sufficient for the stability of (1.6), when $q = 0$. Then we observe that when q goes exponentially to zero, the perturbed system (1.6) is still stable.

Remark 1.1. The main idea behind considering (1.6) is to exploit the fact that the stability of $\dot{Q} + AQ = 0$ increases with M: $\frac{\mathrm{d}}{\mathrm{d}t}\left|Q\right|_H^2 = -2\langle AQ, Q\rangle_{V',V} \le -2\alpha_{M+1}\left|Q\right|_H^2$. To derive (1.6) from (1.4), we have used the fact that $P^{\mathcal{U}}_{E_M^\perp} AQ = AQ$, which holds because both A and $P^{\mathcal{U}}_{E_M^\perp}$ map $E_M^\perp$ into itself.

1.1. An application

Our results can be applied (but, are not restricted) to the parabolic equation

$$\dot{y}(x,t) - \nu\Delta y(x,t) + a(x,t)y(x,t) + \nabla\cdot\big(b(x,t)y(x,t)\big) - \sum_{i=1}^{M} u_i(t)\Psi_i(x) = 0, \quad y(0) = y_0, \quad y|_{\partial\Omega} = 0 \tag{1.7}$$

where $\nu > 0$ and the given functions a and b, defined for $(x,t) \in \Omega \times [0,+\infty)$, take values in $\mathbb{R}$ and $\mathbb{R}^d$, respectively. In this case $H = L^2(\Omega)$, $V = H_0^1(\Omega)$, $Ay = -\nu\Delta y$, $A_{\mathrm{rc}}y = ay + \nabla\cdot(by)$, where the pair (a,b) must be such that $A_{\mathrm{rc}} \in L^\infty((0,+\infty), \mathcal{L}(H,V'))$. We are particularly interested in the situation where the actuators are indicator functions, $\Psi_i = 1_{\omega_i}$, of small domains $\omega_i \subset \Omega$.

1.2. Motivation and previous works

The motivation for considering a system like (1.1) comes mainly from stabilization to time-dependent trajectories, where a system in the form (1.1) arises from the linearization around the targeted trajectory. *Local* stabilization to time-dependent trajectories of a given *nonlinear* system will follow from the stabilization to zero of system (1.1), together with a suitable fixed point argument.

Previous works on stabilization to time-dependent trajectories include [8, 12, 22, 23, 30]. In these references, the feedback law is not given explicitly, but rather through the solution of a differential Riccati equation. See also [1]. In [8], the controllers Ψ_i are localized in a small subset of $\omega \subset \Omega$ and are constructed from suitably truncated eigenfunctions of the Dirichlet Laplacian operator in $L^2(\Omega)^3$. It is proven that for a sufficiently large number of eigenfunctions the system can be stabilized. In [23], the controllers are constructed in a similar way (though the eigenfunctions are those of the Dirichlet Laplacian in $L^2(\mathcal{O})$ for an interval $\mathcal{O} \subseteq \Omega$ containing the support of the control) and estimates on the number of actuators are given, which depend exponentially on the norm $\left|A_{\mathrm{rc}}\right|_X$ of the operator A_{rc}, where $X = L^\infty((0,+\infty), \mathcal{L}(H,V'))$. The authors of [23] perform numerical simulations, which suggest that a better estimate might exist. They also show that taking eigenfunctions of the Laplacian in $L^2(\Omega)$ as actuators leads to an estimate which depends polynomially on $\left|A_{\mathrm{rc}}\right|_X$. In this case, the support $\overline{\omega}$ of the controller is $\overline{\Omega}$.

In [12, 22, 30], the actuators are not anymore necessarily constructed explicitly from the truncated eigenfunctions of the Laplacian operator. Moreover, a sufficient condition for stabilizability is given depending on a suitable norm $\left|1_\omega(1 - P_{\mathcal{U}})1_\omega\right|_X$ where $P_{\mathcal{U}}$ is the orthogonal projection onto $\mathcal{U} \coloneqq \operatorname{span}\{\Psi_i \mid i \in \{1,2,\dots,M\}\}$. Also in this case, estimates on the number M of actuators can be given, for suitably chosen piecewise constant actuators, but again depending *exponentially* on $\left|A_{\mathrm{rc}}\right|_X$. The numerical simulations we find in these works also suggest that a better estimate for the number of actuators could exist. The reason for the exponential dependence of the estimates on M in the above works is due to the fact that the null controllability of the system (for controls in $L^2(\omega)$) is used, for which the norm of the associated control depends exponentially on $\left|A_{\mathrm{rc}}\right|_X$. We refer the reader to [16–19, 35, 36, 41] and [4, 15, 26, 39, 42] for results related to controllability properties of parabolic-like equations. In the present manuscript, we follow a different procedure where we do not use/need the null controllability of the system and in this way we can remove the exponential dependence of the numbers of controls on $\left|A_{\mathrm{rc}}\right|_X$.

Of course, the number of required actuators will depend on the type of actuators we have at our disposal. We will keep this in mind hereafter (see, in particular, the condition for stabilizability (3.2:**CS**a)). In fact, for $A_{\rm rc}y = \rho y$ with $\rho < 0$ we cannot stabilize (1.1) if all of our actuators are orthogonal to a given eigenfunction ϕ of A whose associated eigenvalue α satisfies $\rho + \alpha < 0$.

We are going to present a setting, for the parabolic equation (1.7), where the number of actuators, which are needed for stabilizability, depends *polynomially* on the norm of $A_{\rm rc}$. Still, the support of the control can be a small subset, and the actuators can be piecewise constant.

Furthermore, while in [12, 22, 23, 30] the stabilizing feedback control operator was taken as the solution of an appropriate differential Riccati equation, here we give an explicit simple form of the feedback operator. In particular, for the simulations we do not need to solve any matrix Riccati equation. We recall that solving a matrix Riccati equation is a difficult numerical task for large scale problems.

1.3. A comment on condition (1.2b)

In the case of the parabolic equation (1.7), we know that $\alpha_{M+1} \to +\infty$ as $M \to +\infty$. Thus (1.2b) holds for sufficiently large M providing that $\mathcal{U} = \mathcal{U}(M)$ can be constructed so that $\left|P_{\mathcal{U}}^{E_M^\perp}\right|_{\mathcal{L}(H)}^2$ increases slower than α_{M+1}, *i.e.* we require that $\lim\limits_{M\to+\infty} \alpha_{M+1}\left|P_{\mathcal{U}}^{E_M^\perp}\right|_{\mathcal{L}(H)}^{-2} = +\infty$. Notice that the particular case $\mathcal{U} = E_M$ results in $P_{E_M}^{E_M^\perp}$ as an orthogonal projection. Thus $\left|P_{E_M}^{E_M^\perp}\right|_{\mathcal{L}(H)} = 1$ and the latter condition reduces to $\lim\limits_{M\to+\infty} \alpha_{M+1} = +\infty$, which is known to hold. Moreover (1.2b) reads $\alpha_{M+1} > 10\left|A_{\rm rc}\right|_X^2$, which will clearly be satisfied for big enough M.

In the case $\mathcal{U} \neq E_M$, the projection $P_{\mathcal{U}}^{E_M^\perp}$ is nonorthogonal and $\left|P_{\mathcal{U}}^{E_M^\perp}\right|_{\mathcal{L}(H)} > 1$. In this case, condition (1.2b) is not trivial and will be checked numerically. The simulations for system (1.7) show that for piecewise constant actuators (indicator functions), and for fixed total volume of the support of the actuators, we can construct $\mathcal{U}(M)$ so that $\left|P_{\mathcal{U}}^{E_M^\perp}\right|_{\mathcal{L}(H)}^2$ remains bounded. Consequently, in case we know the asymptotic behavior of the eigenvalues α_M, we can also find an estimate for the number M of actuators. In particular, for a parabolic equation as (1.7) we know that the eigenvalues of the Laplacian satisfy $\alpha_M \geq C_2 M^{\frac{2}{d}}$, see [25]. Then if $\left|P_{\mathcal{U}}^{E_M^\perp}\right|_{\mathcal{L}(H)} \leq C_1$ for all $M \geq 1$, condition (1.2b) will follow from

$$(M+1) > C_2^{-\frac{d}{2}}\left(6 + 4C_1\right)^{\frac{d}{2}}\left|A_{\rm rc}\right|_{L^\infty((0,+\infty),\mathcal{L}(H,V'))}^d. \tag{1.8}$$

Thus, M depends polynomially on the norm of $A_{\rm rc}$, which improves the exponential dependence derived in [12, 22, 23, 30]. Differently from those previous works, here the support $\overline{\omega} = \bigcup\limits_{i=1}^{M} \operatorname{supp}\Psi_i$ of the controller is not fixed *a priori* (still, its volume $|\omega|$ can be small).

While we are particularly interested in the nonautonomous case, the results are, of course, valid also in the autonomous case. However, we recall that in the autonomous case other tools, like the spectral properties of the system operator $A + A_{\rm rc}$ can be used to construct a stabilizing controller and to give estimates on the dimension of its range (in both cases of internal and boundary controls). We refer to the works [2, 5–7, 9, 33] and references therein. Unfortunately, the spectral properties of $A + A_{\rm rc}(t)$ seem to be (at least, by themselves) not appropriate for studying the stability of the corresponding nonautonomous system, see [40]. In [5, 6, 20], the feedbacks are constructed explicitly, while in [2, 7, 9, 33] they are based on Riccati equations.

The results in this work will be applied to internal controls for parabolic equations. The extension of our procedure to the case of boundary controls is not clear yet and could be the subject of a future work. Stabilizability and controllability to time-dependent trajectories by Riccati-based boundary feedback controls for parabolic-like equations was investigated in [30, 34]. See also [29] where an explicit boundary feedback is proposed for the one-dimensional case.

The rest of the paper is organized as follows. In Section 2, we recall some results concerning weak solutions for parabolic-like systems and the properties of nonorthogonal projections. In Section 3, we prove our main result. In Section 4, we discuss the sufficient condition (1.2b). In Section 5, we present the results of some numerical simulations showing the performance of our feedback and its robustness against estimation errors. Finally, in Section 6, we present additional remarks concerning our results.

1.4. Notation

We follow [12, 22, 23, 30]. We write $\mathbb{R}$ and $\mathbb{N}$ for the sets of real numbers and nonnegative integers, respectively, and we define $\mathbb{R}_r := (r, +\infty)$, for $r \in \mathbb{R}$, and $\mathbb{N}_0 := \mathbb{N} \setminus \{0\}$. We denote by $\Omega \subset \mathbb{R}^d$ a bounded open connected subset, with $d \in \mathbb{N}_0$.

For a normed space X, we denote by $|\cdot|_X$ the corresponding norm, by X' its dual, and by $\langle\cdot,\cdot\rangle_{X',X}$ the duality between X' and X. The dual space is endowed with the usual dual norm: $|f|_{X'} := \sup\{\langle f, x\rangle_{X',X} \mid x \in X \text{ and } |x|_X = 1\}$. In case X is a Hilbert space we denote the inner product by $(\cdot,\cdot)_X$.

Given an open interval $I \subseteq \mathbb{R}$ and two Banach spaces X, Y, we write $W(I, X, Y) := \{f \in L^2(I, X) \mid \partial_t f \in L^2(I, Y)\}$, where the derivative $\partial_t f$ is taken in the sense of distributions. This space is endowed with the natural norm $|f|_{W(I, X, Y)} := \bigl(|f|^2_{L^2(I, X)} + |\partial_t f|^2_{L^2(I, Y)}\bigr)^{1/2}$. In the case $X = Y$, we write $H^1(I, X) := W(I, X, X)$.

If the inclusions $X \subseteq Z$ and $Y \subseteq Z$ are continuous, where Z is a Hausdorff topological space, then we can define the Banach spaces $X \times Y$, $X \cap Y$, and $X + Y$, endowed with the norms $|(a, b)|_{X\times Y} := \bigl(|a|^2_X + |b|^2_Y\bigr)^{\frac{1}{2}}$; $|a|_{X\cap Y} := |(a, a)|_{X\times Y}$; and $|a|_{X+Y} := \inf_{(a^X, a^Y)\in X\times Y}\bigl\{|(a^X, a^Y)|_{X\times Y} \mid a = a^X + a^Y\bigr\}$, respectively. We can show that, if X and Y are endowed with a scalar product, then also $X \times Y$, $X \cap Y$, and $X + Y$ are. In case we know that $X \cap Y = \{0\}$, we say that $X + Y$ is a direct sum and we write $X \oplus Y$ instead.

Again, if X and Y are endowed with a scalar product, then also $W(I, X, Y)$ is. The space of continuous linear mappings from X into Y will be denoted by $\mathcal{L}(X,Y)$. When $X = Y$ we simply write $\mathcal{L}(X) := \mathcal{L}(X,X)$.

If the inclusion $X \subseteq Y$ is continuous, we write $X \hookrightarrow Y$. We write $X \xhookrightarrow{\text{d}} Y$, respectively $X \xhookrightarrow{\text{c}} Y$, if the inclusion is also dense, respectively compact.

The kernel and range of a linear mapping $A\colon Z \to W$, between vector spaces Z and W, will be denoted $\operatorname{Ker}(A) := \{x \in Z \mid Ax = 0\}$ and $\operatorname{Ran}(A) := \{Ax \mid x \in Z\}$, respectively.

$\overline{C}_{[a_1,\dots,a_k]}$ denotes a nonnegative function of nonnegative variables a_j that increases in each of its arguments.

Finally, C, C_i, $i = 0, 1, \dots$, stand for unessential positive constants.

2. Preliminaries

Here, we introduce the general properties we ask for the operators A and A_{rc}, and derive some results on the regularity of the solutions for system (1.1).

2.1. Assumptions on the state operators

Let H be a Hilbert space H that we will consider as pivot space, $H' = H$, and let V be another Hilbert space with $V \subseteq H$.

Assumption 2.1. $A \in \mathcal{L}(V, V')$ is an isomorphism from V onto V', A is symmetric, and $(y, z) \mapsto \langle Ay, z\rangle_{V',V}$ is a complete scalar product on V.

From now on we suppose that V is endowed with the scalar product $(y, z)_V := \langle Ay, z\rangle_{V',V}$, which still makes V a Hilbert space. Therefore, $A\colon V \to V'$ is an isometry.

Assumption 2.2. The inclusion $V \subseteq H$ is dense, continuous, and compact. That is, $V \xhookrightarrow{\text{d, c}} H$.

Necessarily, we have that

$$\langle y,\, z\rangle_{V',\,V} = (y,\, z)_H, \quad \text{for all } (y,\, z) \in H \times V,$$

and also that the operator A is densely defined in H, with domain $\mathrm{D}(A) \coloneqq \{u \in V \mid Au \in H\}$ endowed with the scalar product $(y,\, z)_{\mathrm{D}(A)} \coloneqq (Ay,\, Az)_H$, and the inclusions

$$\mathrm{D}(A) \xhookrightarrow{\mathrm{d,\,c}} V \xhookrightarrow{\mathrm{d,\,c}} H \xhookrightarrow{\mathrm{d,\,c}} V' \xhookrightarrow{\mathrm{d,\,c}} \mathrm{D}(A)'.$$

Further, A has a compact inverse $A^{-1}\colon H \to \mathrm{D}(A)$, and we can find a nondecreasing system of (repeated) eigenvalues $(\alpha_n)_{n\in\mathbb{N}_0}$ and a corresponding complete basis of eigenfunctions $(e_n)_{n\in\mathbb{N}_0}$:

$$0 < \alpha_1 \le \alpha_2 \le \dots \le \alpha_n \le \alpha_{n+1} \to +\infty \quad \text{and} \quad Ae_n = \alpha_n e_n.$$

For every $\beta \in \mathbb{R}$, we define the fractional powers A^β of A by

$$A^\beta \sum_{n=1}^{+\infty} y_n e_n \coloneqq \sum_{n=1}^{+\infty} \alpha_n^\beta y_n e_n,$$

and the corresponding domains $\mathrm{D}(A^{|\beta|}) \coloneqq \{y \in H \mid A^{|\beta|} y \in H\}$, and $\mathrm{D}(A^{-|\beta|}) \coloneqq \mathrm{D}(A^{|\beta|})'$. We have that $\mathrm{D}(A^\beta) \xhookrightarrow{\mathrm{d,\,c}} \mathrm{D}(A^{\beta_1})$, for all $\beta > \beta_1$, and we can see that $\mathrm{D}(A^0) = H$, $\mathrm{D}(A^1) = \mathrm{D}(A)$, $\mathrm{D}(A^{\frac{1}{2}}) = V$.

For the time-dependent operators we assume the following:

Assumption 2.3. For almost every $t > 0$ we have $A_{\mathrm{rc}}(t) \in \mathcal{L}(H, V')$, and there is a nonnegative constant C_{rc} such that, $|A_{\mathrm{rc}}|_{L^\infty(\mathbb{R}_0, \mathcal{L}(H,V'))} \le C_{\mathrm{rc}}$.

2.2. Weak solutions

We recall a regularity result for system

$$\dot{y}(t) + Ay(t) + A_{\mathrm{rc}}(t)y(t) + f(t) = 0, \quad y(s_0) = y_0, \tag{2.1}$$

with a general external force in place of the control, and with the initial time shifted to $t = s_0 \ge 0$. In what follows s_0 and s_1 stand for two nonnegative real numbers and I for a finite interval as follows.

$$I \coloneqq (s_0,\, s_1), \quad 0 \le s_0 < s_1. \quad \text{Further } |I| \coloneqq s_1 - s_0. \tag{2.2}$$

Throughout the paper we assume that $y_0 \in H$.

Lemma 2.4. *Given $f \in L^2(I,\, V')$, there is a weak solution $y \in W(I,\, V,\, V')$ for (2.1). Moreover, y is unique and depends continuously on the data:*

$$|y|^2_{W(I,\,V,\,V')} \le \overline{C}_{[|I|,\,C_{\mathrm{rc}}]} \left(|y(s_0)|_H^2 + |f|^2_{L^2(I,\,V')} \right).$$

The proof is omitted since it follows by well known arguments. Finally, recall that for $f \in L^2_{\mathrm{loc}}(\mathbb{R}_{s_0},\, V')$, the function y defined in $\mathbb{R}_{s_0}$ by the property that $y|_{(s_0,\,s)}$ coincides with the weak solution of (2.1) in $(s_0,\, s)$ for all $s > s_0$, is well defined, and is called the global weak solution of (2.1) in the half-line $\mathbb{R}_{s_0} = (s_0,\, +\infty)$.

2.3. Nonorthogonal projections

We are going to use nonorthogonal projection operators associated with a suitable direct sum splitting the Hilbert space H.

Definition 2.5. Two closed subspaces $\mathcal{F} \subset H$ and $\mathcal{E} \subset H$ are said complementary in the Hilbert space H if we have the direct sum $H = \mathcal{F} \oplus \mathcal{E}$. The projection onto $\mathcal{F}$ along $\mathcal{E}$ will be denoted

$$P_{\mathcal{F}}^{\mathcal{E}} \colon H \to \mathcal{F}, \quad x \mapsto x_{\mathcal{F}}$$

where $x_{\mathcal{F}}$ is defined by

$$x = x_{\mathcal{F}} + x_{\mathcal{E}} \quad \text{and} \quad (x_{\mathcal{F}}, x_{\mathcal{E}}) \in \mathcal{F} \times \mathcal{E}.$$

Remark 2.6. The continuity of the projection $P_{\mathcal{F}}^{\mathcal{E}}$ is well known (see [13], Sect. 2.4, Thm. 2.10). Notice that $P_{\mathcal{E}}^{\mathcal{F}} = 1 - P_{\mathcal{F}}^{\mathcal{E}}$. Here, "1" is understood to be the identity operator on H.

The projection $P_{\mathcal{F}}^{\mathcal{E}}$ is orthogonal if $\mathcal{E} = \mathcal{F}^{\perp}$. We shall denote orthogonal projections simply as

$$P_{\mathcal{F}} \coloneqq P_{\mathcal{F}}^{\mathcal{F}^{\perp}}.$$

Henceforth, let us fix two sets $\{f_1, f_2, \dots, f_M\} \subset H$ and $\{g_1, g_2, \dots, g_M\} \subset H$ in the Hilbert space H. We assume that the vectors of each set are linearly independent and consider the M-dimensional subspaces $\mathcal{F} \coloneqq \operatorname{span}\{f_1, f_2, \dots, f_M\}$ and $\mathcal{G} \coloneqq \operatorname{span}\{g_1, g_2, \dots, g_M\}$.

We denote the "coordinates to span" mapping as follows

$$[\mathcal{F}] \colon \mathcal{M}_{M\times 1} \to \mathcal{F}, \qquad [\mathcal{F}]v = [\mathcal{F}] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_M \end{bmatrix} \coloneqq \sum_{i=1}^{M} v_i f_i,$$

where $\mathcal{M}_{m\times n}$ denotes the space of $(m \times n)$-matrices with real entries. Notice that $[\mathcal{F}]$ depends on the (ordered) basis $\{f_1, \dots, f_M\}$. We define $[\mathcal{G}] \colon \mathcal{M}_{M\times 1} \to \mathcal{G}$ analogously. We will also denote the matrix

$$[(\mathcal{G}, \mathcal{F})_H] \coloneqq [(g_i, f_j)_H] \in \mathcal{M}_{M\times M}$$

whose entry in the ith row and jth column is $(g_i, f_j)_H$. Further for a given vector $y \in H$, we introduce the vectors

$$[(\mathcal{G}, y)_H] \coloneqq \begin{bmatrix} (g_1, y)_H \\ (g_2, y)_H \\ \vdots \\ (g_M, y)_H \end{bmatrix} \quad \text{and} \quad [(y, \mathcal{F})_H] \coloneqq \begin{bmatrix} (y, f_1)_H & (y, f_2)_H & \dots & (y, f_M)_H \end{bmatrix}.$$

Lemma 2.7. *The following conditions are equivalent:*

(a): $H = \mathcal{F} \oplus \mathcal{G}^{\perp}$,

(b): $[(\mathcal{G}, \mathcal{F})_H]$ is invertible,

(c): $P_{\mathcal{G}}\mathcal{F} = \mathcal{G}$.

Proof. We will show the implications (a)⇒(c)⇒(b)⇒(a).
(a)⇒(c): $\mathcal{G} = P_{\mathcal{G}}H = P_{\mathcal{G}}(\mathcal{F} \oplus \mathcal{G}^\perp) = P_{\mathcal{G}}\mathcal{F}$.
(c)⇒(b): Given $v \in \mathrm{Ker}[(\mathcal{G},\mathcal{F})_H]$, we have $y = [\mathcal{F}]v \in \mathcal{F} \bigcap \mathcal{G}^\perp$, which implies $P_{\mathcal{G}}y = 0$. Since $(P_{\mathcal{G}}|_{\mathcal{F}}) \colon \mathcal{F} \to \mathcal{G}$ is necessarily an isomorphism, and $0 = P_{\mathcal{G}}y = (P_{\mathcal{G}}|_{\mathcal{F}})y$, it follows that $y = 0$ and $v = [\mathcal{F}]^{-1}y = 0$.
(b)⇒(a): Let $h \in H$. We can write

$$h = z + w, \quad \text{with} \quad (z, w) \in (\mathcal{F} + \mathcal{G}^\perp) \times (\mathcal{F} + \mathcal{G}^\perp)^\perp.$$

Therefore, we have that $w \in \mathcal{F}^\perp \bigcap \mathcal{G}$, which implies that $0 = [(w,\mathcal{F})_H] = ([\mathcal{G}]^{-1}w)^\top [(\mathcal{G},\mathcal{F})_H]$, where $B^\top$ denotes the transpose of the matrix (vector) B. Necessarily $[\mathcal{G}]^{-1}w = 0$ and so $w = 0$, which gives us $h = z \in \mathcal{F} + \mathcal{G}^\perp$. We can conclude that $H = \mathcal{F} + \mathcal{G}^\perp$. Finally, if $v \in \mathcal{F} \bigcap \mathcal{G}^\perp$ we find that $0 = [(\mathcal{G}, v)_H] = [(\mathcal{G},\mathcal{F})_H]([\mathcal{F}]^{-1}v)$, which implies that $[\mathcal{F}]^{-1}v = 0$, hence $v = 0$. □

Lemma 2.8. *If $H = \mathcal{F} \oplus \mathcal{G}^\perp$, then the associated projection onto $\mathcal{F}$ along $\mathcal{G}^\perp$ is given by*

$$P_{\mathcal{F}}^{\mathcal{G}^\perp} y = \left([\mathcal{F}] \circ [(\mathcal{G},\mathcal{F})_H]^{-1}\right)[(\mathcal{G}, y)_H], \quad \textit{for all} \quad y \in H. \tag{2.3a}$$

If in addition each of the sets $\{f_1, f_2, \dots, f_M\}$ and $\{g_1, g_2, \dots, g_M\}$ is orthonormal, then

$$P_{\mathcal{F}}^{\mathcal{G}^\perp} = P_{\mathcal{F}} \circ [\mathcal{G}] \circ ([(\mathcal{G},\mathcal{F})_H][(\mathcal{F},\mathcal{G})_H])^{-1} \circ [\mathcal{G}]^{-1} \circ P_{\mathcal{G}}. \tag{2.3b}$$

Proof. Let us denote the mapping $y \mapsto \mathcal{P}y := \left([\mathcal{F}] \circ [(\mathcal{G},\mathcal{F})_H]^{-1}\right)[(\mathcal{G}, y)_H]$. It follows that for any pair $(f, h) \in \mathcal{F} \times \mathcal{G}^\perp$,

$$\mathcal{P}(f + h) = \mathcal{P}f.$$

Next notice that $[(\mathcal{G},\mathcal{F})_H]v = [(\mathcal{G}, [\mathcal{F}]v)_H]$, which implies $v = [(\mathcal{G},\mathcal{F})_H]^{-1}[(\mathcal{G}, [\mathcal{F}]v)_H]$, for any given vector $v \in \mathcal{M}_{M\times 1}$. Since $f = [\mathcal{F}][\mathcal{F}]^{-1}f$, we have

$$\mathcal{P}(f + h) = \left([\mathcal{F}] \circ [(\mathcal{G},\mathcal{F})_H]^{-1}\right)[(\mathcal{G}, [\mathcal{F}][\mathcal{F}]^{-1}f)_H] = [\mathcal{F}][\mathcal{F}]^{-1}f = f$$

and thus necessarily $P_{\mathcal{F}}^{\mathcal{G}^\perp} = \mathcal{P}$.

If the sets $\{f_1, f_2, \dots, f_M\}$ and $\{g_1, g_2, \dots, g_M\}$ are orthonormal, then it is clear that for all $(y, v) \in H \times \mathcal{M}_{M\times 1}$,

$$[(\mathcal{G}, y)_H] = ([\mathcal{G}]^{-1} \circ P_{\mathcal{G}})y \quad \text{and} \quad P_{\mathcal{F}}([\mathcal{G}]v) = [\mathcal{F}]([(\mathcal{F},\mathcal{G})_H]v)$$

which, together with (2.3a), leads us to

$$P_{\mathcal{F}}^{\mathcal{G}^\perp} y = \left([\mathcal{F}] \circ [(\mathcal{G},\mathcal{F})_H]^{-1}\right)[(\mathcal{G}, y)_H] = (P_{\mathcal{F}} \circ [\mathcal{G}] \circ [(\mathcal{F},\mathcal{G})_H]^{-1} \circ [(\mathcal{G},\mathcal{F})_H]^{-1} \circ [\mathcal{G}]^{-1} \circ P_{\mathcal{G}})y,$$

which is equivalent to (2.3b). □

Now we present a corollary on the computation of the norm $\left|P_{\mathcal{F}}^{\mathcal{G}^\perp}\right|_{\mathcal{L}(H)}$, which we will use in the numerical simulations.

Corollary 2.9. *If each of the sets* $\{f_1, f_2, \dots, f_M\}$ *and* $\{g_1, g_2, \dots, g_M\}$ *is orthonormal and if* $H = \mathcal{F} \oplus \mathcal{G}^\perp$*, then the projection onto* $\mathcal{F}$ *satisfies*

$$\left|P_{\mathcal{F}}^{\mathcal{G}^\perp}\right|_{\mathcal{L}(H)}^2 = \left(\min_\theta\{\theta \text{ is an eigenvalue of } [(\mathcal{G},\mathcal{F})_H][(\mathcal{F},\mathcal{G})_H]\}\right)^{-1}.$$

Proof. First of all, note that it follows, from Lemma 2.7, that $[(\mathcal{G},\mathcal{F})_H][(\mathcal{F},\mathcal{G})_H]$ is symmetric and positive definite. It is well known that the orthogonal projections $P_{\mathcal{F}}$ and $P_{\mathcal{G}}$ have norm 1. It is also true that the mappings $[\mathcal{G}] \in \mathcal{L}(\mathcal{M}_{M\times 1}, \mathcal{G})$ and $[\mathcal{F}] \in \mathcal{L}(\mathcal{M}_{M\times 1}, \mathcal{F})$ are bijective isometries (we suppose $\mathcal{M}_{M\times 1} \sim \mathbb{R}^M$ endowed with the usual Euclidean scalar product). Now we observe that the mapping $[(\mathcal{G},\mathcal{F})_H]^{-1} \in \mathcal{L}(\mathcal{M}_{M\times 1})$ is exactly the mapping sending the coordinates k of a vector field $g = [\mathcal{G}]k \in \mathcal{G}$ to the coordinates of the projection $P_{\mathcal{F}}^{\mathcal{G}^\perp} g \in \mathcal{F}$. Therefore, we have that

$$\begin{aligned}\left|P_{\mathcal{F}}^{\mathcal{G}^\perp}\right|_{\mathcal{L}(H)}^2 &= \sup_{k\in\mathcal{M}_{M\times 1}\setminus\{0\}} \frac{\left|[(\mathcal{G},\mathcal{F})_H]^{-1}k\right|_{\mathcal{M}_{M\times 1}}^2}{|k|_{\mathcal{M}_{M\times 1}}^2} = \sup_{k\in\mathcal{M}_{M\times 1}\setminus\{0\}} \frac{([(\mathcal{G},\mathcal{F})_H]^{-1}k)^\top [(\mathcal{G},\mathcal{F})_H]^{-1}k}{k^\top k}\\ &= \sup_{k\in\mathcal{M}_{M\times 1}\setminus\{0\}} \frac{k^\top([(\mathcal{G},\mathcal{F})_H][(\mathcal{F},\mathcal{G})_H])^{-1}k}{k^\top k} = \max_\beta\{\beta \text{ is an eigenvalue of } ([(\mathcal{G},\mathcal{F})_H][(\mathcal{F},\mathcal{G})_H])^{-1}\}.\end{aligned}$$

The proof is finished. □

3. The stabilizing feedback control

Here, we present the stabilizing control for system (1.1), provided a general condition is satisfied by the set of actuators.

Given $M \in \mathbb{N}_0$, let E_M be the space spanned by the eigenfunctions associated with the first M eigenvalues of A:

$$E_M \coloneqq \operatorname{span}\{e_n \mid Ae_n = \alpha_n e_n \text{ and } n \in \{1,2,\dots,M\}\}, \quad M \ge 1. \tag{3.1}$$

Remark 3.1. The spaces $E_M = P_{E_M} H$ are well defined as soon as the complete basis of eigenfunctions $\{e_n \mid Ae_n = \alpha_n e_n,\ n \in \mathbb{N}_0\}$ has been fixed (and further ordered for eigenfunctions corresponding to the same eigenvalue).

3.1. The condition for stabilizability

We will show that a sufficient condition for the existence of a stabilizing control taking values in a subspace $\mathcal{U} = \operatorname{span}\{\Psi_1, \Psi_2, \dots, \Psi_M\} \subset H$ is given by

$$H = \mathcal{U} \oplus E_M^\perp, \tag{3.2:$\mathbf{CS}$a}$$

$$\alpha_{M+1} > \inf_{\substack{\gamma\in\mathbb{R}_0^2,\\ (2-\gamma_1-\gamma_2)>0}} \frac{1}{(2-\gamma_1-\gamma_2)} \left(\gamma_1^{-1}\Xi_1 + \gamma_2^{-1}\left(2 + 2\left|P_{\mathcal{U}}^{E_M^\perp}\right|_{\mathcal{L}(H)}^2\right)\Xi_2\right), \tag{3.2:$\mathbf{CS}$b}$$

where

$$\Xi_1 \coloneqq \sup_{(t,Y)\in\mathbb{R}_0\times(E_M^\perp\cap V)} \frac{|\langle A_{\rm rc}(t)Y, Y\rangle_{V',V}|_{\mathbb{R}}^2}{|Y|_H^2\,|Y|_V^2} \le \left|P_{E_M^\perp} A_{\rm rc} P_{E_M^\perp}\right|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H,V'))}^2, \tag{3.3a}$$

$$\Xi_2 \coloneqq \left|P_{E_M} A_{\rm rc} P_{E_M^\perp}\right|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H,V'))}^2. \tag{3.3b}$$

In particular, we note that when $\Xi_2 = 0$, then the norm $\left|P_{\mathcal{U}}^{E_M^\perp}\right|_{\mathcal{L}(H)}^2$ of the projection plays no role in (3.2:**CS**b). This occurs, for example, in the case $A_{\rm rc} y = \rho y$, with $\rho \in \mathbb{R}$,

Observe that from (3.2:**CS**a) it necessarily follows that $\mathcal{U}$ is an M-dimensional space, because (3.2:**CS**a) implies $E_M = P_{E_M} H = P_{E_M}(\mathcal{U} \oplus E_M^\perp) = P_{E_M}\mathcal{U}$. As a consequence, the vectors in $\{\Psi_1, \Psi_2, \dots, \Psi_M\}$ are necessarily linearly independent.

Lemma 3.2. *Denoting by $E_M^{\perp,V'}$ the orthogonal complement of E_M in V', we have $E_M^\perp = H \bigcap E_M^{\perp,V'}$. Further if $\mathcal{U}$ satisfies (3.2:**CS**a), then we also have $V' = \mathcal{U} \oplus E_M^{\perp,V'}$.*

Proof. Notice that A is a bijection from E_M onto itself. For any $h \in H$ and any $\phi \in E_M$, it follows that $(h, A^{-1}\phi)_H = (A^{-\frac{1}{2}}h, A^{-\frac{1}{2}}\phi)_H = (h,\phi)_{V'}$, from which we conclude that $E_M^\perp = E_M^{\perp,V'} \bigcap H$. Now, given $x \in V'$, we may write $x \in E_M \oplus E_M^{\perp,V'} \subseteq \mathcal{U} + E_M^\perp + E_M^{\perp,V'} \subseteq \mathcal{U} + E_M^{\perp,V'}$, and we can conclude that $V' = \mathcal{U} + E_M^{\perp,V'}$. Finally, since $\mathcal{U} \subset H$, it follows $\mathcal{U} \bigcap E_M^{\perp,V'} = (\mathcal{U} \bigcap E_M^{\perp,V'}) \bigcap H = \mathcal{U} \bigcap E_M^\perp = \{0\}$. □

Lemma 3.3. *With $\mathcal{F} \in \{E_M, \mathcal{U}\}$, the projection $P_{\mathcal{F}}^{E_M^{\perp,V'}}$, in V' onto $\mathcal{F}$ along $E_M^{\perp,V'}$, is an extension of $P_{\mathcal{F}}^{E_M^\perp}$. Thus, also $P_{E_M^{\perp,V'}}^{\mathcal{F}}$ is an extension of $P_{E_M^\perp}^{\mathcal{F}}$.*

Proof. Let $h \in H$. We have that $P_{E_M}^{E_M^{\perp,V'}} h \in E_M$, and from Lemma 3.2, $(1 - P_{E_M}^{E_M^{\perp,V'}})h \in E_M^{\perp,V'} \bigcap H = E_M^\perp$, which imply that $P_{E_M}^{E_M^{\perp,V'}} h = P_{E_M}^{E_M^\perp} h$ and $P_{E_M^{\perp,V'}}^{E_M} h = P_{E_M^\perp}^{E_M} h$. Analogously, from $P_{\mathcal{U}}^{E_M^{\perp,V'}} h \in \mathcal{U}$ and $(1 - P_{\mathcal{U}}^{E_M^{\perp,V'}})h \in E_M^{\perp,V'} \bigcap H = E_M^\perp$, we conclude that $P_{\mathcal{U}}^{E_M^{\perp,V'}} h = P_{\mathcal{U}}^{E_M^\perp} h$ and $P_{E_M^{\perp,V'}}^{\mathcal{U}} h = P_{E_M^\perp}^{\mathcal{U}} h$. □

Motivated by Lemma 3.3 and for simplicity, hereafter the projections $P_{\mathcal{U}}^{E_M^{\perp,V'}} \in \mathcal{L}(V')$ and $P_{E_M^{\perp,V'}}^{\mathcal{U}} \in \mathcal{L}(V')$, in Lemma 3.3, will be denoted by $P_{\mathcal{U}}^{E_M^\perp}$ and $P_{E_M^\perp}^{\mathcal{U}}$. Similarly, we denote the orthogonal projections $P_{E_M}^{E_M^{\perp,V'}} \in \mathcal{L}(V')$ and $P_{E_M^{\perp,V'}}^{E_M} \in \mathcal{L}(V')$ simply by $P_{E_M} = P_{E_M}^{E_M^\perp}$ and $P_{E_M^\perp} = P_{E_M^\perp}^{E_M}$.

Lemma 3.4. *Assume that $\mathcal{U}$ satisfies (3.2:**CS**a), then we have the following properties.*

$$P_{E_M} = P_{E_M} P_{\mathcal{U}}^{E_M^\perp}, \qquad P_{\mathcal{U}}^{E_M^\perp} = P_{\mathcal{U}}^{E_M^\perp} P_{E_M}, \qquad \text{and} \qquad P_{E_M^\perp}^{\mathcal{U}} = P_{E_M^\perp} + P_{E_M^\perp}^{\mathcal{U}} P_{E_M}.$$

Proof. The proof is straightforward. □

We recall the interval $I = (s_0, s_1)$ in (2.2).

Lemma 3.5. *Let $\mathcal{U} \subset H$ satisfy (3.2:**CS**a), $w_0 \in E_M^\perp$, and $q \in H^1(I, E_M)$. Then there exists a weak solution, taking its values in $E_M^\perp \subset H$, for the system*

$$\dot{Q} + P_{E_M^\perp}^{\mathcal{U}}\Big((A + A_{\rm rc})(Q + q)\Big) + P_{E_M^\perp}^{\mathcal{U}}\dot{q} = 0, \qquad Q(s_0) = w_0. \tag{3.4}$$

Moreover, the solution is unique and depends continuously on the data as follows:

$$|Q|_{W(I,V,V')}^2 \le \overline{C}_{\left[|I|,\, M,\, C_{\rm rc},\, \left|P_{E_M^\perp}^{\mathcal{U}}\right|_{\mathcal{L}(V')},\, \left|P_{\mathcal{U}}^{E_M^\perp}\right|_{\mathcal{L}(H)}\right]} \Big(|w_0|_H^2 + |q|_{H^1(I,H)}^2\Big). \tag{3.5}$$

Proof. Existence of a solution satisfying (3.4) follows by standard arguments. We restrict ourselves to the following estimates, which will in particular show the structure of the constant $\overline{C}$ in (3.5). We look for Q taking its values in $V\bigcap E_M^\perp = V\bigcap P^{\mathcal U}_{E_M^\perp}H$. Observe that A maps $V\bigcap E_M^\perp$ into $E_M^{\perp,V'}\subset V'$. For $Q\in V\bigcap E_M^\perp$ we have that $AQ\in E_M^{\perp,V'}$, $P^{\mathcal U}_{E_M^\perp}AQ=AQ$, and in particular $\langle P^{\mathcal U}_{E_M^\perp}AQ,\,Q\rangle_{V',V}=|Q|_V^2$. Taking the duality product with $2Q$ in (3.4), and using Lemma 3.4, we obtain

$$\begin{aligned}\frac{\mathrm d}{\mathrm dt}|Q|_H^2 &= -2|Q|_V^2 - 2\Big\langle P^{\mathcal U}_{E_M^\perp}Aq,Q\Big\rangle_{V',V} - 2\Big\langle P^{\mathcal U}_{E_M^\perp}A_{\rm rc}(Q+q),Q\Big\rangle_{V',V} - 2\Big\langle P^{\mathcal U}_{E_M^\perp}\dot q,Q\Big\rangle_{V',V}\\ &= -2|Q|_V^2 - 2\Big\langle P_{E_M^\perp}A_{\rm rc}Q,Q\Big\rangle_{V',V} - 2\Big\langle P^{\mathcal U}_{E_M^\perp}P_{E_M}A_{\rm rc}Q,Q\Big\rangle_{V',V}\\ &\quad - 2\Big\langle P^{\mathcal U}_{E_M^\perp}Aq,Q\Big\rangle_{V',V} - 2\Big\langle P^{\mathcal U}_{E_M^\perp}A_{\rm rc}q,Q\Big\rangle_{V',V} - 2\Big\langle P^{\mathcal U}_{E_M^\perp}\dot q,Q\Big\rangle_{V',V}\end{aligned}$$

and for any given positive constants γ_1,γ_2, and γ_3,

$$\begin{aligned}\frac{\mathrm d}{\mathrm dt}|Q|_H^2 &\le -\Big(2-\gamma_1-\gamma_2-3\gamma_3\Big)|Q|_V^2 + \gamma_1^{-1}\Xi_1|Q|_H^2 + \gamma_2^{-1}\Big|P^{\mathcal U}_{E_M^\perp}P_{E_M}\Big|^2_{\mathcal L(V')}\Xi_2|Q|_H^2\\ &\quad + \gamma_3^{-1}\Big|P^{\mathcal U}_{E_M^\perp}P_{E_M}\Big|^2_{\mathcal L(V')}\Big(|q|_V^2+|\dot q|_{V'}^2\Big) + \gamma_3^{-1}\Big|P^{\mathcal U}_{E_M^\perp}\Big|^2_{\mathcal L(V')}\Xi_3|q|_H^2\end{aligned}$$

with Ξ_1 and Ξ_2 as in (3.3a)–(3.3b) and with

$$\Xi_3 := |P_{\mathcal U^\perp}A_{\rm rc}P_{E_M}|^2_{\mathcal L(V')}.$$

For $X\hookrightarrow Y$, let us denote by $|\iota|_{\mathcal L(X,Y)}$ the norm of the inclusion mapping $\iota(x)=x$ for all $x\in X$. If $X_0\subseteq X$ is a subspace of X we also denote by $\big|\iota|_{X_0}\big|_{\mathcal L(X,Y)}$ the norm of the inclusion $X_0\hookrightarrow Y$. Recall that for $p\in E_M$, we have $|p|_H^2\le\alpha_M|p|_{V'}^2$. Also, for $z\in E_M^\perp$ we have $|z|_{V'}^2\le\alpha_{M+1}^{-1}|z|_H^2$. Therefore, using Lemma 3.4 and the identity $1=P_{\mathcal U}^{E_M^\perp}+P^{\mathcal U}_{E_M^\perp}$, we find

$$\begin{aligned}\Big|P^{\mathcal U}_{E_M^\perp}P_{E_M}\Big|^2_{\mathcal L(V')} &\le \Big|\iota|_{E_M^\perp}\Big|^2_{\mathcal L(H,V')}\Big|P^{\mathcal U}_{E_M^\perp}P_{E_M}\Big|^2_{\mathcal L(H,H)}\big|\iota|_{E_M}\big|^2_{\mathcal L(V',H)}|P_{E_M}|^2_{\mathcal L(V',V')}\\ &\le \alpha_{M+1}^{-1}\Big|P^{\mathcal U}_{E_M^\perp}P_{E_M}\Big|^2_{\mathcal L(H)}\alpha_M \le \Big|P_{E_M}-P_{\mathcal U}^{E_M^\perp}\Big|^2_{\mathcal L(H)} \le 2+2\Big|P_{\mathcal U}^{E_M^\perp}\Big|^2_{\mathcal L(H)},\end{aligned}$$

which, since $|\iota|^2_{\mathcal L(H,V')}\le\alpha_1^{-1}$, leads us to

$$\begin{aligned}\frac{\mathrm d}{\mathrm dt}|Q|_H^2 &\le -\Big(2-\gamma_1-\gamma_2-3\gamma_3\Big)|Q|_V^2 + \left(\gamma_1^{-1}\Xi_1+\gamma_2^{-1}\left(2+2\Big|P_{\mathcal U}^{E_M^\perp}\Big|^2_{\mathcal L(H)}\right)\Xi_2\right)|Q|_H^2\\ &\quad + \gamma_3^{-1}\left(\Big|P^{\mathcal U}_{E_M^\perp}P_{E_M}\Big|^2_{\mathcal L(V')}+\Big|P^{\mathcal U}_{E_M^\perp}\Big|^2_{\mathcal L(V')}\Xi_3\right)\Big(\alpha_M|q|_H^2+|q|_H^2+\alpha_1^{-1}|\dot q|_H^2\Big).\end{aligned}\tag{3.6}$$

For any time $t\in I$ and any triple $(\gamma_1,\gamma_2,\gamma_3)\in\mathbb R_0^3$ such that $2-\gamma_1-\gamma_2-3\gamma_3>0$, by Gronwall's inequality we obtain

$$|Q(t)|_H^2 \le \mathrm e^{D_1(t-s_0)}\Big(|w_0|_H^2+D_2|q|^2_{H^1(I,H)}\Big)\tag{3.7}$$

with

$$D_1 := C_{\rm rc}^2\left(\gamma_1^{-1} + 2\gamma_2^{-1} + 2\gamma_2^{-1}\left|P_{\mathcal U}^{E_M^\perp}\right|_{\mathcal L(H)}^2\right),$$
$$D_2 := \gamma_3^{-1}\left(\left|P_{E_M^\perp}^{\mathcal U}P_{E_M}\right|_{\mathcal L(V')}^2 + \left|P_{E_M^\perp}^{\mathcal U}\right|_{\mathcal L(V')}^2\Xi_3\right)\left(1+\alpha_M+\alpha_1^{-1}\right),$$

where $C_{\rm rc}^2$ was defined in Assumption 2.3. Some more standard estimates lead us to (3.5).

Finally, the uniqueness of Q follows from the fact that the difference between two solutions will solve (3.4) with $q = 0$ and $w_0 = 0$, and in that case the right hand side of (3.7) vanishes. □

3.2. The explicit closed loop system

Hereafter, we suppose that Assumptions 2.1, 2.2, and 2.3 are satisfied. The following theorem shows that condition (3.2:**CS**) guarantees the existence of a finite dimensional explicit stabilizing feedback operator.

Theorem 3.6. *Let $\lambda > 0$. If $\mathcal U$ satisfies (3.2:**CS**), then the system*

$$\dot y(t) + Ay(t) + A_{\rm rc}(t)y(t) - P_{\mathcal U}^{E_M^\perp}\left(Ay(t) + A_{\rm rc}(t)y(t) - \lambda y(t)\right) = 0, \tag{3.8a}$$
$$y(0) = y_0, \tag{3.8b}$$

has a unique weak solution. Moreover, there is a pair of constants $D \ge 1$ and $\mu > 0$ such that

$$|y(t)|_H^2 \le D\mathrm{e}^{-\mu(t-s_0)}\,|y(s_0)|_H^2\,, \quad \textit{for all} \quad t \ge s_0 \ge 0. \tag{3.9}$$

Proof. Let us take $q = \mathrm{e}^{-\lambda t}P_{E_M}y_0$ and $w_0 = P_{E_M^\perp}y_0$, and let Q be the corresponding solution to system (3.4). Then, we observe that $y = Q + q$ solves (3.8). In fact

$$\begin{aligned}\dot y &= -P_{E_M^\perp}^{\mathcal U}\left((A + A_{\rm rc})\,(Q + q)\right) - P_{E_M^\perp}^{\mathcal U}\dot q + \dot q = -P_{E_M^\perp}^{\mathcal U}\left((A + A_{\rm rc})\,y\right) + P_{\mathcal U}^{E_M^\perp}\dot q\\ &= -\left(1 - P_{\mathcal U}^{E_M^\perp}\right)\left((A + A_{\rm rc})\,y\right) - \lambda P_{\mathcal U}^{E_M^\perp}q,\end{aligned}$$

where we have used $1 = P_{\mathcal U}^{E_M^\perp} + P_{E_M^\perp}^{\mathcal U}$ (*cf.* Rem. 2.6). Therefore (3.8a) follows. Clearly we also have $y(0) = Q(0) + q(0) = w_0 + P_{E_M}y_0 = y_0$.

From (3.2:**CS**b) we can choose a pair $(\widetilde\gamma_1, \widetilde\gamma_2)$ such that $2 - \widetilde\gamma_1 - \widetilde\gamma_2 > 0$ and

$$\mu := (2 - \widetilde\gamma_1 - \widetilde\gamma_2)\alpha_{M+1} - \left(\widetilde\gamma_1^{-1}\Xi_1 + \widetilde\gamma_2^{-1}\left(2 + 2\left|P_{\mathcal U}^{E_M^\perp}\right|_{\mathcal L(H)}^2\right)\Xi_2\right) > 0.$$

Then we can also choose a small enough $\widetilde\gamma_3 > 0$ such that

$$\mu_{\widetilde\gamma} := (2 - \widetilde\gamma_1 - \widetilde\gamma_2 - 3\widetilde\gamma_3)\alpha_{M+1} - \left(\widetilde\gamma_1^{-1}\Xi_1 + \widetilde\gamma_2^{-1}\left(2 + 2\left|P_{\mathcal U}^{E_M^\perp}\right|_{\mathcal L(H)}^2\right)\Xi_2\right) > 0.$$

Therefore, from (3.6) and the Gronwall's inequality, it follows that, for any $t \ge s_0 \ge 0$,

$$\begin{aligned} |Q(t)|_H^2 &\le \mathrm{e}^{-\mu_{\widetilde{\gamma}}(t-s_0)}|Q(s_0)|_H^2 + \widetilde{D}_2 \int_{s_0}^t \mathrm{e}^{-\mu_{\widetilde{\gamma}}(t-s)}(|q(s)|_H^2 + |\dot{q}(s)|_H^2 \,\mathrm{d}s \\ &\le \mathrm{e}^{-\mu_{\widetilde{\gamma}}(t-s_0)}|Q(s_0)|_H^2 + \widetilde{D}_2(1+\lambda^2) \int_{s_0}^t \mathrm{e}^{-\mu_{\widetilde{\gamma}}(t-s)}\mathrm{e}^{-2\lambda(s-s_0)}|q(s_0)|_H^2 \,\mathrm{d}s \end{aligned} \tag{3.10}$$

with $\widetilde{D}_2 \coloneqq \widetilde{\gamma}_3^{-1}\left(\left|P_{E_M^\perp}^{\mathcal{U}} P_{E_M}\right|_{\mathcal{L}(V')}^2 + \left|P_{E_M^\perp}^{\mathcal{U}}\right|_{\mathcal{L}(V')}^2 \Xi_3\right)(1+\alpha_M+\alpha_1^{-1})$.

Now, if $2\lambda \ne \mu_{\widetilde{\gamma}}$, we obtain

$$\begin{aligned} |Q(t)|_H^2 &\le \mathrm{e}^{-\mu_{\widetilde{\gamma}}(t-s_0)}|Q(s_0)|_H^2 + \widetilde{D}_2(1+\lambda^2)|q(s_0)|_H^2 \mathrm{e}^{-\mu_{\widetilde{\gamma}}t+2\lambda s_0}\frac{\mathrm{e}^{(\mu_{\widetilde{\gamma}}-2\lambda)t} - \mathrm{e}^{(\mu_{\widetilde{\gamma}}-2\lambda)s_0}}{\mu_{\widetilde{\gamma}}-2\lambda} \\ &\le \mathrm{e}^{-\mu_{\widetilde{\gamma}}(t-s_0)}|Q(s_0)|_H^2 + \widetilde{D}_2(1+\lambda^2)|q(s_0)|_H^2|\mu_{\widetilde{\gamma}}-2\lambda|^{-1}\mathrm{e}^{-\mu(t-s_0)} \end{aligned}$$

with $\mu \coloneqq \min\{\mu_{\widetilde{\gamma}}, 2\lambda\} > 0$.

On the other hand, if $2\lambda = \mu_{\widetilde{\gamma}}$ we obtain

$$\begin{aligned} |Q(t)|_H^2 &\le \mathrm{e}^{-\mu_{\widetilde{\gamma}}(t-s_0)}|Q(s_0)|_H^2 + \widetilde{D}_2(1+\lambda^2)|q(s_0)|_H^2 \int_{s_0}^t \mathrm{e}^{-\mu_{\widetilde{\gamma}}(t-s_0)} \,\mathrm{d}s \\ &\le \mathrm{e}^{-\mu_{\widetilde{\gamma}}(t-s_0)}|Q(s_0)|_H^2 + \widetilde{D}_2(1+\lambda^2)|q(s_0)|_H^2\mathrm{e}^{-\mu_{\widetilde{\gamma}}(t-s_0)}(t-s_0) \\ &\le \mathrm{e}^{-\mu_{\widetilde{\gamma}}(t-s_0)}|Q(s_0)|_H^2 + \widetilde{D}_2(1+\lambda^2)|q(s_0)|_H^2\mathrm{e}^{-\mu(t-s_0)}e^{-1}(\mu_{\widetilde{\gamma}}-\mu)^{-1} \end{aligned}$$

for any $\mu < \mu_{\widetilde{\gamma}}$. Notice that $e^{-1}(\mu_{\widetilde{\gamma}}-\mu)^{-1} = \max_{s>0} \mathrm{e}^{-(\mu_{\widetilde{\gamma}}-\mu)s}s$.

Thus, in either case, there exists a constant D_3 such that

$$|Q(t)|_H^2 \le \mathrm{e}^{-\mu_{\widetilde{\gamma}}(t-s_0)}|Q(s_0)|_H^2 + D_3\mathrm{e}^{-\mu(t-s_0)}|q(s_0)|_H^2,$$

with $\mu < \min\{\mu_{\widetilde{\gamma}}, 2\lambda\}$. This implies that

$$\begin{aligned} |y(t)|_H^2 = |Q(t)|_H^2 + |q(t)|_H^2 &\le \mathrm{e}^{-\mu_{\widetilde{\gamma}}(t-s_0)}\left|Q(s_0)\right|_H^2 + (1+D_3)\mathrm{e}^{-\mu(t-s_0)}\left|q(s_0)\right|_H^2 \\ &\le D\mathrm{e}^{-\mu(t-s_0)}\left|y(s_0)\right|_H^2 \end{aligned} \tag{3.11}$$

with $\mu < \min\{\mu_{\widetilde{\gamma}}, 2\lambda\} > 0$ and $D \coloneqq 1 + D_3 \ge 1$. □

Next we prove that our feedback operator in Theorem 3.6 is bounded.

Theorem 3.7. *The feedback operator in Theorem 3.6*

$$y \to \mathcal{K}(t)y \coloneqq P_{\mathcal{U}}^{E_M^\perp}(Ay + A_{\mathrm{rc}}(t)y - \lambda y)$$

and its associated control $\eta(t) \coloneqq \mathcal{K}(t)y(t)$ are bounded:

$$|\mathcal{K}|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H))} \le \left|P_{\mathcal{U}}^{E_M^\perp}\right|_{\mathcal{L}(V',H)}\left(\alpha_M^{\frac{1}{2}} + |A_{\mathrm{rc}}|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H,V'))} + \lambda\alpha_1^{-\frac{1}{2}}\right), \tag{3.12a}$$

$$\left|\mathrm{e}^{\frac{\varepsilon}{2}(t-s_0)}\eta\right|_{L^2(\mathbb{R}_{s_0},H)}^2 \le |\mathcal{K}|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H))}^2 \frac{D}{\mu-\varepsilon}\left|y(s_0)\right|_H^2, \quad \textit{for any} \quad \varepsilon < \mu, \tag{3.12b}$$

with μ as in (3.9).

Proof. Recalling that $P_{\mathcal{U}}^{E_M^\perp} = P_{\mathcal{U}}^{E_M^\perp} P_{E_M}$ and $P_{E_M} A = A P_{E_M}$, we find

$$|\mathcal{K}(t)|_{\mathcal{L}(H)} \le \left| P_{\mathcal{U}}^{E_M^\perp} \right|_{\mathcal{L}(V',H)} \left(\left|A|_{E_M}\right|_{\mathcal{L}(H,V')} + |A_{\mathrm{rc}}(t)|_{\mathcal{L}(H,V')} + \lambda \alpha_1^{-\frac{1}{2}} \right),$$

which implies that

$$|\mathcal{K}|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H))} \le \left| P_{\mathcal{U}}^{E_M^\perp} \right|_{\mathcal{L}(V',H)} \left(\alpha_M^{\frac{1}{2}} + |A_{\mathrm{rc}}|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H,V'))} + \lambda \alpha_1^{-\frac{1}{2}} \right).$$

Now from (3.9) we find that the associated control $\eta(t)$ satisfies, for any $\varepsilon < \mu$,

$$\begin{aligned} \left| \mathrm{e}^{\frac{\varepsilon}{2}(t-s_0)} \eta \right|^2_{L^2(\mathbb{R}_{s_0},H)} &\le |\mathcal{K}|^2_{L^\infty(\mathbb{R}_0,\mathcal{L}(H))} \left| \mathrm{e}^{\frac{\varepsilon}{2}(t-s_0)} y \right|^2_{L^2(\mathbb{R}_{s_0},H)} \\ &\le |\mathcal{K}|^2_{L^\infty(\mathbb{R}_0,\mathcal{L}(H))} \frac{D}{\mu-\varepsilon} |y(s_0)|^2_H , \end{aligned}$$

which completes the proof. □

Remark 3.8. We can prove that under the same condition (3.2:**CS**) the feedback given by $y \to \mathcal{K}_*(t)y := -P_{\mathcal{U}}^{E_M^\perp}(A_{\mathrm{rc}}(t)y - \lambda y)$ is also stabilizing (which corresponds to take $q_* = P_{E_M} y$ as the solution of the system $\dot{q}_* = -Aq_* - \lambda q_*, \quad q_*(0) = P_{E_M} y(0)$). In this case, in (3.12a) we obtain a better estimate $|\mathcal{K}_*|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H))} \le \left| P_{\mathcal{U}}^{E_M^\perp} \right|_{\mathcal{L}(V',H)} \left(|A_{\mathrm{rc}}|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H,V'))} + \lambda \alpha_1^{-\frac{1}{2}} \right)$. However, from simulations we have performed (and will not present here) it is not clear whether the associated bound constant D in (3.9) (and in (3.12b)) will be smaller for $\mathcal{K}_*$ than for $\mathcal{K}$. Thus, we have chosen to consider the feedback $\mathcal{K}$ only, whose associated projection $q = P_{E_M} y$ is more explicit.

Remark 3.9. We observe that the stabilization result in Theorem 3.6 still holds true when E_M is replaced by a space spanned by any set of M linearly independent eigenfunctions, say by $\widetilde{E}_M = \mathrm{span}\{e_{\sigma(i)} \mid i \in \{1,2,\dots,M\}\}$ with $\sigma\colon \mathbb{N}_0 \to \mathbb{N}_0$ being an increasing function. For that we just need to replace the condition (3.2:**CS**) by the analogous one with

$$\left(\widetilde{E}_M, \min\{\alpha_j \mid \alpha_j \notin \{\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \dots \alpha_{\sigma(M)}\}\} \right)$$

in the place of (E_M, α_{M+1}). This will be used in Section 4.8.

4. Remarks on the proposed sufficient condition for stability

First we investigate condition (3.2:**CS**) in a few particular situations. Then we explain how we may check our condition numerically in a general situation. Finally, numerical simulations are presented for parabolic equations.

4.1. The case $M = 0$. Free dynamics

Taking no control in (3.8a) is equivalent to taking $\mathcal{U} = \{0\}$. That is, the free dynamics is followed. This corresponds to take $M = 0$, once we define $E_0 := \{0\}$. Indeed, we have $H = \mathcal{U} \oplus H = \mathcal{U} \oplus E_M^\perp$, and $P_{E_M} = 0 = P_{\mathcal{U}}^{E_M^\top}$. In particular (3.2:**CS**a) is satisfied.

Since, for $M = 0$ we have that $\Xi_2 = 0$, the condition for stability (3.2:**CS**b) reads for free dynamics

$$\alpha_1 > \inf_{\substack{\gamma_1\in\mathbb{R}_0,\\ (2-\gamma_1)>0}} \frac{1}{(2-\gamma_1)\gamma_1}\Xi_1 = \Xi_1 = \sup_{(t,Y)\in\mathbb{R}_0\times V} \frac{|\langle A_{\rm rc}(t)Y, Y\rangle_{V',V}|_{\mathbb{R}}^2}{|Y|_H^2\,|Y|_V^2}.$$

4.2. The particular case of a constant reaction

From condition (3.2:**CS**) we see that $\left|P_{\mathcal{U}}^{E_M^\perp}\right|_{\mathcal{L}(H)}$ will play no role when $\Xi_2 = \left|P_{E_M} A_{\rm rc} P_{E_M^\perp}\right|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H,V'))}^2 = 0$. This is the case for a constant reaction $A_{\rm rc}y = \rho y$.

With $\rho < 0$, the uncontrolled system

$$\dot{y} + Ay + \rho y = 0$$

is not stable if $\rho < -\alpha_1$. In this case $\left|P_{E_M^\perp} A_{\rm rc} P_{E_M^\perp}\right|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H,V'))}^2 = \alpha_{M+1}^{-1}\rho^2$, and the sufficient condition (3.2:**CS**) reduces to

$$\begin{aligned} &H = \mathcal{U} \oplus E_M^\perp\\ &\alpha_{M+1} > \inf_{\substack{\gamma_1\in\mathbb{R}_0,\\ (2-\gamma_1)>0}} \frac{1}{(2-\gamma_1)}\gamma_1^{-1}\alpha_{M+1}^{-1}\rho^2 = \alpha_{M+1}^{-1}\rho^2. \end{aligned}$$

In particular, here the norm $\left|P_{\mathcal{U}}^{E_M^\perp}\right|_{\mathcal{L}(H)}$ plays no role. That is, it is sufficient to take M actuators such that $\alpha_{M+1} > \rho$ and $H = \mathcal{U} \oplus E_M^\perp$. Notice, however that $\left|P_{E_M^\perp}^{\mathcal{U}} P_{E_M}\right|_{\mathcal{L}(V')}^2$ and $\left|P_{E_M^\perp}^{\mathcal{U}}\right|_{\mathcal{L}(V')}^2 |P_{\mathcal{U}^\perp} A_{\rm rc} P_{E_M}|_{\mathcal{L}(V')}^2$ still play a role to the constant D in (3.11), and thus also in the norm of the feedback control in (3.12b).

4.3. The particular case of a conservative convection

It is well known that the parabolic system (1.7), under homogeneous Dirichlet boundary conditions is stable with $a = 0$ and with a general $b \in L^\infty(\mathbb{R}_0 \times \Omega, \mathbb{R}^d)$ satisfying $\nabla \cdot b = 0$. We can see that condition (3.2:**CS**) reflects this fact. Indeed, in this case we have $\langle A_{\rm rc}y, y\rangle_{V',V} = \langle \nabla \cdot (by), y\rangle_{V',V} = 0$. Thus, for $M = 0$, we obtain that $\Xi_1 = 0$. From Section 4.1 we conclude that the uncontrolled system is stable, since $\alpha_1 > 0$.

4.4. Checking the proposed sufficient condition numerically

By considering the values $(\gamma_1, \gamma_2) = (\frac{1}{2}, \frac{1}{2})$ we conclude that (3.2:**CS**b) holds if

$$\alpha_{M+1} > \left(6 + 4\left|P_{\mathcal{U}}^{E_M^\perp}\right|_{\mathcal{L}(H)}^2\right) |A_{\rm rc}|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H,V'))}^2 . \tag{4.1}$$

which is exactly (1.2b) in Section 1.

Once we know $|A_{\rm rc}|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H,V'))}^2$ and α_{M+1}, we can investigate (4.1), by computing $\left|P_{\mathcal{U}}^{E_M^\perp}\right|_{\mathcal{L}(H)}^2$.

In applications the actuators will be in a suitable class $\mathbb{U}$ (of functions). For example, for parabolic equations (1.7), we could consider actuators which are supported in an *a priori* given subset, or we could consider piecewise constant actuators.

The question is: given $|A_{\rm rc}|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H,V'))}^2$ can we find M and actuators $\{\Psi_i^M \mid i = 1, 2, \dots, M\} \subset \mathbb{U}$ so that $\mathcal{U} = \mathcal{U}(M) = \operatorname{span}\{\Psi_i^M \mid i = 1, 2, \dots, M\}$ satisfies (4.1)?

For a general $|A_{\rm rc}|^2_{L^\infty(\mathbb{R}_0,\mathcal{L}(H,V'))}$, we can say that (4.1) will hold for big enough M if we know that

$$\lim_{M\to+\infty} \alpha_{M+1} \left| P_{\mathcal{U}}^{E_M^\perp} \right|^{-2}_{\mathcal{L}(H)} = +\infty. \tag{4.2}$$

This results in the question: can we find a large enough M and a set of actuators $\{\Psi_i^M \mid i = 1, 2, \dots, M\}$ so that their span, $\mathcal{U} = \mathcal{U}(M) = \operatorname{span}\{\Psi_i^M \mid i = 1, 2, \dots, M\}$, satisfies (4.2)? We recall that the answer is not trivial when $\mathcal{U} \neq E_M$ because $\left| P_{\mathcal{U}}^{E_M^\perp} \right|_{\mathcal{L}(H)} > 1$ and, depending on $\mathcal{U}$, may take arbitrary large values.

We assume here that the complete system of eigenfunctions $\{e_i \mid i \in \mathbb{N}_0\}$ of A is orthonormal in the H-scalar product. We can also suppose that $\{\Psi_i^M \mid i = 1, 2, \dots, M\}$ is an orthonormal family in the H-scalar product, otherwise we just orthonormalize it.

From Corollary 2.9, condition (4.2) will follow from

$$\lim_{M\to+\infty} \alpha_{M+1} \min \operatorname{Eig}([\Theta(M)]) = +\infty, \tag{4.3}$$

where for simplicity we have denoted by $\operatorname{Eig}(([\Theta(M)])$ the set of eigenvalues of the symmetric matrix

$$[\Theta(M)] \coloneqq [(E_M, \mathcal{U})_H][(\mathcal{U}, E_M)_H]. \tag{4.4}$$

Now we show how we are going to check (4.3) numerically, in the case of a parabolic equation as in (1.7) under homogeneous Dirichlet boundary conditions, with $A = -\nu\Delta\colon V \to V'$, $\nu > 0$. In this setting, we will have the spaces $H = L^2(\Omega)$, $V = H^1_0(\Omega)$, and $V' = H^{-1}(\Omega)$.

We will consider a given triangulation Ω_D of the domain Ω with nodes $\mathbf{x}$. As basis functions we take the usual piecewise linear (hat) functions. For a given (regular enough) function u, let $u(\mathbf{x})$ denote the column vector of the values of u at the nodes. We denote by $\mathbf{M}$ the mass matrix associated to our finite-element subspace in H and take the following discrete approximations:

$$[(\mathcal{U}, E_M)_H] \approx \mathbf{P}(M) \coloneqq \begin{bmatrix} \Psi_1^M(\mathbf{x})^\top \\ \Psi_2^M(\mathbf{x})^\top \\ \vdots \\ \Psi_M^M(\mathbf{x})^\top \end{bmatrix} \mathbf{M} \begin{bmatrix} e_1(\mathbf{x}) & e_2(\mathbf{x}) & \dots & e_M(\mathbf{x}) \end{bmatrix},$$
$$[\Theta(M)] \approx [\boldsymbol{\Theta}(M)] \coloneqq \mathbf{P}(M)^\top \mathbf{P}(M),$$

and

$$\min \operatorname{Eig}([\Theta(M)]) \approx \min \operatorname{Eig}([\boldsymbol{\Theta}(M)]).$$

The matrix $\mathbf{P}(M)^\top \mathbf{P}(M)$ is symmetric and positive semidefinite. By (3.2:**CS**a) and Lemma 2.7 the matrix $[\Theta(M)]$ is invertible. Up to discretization, this translates to the assumption that $[\boldsymbol{\Theta}(M)]$ is positive definite, that is, $\min \operatorname{Eig}([\boldsymbol{\Theta}(M)]) > 0$.

Denoting $\vartheta_M \coloneqq \min \operatorname{Eig}([\boldsymbol{\Theta}(M)])$, condition (4.3) will read

$$\lim_{M\to+\infty} \alpha_{M+1} \vartheta_M = +\infty. \tag{4.5}$$

4.5. Construction of the actuators in one dimension

We will perform some 1D simulations on the interval $\Omega = (0, L)$, for piecewise constant actuators and $A = -\nu\Delta = -\nu\partial_{xx}$. For a suitable chosen $\mathcal{U} = \mathcal{U}(M)$, the simulations suggest that $\vartheta_M \geq \delta > 0$ remains away from zero, with δ independent of M, when the total length of the actuators support is fixed. This confirms (4.5).

In this 1D case, the finite elements are chosen with respect to the partition $\mathbf{x} = \begin{bmatrix}0h & 1h & \dots & Nh\end{bmatrix}^\top$ of Ω, where $h = L/N$. We fix the total length of the support of the control, by setting the total volume to be rL, with $r \in (0, 1)$.

Piecewise constant actuators $\{\Psi_i^M \mid i = 1, 2, \dots, M\}$ are constructed as follows: consider the Mth normalized eigenfunction $e_M(x) = (\frac{2}{L})^{\frac{1}{2}} \sin(\frac{M\pi x}{L})$ whose extrema are located at

$$q_j = (2j - 1)\frac{L}{2M}, \quad j = 1, 2, \dots, M.$$

Now we set the actuators as the indicator functions

$$\Psi_i^M(x) \coloneqq 1_{\omega_i}(x) \coloneqq \begin{cases} 1, & \text{if } x \in \omega_i; \\ 0, & \text{if } x \in (0, L) \setminus \overline{\omega_i}; \end{cases},$$

with

$$\omega_i \coloneqq (q_j - \frac{rL}{2M}, q_j + \frac{rL}{2M}), \quad i = 1, 2, \dots, M.$$

Note that all actuators have the same support length: $\mathrm{length}(\mathrm{supp}(\Psi_i^M)) = \frac{rL}{M}$, and the total length of the support of the control is independent of M: $\mathrm{length}\left(\bigcup\limits_{i=1}^M \mathrm{supp}(\Psi_i^M)\right) = \sum\limits_{i=1}^M \mathrm{length}(\mathrm{supp}(\Psi_i^M)) = rL$.

4.6. Numerical examples

Here, we perform simulations for the computation of $\vartheta_M = \left|P_{\mathcal{U}}^{E_M^\perp}\right|_{\mathcal{L}(H)}^{-2}$ appearing in the sufficient condition (4.5). We would like to have $\vartheta_M > 0$ as large as possible, and remaining away from 0 as M increases.

Our actuators $1_{\omega_i} = 1_{\omega_i}(x)$ will be constructed as in Section 4.5 (in the figures "$\mathcal{D}_{\mathrm{act}} = \mathrm{mxe}$" underlines that the actuators are distributed following the extrema of the eigenfunction e_M). Figure 1 shows those actuators for selected values of M, for $L = 1$.

To compute ϑ_M we proceeded as follows. We considered the basis $\{\sin(\frac{i\pi x}{L}) \mid i \in \{1, 2, \dots, M\}\}$ for E_M and the basis $\{1_{\omega_i} \mid i \in \{1, 2, \dots, M\}\}\}$ for $\mathcal{U}$. Then we orthonormalized these bases, constructed $[\mathbf{\Theta}(M)]$, and computed ϑ_M. The orthonormalization was done numerically through the Gram–Schmidt procedure (in the "mass" scalar product $(v, w) \mapsto w^\top \mathbf{M} v$).

In Figure 2a, we see how ϑ_M depends on M, for several rates r of the control support volume rL. It seems that, for fixed r, ϑ_M tends to a positive constant, thus remaining away from zero.

Further ϑ_M increases with $r \in (0, \frac{1}{2})$. That is, as we would intuitively expect, the norm of $P_{\mathcal{U}}^{E_M}$ gets larger as the volume of the control support gets smaller.

In Figure 2b, we see how ϑ_M depends on r, for a fixed given pair (M, L).

The simulations suggest that there exist a constant $C = C(r) > 0$ so that $\left|P_{\mathcal{U}}^{E_M^\perp}\right|_{\mathcal{L}(H)}^2 \leq C(r)$. Notice that for $\Omega = (0, L)$, the eigenvalues of $A = -\nu\Delta$ are $\nu\frac{\pi^2}{L^2}M^2$ which allows us to obtain an estimate

$$M > \nu^{-\frac{1}{2}} \tfrac{L}{\pi} \left(6 + 4C(r)\right)^{\frac{1}{2}} \left|A_{\mathrm{rc}}\right|_{L^\infty(\mathbb{R}_0, \mathcal{L}(H, V'))}$$

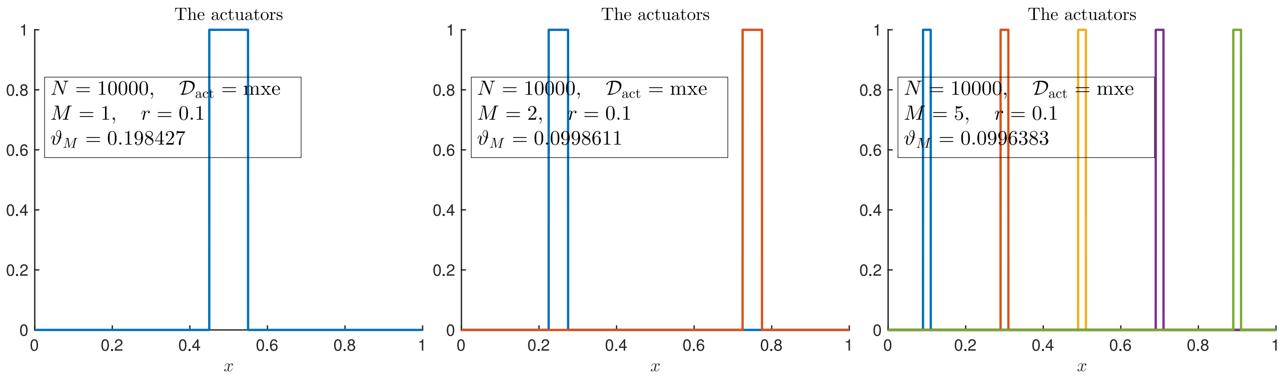
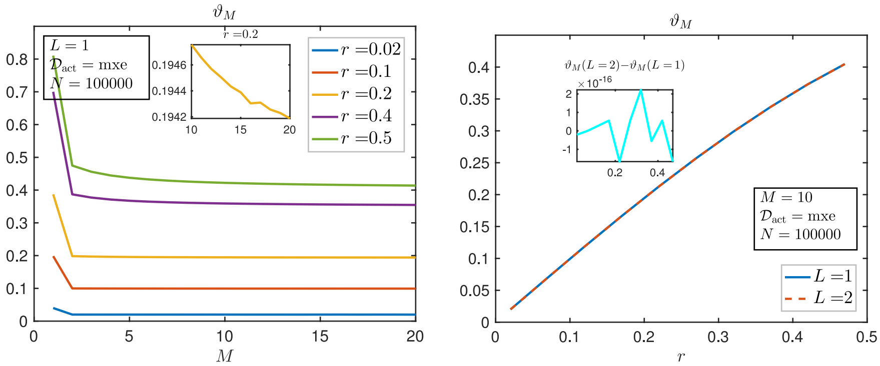


FIGURE 1. The family of actuators.



(a) Dependence on M. (b) Dependence on r

FIGURE 2. The smallest eigenvalue of $[\Theta(M)]$.

on the number of actuators which allow us to stabilize the system.

In Figure 2b, we can also see that such dependence does not depend on L, once (r, M) is given. Indeed this follows from the fact that for two given intervals $(0, L_1)$ and $(0, L_2)$, for normalized eigenfunctions $e_i^{L_n} = (\frac{2}{L_n})^{\frac{1}{2}} \sin(\frac{i\pi x}{L_n}) \in L^2((0, L_n))$, and for normalized actuators $\psi_{a,b}^{L_n}(x) = (\frac{1}{L_n(b-a)})^{\frac{1}{2}} 1_{(L_n a, L_n b)}(x) \in L^2((0, L_n))$, with $1 > b > a > 0$, we have that,

$$\left(e_i^{L_1}, \psi_{a,b}^{L_1}\right)_{L^2((0,L_1))} = \left(e_i^{L_1}, \psi_{a,b}^{L_2}\right)_{L^2((0,L_2))}. \tag{4.6}$$

This means that, for normalized eigenfunctions and actuators, the entries in $[(E_M, \mathcal{U})_H]$ and in $[\Theta(M)]$ do not depend on the length L_n provided, when taking actuators $1_{(L_1 a, L_1 b)}$ in $(0, L_1)$, we take the corresponding actuators $1_{(L_2 a, L_2 b)}$ on $(0, L_2)$.

4.7. On the placement of the actuators

Concerning (3.2:**CS**b), intuitively we expect that distributing the actuators over the interval $\Omega = (0, L)$ is better than concentrating them in some region. This is confirmed in Figures 3 and 4. The uniformly distributed actuators 1_{ω_i} ("$\mathcal{D}_{\rm act} = $ uni" in figures) correspond to

$$\omega_i = \left(\tfrac{iL}{M+1} - \tfrac{rL}{2M}, \tfrac{iL}{M+1} + \tfrac{rL}{2M}\right), \quad i = 1, \dots, M,$$

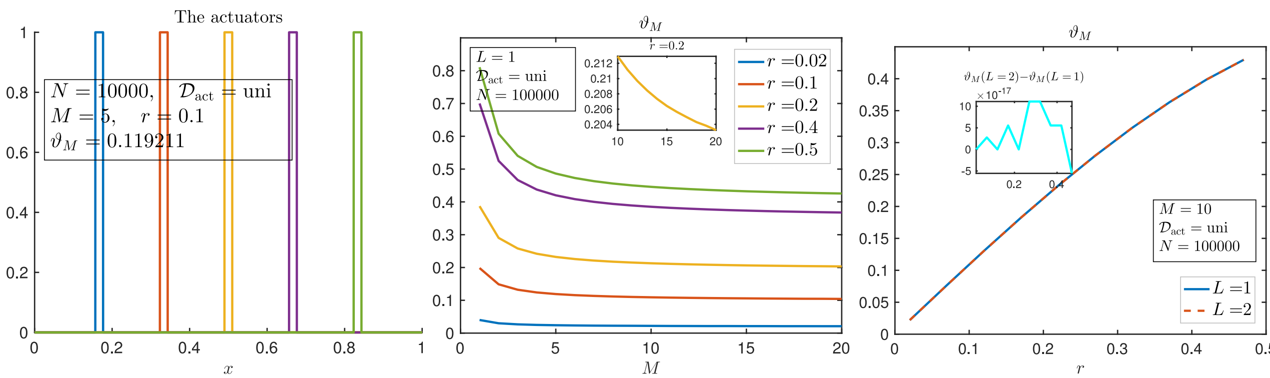


FIGURE 3. Uniformly distributed actuators.

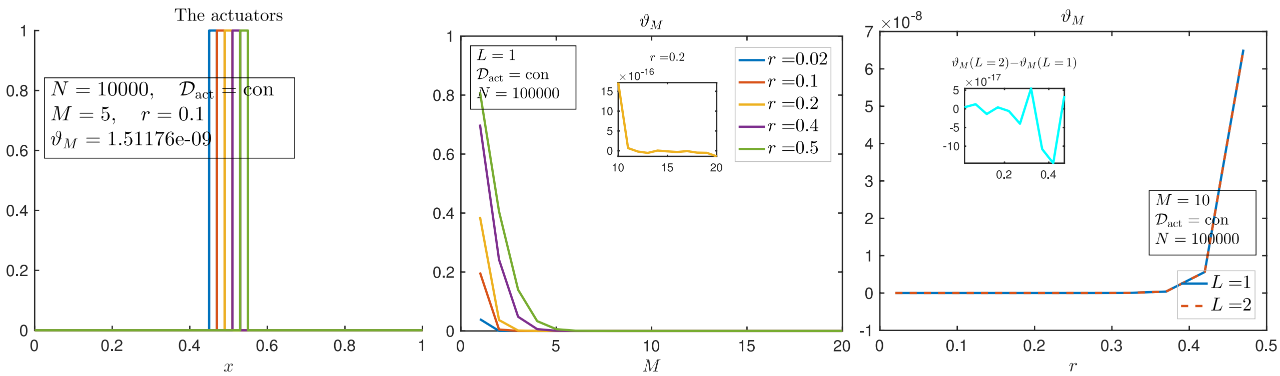


FIGURE 4. Concentrated actuators.

while the concentrated actuators ("$\mathcal{D}_{\rm act} = {\rm con}$" in figures) correspond to a uniform partition of $\left(\frac{L}{2} - \frac{rL}{2}, \frac{L}{2} + \frac{rL}{2}\right)$:

$$\omega_j = \left(\tfrac{L}{2} - \tfrac{rL}{2} + (j-1)\tfrac{rL}{M}, \tfrac{L}{2} - \tfrac{rL}{2} + j\tfrac{rL}{M}\right), \quad j = 1, \dots, M.$$

We see that the distributed actuators present an analogous behavior as those constructed as in Section 4.5, and ϑ_M remains away from 0. For the concentrated actuators ϑ_M reaches values very close to zero. The simulations were done in Matlab with machine "precision" `eps` $\approx \frac{2.2204}{10^{16}}$.

On the other hand, Lemma 4.2 presented below shows that (3.2:**CS**a) is always satisfied for the actuator placements as in Figures 1, 3, and 4. Let

$$\omega_i = (\omega_i^l, \omega_i^r) \subset (0, L), \quad i \in \{1, 2, \dots, M\}, \quad \text{with} \quad \omega_j^l < \omega_j^r \le \omega_{j+1}^l \quad \text{and} \quad j \in \{1, 2, \dots, M-1\}, \tag{4.7}$$

and let us set the actuators $\Psi_i^M = 1_{\omega_i}$, its linear span $\mathcal{U} = \operatorname{span}\{\Psi_i^M \mid i \in \{1, 2, \dots, M\}\}$, and the spaces $H = L^2(\Omega) = L^2((0, L))$, $E_M = \operatorname{span}\{(\frac{2}{L})^{\frac{1}{2}} \sin(\frac{i\pi x}{L}) \mid i \in \{1, 2, \dots, M\}\}$, and $E_M^\perp = \operatorname{span}\{(\frac{2}{L})^{\frac{1}{2}} \sin(\frac{i\pi x}{L}) \mid i \in \mathbb{N},\ i \ge M+1\}$. We will need the following result whose proof is given in Appendix A.1.

Proposition 4.1. *Any given function $h \in E_M \setminus \{0\}$ has at most $M-1$ zeros in $(0, L)$.*

Lemma 4.2. *With the domains as in (4.7), we have that $H = \mathcal{U} \oplus E_M^\perp$.*

Proof. By Lemma 2.7 and identity (4.6), it is enough to consider the case $L=\pi$ and prove that the matrix $[(\frac{2}{L})^{\frac{1}{2}}\sin(ix),\Psi_j^M)_H]$ is invertible.

Let us be given a row matrix vector $v=\begin{bmatrix}v_1 & v_2 & \dots & v_M\end{bmatrix}\in\mathcal{M}_{1\times M}$ so that $v[\sin(ix),\Psi_j^M)_H]=0$.

From $0=v[\sin(ix),\Psi_j^M)_H]=\left[\left(\sum\limits_{i=1}^M v_i\sin(ix),\Psi_j^M\right)_H\right]$ we have that $h\coloneqq\sum\limits_{i=1}^M v_i\sin(ix)$ satisfies

$$\int_{\omega_j} h(x)\,\mathrm{d}x=\left(h,\Psi_j^M\right)_H=0,\quad\text{for all}\quad j\in\{1,2,\dots,M\},$$

which implies that h has at least one zero in each open subset w_j. That is, h has at least M zeros in $(0,L)$ and, by Proposition 4.1, it follows that necessarily $h=0$, which in turn implies that $v=0$. □

4.8. Construction of the actuators in higher dimensional domains

For general domains $\Omega\subset\mathbb{R}^d$ the eigenvalues of the Dirichlet Laplacian (see [25], Cor. 1). Satisfy $\alpha_M\ge C_dM^{\frac{2}{d}}$, with $C_d=(2\pi)\left|\mathcal{B}_d\right|^{-\frac{2}{d}}\left|\Omega\right|^{-\frac{2}{d}}\frac{d}{d+2}$, where $|\mathcal{B}_d|$ denotes the volume of the unit ball in $\mathbb{R}^d$ and $|\Omega|$ the volume of Ω. Therefore, condition (4.1) follows from

$$M+1>{C_d}^{-\frac{d}{2}}\left(6+4\left|P_{\mathcal{U}}^{E_M^\perp}\right|_{\mathcal{L}(H)}^2\right)^{\frac{d}{2}}\left|A_{\mathrm{rc}}\right|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H,V'))}^d,\tag{4.8}$$

which gives us an estimate for M which depends linearly on $\left|A_{\mathrm{rc}}\right|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H,V'))}^d$.

Now the question we may ask is whether we can also construct the actuators $\mathcal{U}(M)$ so that the norm $\left|P_{\mathcal{U}}^{E_M^\perp}\right|_{\mathcal{L}(H)}^2$ remains bounded as M increases.

4.8.1. The case of a rectangle

In the case of a rectangle $\Omega^\times=\bigtimes\limits_{n=1}^d(0,L_n)$, we know that the eigenfunctions of Δ are the products of the 1D eigenfunctions. Now we fix $r\in(0,1)$ and define, in each interval $(0,L_n)$, the 1D actuators as in Section 4.5 (say covering a total region of volume rL_n), and their linear span $\mathcal{U}=(\mathcal{U})_n$. The notation $(\cdot)_n$ simply means that we are referring to the domain $(0,L_n)$. Next, we consider actuators defined in $\Omega^\times$ which are the products of those 1D actuators. That is, normalized eigenfunctions and actuators read, respectively

$$e_{[j]}^\times=e_{j_1}^1(x_1)e_{j_2}^2(x_2)\dots e_{j_d}^d(x_d)\quad\text{and}\quad\Psi_{[j]}^\times=\overline{\Psi}_{j_1}^1(x_1)\overline{\Psi}_{j_2}^2(x_2)\dots\overline{\Psi}_{j_d}^d(x_d),\qquad j\in\mathbb{N}_0^d,$$

where $x=(x_1,x_2,\dots,x_d)\in\Omega^\times$. For example, we have $e_{j_2}^2(x_2)=(\frac{2}{L_2})^{\frac{1}{2}}\sin(\frac{j_2\pi x_2}{L_2})\in(E_M)_2$ and $\overline{\Psi}_{j_3}^3(x_3)=\frac{1}{|\omega_{j_3}|}1_{\omega_{j_3}}(x_3)\in(\mathcal{U})_3$.

Now we recall Remark 3.9 and consider the spaces

$$\widetilde{E}_{M^d}=E_M^\times\coloneqq\operatorname{span}\left\{e_{[j]}^\times\mid j\in\{1,2,\dots,M\}^d\right\}\quad\text{and}\quad\mathcal{U}^\times=\operatorname{span}\left\{\Psi_{[j]}^\times\mid j\in\{1,2,\dots,M\}^d\right\}.$$

Let us denote $H_n\coloneqq L^2((0,L_n))$, for $n\in\{1,2,\dots,d\}$, and $H^\times\coloneqq L^2(\Omega^\times)$. We will also use the notation $(\cdot)_n$ for operators, to underline that the operator $(P)_n$ is understood to be in $\mathcal{L}(H_n)$. Namely, the projections $(P_{\mathcal{U}}^{E_M^\perp})_n$ and $(P_{E_M})_n$ are in $\mathcal{L}(H_n)$.

We "extend" the projections $(P_{\mathcal{U}}^{E_M^\perp})_n$ to operators $(P_{\mathcal{U}}^{E_M^\perp})_n^\times$ in $\mathcal{L}(H^\times)$ defined as

$$\left(P_{\mathcal{U}}^{E_M^\perp}\right)_n^\times f(x) \coloneqq \sum_{j_n=1}^{M} \left(\int_0^{L_n} f(x_1, x_2, \dots, x_n, \dots, x_d) e_{j_n}^n(x_n) \,\mathrm{d}x_n \right) \left(P_{\mathcal{U}}^{E_M^\perp}\right)_n e_{j_n}^n(x_n). \tag{4.9}$$

We have the following Lemmas 4.3 and 4.4, whose proofs are given in the Appendix.

Lemma 4.3. *We have $H^\times = \mathcal{U}^\times \oplus E_M^{\times\perp}$, and the projection $P_{\mathcal{U}^\times}^{E_M^{\times\perp}}$ coincides with the composition*

$$\mathcal{P}_\circ \coloneqq \left(P_{\mathcal{U}}^{E_M^\perp}\right)_d^\times \circ \left(P_{\mathcal{U}}^{E_M^\perp}\right)_{d-1}^\times \circ \dots \circ \left(P_{\mathcal{U}}^{E_M^\perp}\right)_2^\times \circ \left(P_{\mathcal{U}}^{E_M^\perp}\right)_1^\times.$$

From Section 4.6, we recall that $\vartheta_M^{-1} = \left|(P_{\mathcal{U}}^{E_M^\perp})_n\right|_{H_n}^2$ is independent of L_n (*cf.* Fig. 2b).

Lemma 4.4. *We have that $|\mathcal{P}_\circ|_{\mathcal{L}(H^\times)}^2 = \vartheta_M^{-d}$.*

By Remark 3.9 we can see that $\widetilde{\alpha}_M \coloneqq \frac{(M+1)^2\pi^2}{L_{\overline{n}}^2} + \sum\limits_{n\in\{1,2,\dots,d\}\setminus\overline{n}} \frac{\pi^2}{L_n^2}$ is the smallest eigenvalue of the Dirichlet Laplacian in $\widetilde{E}_{M^d}^\perp = E_M^{\times\perp}$, corresponding to the eigenfunction $e_{[j]}^\times$ with $[j]$ defined by $j_n = \begin{cases} 1, & \text{if } n \neq \overline{n}, \\ M+1, & \text{if } n = \overline{n}, \end{cases}$ where $\overline{n} \coloneqq \min\left\{ n \in \{1,2,\dots,d\} \mid L_n = \max\limits_{m\in\{1,2,\dots,d\}} L_m \right\}$.

Hence, for the parabolic equation (1.7) in the rectangle $\Omega^\times$, recalling (4.1) and Remark 3.9, we have that the stability of the corresponding system (3.8) follows from $\widetilde{\alpha}_M > \left(6 + 4\vartheta_M^{-d}\right) |A_{\mathrm{rc}}|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H,V'))}^2$. Since $\widetilde{\alpha}_M \geq \frac{\pi^2}{L_{\overline{n}}^2}\left((M+1)^2 + d - 1\right)$, the relation $M+1 \geq \frac{L_{\overline{n}}}{\pi}\left(6 + 4\vartheta_M^{-d}\right)^{\frac{1}{2}} |A_{\mathrm{rc}}|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H,V'))}$ implies stability of system (3.8). Recalling that there are M^d actuators in $E_M^{\times\perp}$, we can derive an estimate on the number of actuators M_{suff} which allow us to stabilize the system as $M_{\mathrm{suff}} \geq \left(\frac{L_{\overline{n}}}{\pi}\right)^d \left(6 + 4\vartheta_M^{-d}\right)^{\frac{d}{2}} |A_{\mathrm{rc}}|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H,V'))}^d$. The latter estimate is analogous to (4.8). Notice also that the eigenvalues $\widetilde{\alpha}_M$ satisfy $\widetilde{\alpha}_M \geq M^2$, and the subsequence of the (ordered and repeated) eigenvalues α_{M^d} satisfy $\alpha_{M^d} \geq C_d(M^d)^{\frac{2}{d}} = C_d M^2$. So, the asymptotic behavior of $\widetilde{\alpha}_M$ and α_{M^d} are analogous.

Notice, however, that in 1D the actuators cover, in each $(0, L_n)$, a total volume $|\omega_{L_n}| = rL_n$, while in dD they cover a volume $|\omega^\times| = r^d \bigtimes_{n=1}^d L_n$. That is, the relative volume covered by the actuators is smaller: $\frac{|\omega^\times|}{|\Omega^\times|} = r^d < r = \frac{|\omega_{L_n}|}{|(0,L_n)|}$.

Summarizing, we have seen that $\left|P_{\mathcal{U}^\times}^{\widetilde{E}_{M^d}^\perp}\right|_{\mathcal{L}(L^2(\Omega^\times))}^2 \leq \vartheta_M^{-d}$ remains bounded, provided the 1D projections do: $\left|(P_{\mathcal{U}}^{E_M^\perp})_n\right|_{\mathcal{L}(L^2((0,L_n)))}^2 = \vartheta_M^{-1} \leq C$ for all $M \in \mathbb{N}$. From the simulations in Section 4.6, $\vartheta_M^{-1} \leq C(r)$ remains bounded. For rectangles $\Omega^\times \subset \mathbb{R}^d$ we can always find/construct a big enough number $M = M_{\mathrm{suff}}$ of actuators depending polynomially on $|A_{\mathrm{rc}}|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H,V'))}$ (linearly on $|A_{\mathrm{rc}}|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H,V'))}^d$), so that system (3.8) is stable.

4.8.2. The case of a general domain

We conjecture that for general (regular enough) domains $\Omega \subset \mathbb{R}^d$, the norm $\left|P_{\mathcal{U}}^{E_M^\perp}\right|_{\mathcal{L}(L^2(\Omega))}^2$ will remain bounded provided we distribute piecewise constant actuators as uniformly as possible over Ω. And thus, we will be able to find a number M, as (4.8), of actuators, so that system (3.8) is stable.

5. Simulations for the closed-loop system

As described in Section 4.2, the spatial discretization is carried out by piecewise linear finite elements. For the full discretization the Crank–Nicolson scheme is used for temporal discretization for the grid defined by $\{t_j = jk : j \in \mathbb{N}\}$, with the time-step $k > 0$.

5.1. A constant reaction in 1D

We consider system (1.7) on the interval $\Omega = (0, L)$ with a constant reaction and no convection

$$a(x,t) = \rho := -35\nu(\tfrac{\pi}{L})^2, \qquad b(x,t) = 0, \tag{5.1a}$$

and the initial condition

$$y_0 = 0.05\sin(\tfrac{1\pi x}{L}). \tag{5.1b}$$

That is, $A_{\rm rc} = \rho < 0$. Under our feedback control, as in Theorem 3.6, the system reads

$$\dot y - \nu\Delta y + \rho y - P_{\mathcal{U}}^{E_M^\perp}(-\nu\Delta y + \rho y - \lambda y) = 0, \tag{5.2}$$

$$y|_{\partial\Omega} = 0, \quad \text{and} \quad y(0) = y_0. \tag{5.3}$$

In this case, we observe that the constants in (3.3) are given by

$$\Xi_1 = \sup_{(t,Y)\in\mathbb{R}_0\times(E_M^\perp\cap V)} \frac{|\langle \rho Y, Y\rangle_{V',V}|_{\mathbb{R}}^2}{|Y|_H^2\,|Y|_V^2} = \sup_{Y\in E_M^\perp\cap V} \frac{|Y|_H^2}{|Y|_V^2}\rho^2 = \alpha_{M+1}^{-1}\rho^2,$$

$$\Xi_2 = 0.$$

Therefore, condition (3.2:**CS**b) reads

$$\alpha_{M+1}^2 > \inf_{\substack{\gamma\in\mathbb{R}_0^2,\\ (2-\gamma_1-\gamma_2)>0}} \frac{\gamma_1^{-1}}{(2-\gamma_1-\gamma_2)}\rho^2 = \inf_{\substack{\gamma_1\in\mathbb{R}_0,\\ (2-\gamma_1)>0}} \frac{\gamma_1^{-1}}{(2-\gamma_1)}\rho^2 = \left(\sup_{\gamma_1\in(0,2)} \gamma_1(2-\gamma_1)\right)^{-1}\rho^2 = \rho^2. \tag{5.4}$$

We will take $L = 1$ and $\nu = 0.1$. For the ordered eigenvalues $\alpha_i = \nu(\frac{\pi}{L})^2 i^2$ of $A = -\nu\Delta$, we can find $-\alpha_1 - \rho > 0$, which implies that the free (uncontrolled) dynamical system $\dot y_{\rm unc} - \nu\Delta y_{\rm unc} + \rho y_{\rm unc} = 0$ is unstable. Indeed, the free solution corresponding to $y_{\rm unc}(0) = y_0$ is $y_{\rm unc}(t) = \mathrm{e}^{(-\rho-\alpha_1)t}y_0$, $t \ge 0$.

We can also see that (5.4) is satisfied for $M \ge 5$, because in this case $\alpha_{M+1}^2 \ge \alpha_6^2 > \rho^2$. Therefore, our closed-loop system (5.2) is stable provided we take $M = 5$ actuators spanning $\mathcal{U}$ such that $H = \mathcal{U} \oplus E_M^\perp$. This is satisfied for actuators as in Figures 1, 3, or 4. The stabilizing effect is confirmed in Figures 5–7. Note, however, that the location of the actuators has a significant influence on the value of ϑ_M and on the magnitudes reached by the (norm of the) solution and by the control. This confirms the fact, already mentioned in Section 4.2, that though in this case the norm of $P_{\mathcal{U}}^{E_M^\perp}$ plays no role in the condition for stability (3.2:**CS**), it still makes a considerable difference concerning the bound D in (3.11), which provides a bound for the magnitudes reached by the norm of the solution.

The projection $P_{\mathcal{U}}^{E_M^\perp}$ has been constructed following (2.3a) from the basis $\{\sin(\frac{i\pi x}{L}) \mid i \in \{1,\dots,M\}\}$ for E_M and from the basis $\{1_{\omega_i} \mid i \in \{1,\dots,M\}\}\}$ for $\mathcal{U}$. The (numerically) orthonormalized bases have been used only for the computation of ϑ_M.

To see the performance of the proposed feedback, the control will be switched on only during a time interval *FeedOn*. That is, for time $t \notin$ *FeedOn* we follow the free dynamics. We plot the results on the behavior of

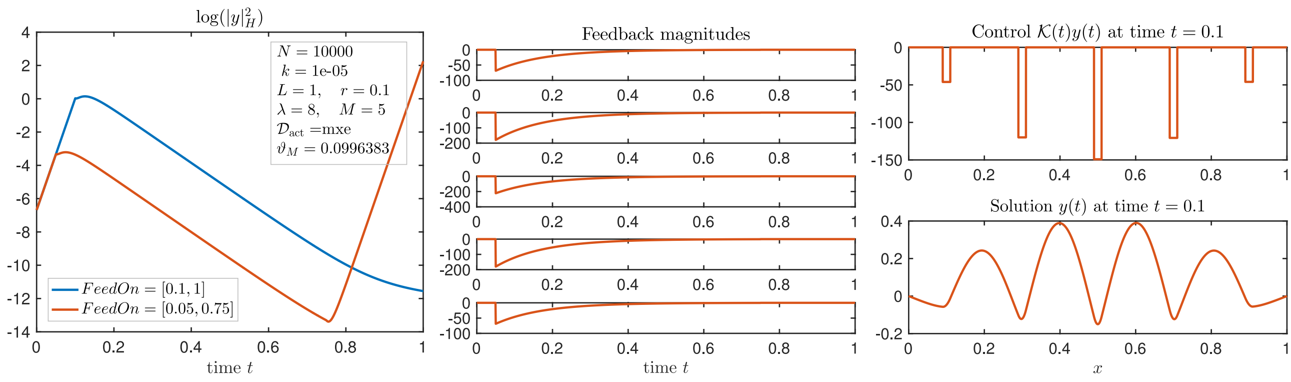


FIGURE 5. Located at extremizers of the Mth eigenfunction.

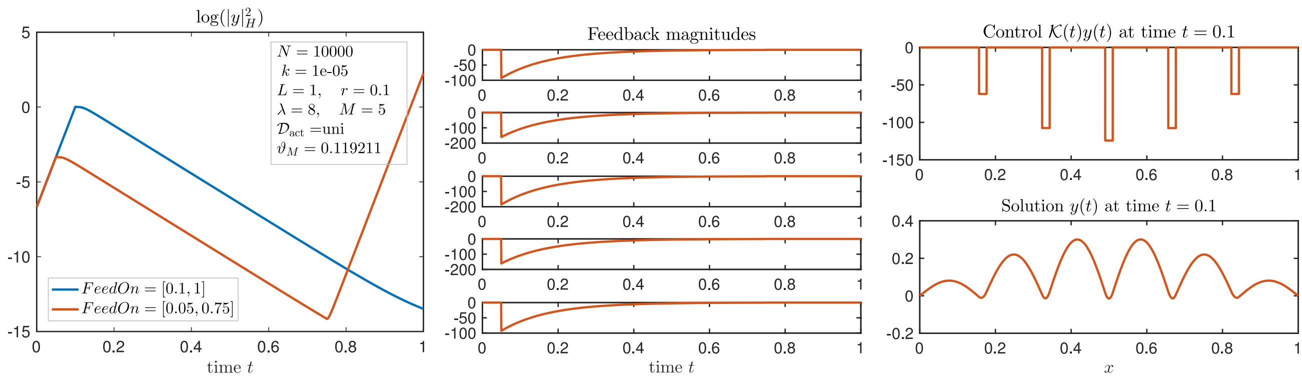


FIGURE 6. Uniformly distributed.

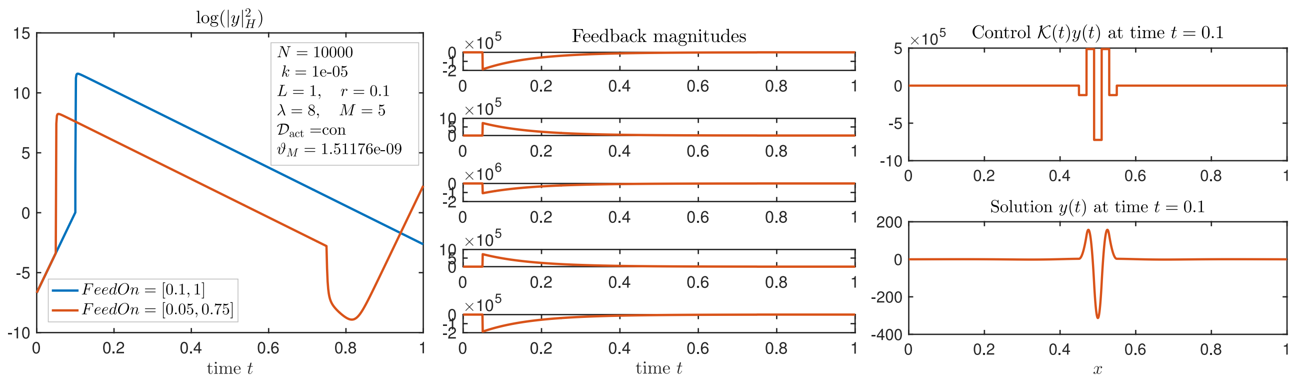


FIGURE 7. Concentrated at the center.

the norm of the solution in two cases, $FeedOn = (0.1, 1)$ and $FeedOn = (0.05, 0.75)$. The magnitudes of the control actuators will be plotted only for the latter case. Recall (from Sects. 4.5 and 4.7) that the actuators 1_{ω_i} are numbered from left to right, that is, $\omega_i = (a_i, b_i)$ with $a_i < b_i \le a_{i+1} < b_{i+1}$ (*cf.* Figs. 1, 3, and 4). In Figures 5–7, their corresponding magnitudes are ordered from top to bottom.

Again for the latter case we also plot the control and solution at a selected instant of time $t = 0.1$.

Finally, in this example, $M=5$ is sharp, that is, with 4 actuators the proposed feedback is not able to stabilize the system, because the solution of (5.2), with $M=4$ and issued from $y_0=e_5$, is given by $y(t)=\mathrm{e}^{(-\alpha_5-\rho)t}y_0$, which coincides with the free dynamics solution and whose norm goes to $+\infty$ with time. Notice that in this case, we even cannot stabilize the system if we take the first 4 eigenfunctions as actuators, because the dynamics onto the subspace $E_4^\perp$ will remain free for any given control v (replacing $P_{\mathcal{U}}^{E_M^\perp}(-\nu\Delta y+\rho y-\lambda y)$ and) taking values in E_4.

5.2. A general reaction–convection term in 1D

In Figure 8, we see the performance of the proposed feedback for a reaction–convection term depending on both space and time variables. More precisely, we have taken

$$a(x,t)=-35\nu(\tfrac{\pi}{L})^2-10\left|\sin(4t)x\cos(xt)\right|_{\mathbb{R}}, \qquad b(x,t)=-4\cos(3\pi t)-5(\tfrac{L-x}{L})^2+2, \tag{5.5a}$$

and the initial condition

$$y_0(x)=\sin(\tfrac{13\pi x}{L}). \tag{5.5b}$$

Observe that for the constants in (3.3) we find

$$\begin{aligned}\Xi_1 &\le \left|P_{E_M^\perp}A_{\mathrm{rc}}P_{E_M^\perp}\right|^2_{\mathcal{L}(H,V')} \le \left(\left|P_{E_M^\perp}\right|_{\mathcal{L}(H,V')}|a|_{L^\infty(\mathbb{R}_0,L^\infty(\Omega))}+|b|_{L^\infty(\mathbb{R}_0,L^\infty(\Omega))}\right)^2\\ &\le \left(\alpha_{M+1}^{-\frac12}|a|_{L^\infty(\mathbb{R}_0,L^\infty(\Omega))}+|b|_{L^\infty(\mathbb{R}_0,L^\infty(\Omega))}\right)^2 \le 2\alpha_{M+1}^{-1}|a|^2_{L^\infty(\mathbb{R}_0,L^\infty(\Omega))}+2|b|^2_{L^\infty(\mathbb{R}_0,L^\infty(\Omega))},\end{aligned} \tag{5.6a}$$

$$\Xi_2=\left|P_{E_M}A_{\mathrm{rc}}P_{E_M^\perp}\right|^2_{L^\infty(\mathbb{R}_0,\mathcal{L}(H,V'))} \le 2\alpha_1^{-1}|a|^2_{L^\infty(\mathbb{R}_0,L^\infty(\Omega))}+2|b|^2_{L^\infty(\mathbb{R}_0,L^\infty(\Omega))}. \tag{5.6b}$$

We also know that (3.2:**CS**b) follows if (taking for simplicity $\gamma_1=\frac12=\gamma_2$)

$$\alpha_{M+1}>\left(2\Xi_1+4\left(1+\left|P_{\mathcal{U}}^{E_M^\perp}\right|^2_{\mathcal{L}(H)}\right)\Xi_2\right). \tag{5.6c}$$

With (a,b) as in (5.5a), with $L=1$, and with $\nu=0.1$, we can find

$$|a|_{L^\infty(\mathbb{R}_0,L^\infty(\Omega))}<34.5437<35, \qquad |b|_{L^\infty(\mathbb{R}_0,L^\infty(\Omega))}\le 11.$$

Further, from Corollary 2.9 and Figure 2 (for $r=0.1$), we have that $\left|P_{\mathcal{U}}^{E_M^\perp}\right|^2_{\mathcal{L}(H)}=\vartheta_M^{-1}\approx 10$. By direct computations that we give in Appendix A.4, the above estimates allow us to conclude:

$$\text{Condition (3.2:\textbf{CS}b) holds true if }\ \alpha_{M+1}>149\,490. \tag{5.7}$$

Accordingly to (3.2:**CS**b), from

$$\alpha_{M+1}>124\,707 \Longleftarrow M\ge -1+(\tfrac{\nu\pi^2}{L^2})^{-\frac12}\sqrt{124\,707}\approx 354.4639,$$

it follows that for $M=355$ actuators our closed loop system is stable. However, from Figure 8 we can see that for $M\ge 7$ we already observe stability of the system, for the initial condition (5.5b). This indicates that the sufficient condition (3.2:**CS**b) is likely not necessary. However, to get better estimates on M we will probably

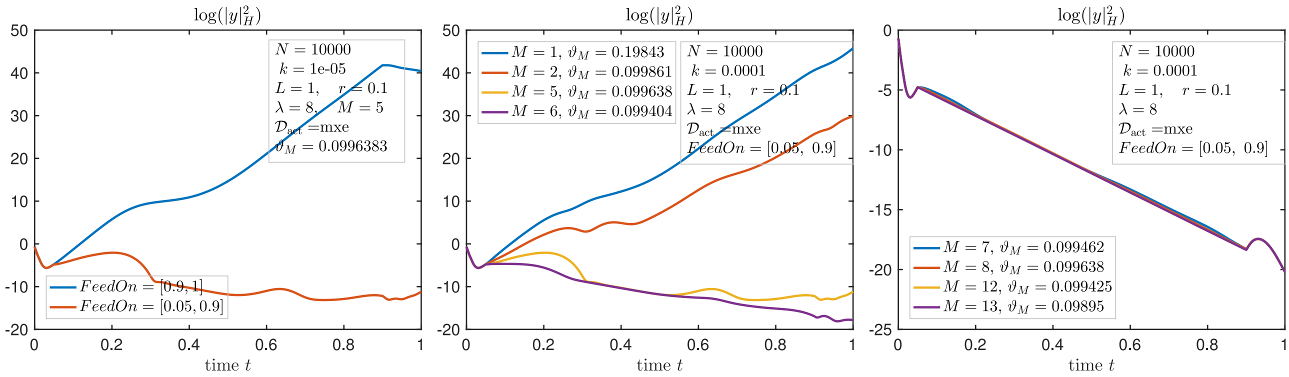


FIGURE 8. Feedback performance. Dependence on the number M of actuators.

need to use more qualitative properties (*e.g.*, as in Sects. 4.2 and 4.3) of the particular pair (a, b) in (5.5a) other than the quantitative properties in (3.2:**CS**b) involving only the norm of $A_{\rm rc}$.

We see that for small M (in $\{0, 1, 2\}$) our closed-loop system is not stable, while it is stable for large M. We also see that for $M \geq 7$ the exponentially decreasing rate of $|y|_H^2$ is close to $16 = 2\lambda$. This rate cannot be improved (*e.g.*, by adding more actuators) because it coincides with the exact decreasing rate we have imposed for the norm $|P_{E_M} y|_H^2$ of the finite-dimensional projection onto E_M (*cf.* (3.11), where $\mu < 2\lambda$).

We see that in a neighborhood of $t = 0$ the norm of the (free dynamics) solution is decreasing. This is because our initial condition is in $E_{\widetilde{M}}^{\perp}$ with large enough $\widetilde{M} = 12$. Notice that we have $\frac{\rm d}{{\rm d}t} |y|_H^2 \leq -2 |y|_V^2 + 2 |A_{\rm rc}|_{\mathcal{L}(H,V')} |y|_H |y|_V$, which implies $\frac{\rm d}{{\rm d}t} |y|_H^2 \leq (-\alpha_{\widetilde{M}+1} + |A_{\rm rc}|_{\mathcal{L}(H,V')}^2) |y|_H^2$, at the initial time.

After some time the norm of the uncontrolled solution is increasing. We can conclude that the reaction–convection operator is necessarily transferring energy to the space $E_{\widetilde{M}}$. In this situation, a control is needed to stabilize the system, and we see that our feedback controller is able to do it (for large enough M).

5.3. Response to measurement errors

To apply a feedback control in applications we will need to know the state $y(t)$ at time t. Then we can compute our control $\mathcal{K}(t)y(t)$ to be input into the system. Often it is not possible to know $y(t)$ exactly, but we can obtain suitable estimates for $y(t)$. For example, from measurements of suitable outputs of the system we can sometimes construct a dynamical observer which provides us with an estimate $\widehat{y}(t)$ for $y(t)$.

Once we have an estimate $\widehat{y}(t)$ for $y(t)$, we can compute an estimate of our control as $\mathcal{K}(t)\widehat{y}(t)$. Feedback controls are demanded in applications, because they are able to respond to (small) measurement/estimation errors. In Figure 9, we confirm that our proposed feedback is robust against such errors. As the magnitude of the noise (measurement error), $\eta \coloneqq \widehat{y}(t) - y(t)$, gets smaller the solution goes to a smaller neighborhood of zero.

We present the results corresponding to simulations of the system (3.8) with a perturbed feedback (*i.e.*, with an estimated feedback control $\mathcal{K}(t)\widehat{y}(t)$):

$$\dot{y} - \nu\Delta y + ay + \nabla\cdot(by) - P_{\mathcal{U}}^{E_M^{\perp}}\Big(-\nu\Delta\widehat{y} + a\widehat{y} + \nabla\cdot(b\widehat{y}) - \lambda\widehat{y}\Big) = 0, \tag{5.8}$$

$$y|_{\partial\Omega} = 0, \quad \text{and} \quad y(0) = y_0. \tag{5.9}$$

where $\widehat{y} \coloneqq y + \eta$.

We will take the noise in the form $\eta = \eta_1 y + \eta_2$ having a component $\eta_1 y$ which is proportional to the state y and a component η_2 which is independent of the state. We will test with three types of hypothetical noises η

("typnoi" in figures):

– $\eta(x,t) = \mathrm{expl}(t,x,\zeta)$
$$:= \mathrm{e}^{\zeta}\Big(\big(\sin(20\pi t) + 0.1\sin(100\pi(t+x)) - 1\big)y(x,t) + \cos(10\pi t) + .1\cos(200\pi(t+x))\Big),$$
– $\eta(x,t) = \mathrm{rndn}(t,x,\zeta)$
$$:= \mathrm{e}^{\zeta}\Big((\min\{1,\max\{-1,\mathrm{v}_{\mathrm{ran}1}(x,t)\}\} - 1)y(x,t) + \min\{1,\max\{-1,\mathrm{v}_{\mathrm{ran}2}(x,t)\}\}\Big),$$
– $\eta(x,t) = \mathrm{rndm}(t,x,\zeta)$
$$:= \mathrm{e}^{\zeta}\Big((\mathrm{v}_{\mathrm{ran}3}(x,t) - 1.5)y(x,t) + \mathrm{v}_{\mathrm{ran}4}(x,t) - \mathrm{v}_{\mathrm{ran}5}(x,t)\Big).$$

The functions rndn and rndm are "random" and are to be understood as follows: once we have solved our system up to time $t_m = mk$, say we have just found $y(t_m)$, then we generate random vectors $\mathrm{v}_{\mathrm{ran}i}(t_m) \in \mathbb{R}^{N+1}$, from which we construct the noise functions rndn and rndm at time $t = t_m$.

For $\mathrm{rndn}(t,x,\zeta)$ the vectors $\mathrm{v}_{\mathrm{ran}i}(t_m)$, are generated by the Matlab function `randn`, while for $\mathrm{rndm}(t,x,\zeta)$ they are generated by the Matlab function `rand`.

In Figure 9, we take $M = 6$ actuators. The simulations correspond to the data (a,b) as in (5.5a), and the initial condition is taken

$$y_0(x) = 10\sin(\tfrac{13\pi x}{L}).$$

We test with several values of ζ ("magnoi" in figures), to see the response of our feedback as the magnitude of the noise decreases. We see that the magnitude of the noise decreases with ζ. When $\zeta = -\infty$ ($-$Inf in the figures) the noise vanishes.

5.4. Numerical simulations in 2D

Now we present the results of a simulation for a parabolic equation in a domain $\Omega \subset \mathbb{R}^2$. In the previous examples for an 1D interval $(0,L)$, we knew the analytic expression for the Dirichlet Laplacian eigenfunctions, $e_n^{1\mathrm{D}} = (\frac{2}{L})^{\frac{1}{2}}\sin(\frac{n\pi x_1}{L})$. Then for the numerical simulations we can just evaluate this functions at the mesh points and construct the numerical projection $P_{\mathcal{U}}^{E_M^\perp}$ from those vectors. Now we are going to consider a domain $\Omega \subset \mathbb{R}^2$ where we do not know the analytic expression for the eigenfunctions. In this case, we have to compute the eigenpairs numerically. In Figure 10, we plot the first 5 numerical eigenpairs (e_i, α_i), $i \in \{1,2,3,4,5\}$ of $-\Delta$. We test with 4 actuators 1_{ω_i} where the location of each ω_i is also plotted in Figure 10, together with the triangular mesh we used in our simulations.

We can see that the first 4 eigenvalues are simple. We have chosen to place the actuators close to the extremizers of the 4th eigenfunction e_4. This was motivated from the results obtained in 1D for the actuators constructed as in Section 4.5 and by trying to place them "as uniformly as possible" (as "suggested" at the end of Sect. 4.8). We have the volume ratio $r = \frac{\left|\bigcup_{i=1}^4 \omega_i\right|}{|\Omega|} = \frac{4}{36+2\pi} \approx 0.0946$.

In Figure 11, we see the response of our feedback to measurement errors. Again the robustness of our feedback is confirmed. The norm of solution converges to a smaller neighborhood of 0 as the magnitude of the noise decreases. We also see that the numerical noisy solution is more sensitive to the time-step k than the unperturbed solution. However, qualitatively the behavior is similar.

Finally, in Figure 11 in the case of the larger time interval, the norm is plotted only for 1001 equidistant time instants, namely for $t = ik_{\mathrm{sub}}$, $i = 0,1,\ldots,1000$.

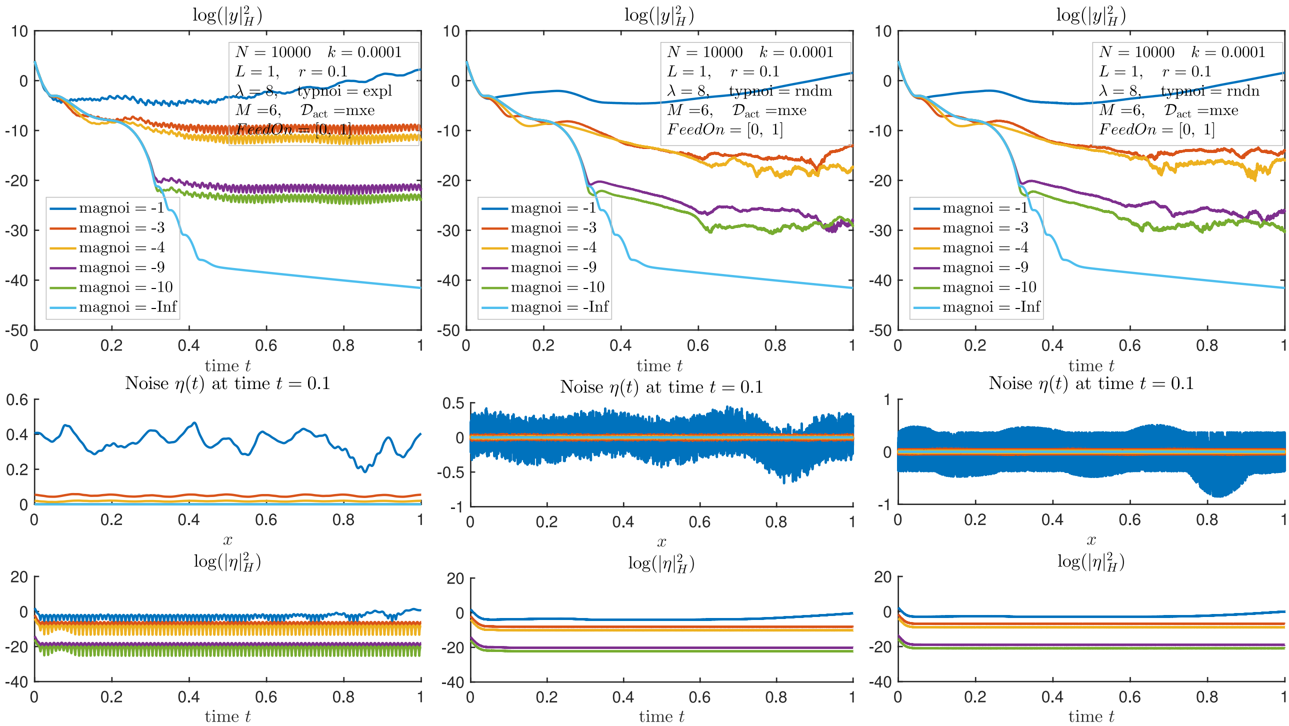


FIGURE 9. Response to measurement errors.

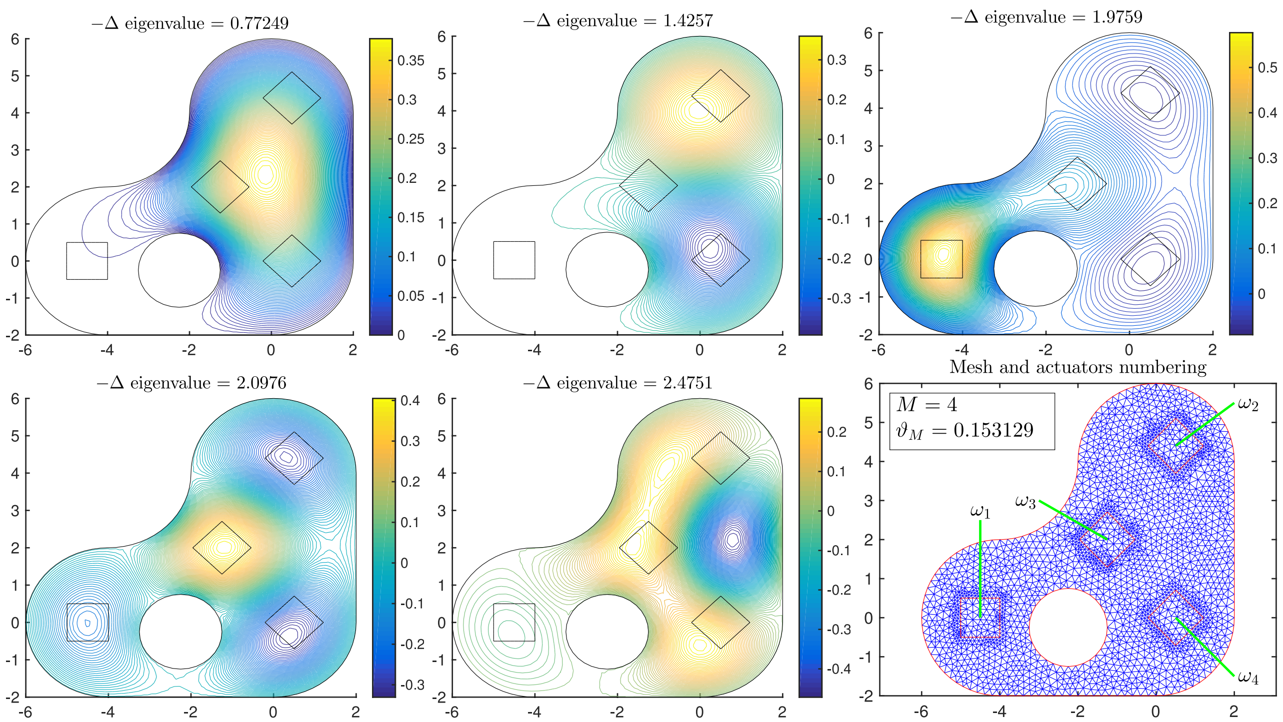


FIGURE 10. Eigenpairs of $-\Delta$ and actuators.

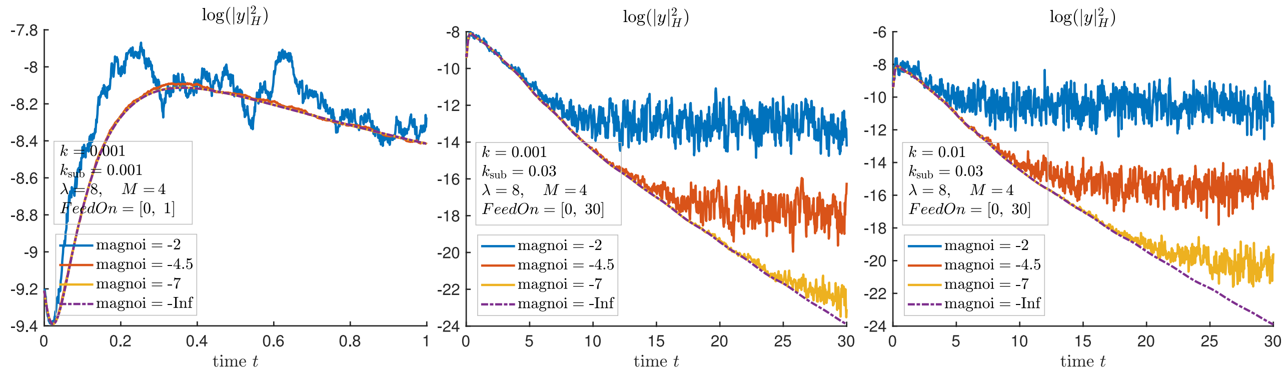


FIGURE 11. Response to measurement errors.

The simulations' results correspond to the parabolic equation (1.7) under Dirichlet boundary conditions and

$$a(t, x_1, x_2) = -0.1 - 0.2|\sin(t + x_1)|_{\mathbb{R}}, \quad b(t, x_1, x_2) = \begin{pmatrix} 0.1(x_1 + x_2) \\ 0.1\cos(t)x_1x_2 \end{pmatrix}, \quad \begin{cases} \nu = 0.1, \\ y_0 = 0.01e_1. \end{cases} \tag{5.10}$$

Notice that the first 5 eigenvalues of $-\nu\Delta$ extend from $\alpha_1 \approx 0.077249$ to $\alpha_5 \approx 0.24751$.
Note that for $x = (x_1, x_2) \in \Omega$, as in Figure 10 we have that

$$\sup_{x\in\Omega} |x_1 + x_2|_{\mathbb{R}^2} \le 8 \quad \text{and} \quad \sup_{x\in\Omega} |x_1x_2|_{\mathbb{R}^2} \le 36.$$

Hence, with (a, b) as in (5.10), we find

$$|a|_{L^\infty(\mathbb{R}_0, L^\infty(\Omega,\mathbb{R}))} < 0.3, \qquad |b|_{L^\infty(\mathbb{R}_0, L^\infty(\Omega,\mathbb{R}^2))} < 0.1(8 + 36)^{\frac{1}{2}} \approx 0.6634 < 0.7.$$

Proceeding as in Section 5.2, for the constants in (3.3) we find, see (5.6),

$$\Xi_1 \le 2\alpha_{M+1}^{-1}(0.3)^2 + 2(0.7)^2, \quad \text{and} \quad \Xi_2 \le 2\alpha_1^{-1}(0.3)^2 + 2(0.7)^2,$$

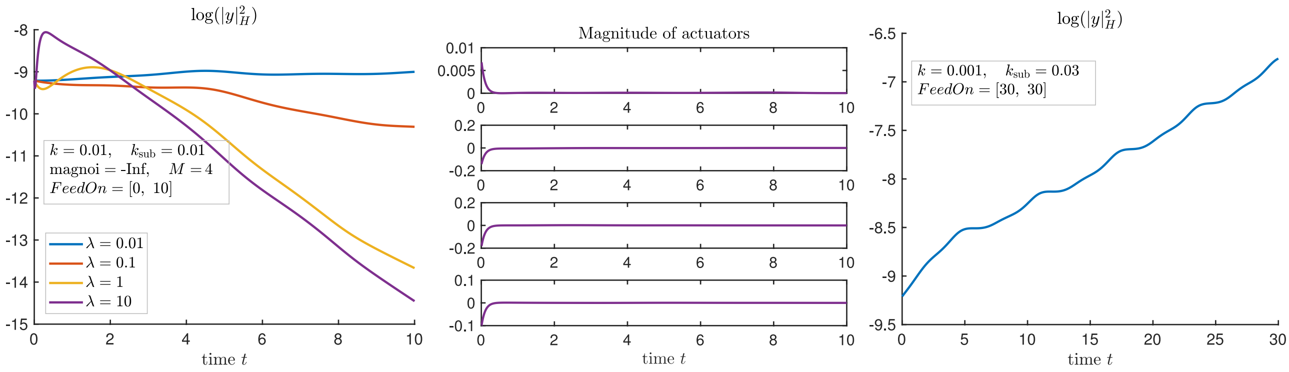
and by direct computations that we present in Appendix A.4, we can show that:

$$\text{Condition (3.2:}\mathbf{CS}\text{b) holds true if } \alpha_{M+1} > 16 + 14 \left| P_{\mathcal{U}}^{E_M^\perp} \right|_{\mathcal{L}(H)}^2. \tag{5.12}$$

Now, for general M, we do not know neither $\left| P_{\mathcal{U}}^{E_M^\perp} \right|_{\mathcal{L}(H)}$ nor α_{M+1}. However, we know that $\left| P_{\mathcal{U}}^{E_M^\perp} \right|_{\mathcal{L}(H)} > 1$ and we can see that with the 4 actuators, as in Figure 10, condition 5.12 is not satisfied, because on one hand

$$\alpha_{4+1} > 16 + 14 \left| P_{\mathcal{U}}^{E_M^\perp} \right|_{\mathcal{L}(H)}^2 \quad \Longrightarrow \quad \alpha_{4+1} > 30,$$

and on the other hand, from Figure 10, $\alpha_{4+1} \approx 2.4751\nu = 0.24751 < 30$. Again as in Section 5.2, this suggests that the condition (3.2:**CS**b) is likely not necessary. Note that Figure 11 shows that with those 4 actuators we have stability, for the data as in (5.10).



(a) Dependence on λ. (b) Magnitude of actuators. (c) Free dynamics.

FIGURE 12. Dependence on λ and free dynamics.

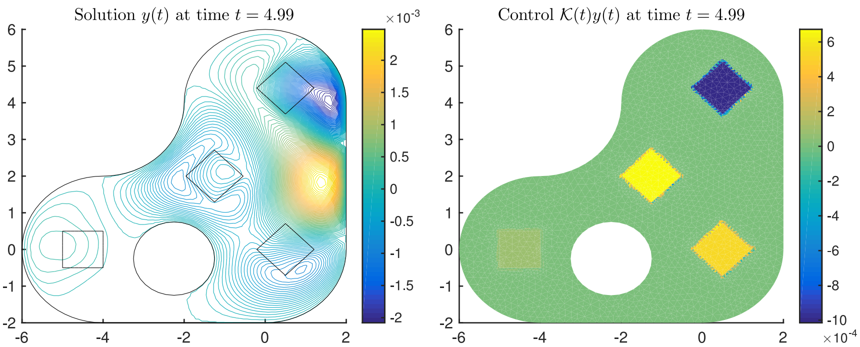


FIGURE 13. Solution and control at a selected time instant.

5.5. On the parameter λ

We see that our feedback essentially imposes/chooses the projection of the (unperturbed) solution to be $q(t) = P_{E_M} y(t) = \mathrm{e}^{-\lambda t} P_{E_M} y(0)$. The parameter λ is at our disposal and its choice plays a role in the behavior of the closed-loop system. In Figure 12a, with the data as in Section 5.4 and (5.10), we can see that a larger $\lambda > 0$ may provide a faster decrease of the norm of the solution. At the same time the transient bound may become larger.

In Figure 12b, we see the magnitudes (ordered from top to bottom) of the actuators corresponding to the last case plotted in Figure 12a (recall the ordering of the actuators in Fig. 10). We can also see, in Figure 12c, that the free dynamics is unstable.

Again for the last case plotted in Figure 12a, we see the corresponding solution and control, at a selected instant of time, in Figure 13.

6. FINAL REMARKS

We present here a short discussion on the manuscript results and on related interesting questions to be investigated in future work.

6.1. On the main result

Recall that in Theorem 3.6 we have the Assumptions 2.1, 2.2, and 2.3 and the sufficient condition (3.2:**CS**) for the stability of the closed-loop system (3.8).

Assumptions 2.1, 2.2, and 2.3 are quite general and are fulfilled for systems other than "pure" parabolic as (1.7). For example, they are satisfied for the Oseen–Stokes system

$$\dot{y} - \nu P_H \Delta y + P_H\big(\langle v \cdot \nabla\rangle y + \langle y \cdot \nabla\rangle v\big) = 0,$$

which can be seen as the linearization of the Navier–Stokes system around a given targeted trajectory v. Here, $P_H\colon L^2(\Omega,\mathbb{R}^d) \to H$ is the Leray orthogonal projection onto the space $H = \{z \in L^2(\Omega,\mathbb{R}^d) \mid \operatorname{div} z = 0 \text{ and } z\cdot \mathbf{n} = 0\}$, where $d \in \{2,3\}$, Ω is an open (smooth) bounded domain, and $\mathbf{n}$ is the unit outward normal to the boundary $\Gamma = \partial\Omega$. For the case of Dirichlet boundary conditions, Assumptions 2.1 and 2.2 are satisfied by the Stokes operator $A = -\nu P_H \Delta$ with $V = H_0^1(\Omega,\mathbb{R}^d) \bigcap H$ and $\mathrm{D}(A) = H^2(\Omega,\mathbb{R}^d) \bigcap V$. Assumption 2.3 will be satisfied for a regular enough function v, by using well known estimates for the term $A_{\mathrm{rc}}(t)y = P_H\big(\langle v(t) \cdot \nabla\rangle y + \langle y \cdot \nabla\rangle v(t)\big)$. We refer the reader to [8, 38]. See also [37] and ([31], Sect. 6) and references therein for other boundary conditions. In particular, the main stabilization results derived in Section 3 are valid for the Oseen–Stokes system. Here, we restrict ourselves to the simulations for our closed loop parabolic equations, but it is interesting, in a future work, to perform some simulations of the corresponding closed loop Oseen–Stokes system. And in particular to check condition (3.2:**CS**).

6.2. On the application to nonlinear systems

For future work it is of interest to investigate the response of our proposed feedback when applied to a nonlinear equation. Due to Theorem 3.7, and following a standard argument as in ([8], Sect. 4) we can stabilize the nonlinear system (for a suitable class of nonlinearities as in [30], and for strong solutions), provided the initial condition is taken in a small neighborhood of 0. In particular, it can be of interest to compare the size of this neighborhood with the size of the corresponding neighborhood associated with the Riccati-based feedback, for example, considered in [12, 22, 23, 30].

6.3. A comparison to Riccati feedback

While in the autonomous case, computing a Riccati-based feedback operator Π can be already a difficult numerical task, it becomes even more involved in our nonautonomous setting where (in theory) we would need to solve backwards in time a differential Riccati equation in the time interval $(0,+\infty)$. This is, of course, unfeasible (in practice). In spite of this, in [23] the authors propose an end point condition $\Pi(T)$ for the differential Riccati equation to be solved in a finite interval $(0,T)$, $T>0$. For works related with the numerical computation/approximation of the Riccati equations we refer the reader to [3, 10, 11, 14, 24].

The construction of the explicit feedback controller $\mathcal{K}$ we propose is in general an easier and faster numerical task. Indeed its construction, with M actuators, boils down to the computation of the first M eigenfunctions of the Laplacian. Furthermore, once those eigenfunctions are available, we can perform the simulations for any other reaction–convection term just by solving the explicit closed-loop system. On the other hand, for Riccati-based feedback we need to solve a differential Riccati equation for each reaction–convection term.

6.4. On the location of the actuators

In view of the discussion in Sections 4.6 and 4.7, we may ask ourselves the question on the best placement or location of the actuators. For example, it is interesting to know what is the *best location* (*i.e.*, the pairs $\{(\text{center,orientation})_i \mid i \in \{1,2,3,4\}\}$) inside Ω of the 4 actuator regions as in Figure 10, *in order to maximize* ϑ_M. We refer to [27, 28] where the question of optimal actuator location is addressed, with the goal of minimizing a quadratic cost functional. See also to the recent works [21, 32] concerning the problem of optimal actuators *design* (where the goal is again to minimize a suitable cost functional, where the shape of the control's (or each actuator's) support is not fixed *a priori*).

6.5. *Ad hoc* explicit feedback for the autonomous symmetric case

We have presented a feedback that addresses the case of a general time-dependent reaction–convection term $A_{\rm rc}$. In such situation (3.2:**CS**) is a sufficient condition for the stability of the closed-loop system (3.8). Of course, in particular situations we may use the specific properties of $A_{\rm rc}$ to construct a different feedback. Here, we will show that in the case of a time-independent reaction we can construct a version of our explicit feedback, whose range has a dimension depending on the spectral properties of $A + A_{\rm rc}$.

We still assume Assumptions 2.1, 2.2, and 2.3. In addition, we also assume that $A_{\rm rc}$ is independent of time and symmetric. It turns out that the operator $\mathcal{A} \coloneqq A + A_{\rm rc}$ is symmetric and has finite number of nonpositive eigenvalues. Let us say that its (repeated) eigenvalues β_i and eigenfunctions θ_i satisfy

$$\mathcal{A}\theta_i = \beta_i\theta_i, \qquad \beta_1 \le \beta_2 \le \cdots \le \beta_N \le 0 < \beta_{N+1} \le \beta_{N+2} \le \dots \qquad \beta_i \to +\infty, \qquad i \in \mathbb{N}_0.$$

Furthermore, let $\Theta_M \coloneqq \operatorname{span}\{\theta_i \mid i \in \{1, 2, \dots, M\}\}$ and let $\Theta_M^\perp$ be the orthogonal complement of Θ_M, in H.

Now, instead of (3.2:**CS**), we assume that our actuators, Ψ_i, are such that that their linear span $\mathcal{U} = \operatorname{span}\{\Psi_i \mid i \in \{1, 2, \dots, M\}\}$ satisfy

$$H = \mathcal{U} \oplus \Theta_M^\perp \quad \text{and} \quad \mathrm{M} \ge N. \tag{6.1}$$

Theorem 6.1. *Let $\lambda > 0$ and $\sigma > -\beta_1$. If (6.1) holds true, then the system*

$$\dot{y} + Ay + A_{\rm rc}y - P_{\mathcal{U}}^{\Theta_M^\perp}(Ay + A_{\rm rc}y - \lambda y) \tag{6.2}$$

is exponentially stable with rate $\mu = \min\{\lambda, \sigma + \beta_{M+1} - \sigma^2(\sigma + \beta_{M+1})^{-1}\} > 0$.

Proof. Let $\mathcal{A}_\sigma y \coloneqq Ay + A_{\rm rc} + \sigma y$. We may write (6.2) in the form

$$\dot{y} + \mathcal{A}_\sigma y - \sigma y - P_{\mathcal{U}}^{\Theta_M^\perp}(\mathcal{A}_\sigma y - \sigma y - \lambda y) \tag{6.3}$$

and we can see that $\mathcal{A}_\sigma$ satisfies Assumptions 2.1 and 2.2, and the constant reaction operator $-\sigma 1$ satisfies Assumption 2.3. Notice that the eigenvalues of $\mathcal{A}_\sigma$ are given by $\sigma + \beta_i > 0$, $i \in \mathbb{N}_0$. That is, we can proceed as in Section 4.2 and conclude that (6.3) is stable provided $\sigma + \beta_{M+1} > \sigma$, that is, provided $\beta_{M+1} > 0$, which is true because $M \ge N$, by assumption (6.1). Furthermore, recalling Theorem 3.6 and the discussion in Section 4.2, exponential stability holds with rate $\mu = \min\{\lambda, \sigma + \beta_{M+1} - \sigma^2(\sigma + \beta_{M+1})^{-1}\} > 0$. □

Above we supposed that the operator $A_{\rm rc}$ is symmetric. The extension to the nonsymmetric case could be possible, but it will involve other tools we do not want to introduce here, because the investigation of the particular time-independent case is not the focus of this paper. In any case we refer the reader to ([4], Sect. 2.2), where a different stabilizability condition is given, but where the proposed feedback is given implicitly through the solution of an algebraic Riccati equation.

Appendix

A.1 Proof of Proposition 4.1

It is enough to consider the case $L = \pi$. Let us be given $w \in \mathbb{R}^M \setminus \{0\}$ and set

$$h(x) = \sum_{i=1}^{M} w_i \sin(ix). \tag{A.1}$$

Lemma A.2. *For any $i \in \mathbb{N}_0$, there exists $\sigma_i = (\sigma_{i,1}, \sigma_{i,2}, \ldots, \sigma_{i,i-1}) \in \mathbb{R}^{i-1}$ so that*

$$\sin(ix) = \sin(x) \left(\sum_{j=1}^{i-1} \sigma_{i,j} \cos^j(x) \right). \tag{A.2}$$

Proof. We proceed by induction. The lemma holds trivially for $j = 1$, and follows easily for $j = 2$ because

$$\sin(2x) = 2\sin(x)\cos(x).$$

Let us fix $s \in \mathbb{N}_0$ and suppose that the lemma holds true for all $i \le s - 1$. We want to prove that it also holds true for $i = s$. We will consider two cases: s odd and s even.
The case $s = 2k + 1$, with $k \in \mathbb{N}$, $k \ge 1$. We write

$$\sin(sx) = \sin(2kx)\cos(x) + \sin(x)\cos(2kx) = \sin(2kx)\cos(x) + \sin(x)\left(1 - 2\sin^2(kx)\right).$$

From induction hypothesis and (A.2), denoting $Q_i(x) \coloneqq \sum_{j=1}^{i-1} \sigma_{i,j} \cos^j(x)$, for all $i < s$, we find that

$$\begin{aligned} \sin(sx) &= \sin(x) Q_{2k}(x)\cos(x) + \sin(x)\left(1 - 2\sin^2(x) Q_k^2(x)\right) \\ &= \sin(x)\Big(Q_{2k}(x)\cos(x) + \left(1 - 2(1 - \cos^2(x)) Q_k^2(x)\right)\Big). \end{aligned}$$

It is clear that $Q_{2k}(x)\cos(x) + \left(1 - 2(1 - \cos^2(x)) Q_k^2(x)\right)$ is a polynomial of degree $2k$ in $\cos(x)$. That is, there exists $\sigma_s = (\sigma_{s,1}, \sigma_{s,2}, \ldots, \sigma_{s,s-1}) \in \mathbb{R}^{s-1}$ so that we can write

$$Q_{2k}(x)\cos(x) + \left(1 - 2(1 - \cos^2(x)) Q_k^2(x)\right) \eqqcolon \sum_{j=1}^{s-1} \sigma_{s,j} \cos^j(x) \eqqcolon Q_s(x).$$

The case $s = 2k$, with $k \in \mathbb{N}$, $k \ge 2$. Similarly, we write

$$\sin(sx) = \sin(2(k-1)x)\cos(2x) + \sin(2x)\cos(2(k-1)x) = \sin(2(k-1)x)\cos(2x) + \sin(2x)\left(1 - 2\sin^2((k-1)x)\right).$$

and find

$$\begin{aligned} \sin(sx) &= \sin(x) Q_{2(k-1)}(x)\cos(2x) + \sin(2x)\left(1 - 2\sin^2(x) Q_{k-1}^2(x)\right) \\ &= \sin(x)\Big(Q_{2(k-1)}(x)(2\cos^2(x) - 1) + 2\cos(x)\left(1 - 2(1 - \cos^2(x)) Q_{k-1}^2(x)\right)\Big). \end{aligned}$$

Since $Q_{2(k-1)}(x)(2\cos^2(x) - 1) + 2\cos(x)\left(1 - 2(1 - \cos^2(x)) Q_{k-1}^2(x)\right)$ is a polynomial of degree $2k - 1$ in $\cos(x)$, we conclude that there exists $\sigma_s = (\sigma_{s,1}, \sigma_{s,2}, \ldots, \sigma_{s,s-1}) \in \mathbb{R}^{s-1}$ so that we can write

$$Q_{2(k-1)}(x)(2\cos^2(x) - 1) + 2\cos(x)\left(1 - 2(1 - \cos^2(x)) Q_{k-1}^2(x)\right) \eqqcolon \sum_{j=1}^{s-1} \sigma_{s,j} \cos^j(x) \eqqcolon Q_s(x).$$

Therefore, the Lemma holds true for $i = s$. □

Now, recalling (A.1), we can write

$$\begin{aligned} h(x) &= \sum_{i=1}^{M} w_i \sin(ix) = \sin(x) \sum_{i=1}^{M} w_i \left(\sum_{j=1}^{i-1} \sigma_{i,j} \cos^j(x) \right) = \sin(x) \sum_{j=1}^{M-1} \left(\sum_{i=1}^{j+1} w_i \sigma_{i,j} \right) \cos^j(x) \\ &= \sin(x) \sum_{j=1}^{M-1} \xi_j \cos^j(x), \end{aligned}$$

and we have that

$$h(x) = 0 \Longleftrightarrow \sum_{j=1}^{M-1} \xi_j \cos^j(x) = 0, \quad \text{for} \quad x \in (0, \pi).$$

Since $w \neq 0$, then h is not identically zero, that is, at least one of the constants ξ_j is nonzero. Notice that the polynomial $y \mapsto p(y) \coloneqq \sum_{j=1}^{M-1} \xi_j y^j$ has at most $M - 1$ roots ρ in $\mathbb{R}$. Thus, we have that for $x \in (0, \pi)$,

$$h(x) = 0 \Longleftrightarrow \cos(x) = \rho, \quad \text{with} \quad p(\rho) = 0.$$

That is,

$$x = \arccos(\rho), \qquad p(\rho) = 0, \qquad \rho \in (-1, 1).$$

Since there are at most $M - 1$ roots ρ for p, we have that there are at most $M - 1$ zeros $x \in (0, \pi)$ for h. $\square$

A.2 Proof of Lemma 4.3

We first show that the range of the composition $\mathcal{P}_\circ$ is contained in $\mathcal{U}^\times$. Given a function $f \in H^\times$ we have that $(P_{\mathcal{U}}^{E_M^\perp})_1^\times f = \sum_{j_1=1}^{M} f_{j_1}^1(x_2, \dots, x_d) \overline{\Psi}_{j_1}^1(x_1)$. Similarly, we find the identities $(P_{\mathcal{U}}^{E_M^\perp})_2^\times \circ (P_{\mathcal{U}}^{E_M^\perp})_1^\times f = \sum_{j_1, j_2=1}^{M} f_{(j_1,j_2)}^1(x_3, \dots, x_d) \overline{\Psi}_{j_1}^1(x_1) \overline{\Psi}_{j_2}^2(x_2)$ and $\mathcal{P}_\circ f = \sum_{[j] \in \{1,2,\dots,M\}^d} f_{[j]}^1 \Psi_{[j]}^\times$. Thus

$$\mathcal{P}_\circ f \in \mathcal{U}^\times, \quad \text{for all} \quad f \in H^\times. \tag{A.3}$$

Next, we show that $(1 - \mathcal{P}_\circ) f \in E_M^{\times\perp}$. From (4.9) and the Fubini's theorem it follows that, for any $f \in H^\times$ and any eigenfunction $e_{[i]}^\times \in E_M^\times$, we have

$$\left(\left(P_{\mathcal{U}}^{E_M^\perp} \right)_n^\times f, e_{[i]}^\times \right)_{H^\times} = \sum_{j_n=1}^{M} \int_0^{L_1} \mathrm{d}x_1 \cdots \int_0^{L_{n-1}} \mathrm{d}x_{n-1} \int_0^{L_{n+1}} \mathrm{d}x_{n+1} \cdots \int_0^{L_d} g_n^{j_n} \mathrm{d}x_d \int_0^{L_n} h_n \mathrm{d}x_n$$

with

$$\begin{aligned} g_n^{j_n} &= \left(\int_0^{L_n} f(x_1, x_2, \dots, x_n, \dots, x_d) e_{j_n}^n(x_n) \, \mathrm{d}x_n \right) \bigtimes_{r \in \{1,\dots,d\} \setminus \{n\}} e_{i_r}^r(x_r), \\ h_n &= \left(\left(P_{\mathcal{U}}^{E_M^\perp} \right)_n e_{j_n}^n(x_n) \right) e_{i_n}^n(x_n). \end{aligned}$$

Since $\left(1-\left(P_{\mathcal{U}}^{E_M^\perp}\right)_n\right)e_{j_n}^n(x_n)\in(E_M^\perp)_n$ and $e_{i_n}^n(x_n)\in(E_M)_n$ it follows that

$$\begin{aligned}\int_0^{L_n} h_n\mathrm{d}x_n &= \int_0^{L_n} h_n+\left(1-\left(P_{\mathcal{U}}^{E_M^\perp}\right)_n\right)e_{j_n}^n(x_n)e_{i_n}^n(x_n)\mathrm{d}x_n=\int_0^{L_n} e_{j_n}^n(x_n)e_{i_n}^n(x_n)\mathrm{d}x_n\\ &=\begin{cases}1, & \text{if } j_n=i_n, \text{ and}\\ 0, & \text{if } j_n\neq i_n,\end{cases}\end{aligned}$$

from which we obtain that, for all $f\in H^\times$ and all $[i]\in\{1,2,\dots,M\}^d$,

$$\left(\left(P_{\mathcal{U}}^{E_M^\perp}\right)_n^\times f,e_{[i]}^\times\right)_{H^\times}=\int_0^{L_1}\mathrm{d}x_1\cdots\int_0^{L_{n-1}}\mathrm{d}x_{n-1}\int_0^{L_{n+1}}\mathrm{d}x_{n+1}\cdots\int_0^{L_d} g_n^{i_n}\mathrm{d}x_d=\left(f,e_{[i]}^\times\right)_{H^\times}.$$

which implies that for all $g\in H^\times$ and all $[i]\in\{1,2,\dots,M\}^d$,

$$\begin{aligned}\left(\mathcal{P}_\circ g,e_{[j]}^\times\right)_{H^\times}&=\left(\left(P_{\mathcal{U}}^{E_M^\perp}\right)_d^\times\left(\left(P_{\mathcal{U}}^{E_M^\perp}\right)_{d-1}^\times\circ\dots\circ\left(P_{\mathcal{U}}^{E_M^\perp}\right)_1^\times g\right),e_{[i]}^\times\right)_{H^\times}\\ &=\left(\left(P_{\mathcal{U}}^{E_M^\perp}\right)_{d-1}^\times\circ\dots\circ\left(P_{\mathcal{U}}^{E_M^\perp}\right)_1^\times g,e_{[i]}^\times\right)_{H^\times}=\dots=(g,e_{[i]}^\times)_{H^\times}.\end{aligned}$$

That is,

$$g-\mathcal{P}_\circ g\in E_M^{\times\perp},\quad\text{for all}\quad g\in H^\times. \tag{A.4}$$

Finally, we show that $H^\times=\mathcal{U}^\times\oplus E_M^{\times\perp}$. From (A.3) and (A.4) it follows that $H^\times=\mathcal{U}^\times+E_M^{\times\perp}$. Let now $v\in\mathcal{U}^\times\bigcap E_M^{\times\perp}$ which we may write as

$$v=\sum_{[j]\in\{1,2,\dots,M\}^d}v_{[j]}\Psi_{[j]}^\times.$$

Therefore, for any eigenfunction $e_{[i]}^\times\in E_M^\times$ we have $0=(v,e_{[i]}^\times)_{H^\times}$, that is,

$$0=\sum_{j\in\{1,2,\dots,M\}^d}v_{[j]}\left(\Psi_{[j]}^\times,e_{[i]}^\times\right)_{H^\times} \tag{A.5}$$

which gives us

$$0=\sum_{[j]\in\{1,2,\dots,M\}^d}v_{[j]}\bigtimes_{n=1}^{d}\left(\overline{\Psi}_{j_n}^n,e_{i_n}^n\right)_{H_n}=\sum_{j_d=1}^{M}w_{j_d}\left(\overline{\Psi}_{j_d}^n,e_{i_d}^n\right)_{H_d}. \tag{A.6}$$

with

$$w_{j_d}:=\sum_{j\in\{1,2,\dots,M\}^{d-1}}v_{[(j,j_d)]}\left(\bigtimes_{n=1}^{d-1}\left(\overline{\Psi}_{j_n}^n,e_{i_n}^n\right)_{H_n}\right).$$

We know that $Z:=[((\mathcal{U})_n,(E_M^\perp)_n)_{H_n}]\in\mathcal{M}_{M\times M}$ is invertible (because so is $[\Theta(M)]=Z^\top Z$, *cf.* (4.4) and Fig. 2).

Now, with $w \coloneqq [w_{j_1}, w_{j_2}, \dots, w_{j_d}] \in \mathcal{M}_{1\times M}$, from (A.5) we arrive at $0 = wZ$, which implies that $w = 0$. That is, $0 = w_{j_d}$, for all $j_d \in \{1, 2, \dots, M\}$.

For a fixed $j_d \in \{1, 2, \dots, M\}$ the equation $0 = w_{j_d}$ is similar to (A.5). Thus, we can repeat the argument to conclude that

$$0 = w_{j_{d-1},j_d} \coloneqq \sum_{j\in\{1,2,\dots,M\}^{d-2}} v_{[(j,j_{d-1},j_d)]} \left(\bigtimes_{n=1}^{d-2} (\overline{\Psi}^n_{j_n}, e^n_{i_n})_{H_n} \right),$$

for all $(j_{d-1}, j_d) \in \{1, 2, \dots, M\}^2$, and

$$0 = w_{j_1,\dots,j_d} \coloneqq v_{[j]}, \quad \text{for all} \quad j \in \{1, 2, \dots, M\}^d.$$

Therefore $v = 0$, and

$$H^\times = \mathcal{U}^\times \oplus E_M^{\times\perp}. \tag{A.7}$$

From (A.3), (A.4), and (A.7), we necessarily have $\mathcal{P}_\circ = P_{\mathcal{U}^\times}^{E_M^{\times\perp}}$. □

A.3 Proof of Lemma 4.4

Recalling (4.9), we denote, for all $m \in \mathbb{N}_0$ and all $f \in H^\times$,

$$I^f_m = I^f_m(x_1, x_2, \dots, x_{n-1}, x_{n+1}, \dots, x_d) \coloneqq \int_0^{L_n} f(x_1, x_2, \dots, x_n, \dots, x_d) e^n_m(x_n)\,\mathrm{d}x_n.$$

Observe that we may write $f = \sum_{m=1}^{+\infty} I^f_m e^n_m(x_n)$,

$$\left(P_{\mathcal{U}}^{E_M^\perp}\right)_n^\times f = \sum_{j_n=1}^{M} I_{j_n} f \left(P_{\mathcal{U}}^{E_M^\perp}\right)_n e^n_{j_n}(x_n), \quad \text{and} \quad \int_0^{L_n} f^2\,\mathrm{d}x_n = \sum_{m=1}^{+\infty} \left(I^f_m\right)^2,$$

from which we obtain, since I^f_m does not depend on x_n,

$$\begin{aligned}
&\left|\left(P_{\mathcal{U}}^{E_M^\perp}\right)_n^\times f\right|_{H^\times}^2 = \int_{\Omega^\times} \left(\left(P_{\mathcal{U}}^{E_M^\perp}\right)_n^\times f\right)^2 \mathrm{d}\Omega^\times \\
&= \int_0^{L_1} \mathrm{d}x_1 \cdots \int_0^{L_{n-1}} \mathrm{d}x_{n-1} \int_0^{L_{n+1}} \mathrm{d}x_{n+1} \cdots \int_0^{L_d} \mathrm{d}x_d \int_0^{L_n} \left(\left(P_{\mathcal{U}}^{E_M^\perp}\right)_n \sum_{j_n=1}^{M} I^f_{j_n} e^n_{j_n}(x_n)\right)^2 \mathrm{d}x_n.
\end{aligned}$$

Then, from

$$\begin{aligned}
\int_0^{L_n} \left(\left(P_{\mathcal{U}}^{E_M^\perp}\right)_n v\right)^2 \mathrm{d}x_n &= \left|\left(P_{\mathcal{U}}^{E_M^\perp}\right)_n v\right|_{H_n}^2 \le \vartheta_M^{-1} |v|_{H_n}^2, \quad \text{for any} \quad v \in H_n, \\
\int_0^{L_n} \left(\sum_{j_n=1}^{M} I^f_{j_n} e^n_{j_n}(x_n)\right)^2 \mathrm{d}x_n &= \sum_{j_n=1}^{M} \left(I^f_{j_n}\right)^2 \le \int_0^{L_n} f^2\,\mathrm{d}x_n,
\end{aligned}$$

it follows that $\left|(P_{\mathcal U}^{E_M^\perp})_n^\times f\right|_{H^\times}^2 \le \vartheta_M^{-1}\left|f\right|_{H^\times}^2$, which leads us to $\left|\mathcal P_\circ\right|_{\mathcal L(H^\times)}^2 \le \bigtimes_{n=1}^{d}\left|(P_{\mathcal U}^{E_M^\perp})_n^\times\right|_{H^\times}^2 \le \vartheta_M^{-d}$. To finish the proof we now find $\zeta\in H^\times$ such that

$$\left|\zeta\right|_{H^\times} = 1 \quad\text{and}\quad \left|\mathcal P_\circ\zeta\right|_{H^\times} = \vartheta_M^{-\frac d2}. \tag{A.8}$$

We start by observing that

$$\begin{aligned}\left|\left(P_{\mathcal U}^{E_M^\perp}\right)_n\right|_{\mathcal L(H_n)} &= \sup_{\phi\in H_n\setminus\{0\}}\frac{\left|\left(P_{\mathcal U}^{E_M^\perp}\right)_n(P_{E_M})_n\phi\right|_{H_n}}{\left|\phi\right|_{H_n}} \le \sup_{\phi\in H_n\setminus\{0\}}\frac{\left|(P_{\mathcal U}^{E_M})_n(P_{E_M})_n\phi\right|_{H_n}}{\left|(P_{E_M})_n\phi\right|_{H_n}}\\ &= \sup_{\xi\in(E_M)_n\setminus\{0\}}\frac{\left|\left(P_{\mathcal U}^{E_M}\right)_n\xi\right|_{H_n}}{\left|\xi\right|_{H_n}} = \sup_{\substack{\xi\in(E_M)_n\\ \left|\xi\right|_{H_n}=1}}\left|(P_{\mathcal U}^{E_M})_n\xi\right|_{H_n}.\end{aligned}$$

Necessarily $\left|(P_{\mathcal U}^{E_M^\perp})_n\right|_{\mathcal L(H_n)} = \sup_{\substack{\xi\in(E_M)_n\\ \left|\xi\right|_{H_n}=1}}\left|(P_{\mathcal U}^{E_M^\perp})_n\xi\right|_{H_n}$ and, since $\{\eta\in(E_M)_n \mid \left|\eta\right|_{H_n}=1\}$ is compact, there exists a maximizer $\overline\xi_n\in(E_M)_n$, with $\left|\overline\xi_n\right|_{H_n}=1$ and $\left|(P_{\mathcal U}^{E_M^\perp})_n\overline\xi_n\right|_{H_n} = \left|(P_{\mathcal U}^{E_M^\perp})_n\right|_{\mathcal L(H_n)} = \vartheta_M^{-\frac12}$. Finally, we see that $\xi^\times(x) \coloneqq \bigtimes_{n=1}^d\overline\xi_n(x_n)$ satisfies

$$\left|\xi^\times\right|_{H^\times}^2 = \bigtimes_{n=1}^{d}\left|\overline\xi_n\right|_{H_n}^2 = 1 \quad\text{and}\quad \left|\mathcal P_\circ\xi^\times\right|_{H^\times}^2 = \bigtimes_{n=1}^{d}\left|(P_{\mathcal U}^{E_M^\perp})_n\overline\xi_n\right|_{H_n}^2 = \vartheta_M^{-d}.$$

That is, $\zeta\coloneqq\xi^\times$ satisfies (A.8). □

A.4 Proof of (5.7) and (5.12)

Proof of (5.7). We know that (3.2:**CS**b) follows from (5.6c), which in turn follows from

$$\alpha_{M+1} > 4\alpha_{M+1}^{-1}(35)^2 + 4(11)^2 + 8\left(1+\left|P_{\mathcal U}^{E_M^\perp}\right|_{\mathcal L(H)}^2\right)\left(\alpha_1^{-1}(35)^2+(11)^2\right).$$

Then, with $q_0\coloneqq 4(35)^2$ and $q_1\coloneqq 4(11)^2 + 8\left(1+\left|P_{\mathcal U}^{E_M^\perp}\right|_{\mathcal L(H)}^2\right)\left((\frac{\nu\pi^2}{L^2})^{-1}(35)^2+(11)^2\right)$, (3.2:**CS**b) follows if

$$\alpha_{M+1}^2 - q_1\alpha_{M+1} - q_0 > 0 \Longleftarrow \alpha_{M+1} > \frac{q_1+\sqrt{q_1^2+4q_0}}{2} \Longleftarrow \alpha_{M+1} > q_1+\sqrt{q_0}.$$

Since $q_1 \approx 484 + 10\,897\left(1+\left|P_{\mathcal U}^{E_M^\perp}\right|_{\mathcal L(H)}^2\right) = 11\,381 + 10\,897\left|P_{\mathcal U}^{E_M^\perp}\right|_{\mathcal L(H)}^2$, and since from Corollary 2.9 and Figure 2 (for $r=0.1$) we have that $\left|P_{\mathcal U}^{E_M^\perp}\right|_{\mathcal L(H)}^2 = \vartheta_M^{-1}\approx 10$, we have that $\alpha_{M+1} > q_1+\sqrt{q_0}$ follows from $\alpha_{M+1} > 113\,81 + 108\,970 = 124\,707$. □

Proof of (5.12). Proceeding as in the proof of (5.7), we see that (3.2:**CS**b) follows if

$$\alpha_{M+1} > 4\alpha_{M+1}^{-1}(0.3)^2 + 4(0.7)^2 + 8\left(1+\left|P_{\mathcal U}^{E_M^\perp}\right|_{\mathcal L(H)}^2\right)\left(\alpha_1^{-1}(0.3)^2+(0.7)^2\right).$$

From Figure 10 we know that $\alpha_1 \approx 0.77249\nu = 0.077249$. Then, with $q_0 \coloneqq 4(0.3)^2$ and $q_1 \coloneqq 4(0.7)^2 + 8\left(1 + \left|P_{\mathcal{U}}^{E_M^\perp}\right|_{\mathcal{L}(H)}^2\right)\left(0.077249^{-1}(0.3)^2 + (0.7)^2\right)$, (3.2:**CS**b) follows if

$$\alpha_{M+1} > q_1 + \sqrt{q_0}.$$

Since $q_1 \approx 1.96 + 13.2405\left(1 + \left|P_{\mathcal{U}}^{E_M^\perp}\right|_{\mathcal{L}(H)}^2\right) = 15.2005 + 13.2405\left|P_{\mathcal{U}}^{E_M^\perp}\right|_{\mathcal{L}(H)}^2$, we conclude that $\alpha_{M+1} > q_1 + \sqrt{q_0}$ follows from $\alpha_{M+1} > 16 + 14\left|P_{\mathcal{U}}^{E_M^\perp}\right|_{\mathcal{L}(H)}^2$. □

Acknowledgements. The work of K. Kunisch was partly supported by the ERC advanced grant 668998 (OCLOC) under the EU's H2020 research program.

References

[1] K. Ammari, T. Duyckaerts and A. Shirikyan, Local feedback stabilisation to a non-stationary solution for a damped non-linear wave equation. *Math. Control Relat. Fields* **6** (2016) 1–25.

[2] M. Badra and T. Takahashi, Stabilization of parabolic nonlinear systems with finite dimensional feedback or dynamical controllers: application to the Navier–Stokes system. *SIAM J. Control Optim.* **49** (2011) 420–463.

[3] H.T. Banks and K. Kunisch, The linear regulator problem for parabolic systems. *SIAM J. Control Optim.* **22** (1984) 684–698.

[4] V. Barbu, Stabilization of Navier–Stokes flows. *Communications and Control Engineering Series.* Springer-Verlag, London (2011).

[5] V. Barbu, Stabilization of Navier–Stokes equations by oblique boundary feedback controllers. *SIAM J. Control Optim.* **50** (2012) 2288–2307.

[6] V. Barbu, Boundary stabilization of equilibrium solutions to parabolic equations. *IEEE Trans. Automat. Control* **58** (2013) 2416–2420.

[7] V. Barbu, I. Lasiecka and R. Triggiani, Abstract settings for tangential boundary stabilization of Navier–Stokes equations by high- and low-gain feedback controllers. *Nonlinear Anal.* **64** (2006) 2704–2746.

[8] V. Barbu, S.S. Rodrigues and A. Shirikyan, Internal exponential stabilization to a nonstationary solution for 3D Navier–Stokes equations. *SIAM J. Control Optim.* **49** (2011) 1454–1478.

[9] V. Barbu and R. Triggiani, Internal stabilization of Navier–Stokes equations with finite-dimensional controllers. *Indiana Univ. Math. J.* **53** (2004) 1443–1494.

[10] P. Benner, A MATLAB repository for model reduction based on spectral projection. In *Proc. of the 2006 IEEE Conference on Computer Aided Control Systems Design, October 4–6* (2006) 19–24.

[11] P. Benner, A.J. Laub and V. Mehrmann, Benchmarks for the numerical solution of algebraic Riccati equations. *IEEE Control Syst. Mag.* **17** (1997) 18–28.

[12] T. Breiten, K. Kunisch and S.S. Rodrigues, Feedback stabilization to nonstationary solutions of a class of reaction diffusion equations of FitzHugh–Nagumo type. *SIAM J. Control Optim.* **55** (2017) 2684–2713.

[13] H. Brezis, Functional analysis, Sobolev spaces and partial differential equations. In *Universitext.* Springer, New York (2011).

[14] J.A. Burns, E.W. Sachs and L. Zietsman, Mesh independence of Kleinman–Newton iterations for Riccati equations in Hilbert space. *SIAM J. Control Optim.* **47** (2008) 2663–2692.

[15] J.-M. Coron, Control and Nonlinearity. Vol. 136 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI (2007).

[16] J.-M. Coron and E. Trélat, Global steady-state controllability of one-dimensional semilinear heat equations. *SIAM J. Control Optim.* **43** (2004) 549–569.

[17] J.-M. Coron and E. Trélat, Global steady-state stabilization and controllability of 1D semilinear wave equations. *Commun. Contemp. Math.* **8** (2006) 535–567.

[18] T. Duyckaerts, X. Zhang and E. Zuazua, On the optimality of the observability inequalities for parabolic and hyperbolic systems with potentials. *Ann. Inst. Henri Poincaré Anal. Non Linéaire* **25** (2008) 1–41.

[19] A.V. Fursikov and O. Yu Imanuvilov, Controllability of Evolution Equations. Vol. 34 of *Lecture Notes Series.* Seoul National University Research Institute of Mathematics Global Analysis Research Center, Seoul (1996).

[20] A. Halanay, C.M. Murea and C.A. Safta, Numerical experiment for stabilization of the heat equation by Dirichlet boundary control. *Numer. Funct. Anal. Optim.* **34** (2013) 1317–1327.

[21] D. Kalise, K. Kunisch and K. Sturm, Optimal actuator design based on shape calculus. *Math. Models Methods Appl. Sci.* **28** (2018) 2667–2717.

[22] A. Kröner and S.S. Rodrigues, Internal exponential stabilization to a nonstationary solution for 1D Burgers equations with piecewise constant controls, in *Proc. of the 2015 European Control Conference (ECC), Linz, Austria* (2015) 2676–2681.

[23] A. Kröner and S.S. Rodrigues, Remarks on the internal exponential stabilization to a nonstationary solution for 1D Burgers equations. *SIAM J. Control Optim.* **53** (2015) 1020–1055.

[24] P. Kunkel and V. Mehrmann, Numerical solution of differential algebraic Riccati equations. *Linear Algebra Appl.* **137/138** (1990) 39–66.

[25] P. Li and S.-T. Yau, On the Schrödinger equation and the eigenvalue problem. *Commun. Math. Phys.* **88** (1983) 309–318.

[26] J.-L. Lions, Optimal control of systems governed by partial differential equations. In Vol. 170 of *Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen.* Springer-Verlag, Berlin, Heidelberg (1971).

[27] K. Morris, Linear-quadratic optimal actuator location. *IEEE Trans. Automat. Control* **56** (2011) 113–124.

[28] K. Morris and S. Yang, A study of optimal actuator placement for control of diffusion, in *Proc. of the 2016 American Control Conference (AAC), Boston, MA, USA* (2016) 2566–2571.

[29] I. Munteanu, Boundary stabilisation to non-stationary solutions for deterministic and stochastic parabolic-type equations. *Int. J. Control* **92** (2019) 1720–1728.

[30] D. Phan and S.S. Rodrigues, Stabilization to trajectories for parabolic equations. *Math. Control Signals Syst.* **30** (2018) 11.

[31] D. Phan and S.S. Rodrigues, Gevrey regularity for Navier–Stokes equations under Lions boundary conditions. *J. Funct. Anal.* **272** (2017) 2865–2898.

[32] Y. Privat, E. Trélat and E. Zuazua, Actuator design for parabolic distributed parameter systems with the moment method. *SIAM J. Control Optim.* **55** (2017) 1128–1152.

[33] J.-P. Raymond and L. Thevenet, Boundary feedback stabilization of the two-dimensional Navier–Stokes equations with finite-dimensional controllers. *Discrete Contin. Dyn. Syst.* **27** (2010) 1159–1187.

[34] S.S. Rodrigues, Feedback boundary stabilization to trajectories for 3D Navier–Stokes equations. *Appl. Math. Optim.* (2018). DOI: 10.1007/s00245-017-9474-5.

[35] D.L. Russell, Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions. *SIAM Rev.* **20** (1978) 639–739.

[36] M. Schmidt and E. Trélat, Controllability of Couette flows. *Commun. Pure Appl. Anal.* **5** (2006) 201–211.

[37] R. Temam, Navier–Stokes equations and nonlinear functional analysis, in Vol. 66 of *CBMS-NSF Regional Conference Series in Applied Mathematics*, *2nd edition, SIAM, Philadelphia* (1995).

[38] R. Temam, Navier–Stokes Equations: Theory and Numerical Analysis. Reprint of the 1984 edition, AMS Chelsea Publishing, Providence, RI (2001).

[39] M. Tucsnak and G. Weiss, Observation and Control for Operator Semigroups. Birkhäuser Basel, Basel (2009).

[40] M.Y. Wu, A note on stability of linear time-varying systems. *IEEE Trans. Automat. Control* **19** (1974) 162.

[41] M. Yamamoto, Carleman estimates for parabolic equations and applications. *Inverse Probl.* **25** (2009) 123013.

[42] J. Zabczyk, Mathematical Control Theory: An Introduction. Systems Theory, Control. Birkhäuser, Boston (1992).