INTERFACIAL ENERGY AS A SELECTION MECHANISM FOR MINIMIZING GRADIENT YOUNG MEASURES IN A ONE-DIMENSIONAL MODEL PROBLEM

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Abstract. Energy functionals describing phase transitions in crystalline solids are often non-quasiconvex and minimizers might therefore not exist. On the other hand, there might be infinitely many gradient Young measures, modelling microstructures, generated by minimizing sequences, and it is an open problem how to select the physical ones.

In this work we consider the problem of selecting minimizing sequences for a one-dimensional three-well problem $\mathcal{E}$. We introduce a regularization $\mathcal{E}^{\varepsilon}$ of $\mathcal{E}$ with an ε-small penalization of the second derivatives, and we obtain as $\varepsilon \downarrow 0$ its Γ-limit and, under some further assumptions, the Γ-limit of a suitably rescaled version of $\mathcal{E}^{\varepsilon}$. The latter selects a unique minimizing gradient Young measure of the former, which is supported just in two wells and not in three. We then show that some assumptions are necessary to derive the Γ-limit of the rescaled functional, but not to prove that minimizers of $\mathcal{E}^{\varepsilon}$ generate, as $\varepsilon \downarrow 0$, Young measures supported just in two wells and not in three.

1. Introduction

A common problem that arises when studying martensitic transformations in the context of nonlinear elasticity (see *e.g.* [4, 5, 7, 17]) is to minimize an energy functional

$$E(y) = \int_{\Omega} \phi(\nabla y(x)) \, \mathrm{d}x,$$

where Ω is an open and bounded Lipschitz domain, and $y \colon \Omega \to \mathbb{R}^3$ is a map in a suitable Sobolev space satisfying $y = \bar{y}$ on $\partial\Omega$, for some smooth enough mapping $\bar{y}$. In this context, the continuous function $\phi \colon \mathbb{R}^{3 \times 3} \to [0, +\infty]$ is generally such that

$$\phi(F) = 0 \qquad \Longleftrightarrow \qquad F \in \mathcal{K} := \sum_{i=1}^{n} SO(3) U_i,$$

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where $n \geq 1$ and U_i are positive definite symmetric matrices representing the different variants of martensite. As in general E is not quasiconvex, minimizers for this energy might not exist. Therefore, following the idea of [4] one can study the behaviour of minimizing sequences, having a gradient that tends in measure to $\mathcal{K}$, and characterized by interesting microstructures. In order to capture the limiting behaviour of the minimizing sequences, one can study the relaxed functional

$$\bar{E}(\nu) = \int_\Omega \int_{\mathbb{R}^{3\times 3}} \phi(F) \, d\nu_x(F) \, dx,$$

where ν is a gradient Young measure containing the information about microstructures in the crystal (see *e.g.*, [5, 17, 20]). Defining $\mathcal{M}_1(\mathbb{R}^{3\times 3})$ as the set of probability measures on $\mathbb{R}^{3\times 3}$, let us consider

$$\mathcal{A} := \left\{ \nu \in L^\infty_{w^*}(\Omega; \mathcal{M}_1(\mathbb{R}^{3\times 3})) \,\middle|\, \begin{array}{c} \operatorname{supp} \nu_x \subset \mathcal{K}, \, \exists y \in W^{1,\infty}(\Omega; \mathbb{R}^3) \text{ s.t. } y = \bar{y} \text{ on } \partial\Omega, \text{ and} \\ \int_{\mathbb{R}^{3\times 3}} F d\nu_x(F) = \nabla y(x) \text{ a.e. in } \Omega \end{array} \right\},$$

and notice that this set is the set of minimizers of $\bar{E}$ whenever $\min \bar{E} = 0$. Here, we denoted by $L^\infty_{w^*}(\Omega; \mathcal{M}_1(\mathbb{R}^{3\times 3}))$ the space $L^\infty(\Omega; \mathcal{M}_1(\mathbb{R}^{3\times 3}))$ endowed with the weak$*$ topology. The solutions constructed in [18] with the technique of convex integration show that the set $\mathcal{A}$ might contain infinitely many minimizers for $\bar{E}$, and its elements might sometimes appear non-physical. In agreement with the physics, many authors in the literature (see *e.g.* [2, 4, 10–12, 15]) have considered a regularization of E that penalizes the second derivatives of y such as

$$E^\varepsilon(y) = \int_\Omega \left(\varepsilon^2 |\nabla^2 y|^2 + \phi(\nabla y(x)) \right) dx \qquad \text{or} \qquad \tilde{E}^\varepsilon(y) = \varepsilon |\nabla^2 y|(\Omega) + \int_\Omega \phi(\nabla y(x)) \, dx. \tag{1.1}$$

Here, $\varepsilon > 0$ is small and $|\nabla^2 y|(\Omega)$ is the norm of $\nabla^2 y$ as a measure on Ω. Many results have been proved in the case $n = 2$ and without boundary conditions. For example, it is proved in [12] that the requirement $\nabla y \in BV(\Omega; \mathcal{K})$ forces the gradient discontinuities to be just on planes that never intersect in Ω. In [10] the limit solutions for E^ε as $\varepsilon \to 0$ when $\mathcal{K} = \{A, B\}$ are characterized *via* a Γ-limit argument. In the two-dimensional setting with $\mathcal{K} = \{SO(2)A, SO(2)B\}$ the generalised Γ-limit has been analysed in [11], and strongly exploits the above-mentioned result of [12].

More generally, we could argue that the physically relevant minimizers of $\bar{E}$ are not those in $\mathcal{A}$, but those belonging to the subset

$$\mathcal{B} := \left\{ \nu \in \mathcal{A} \,\middle|\, \begin{array}{c} \exists \text{ minimizers } u^{\varepsilon_j} \text{ of } E^{\varepsilon_j}, \text{ with } \varepsilon_j \downarrow 0, \text{ such that } \delta_{\nabla u^{\varepsilon_j}} \to \nu \\ \text{in } L^\infty_{w^*}(\Omega; \mathcal{M}_1(\mathbb{R}^{3\times 3})) \end{array} \right\},$$

or equivalently $\tilde{\mathcal{B}}$ where E^{ε_j} is replaced by $\tilde{E}^{\varepsilon_j}$.

Finding an explicit characterization for $\mathcal{B}$ seems however out of reach for the general three-dimensional problem. For this reason, in this work we focus on the one-dimensional energy functional

$$\mathcal{E}(u) = \int_0^1 \left(W(u_x) + u^2 \right) dx, \tag{1.2}$$

which has been often considered in the literature (see *e.g.* [3, 15, 16, 19]) as a one-dimensional prototype for E. Indeed, the role of the boundary conditions in more dimensions is played here by the term u^2 in the energy, which forces the L^2-norm of the minimizers (or of the minimizing sequences) to be close to a prescribed value, which is chosen to be null for simplicity, and whose gradient does not sit on the wells.
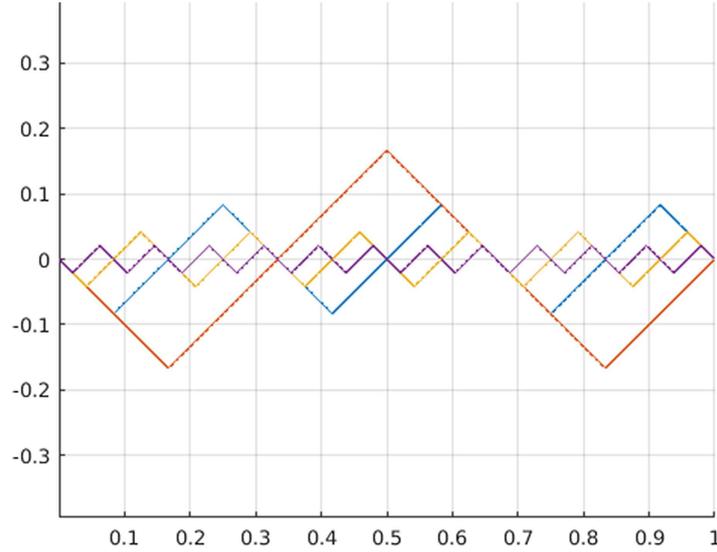
FIGURE 1. Minimizing sequences for $\mathcal{E}$ when $0 \notin \mathcal{Z}$.

Suppose W satisfies

(H1) $W : \mathbb{R} \mapsto \mathbb{R}_+$ is a continuous non-negative function;

(H2) there exist $c_1, c_2, c_3 > 0$ and $p \in (1, \infty)$ such that

$$c_1|s|^p - c_2 \leq W(s) \leq c_3(|s|^p + 1), \qquad \forall s \in \mathbb{R};$$

(H3) $W(s) = 0$, for each $s \in \mathcal{Z}$, and $W(s) > 0$, otherwise, where

$$\mathcal{Z} := \{s \in \mathbb{R} : s \in \operatorname{argmin}(W)\}.$$

If $\mathcal{Z}$ has a finite number of elements, if there exist $z_1, z_2 \in \mathcal{Z}$ with $z_1 < 0 < z_2$, and if $0 \notin \mathcal{Z}$, then W is not convex and $\mathcal{E}$ does not have minimizers in $W_0^{1,p}(0,1)$. Indeed, by constructing arbitrarily small saw-tooth functions (*cf.* Fig. 1) with gradient in $\mathcal{Z}$ one can show that $\inf \mathcal{E} = 0$. Therefore, the existence of a minimizer $u \in W_0^{1,p}(0,1)$ would imply $u = 0$, and hence $u_x = 0$ a.e. in $(0,1)$, which is in contradiction with the fact that, by (H3), $W(0) \neq 0$. For this reason, we consider the regularized problem

$$\mathcal{E}^\varepsilon(u) = \begin{cases} \int_0^1 \left(\varepsilon^6 |u_{xx}|^2 + W(u_x) + u^2\right) \mathrm{d}x, & \text{if } u \in W_0^{1,p}(0,1) \cap H^2(0,1), \\ +\infty, & \text{otherwise,} \end{cases} \tag{1.3}$$

which is the one-dimensional analogue of (1.1). $\mathcal{E}^\varepsilon$ can also be rewritten by using gradient Young measures (see *e.g.*, [17, 20]) as

$$\bar{\mathcal{E}}^\varepsilon(u, \nu) = \begin{cases} \mathcal{E}^\varepsilon(u), & \text{if } u \in W_0^{1,p}(0,1) \cap H^2(0,1) \text{ and } \nu_x = \delta_{u_x(x)} \text{ a.e. in } (0,1), \\ +\infty, & \text{otherwise,} \end{cases} \tag{1.4}$$

with δ_s denoting the Dirac mass at s. In this case the problem admits a solution in $W_0^{1,p}(0,1) \cap H^2(0,1)$ and the question arises as to what happens to the limit of the minimizers u^ε as $\varepsilon \downarrow 0$. For every $u \in W_0^{1,p}(0,1)$, let

us define the set of its gradient Young measures:

$$\mathrm{GYM}^p(u) := \left\{ \nu \in L^\infty_{w^*}(0,1;\mathcal{M}_1(\mathbb{R})) \middle| \begin{array}{c} \int_{\mathbb{R}} s\,\mathrm{d}\nu_x(s) = u_x(x) \text{ a.e. } x \in (0,1), \\[2mm] \int_0^1 \int_{\mathbb{R}} |s|^p \,\mathrm{d}\nu_x(s)\,\mathrm{d}x < \infty \end{array} \right\},$$

$$\mathrm{GYM}^\infty(u) := \left\{ \nu \in L^\infty_{w^*}(0,1;\mathcal{M}_1(\mathbb{R})) \middle| \begin{array}{c} \int_{\mathbb{R}} s\,\mathrm{d}\nu_x(s) = u_x, \text{ supp}\,\nu_x \subset K \text{ a.e. in } (0,1), \\[2mm] K \subset \mathbb{R} \text{ compact} \end{array} \right\}.$$

Here, $\mathcal{M}(\mathbb{R})$ and $\mathcal{M}_1(\mathbb{R})$, often abbreviated below by $\mathcal{M}$ and $\mathcal{M}_1$, are respectively the space of bounded Radon measures μ on $\mathbb{R}$, and its subset of probability measures. A preliminary result that is proved later in Section 2 is the following

Proposition 1.1. *Let W satisfy* (H1)–(H3). *Then, $\bar{\mathcal{E}}^\varepsilon$ Γ-converges to*

$$\mathcal{E}^0(u,\nu) = \begin{cases} \int_0^1 \big(\langle \nu_x, W \rangle + u^2 \big)\,\mathrm{d}x, & \text{if } u \in W_0^{1,p}(0,1), \nu \in \mathrm{GYM}^p(u), \\ +\infty, & \text{otherwise,} \end{cases}$$

in the $L^2(0,1) \times L^\infty_{w^*}(0,1;\mathcal{M})$ topology as ε tends to 0.

If $\mathcal{Z} = \{z_1, z_2\}$ with $z_1 < 0 < z_2$, then under (H1)–(H3) minimizers (u,ν) of $\mathcal{E}^0$ must satisfy

$$u(x) = 0, \qquad \nu \in \mathrm{GYM}^p(0), \qquad \text{supp}\,\nu_x \in \mathcal{Z}, \qquad \text{a.e. in } (0,1). \tag{1.5}$$

These conditions determine a unique minimizer (u,ν) to $\mathcal{E}^0$, namely

$$u(x) = 0, \qquad \nu_x = \frac{z_2}{z_2 - z_1}\delta_{z_1} - \frac{z_1}{z_2 - z_1}\delta_{z_2}, \qquad \text{a.e. } x \text{ in } (0,1).$$

Let us assume

(H4) $\mathcal{Z} = \{z_1, z_2, z_3\}$, and, without loss of generality, that $z_1 < 0 < z_2 < z_3$.

In this case, given any arbitrary measurable

$$\lambda \colon (0,1) \to \left[0, \left(1 - \frac{z_2}{z_1}\right)^{-1}\right],$$

the pair (u,ν) defined for almost every $x \in (0,1)$ by $u(x) = 0$ and

$$\nu_x = -\frac{z_3 + \lambda(x)(z_2 - z_3)}{z_1 - z_3}\delta_{z_1} + \lambda(x)\delta_{z_2} + \frac{z_1 + \lambda(x)(z_2 - z_1)}{z_1 - z_3}\delta_{z_3}, \tag{1.6}$$

minimizes $\mathcal{E}^0$. As a consequence, by assuming (H1)–(H4), uniqueness of minimizers for $\mathcal{E}^0$ is lost, that is, the gradient of the minimizing sequences for $\mathcal{E}$ oscillate, and converge in measure to $\{z_1, z_2, z_3\}$ without any particular preference. The aim of this work is to prove that minimizers of $\mathcal{E}^\varepsilon$ generate gradient Young measures supported in $\{z_1, z_2\}$, but not in z_3. Therefore, by choosing minimizers of $\mathcal{E}^\varepsilon$ with $\varepsilon \downarrow 0$ as minimizing sequences for $\mathcal{E}$, we can select a unique minimizing gradient Young Measure, out of the infinitely many given above.

Let

$$V := H^2(0,1) \cap W_0^{1,p}(0,1).$$

Then we define I^ε by

$$I^\varepsilon(u) = I^\varepsilon(u,\nu) := \begin{cases} \varepsilon^{-2} \int_0^1 \left(\varepsilon^6 u_{xx}^2 + W(u_x) + u^2\right) \mathrm{d}x, & \text{if } u \in V, \ \nu_x = \delta_{u_x(x)}, \\ +\infty, & \text{otherwise.} \end{cases}$$

We remark that this problem was thoroughly studied in [15, 16], under the assumption that W is a double-well potential, and where quasi-periodicity of the minimizers was also proved. As shown below, however, generalization to a three well problem is non-trivial and requires a good understanding on the possible shape of the minimizing sequences. We also point out that the behaviour of I^ε is different from the one of Modica–Mortola-type functionals (see *e.g.* [9, 14]) as $\varepsilon \downarrow 0$. Indeed, in our case the term in u^2 forces minimizers of I^ε to oscillate faster and faster as $\varepsilon \downarrow 0$, making the number of oscillations in the gradient tend to infinity. In what follows we define

$$E_0 := 2 \int_{z_1}^{z_2} |W(s)|^{\frac{1}{2}} \mathrm{d}s, \qquad E_1 := 2 \int_{z_2}^{z_3} |W(s)|^{\frac{1}{2}} \mathrm{d}s,$$

and

$$A_0 := \inf_d \left(3^{-1} z_2^2 z_{21} d^2 + E_0 d^{-1}\right) = (2^{-1}3)^{\frac{2}{3}} E_0^{\frac{2}{3}} \left(z_2^2 z_{21}\right)^{\frac{1}{3}},$$

$$B_0 := \inf_d \left(3^{-1} z_3^2 z_{31} d^2 + (E_0 + E_1) d^{-1}\right) = (2^{-1}3)^{\frac{2}{3}} (E_0 + E_1)^{\frac{2}{3}} \left(z_3^2 z_{31}\right)^{\frac{1}{3}}.$$

where $z_{i1} := (1 - \frac{z_i}{z_1})$ for $i = 2,3$. Further assumptions on W are as follows:

(H5) (Coercivity) There exist $\eta_0 \in (0, \min\{1, -z_1, z_2, \frac{z_3 - z_2}{2}\})$, $c_0 > 0$, $q > 0$ such that

$$W(s) \geq c_0 \min\left\{\min_i |s - z_i|^q, |\eta_0|^q\right\}, \qquad \forall s \in \mathbb{R}.$$

(H6) Let $f_6(y) := 9(E_0 + E_1)^2 \left(z_2^2 + y^3 z_3^2 + 3yz_2(yz_3 + z_2)\right)$, then

$$f_6(y) - \left(A_0 + B_0 y\right)^3 \geq 0, \quad \text{for every } y \geq 0.$$

(H7) Let $f_7(y) := \frac{9}{4}(E_0 + 2E_1)^2 \left(z_2^2 z_{21} + y^3 z_3^2 z_{31} + 3yz_2 z_{31}(yz_3 + z_2)\right)$, then

$$f_7(y) - \left(A_0 + B_0 y\right)^3 \geq 0, \quad \text{for every } y \geq 0.$$

(H8) Let

$$f_8(y) := 9(E_0 + E_1)^2 \left(z_2^2 z_{21} + y^3 z_3^2 z_{31} - 3\frac{(y^2 z_{31} z_3 - z_2 z_{21})^2}{4(z_{21} + yz_{31})}\right),$$

then,

$$f_8(y) - \left(A_0 + B_0 y\right)^3 \geq 0, \quad \text{for every } y \geq 0.$$

These technical assumptions are used to guarantee that the microstructures constructed in Section 3 are energetically preferable to those constructed in Propositions 7.1 and 7.2 (see also Fig. 7). Here by microstructure we mean the shape of a building block which is repeated quasi-periodically in configurations of low energy for I^ε. The period gets smaller with ε. The preferred microstructure clearly depends on the position of the wells, that is on z_1, z_2, z_3, and on the cost of passing from one well to the other, that is on E_0, E_1. (H6) and (H7) reduce to checking that two cubic polynomials are non-negative on $\mathbb{R}_+$. (H6)–(H8) can be verified easily with a computer and hold in a wide range of cases. We refer the reader to Section 7.1 for more details and for a couple of examples.

The first result that we prove is a second Γ-limit for $\bar{\mathcal{E}}$, that is a Γ-limit result for I^ε.

Theorem 1.2. *Assume* (H1)–(H8). *Then* $I^\varepsilon(u, \nu)$ Γ-*converges in the* $L^2(0,1) \times L^\infty_{w*}(0,1;\mathcal{M})$ *topology to*

$$I^0(u,\nu) = \begin{cases} A_0 \int_0^1 \nu_x(z_2)\,\mathrm{d}x + B_0 \int_0^1 \nu_x(z_3)\,\mathrm{d}x, & \text{if } u = 0, \ \nu \in \mathrm{GYM}^\infty(0), \ \mathrm{supp}\,\nu_x \subset \mathcal{Z} \ a.e., \\ +\infty, & \text{otherwise.} \end{cases}$$

We remark that, as $\nu \in \mathrm{GYM}^\infty(0)$, and $\mathrm{supp}\,\nu \subset \mathcal{Z}$ a.e., we must have

$$\int_0^1 \nu_x(z_3)\,\mathrm{d}x = z_{31}^{-1} - \frac{z_{21}}{z_{31}} \int_0^1 \nu_x(z_2)\,\mathrm{d}x.$$

On the other hand $A_0 < \frac{z_{21}}{z_{31}} B_0$, so that $I^0(0,\nu)$ is a linearly decreasing function of $\int_0^1 \nu_x(z_2)$. Therefore, the minimum of I^0 is attained at

$$\int_0^1 \nu_x(z_3)\,\mathrm{d}x = 0, \qquad \int_0^1 \nu_x(z_2)\,\mathrm{d}x = z_{21}^{-1}.$$

Thus, minimizing sequences for $\mathcal{E}^\varepsilon$ have gradients tending in measure to $\{z_1, z_2\}$, and z_3 is not seen in the limit. That is, the vanishing interfacial energy limit selects a unique minimizer out of the infinitely many minimizers of $\mathcal{E}^0$.

As shown in Section 7, (H7) and (H8) are necessary conditions to prove the above Γ-limit result. Nonetheless, it turns out that we can characterize the set of gradient Young measures generated by minimizing sequences for I^ε, even without the second Γ-limit for $\bar{\mathcal{E}}$. This is the result of the following theorem, where also (H6) is relaxed:

Theorem 1.3. *Assume* (H1)–(H5) *and* $z_3 \le 3|z_1|$. *Then any sequence* $u^j \in V$ *of minimizers for* $\mathcal{E}^{\varepsilon_j}$, *with* $\varepsilon_j \to 0$, *is such that* $u^j \to 0$ *in* $L^2(0,1)$, $\delta_{u^j_x} \to \nu$ *in* $L^\infty_{w*}(0,1;\mathcal{M})$, *and* $\nu \in \mathrm{GYM}^\infty(0)$ *satisfies*

$$\mathrm{supp}\,\nu_x \in \{z_1, z_2\}, \qquad \nu_x = \frac{z_2}{z_2 - z_1}\delta_{z_1} - \frac{z_1}{z_2 - z_1}\delta_{z_2}, \qquad a.e. \ x \in (0,1).$$

In this way we have shown that, in our case, even if the set of gradient Young measures minimizing $\mathcal{E}^0$ has infinitely many elements, its subset generated by minimizers for $\mathcal{E}^\varepsilon$, which are also minimizers for the regularized and rescaled problem I^ε, contains just one element.

Therefore, the one-dimensional model problem studied in this paper confirms that vanishing interface energy can be used as a tool to select minimizing gradient Young measures. This suggests that for the three-dimensional problem E the set $\mathcal{B}$ is actually much smaller than $\mathcal{A}$. Furthermore, our results show that the shape of the second Γ-limit for E^ε might change with the shape of ϕ. Nonetheless, as in our model problem, it might be possible to characterize $\mathcal{B}$ independently of the second Γ-limit for E^ε.

The plan for the paper is the following: in Section 2 we prove Proposition 1.1, in Sections 3 and 4 we compute some upper and lower bounds for I^ε. Section 5 is devoted to prove Theorem 1.2, while Section 6 is devoted to

prove Theorem 1.3. Finally, in Section 7 we sketch necessity of (H7)–(H8) and give an example where (H7)–(H8) hold, and one where they don't.

In the following sections we will denote by c a generic positive constant depending only on the parameters of the problem, and not on the quantities $N, M, N_\varepsilon, M_\varepsilon, \eta, \varepsilon, \mu, j, \sigma$ appearing below. Its value may change from line to line or even within the same line.

2. Proof of the first Γ-limit

In this section we prove Proposition 1.1.

We first observe that, as $\bar{\mathcal{E}}^\varepsilon(u, \nu)$ is a monotone sequence in ε, the Γ-limit exists and is given by the lower semicontinuous envelope of the pointwise limit of the sequence (*cf.* [8], Rem. 1.40). That is, the Γ-limit is given by

$$
sc \begin{cases} \int_0^1 \big(\langle \nu_x, W\rangle + u^2\big)\mathrm{d}x, & \text{if } u \in V, \nu_x = \delta_{u_x(x)} \text{ a.e. in } (0,1), \\ +\infty, & \text{otherwise,} \end{cases} \tag{2.1}
$$

where sc denotes the lower semicontinuous envelope with respect to the topology $L^2(0,1) \times L^\infty_{w*}(0,1;\mathcal{M})$. We first claim that (2.1) is equal to

$$
sc \begin{cases} \int_0^1 \big(\langle \nu_x, W\rangle + u^2\big)\mathrm{d}x, & \text{if } u \in W^{1,p}(0,1), \nu_x \in \text{GYM}^p(u), \\ +\infty, & \text{otherwise,} \end{cases} \tag{2.2}
$$

that is we can relax the requirements $u \in V, \nu_x = \delta_{u_x(x)}$ a.e. in $(0,1)$. Indeed, given an $u \in W^{1,p}_0(0,1)$, we can approximate it by $u^j \in H^2(0,1)$ such that $u^j \to u$ strongly in $W^{1,p}_0(0,1)$. Therefore, by passing into the limit as j tends to ∞ we can drop the requirement $u \in H^2(0,1)$ in (2.1). Now, let $\nu \in \text{GYM}^p(u)$ for some $u \in W^{1,p}_0(0,1)$. Then by ([20], Thm. 8.7) we know the existence of a sequence $u^j \in W^{1,p}(0,1)$ converging weakly to u in $W^{1,p}(0,1)$, strongly in $L^2(0,1)$, such that $\delta_{u^j_x}$ converges to ν in $L^\infty_{w*}(0,1;\mathcal{M})$. Thanks to ([20], Lem. 8.3), the sequence can actually be chosen in $W^{1,p}_0(0,1)$. Therefore, the fact that (*cf.* [20], Thm. 6.11)

$$
\liminf_j \int_0^1 \langle \delta_{u^j_x}, W\rangle \,\mathrm{d}x \geq \int_0^1 \langle \nu_x, W\rangle \,\mathrm{d}x,
$$

allows us to drop also the requirement on ν that $\nu_x = \delta_{u_x(x)}$ for a.e. $x \in (0,1)$, concluding the proof that (2.1) is equal to (2.2). We now claim that we can drop sc from (2.2), which means that

$$
\begin{cases} \int_0^1 \big(\langle \nu_x, W\rangle + u^2\big)\mathrm{d}x, & \text{if } u \in W^{1,p}(0,1), \nu_x \in \text{GYM}^p(u), \\ +\infty, & \text{otherwise,} \end{cases} \tag{2.3}
$$

is already lower semicontinuous in the $L^2(0,1) \times L^\infty_{w*}(0,1;\mathcal{M})$ topology. To prove this claim, it is sufficient to show that for every sequence $(u_j, \nu^j) \in L^2(0,1) \times \text{GYM}(u_j)$ converging to (u, ν) in $L^2(0,1) \times L^\infty_{w*}(0,1;\mathcal{M})$, we have $\liminf_j \mathcal{E}^0(u_j, \nu^j) \geq \mathcal{E}^0(u, \nu)$. We will follow the approach devised in [6]. If $\liminf_j \mathcal{E}^0(u_j, \nu^j) = \infty$, the thesis follows trivially. Therefore, by passing without loss of generality to a subsequence, we can assume $\mathcal{E}^0(u_j, \nu^j) \leq C$. By (H2), this implies that

$$
\int_0^1 \langle \nu^j_x, |\cdot|^p\rangle \,\mathrm{d}x \leq C,
$$

and, by ([21], Thm. 3.6), we deduce that ν_x is a probability measure for almost every $x \in (0,1)$. Jensen's inequality and the fact that $|\cdot|^p$ is convex yield

$$\int_0^1 |\bar{\nu}_x^j|^p \, \mathrm{d}x \le \int_0^1 \langle \nu_x^j, |\cdot|^p \rangle \, \mathrm{d}x \le C, \qquad \text{where} \qquad \bar{\nu}^j := \int_{\mathbb{R}} s \, \mathrm{d}\nu^j(s).$$

It follows that $\bar{\nu}^j \rightharpoonup \bar{\nu}$ in $L^p(0,1)$ and, therefore, that $u_j \rightharpoonup u$ in $W^{1,p}(0,1)$, where $u_x = \bar{\nu}$. A result like the one in ([21], Prop. 4.5) finally gives us that $\nu \in \mathrm{GYM}^p(u)$. At this point, an application of ([21], Prop. 3.7) allows us to deduce that $\liminf_j \mathcal{E}^0(u_j, \nu^j) \ge \mathcal{E}^0(u, \nu)$, thus concluding the proof.

Remark 2.1. Following the same strategy it is actually possible to prove that E^ε and $\tilde{E}^\varepsilon$ Γ-converge in the $L^1(\Omega) \times L^\infty_{w^*}(\Omega; \mathcal{M}_1(\mathbb{R}^{3\times3}))$ topology to $\bar{E}$ as $\varepsilon \to 0$.

3. Construction of an upper bound

In this section we prove the following proposition:

Proposition 3.1. *Assume* (H1)–(H5), *let* $n \in \mathbb{N}$, $n \ge 2$, *and let* $0 = x_1 < x_2 < \cdots < x_n = 1$ *be a partition of* $[0,1]$. *There exist* $\zeta > 0$ *and* $\varepsilon_0 = \varepsilon_0(\min_i(x_{i+1} - x_i)) > 0$, *such that for every* $\varepsilon \le \varepsilon_0$ *we can find* $u \in V$ *with*

$$
\begin{aligned}
\int_{x_i}^{x_{i+1}} \left(\varepsilon^4 u_{xx}^2 + \varepsilon^{-2} W(u_x) + \varepsilon^{-2} u^2 \right) \mathrm{d}x &\le A_0 z_{21}^{-1}(x_{i+1} - x_i) + c\varepsilon^\zeta, &&\text{for } i \text{ odd,} \\
\int_{x_i}^{x_{i+1}} \left(\varepsilon^4 u_{xx}^2 + \varepsilon^{-2} W(u_x) + \varepsilon^{-2} u^2 \right) \mathrm{d}x &\le B_0 z_{31}^{-1}(x_{i+1} - x_i) + c\varepsilon^\zeta, &&\text{for } i \text{ even.}
\end{aligned}
\tag{3.1}
$$

Furthermore, for every $\sigma \in (\varepsilon^{\frac{1}{\max\{3,q\}}}, \eta_0)$,

$$
\begin{aligned}
\left| \mathcal{L}\big((x_i, x_{i+1}) \cap \{|u_x - z_2| \le \sigma\}\big) - z_{21}^{-1}(x_{i+1} - x_i) \right| + \left| \mathcal{L}\big((x_i, x_{i+1}) \cap \{|u_x - z_3| \le \sigma\}\big) \right| &\le c\varepsilon^\zeta, \\
\left| \mathcal{L}\big((x_i, x_{i+1}) \cap \{|u_x - z_2| \le \sigma\}\big) \right| + \left| \mathcal{L}\big((x_i, x_{i+1}) \cap \{|u_x - z_3| \le \sigma\}\big) - z_{31}^{-1}(x_{i+1} - x_i) \right| &\le c\varepsilon^\zeta,
\end{aligned}
\tag{3.2}
$$

respectively when i *is odd and* i *is even.*

Proof. Here we generalize the approach devised in [15]. For simplicity, we prove the statement assuming $n = 3$ and $x_2 = l_0$ for some $l_0 \in (0,1)$. Let us also define λ_2, λ_3 as $\lambda_2 := z_{21}^{-1} l_0$ and $\lambda_3 := z_{31}^{-1}(1 - l_0)$. We first construct the bit of u with energy $A_0\lambda_2$ in $(0, l_0)$, and then use the same argument to construct on $(l_0, 1)$ the bit of u which has energy $B_0\lambda_3$.

We start by splitting the interval $(0, l_0)$ into N pieces of length $l_N := \frac{z_{21}\lambda_2}{N} = \frac{l_0}{N}$. Let us also consider $\hat{w}(x)$, solution of

$$\varepsilon^3 \hat{w}_x = \sqrt{W(\hat{w})}, \qquad \hat{w}(0) = 0. \tag{3.3}$$

Standard ODE theory tells us that $\hat{w}$ exists, and that $\hat{w}$ is strictly increasing with x when $\hat{w}(x) \in (z_1, z_2)$. We point out that, in case $q < 2$, the solution might not be unique. In this case, when solutions encounter z_1 or z_2 we choose the one that stays bounded in $[z_1, z_2]$ and does not decrease/increase further. As $\hat{w}(x - \omega)$ still satisfies the equation in (3.3) for every $\omega \in \mathbb{R}$, we will choose $\omega = \omega^*$ so that

$$F(\omega^*) := \int_0^{l_N} \hat{w}(s - \omega^*) \, \mathrm{d}s = 0. \tag{3.4}$$

Indeed, this is possible as F is negative for $\omega \to \infty$, positive when $\omega \to -\infty$, continuous and decreasing. Now we define w as

$$w(x) = \begin{cases} \hat{w}(x - \omega^* - x_i), & \text{if } x \in (x_i, x_{i+1}) \text{ when } i \text{ even,} \\ \hat{w}(x_{i+1} - \omega^* - x), & \text{if } x \in (x_i, x_{i+1}) \text{ when } i \text{ odd,} \end{cases} \tag{3.5}$$

where $x_i := i l_N$ for $i = 0, \ldots, N$. We are now ready to construct u as

$$u(x) := \int_0^x w(s)\, \mathrm{d}s, \tag{3.6}$$

and to notice that, by (3.4), $u(x_i) = 0$ for each $i = 0, \ldots, N$. By (3.3) we have

$$\int_{x_i}^{x_{i+1}} \left(\varepsilon^4 u_{xx}^2 + \varepsilon^{-2} W(u_x) \right) \mathrm{d}x = 2\varepsilon \int_{x_i}^{x_{i+1}} |W(u_x)|^{\frac{1}{2}} |u_{xx}|\, \mathrm{d}x \leq 2\varepsilon \int_{z_1}^{z_2} |W(s)|^{\frac{1}{2}} \mathrm{d}s = \varepsilon E_0.$$

On the other hand, called x_i^* the point in (x_i, x_{i+1}) such that $u_x(x_i^*) = 0$, and assuming without loss of generality that $u_x > 0$ in (x_i, x_i^*) (the case $u_x < 0$ is similar), we have

$$\int_{x_i}^{x_{i+1}} u^2\, \mathrm{d}x \leq \int_{x_i}^{x_i^*} \left(z_2(x - x_i) \right)^2 \mathrm{d}x + \int_{x_i^*}^{x_{i+1}} \left(z_1(x_{i+1} - x) \right)^2 \mathrm{d}x = 3^{-1}\left(z_2^2 \alpha^3 + z_1^2 \gamma^3 \right) l_N^3, \tag{3.7}$$

where

$$\alpha := \mathscr{L}\left((x_0, x_1) \cap \{w \geq 0\} \right) l_N^{-1}, \qquad \gamma := \mathscr{L}\left((x_0, x_1) \cap \{w < 0\} \right) l_N^{-1}.$$

Therefore,

$$\int_0^{l_0} \left(\varepsilon^4 u_{xx}^2 + \varepsilon^{-2} W(u_x) + \varepsilon^{-2} u^2 \right) \mathrm{d}x \leq N \left(3^{-1} \varepsilon^{-2} \left(z_2^2 \alpha^3 + z_1^2 \gamma^3 \right) l_N^3 + \varepsilon E_0 \right)$$
$$= l_0 \left(3^{-1} \left(z_2^2 \alpha^3 + z_1^2 \gamma^3 \right) d_\varepsilon^2 + E_0 d_\varepsilon^{-1} \right), \tag{3.8}$$

where $d_\varepsilon = \frac{l_N}{\varepsilon}$. Now, chosen $\eta \in (0, \eta_0)$, with η_0 as in (H4), we notice that

$$0 = \int_0^{l_N} \hat{w}(s - \omega^*)\, \mathrm{d}s \leq (z_2 \alpha + (z_1 + \eta)\gamma) l_N + r, \tag{3.9}$$

with $r := -(z_1 + \eta)\mathscr{L}(\{s \colon \hat{w}(s - \omega^*) \in (z_1 + \eta, 0)\})$. But r can be estimated as follows: we can rewrite (3.3) in terms of $\hat{v} := \hat{w} - z_1$ as

$$\varepsilon^3 \hat{v}_y = -\sqrt{W(\hat{v} + z_1)}, \qquad \hat{v}(0) = -z_1,$$

where we also made the change of variable $y = -x$. Now, called y^* the point in $\mathbb{R}_+$ where $\hat{v}(y^*) = \eta_0$, by (H5) we have

$$\varepsilon^3 \hat{v}_y(y) \leq -\hat{c}, \qquad \text{for all } y \in (0, y^*],$$

for some $\hat{c} > 0$. After an integration in y between 0 and y^*, this leads to $y^* \leq c\varepsilon^3$. In the same way, when $\eta \leq \hat{v} < \eta_0$, (H5) implies

$$\varepsilon^3 \hat{v}_y \leq -c_0|\hat{v}|^{\frac{q}{2}} \leq -c_0|\hat{v}|^{\frac{\max\{3,q\}}{2}}.$$

Let us now denote by $\tilde{y}$ the point in $\mathbb{R}^+$ such that $\hat{v}(\tilde{y}) = \eta$. An integration between y^* and $\tilde{y}$ yields $\tilde{y} - y^* \leq c\varepsilon^3 \eta^{1-\frac{\max\{3,q\}}{2}}$. Thus, as $\mathscr{L}(\{s \colon \hat{w}(s - \omega^*) \in (z_1 + \eta, 0)\}) = \tilde{y}$, we have obtained

$$|r| \leq |z_1|\tilde{y} \leq c\varepsilon^3 \eta^{1-\frac{\max\{3,q\}}{2}}. \tag{3.10}$$

This together with (3.9) thus imply

$$\gamma \leq \frac{z_2}{|z_1|}\alpha + c\bar{r}, \tag{3.11}$$

where $\bar{r} := \eta + N\varepsilon^3 \eta^{1-\frac{\max\{3,q\}}{2}}$. On the other hand,

$$0 = \int_0^l \hat{w}(s - \omega^*)\,\mathrm{d}s \geq ((z_2 - \eta)\alpha + z_1\gamma)l_N + r_1,$$

where now $r_1 := (z_2 - \eta)\mathscr{L}(\{s \colon \hat{w}(s - \omega^*) \in (0, z_2 - \eta)\})$. By arguing as to get (3.11), we have

$$\frac{z_2}{|z_1|}\alpha \leq \gamma + c\bar{r},$$

and, as $\alpha + \gamma = 1$, by (3.11) we thus deduce $\left|\alpha - \frac{z_1}{z_1 - z_2}\right| \leq c\bar{r}$. The fact that, by construction, $l_0 \frac{z_1}{z_1 - z_2} = \lambda_2$ also implies

$$\lambda_2 - c\bar{r} \leq \alpha l_0 \leq \lambda_2 + c\bar{r}. \tag{3.12}$$

We now choose N as the smallest even integer larger than $l_0(\varepsilon d^*)^{-1}$, where

$$d^* = (3E_0)^{\frac{1}{3}}\left(2\lambda_2^3 l_0^{-3} z_2^2 \left(1 - \frac{z_2}{z_1}\right)\right)^{-\frac{1}{3}}.$$

In this way, $d_\varepsilon \leq d^*$, $d_\varepsilon^{-1} \leq c\varepsilon + (d^*)^{-1}$ and $N \leq c\varepsilon^{-1}$. Let us also choose $\eta = \varepsilon^{\frac{4}{\max\{3,q\}}}$, so that $\bar{r} \leq c\varepsilon^{\frac{4}{\max\{3,q\}}}$ and $\varepsilon^{\frac{4}{\max\{3,q\}}} < 1$ for each $\varepsilon \leq \varepsilon_0 < 1$. By exploiting (3.11)–(3.12) in (3.8) we thus get

$$\begin{aligned}
\int_0^{l_0} \left(\varepsilon^4 u_{xx}^2 + \varepsilon^{-2}W(u_x) + \varepsilon^{-2}u^2\right)\mathrm{d}x &\leq l_0\left(3^{-1}\alpha^3 z_2^2\left(1 - \frac{z_2}{z_1}\right)d_\varepsilon^2 + E_0 d_\varepsilon^{-1}\right) + c\tilde{r} \\
&\leq l_0\left(3^{-1}\lambda_2^3 l_0^{-3} z_2^2\left(1 - \frac{z_2}{z_1}\right)d_\varepsilon^2 + E_0 d_\varepsilon^{-1}\right) + c\tilde{r} \\
&\leq l_0\left(3^{-1}\lambda_2^3 l_0^{-3} z_2^2\left(1 - \frac{z_2}{z_1}\right)(d^*)^2 + E_0(d^*)^{-1}\right) + c\tilde{r} \\
&\leq \lambda_2 A_0 + c\tilde{r}.
\end{aligned} \tag{3.13}$$

Here and below $\tilde{r} := \varepsilon^{\frac{4}{\max\{3,q\}}} + \varepsilon$. We remark that α depends on ε, but for every $\sigma \in (\varepsilon^{\frac{1}{\max\{3,q\}}}, \eta_0)$, we have that

$$\mathscr{L}\big((0, l_0) \cap \{|u_x - z_2| \le \sigma\}\big) = \alpha l_0 - R, \tag{3.14}$$

where $R = N\mathscr{L}((x_0, x_1) \cap \{s \colon 0 < w(s) < z_2 - \sigma\})$. By arguing as in the proof of (3.10) with η replaced by σ, we have that $|R| \le cN\sigma^{1-\max\{3,q\}}\varepsilon^3 \le c\sigma^{-\max\{3,q\}}\varepsilon^2$. Thus, since we assumed $\sigma \ge \varepsilon^{\frac{1}{\max\{3,q\}}}$, we deduce $|R| \le c\varepsilon$. Therefore, by recalling (3.12) with $\eta = \varepsilon^{\frac{4}{\max\{3,q\}}}$, from (3.14) we finally obtain

$$\big|\mathscr{L}\big((0, l_0) \cap \{|u_x - z_2| \le \sigma\}\big) - \lambda_2\big| \le c\varepsilon^{\frac{4}{\max\{3,q\}}} + c\varepsilon. \tag{3.15}$$

Let us now focus on the interval $(l_0, 1)$, where we want to construct the part of u related to the B_0−term of the energy in (3.1). This part of the argument is very similar to the one above, however, as there might be no solution to (3.3) connecting z_1 to z_3, this time we need to construct an u whose gradient is slightly more complicated. Below, we try to highlight the differences from the case above without incurring into many repetitions. Let us consider $\tilde{w}$ to be the solution to

$$\varepsilon^3 \tilde{w}_x = \sqrt{W(\tilde{w})}, \qquad \tilde{w}(s_0 + 2\mu^{\theta+1}) = z_2 + \mu, \tag{3.16}$$

where $s_0 > 0$ is such that $\hat{w}(s_0) = z_2 - \mu$, $\hat{w}$ is as in (3.3), and $\theta = \frac{3}{2}(\max\{q, 3\} - 2)$. Here and below $\mu = \varepsilon^{\frac{2}{\max\{3,q\}-2}}$, so that $\mu^\theta = \varepsilon^3$. We remark that an argument as the one to prove (3.10) yields

$$s_0 \le c\varepsilon^3 \mu^{1-\frac{\max\{3,q\}}{2}} \le c\varepsilon^2,$$

so that s_0 does not explode but actually goes to zero faster than ε. Again, if $q < 2$ $\tilde{w}$ might not be unique, but we choose the one which stays bounded in $[z_2, z_3]$. Let us define v as

$$v(s) = \begin{cases} \hat{w}(s), & \text{if } s \le s_0, \\ \mu^{-\theta}(s - s_0) + (z_2 - \mu), & \text{if } s_0 < s \le s_0 + 2\mu^{\theta+1}, \\ \tilde{w}(s), & \text{if } s_0 + 2\mu^{\theta+1} < s. \end{cases}$$

Again, we divide $(l_0, 1)$ into M subintervals of equal length $l_M := M^{-1}(1 - l_0)$, and notice that, as v is monotone, we can find ω_* such that

$$\int_0^{l_M} v(s - \omega_*) \, \mathrm{d}s = 0.$$

As in the previous part of the proof, we construct

$$w(x) = \begin{cases} v(x - \omega_* - y_i), & \text{if } x \in (y_i, y_{i+1}) \text{ when } i \text{ even and } i \ne 0, \\ v(y_{i+1} - \omega_* - x), & \text{if } x \in (y_i, y_{i+1}) \text{ when } i \text{ odd,} \end{cases} \tag{3.17}$$

with $y_i := l_0 + il_M$, for $i = 0, \dots, M$, and u as in (3.6). We remark that, as in general

$$\lim_{s \uparrow l_0} w(s) = \hat{w}(-\omega^*) \ne \hat{w}(-\omega_*),$$

w needs to be defined differently in (y_0, y_1) in order to be continuous and to have $u \in H^2(0,1)$. For this reason, we construct w as follows in (y_0, y_1):

$$w(x) = \begin{cases} \hat{w}(-\omega^*) + \frac{s-y_0}{\varepsilon^3}(z_1 - \hat{w}(-\omega^*)), & \text{if } x \in (y_0, y_0 + \varepsilon^3), \\ z_1, & \text{if } x \in (y_0 + \varepsilon^3, y_0 + \varepsilon^3 + a), \\ z_1 + \frac{s-y_0-\varepsilon^3-a}{\varepsilon^3}(z_3 - z_1), & \text{if } x \in (y_0 + \varepsilon^3 + a, y_0 + 2\varepsilon^3 + a), \\ z_3, & \text{if } x \in (y_0 + 2\varepsilon^3 + a, y_1 - \varepsilon^3), \\ v(l_M - \omega_*) + \frac{y_1-s}{\varepsilon^3}(z_3 - v(l_M - \omega_*)), & \text{if } x \in (y_1 - \varepsilon^3, y_1), \end{cases}$$

where a is such that $\int_{y_0}^{y_1} w(s)\,\mathrm{d}s = 0$. We point out that such a exists for each $\varepsilon \leq \varepsilon_0$, for some $\varepsilon_0 < 1$ depending on z_1, z_3 and l_0 only. After defining u in (y_0, y_1) as in (3.6), we have $u(y_0) = u(y_1) = 0$ and

$$\int_{y_0}^{y_1} u^2(s)\,\mathrm{d}s \leq \left(\max\{|z_1|, z_3\}\right)^2 \int_{y_0}^{y_1} (s-y_0)^2\,\mathrm{d}s \leq c l_M^3, \varepsilon^{-3}.$$

Thus,

$$\int_{y_0}^{y_1} \left(\varepsilon^4 u_{xx}^2 + \varepsilon^{-2} W(u_x) + \varepsilon^{-2} u^2\right)\,\mathrm{d}s \leq c(\varepsilon + \varepsilon^{-2} M^{-3}).$$

On the other hand, if $i > 0$, by the definition of E_0, E_1 and by the way we constructed u we have

$$\int_{y_i}^{y_{i+1}} \left(\varepsilon^4 u_{xx}^2 + \varepsilon^{-2} W(u_x)\right)\,\mathrm{d}x \leq 2\varepsilon\left(\int_{z_1}^{z_2} |W(s)|^{\frac{1}{2}}\,\mathrm{d}s + \int_{z_2}^{z_3} |W(s)|^{\frac{1}{2}}\,\mathrm{d}s\right) + c\mu^{\theta+1}(\varepsilon^4 \mu^{-2\theta} + \varepsilon^{-2})$$

$$\leq \varepsilon(E_0 + E_1) + c\mu\varepsilon.$$

Furthermore, once defined

$$\beta := \mathscr{L}\big((y_1, y_2) \cap \{w \geq 0\}\big) l_M^{-1}, \qquad \gamma := \mathscr{L}\big((y_1, y_2) \cap \{w < 0\}\big) l_M^{-1},$$

by arguing as in the proof of (3.7) we deduce

$$\int_{y_i}^{y_{i+1}} u^2 \leq 3^{-1}\big(z_3^2 \beta^3 + z_1^2 \gamma^3\big) l_M^3.$$

Therefore, collecting the inequalities above

$$\begin{aligned} &\int_{l_0}^{1} \left(\varepsilon^4 u_{xx}^2 + \varepsilon^{-2} W(u_x) + \varepsilon^{-2} u^2\right)\,\mathrm{d}x \\ &\leq c(\mu\varepsilon M + \varepsilon + \varepsilon^{-2} M^{-3}) + (1 - l_0)\Big((E_0 + E_1)h_\varepsilon^{-1} + 3^{-1}(z_3^2 \beta^3 + z_1^2 \gamma^3)h_\varepsilon^2\Big), \end{aligned} \tag{3.18}$$

where now $h_\varepsilon = \frac{l_M}{\varepsilon}$. As in (3.9) we have

$$0 = \int_0^{l_M} v(s - \omega^*)\,\mathrm{d}s \geq (z_1\gamma + (z_3 - \mu)\beta)l_M - r_2, \tag{3.19}$$

with $r_2 := (z_3 - \mu)\mathscr{L}\big(\{s : v(s - \omega^*) \in (0, z_3 - \mu)\}\big)$. We first notice that

$$r_2 = (z_3 - \mu)\Big(\mu^{\theta+1} + s_0 + \mathscr{L}\big(\{s : \tilde{w}(s) \in (z_2 + \mu, z_3 - \mu)\}\big)\Big).$$

Thus, by arguing as in the proof of (3.10), we first deduce

$$\mathscr{L}\big(\{s : \tilde{w}(s) \in (z_2 + \mu, z_3 - \mu)\}\big) \le c\varepsilon^3 \mu^{1 - \frac{\max\{3,q\}}{2}},$$

and therefore

$$|r_2| \le c(\varepsilon^3 \mu^{1 - \frac{\max\{3,q\}}{2}} + s_0 + \mu^{\theta+1}) \le c\varepsilon^2. \tag{3.20}$$

Define $\bar{r}_M := M\varepsilon^2 + \mu$, then (3.19)–(3.20) imply

$$\frac{z_3}{|z_1|}\beta \le \gamma + c\bar{r}_M. \tag{3.21}$$

In the same way, we can prove that

$$\gamma \le \frac{z_3}{|z_1|}\beta + c\bar{r}_M, \tag{3.22}$$

and, recalling that $\beta + \gamma = 1$, $(1 - l_0)\frac{z_1}{z_1 - z_3} = \lambda_3$, by (3.21) we obtain

$$\lambda_3 - c\bar{r}_M \le \beta(1 - l_0) \le \lambda_3 + c\bar{r}_M. \tag{3.23}$$

Then, after choosing M to be the smallest integer larger than $(1 - l_0)(\varepsilon h^*)^{-1}$, with

$$h^* := \big(3(E_0 + E_1)\big)^{\frac{1}{3}} \Big(2\lambda_3^3(1 - l_0)^{-3}z_3^2\Big(1 - \frac{z_3}{z_1}\Big)\Big)^{-\frac{1}{3}},$$

and exploiting (3.22)–(3.23), (3.18) becomes

$$\int_{l_0}^{1} \big(\varepsilon^4 u_{xx}^2 + \varepsilon^{-2}W(u_x) + \varepsilon^{-2}u^2\big)\,\mathrm{d}x \le (1 - l_0)\Big((E_0 + E_1)h_\varepsilon^{-1} + 3^{-1}z_3^2\lambda_3^3(1 - l_0)^{-3}\Big(1 - \frac{z_3}{z_1}\Big)h_\varepsilon^2\Big) + c\hat{r}$$

$$\le \beta B_0(1 - l_0) + c\hat{r} \le B_0\lambda_3 + c\hat{r},$$

where $\hat{r} = \varepsilon + \mu$. Here, we repeatedly used the fact that $M \le c\varepsilon^{-1}, M^{-1} \le c\varepsilon$ and that $\mu, \varepsilon < 1$ in order to estimate the above error. This together with (3.13) proves (3.1).

Now, since $\sigma > \varepsilon^{\frac{1}{\max\{3,q\}}} > \mu$,

$$\mathscr{L}\big((l_0, 1) \cap \{|u_x - z_2| \le \sigma\}\big) \le \mathscr{L}\big((l_0, 1) \cap \{|u_x - z_2| \le \mu\}\big) + \mathscr{L}\big((l_0, 1) \cap \{\mu \le |u_x - z_2| \le \sigma\}\big)$$

$$\le cM(\mu^{\theta+1} + \varepsilon^3\sigma^{1 - \frac{\max\{3,q\}}{2}}) + c\varepsilon^3 \le c\varepsilon,$$

where we argued as to get (3.10) in order to bound $\mathscr{L}\big((l_0, 1) \cap \{\mu \le |u_x - z_2| \le \sigma\}\big)$. Furthermore,

$$\mathscr{L}\big((l_0, 1) \cap \{|u_x - z_3| \le \sigma\}\big) = \beta(1 - l_0) + R_2 - \beta l_M + \mathscr{L}\big((y_0, y_1) \cap \{|u_x - z_3| \le \sigma\}\big),$$
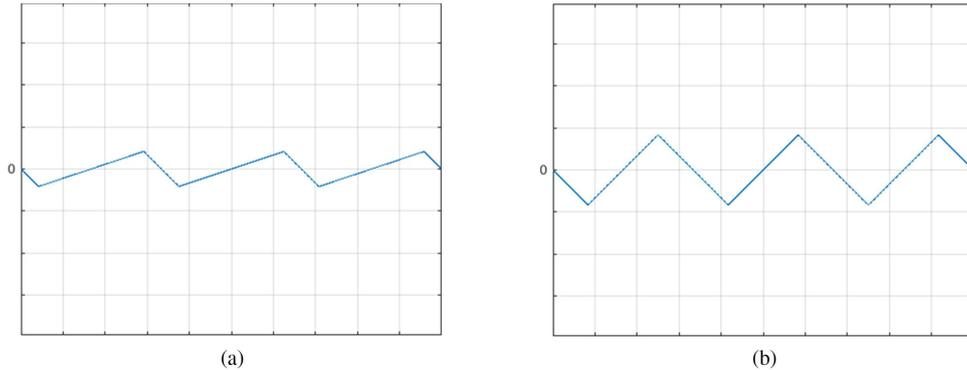
FIGURE 2. Piecewise approximation of the constructed function. In *panel a* we show the function constructed in $(0, l_0)$, whose gradient oscillates between z_1 and z_2. In *panel b* we show the function constructed in $(l_0, 1)$, whose gradient oscillates between z_1 and z_3.

where $R_2 := (M-1)\mathscr{L}\big((y_1, y_2) \cap \{v(s - \omega^*) \in (0, z_3 - \sigma)\}\big)$. As $R_2 \leq cM|r_2| \leq c\varepsilon$ and $l_M \leq c\varepsilon$ we have

$$\big|\mathscr{L}\big((l_0, 1) \cap \{|u_x - z_3| \leq \sigma\}\big) - \beta(1 - l_0)\big| \leq c\varepsilon. \tag{3.24}$$

Now, recalling that (3.23) implies $|\lambda_3 - \beta(1 - l_0)| \leq c\hat{r}$, (3.24) and the triangular inequality imply

$$\big|\mathscr{L}\big((l_0, 1) \cap \{|u_x - z_3| \leq \sigma\}\big) - \lambda_3\big| \leq c\hat{r}.$$

This together with (3.15) lead to the second statement of the result.   □

4. CONSTRUCTION OF A LOWER BOUND

This section is the core of this paper, and is where we prove a lower bound for the energy depending on the global volume fractions $\lambda_k^\eta(v)$ defined in (4.2) below, and representing the one-dimensional Lebesgue measure of the set where v_x is in an η-neighbourhood of z_k. Here, $\eta \in (\varepsilon^{\frac{1}{q+1}}, \eta_0)$, η_0 is as in (H4) and $v \in V$. We point out that the presence of a third well gives the possibility of many different microstructures (see *e.g.*, Figs. 2a, 2b and 7), and makes the estimates below long and technical.

The strategy to prove our lower bounds is the following: for every $v \in V$ of finite energy we identify L-intervals (see Def. 4.3), sets in which $v_x > z_1 + \eta$ and containing at least a point where $v_x \geq z_2 + \eta$. By Lemma 4.2, the number N_v of L-intervals is finite, and can be bounded by a constant times $\frac{1}{\varepsilon}$. In Proposition 4.8 we estimate from below the L^2-norm of v in the L-intervals $L_i \subset (0, 1)$, with $i = 1, \ldots, N_v$. We highlight that the sharp estimates are different for different types of microstructures (see Def. 4.7 and Fig. 5). We then identify (possibly empty) regions $\Sigma_i \subset (0, 1)$ in the set where v_x is in an η-neighbourhood of z_1, and in these sets we estimate the L^2-norm of v. The lower bounds for the L^2-norm in the sets Σ_i are combined with the L^2-estimates in the L_i's to obtain good lower bounds for the L^2-norm of v on every disjoint set $F_i := L_i \cup \Sigma_i$. The interface energy, that is, the energy necessary for the transition of v_x from one well of W to another, can be bounded *via* the Modica–Mortola estimate

$$\int_a^b \big(\varepsilon^4 v_{xx}^2 + \varepsilon^{-2} W(v_x)\big) \,\mathrm{d}x \geq 2\varepsilon \int_a^b |\sqrt{W(v_x)} v_{xx}| \,\mathrm{d}x \geq 2\varepsilon \bigg|\int_{v_x(a)}^{v_x(b)} \sqrt{W(s)} \,\mathrm{d}s\bigg|,$$
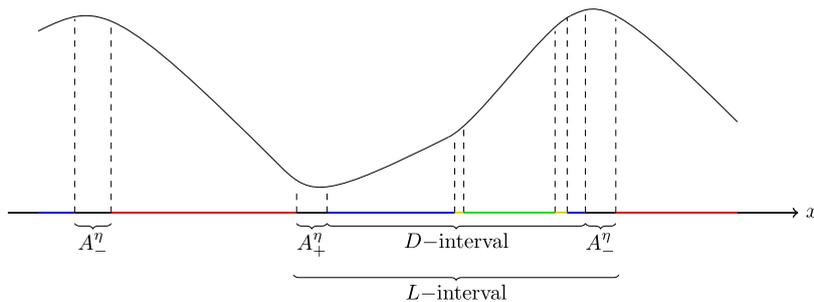
FIGURE 3. Example of L-interval and D-interval as definded in Definition 4.3. In this picture, the red, the blue and the green intervals are respectively the sets of points where $|v_x - z_1| \leq \eta$, $|v_x - z_2| \leq \eta$, and $|v_x - z_3| \leq \eta$. The $B_\pm^\eta$-transition layers are coloured in yellow.

valid for every $0 \leq a \leq b \leq 1$. We show that for each $i = 1, \ldots, N_v$

$$\text{energy of v in } F_i + \text{small error} \geq \varepsilon(\text{interface energy in } F_i) + \frac{1}{\varepsilon^2}(L^2\text{-norm of } v \text{ in } F_i) \tag{4.1}$$

$$\geq \min_{d>0}\Big\{ d(\text{interface energy in } F_i) + \frac{1}{d^2}(L^2\text{-norm of } v \text{ in } F_i)\Big\}.$$

In the two-well case (see [15]), it is possible to sum the resulting lower bounds over $i = 1, \ldots, N_v$, and to obtain a lower bound depending on global quantities only. In our case, however, the lower bounds deduced *via* (4.1) are nonlinear in the volume fractions α_i, β_i (see (4.6) below), defined respectively as the Lebesgue measures of the regions of F_i where v_x is close to z_2, z_3. Furthermore, we get lower bounds which are different depending on the different microstructures in the interval (see *e.g.*, Figs. 2a, 2b and 7). This means that different microstructures give a different dependence of the lower bound on the volume fractions α_i, β_i. These facts increase the complexity of the problem, as they do not allow one, in general, to collect the estimates for the different F_i and to obtain a lower bound depending only on the global volume fractions $\lambda_k^\eta(v)$, $k = 1, 2, 3$.

Finally, in Theorem 4.10 we use assumptions (H6)–(H8) to bound from below the estimates obtained in Proposition 4.8 with the linear function $A_0\alpha_i + B_0\beta_i$. We can hence sum the contribution of every disjoint set F_i and obtain the final lower bound $A_0\lambda_2^\eta(v) + B_0\lambda_3^\eta(v)$. The final estimate looks independent of $\lambda_1^\eta(v)$, but this is because we implicitly make use of

$$\sum_{k=1}^{3} \lambda_k^\eta(v) = 1 + \text{small error}.$$

Let $\eta_0 > 0$ be as in (H4). Given a generic $v \in H^2(0, 1)$, $\eta \in (0, \eta_0)$, let us define the kth global volume fraction for v as

$$\lambda_k^\eta(v) := \mathscr{L}\big(\{x \in (0, 1) \colon |v_x(x) - z_k| \leq \eta\}\big), \qquad k = 1, 2, 3, \tag{4.2}$$

and let us also generalize the definition of transition layers given in [15] (*cf.* also Fig. 3)

Definition 4.1. Let $v \in H^2(a, b)$ and $\eta \in (0, \eta_0)$. An interval (x^-, x^+) is called an A_+^η-transition (resp. an A_-^η-transition) layer for v if

$$\begin{aligned}
v_x(x) &\in (z_1 + \eta, z_2 - \eta), &\forall x \in (x^-, x^+), \\
v_x(x^-) &= z_1 + \eta, &(\text{resp. } v_x(x^+) = z_1 + \eta), \\
v_x(x^+) &= z_2 - \eta, &(\text{resp. } v_x(x^-) = z_2 - \eta).
\end{aligned}$$

An interval (x^-, x^+) is called a B_+^η-transition (resp. a B_-^η-transition) layer for v if

$$
\begin{aligned}
v_x(x) &\in (z_2 + \eta, z_3 - \eta), &&\forall x \in (x^-, x^+), \\
v_x(x^-) &= z_2 + \eta, &&(\text{resp. } v_x(x^+) = z_2 + \eta), \\
v_x(x^+) &= z_3 - \eta, &&(\text{resp. } v_x(x^-) = z_3 - \eta).
\end{aligned}
$$

Given a function $v \in H^2(0,1)$ and $\eta \in (0, \eta_0)$ we denote by $\#A_+^\eta$ (or by $\#A_-^\eta, \#B_+^\eta, \#B_-^\eta$) the number of A_+^η-transition layers for v (resp. $\#A_-^\eta, \#B_+^\eta, \#B_-^\eta$-transition layers for v) in the interval $(0,1)$. The number of transition layers of a function v with bounded energy can be controlled by a constant times ε^{-1}, as stated in the following lemma:

Lemma 4.2. *Assume* (H1)–(H5), $\eta < \eta_0$, *and let* $\varepsilon > 0$ *and* $v \in H^2(0,1)$ *be such that* $I^\varepsilon(v) \le C$. *Then, there exists* $c = c(C) > 0$ *such that*

$$
\max\{\#A_+^\eta, \#A_-^\eta, \#B_+^\eta, \#B_-^\eta\} \le c\varepsilon^{-1}.
$$

Proof. Let us first recall that, given $0 \le a \le b \le 1$ we have

$$
\int_a^b \left(\varepsilon^4 v_{xx}^2 + \varepsilon^{-2} W(v_x)\right) \mathrm{d}x \ge 2\varepsilon \int_a^b \left|\sqrt{W(v_x)} v_{xx}\right| \mathrm{d}x \ge 2\varepsilon \left|H(v_x(b)) - H(v_x(a))\right|, \tag{4.3}
$$

where $H(s) = \int_0^s \sqrt{W(r)}\,\mathrm{d}r$. Now, let us restrict ourselves to the case of the $A_\pm^\eta$-transition layers, as the proof for the $B_\pm^\eta$-transition layers follows the same strategy. Let (x^-, x^+) be an $A_\pm^\eta$-transition layer, then by (4.3) and the fact that $\eta < \eta_0$ we have

$$
\int_{x^-}^{x^+} \left(\varepsilon^4 v_{xx}^2 + \varepsilon^{-2} W(v_x)\right) \mathrm{d}x \ge 2\varepsilon(H(z_2 - \eta_0) - H(z_1 + \eta_0)) > \varepsilon\tilde{c}, \tag{4.4}
$$

for some positive constant $\tilde{c}$. Summing all the $A_\pm^\eta$-transition layers we thus get

$$
C \ge I^\varepsilon(v) \ge \varepsilon\tilde{c}(\#A_+^\eta + \#A_-^\eta),
$$

which concludes the proof. □

We can now introduce also the D-intervals, which are the intervals between an A_+^η-transition layer (y^-, x^-) and the first A_-^η-transition layer (y^+, x^+) in order of appearance in $(0,1)$ after (y^-, x^-) (see Fig. 3):

Definition 4.3. Let $v \in H^2(a,b)$ and $\eta \in (0, \eta_0)$. Let (y^-, x^-) be an A_+^η-transition layer for v and (y^+, x^+) be an A_-^η-transition layer for v, with $x^- \le x^+$. We say that (y^-, y^+) is a L-interval for v, if

$$
v_x(x) > z_1 + \eta, \quad \text{for each } x \in (y^-, y^+), \quad \text{and} \quad v_x(y^+) = v_x(y^-) = z_1 + \eta. \tag{4.5}
$$

If (4.5) holds, the interval (x^-, x^+) is called a D-interval for v.

It is important to notice that v_x might take negative values in a D-interval. For every $v \in V$, the number

$$
N_v = \text{number of } D\text{-intervals for } v \text{ in } (0,1),
$$

is finite. Indeed, for every $v \in V$, v_x is continuous and N_v is equal to the number of A_+^η-transition layers, which is finite by Lemma 4.2. We denote by D_i the ith D-interval in order of appearance in the interval $(0,1)$, where i goes from 1 to N_v. This means that, given two D-intervals $D_i = (x_i^-, x_i^+)$ and $D_j = (x_j^-, x_j^+)$, we have $x_i^+ < x_j^-$
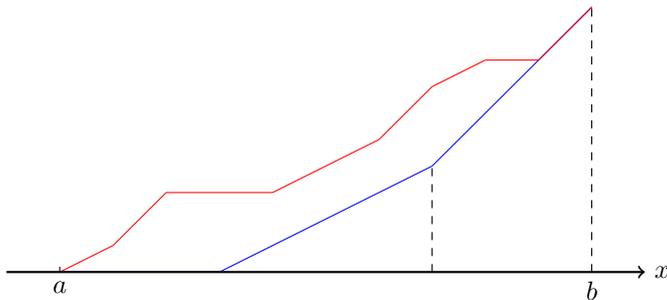
FIGURE 4. Representation of Lemma 4.4. In blue the optimal-function $u \in W^{1,1}(a,b)$, which is convex, in red a generic function $v \in W^{1,1}(a,b)$ such that $u(a) = v(a)$ and (4.8) holds.

if and only if $i < j$. The same can be done for L-intervals. Given $v \in H^2(0,1)$ we define also the following quantities

$$\alpha_i^\eta(v) := \mathscr{L}\big(\{x \in D_i \colon |v_x(x) - z_2| \le \eta\}\big), \qquad \beta_i^\eta(v) := \mathscr{L}\big(\{x \in D_i \colon |v_x(x) - z_3| \le \eta\}\big). \tag{4.6}$$

measuring the subset of D_i where v_x is respectively close to z_2 and z_3. For ease of notation, we omit the dependence on v of $\alpha_i^\eta, \beta_i^\eta$ and λ_k^η. In what follows we will also drop the η from $\alpha_i^\eta, \beta_i^\eta$, keeping their dependence from this variable implicit. We remark that, denoting $D_1 = (x_1^-, x_1^+)$, $D_{N_v} = (x_{N_v}^-, x_{N_v}^+)$, we have

$$\sum_{i=1}^{N_v} \alpha_i + \mathscr{L}\big(\{x \in (0, x_1^-) \cup (x_{N_v}^+, 1) \colon |v_x(x) - z_2| \le \eta\}\big) = \lambda_2^\eta,$$

$$\sum_{i=1}^{N_v} \beta_i + \mathscr{L}\big(\{x \in (0, x_1^-) \cup (x_{N_v}^+, 1) \colon |v_x(x) - z_3| \le \eta\}\big) = \lambda_3^\eta, \tag{4.7}$$

and that, in general, $\alpha_i + \beta_i < \mathscr{L}(D_i)$. Below, we estimate the energy of a generic $v \in V$ on every D-interval in terms of the quantities α_i, β_i. In order to do that, we first need to prove the following lemma, which is graphically explained in Figure 4.

Lemma 4.4. *Let $0 \le a \le b \le 1$ and let $u, v \in W^{1,1}(a,b)$ be two non-decreasing functions such that $u(a) = v(a)$ and*

$$\mathscr{L}\big(\{x \in (a,b) \colon u_x \ge \rho\}\big) = \mathscr{L}\big(\{x \in (a,b) \colon v_x \ge \rho\}\big), \tag{4.8}$$

for every $\rho \ge 0$. If u is the optimal function, that is if u_x is non-decreasing in (a,b), then $u(x) \le v(x)$ for every $x \in [a,b]$.

Proof. We first notice that, as u_x is nondecreasing, $\{u_x \ge \rho\}$ is either empty, or an interval containing b. Thus, for every $x \in (a,b)$, we have

$$\mathscr{L}\big(\{u_x \ge \rho\} \cap (a,x)\big) = \big(\mathscr{L}(\{u_x \ge \rho\}) + x - b\big)_+ = \big(\mathscr{L}(\{v_x \ge \rho\}) + x - b\big)_+ \le \mathscr{L}\big(\{v_x \ge \rho\} \cap (a,x)\big),$$

where we denoted by $(\cdot)_+ := \max\{0, \cdot\}$, and where we used $\mathscr{L}(A \cap B) = \mathscr{L}(A) + \mathscr{L}(B) - \mathscr{L}(A \cup B)$ in the first and last passage. Therefore,

$$u(x) - u(a) = \int_a^x u_x(s)\,\mathrm{d}s = \int_0^\infty \mathscr{L}\big(\{u_x \ge \rho\} \cap (a,x)\big)\,\mathrm{d}\rho \le \int_0^\infty \mathscr{L}\big(\{v_x \ge \rho\} \cap (a,x)\big)\,\mathrm{d}\rho = v(x) - v(a),$$

for every $x \in [a,b]$. As $u(a) = v(a)$, the claimed is proved.   □

Remark 4.5. It follows from Lemma 4.4 that, given $0 \leq a < b \leq 1$, and two Borel sets $\mathcal{C}_1, \mathcal{C}_2 \subset (a,b)$ such that $\mathcal{C}_1 \cap \mathcal{C}_2 = \varnothing$, then

$$\int_a^b \left(\tau_0 + \sum_{i=1,2} \tau_i \mathscr{L}\big((a,x) \cap \mathcal{C}_i\big) \right)^2 \mathrm{d}x \geq \int_0^{\mathscr{L}(\mathcal{C}_1)} (\tau_0 + \tau_1 x)^2 \, \mathrm{d}x + \int_0^{\mathscr{L}(\mathcal{C}_2)} (\tau_0 + \tau_1 \mathscr{L}(\mathcal{C}_1) + \tau_2 x)^2 \, \mathrm{d}x$$

for any $\tau_0 \geq 0$, $\tau_2 > \tau_1 \geq 0$.

We can now start to estimate the energy in the L-intervals. We start by obtaining the desired lower bound for all the L-intervals in which α_i and β_i are either too small or too large.

Lemma 4.6. *Assume* (H1)–(H5), *and let* $v \in H^2(0,1)$, $\eta \in (0, \eta_0)$ *and* $\varepsilon \leq \eta^q$ *be such that* $I^\varepsilon(v) \leq C$, *with* $C > 0$. *Then, there exist* $R_*, R^*(C) > 0$ *such that for any* $L_i = (y_i^-, y_i^+)$, *with* $\max\{\alpha_i, \beta_i\} \geq R^* \varepsilon$ *or* $\max\{\alpha_i, \beta_i\} \leq R_* \varepsilon$,

$$\int_{y_i^-}^{y_i^+} \left(\varepsilon^4 v_{xx}^2 + \varepsilon^{-2} W(v_x) + \varepsilon^{-2} v^2 \right) \mathrm{d}x \geq A_0 \alpha_i + B_0 \beta_i. \tag{4.9}$$

Proof. First we want to prove that if either α_i or β_i is too large, then v also becomes large, and hence its L^2-norm on D_i is bigger than $A_0 \alpha_i + B_0 \beta_i$. In order to do this, let $\max\{\alpha_i, \beta_i\} = R\varepsilon$ for some $R > 0$. Let $D_i = (x_i^-, x_i^+)$. We assume the existence of $x_i^* \in (x_i^-, x_i^+)$ such that $v(x_i^*) = 0$, but the following estimates hold in the case $v > 0$ (or $v < 0$) in (x_i^-, x_i^+) by taking $x_i^* = x_i^-$ (resp. $x_i^* = x_i^+$). Assume also without loss of generality that

$$\max\left\{ \mathscr{L}\big((x_i^*, x_i^+) \cap \{|v_x - z_2| \leq \eta\}\big), \mathscr{L}\big((x_i^*, x_i^+) \cap \{|v_x - z_3| \leq \eta\}\big) \right\} \geq \varepsilon \frac{R}{2}, \tag{4.10}$$

the alternative case can be proved similarly by replacing below (x_i^*, x_i^+) with (x_i^-, x_i^*). We now approximate from below v in (x_i^*, x_i^+) with a piecewise linear function minus a small error proportional to ε. Later, we use Lemma 4.4 to estimate the L^2-norm of v from below with the L^2-norm of its piecewise linear lower bound. We remark that, as (y_i^-, y_i^+) is a L-interval for v, $v_x(x) > z_1 + \eta$ for each $x \in (y_i^-, y_i^+)$. Therefore,

$$v(x) \geq \int_{x_i^*}^x v_x \, \mathrm{d}x \geq (z_2 - \eta) \mathscr{L}\big((x_i^*, x) \cap \{v_x - z_2 \geq -\eta\}\big) + (z_1 + \eta) \mathscr{L}\big((x_i^*, x) \cap \{z_1 + \eta < v_x \leq 0\}\big). \tag{4.11}$$

The last term in (4.11) can be controlled from below by $(z_1 + \eta) \mathscr{L}(\Sigma^\eta)$, where

$$\Sigma^\eta := \{x \in (0,1) : |v_x - z_k| > \eta, \, \forall k = 1,2,3\}. \tag{4.12}$$

By the boundedness of $I^\varepsilon(v)$ and (H5), we can write

$$c_0 \mathscr{L}(\Sigma^\eta) \eta^q \leq \int_{\Sigma^\eta} W(v_x) \, \mathrm{d}x \leq \int_0^1 W(v_x) \, \mathrm{d}x \leq C\varepsilon^2,$$

which implies

$$\mathscr{L}(\Sigma^\eta) \leq c\varepsilon^2 \eta^{-q}. \tag{4.13}$$

It follows then from (4.11), (4.13) and $\varepsilon \leq \eta^q$ that

$$v(x) \geq \hat{c}\mathscr{L}\big((x_i^*, x) \cap \{v_x - z_2 \geq -\eta\}\big) - c\varepsilon, \tag{4.14}$$

for every $x \in (x_i^*, x_i^+)$ and some positive constant $\hat{c}$. Therefore, thanks to (4.10) and Lemma 4.4,

$$2\int_{x_i^*}^{x_i^+} v^2 \, dx + c\varepsilon^2(x_i^+ - x_i^*) \geq \hat{c}^2 \int_{x_i^*}^{x_i^+} \Big(\mathscr{L}\big((x_i^*, x) \cap \{v_x - z_2 \geq -\eta\}\big)\Big)^2 dx \geq \hat{c}^2 \int_0^{\varepsilon\frac{R}{2}} x^2 \, dx \geq \frac{\hat{c}^2}{8}\varepsilon^3 R^3,$$

which, by using the fact that

$$(x_i^+ - x_i^*) \leq 2R\varepsilon + \mathscr{L}(\Sigma^\eta) \leq c\varepsilon(1 + R), \tag{4.15}$$

yields

$$\int_{y_i^-}^{y_i^+} \big(\varepsilon^4 v_{xx}^2 + \varepsilon^{-2}W(v_x) + \varepsilon^{-2}v^2\big) \, dx \geq \varepsilon^{-2} \int_{x_i^-}^{x_i^+} v^2 \, dx \geq \bar{c}\varepsilon R^3 - c\varepsilon(1 + R), \tag{4.16}$$

for some $\bar{c} > 0$. On the other hand, there exists $c^* > 0$ such that

$$A_0\alpha_i + B_0\beta_i \leq c^* R\varepsilon. \tag{4.17}$$

Therefore, setting R^* as the biggest root of $\bar{c}R^3 = (c + c^*)(R + 1)$, we deduce that, if $R \geq R^*$, (4.16)–(4.17) imply (4.9).

In order to show that R cannot be too small, we recall that in every L-interval there are exactly two $A_\pm^\eta-$ transition layers. By (4.3)–(4.4) we thus have

$$\int_{y_i^-}^{y_i^+} \big(\varepsilon^4 v_{xx}^2 + \varepsilon^{-2}W(v_x) + \varepsilon^{-2}v^2\big) \, dx \geq \varepsilon\tilde{c},$$

for some $\tilde{c} > 0$. Hence, if we set $R_* = \frac{\tilde{c}}{c^*}$, by (4.17) we deduce that (4.9) holds for every $R < R_*$. We remark that R^*, R_* do not depend on i, v, ε, η in any way. R_* does not depend on C either. $\qquad\square$

Let n_i be the even number of $B_\pm^\eta$-transition layers for v in D_i. If $n_i = 2$, we denote the B_+^η and the B_-^η-transition layers for v in D_i respectively by $(z_{i,1}^-, z_{i,1}^+)$ and $(z_{i,2}^-, z_{i,2}^+)$, and we define E_i as $E_i := (z_{i,1}^+, z_{i,2}^-)$. We remark that, in general, $\beta_i < \mathscr{L}(E_i)$, but that $v_x(x) > z_2 + \eta$ for every $x \in E_i$. The energy estimates in Proposition 4.8 below are different for the different types of D-intervals given in the following definition (see Fig. 5)

Definition 4.7. Let $v \in V$, $\eta \in (0, \eta_0)$ and $\varepsilon \leq \eta^{q+1}$ be such that $I^\varepsilon(v) \leq C$, with $C > 0$. Let $D_i = (x_i^-, x_i^+)$ be the ith D-interval for v, and $n_i \in 2\mathbb{N}$ be the number of $B_\pm^\eta$-transition layers for v in D_i. Then we say that D_i is of

- **type 0**: if $\max\{\alpha_i, \beta_i\} \notin (\varepsilon R_*, \varepsilon R^*)$;

(a) D−interval of type I

(b) D−interval of type II

(c) D−interval of type III
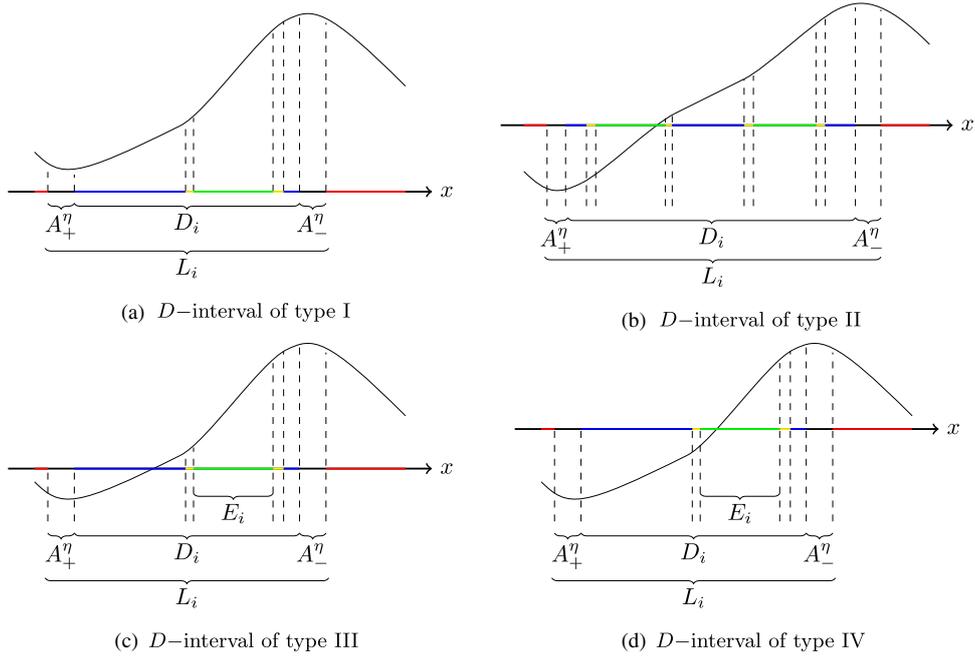
(d) D−interval of type IV

FIGURE 5. Classification of D-intervals (*cf.* Def. 4.7). In this picture, the red, the blue and the green intervals are respectively the sets of points where $|v_x - z_1| \leq \eta$, $|v_x - z_2| \leq \eta$, and $|v_x - z_3| \leq \eta$. The $B_\pm^\eta$-transition layers are coloured in yellow.

- **type I**: if $\max\{\alpha_i, \beta_i\} \in (\varepsilon R_*, \varepsilon R^*)$ and $v(x_i^-)v(x_i^+) \geq 0$;
- **type II**: if $\max\{\alpha_i, \beta_i\} \in (\varepsilon R_*, \varepsilon R^*)$, $v(x_i^-)v(x_i^+) < 0$ and either $n_i = 0$ or $n_i \geq 4$;
- **type III**: if $\max\{\alpha_i, \beta_i\} \in (\varepsilon R_*, \varepsilon R^*)$, $v(x_i^-)v(x_i^+) < 0$, $n_i = 2$ and there exists no $x_i^* \in E_i$ such that $v(x_i^*) = 0$;
- **type IV**: if $\max\{\alpha_i, \beta_i\} \in (\varepsilon R_*, \varepsilon R^*)$, $v(x_i^-)v(x_i^+) < 0$, $n_i = 2$ and there exists $x_i^* \in E_i$ such that $v(x_i^*) = 0$.

In this definition R_*, R^* are as in the statement of Lemma 4.6.

In Proposition 4.8 below we prove some lower bounds for the L^2-norm of v in the D_i's. Then, we identify disjoint sets Σ_i, $i = 1, \ldots, N_v$ in which $|v_x - z_1| \leq \eta$, and in these sets we estimate from below the L^2-norm of v in terms of α_i, β_i. We then combine the estimates in the Σ_i's with the estimates in the L_i's, and we argue as in (4.1) to obtain a lower bound for the energy on the sets $F_i = L_i \cup \Sigma_i$. The results of Proposition 4.8 are the basic tool to prove both Theorem 4.10, from which follows Theorem 1.2, and Theorem 6.1, from which follows Theorem 1.3.

Proposition 4.8. *Assume* (H1)–(H5) *and let $C > 0$. Then, there exists $\eta_1 = \eta_1(C) \in (0, \eta_0)$ such that, for every $\eta \in (0, \eta_1)$, $\varepsilon \leq \eta^{q+1}$ and $v \in V$ satisfying $I^\varepsilon(v) \leq C$, there exists a collection of (possibly empty) Borel sets Σ_i, $i = 1, \ldots, N_v$ such that*

$$\Sigma_i \cap \Sigma_j = \Sigma_i \cap L_j = \Sigma_i \cap L_i = \varnothing, \qquad \text{for every } j \neq i \in \{1, \ldots, N_v\},$$

and, for every $i = 1, \ldots, N_v$,

$$
\begin{aligned}
&\int_{L_i \cup \Sigma_i} \left(\varepsilon^4 v_{xx}^2 + \varepsilon^{-2} W(v_x) + \varepsilon^{-2} v^2 \right) \mathrm{d}x \geq A_0 \alpha_i + B_0 \beta_i, && \text{if } D_i \text{ is type 0,} \\
&\int_{L_i \cup \Sigma_i} \left(\varepsilon^4 v_{xx}^2 + \varepsilon^{-2} W(v_x) + \varepsilon^{-2} v^2 \right) \mathrm{d}x \geq \alpha_i f_6^{\frac{1}{3}} \left(\frac{\beta_i}{\alpha_i} \right) - c\varepsilon\eta, && \text{if } D_i \text{ is type I,} \\
&\int_{L_i \cup \Sigma_i} \left(\varepsilon^4 v_{xx}^2 + \varepsilon^{-2} W(v_x) + \varepsilon^{-2} v^2 \right) \mathrm{d}x \geq \alpha_i f_7^{\frac{1}{3}} \left(\frac{\beta_i}{\alpha_i} \right) - c\varepsilon\eta, && \text{if } D_i \text{ is type II,} \\
&\int_{L_i \cup \Sigma_i} \left(\varepsilon^4 v_{xx}^2 + \varepsilon^{-2} W(v_x) + \varepsilon^{-2} v^2 \right) \mathrm{d}x \geq \alpha_i f_8^{\frac{1}{3}} \left(\frac{\beta_i}{\alpha_i} \right) - c\varepsilon\eta, && \text{if } D_i \text{ is type III/IV}
\end{aligned}
\tag{4.18}
$$

where f_6, f_7, f_8 are as in (H6)–(H8).

Remark 4.9. The first, the third and the fourth lower bounds in Proposition 4.8 are sharp up to an error proportional to some positive power of ε. Sharpness of the first bound is given by Proposition 3.1. For the third and the fourth bound we refer the reader to the proofs of Propositions 7.1 and 7.2 respectively.

Proof. Let $C > 0$, $v \in V$, $\varepsilon \leq \eta^{q+1}$ with $I^\varepsilon(v) \leq C$ and $\eta \in (0, \eta_1)$ with η_1 to be determined later. We divide the proof in three steps: in the first we prove the estimates (4.26), (4.28), (4.29) and (4.32) for the L^2-norm of v in the D-intervals. As explained above, the estimates are different for different types of D-intervals. In step two we construct the sets Σ_i for every $i = 1, \ldots, N_v$, and we estimate from below the L^2-norm of v on these sets in terms of α_i, β_i. Finally, in the last step we combine the estimates for the L^2-norm of v, with the estimates for the interfacial energy, and deduce (4.18) by means of (4.1).

Step 1: The strategy to prove estimates for the L^2-norm of v in the D-intervals is the following: for $i = 1, \ldots, N_v$ we divide D_i into two intervals (one is actually empty if D_i is of type I), one in which v is bigger or equal than $-c\varepsilon\eta$, one in which v is smaller or equal than $c\varepsilon\eta$. As shown below, for D-intervals of type II/IV choosing $x_i^* \in D_i$ such that $v(x_i^*) = 0$ and taking $D_i \cap \{x \geq x_i^*\}, D_i \cap \{x \leq x_i^*\}$ as sub-intervals suffices. In each interval we approximate v with a suitable continuous piecewise linear function with gradient a.e. in $\{0, z_2, z_3\}$, namely $\sum_{k=2,3} z_k \mathscr{L} \left((x_i^*, x) \cap \{|v_x - z_k| \leq \eta\} \right)$, and use Lemma 4.4 to bound its L^2-norm from below. In conclusion we combine the estimates obtained in the two sub-intervals of D_i. The estimates (4.29) and (4.32) for type III/IV D-intervals depend on the quantities $\omega_i^a, \omega_i^b \in [0, 1]$ (defined in (4.23) below). Estimates (4.29) and (4.32) need to be combined with the estimates of Step 2 before minimizing over ω_i^a, ω_i^b.

Let $D_i = (x_i^-, x_i^+)$, and let us first focus on the case $v(x_i^-) v(x_i^+) < 0$. We notice that the continuity of v, implies the existence of $x_i^* \in (x_i^-, x_i^+)$ such that $v(x_i^*) = 0$. We estimate separately $\int_{x_i^*}^{x_i^+} v^2 \, \mathrm{d}s$ and $\int_{x_i^-}^{x_i^*} v^2 \, \mathrm{d}s$. Let us start with the first. As $v_x > z_1 + \eta$ in (x_i^-, x_i^+), we have

$$
v(x) \geq \int_{x_i^*}^{x} v_x \, \mathrm{d}x \geq \sum_{k=2,3} (z_k - \eta) \mathscr{L} \left((x_i^*, x) \cap \{|v_x - z_k| \leq \eta\} \right) + (z_1 + \eta) \mathscr{L} \left(\Sigma^\eta \right)
\tag{4.19}
$$

where Σ^η is as in (4.12) and satisfies (4.13). It follows then from (4.19) and the assumption $\max\{\alpha_i, \beta_i\} \leq R^* \varepsilon$ that

$$
v(x) \geq \sum_{k=2,3} z_k \mathscr{L} \left((x_i^*, x) \cap \{|v_x - z_k| \leq \eta\} \right) - c \frac{\varepsilon^2}{\eta^q} - c\varepsilon\eta R^*,
\tag{4.20}
$$

for every $x \in (x_i^*, x_i^+)$. We now use the fact that, as $\eta \in (0,1)$, $(1-\eta)(a+b)^2 \le a^2 + 2\eta^{-1}b^2$. This yields

$$(1-\eta)\int_{x_i^*}^{x_i^+} \big(v(x) + c(\varepsilon^2\eta^{-q} + \varepsilon\eta)\big)^2 \mathrm{d}x \le \int_{x_i^*}^{x_i^+} v^2 \, \mathrm{d}x + c\hat{r}(x_i^+ - x_i^*), \tag{4.21}$$

with $\hat{r} := \eta^{-1}(\varepsilon^2\eta^{-q} + \varepsilon\eta)^2$. On the other hand, by Lemma 4.4,

$$
\begin{aligned}
\int_{x_i^*}^{x_i^+} \big(v(x) + c(\varepsilon^2\eta^{-q} + \varepsilon\eta)\big)^2 \mathrm{d}x &\ge \int_{x_i^*}^{x_i^+} \Big(\sum_{k=2,3} z_k \mathscr{L}\big((x_i^*, x) \cap \{|v_x - z_k| \le \eta\}\big) \Big)^2 \mathrm{d}x \\
&\ge z_2^2 \int_0^{\omega_i^a \alpha_i} x^2 \, \mathrm{d}x + \int_0^{\omega_i^b \beta_i} (z_3 x + z_2 \omega_i^a \alpha_i)^2 \mathrm{d}x \\
&\ge 3^{-1}\big(z_2^2(\omega_i^a \alpha_i)^3 + z_3^2(\omega_i^b \beta_i)^3 + g_0(\alpha_i, \beta_i, \omega_i^a, \omega_i^b)\big)
\end{aligned}
\tag{4.22}
$$

where $\omega_i^a, \omega_i^b \in [0,1]$ are such that

$$\mathscr{L}\big(D_i \cap \{x > x_i^*\colon |v_x - z_2| \le \eta\}\big) = \omega_i^a \alpha_i, \qquad \mathscr{L}\big(D_i \cap \{x > x_i^*\colon |v_x - z_3| \le \eta\}\big) = \omega_i^b \beta_i, \tag{4.23}$$

and $g_0(a,b,\omega^a,\omega^b) = 3z_2 ab\omega^a\omega^b(z_2 a\omega^a + z_3 b\omega^b)$. Therefore, putting together (4.21)–(4.22), by (4.15) we obtain

$$
\begin{aligned}
\int_{x_i^*}^{x_i^+} v^2 \, \mathrm{d}x &\ge (1-\eta)3^{-1}\big(z_2^2(\omega_i^a \alpha_i)^3 + z_3^2(\omega_i^b \beta_i)^3 + g_0(\alpha_i, \beta_i, \omega_i^a, \omega_i^b)\big) - c\varepsilon^3\eta \\
&\ge 3^{-1}\big(z_2^2(\omega_i^a \alpha_i)^3 + z_3^2(\omega_i^b \beta_i)^3 + g_0(\alpha_i, \beta_i, \omega_i^a, \omega_i^b)\big) - c\varepsilon^3\eta.
\end{aligned}
\tag{4.24}
$$

Here we also used $\max\{\alpha_i, \beta_i\} \le R^*\varepsilon$ and $\varepsilon \le \eta^{q+1}$. In the same way, we prove

$$\int_{x_i^-}^{x_i^*} v^2 \, \mathrm{d}x + c\varepsilon^3\eta \ge 3^{-1}\big(z_2^2((1-\omega_i^a)\alpha_i)^3 + z_3^2((1-\omega_i^b)\beta_i)^3 + g_0(\alpha_i, \beta_i, 1-\omega_i^a, 1-\omega_i^b)\big). \tag{4.25}$$

It turns out that if we sum (4.24) to (4.25) the right hand side is a convex quadratic polynomial in ω_i^a, ω_i^b with minimum in $\omega_i^a = \omega_i^b = \frac{1}{2}$. Therefore, from (4.24) and (4.25) we finally get

$$\int_{D_i} v^2 \, \mathrm{d}x \ge 12^{-1} h(\alpha_i, \beta_i) - c\varepsilon^3\eta, \qquad \text{for all } D_i \text{ of type II}, \tag{4.26}$$

where

$$h(a,b) := z_2^2 a^3 + z_3^2 b^3 + 3\big(abz_2(z_2\alpha_i + z_3\beta_i)\big). \tag{4.27}$$

The estimate

$$\int_{D_i} v^2 \, \mathrm{d}x \ge 3^{-1} h(\alpha_i, \beta_i) - c\varepsilon^3\eta, \qquad \text{for all } D_i \text{ of type I}, \tag{4.28}$$

follows again from (4.24) (or (4.25)) if $v(x_i^-) > 0$ (resp. $v(x_i^+) < 0$) which can be deduced *via* the same argument by setting $x_i^* = x_i^-$ (resp. $x_i^* = x_i^+$). In this case, the above estimates hold with $\omega_i^a, \omega_i^b = 1$ (resp. $\omega_i^a, \omega_i^b = 0$).

Let us suppose now that $v(x_i^-)v(x_i^+) < 0$ and that $n_i = 2$. If D_i is of type III, a simple combination of (4.24)–(4.25) leads to

$$\int_{D_i} v^2 \, \mathrm{d}x \geq 3^{-1}\hat{h}_2(\alpha_i, \beta_i, \omega_i^a, \omega_i^b) - c\varepsilon^3\eta, \qquad \text{for all } D_i \text{ of type III}, \tag{4.29}$$

where

$$\begin{aligned}
\hat{h}_k(a, b, \omega^a, \omega^b) &:= z_2^2 a^3 + z_3^2 b^3 + 3\big(z_2^2\omega^a(\omega^a - 1)a^3 + z_3^2\omega^b(\omega^b - 1)b^3\big) \\
&\quad + z_k 3ab\big(\omega^a\omega^b(z_2\omega^a a + z_3\omega^b b) + (1 - \omega^a)(1 - \omega^b)(z_2(1 - \omega^a)a + z_3(1 - \omega^b)b)\big)
\end{aligned} \tag{4.30}$$

If D_i is of type IV, we can assume that $x_i^* \in E_i$, and the above estimates can be improved. Indeed, by using Remark 4.5 first with $\tau_2 = z_3, \tau_1 = \tau_0 = 0, \mathcal{C}_2 = \{x \in (x_i^*, x_i^+) \colon |v_x(x) - z_3| \leq \eta\}$, $(a, b) = E_i \cap (x_i^*, x_i^+)$ and then with $(a, b) = (x_i^*, x_i^+) \setminus E_i$ $\tau_2 = z_2, \tau_1 = 0, \tau_0 = z_3\omega_i^b\beta_i, \mathcal{C}_2 = \{x \in (x_i^*, x_i^+) \colon |v_x(x) - z_2| \leq \eta\}$, we can modify (4.22) as follows:

$$\begin{aligned}
\int_{x_i^*}^{x_i^+} \Big(\sum_{k=2,3} z_k \mathscr{L}\big((x_i^*, x) \cap \{|v_x - z_k| \leq \eta\}\big) \Big)^2 \mathrm{d}x &\geq z_3^2 \int_0^{\omega_i^b\beta_i} x^2 \, \mathrm{d}x + \int_0^{\omega_i^a\alpha_i} (z_2 x + z_3\omega_i^b\beta_i)^2 \, \mathrm{d}x \\
&\geq 3^{-1}\big(z_2^2(\omega_i^a\alpha_i)^3 + z_3^2(\omega_i^b\beta_i)^3 + \frac{z_3}{z_2}g_0(\alpha_i, \beta_i, \omega_i^a, \omega_i^b)\big),
\end{aligned}$$

which by (4.20)–(4.21) yields

$$\int_{x_i^*}^{x_i^+} v^2 \, \mathrm{d}x + c\varepsilon^3\eta \geq 3^{-1}\big(z_2^2(\omega_i^a\alpha_i)^3 + z_3^2(\omega_i^b\beta_i)^3 + \frac{z_3}{z_2}g_0(\alpha_i, \beta_i, \omega_i^a, \omega_i^b)\big).$$

In the same way, we prove

$$\int_{x_i^-}^{x_i^*} v^2 \, \mathrm{d}x + c\varepsilon^3\eta \geq 3^{-1}\big(z_2^2((1 - \omega_i^a)\alpha_i)^3 + z_3^2((1 - \omega_i^b)\beta_i)^3 + \frac{z_3}{z_2}g_0(\alpha_i, \beta_i, 1 - \omega_i^a, 1 - \omega_i^b)\big). \tag{4.31}$$

By summing up the last two inequalities, we hence get

$$\int_{D_i} v^2 \, \mathrm{d}x \geq 3^{-1}\hat{h}_3(\alpha_i, \beta_i, \omega_i^a, \omega_i^b) - c\varepsilon^3\eta, \qquad \text{for all } D_i \text{ of type IV}. \tag{4.32}$$

We remark that, in this case, the determinant of the Hessian matrix of $\hat{h}$ with respect to ω_i^a, ω_i^b is negative, and hence $\hat{h}$ cannot be bounded from below by choosing $\omega_i^a = \omega_i^b = \frac{1}{2}$.

Step 2: For every $i = 1, \dots, N_v$, we now construct Σ_i and estimate $\int_{\Sigma_i} v^2 \, \mathrm{d}x$ from below. Finally, we show that the Σ_i's are disjoint. As shown in Figure 6, $\Sigma_i = \Sigma_i^d \cup \Sigma_i^m$, where Σ_i^d, Σ_i^m are subsets of $\{|v_x - z_1| \leq \eta\}$ and respectively of $\{v > 0\}$ and $\{v < 0\}$.

We first set $\Sigma_i = 0$ whenever D_i is of type 0 or of type I. We can hence focus on the i's where $D_i = (x_i^-, x_i^+)$ is such that $v(x_i^+)v(x_i^-) < 0$ and $\max\{\alpha_i, \beta_i\} \in (\varepsilon R_*, \varepsilon R^*)$, that is on type II–IV D-intervals. The idea is to bound from below v (or from above) on the set where $|v_x - z_1| \leq \eta$ and $v \geq 0$ (resp. $v \leq 0$) with a continuous piecewise-linear function minus (resp. plus) a small error. We then estimate the L^2-norm of the piecewise linear approximation of v and express it in terms of α_i, β_i. We denote by x_i^* a point in D_i (the same that was chosen
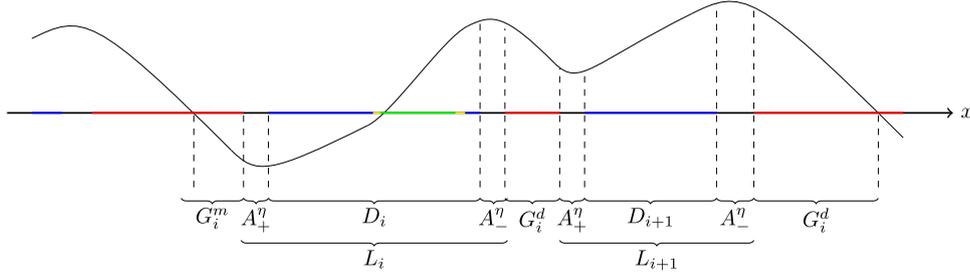
FIGURE 6. Examples of sets Σ_i constructed in the proof of Proposition 4.8. Here, $\Sigma_{i+1} = \varnothing$ because D_{i+1} is of type I, while $\Sigma_i = \Sigma_i^d \cup \Sigma_i^m$, where $\Sigma_i^d \subset G_i^d$, $\Sigma_i^m \subset G_i^m$. The sets G_i^d, G_i^m are constructed as follows: let $D_i = (x_i^-, x_i^+)$, then G_i^d (resp. G_i^m) is the intersection of $\{|v_x - z_1| \leq \eta\}$ with the largest interval (x_i^+, p^+), $x_i^+ \leq p^+$ (resp. (p^-, x_i^-), $x_i^- \geq p^-$), where v is strictly positive (resp. negative). The red, the blue and the green intervals are respectively the sets of points where $|v_x - z_1| \leq \eta$, $|v_x - z_2| \leq \eta$, and $|v_x - z_3| \leq \eta$. The $B_\pm^\eta$-transition layers are coloured in yellow.

in Step 1) such that $v(x_i^*) = 0$ and such that $x_i^* \in E_i$ if D_i is of type IV. From (4.20) we have

$$v(x_i^+) \geq z_2 \omega_i^a \alpha_i + z_3 \omega_i^b \beta_i - c\eta\varepsilon, \tag{4.33}$$

and, in a similar way, we can prove

$$v(x_i^-) \leq -(z_2(1 - \omega_i^a)\alpha_i + z_3(1 - \omega_i^b)\beta_i) + c\eta\varepsilon, \tag{4.34}$$

where ω_i^a, ω_i^b are as in (4.23). We now claim that there exist $\eta_1 \in (0, \eta_0]$ depending just on R_*, R^* and C, such that, for every $\eta < \eta_1$, $v(x_i^+) > 0$ and $v(x_i^-) < 0$. Indeed, we recall that we are working under the assumption $\max\{\alpha_i, \beta_i\} \geq R_*\varepsilon$, and we suppose without loss of generality that $\alpha_i \geq R_*\varepsilon$; the case $\beta_i \geq R_*\varepsilon$ can be treated similarly. Suppose first that $\omega_i^a \geq \frac{1}{2}$, then (4.33) implies $v(x_i^+) \geq \frac{1}{2}z_2\varepsilon R_* - c\eta\varepsilon$. Thus, choosing η small enough, we get $v(x_i^+) > 0$, and, by the fact that $v(x_i^+)v(x_i^-) < 0$, also that $v(x_i^-) < 0$. If $\omega_i^a < \frac{1}{2}$, then $(1 - \omega_i^a) > \frac{1}{2}$, and (4.34) yields $v(x_i^-) \leq -\frac{1}{2}z_2\varepsilon R_* + c\eta\varepsilon$, so that for every η small enough $v(x_i^-) < 0$, and hence $v(x_i^+) > 0$ as claimed. Now, in the same spirit of (4.33), we have

$$v(x) - v(x_i^+) \geq \int_{x_i^+}^{x} v_x \, dx \geq (z_1 - \eta)\mathscr{L}\left((x_i^+, x) \cap \{|v_x - z_1| \leq \eta\}\right) - \tilde{r}, \qquad x \geq x_i^+, \tag{4.35}$$

$$v(x_i^+) \leq \int_{x_i^*}^{x_i^+} v_x \, dx \leq \sum_{k=2,3} z_k \mathscr{L}\left((x_i^*, x_i^+) \cap \{|v_x - z_k| \leq \eta\}\right) + \tilde{r} + c\eta\varepsilon R^*, \tag{4.36}$$

where

$$\tilde{r} = \int_{\Sigma^\eta} |v_x| \, dx, \qquad \Sigma^\eta = \left\{x \in (0, 1) \colon |v_x(x) - z_k| > \eta, \, k = 1, 2, 3\right\}. \tag{4.37}$$

We now give an estimate for $\tilde{r}$. To this aim, we split Σ^η into

$$\Sigma_1^\eta := \left\{x \in \Sigma^\eta \colon |v_x(x)| \leq t_0\right\}, \qquad \Sigma_2^\eta := \left\{x \in \Sigma^\eta \colon t_0 < |v_x(x)|\right\},$$

where t_0 is such that $W(s) \geq \frac{c_1}{2}|s|^p$ for each s satisfying $|s| > t_0$, and its existence is guaranteed by (H2). By (4.13), we have

$$\int_{\Sigma_1^\eta} |v_x| \, \mathrm{d}x \leq t_0 \mathscr{L}(\Sigma^\eta) \leq c\frac{\varepsilon^2}{\eta^q}. \tag{4.38}$$

On the other hand,

$$\frac{c_1}{2}\mathscr{L}(\Sigma_2^\eta)|t_0|^p \leq \int_{\Sigma_2^\eta} W(v_x) \, \mathrm{d}x \leq C\varepsilon^2,$$

implies $\mathscr{L}(\Sigma_2^\eta) \leq c\varepsilon^2$. Therefore,

$$\int_{\Sigma_2^\eta} |v_x(x)| \, \mathrm{d}x \leq c\int_{\Sigma_2^\eta} W^{\frac{1}{p}}(v_x(x)) \, \mathrm{d}x \leq c\big(\mathscr{L}(\Sigma_2^\eta)\big)^{\frac{p-1}{p}}\left(\int_{\Sigma_2^\eta} W(v_x(x)) \, \mathrm{d}x\right)^{\frac{1}{p}} \leq c\varepsilon^2, \tag{4.39}$$

where we also made use of the fact that $I^\varepsilon(v) \leq C$. Collecting (4.38)–(4.39) we thus get

$$\tilde{r} \leq \int_{\cup_k \Sigma_k^\eta} |v_x| \, \mathrm{d}x \leq c(\varepsilon^2 \eta^{-q} + \varepsilon^2) \leq cr^*, \tag{4.40}$$

with $r^* := \varepsilon\eta$. By (4.35)–(4.36), (4.40) together with $\max\{\alpha_i, \beta_i\} \leq R^*\varepsilon$, we obtain

$$v(x) - v(x_i^+) \geq (z_1 - \eta)\mathscr{L}\big((x_i^+, x) \cap \{|v_x - z_1| \leq \eta\}\big) - cr^*, \qquad x \geq x_i^+, \tag{4.41}$$

and

$$0 \leq v(x_i^+) \leq c(\varepsilon\eta + r^* + \varepsilon) \leq c\varepsilon. \tag{4.42}$$

Now let

$$g_b(x) = v(x_i^+) - cr^* - (|z_1| + \eta)\mathscr{L}\big((x_i^+, x) \cap \{|v_x - z_1| \leq \eta\}\big).$$

Let $\bar{x}_i \in [x_i^+, 1]$ be the smallest x in $[x_i^+, 1]$ such that $g_b(x) \leq 0$, and let us set

$$\Sigma_i^d := (x_i^+, \bar{x}_i) \cap \{|v_x - z_1| \leq \eta\}.$$

The existence of $\bar{x}_i$ is guaranteed by the continuity of v and the fact that $v(1) = 0$ together with (4.41) imply $g_b(1) \leq 0$. Thus, $\Sigma_i^d = \varnothing$ if and only if $g_b(x_i^+) \leq 0$. Now, if $\Sigma_i^d \neq \varnothing$, that is $v(x_i^+) > cr^*$, by (4.41) we have that

$$v(x) \geq g_b(x) > 0, \qquad \text{for ever } x \in (x_i^+, \bar{x}_i). \tag{4.43}$$

Now, from (4.43) we deduce that

$$\int_{\Sigma_i^d} v^2 \, \mathrm{d}x \geq \int_{\Sigma_i^d} g_b^2(x) \, \mathrm{d}x \geq (|z_1| + \eta)^{-1}\int_0^{v(x_i^+) - r^*} x^2 \, \mathrm{d}x \geq (|z_1| + \eta)^{-1}v^3(x_i^+) - c\varepsilon^3\eta.$$

Here we have used a change of variable $y = g_b(x)$, and, in the last inequality, we exploited (4.42). The same lower bound holds trivially if $\Sigma_i^d = \varnothing$, and hence $v(x_i^+) \leq cr^*$. Therefore, as $(|z_1| + \eta)^{-1} \geq |z_1|^{-1} - |z_1|^{-2}\eta$, by (4.33) we obtain

$$\int_{\Sigma_i^d} v^2 \, \mathrm{d}x \geq 3^{-1}|z_1|^{-1}\big(z_2\omega_i^a\alpha_i + \omega_i^b\beta_i z_3\big)^3 - cr_b, \tag{4.44}$$

where $r_b := \varepsilon^3 \eta$ and we made use of (4.42) and the fact that $\max\{\alpha_i, \beta_i\} \leq \varepsilon R^*$. In the same way, letting

$$g_m(x) = v(x_i^-) + cr^* + (|z_1| + \eta)\mathscr{L}\big((x, x_i^-) \cap \{|v_x - z_1| \leq \eta\}\big),$$

and $\tilde{x}_i \in [0, x_i^-]$ be the largest $x \in [0, x_i^-]$ such that $g_m(x) \geq 0$, we can define

$$\Sigma_i^m := (\tilde{x}_i, x_i^-) \cap \{|v_x - z_1| \leq \eta\}$$

and deduce

$$\int_{\Sigma_i^m} v^2 \, \mathrm{d}x \geq 3^{-1}|z_1|^{-1}\big(z_2(1 - \omega_i^a)\alpha_i + (1 - \omega_i^b)\beta_i z_3\big)^3 - cr_b. \tag{4.45}$$

Define now $\Sigma_i := \Sigma_i^m \cup \Sigma_i^d$, then by (4.44)–(4.45) we deduce

$$\int_{\Sigma_i} v^2 \, \mathrm{d}x + cr_b \geq 3^{-1}|z_1|^{-1}\big(z_2\omega_i^a\alpha_i + \omega_i^b\beta_i z_3\big)^3 + 3^{-1}|z_1|^{-1}\big(z_2(1 - \omega_i^a)\alpha_i + (1 - \omega_i^b)\beta_i z_3\big)^3. \tag{4.46}$$

We remark that $v_x(x) \leq z_1 + \eta$ for each $x \in \Sigma_i$, while $v_x(x) > z_1 + \eta$ in every L_j, $j = 1, \ldots, N_v$. In this way $L_j \cap \Sigma_i = \varnothing$ for every $i, j = 1, \ldots, N_v$. We now claim that $\Sigma_j \cap \Sigma_i = \varnothing$ for every $j = 1, \ldots, N_v$ with $i \neq j$. Indeed, let x_j^-, x_j^+ be such that $D_j = (x_j^-, x_j^+)$. In the case $v(x_j^-)v(x_j^-) \geq 0$ we defined $\Sigma_j = \varnothing$ and the conclusion follows trivially. We can hence focus on the case $v(x_j^-)v(x_j^+) < 0$, and recall that $\eta < \eta_1$ implies $v(x_j^+) > 0$ and $v(x_j^-) < 0$. Now, the construction of the Σ_j^d, Σ_j^m is such that $\Sigma_j^d \subset (x_j^+, \bar{x}_j)$ (resp. $\Sigma_j^m \subset (\tilde{x}_j, x_j^-)$). Furthermore, as stated in (4.43), v is strictly positive in $(x_i^+, \bar{x}_i)$ (resp. strictly negative in $(\tilde{x}_i, x_i^-)$). Therefore, $\Sigma_i^d \cap \Sigma_j^m = \varnothing$ for every $i, j = 1, \ldots, N_v$. Finally, assuming without loss of generality $i < j$, we have that $\Sigma_i^d \subset (x_i^+, \bar{x}_i)$ and $\Sigma_j^d \subset (x_j^+, \bar{x}_j)$ with $v(x) > 0$ for every $x \in (x_i^+, \bar{x}_i) \cup (x_j^+, \bar{x}_j)$. But as $v(x_j^-) < 0$ and $x_j^- \in (x_i^+, x_j^+)$, $(x_i^+, \bar{x}_i)$ and $(x_j^+, \bar{x}_j)$ have to be disjoint, and so must be Σ_i^d and Σ_j^d. In the same way we prove $\Sigma_i^m \cap \Sigma_j^m = \varnothing$ for every $j = 1, \ldots, N_v$, $i \neq j$, concluding the proof of the claim.

Step 3: We now combine the estimates of Step 1 with the estimates of Step 2, and use an argument as the one in (4.1) to deduce the bounds in (4.18) for type I–IV D-intervals. In the case of type II/IV D-intervals a minimization over ω_i^a, ω_i^b is performed to get lower bounds independent of these parameters.

We start by noticing that Lemma 4.6 leads to (4.18) in the case of type 0 D-intervals. Now, we notice that, by (4.3),

$$\begin{aligned}
\int_{L_i \cup \Sigma_i} \big(\varepsilon^4 v_{xx}^2 + \varepsilon^{-2}W(v_x)\big) \, \mathrm{d}x &\geq \int_{L_i} \big(\varepsilon^4 v_{xx}^2 + \varepsilon^{-2}W(v_x)\big) \, \mathrm{d}x \\
&\geq 2\varepsilon\big(H(z_2 - \eta) - H(z_1 + \eta)\big) + n_i\varepsilon\big(H(z_3 - \eta) - H(z_2 + \eta)\big) \\
&\geq \varepsilon(2E_0 + n_iE_1) - c\eta\varepsilon, \tag{4.47}
\end{aligned}$$

for every $i = 1, \ldots, N_v$. For the i's where D_i is of type I, we set $\Sigma_i = \varnothing$, and thus, by (4.28),

$$\int_{L_i \cup \Sigma_i} \varepsilon^{-2} v^2 \, \mathrm{d}x + c\varepsilon\eta \geq 3^{-1}\varepsilon^{-2}h(\alpha_i, \beta_i), \qquad \text{if } D_i \text{ is type I.} \tag{4.48}$$

If $n_i = 0$ or $n_i \geq 4$, we can minimize the right hand side of (4.46) over $\omega_i^a, \omega_i^b \in [0,1]$. This is a convex quadratic function attaining its minimum at $\omega_i^a = \omega_i^b = \frac{1}{2}$. Thus, by (4.26) we get

$$\int_{L_i \cup \Sigma_i} \varepsilon^{-2} v^2 \, \mathrm{d}x + c\varepsilon\eta \geq \frac{1}{12\varepsilon^2}\Big(z_2^2 z_{21}\alpha_i^3 + z_3^2 z_{31}\beta_i^3 + 3\alpha_i\beta_i z_2 z_{31}(\beta_i z_3 + \alpha_i z_2)\Big), \qquad \text{if } D_i \text{ is type II.} \tag{4.49}$$

In case of type III D-intervals, we recall that $\omega_i^b \in \{0,1\}$. Therefore, we assume without loss of generality that $\omega_i^b = 1$ (the case $\omega_i^b = 0$ can be treated similarly) and by (4.46) we deduce

$$
\begin{aligned}
cr_b + \int_{\Sigma_i \cup D_i} v^2 \, \mathrm{d}x &\geq 3^{-1}\Big(\hat{h}_2(\alpha_i, \beta_i, \omega_i^a, 1) + |z_1|^{-1}\big(z_2\alpha_i\omega_i^a + z_3\beta_i\big)^3 + |z_1|^{-1}\big(z_2\alpha_i(1 - \omega_i^a)\big)^3\Big) \\
&\geq 3^{-1}\Big(z_2^2 z_{21}\alpha_i^3 + z_3^2 z_{31}\beta_i^3 - 3\alpha_i^3 f_0\Big(\frac{\beta_i}{\alpha_i}\Big)\Big),
\end{aligned}
\tag{4.50}
$$

where f_0 is defined by

$$f_0(y) = \frac{(y^2 z_{31} z_3 - z_2 z_{21})^2}{4(z_{21} + y z_{31})}. \tag{4.51}$$

We remark that the last lower bound in (4.50) is sharp if and only if $\frac{\beta_i}{\alpha_i} \leq \sqrt{\frac{z_2 z_{21}}{z_3 z_{31}}}$. For type IV D-intervals, (4.46), together with (4.32) yield

$$cr_b + \int_{\Sigma_i \cup D_i} v^2 \, \mathrm{d}x \geq h^*(\alpha_i, \beta_i, \omega_i^a, \omega_i^b) \tag{4.52}$$

where

$$
\begin{aligned}
h^*(\alpha_i, \beta_i, \omega_i^a, \omega_i^b) := \ &3^{-1}|z_1|^{-1}\big(z_2\alpha_i(1 - \omega_i^a) + z_3\beta_i(1 - \omega_i^b)\big)^3 \\
&+ 3^{-1}\Big(\hat{h}_3(\alpha_i, \beta_i, \omega_i^a, \omega_i^b) + |z_1|^{-1}\big(z_2\alpha_i\omega_i^a + z_3\beta_i\omega_i^b\big)^3\Big).
\end{aligned}
$$

We claim that

$$\min_{(\omega^a, \omega^b) \in [0,1]^2} h^*(\alpha_i, \beta_i, \omega^a, \omega^b) \geq 3^{-1}\Big(z_2^2 z_{21}\alpha_i^3 + z_3^2 z_{31}\beta_i^3 - 3\alpha_i^3 f_0\Big(\frac{\beta_i}{\alpha_i}\Big)\Big), \tag{4.53}$$

with f_0 is as in (4.51). Indeed, h^* is a second-order polynomial in ω_i^a, ω_i^b with negative Hessian determinant. Therefore, the minimum among the $\omega_i^a, \omega_i^b \in [0,1]$ is attained at $\omega_i^a \in \{0,1\}$, or $\omega_i^b \in \{0,1\}$. More precisely, if $\frac{\beta_i}{\alpha_i} \geq \sqrt{\frac{z_2 z_{21}}{z_3 z_{31}}}$, the minimum is attained at $\omega_i^a \in \{0,1\}$ and a minimization over $\omega_i^b \in [0,1]$ gives (4.53). The same lower bound can be achieved when $\frac{\beta_i}{\alpha_i} < \sqrt{\frac{z_2 z_{21}}{z_3 z_{31}}}$. Indeed, in this case, the minimum is attained at $\omega_i^b \in \{0,1\}$, and, by using the fact that $z_2 < z_3$, we can bound from below $h^*(\alpha_i, \beta_i, \omega_i^a, \omega_i^b)$ with

$$3^{-1}\Big(\hat{h}_2(\alpha_i, \beta_i, \omega_i^a, \omega_i^b) + |z_1|^{-1}\big(z_2\alpha_i\omega_i^a + z_3\beta_i\big)^3 + |z_1|^{-1}\big(z_2\alpha_i(1 - \omega_i^a)\big)^3\Big),$$

which by (4.50) yields again to (4.53). Therefore, for type III/IV D-intervals (4.50)–(4.53) imply

$$\int_{L_i \cup \Sigma_i} \varepsilon^{-2} v^2 \, \mathrm{d}x + c\varepsilon\eta \geq 3^{-1} \Big(z_2^2 z_{21} \alpha_i^3 + z_3^2 z_{31} \beta_i^3 - 3\alpha_i^3 f_0 \Big(\frac{\beta_i}{\alpha_i}\Big) \Big), \qquad \text{if } D_i \text{ is type III/IV.} \tag{4.54}$$

Finally, by combining (4.47)–(4.49) and (4.54), and by arguing as in (4.1), we obtain (4.18). $\qquad\square$

As a corollary of the previous result, we can prove

Theorem 4.10. *Assume* (H1)–(H8), *and let* $C > 0$. *Then there exists* $\eta_1 = \eta_1(C) \in (0, \eta_0)$ *such that, if* $\eta \in (0, \eta_1)$, $\varepsilon \leq \eta^{q+1}$ *and* $v \in V$ *satisfies* $I^\varepsilon(v) \leq C$, *it holds*

$$\int_0^1 \big(\varepsilon^4 v_{xx}^2 + \varepsilon^{-2} W(v_x) + \varepsilon^{-2} v^2\big) \, \mathrm{d}x \geq \big(A_0 \lambda_2^\eta + B_0 \lambda_3^\eta\big) - c\eta. \tag{4.55}$$

Proof. Thanks to Proposition 4.8 and (H6)–(H8), we have

$$\int_{L_i \cup \Sigma_i} \big(\varepsilon^4 v_{xx}^2 + \varepsilon^{-2} W(v_x) + \varepsilon^{-2} v^2\big) \, \mathrm{d}x \geq A_0 \alpha_i + B_0 \beta_i - c\eta\varepsilon,$$

for every $i = 1, \dots, N_v$. It just remains to provide an estimate for the intervals $D_0 := (0, y_1^-)$ and $D_{N_v+1} := (y_{N_v}^+, 1)$. We deal with the first case, as the second can be treated similarly. If $\mathscr{L}(\{x \in D_0 \colon |v_x(x) - z_k| \leq \eta\}) > R^*\varepsilon$ for some $k = 2, 3$, then, by arguing as in the proof of Lemma 4.6 (*cf.* (4.10)–(4.17)) we deduce

$$\int_{D_0} \big(\varepsilon^4 v_{xx}^2 + \varepsilon^{-2} W(v_x) + \varepsilon^{-2} v^2\big) \, \mathrm{d}x \geq A_0 \alpha_0 + B_0 \beta_0.$$

On the other hand, if $\mathscr{L}(\{x \in D_0 \colon |v_x(x) - z_k| \leq \eta\}) \leq R^*\varepsilon$ for $k = 1, 2$, then

$$\int_{D_0} \big(\varepsilon^4 v_{xx}^2 + \varepsilon^{-2} W(v_x) + \varepsilon^{-2} v^2\big) \, \mathrm{d}x \geq A_0 \alpha_0 + B_0 \beta_0 - (A_0 + B_0) R^* \varepsilon. \tag{4.56}$$

Therefore, recalling that $N_v \leq c\varepsilon^{-1}$ (see Lem. 4.2)

$$\int_0^1 \big(\varepsilon^4 v_{xx}^2 + \varepsilon^{-2} W(v_x) + \varepsilon^{-2} v^2\big) \, \mathrm{d}x \geq A_0 \sum_{i=0}^{N_v+1} \alpha_i + B_0 \sum_{i=0}^{N_v+1} \beta_i - c\eta,$$

which, by the definition of the α_i's, β_i's, and of $\lambda_2^\eta, \lambda_3^\eta$ (see (4.7)) coincides with (4.55). $\qquad\square$

5. The second Γ-limit

In this section we prove the Γ-limit for I^ε, that is a second Γ-limit for $\mathcal{E}^\varepsilon$, as stated in Theorem 1.2. The first step is to prove compactness for the family of energy functionals I^ε.

Proposition 5.1. *Assume* (H1)–(H5). *Let* $C > 0$, $\varepsilon_j \downarrow 0$ *and* $(u_j, \nu_j) \in L^2(0, 1) \times L^\infty_{w^*}(0, 1; \mathcal{M})$ *be such that*

$$I^{\varepsilon_j}(u_j, \nu_j) \leq C, \qquad \forall j \in \mathbb{N}. \tag{5.1}$$

Then, up to a subsequence, (u_j, ν_j) converges to (u, ν) in $L^2(0,1) \times L^\infty_{w*}(0,1;\mathcal{M})$. Furthermore, $u = 0$ and $\nu \in \mathrm{GYM}^\infty(u)$ satisfies

$$\nu_x = \lambda_1(x)\delta_{z_1} + \lambda_2(x)\delta_{z_2} + \lambda_3(x)\delta_{z_3}, \qquad a.e.\ x \in (0,1), \tag{5.2}$$

where $\lambda_1, \lambda_2, \lambda_3 \in L^\infty(0,1;[0,1])$ are such that

$$\lambda_1(x) + \lambda_2(x) + \lambda_3(x) = 1, \qquad and \qquad \sum_{k=1}^3 z_k \lambda_k(x) = 0, \tag{5.3}$$

for a.e. $x \in (0,1)$.

Proof. We first notice that (5.1) implies strong convergence of u_j to $u = 0$ in $L^2(0,1)$. Furthermore, as $W(s) \geq c_1|s|^p - c_2$, we also have

$$\|u_{j,x}\|_{L^p} \leq c, \tag{5.4}$$

and, therefore, up to a subsequence $u_j \to 0$, weakly in $W^{1,p}_0(0,1)$. In fact, (5.4) also implies that, up to a further non-relabelled subsequence, $u_{j,x}$ generates a gradient Young measure ν_x, weak* limit of ν_j in $L^\infty_{w*}(0,1;\mathcal{M})$. Defined Σ^η_j as

$$\Sigma^\eta_j := \left\{ x \in (0,1) \colon |u_{j,x}(x) - z_k| > \eta,\ k = 1,2,3 \right\}, \tag{5.5}$$

for some $\eta \in (0, \eta_0)$, by (H5) we have

$$C\varepsilon_j^2 \geq \int_0^1 W(u_{j,x})\,\mathrm{d}x \geq \int_{\Sigma^\alpha_j} W(u_{j,x})\,\mathrm{d}x \geq c_0 \eta^q \mathscr{L}(\Sigma^\eta_j).$$

This implies

$$\mathscr{L}(\Sigma^\eta_j) \leq c\frac{\varepsilon_j^2}{\eta^q}, \tag{5.6}$$

which is convergence in measure of $u_{j,x}$ to $\mathcal{Z}$. Therefore, ν_x is a probability measure supported on $\mathcal{Z}$ (see e.g., [1]), and hence $\nu_x = \lambda_1(x)\delta_{z_1} + \lambda_2(x)\delta_{z_2} + \lambda_3(x)\delta_{z_3}$ for a.e. $x \in (0,1)$, as claimed. The fact that ν is a probability measure implies the first identity in (5.3). By ([20], Thm. 8.7), we also know that ν is the gradient Young measure related to $u = 0$, and therefore the average of ν must be 0, that is $\sum_{k=1}^3 \lambda_k(x)z_k = 0$ for a.e. $x \in (0,1)$, which is the last identity in (5.3). $\square$

Given a sequence $u_j \in W^{1,p}_0(0,1)$ and $\eta > 0$, let us define

$$\lambda^\eta_{k,j} := \mathscr{L}\left(\left\{x \in (0,1)\colon |u_{j,x}(x) - z_k| \leq \eta\right\}\right), \qquad k = 1,2,3.$$

The following result is used below:

Lemma 5.2. *Assume (H1)–(H5). Let $C > 0$, $\eta \in (0,\eta_0)$, $\varepsilon_j \downarrow 0$ and $(u_j, \nu_j) \in L^2(0,1) \times L^\infty_{w*}(0,1;\mathcal{M})$ be a sequence converging to $(0,\nu)$ in $L^2(0,1) \times L^\infty_{w*}(0,1;\mathcal{M})$ such that $I^{\varepsilon_j}(u_j) \leq C$ for each j. Then ν satisfies*

(5.2)–(5.3) *and*

$$\lim_j \lambda_{k,j}^{\eta} = \int_0^1 \lambda_k \, \mathrm{d}x. \tag{5.7}$$

Proof. The fact that ν satisfies (5.2)–(5.3) follows directly from Proposition 5.1. We just need to prove (5.7). Let us consider a continuous function $f_k \colon \mathbb{R} \to [0,1]$, which is equal to 1 for those s such that $|s - z_k| \leq \eta$, and equal to 0 for $|s - z_k| \geq \eta_0$. We have

$$\int_0^1 \lambda_k \, \mathrm{d}x = \int_0^1 \langle \nu, f_k \rangle \, \mathrm{d}x = \lim_j \int_0^1 \langle \nu_j, f_k \rangle \, \mathrm{d}x = \lim_j \int_0^1 f_k(u_{j,x}) \, \mathrm{d}x.$$

Now, we notice that, as $\eta_0 < \frac{|z_k - z_h|}{2}$ for each $h \neq k \in \{1,2,3\}$ (*cf.* (H5)),

$$\int_0^1 f_k(u_{j,x}) \, \mathrm{d}x = \lambda_k^{\eta}(u_j) + r,$$

where $0 < r \leq \mathscr{L}\big(\Sigma_j^{\eta}\big) \leq c\varepsilon_j^2 \eta^{-q}$. Here, Σ_j^{η} is as in (5.5), and was estimated by means of (5.6). Therefore, collecting all previous identities, we finally get

$$\lim_j \lambda_k^{\eta}(u_j) = \int_0^1 \lambda_k \, \mathrm{d}x,$$

which concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

5.1. Proof of Theorem 1.2

By ([8], Rem. 1.29) we just need to show the Γ-$\limsup$ inequality for every $\nu \in X$, where X is the set containing all $\nu \in \mathrm{GYM}^{\infty}(0)$, with $\operatorname{supp}\nu \subset \mathcal{Z}$, and such that $\lambda_2 := \nu(z_2) \colon (0,1) \to \{0, z_{21}^{-1}\}$ is constant on every sub-interval (x_i, x_{i+1}), $i = 1, \ldots, n-1$, for some partition $0 = x_1 \leq \cdots \leq x_n = 1$ of $(0,1)$ and some $n \in \mathbb{N}$. Indeed, X is dense with respect to the weak$*$ topology of $L_{w*}^{\infty}(0,1;\mathcal{M})$ in the set containing all $\nu \in \mathrm{GYM}^{\infty}(0)$ such that $\operatorname{supp}\nu \subset \mathcal{Z}$. This is because the space of piecewise constant functions in $L^{\infty}(0,1;\{0,z_{21}^{-1}\})$ is weak$*$ dense in the space of piecewise constant functions in $L^{\infty}(0,1;(0,z_{21}^{-1}))$ (*cf.* [13], Pb. 1, Sect. 8.6), which is weakly$*$ dense in $L^{\infty}(0,1;(0,z_{21}^{-1}))$. On X the Γ-$\limsup$ follows directly by Proposition 3.1 and Lemma 5.2. Therefore, we just need to prove the Γ-$\liminf$ inequality. In order to do that, we need to consider a generic sequence ε_j converging to 0, a sequence $u_j \in V$ converging strongly in $L^2(0,1)$ to $u \in L^2(0,1)$ and a sequence of parametrized measures $\nu_j \in L_{w*}^{\infty}(0,1;\mathcal{M})$ converging weakly$*$ to ν in $L_{w*}^{\infty}(0,1;\mathcal{M})$. If $\liminf_j I^{\varepsilon_j}(u_j) = \infty$, the $\liminf$ inequality is trivial. Otherwise, up to a subsequence, we can assume the existence of $C > 0$, independent of ε_j, such that

$$I^{\varepsilon_j}(u_j) \leq C, \qquad \forall j \in \mathbb{N}.$$

In this case, Proposition 5.1 implies that $u = 0$ and that $\nu \in \mathrm{GYM}^{\infty}(0)$ satisfies (5.2)–(5.3). Now, Theorem 4.10 guarantees the existence of $\eta_1 > 0$ such that, fixed $\eta \in (0, \eta_1)$,

$$\int_0^1 \big(\varepsilon_j^4 u_{j,xx}^2 + \varepsilon_j^{-2} W(u_{j,x}) + \varepsilon_j^{-2} u_j^2\big) \, \mathrm{d}x \geq \big(A_0 \lambda_{2,j}^{\eta} + B_0 \lambda_{3,j}^{\eta}\big) - c\eta,$$

for all $\varepsilon_j < \eta^{q+1}$. Now, by taking the $\liminf$ on both sides, and recalling Lemma 5.2, we deduce

$$\liminf_{j} \int_0^1 \left(\varepsilon_j^4 u_{j,xx}^2 + \varepsilon_j^{-2} W(u_{j,x}) + \varepsilon_j^{-2} u_j^2 \right) dx \geq A_0 \int_0^1 \lambda_2 \, dx + B_0 \int_0^1 \lambda_3 \, dx - c\eta.$$

The arbitrariness of η yields to the desired Γ-$\liminf$ inequality.

6. Selecting minimizing sequences without Γ-convergence

In this section we prove Theorem 1.3. In order to do this we strongly rely on the estimates of Section 4, to which we refer the reader for the notation. We start with the following theorem:

Theorem 6.1. *Assume* (H1)–(H5) *and* $z_3 \leq 3|z_1|$. *Then, there exist* $\varepsilon_1 > 0$ *and* $\xi > 0$ *such that, if* $\varepsilon < \varepsilon_1$

$$\inf_{v \in V} \int_0^1 \left(\varepsilon^4 v_{xx}^2 + \varepsilon^{-2} W(v_x) + \varepsilon^{-2} v^2 \right) dx = A_0 z_{21}^{-1} + o(\varepsilon^{\xi}). \tag{6.1}$$

Furthermore, every minimizer u *of* I^{ε} *satisfies*

$$\mathscr{L}\left(\{ x \in (0,1) \colon |u_x(x) - z_3| \leq \varepsilon^{\frac{1}{q+1}} \} \right) \leq c\varepsilon^{\xi}. \tag{6.2}$$

Proof. We first notice that Proposition 3.1 implies

$$\inf_{v \in V} \int_0^1 \left(\varepsilon^4 v_{xx}^2 + \varepsilon^{-2} W(v_x) + \varepsilon^{-2} v^2 \right) dx \leq A_0 z_{21}^{-1} + c\varepsilon^{\zeta}. \tag{6.3}$$

Let us assume $\eta \in (0, \eta_1)$, with η_1 as in the statement of Proposition 4.8, $\varepsilon \leq \eta^{q+1}$, and define

$$K := \frac{z_{31}}{z_{21}} \sqrt[3]{\frac{(E_0 + E_1)^2}{E_0^2}} > 1.$$

We notice that $B_0 > K A_0$. We now look for a lower bound for the energy I^{ε} of the type (4.55), but with B_0 replaced by $K A_0$. This new bound does not rely on (H6)–(H8), but is deduced by strongly exploiting the estimates in Section 4.

It can be checked by using $z_2 < z_3$, $z_{21} < z_{31}$, $z_2 z_{31} < z_3 z_{31}$, that f_7, f_8 given in (H7)–(H8) satisfy

$$f_7^{\frac{1}{3}}(y) \geq A_0 + K A_0 y, \qquad f_8^{\frac{1}{3}}(y) \geq A_0 + K A_0 y, \qquad \text{for every } y \geq 0. \tag{6.4}$$

Furthermore, as we assumed $z_3 \leq 3|z_1|$, we also have

$$f_6^{\frac{1}{3}}(y) \geq A_0 + K A_0 y, \qquad \text{for every } y \geq 0. \tag{6.5}$$

Therefore, collecting the estimates for every L-interval, from Proposition 4.8 together with (6.4) and (6.5) and the fact that $B_0 > K A_0$, we deduce

$$\int_{\sum_i (L_i \cup \Sigma_i)} \left(\varepsilon^4 v_{xx}^2 + \varepsilon^{-2} W(v_x) + \varepsilon^{-2} v^2 \right) dx + c\eta \geq A_0 \sum_i (\alpha_i + K \beta_i).$$

Here we also made use of Lemma (4.2) to bound N_v with $c\varepsilon^{-1}$. Finally, recalling (4.56) we deduce

$$\int_0^1 \left(\varepsilon^4 v_{xx}^2 + \varepsilon^{-2} W(v_x) + \varepsilon^{-2} v^2 \right) \mathrm{d}x + c\eta \geq A_0 (\lambda_2^\eta + K \lambda_3^\eta). \tag{6.6}$$

Now, thanks to (4.12) and (4.13), and the fact that $\varepsilon \leq \eta^{q+1}$, we can write

$$1 \geq \lambda_1^\eta + \lambda_2^\eta + \lambda_3^\eta = 1 - \mathscr{L}(\Sigma^\eta) \geq 1 - c\varepsilon\eta, \tag{6.7}$$

while, on the other hand,

$$0 = \int_0^1 v_x \, \mathrm{d}x \leq \sum_{k=1}^3 (z_k + \eta) \lambda_k^\eta + \tilde{r} \leq \sum_{k=1}^3 z_k \lambda_k^\eta + c\eta, \qquad 0 = \int_0^1 v_x \, \mathrm{d}x \geq \sum_{k=1}^3 z_k \lambda_k^\eta - c\eta \tag{6.8}$$

where $\tilde{r}$ is defined as in (4.37), and has been bounded according to the estimate in (4.40). By combining (6.7) and (6.8) we are led to

$$z_{31}^{-1}(1 - z_{21}\lambda_2^\eta) + c\eta \geq \lambda_3^\eta \geq z_{31}^{-1}(1 - z_{21}\lambda_2^\eta) - c\eta, \tag{6.9}$$

so that, by (6.6),

$$\int_0^1 \left(\varepsilon^4 v_{xx}^2 + \varepsilon^{-2} W(v_x) + \varepsilon^{-2} v^2 \right) \mathrm{d}x + c\eta \geq A_0 \left(\lambda_2^\eta + \frac{K}{z_{31}}(1 - z_{21}\lambda_2^\eta) \right). \tag{6.10}$$

The right-hand side of the above inequality is a decreasing function of λ_2^η so minimized by the biggest admissible λ_2^η. But as $\lambda_3^\eta \geq 0$, (6.9) entails $\lambda_2^\eta \leq z_{21}^{-1} + c\eta$, thus implying

$$\int_0^1 \left(\varepsilon^4 v_{xx}^2 + \varepsilon^{-2} W(v_x) + \varepsilon^{-2} v^2 \right) \mathrm{d}x + c\eta \geq A_0 z_{21}^{-1}. \tag{6.11}$$

Choosing $\eta = \varepsilon^{\frac{1}{q+1}}$ and $\varepsilon_1 = \min\{\eta_1^{q+1}, \varepsilon_0\}$, where ε_0 is as in Proposition 3.1, we complete the proof of (6.1). Finally, combining (6.3) and (6.10) and (6.11) we get

$$c\varepsilon^\xi + A_0 z_{21}^{-1} \geq A_0 \lambda_2^\eta + z_{31}^{-1} K A_0 (1 - z_{21}\lambda_2^\eta) \geq A_0 z_{21}^{-1} - c\varepsilon^\xi,$$

which is

$$\left| (\lambda_2^\eta - z_{21}^{-1})(1 - K\frac{z_{21}}{z_{31}}) \right| \leq c\varepsilon^\xi.$$

This implies $|\lambda_2^\eta - z_{21}^{-1}| \leq c\varepsilon^\xi$ which, by (6.9), concludes the proof. $\qquad\square$

6.1. Proof of Theorem 1.3

This follows as a corollary of Theorem 6.1.

By Proposition 5.1 together with (6.1) we know that every $u_{\varepsilon_j} \in V$ sequence of minimizers for I^{ε_j}, and hence of minimizers for $\mathcal{E}^{\varepsilon_j}$, generates, up to a subsequence, a gradient Young measure $\nu \in \mathrm{GYM}^p(0)$ as $\varepsilon_j \to 0$, and

that $\operatorname{supp} \nu_x \subset \mathcal{Z}$ almost everywhere in $(0,1)$. As a consequence, ν_x is of the form

$$\nu_x = \lambda_1(x)\delta_{z_1} + \lambda_2(x)\delta_{z_2} + \lambda_3(x)\delta_{z_3},$$

with the λ_i's satisfying (5.3) for a.e. $x \in (0,1)$. Therefore we just need to show that $\lambda_3 = 0$ a.e. in $(0,1)$. Let us consider a continuous function $f_3 \colon \mathbb{R} \to [0,1]$ which is equal to 1 for all $s \in \mathbb{R}$ such that $|s - z_3| \leq \frac{\eta_0}{2}$, and 0 if $|s - z_3| \geq \eta_0$. By arguing as in the proof of Lemma 5.2, we get

$$0 \leq \int_0^1 \lambda_3 \, \mathrm{d}x = \lim_j \int_0^1 f_3(u_{\varepsilon_j, x}) \, \mathrm{d}x \leq \lim_j \left(\lambda_3^\eta(u_{\varepsilon_j}) + c\varepsilon_j^2 \eta^{-q} \right).$$

After choosing $\eta = \varepsilon_j^{\frac{1}{q+1}}$, (6.2) gives the sought result.

7. Some remarks on the assumptions

It is worth spending some words on assumptions (H6)–(H8). Hypothesis (H6) is needed in our construction of a lower bound, but it might be possible to remove it by making the arguments of Section 4 more involved. It is easy to check that it fails whenever $z_3 > 3|z_1|$. On the other hand, as mentioned in the introduction, it turns out that (H7)–(H8) are necessary conditions in order to prove Theorem 1.2, and the second Γ-limit would have a different form without these assumptions. Indeed, as explained in the introduction, these hypotheses guarantee that the microstructures used in the construction of Proposition 3.1 are energetically preferable to those constructed in the following Propositions, and shown in Figure 7.

Proposition 7.1. *Assume* (H7) *is not satisfied. Then, there exist* $\lambda_1, \lambda_2, \lambda_3 \in (0,1)$ *satisfying* (5.3), $u_\varepsilon \in V$ *such that* $(u_\varepsilon, \delta_{u_{\varepsilon,x}}) \to (0, \nu)$ *in* $L^2(0,1) \times L^\infty_{w^*}(0,1;\mathcal{M})$, *where* $\nu_x = \lambda_1 \delta_{z_1} + \lambda_2 \delta_{z_2} + \lambda_3 \delta_{z_3}$ *a.e. in* $(0,1)$, *and*

$$\limsup_{\varepsilon \downarrow 0} I^\varepsilon(u_\varepsilon) < A_0 \lambda_2 + B_0 \lambda_3.$$

Proof. The proof of the Proposition is very similar to the one of Proposition 3.1 in many details. For this reason we skip some long computation and just give the idea of the proof.

Let $\hat{y} \geq 0$ such that the inequality in (H7) does not hold. Let us choose

$$\lambda_2 = (\hat{y}z_{31} + z_{21})^{-1}, \qquad \lambda_3 = z_{31}^{-1} - \frac{z_{21}}{z_{31}}\lambda_2, \qquad \lambda_1 = 1 - \lambda_2 - \lambda_3, \tag{7.1}$$

so that $\frac{\lambda_3}{\lambda_2} = \hat{y}$. It is easy to check that $\lambda_1, \lambda_2, \lambda_3 \in (0,1)$ and satisfy (5.3). We divide $(0,1)$ in N_ε subintervals (x_i, x_{i+1}) of length N_ε^{-1}, where $x_i = iN_\varepsilon^{-1}$ for $i = 0, \dots, N_\varepsilon$. On every interval we construct v_ε as a suitable continuous approximation (see the proof of Prop. 3.1) of the function (see the derivative of the function in Fig. 7a)

$$\hat{v}_\varepsilon(s) = \begin{cases} z_2, & \text{if } 0 \leq |s - (2N_\varepsilon)^{-1}| \leq \lambda_2(2N_\varepsilon)^{-1}, \\ z_3, & \text{if } \lambda_2 N_\varepsilon^{-1} < |s - (2N_\varepsilon)^{-1}| \leq \lambda_2(2N_\varepsilon)^{-1} + \lambda_3(2N_\varepsilon)^{-1}, \\ z_1, & \text{if } \lambda_2(2N_\varepsilon)^{-1} + \lambda_3(2N_\varepsilon)^{-1} \leq |s - (2N_\varepsilon)^{-1}| \leq (2N_\varepsilon)^{-1}. \end{cases}$$

We remark that, as in the proof of Proposition 3.1, v_ε must satisfy $\int_0^{N_\varepsilon^{-1}} v_\varepsilon(s) \, \mathrm{d}s = 0$, and $v_\varepsilon(0) = v_\varepsilon(N_\varepsilon^{-1})$. Therefore, after defining w_ε as the N_ε^{-1}-periodic extension of v_ε, we construct u_ε as in (3.6). Now, an argument
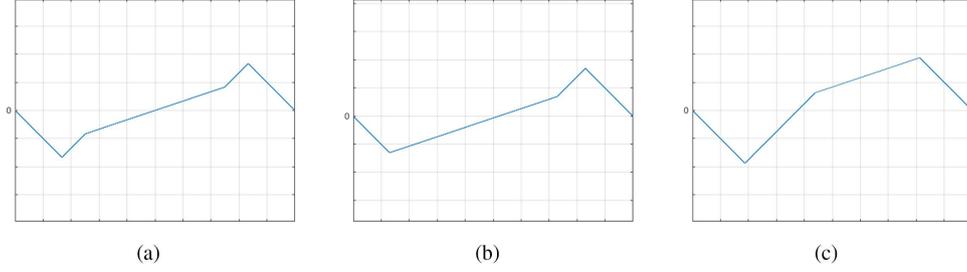
FIGURE 7. Microstructure of lower energy in case (H7) does not hold (*panel a*). Microstructures of low energy in case (H8) does not hold are represented in *panels b and c*, respectively, in the case where $\hat{y} \leq \sqrt{\frac{z_2 z_{21}}{z_3 z_{31}}}$ and $\hat{y} > \sqrt{\frac{z_2 z_{21}}{z_3 z_{31}}}$.

as the one in the proof of Proposition 3.1 allows us to prove that

$$I^\varepsilon(u_\varepsilon) \leq 3^{\frac{2}{3}} 2^{-\frac{2}{3}} (E_0 + 2E_1)^{\frac{2}{3}} \left(z_2^2 z_{21} \lambda_2^3 + z_3^2 z_{31} \lambda_3^3 + 3\lambda_2 \lambda_3 z_2 z_{21} (\lambda_3 z_3 + \lambda_2 z_2) \right)^{\frac{1}{3}} + c\varepsilon^\xi = \lambda_2 f_7(\hat{y}) + c\varepsilon^\xi$$

for some $\xi > 0$, and that (3.2) holds. Here we have used that, by construction, $\frac{\lambda_3}{\lambda_2} = \hat{y}$. As $\hat{y}$ contradicts (H7), we have

$$I^\varepsilon(u_\varepsilon) < A_0 \lambda_2 + B_0 \lambda_3 + c\varepsilon^\xi. \tag{7.2}$$

Furthermore, by arguing as to get (3.2), we have

$$\left| \mathscr{L}\big((0,1) \cap \{|u_x - z_2| \leq \sigma\}\big) - \lambda_2 \right| + \left| \mathscr{L}\big((0,1) \cap \{|u_x - z_3| \leq \sigma\}\big) - \lambda_3 \right| \leq c\varepsilon^\zeta. \tag{7.3}$$

Thus, taking the lim sup in (7.2), by Lemma 5.2 we obtain the sought result. □

Proposition 7.2. *Assume* (H8) *is not satisfied. Then, there exist* $\lambda_1, \lambda_2, \lambda_3 \in (0,1)$ *satisfying* (5.3), $u_\varepsilon \in V$ *such that* $(u_\varepsilon, \delta_{u_{\varepsilon,x}}) \to (0, \nu)$ *in* $L^2(0,1) \times L^\infty_{w^*}(0,1; \mathcal{M})$, *where* $\nu_x = \lambda_1 \delta_{z_1} + \lambda_2 \delta_{z_2} + \lambda_3 \delta_{z_3}$ *a.e. in* $(0,1)$, *and*

$$\limsup_{\varepsilon \downarrow 0} I^\varepsilon(u_\varepsilon) < A_0 \lambda_2 + B_0 \lambda_3.$$

Proof. Again, the proof of this proposition is very similar to, Propositions 3.1 and 7.1 in many details. For this reason we skip some long computation and just give the idea of the proof.

Let $\hat{y} \geq 0$ such that (H8) does not hold. Let us choose $\lambda_1, \lambda_2, \lambda_3$ that satisfy (5.3) as in (7.1). Again, we divide $(0,1)$ in M_ε subintervals (x_i, x_{i+1}) of length M_ε^{-1}, where $x_i = iM_\varepsilon^{-1}$ for $i = 0, \ldots, M_\varepsilon$. We have two cases:

$$0 \leq \hat{y} \leq \sqrt{\frac{z_2 z_{21}}{z_3 z_{31}}} \qquad \text{and} \qquad \sqrt{\frac{z_2 z_{21}}{z_3 z_{31}}} < \hat{y}.$$

In the first case (see the derivatives of the function in Fig. 7b), we define $\omega^a := (z_2 z_{21} - z_3 z_{31} \hat{y}^2)(2z_2(z_{21} + \hat{y} z_{31}))^{-1}$ and

$$
\hat{v}_a^\varepsilon(s) = \begin{cases} z_1, & \text{if } 0 \leq s < \frac{z_2}{z_1}(1 - \omega^a)\lambda_2 M_\varepsilon^{-1}, \\ z_2, & \text{if } \frac{z_2}{z_1}(1 - \omega^a)\lambda_2 M_\varepsilon^{-1} \leq s < \left(\frac{z_2}{z_1}(1 - \omega^a) + 1\right)\lambda_2 M_\varepsilon^{-1}, \\ z_3, & \text{if } \left(\frac{z_2}{z_1}(1 - \omega^a) + 1\right)\lambda_2 M_\varepsilon^{-1} \leq s < \left(\frac{z_2}{z_1}(1 - \omega^a) + 1\right)\lambda_2 M_\varepsilon^{-1} + \lambda_3 M_\varepsilon^{-1}, \\ z_1, & \text{if } \left(\frac{z_2}{z_1}(1 - \omega^a) + 1\right)\lambda_2 M_\varepsilon^{-1} + \lambda_3 M_\varepsilon^{-1} \leq s. \end{cases}
$$

In the second (see the derivatives of the function in Fig. 7c), we define $\omega^b := -\omega^a$ and

$$
\hat{v}_b^\varepsilon(s) = \begin{cases} z_1, & \text{if } 0 \leq s < \frac{z_3}{z_1}(1 - \omega^b)\lambda_3 M_\varepsilon^{-1}, \\ z_3, & \text{if } \frac{z_3}{z_1}(1 - \omega^b)\lambda_3 M_\varepsilon^{-1} \leq s < \left(\frac{z_3}{z_1}(1 - \omega^b) + 1\right)\lambda_3 M_\varepsilon^{-1}, \\ z_2, & \text{if } \left(\frac{z_3}{z_1}(1 - \omega^b) + 1\right)\lambda_3 M_\varepsilon^{-1} \leq s < \left(\frac{z_3}{z_1}(1 - \omega^b) + 1\right)\lambda_3 M_\varepsilon^{-1} + \lambda_2 M_\varepsilon^{-1}, \\ z_1, & \text{if } \left(\frac{z_3}{z_1}(1 - \omega^b) + 1\right)\lambda_3 M_\varepsilon^{-1} + \lambda_2 M_\varepsilon^{-1} \leq s. \end{cases}
$$

Now, let us consider suitable continuous approximations v_a^ε and v_b^ε of $\hat{v}_a^\varepsilon$ and $\hat{v}_b^\varepsilon$, which can be obtained in the same way as the one in Proposition 3.1. Again, we remark that v_l^ε for $l = a, b$ must satisfy $v_l^\varepsilon(0) = v_l^\varepsilon(M_\varepsilon^{-1})$ and $\int_0^{M_\varepsilon^{-1}} v_l^\varepsilon(s)\,\mathrm{d}s = 0$. Let $w_a^\varepsilon, w_b^\varepsilon$ be the M_ε^{-1}-periodic extensions of v_a^ε and v_b^ε, respectively, and define u_ε as in (3.6). An argument as the one in Proposition 3.1 hence allows proving

$$
I^\varepsilon(u_\varepsilon) \leq 3^{\frac{2}{3}}(E_0 + E_1)^{\frac{2}{3}}\left(z_2^2 z_{21}\lambda_2^3 + z_3^2 z_{31}\lambda_3^3 - 3\lambda_2^3 f_0(\hat{y})\right)^{\frac{1}{3}} + c\varepsilon^\xi = \lambda_2 f_8^{\frac{1}{3}}(\hat{y}) + c\varepsilon^\xi,
$$

for some $\xi > 0$, for some $\varepsilon_0 > 0$, and for every $\varepsilon \in (0, \varepsilon_0)$. Here f_0 is as in (4.51), and we used the fact that, by construction, $\frac{\lambda_3}{\lambda_2} = \hat{y}$. The fact that $\hat{y}$ violates the inequality in (H8) yields

$$
I^\varepsilon(u_\varepsilon) < A_0\lambda_2 + B_0\lambda_3 + c\varepsilon^\xi. \tag{7.4}
$$

Furthermore, by arguing as in the proof of Proposition 3.1 we can prove estimates as the ones in (7.3). Therefore, by taking the $\limsup$ in (7.4) and exploiting Lemma 5.2 we conclude the proof of the proposition. $\square$

7.1. Two examples

An easy example where hypotheses (H1)–(H8) hold is when

$$
W(s) = (s - 1)^2(s + 1)^2(s - 3^{-1})^2.
$$

Indeed, in this case $E_0 \approx 1.054$, $E_1 \approx 0.165$, $A_0 \approx 0.718$, $B_0 \approx 1.883$, $z_1 = -1$, $z_2 = \frac{1}{3}$, $z_3 = 1$. It is trivial to check that in this context (H1)–(H5) hold. Hypotheses (H6)–(H8) are here verified graphically (*cf.* Fig. 8). It can be proved that (H6)–(H8) hold for W of the form

$$
W(s) = (s - 1)^2(s + 1)^2(s - z_2)^2,
$$

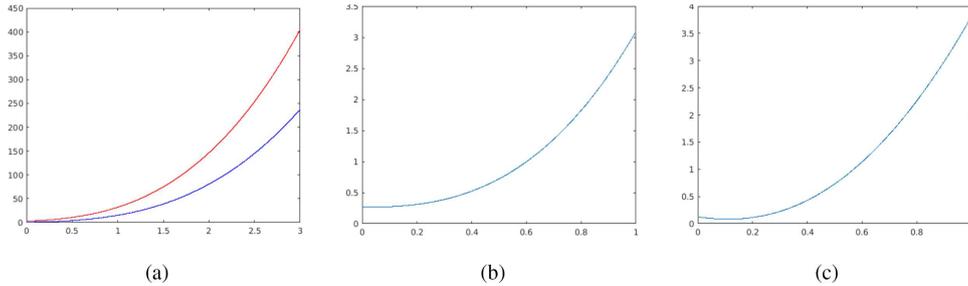whenever $z_2 \in (-0.49, 0.49)$. The bound on z_2 is not sharp.

FIGURE 8. Verification of the hypotheses (H6)–(H8) for the examples of Section 7.1. *Panel a*: the graphical verification of (H6) for the two examples: in blue the case $z_2 = \frac{1}{3}$, in red the one with $z_2 = \frac{1}{2}$. *Panels b and c* verify (H7) and (H8) in the example with $z_2 = \frac{1}{3}$.

On the other hand, let us consider

$$W(s) = (s-1)^2(s+1)^2(s-2^{-1})^2,$$

where, in our notation, $E_0 \approx 1.406$, $E_1 \approx 0.073$, $A_0 \approx 1.186$, $B_0 \approx 2.143$, $z_1 = -1$, $z_2 = \frac{1}{2}$, $z_3 = 1$. In this case (H1)–(H6) hold (*cf.* Fig. 8a). However, hypotheses (H7) and (H8) fail, respectively, in a neighbourhood of $\hat{y}_7 = 0.585$ and $\hat{y}_8 = 0.204$. Here, it is energetically very cheap to pass from z_2 to z_3, so other microstructures are energetically favourable for $\frac{\lambda_3}{\lambda_2}$ close to $\hat{y}_7$ or $\hat{y}_8$. Nonetheless, as $z_3 \leq 3|z_1|$, thanks to Theorem 1.3, we can still select minimizing gradient Young measures for $\mathcal{E}^0$ by means of vanishing interfacial energy.

References

[1] J.M. Ball, A version of the fundamental theorem for Young measures. PDEs and Continuum Models of Phase Transitions. Vol. 344 of *Lecture Notes in Physics*. Springer, Berlin (1989) 207–215.

[2] J.M. Ball and E.C.M. Crooks, Local minimizers and planar interfaces in a phase-transition model with interfacial energy. *Calc. Var. Partial Differ. Equ.* **40** (2011) 501–538.

[3] J.M. Ball, P.J. Holmes, R.D. James, R.L. Pego and P.J. Swart, On the dynamics of fine structure. *J. Nonlinear Sci.* **1** (1991) 17–70.

[4] J.M. Ball and R.D. James, Fine phase mixtures as minimizers of energy. *Arch. Ratio. Mech. Anal.* **100** (1987) 13–52.

[5] J.M. Ball and R.D. James, Proposed experimental tests of a theory of fine microstructure and the two-well problem. *Philos. Trans. R. Soc. Lond. A* **338** (1992) 389–450.

[6] J.M. Ball and K. Koumatos, An investigation of non-planar austenite-martensite interfaces. *Math. Models Methods Appl. Sci.* **24** (2014) 1937–1956.

[7] K. Bhattacharya, Microstructure of Martensite. Why It Forms and How It Gives Rise to the Shape-Memory Effect, Oxford Series on Materials Modelling. Oxford University Press, Oxford (2003).

[8] A. Braides, Γ-convergence for beginners. Vol. 22 of *Oxford Lecture Series in Mathematics and Its Applications.* Oxford University Press, Oxford (2002).

[9] M. Cicalese, E.N. Spadaro and C.I. Zeppieri, Asymptotic analysis of a second-order singular perturbation model for phase transitions. *Calc. Var. Partial Diff. Equ.* **41** (2011) 127–150.

[10] S. Conti, I. Fonseca and G. Leoni, A Γ-convergence result for the two-gradient theory of phase transitions. *Commun. Pure Appl. Math.* **55** (2002) 857–936.

[11] S. Conti and B. Schweizer, Rigidity and gamma convergence for solid–solid phase transitions with SO(2) invariance. *Commun. Pure Appl. Math.* **59** (2006) 830–868.

[12] G. Dolzmann and S. Müller, Microstructures with finite surface energy: the two-well problem. *Arch. Ratio. Mech. Anal.* **132** (1995) 101–141.

[13] L.C. Evans, Partial Differential Equations, 2nd edn. Vol. 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI (2010).

[14] L. Modica and S. Mortola, Un esempio di Γ-convergenza. *Boll. Un. Mat. Ital. B* **14** (1977) 285–299.

[15] S. Müller, Minimizing sequences for nonconvex functionals, phase transitions and singular perturbations. Problems Involving Change of Type (Stuttgart, 1988). Vol. 359 of *Lecture Notes in Physics*. Springer, Berlin (1990) 31–44.

[16] S. Müller. Singular perturbations as a selection criterion for periodic minimizing sequences. *Calc. Var. Partial Diff. Equ.* **1** (1993) 169–204.

[17] S. Müller, Variational models for microstructure and phase transitions. Calculus of Variations and Geometric Evolution Problems. Vol. 1713 of *Lecture Notes in Mathematics*. Springer, Berlin (1999) 85–210.

[18] S. Müller and V. Šverák, Convex integration with constraints and applications to phase transitions and partial differential equations. *J. Eur. Math. Soc.* **1** (1999) 393–422.

[19] R.A. Nicolaides and N.J. Walkington, Strong convergence of numerical solutions to degenerate variational problems. *Math. Comput.* **64** (1995) 117–127.

[20] P. Pedregal, Parametrized measures and variational principles. Vol. 30 of *Progress in Nonlinear Differential Equations and Their Applications*. Birkhäuser Verlag, Basel (1997).

[21] M.A. Sychev, A new approach to Young measure theory, relaxation and convergence in energy. *Ann. Inst. Henri Poincaré Anal. Non Linéaire.* **16** (1999) 773–812.